A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF ÇANKIRI KARATEKİN UNIVERSITY

MODIFICATION AND SOLUTION OF THE DYNAMICAL SYSTEM OF COVID-19

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

BY

ALYAA MHMOOD FARAJ FARAJ

ÇANKIRI

2023

MODIFICATION AND SOLUTION OF THE DYNAMICAL SYSTEM OF COVID-19

By Alyaa Mhmood Faraj FARAJ

January 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

Advisor **:** Assoc. Prof. Dr. Ufuk ÖZTÜRK
Co-Advisor **:** Prof. Dr. Saad Naji AL-AZZAWI

Examining Committee Members:

Chairman **:** Assoc. Prof. Dr. Ufuk ÖZTÜRK
Mathematics
Bolu Abant İzzet Baysal University

Member **:** Prof. Dr. Esra Betül KOÇ ÖZTÜRK
Mathematics
Bolu Abant Izzet Baysal University

Member **:** Asst. Prof. Dr. Kahraman Esen ÖZEN
Mathematics
Çankırı Karatekin University

Approved for the Graduate School of Natural and Applied Sciences

Prof. Dr. İbrahim ÇİFTÇİ
Director of Graduate School

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Alyaa Mhmood Faraj FARAJ

ABSTRACT

MODIFICATION AND SOLUTION OF THE DYNAMICAL SYSTEM OF COVID-19

Alyaa Mhmood Faraj FARAJ
Master of Science in Mathematics
Advisor: Assoc. Prof. Dr. Ufuk ÖZTÜRK
Co-Advisor: Prof. Dr. Saad Naji AL-AZZAWI
January 2023

On January 30, 2020, the first instance of COVID-19, a new coronavirus disease for 2019, was found. The most dangerous coronavirus, later known as COVID-19, is the name of the virus. This thesis aims to make predictions about disease control and disease regression in the future. This thesis uses the COVID model considered by Shaikh *et al.* (2020). The modification of this model is accomplished by utilizing the iterative Laplace transform method and a comparative study of different fractional differential operators. The impacts of various biological parameters on the transmission dynamics of COVID-19 are investigated where the medical model is. Where the medical model is, we will modify it, find the equilibrium point, and test the stability, and according to Hartmann's theory, they have the same behavior. We will then solve the modified model by the Sumudu transformation method. The results obtained will be graphically illustrated.

2022, 64 pages

Keywords: COVID-19, Fractional, Calculus, Dynamical system

ÖZET

COVID-19'UN DİNAMİK SİSTEMİNİN DEĞİŞİKLİĞİ VE ÇÖZÜMÜ

Alyaa Mhmood Faraj FARAJ
Matematik, Yüksek Lisans
Tez Danışmanı: Doç. Dr. Ufuk ÖZTÜRK
Eş Danışman: Prof. Dr. Saad Naji AL-AZZAWI
Ocak 2023

30 Ocak 2020'de 2019'un yeni koronavirüs hastalığı olan COVID-19'un ilk örneği bulunmuş olup, daha sonra COVID-19 olarak bilinen en tehlikeli koronavirüs, virüsün adıdır. Bu tez, gelecekte hastalık kontrolü ve hastalık gerilemesi hakkında öngörülerde bulunmayı amaçlamaktadır. Bu tezde Shaikh *et al.* (2020) tarafından ele alınan COVID modeli kullanılmaktadır. Bu modelin modifikasyonu, yinelemeli Laplace dönüşüm yöntemi ve farklı kesirli diferansiyel operatörlerin karşılaştırmalı bir çalışması kullanılarak gerçekleştirilir. Tıbbi modelin olduğu yerde çeşitli biyolojik parametrelerin COVID-19'un bulaşma dinamikleri üzerindeki etkileri araştırılmaktadır. Tıbbi modeli değiştireceğiz, denge noktasını bulacağız ve kararlılığı test edeceğiz ve Hartmann'ın teorisine göre aynı davranışa sahiptirler. Daha sonra değiştirilmiş modeli Sumudu dönüşüm yöntemi ile çözeceğiz. Elde edilen sonuçlar grafiksel olarak gösterilecektir.

2022, 64 sayfa

Anahtar Kelimeler: COVID-19, Kesirli, Analiz, Dinamik sistem

PREFACE AND ACKNOWLEDGEMENTS

First of all, we thank God for the Almighty and praise Him for the blessings He has given us. As for after.

I extend my thanks and appreciation to everyone who extended a helping hand to me on the completion of this thesis, especially to Assoc. Prof. Dr. Ufuk ÖZTÜRK (Bolu Abant İzzet Baysal University, Department of Mathematics) and Prof. Dr. Saad Naji Al-Azzawi (Baghdad University, Department of Mathematics), for their supervisors and guidance on the completion of this research.

I also extend my thanks to my family and every professor who contributed to my education.

Alyaa Mhmood Faraj FARAJ
Çankırı-2022

CONTENTS

LIST OF SYMBOLS

(CF)	Caputo–Fabrizio
$f^{n}(t)_{sup}$	Supremum
$E_{n,m}(z)$	Two parameter mittag-leffler function
$E_{n}(z)$	One parameter mittag-leffler function
$G_{n}(u)$	Sumudu transform of the derivative
$F(s)$	Laplace transform
$G(u)$	Sumudu transform
$\Gamma(z)$	Gamma function
$\beta(m,n)$	Beta function

LIST OF ABBREVIATIONS

ADM	Adomian decomposition method
DTM	Differential transformation method
GEM	Generalized euler method
HAM	Homotopy analysis method
HPM	Homotopy perturbation method
ILTM	Interative laplace transform method
LADM	Laplace adomian decomposition method
RK4	4th order runge kutta method
SADM	Sumudu transform with adomian decomposition method
WHO	World health organization

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

Fractional calculus is one of the branches of mathematics that deals with the derivative and integration of arbitrary orders. Fractional calculus was discovered in the 17th century by Leibniz. Since then, this topic has been developed through effective contributions by Leibniz, Newton, Euler, Fourier, and others. The two major players in fractional calculus are the Caputo derivative and derivative of Riemann-Liouville theory. Derivative of Riemann-Liouville appears for the first time was in Abel's work on the problem of time, and from here it became clear that the concepts of fractional calculus are not just mathematical operations but rather have enormous uses in applications. It has been applied in the fields of physics, engineering, medicine, and other sciences (Daftardar-Gejji 2014). Hence the importance of mathematical modeling as a tool for solving many problems in life. One of these problems is the analysis of the prevalence and fractional differential equations have the property of describing the memory of different processes, making them useful in the treatment of infectious diseases and other scientific fields as they appear as fractional differential equations. (Lichae *et al.* 2019).

In the last 20 years, numerous mathematical models have been created to manage and stop the spread of infectious diseases like HIV, ADIS, and hepatitis B virus cough. This infection, if left unchecked, can spread and eventually cause acute respiratory distress syndrome, pneumonia, and death. Corruption-associated virus type 19 (COVID-19) spreads rapidly in close quarters (within about 2 meters). We extend a basic dynamic process for adaptive "social distancing" measures from a standard SIR model of the COVID pandemic. This process leads to widespread instability or fluctuations in the effective reproduction number around its bifurcation value due to short attention spans and a lack of a consistent long-term pandemic strategy. However, this process is inherently fragile and does not constitute a long-term strategy that societies could follow, even if it does help to slow the spread of the pandemic in the short term. Normal incubation lasts anywhere from 1 week to 14 days (Organization 2020).

Without any conclusive treatment methodology like an immunization, physical separation has been acknowledged universally as the most proficient system for diminishing the seriousness of the illness and succeeding in mastering it (Ferguson *et al.* 2020). The point of these precautions is to hide the fact that people are touching each other in public places like workplaces, classrooms, and other public places. There is growing evidence that coronaviruses cause disease in both humans and animals. We evaluate the feasibility of preventative measures, future episodes, and possible control procedures for the disease using the proposed model based on the awareness investigations and elements of the limit boundary. A mathematical calculation is finished using the iterative Laplace change technique, and a relative investigation of various partial differential administrators is finished. The effects of different organic boundaries on the transmission elements of coronavirus are being explored. To give a reasonable image of the idea of the proposition and its items, it has been partitioned into three sections. The primary part contains a basic prologue to the historical backdrop of fragmentary math and a short synopsis of the clever Covid. The subsequent section incorporates fundamental definitions and ideas. The third section incorporates providing a fragmentary request change of "A numerical model of Coronavirus utilizing partial subordinate: transmission and control elements fuel India's latest outbreak " (Shaikh *et al.* 2020), and it utilizes the Sumudu change with the Adomian technique, and afterward, a correlation is given with different strategies. With respect to the third section, it contains two sections.

The principal area manages the arrangement of the "A fractional derivative-based mathematical model of the COVID-19 epidemic in India, including transmission dynamics and potential for containment" model provided by Shaikh *et al.* (2020), using the Sumudu transform with the Adomian method of Caputo derivative. The second section of this chapter solves the same model using the iteration form method by using the Sumudu transform with the Caputo-Fabrizio fractional derivative.

2. PRELIMINARIES

In this chapter, a number of definitions and concepts are given to use in the following chapter.

2.1 Fractional Calculus

Integrals and derivatives of any real or complex order are the subject of study in fractional calculus. It is a generalization of classical calculus and thus retains many of its basic properties. As a rapidly developing area of calculus over the last few decades, it has provided tremendous new features for research and thus has become increasingly popular in a variety of applications. The theory of integrals and derivatives of arbitrary order (also known as fractional integrals and derivatives) unifies and generalizes n-fold integration and integer-order differentiation and is known by the name (Podlubny 1998). To put it another way, the infinite sequence of classical n-fold integrals and n-fold derivatives can be thought of as a "interpolation" for fractional derivatives and integrals.

$$f(t)\,,\frac{d\,f(t)}{dt}\;\;,\frac{d^2\,f(t)}{dt^2},\cdots$$

$$\int_a^t f(\tau_1)d\tau_1\,,\int_a^t\int_a^{\tau_1} f(\tau_2)d\tau_2\,d\tau_1,\cdots$$

2.2 Dynamical system

Definition 2.1 (Barreira and Valls 2013) A dynamical system is a pair $(\mathrm{X_t}, \mathrm{p})$ where p is the state space or phase space, and $\mathrm{X_t}$ is a set of transformations from p to p, i.e. $\mathrm{X_t}: \mathrm{p} \rightarrow \mathrm{p}$

1. $\mathrm{X_0(x)} = \mathrm{x}$

2. $X_t(X_s(x)) = X_{t+s}(x)$

Note 2.1

(1) If the time $t \in Z$ (integers), then it is called discrete dynamical system.

(2) If $t \in R$, then it is called continuous dynamical system.

2.3 Equilibrium Points

Definition 2.2 (Boyce and DiPrima 2012) Let $\dot{X} = F(X)$ be a dynamical system where $X \in R^n, F \in R^n, \quad F: R^n \text{ X } R \rightarrow R^n, F(t, x_1, \ldots, x_n)$

A point X_0 is called an equilibrium points if $F(X_0) = 0$

Example 2.1

Let $\dot{x} = \sin y$, $\dot{y} = \cos x$ be a dynamical system. Its equilibrium points are given by

$$\cos y = 0 \rightarrow y = \frac{2n+1}{2}\pi, n \in Z$$

$$\sin x = 0 \rightarrow x = m\pi , m \in Z$$

which forms a grid.

2.3.1 Stability of Equilibrium Point Stable

Definition 2.3 (Derrick and Grossman 1976) It is said that X_0, the equilibrium point, is stable if and only if...

$\forall \varepsilon > 0, \exists \delta > 0 \ S.t\ if \ \|X(t_0) - X_0\| < \delta$, then $\|X(t) - X_0\| < \varepsilon, \forall t \geq 0$

where $X(t)$ is the solution otherwise unstable.

2.3.2 Asymptotically stable

Definition 2.4 (Derrick and Grossman 1976) If an equilibrium point is stable in the long run, we say that it is asymptotically stable and $\exists A > 0\ s.t\ if\ |X(t_0) - X_0| < A\ then\ \lim_{t\to\infty} X(t) = X_0$

2.4 Hartman–Grobman theorem

Let x* represent a hyperbolic equilibrium point in the system.Then there is a homeomorphism h defined on some neighborhood U of x^* in $\mathbb{R}^n$ taking nonlinear orbits locally system's flow $\emptyset(t)$ to that of a linear flow systemic e^{Jt} (Guckenheimer and Holmes 1983). Homeomorphism preserves the sense of can also be chosen to preserve parameterization by time. Consequently, When the system is in hyperbolic equilibrium, the orbit structure is identical to that given by the associated linearized dynamical system. Accordingly, it is clear that the local stability of a hyperbolic equilibrium point x^* of The system is sensitive to the direction of the real part of the eigenvalues of $J = dF(x^*)$. Therefore, in the following, We provide the most widely-used criteria for determining the direction of the eigenvalues.

Example 2.2

An online service that calculates normal-form coordinate transformations of autonomous or non-autonomous, deterministic or stochastic systems of differential equations can easily handle the algebra necessary for this example (Roberts 2009).

Let's pretend that a pair of coupled differential equations describing the evolution of a 2D system in variables $u = (y, z)$ are true.

$$\frac{dy}{dt} = -3y + yz \;\; and \;\; \frac{dz}{dt} = z + y^2$$

This system's one and only equilibrium point is at the origin, as shown by a simple calculation,that is $u^* = 0 \quad u = h^{-1}(U) \;\; where \; U = (Y, Z)$ given by

$$y \approx Y + YZ + \frac{1}{42}Y^3 + \frac{1}{2}YZ^2$$

$$z \approx Z - \frac{1}{7}Y^2 - \frac{1}{3}Y^2Z$$

$\frac{dy}{dt} = \; -3Y$ and $\frac{dz}{dt} = \; -3Y$

That is, a distorted version of the linearization gives the original dynamics in some finite neighborhood.

2.5 Gronwall Inequality

Theorem 2.1 Let $\mathrm{f}, \mathrm{g}\colon [0, \alpha] \; \rightarrow \; [0, \infty)$ be continuous and let c be a nonnegative number (Burton 2005). If $\quad f(t) \leq c + g(s)f(s)ds, \quad (0 \leq t < a)$, then

$$f(t) \leq c \;\; \exp \int_0^t g(s)ds \;\; , \quad (0 \leq t < a).$$

Proof

Suppose first that $c > 0$ Divide by $c \; + \; \int_0^t g(s)f(s)ds$ and multiply by $g(t)$ to obtain

$$\mathrm{f(t)g(t)/[c} + \int_0^{\mathrm{t}} \mathrm{g(s)f(s)ds]} \leq \mathrm{g(t)}.$$

An integration from 0 to t yields

$$In\left\{\left[c+\int_0^t g(s)f(s)ds\right]/c\right\} \leq \int_0^t g(s)ds$$

or

$$f(t) \leq \; c+\int_0^t g(s)f(s)ds \leq c\,\exp\int_0^t g(s)ds$$

If $\boldsymbol{c} \;=\; 0$, take the limit as $\boldsymbol{c} \;\rightarrow\; 0$ through positive values. This completes the proof.

2.6 Routh Hurwitz Criterion

To guarantee that all eigenvalues (roots of the characteristic polynomial equation) of the Jacobian matrix $J \;=\; dF(x^*\,)$ lie in the left half of the complex plane, the Routh-Hurwitz criterion imposes a number of constraints on the coefficients $a_1, a_2 \ldots, a_n$,, Therefore, the system will be asymptotically stable at $\boldsymbol{x}^*$ if and only if the Routh Jacobian Hurwitz constraints hold. Nonetheless, if any of these conditions are disregarded, the unstable point (May 2001).

$$\boldsymbol{P_n(y) = y" \; + a_1 y^{n-1} \; + a_2 y^{n-2}} + \ldots + \boldsymbol{a_n}$$

Let $D_1 = a_1$, $D_2 = \det\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}, \ldots \ldots \ldots \ldots \ldots .., D_k = det \begin{bmatrix} a_1 & a_3 & a_5 \ldots & a_{2k-1} \\ 1 & a_2 & a_4 \ldots & a_{2k-2} \\ 0 & a_1 & a_3 \ldots & a_{2k-3} \\ 0 & 1 & a_2 & a_{2k-4} \\ .. & .. & .. & .. \\ 0 & 0 & 0 & a_k \end{bmatrix}$

Where $a_k = 0 \; if \; k > n$. If so, that's because $P_n(y) = 0$ if and only if they have actual negative components $D_k \; > 0$, $for \; all \; k = 1,2, \ldots \ldots ., n$.

Now, by applying this criterion when $n = 2$ we have

$$P_2(y) = y^2 + a_1 y + a_2 = 0$$

Hence $D_1 = a_1$, $D_2 = \det \begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix} = a_1 a_2$

Thus, if $n = 2$,, $a_1 > 0$ and $a_2 > 0$ are both true, then roots with negative real parts are guaranteed.

For n = 3 , we have

$$P_3(y) = y^3 + a_1 y^2 + a_2 y + a_3 = 0$$

And hence

$$D_1 = a_1 , \qquad D_2 = \det \begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix} = a_1 a_2 - a_3$$

$$D_3 = \det \begin{bmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & 0 \\ 0 & a_1 & a_3 \end{bmatrix} = (a_1 a_2 - a_3) a_3$$

Therefore, if n=3, $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$ are all necessary and sufficient conditions for having roots with negative real parts, then n=3 has roots with negative real parts.

2.7 Some Special Functions

In this section, we will handle some special functions and their properties

2.7.1 Gamma function

It is known that the $n \in \mathbb{N}$ fold integral $\int_a^x \int_a^x \dots \int_a^x f(t)dt_1 dt_2 \dots\ dt_n$

can be written in the from $\int_a^x \frac{(x-t)^{n-1} f(t)}{n!}\ dt$

and what happens if n is any number (natural, integer, real, complex). Their equation means that there is a generalization for $n!$. This is called gamma function $\Gamma(n)$.

Definition 2.5 (Srivastava and Choi 2012): As such, we can define the Gamma Function as

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t} dt\ \ , Re\ (z) > 0$$

Intuitive Gamma as a Function of the Imaginary Unit $i = \sqrt{-1}$ gives

$$\Gamma(i) = (-1+i)! \approx -0.1549 + 0.498i$$

The Gamma function $\Gamma(z)$ has multiple equivalent definitions, one of which is given by Weierstrass:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} (1+\frac{z}{k})^{-1} e^{\frac{z}{k}}\ , (z \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, \dots\}),$$

The constant of the Euler-Mascheroin type γ, is defined as

$$\gamma =\ lim_{\ n\to\infty} (\sum_{k=1}^{n} \frac{1}{k} - \log n) \cong 0.5772166$$

As for the Legendre's duplication formula of the Gamma function, viz (Srivastava and Choi 2012)

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad \left(z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\right)$$

for every positive integer m, we have (Srivastava and Choi 2012)

$$(x)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{x+j-1}{m}\right)_n \quad (m \in \mathbb{N}; n \in \mathbb{N}_0),$$

Now we give some Properties of the Gamma Function, including the following (Podlubny 1998).

1. $\Gamma(z+1) = z\Gamma(z)$,

which can be easily proved by integrating by parts

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt = [-e^{-t}t^z]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}\,dt = z\Gamma(z)$$

obviously, $\Gamma(1) = 1$ and using the previous property we obtain for $z = 1,2,3, \ldots$

$$\Gamma(2) = 1.\ \Gamma(1) = 1 = 1!$$

$$\Gamma(3) = 2.\ \Gamma(2) = 2.1! = 2!$$

$$\Gamma(4) = 3.\ \Gamma(3) = 3.2! = 3!$$

$$\Gamma(n+1) = n.\ \Gamma(n) = n(n-1)! = n!$$

2. $\Gamma\frac{1}{2} = \sqrt{\pi}$

3. $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi Z}$, $\qquad 0 < Re\,(z) < 1$

4. $\sqrt{\pi}\Gamma(2z) = 2^{2Z-1}\,\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$, $\qquad (z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots)$

The function it defines in the positive reals has a unique analytic continuation in the negative reals even though $z \leq 0$ does not converge.

Using Euler's integral for positive arguments and the identity function to extend the domain to negative numbers is one way to find that analytic continuation (Das 2020)

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\dots(z+n-1)}$$

and also

$$\Gamma(z) = lim_{\,n\to\infty} \frac{n!\,n^2}{z(z+1)\dots(z+n)}$$

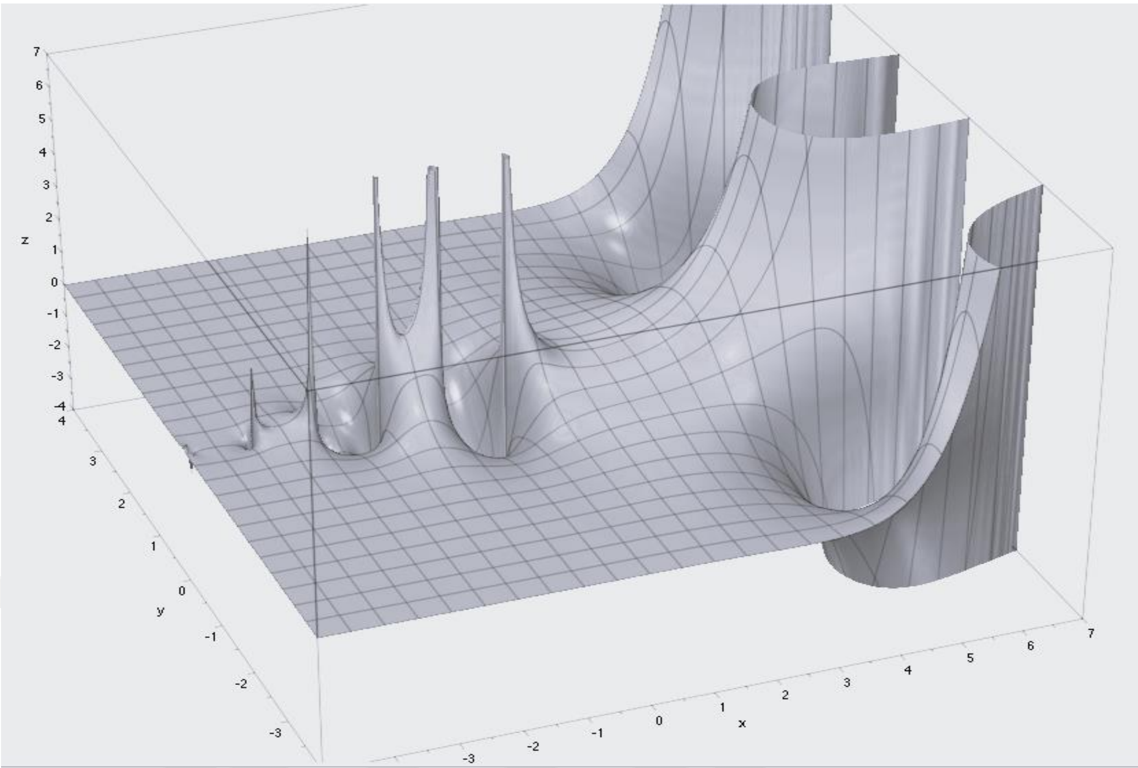

Figure 2.1 Gamma function

2.7.2 Beta function

One of the basic functions of the fractional calculus is beta function.

Definition 2.6 (Milici *et al.* 2019) Beta Function is denoted by $\beta(n, m)$ is

$$\beta(n, m) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad m > 0\,, n > 0$$

The Gamma function and the Beta function are very similar to one another. We have

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad m > 0\,, n > 0$$

Now we give some Properties of the Beta Function

1. $\beta(m,n) = \beta(n,m)$

2. $If\ x = (\sin\theta)^2,\ then\ \beta(m,n) = 2\int_0^{\frac{\pi}{2}} (sin\,\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$

3. $If\ x = \frac{t}{1+t}$, we get the following representation of $\beta(m,n)$ as an infinite integral

$$\beta(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\,dt \qquad m > 0\,, n > 0$$

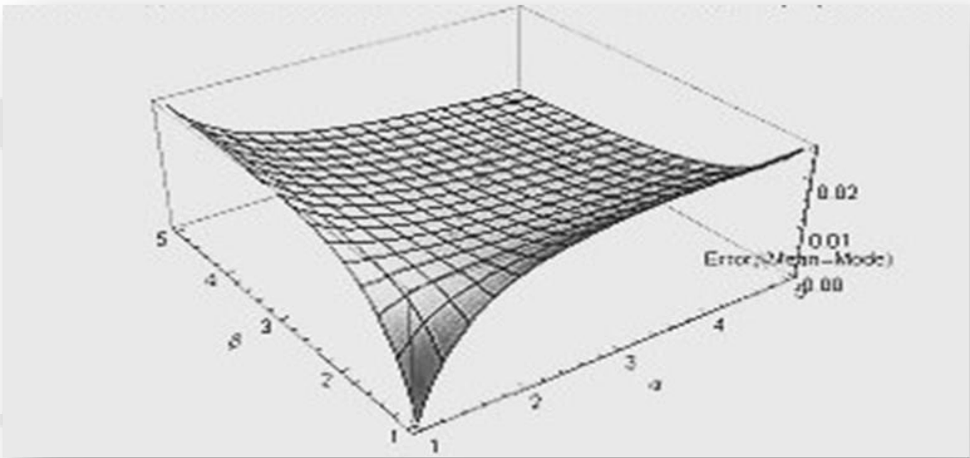

Figure 2.2 Beta function

2.7.3 Mittag- leffler function

Definition 2.7 (Gorenflo *et al.* 2014) Mittag function is denoted by $E_n(z)$, and depends on one parameter n, where n may be complex number with strictly positive real part, from the power series

$$E_n(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(nk+1)} \quad , Re(n) > 0, z \in \mathbb{C}$$

is called one parameter Mittag- Leffler function.

Definition 2.8 The Mittag-Leffler with two parameter is

$$E_{n,m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(nk+m)} \quad , \; z \in \mathrm{C}\,, n > 0, m > 0$$

If $m = 1$, two-parameter version of the Mittag-Leffler function coincides with one parameter

$$E_{n,1}(z) = E_n(z)$$

It follows the previous definition that is:

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(z+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(z+1)!} = \frac{e^z-1}{z}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(z+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(z+2)!} = \frac{e^z-1-z}{z^2}$$

And in general

$$E_{1,m}(z) = \frac{1}{z^{m-1}}\left(e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right)$$

One-parameter Mittag-Leffler functions are thus a special case of the two-parameter versions of these functions. You can also derive other special cases of the Mittag-Leffler function.

Several characteristics of the Mittag-Leffler function (Gorenflo *et al.* 2014).

$$E_2(-z^2) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+1)} = \cos z$$

$$E_2(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \cosh z$$

$$E_2\left(\mp z^{\frac{1}{2}}\right) = \sum_{k=0}^{\infty} \frac{(\mp 1)^k z^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}k+1\right)} = e^z \left[1 + \mathrm{erf}\left(\mp z^{\frac{1}{2}}\right)\right] = e^z\ erfc\left(\mp z^{\frac{1}{2}}\right)$$

The error function is denoted by erf(erf$c$) here (complementary error function) and is given by:

$$\mathrm{erf}\ (z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du \qquad\qquad erfc(z) = 1 - \mathrm{erf}\ (z), z \in \mathbb{C}$$

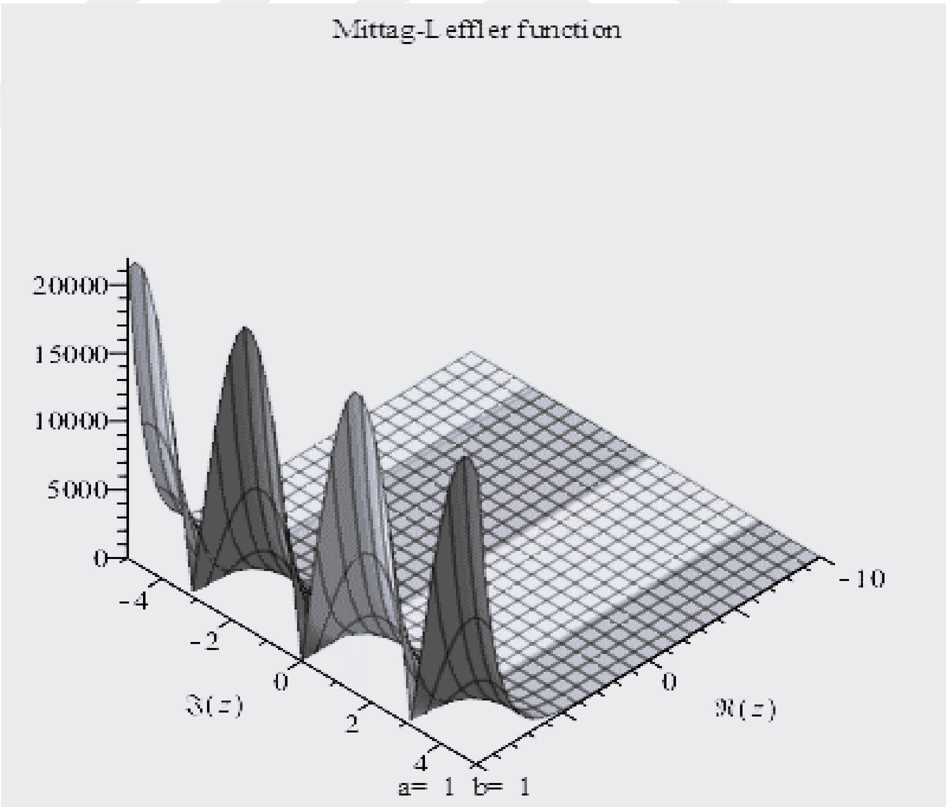

Figure 2.3 Mittag-leffer function

2.8. Fractional derivative

Fractional derivatives and integrals are defined, and the connection between the Riemann-Liouville derivative and the Caputo derivative is explained.

2.8.1 Riemann-Liouville fractional derivative

Definition 2.9 (Ge *et al.* 2018) In calculus, if a function $f\colon [0,\infty) \rightarrow R$ has a Riemann-Liouville fractional integral of order $\alpha > 0$, it is written as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) dt$$

Riemann-Liouville integrals achieves properties according to the following theorem.

Theorem 2.2 (Mirevski *et al.* 2007) Let $f(x)$ and $g(x)$ be such that both $I^m f(x)$and $I^m g(x)$ exist.

Then, satisfy the fundamental properties of Riemann-Liouville integrals, which are:

1. $\lim_{n\to m} I^n f(x) = I^m f(x)$ (continuity)

2. $\mathrm{I^n} [\alpha \mathrm{f(x)} + \mathrm{g(x)}] = \alpha \mathrm{I^n\, f(x)} + \mathrm{I^n g(x)}$ (Linearity)

3. $\mathrm{I^n} [\mathrm{I^m\, f(x)}] = \mathrm{I^{n+m}\, f(x)}$ (Law of exponents)

4. $\mathrm{I^n I^m} = \mathrm{I^m I^n}$ (commutatively).

Definition 2.10 (Heymans and Podlubny 2006) Derivative of the Riemann-Liouville fractional order

n, $m-1 \leq n < m, m \in N$ represents the definition of

$$D_{\alpha}^{n} f(x) = \frac{d^m}{dx^m}\left[\int_{\alpha}^{x} \frac{(x-t)^{m-n-1}}{\Gamma(m-n)} f(t)dt\right]$$

Derivative of Riemann-Liouville achieves properties according to the following theorem.

Theorem 2.3 (Mirevski *et al.* 2007) Let $f(x)$ and $g(x)$ be such that both $Daf(x)$ and $Dag(x)$ exist. Then, the basic properties of the Riemann Liouville derivative satisfy:

1. $\lim_{\alpha \to \beta} D^{\alpha}[f(x)] = f^{(\beta)}(x)$ (continuity)

2. $D^{\alpha}[f(x) + g(x)] = \mu D^{\alpha} f(x) + D^{\alpha} g(x)$ (Linearity)

3. $D^{\alpha}[D^{\mu} f(x)] \neq D^{\alpha+\mu} f(x)$. (The semi- group property)

4. $D^{\alpha} D^{\beta} \neq D^{\beta} D^{\alpha}$. (Non-commutatively)

Example 2.3 Let $n = 1$ and $f(t) = C$, where c is constant. Find the Riemann-Liouville derivative of the function $f(x)$.

Solution. By applying the definition of Riemann-Liouville integration

$$I_{x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t)dt$$

$$I^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} c\,(x-t)^{-\alpha}\, dt = \frac{c}{\alpha\Gamma(1-\alpha)}\, x^{\alpha}$$

Now $D^{\frac{1}{2}}f(x) = \frac{d}{dx}\,[I^{1-\alpha}f(x)]$

$$= \frac{d}{dx}\left[\frac{cx^{\frac{1}{2}}}{\alpha\Gamma(1-\alpha)}\right] = \frac{cx^{\frac{-1}{2}}}{\frac{1}{2}\Gamma(\frac{1}{2})}\ Type\ equation\ here.$$

$$D^{\frac{1}{2}}f(x) = \frac{2c}{\sqrt{\pi}}\,x^{\frac{-1}{2}}$$

The following theorem shows the interplay between the differentiation and integration.

Theorem 2.4 (Mirevski *et al.* 2007) $For\ \mu > 0, x > 0\ and -1 \leq \alpha < \mu -$

1. $D^{\alpha}\,[I^{\alpha}\,f(x)] = f(x)$

2. If the fractional derivative $D^{\alpha}f(x)$ is integrable, then

$$I^{\alpha}[D^{\alpha}f(x)] = f(x) - \sum_{k=1}^{\mu}[D^{\alpha-K}f(x)]_{x=0}\,\frac{x^{\alpha-K}}{\Gamma(\alpha-k+1)}$$

2.8.2 Caputo fractional derivative

Definition 2.11 (Xue 2017) Derivatives of the fractional order, as defined by Caputo, are as follows:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt, n-1 < \alpha \leq n\,, n \in N$$

where $\alpha > 0$ is the derivative order and n is the smallest integer greater than α. To get the Caputo derivative, we use

$$D_c^{\alpha}c = 0\ (c\ is\ a\ constant)$$

$$D^{\alpha}x^{\beta} = \begin{cases} 0 & ,for\ \beta \in \mathbb{N}_0 and\ \beta < \lceil\alpha\rceil \\ \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}\ x^{\beta-\alpha} & ,for\ \beta \in \mathbb{N}_0 and\ \beta \geq \lceil\alpha\rceil or\ \beta \notin \mathbb{N}\ and\ \beta > \ \lfloor\alpha\rfloor \end{cases}$$

Where$\begin{Bmatrix}\lceil\alpha\rceil \\ \lfloor\alpha\rfloor\end{Bmatrix}$ is the $\begin{Bmatrix}smallest \\ biggest\end{Bmatrix}$ natural number greater than α .

$$D_*^{\alpha} f = \frac{d^n f}{dt^n}\ where\ \alpha = n$$

$$D_*^{\alpha}(c_1 f + c_2 g) = c_1 D_*^{\alpha} f + c_2 D_*^{\alpha} g$$

Where c_1 and c_2 are constant

Example 2.4 Let $\alpha\ =\ \frac{1}{2}$, n = 1 $and\ f(t)\ = t^2$. Find the Caputo derivative.

Solution. By applying the formula of Caputo derivative

$$\frac{d^{\frac{1}{2}}}{d^{\frac{1}{2}}}\ f(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\ \int_0^t \frac{2\tau}{(t-\tau)^{\frac{1}{2}}}\ d\tau$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t \tau(t-\tau)^{\frac{-1}{2}}\ d\tau = \frac{-4}{\sqrt{\pi}}\ \tau\,(t-\tau)^{\frac{1}{2}}\Big]_0^t + \frac{4}{\sqrt{\pi}}\int_0^t (t-\tau)^{\frac{1}{2}}\ dt$$

$$\frac{d^{\frac{1}{2}}}{d^{\frac{1}{2}}}\ f(t) =\ \frac{8}{3\sqrt{\pi}} t^{3/2}$$

2.8.3 The relation between Caputo and Riemann-Liouville fractional derivative

In this item, We present an alternative method of illustrating the connection between the Riemann-Liouville operator and the Caputo operator.

Lemma 2.1 (Dielthelm 2010) Let $\alpha \geq 0$ and $\beta = \lceil \alpha \rceil$. Assume that f is such that both

$$D_{*_a}^{\alpha} f(x) = D_a^{\alpha} f(x) - \sum_{k=0}^{\beta-1} \frac{D^k f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}$$

In this case, $D_{*_a}^{\alpha}$ is the Riemann-Liouville fractional differential operator of order α and D_a^{α} is the Caputo differential operator of order α.

Proof

Applying the Caputo derivational definition and let

$$f(x) = (x-a)^{\gamma},\ \gamma > -1 \ and\ \alpha > 0$$

$$D_a^{\alpha} f(x) = D^{\lceil \alpha \rceil} I_a^{\lceil \alpha \rceil - \alpha} f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\lceil \alpha \rceil - \alpha + \gamma + 1)} D^{\lceil \alpha \rceil} \left[(x-a)^{\lceil \alpha \rceil - \alpha + \gamma} \right] I(x)$$

If $\alpha - \gamma \in N$ on the right you'll find the $\lceil \alpha \rceil -$th derivative of a classical scaled polynomial $\lceil \alpha \rceil - (\alpha - \gamma) \in \{0, 1, \ldots\ldots, \lceil \alpha \rceil - 1\}$, then

$$D_a^{\alpha} \left[(\alpha - a)^{\alpha - \beta} \right] = 0, \alpha > 0, \beta \in \{1, 2, \ldots\ldots\ldots, \lceil \alpha \rceil\}$$

On the other hand, if $\alpha - \gamma \notin N$

$$D_{*_a}^{\alpha} [(x-a)^{\gamma}](x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} (x-a)^{\gamma-\alpha}$$

$$D_{*_a}^{\alpha} f(x) = D_a^{\alpha} f(x) - \sum_{k=0}^{\beta-1} \frac{D^k f(a)}{k!} D_a^{\alpha} [(x-a)^k]$$

$$= D_a^{\alpha} f(x) - \sum_{k=0}^{\beta-1} \frac{D^k f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}$$

2.8.4 Caputo-Fabrizio fractional derivative

Caputo-Fabrizio is one of the current proposals for fractional derivatives with a smooth kernel that relies on two different representations in terms of time variables that transform, which makes it appropriate a Laplace transform is used.

Two is more Fourier transform in nature and is connected to the spatial variable (Owolabi and Atangana 2019).

Definition 2.13 (Caputo and Fabrizio 2015) Let $\psi \in L^1(a,b)\,, b > a \; and \; 0 < \alpha < 1$ then the formula for the Caputo-Fabrizio derivative of the function $\psi(t)$ of order α is

$$D_t^{(\alpha)}\big(\psi(t)\big) = \frac{M(\alpha)}{1-\alpha}\int_0^t \acute{\psi}\,(x) exp\left[\frac{-\alpha(t-x)}{1-\alpha}\right] dx$$

In which the normalization function $M(\alpha)$ is used with $M(0) = M(1) = 1$. But if the function does not belong to $L^1(a,b)$, then the derivative is

$$D_t^{(\alpha)}\big(\psi(t)\big) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (\psi(t) - \psi\,(x)) exp\left[\frac{-\alpha(t-x)}{1-\alpha}\right] dx$$

Definition 2.14 Let $\alpha \in (0,1)$ then the fractional integral of Caputo-Fabrizio order α is defined by

$$I_t^{\alpha}\big(\psi(t)\big) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\psi(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \psi(s) ds \; t \geq 0$$

To clarify, the Caputo-Fabrizio fractional integration is a function of order $\alpha \in (0,1)$ that is the mean between the function ψ and its integral of order one, as defined above (Losada and Nieto 2015). Then imposes

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1$$

We obtain a formula for $M(\alpha)$, $M(\alpha) = \frac{2}{2-\alpha}$, $\alpha \in [0,1]$

2.8.5 System of fractional derivative

The main feature of derivatives of fractional order is the description of memory and the genetic characteristics of different processes. The following is an expression for the fractional differential system

$$\begin{cases} D^{\alpha}\, u(\beta) - f\big(c, u(\beta), D^{\alpha}\, u(\beta)\big) = \theta(\beta, u(\beta), v(\beta) \\ \beta \in [0,T], T > 0, 0 < \alpha < 1, \\ D^{\alpha}\, v(\beta) - f\big(c, v(\beta), D^{\alpha}\, v(\beta)\big) = \vartheta(\beta, u(\beta), v(\beta) \\ u(0) + h(u) = u_0\, , v(0) + h(v) = v_0 \end{cases}$$

In which β is a positive real number and $D^{\alpha}\, u(\beta)\ and\ \ D^{\alpha}\, v(\beta)$ are derivatives in terms of fractions, respectively. The functions f, g have a complete and finite set of boundaries and ϑ, θ are continuous functions that are not local (Ahmad *et al.* 2020).

Definition 2.15 (Zhang 2005) A point$X_0 \in R^n$ is called an equilibrium point of a function $f(x)\ if\ f(x)\ = 0$

Definition 2.16 (Dielthelm 2010) It is said that the solution $y(x)\ = 0$ to the differential equation $D^n\, y(x) = f(x, y(x))$ is stable if and only if, for $\varepsilon > 0$ that there is some $\delta > 0$ in a way that the differential equation describing the initial value problem can be $D^n\, y(x) = f(x, y(x))$ in addition to the initial state $y(0) = y_0$ satisfies $\|y(x)\| < \varepsilon$ for all $x \geq o$ whenever $\|y_0\| < \delta.$ (b): The solution $y(x)\ = 0$ of homogenous differential equation $D^n\, y(x) = f(x, y(x))$ can be said to be asymptotically stable if and only if it is stable and there is some $\gamma > 0$ such that $\lim\limits_{x\to\infty} \|y(x)\ = 0\|$ whenever $\|y_0\| < \gamma$.

2.9 Some type of transforms

This section examines the Laplace transform, the Sumudu transform, and their properties.

2.9.1 Laplace transformation

Definition 2.17 (Cohen 2007) Consider the function f (t) defined on the positive half of the real axis ℝ, which may be real or complex in value. For any function $f\ (t)$, we have the Laplace transform as the function $F\ (s)$

$$F\ (s) = \int_0^\infty e^{-st} f(t)dt$$

Where f is exponentially bounded

The inverse of the Laplace transform is defined

$$f\ (t) = L^{-1}[F\ (s)] = \frac{1}{2\pi i} \lim_{t\to\infty} \int_{\sigma-i\tau}^{\sigma+i\tau} e^{sx} F\ (s)ds,$$

Where σ is large enough that $F(s)$ is defined for the real part of $S \geq \sigma$.

2.9.1.1 Some properties of laplace transform

There are many properties of the Laplace transform, but we will refer to some special characteristics of the Laplace transform as following (Cohen 2007, Kazem 2013):

Linearity, i.e.

$$L[\sum_{k=1}^{n} c_k f_k\ (t)] = \sum_{k=1}^{n} c_k\, L[f_k(t)]$$

The integration rule $f(t) = \int_0^t g(x)dx \; then \; L[f(t)] = \frac{F(s)}{s}$

Then by iterating this rule, we obtain $\int_0^{x_n} \dots \int_0^{x_3} \int_0^{x_2} f(x_1)dx_1 dx_2 \dots dx_n = \frac{F(s)}{s^n}$

If $f(t) = t^n$ then $L[f(t)] = F(s) = \frac{\Gamma(n+1)}{s^{n+1}}$

Differentiation rules

$$L\left[f^{(n)}(x)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \grave{f}(0) - \dots - f^{(n-1)}(0)$$

$$= s^n F(x) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$$

The image shift rule for any complex number it holds that

$$L[e^{at} f(t)] = F(s-a)$$

1. The t- multiplication rule

$$L[t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$L[f(t+c)u(t)] = e^{cs}\left[F(s) - \int_0^c e^{-st} f(t)dt\right], c > 0$$

The convolution integration rule

$$(f * g)(t) = \int_0^\infty f(\tau)g(t-\tau)d\tau \,, L[(f * g)(t)] = F(s)G(s)$$

2.9.2 Sumudu transformation

Definition 2.18 (Belgacem and Karaballi 2006) The space of functions over which the Sumudu transform is defined

$$A = \left\{ f(t) | \exists M, \tau_1 \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

according to this equation

$$G(u) = \int_0^\infty g(ut) e^{-t}\, dt\,, \qquad or \qquad G(u) = \frac{1}{u} \int_0^\infty e^{\left(\frac{-t}{u}\right)} g(t) dt,$$

For any function $f\ (t)$ and $-\tau_1 < u < \tau_2$.

Definition 2.19 Denoted by $g(t)$, the inverse Sumudu transform of $G(u)$ is defined as follows.

$$g(t) = S^{-1}[G(u)] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{u} e^{\frac{t}{u}} G(u) du, where\ Re\left(\frac{1}{u}\right) > \gamma\ and\ \gamma \in \mathbb{C}$$

the essential feature of the sumudu transform is $S\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] = u^\alpha, \alpha > 0$

then sumudu inverse transforms of $u^\alpha is\ S^{-1}[u^\alpha] = \frac{t^\alpha}{\Gamma(\alpha+1)}$

Definition 2.20 (Darzi *et al.* 2013) As an example, here is how to write down the Sumudu transform of the Caputo's derivative of a function f:

$$S[D_t^\alpha f(t); u] = u^{-\alpha} G(u) - \sum_{k=0}^{m-1} u^{k-\alpha} \left[f^{(k)}(t)\right]_{t=0}\,, \quad m - 1 < \alpha \le m$$

Definition 2.21 Atangana and Alkahtani defined the Sumudu transformation of Caputo-Fabrizio Derivative of Fractions of $f(t)$ as

$$S(D^{\alpha})G(t) = \frac{M(\alpha)}{1-\alpha+\alpha u}[S(G(t) - G(0)]$$

In which the normalization function $M(\alpha)$ is used

Theorem 2.5 (Belgacem and Karaballi 2006) Let $G(u)$ is Sumudu transform of the function $f(t)$, then the Sumudu transform of derivative $f^{(n)}(t), n \geq 1$ is

$$G_n(u) = u^{-n}\left[G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}?(0)\right], n \geq 1$$

Where $G_n(t)$ is the Sumudu transform of the derivative $f^{(n)}(t)$.

2.9.2.1 Some Properties of Sumudu Transform

There are many properties of the Sumudu transform, but we will refer to some special characteristics of the Sumudu transform as following (Latifi 2020):

Linearity The Sumudu transform is a linear transform which is necessary in transform inversion and this propriety can be expressed as follows:

$$S[\alpha f(t) + \beta g(t)] = \alpha S[f(t)] + \beta S[g(t)]$$

Let $f(t), g(t) \in A$, and $F(u), G(u)$ be the Sumudu transform of $f(t), g(t)$, respectively then the Sumudu of the convolution

$$(f * g)(t) = \int_0^{\infty} f(\tau)g(t-\tau)d\tau, \quad is\ S[(f * g)(t)] = uF(u)G(u)$$

The power series $f(t) = \sum_{v=0}^{\infty}(-1)^{v}\frac{(\alpha t)^{\tau}}{v!} = e^{-\alpha t}$.

Transforms to the geometric series

$$S[f(t)] = \sum_{v=0}^{\infty}(-1)^{v}(\alpha u)^{v} = \frac{1}{1+\alpha u}\ ,\ u \in \left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right).$$

The Sumudu and Laplace transforms exhibit a duality relation as follows (Belgacem and Karaballi 2006)

$$G\left(\frac{1}{s}\right) = sF(s) \quad , \quad F\left(\frac{1}{u}\right) = uG(u)$$

Similarly, the Sumudu and Laplace transform of the $\sin(t)$and $\cos(t)$ is (Belgacem and Karaballi 2006)

$$S[\cos(t)] = L[\sin(t)] = \frac{1}{1+u^2}\ , \qquad S[\sin(t)] = L[\cos(t)] = \frac{u}{1+u^2}$$

Where S is Sumudu transform and L is Laplace transform.

Example 2.5 (Tuluce *et al.* 2015) Let us solve the homogeneous fractional ordinary differential equation by Sumudu transform.

$$\frac{dg(t)}{dt} + \frac{d^{\frac{1}{2}}g(t)}{dt^{1/2}} - 2g(t) = 0\ , \qquad \frac{d^{\frac{1}{2}}g(0)}{dt^{1/2}} = c\ \ , t > 0$$

With initial condition $g(0) = 0$

Solution. We first apply the Sumudu transform

$$S\left(\frac{dg(t)}{dt}\right) + S\left(\frac{d^{\frac{1}{2}}g(t)}{dt^{1/2}}\right) - 2S\big(g(t)\big) = 0$$

$$\frac{1}{u}[G(u) - G(0)] + \frac{G(u)}{u^{\frac{1}{2}}} - \frac{D^{\frac{-1}{2}}|g(t)|}{u}|_{t=o} - 2G(u) = 0$$

$$\frac{G(u)}{u} + \frac{G(u)}{u^{\frac{1}{2}}} - \frac{c}{u} - 2G(u) = 0$$

$$\left(1 + u^{\frac{1}{2}} - 2u\right)G(u) = c$$

$$G(u) = \frac{c}{1+u^{\frac{1}{2}}-2u} = \frac{2c}{3(1+2\sqrt{u})} + \frac{c}{3(1-\sqrt{u})}$$

$$G(u) = \frac{2c}{3}\left(\frac{1-2\sqrt{u}}{1-4u}\right) + \frac{c}{3}\left(\frac{1+\sqrt{u}}{1-u}\right)$$

$$G(u) = \frac{2c}{3}\left(\frac{1}{1-4u} - \frac{2\sqrt{u}}{1-4u}\right) + \frac{c}{3}\left(\frac{1}{1-u} + \frac{\sqrt{u}}{1-u}\right)$$

When applied the inverse Sumudu transform of $G(u)$, we obtain

$$S^{-1}[G(u)] = S^{-1}\left[\frac{2c}{3}\left(\frac{1}{1-4u} - \frac{2\sqrt{u}}{1-4u}\right) + \frac{c}{3}\left(\frac{1}{1-u} + \frac{\sqrt{u}}{1-u}\right)\right]$$

$$g(t) = \frac{2c}{3}\left[e^{4t} - e^{4t}\,\mathrm{erf}(2\sqrt{t})\right] + \frac{c}{3}\left[e^{t} + e^{t}\,erf(\sqrt{t})\right].$$

Example 2.6 (Tuluce *et al.* 2015) Let us solve the nonhomogeneous fractional ordinary differential equation by Sumudu transform

$$D^2 z(t) + D^{\frac{3}{2}} z(t) + z(t) = t + 1, \qquad \left[D_t^{\frac{-1}{2}} z(t)\right]_{t=0} = 0$$

With initial conditions $z(0) = \acute{z}(0) = 1$.

Solution. Take Sumudu transform of both sides

$$S\left[D^2 z(t) + D^{\frac{3}{2}} z(t) + z(t)\right] = S[t+1]$$

$$\frac{1}{u^2}\left[Z(u) - z(0) - u\frac{dz(t)}{dt}\right]_{t=0} + \frac{Z(u)}{u^{3/2}} - \left[\frac{D^{\frac{1}{2}} z(t)}{u}\right]_{t=0} + Z(u) = u + 1$$

$$\frac{1}{u^2}[Z(u) - 1 - u] + \left[\frac{D^{\frac{1}{2}} z(t)}{u}\right]_{t=0} + Z(u) = u + 1$$

$$Z(u) - 1 - u + u^2 Z(u) = u^3 + u^2$$

$$Z(u)[u^2 + 1] = u^3 + u^2 + u + 1$$

$$Z(u) = \frac{u^3 + u^2 + u + 1}{u^2 + 1}$$

$$Z(u) = \frac{(u+1)(u^2+1)}{(u^2+1)} = (u+1)$$

$$S^{-1}[Z(u)] = S^{-1}(u+1)$$

$$z(t) = t + 1$$

3. MODIFICATION AND SOLUTION THE INDIA MODEL OF SHAIKH, SHAIKH AND NISAR

Over the course of the planet's history, COVID disease is likely to emerge as a watershed moment. The severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) (Organization 2020) caused COVID-19, the abbreviated form of COVID-19. In December of 2019, the first case was found in Wuhan, Hubei Province, China (Chen *et al.* 2020). The discovery of the COVID-19 virus on January 30, 2020, prompted the World Health Organization (WHO) to issue a public health emergency declaration. A pandemic has been declared as of March 11, 2020. There is a lack of specificity in the symptoms associated with COVID-19, and many cases have shown that a person infected with the virus may exhibit no symptoms at all. A dry cough (present in 68%) and fever (88% of cases) are the most frequently reported symptoms. Fatigue, muscle and joint pain, sputum production in the respiratory system, sore throat, diminished sense of smell, fever, chills, and shortness of breath are just some of the symptoms that may be present. The progression of this infection can also cause severe respiratory distress syndrome, severe pneumonia, and death. Coxsackievirus type 19 (COVID-19) spreads rapidly amongst neighbors (within approximately 2 m). From one week to two weeks is the average incubation time (Organization 2020). It is generally agreed that, in the absence of a cure, the most effective method of reducing the severity of a disease and gaining control over it is physical distancing, which can take the form of a vaccine (Ferguson *et al.* 2020). The point of these safeguards is to make actual contact in open settings like workplaces, schools, and other public places unnoticeable. The development of the vaccine was time-consuming, but was necessary to prevent the virus from spreading. Protecting against infection and bolstering the immune system, the Corona vaccine is a must-have. As soon as a virus enters the body, the immune system recognizes it, launches an antibody production response, and stores this information for future use. Therefore, the function of the immune system is to defend the body against harm. Since the vaccine has undergone extensive testing, produces a robust immune response and continuous antibodies, and is administered in two doses every three weeks through muscle injection, it is considered safe. You can see some pictures from COVID-19 down below.

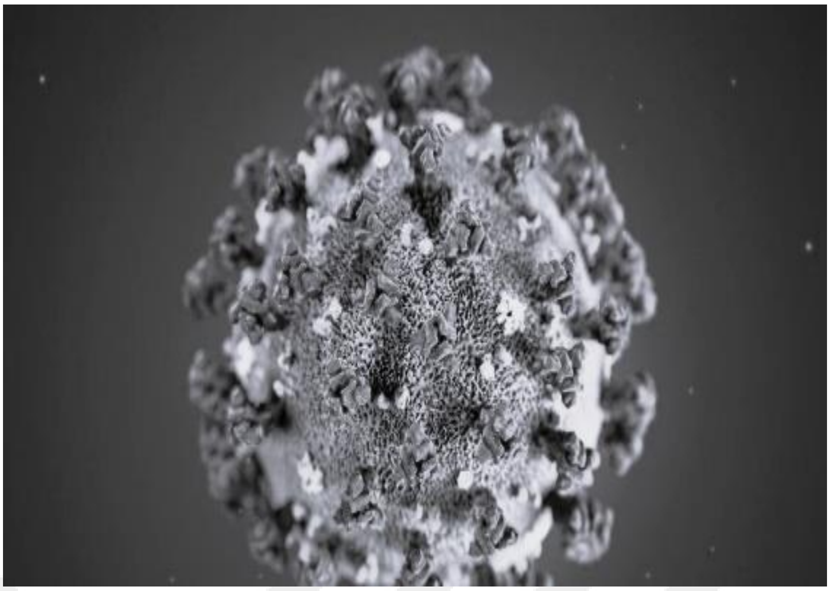

Figure 3.1 Coronavirus in blood

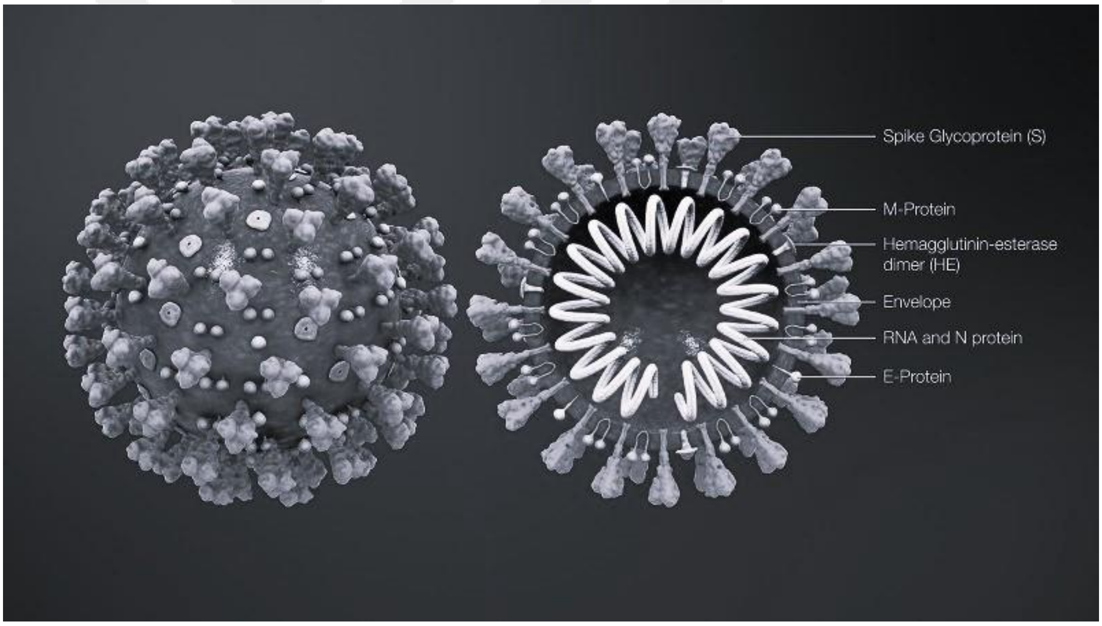

Figure 3.2 Synthesis of coronavirus

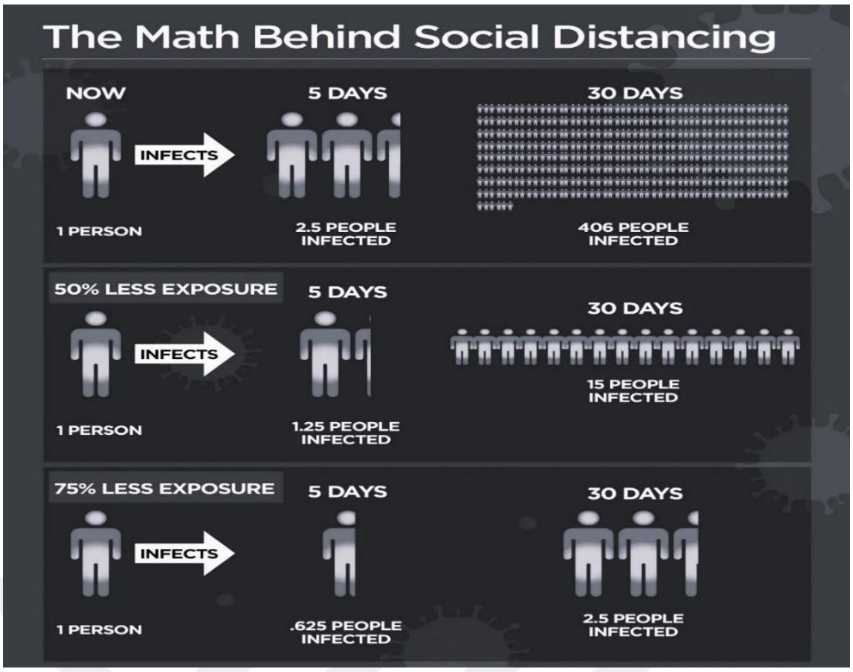

Figure 3.3 A look at the math behind social distancing

3.1 Mathematical Model of Shaikh, Shaikh and Nisar

In order to numerically simulate outbreaks in populations, computational mathematical methods are used. More than a century has passed since the first mathematical models of the dynamics of infectious diseases were developed.

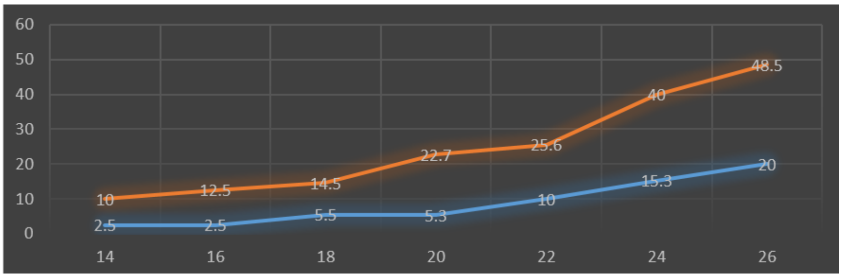

Figure 3.4 Numbers of recovered people and deaths from 14 March 2020 to 26 March 2020 in India

Recovered people, death people:

This paper by Chen *et al.* (2020) explores a mathematical model for recreating the stage-based transmissibility of a novel coronavirus (Chen *et al.* 2020, Khan and Atangana 2020) modelled the interaction between the coronavirus and human populations mathematically. Equation 3.1 presents the fundamental model of (Amjad *et al.* 2010).

$$
\begin{aligned}
D_t S(t) &= \Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma S Q \\
D_t E(t) &= \frac{\alpha S(I+\beta A)}{N} + \gamma S Q - (1-\phi)\delta E - \phi\mu E - \lambda E \\
D_t I(t) &= (1-\phi)\delta E - (\sigma+\lambda)\, I \\
D_t A(t) &= \phi\mu E - (\rho+\lambda)\, A \\
D_t R(t) &= \sigma I + \rho A - \lambda R \\
D_t Q(t) &= \kappa I + \upsilon A - \eta Q
\end{aligned}
\tag{3.1}
$$

The number of populations, N, is divided by the number of time intervals, t, into six equal parts: $S(t)$ represents the susceptibility distribution, $E(t)$ the exposure distribution, $I(t)$ the infectiousness distribution, $A(t)$ the asymptotically infected distribution, $R(t)$ the recovery distribution, and $Q(t)$ the reservoir. This model's parameters, along with their descriptions, values, and citations, can be found in Table 3.1 (Khan and Atangana 2020).

Table 3.1 Description of parameter and notation

parameter and notation description		value
Estimated Annual Birthrate	Δ	53,320.19
Statistically Significant Percentage of Contact	α	0.05
Normalized Death Rate	λ	$\frac{1}{69.50 * 369}$
Fastest possible rate of transmission	β	0.02844
Incubation period of I	δ	0.0717876
duration of A's incubation period	μ	0.05
The percentage of infections that cause no symptoms	ϕ	0.8243
coefficient of transmission for disease	γ	$0.121 * 10^{-7}$
Recovery, reduction, rate, or I	σ	0.09871
Recovery, Elimination, Rate, and A	ρ	0.854302
Effects of I's Virus on Q	κ	0.000398
An I virus-related contribution to Q	ν	0.001
Rate of Virus Elimination from Q	η	0.01

3.2 System adjustment of Mathematical Model of Shaikh, Shaikh and Nisar

If no virus is detected, we'll figure out a solution here.

$$D_t S(\mathrm{t}) = = \Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma S Q$$

$$\frac{ds}{dt} = \Delta - \lambda S$$

$$\frac{ds}{\lambda S - \Delta} = - dt \quad \Rightarrow \quad \frac{ds}{\lambda S - \Delta} = - dt$$

$$\frac{1}{\lambda} \ln|\lambda S - \Delta| = -t + C$$, Where C arbitrary constant

When integrating the above equation, we get

$$S(t) = \frac{\Delta - e^{-\lambda t + c\lambda}}{\lambda}$$

$S(0) = k_1 \quad \Rightarrow \dfrac{1}{\lambda}\ln|\lambda S_0 \; - D| = C = C_0$

$\ln|\lambda S - D| = -\lambda t + \lambda \, C_0$

$\lambda S - \Delta = e^{-\lambda t + \lambda C_0} \quad \Rightarrow A_0 \, e^{-\lambda t}$

$\therefore S(t) = \frac{\Delta - e^{-\lambda t + c\lambda}}{\lambda} \; in \; the \; solution \; Now \;\; t \; \rightarrow \infty \Rightarrow \lim\limits_{t \rightarrow \infty} S(t) = \frac{\Delta}{\lambda}$

If $S(0) = k_1$ then

$c = \dfrac{1}{\lambda}\ln(\mu k_1 - \Delta)$

$S(t) = \dfrac{\Delta + e^{-\lambda t + \ln(\lambda k_1 - \Delta)}}{\lambda}$

$\lim\limits_{t \rightarrow \infty} S(t) = \frac{\Delta}{\lambda}$

Using the following values as substitutes $\Delta = 53320.19$, $\lambda = 0.00004$

$\lim\limits_{t \rightarrow \infty} S(t) = 1333004750$ While $S(t)$ from 1350900000 to 3242160000

In this case, we found that there is a decrease so we will change the equation to the following equation:

$D_\varepsilon S = \Delta - \lambda S$

Use the Laplace transform on both sides of each equation $D_\varepsilon S = \Delta - \lambda S$

of $L(D_{\varepsilon}S) = L(\Delta) - \lambda L(S)$

$$\Delta^{\varepsilon} L(S) - \Delta^{\varepsilon-1} S(0) = \frac{\Delta}{\Delta} - \lambda L(S)$$

$$L(S) = \frac{\Delta + \Delta^{\varepsilon} S(0)}{\Delta(\Delta^{\varepsilon} - \lambda)} = \frac{\Delta}{\Delta(\Delta^{\varepsilon} - \lambda)} + \frac{\Delta^{\varepsilon} S(0)}{\Delta(\Delta^{\varepsilon} - \lambda)}$$

Substitution $S(0) = 1350900000$ and use of the converse Equation with a Laplace transform

$$S(t) = \Delta L^{-1}\left(\frac{\Delta^{-1}}{\Delta^{\varepsilon} - \lambda}\right) + 1350900000 L^{-1}\left(\frac{\Delta^{\varepsilon-1}}{\Delta^{\varepsilon} - \lambda}\right)$$

$$S(t) = \Delta t^{\varepsilon} E_{\varepsilon,\varepsilon+1}(\lambda t^{\varepsilon}) + 1350900000 E_{\varepsilon,1} \frac{(\lambda t^{\varepsilon})^k}{\Gamma(\varepsilon k+1)}$$

$$S(t) = 53320.19 t^{\varepsilon} \sum_{k=0}^{\infty} \frac{(\lambda t^{\varepsilon})^k}{\Gamma(\varepsilon k + \varepsilon + 1)} + 1350900000 \sum_{k=0}^{\infty} \frac{(\lambda t^{\varepsilon})^k}{\Gamma(\varepsilon k + 1)}$$

$$S_0 = \frac{53320.19 t^{\alpha}}{\Gamma(\alpha+1)} + 1350900000$$

$$S_1 = \frac{2.133 t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + \frac{54036 t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

Good accuracy can be achieved with only a few terms due to the uniform convergent (Rahman 2007).

$$S(t) \approx S_0 + S_1 + S_2$$

Then $0 < \varepsilon_i < 1 \Rightarrow \; 1350900000 < S(t) < 3242160000 \;,\; i = 1,2,3, \ldots.$

Differential equations of fractional order are a common mathematical model for problems in biology and related fields. As a result, the system is as follows Equation 3.2:

$$D_\varepsilon(S) = \Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma SQ$$

$$D_\varepsilon(E) = \frac{\alpha S(I+\beta A)}{N} + \gamma SQ - (1-\phi)\delta E - \phi\mu E - \lambda E$$

$$D_\varepsilon(I) = (\delta - \phi\delta)E - (\sigma + \lambda)\, I \qquad (3.2)$$

$$D_\varepsilon(A) = \phi\mu E - (\rho + \lambda)\, A$$

$$D_\varepsilon(R) = \sigma I + \rho A - \lambda R$$

$$D_\varepsilon(Q) = \kappa I + vA - \eta Q$$

Based on the same initial conditions These mathematical models can be solved by a wide variety of numerical methods. Adomian showed a decomposition method (ADM), a potent technique for approximating analytical solutions to differential equations. To get a handle on this system, we employ the Sumudu transform technique in conjunction with the Anomaly Detection Method (SADM).

3.3 Equilibrium Points and Stability Analysis

$$\Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma SQ = 0$$

$$\frac{\alpha S(I+\beta A)}{N} + \gamma SQ - (1-\phi)\delta E - \phi\mu E - \lambda E = 0$$

$$(\delta - \phi\delta)E - (\sigma + \lambda)I = 0$$

$\phi\mu E - (\rho + \lambda)A = 0$

$\sigma I + \rho A - \lambda R = 0$

$\kappa I + vA - \eta R = 0$

Here we find eigenvalues of the Type equation, which are used in the equilibrium point stability test.

$$X = \begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & (-\delta + \emptyset\delta - \phi\mu - \lambda) & 0 & 0 & 0 & 0 \\ 0 & \delta - \emptyset\delta & -(\sigma + \lambda) & 0 & 0 & 0 \\ 0 & \phi\mu & 0 & -(\rho + \lambda) & 0 & 0 \\ 0 & 0 & \sigma & \rho & -\lambda & 0 \\ 0 & 0 & \kappa & \nu & 0 & -\eta \end{bmatrix}$$

The linearization of A1 is in the form

$$\dot{\mathrm{X}} = \begin{bmatrix} \grave{\mathrm{S}} \\ \grave{\mathrm{E}} \\ \grave{\mathrm{I}} \\ \grave{\mathrm{A}} \\ \grave{\mathrm{R}} \\ \grave{\mathrm{Q}} \end{bmatrix} = \mathrm{M} \begin{bmatrix} S \\ E \\ I \\ A \\ R \\ Q \end{bmatrix} + O(\lambda) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (A_2)$$

(Equations of the second degree so that it approaches zero when the point approaches the equilibrium point)

Given that the eigenvalues are not completely imaginary ($\lambda \neq bi$) then the behavior of $(A1$ & $A2)$ are qualitatively the same in the neighbourhood of (N,0,0,0,0,0,0) (Hartman theorem or linearization theory). Therefore, M is similar to:

$$M^* = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{bmatrix}$$

$$\dot{X} = M^* X$$

$$\grave{S} = \lambda_1 S$$

$$\grave{E} = \lambda_2 E$$

$$\grave{I} = \lambda_3 I$$

$$\grave{A} = \lambda_4 A$$

$$\grave{R} = \lambda_5 R$$

$$\grave{Q} = \lambda_6 Q$$

$$|X - \lambda I| =$$

$$\begin{bmatrix} -\lambda - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & (-\delta + \emptyset\delta - \phi\mu - \lambda) - \lambda & 0 & 0 & 0 & 0 \\ 0 & \delta - \emptyset\delta & -(\sigma + \lambda) - \lambda & 0 & 0 & 0 \\ 0 & \phi\mu & 0 & -(\rho + \lambda) - \lambda & 0 & 0 \\ 0 & 0 & \sigma & \rho & -\lambda - \lambda & 0 \\ 0 & 0 & \kappa & \nu & 0 & -\eta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{25000}-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-2687}{50000}-\lambda & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{80} & \frac{-4937}{50000}-\lambda & 0 & 0 & 0 \\ 0 & \frac{103}{2500} & 0 & \frac{-427}{500}-\lambda & 0 & 0 \\ 0 & 0 & \frac{987}{10000} & \frac{427}{500} & -\lambda-\lambda & 0 \\ 0 & 0 & \frac{3}{10000} & \frac{1}{1000} & 0 & \frac{-1}{100}-\lambda \end{bmatrix}$$

Its eigenvalues are

$$\lambda_1 = -0.01\,, \quad \lambda_2 \approx -0.00004 < 0\,, \quad \lambda_3 \approx -0.05374 < 0\,,$$
$$\lambda_4 \approx -0.09874 < 0 \quad \lambda_5 \approx -0.8539 < 0 \quad \lambda_6 \approx -0.00008 < 0$$

Therefore, the point of equilibrium is not stable.

3.4 Solution of the Model

The system of initial-valued fractional equations is solved using SADM in this subsection.

$$D_\varepsilon S(t) = \Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma SQ$$

$$D_\varepsilon E(t) = \frac{\alpha S(I+\beta A)}{N} + \gamma SQ - (1-\phi)\delta E - \phi\mu E - \lambda E$$

$$D_\varepsilon I(t) = (\delta - \phi\delta)E - (\sigma+\lambda)\, I$$

$$D_\varepsilon A(t) = \phi\mu E - (\rho+\lambda)\, A$$

$$D_\varepsilon R(t) = \sigma I + \rho A - \lambda R$$

$D_\varepsilon Q(t) = \kappa I + vA - \eta Q$

$S(0) = S_0\,,\; E(0) = E_0\,, I(0) = I_0\,, A(0) = A_0\,,\; R(0) = R_0\,, Q(0) = Q_0$

On both sides of each model equation, we use the sumudu transform.

$$S[D_\varepsilon S(t)] = S\left[\Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma SQ\right]$$

$$S[D_\varepsilon E(t)] = S\left[\frac{\alpha S(I+\beta A)}{N} + \gamma SQ - (1-\phi)\delta E - \phi\mu E - \lambda E\right]$$

$$S[D_\varepsilon I(t)] = S[(\delta - \phi\delta)E - (\sigma + \lambda)\, I]$$

$$S[D_\varepsilon A(t)] = S[\phi\mu E - (\rho + \lambda)\, A]$$

$$S[D_\varepsilon R(t)] = S[\sigma I + \rho A - \lambda R]$$

$$S[d_\varepsilon Q(t)] = S[\kappa I + vA - \eta Q]$$

$$u^{-\varepsilon}[S(t) - S(0)] = S\left[\Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma SQ\right]$$

$$u^{-\varepsilon}[E(t) - E(0)] = S\left[\frac{\alpha S(I+\beta A)}{N} + \gamma SQ - (1-\phi)\delta E - \phi\mu E - \lambda E\right]$$

$$u^{-\varepsilon}[I(t) - I(0)] = S[(\delta - \phi\delta)E - (\sigma + \lambda)\, I]$$

$$u^{-\varepsilon}[A(t) - A(0)] = S[\phi\mu E - (\rho + \lambda)\, A]$$

$$u^{-\varepsilon}[R(t) - R(0)] = S[\sigma I + \rho A - \lambda R]$$

$$u^{-\varepsilon}[Q(t) - Q(0)] = S[\kappa I + vA - \eta Q]$$

The inverse Sumudu transform is applied to both sides of the equation, and the initial conditions are substituted.

$$S(t) = S(0) + S^{-1}$$

$$E(t) = E(0) + S^{-1}$$

$$I(t) = I(0) + S^{-1}$$

$$A(t) = A(0) + S^{-1}$$

$$R(t) = R(0) + S^{-1}$$

$$Q(t) = Q(0) + S^{-1}$$

$$S = w_1 + S^{-1}\left[u^{\varepsilon} S\left\{\Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \gamma SQ\right\}\right]$$

$$E = w_2 + S^{-1}\left[u^{\varepsilon} S\left\{\frac{\alpha S(I+\beta A)}{N} + \gamma SQ - (1-\phi)\delta E - \phi\mu E - \lambda E\right\}\right]$$

$$I = w_3 + S^{-1}[u^{\varepsilon} S\{(\delta - \phi\delta)E - (\sigma + \lambda)\, I\}]$$

$$A = w_4 + S^{-1}[u^{\varepsilon} S\{\phi\mu E - (\rho + \lambda)A\}]$$

$$R = w_5 + S^{-1}[u^{\varepsilon} S\{\sigma I + \rho A - \lambda R\}]$$

$$Q = w_6 + S^{-1}[u^{\varepsilon} S\{\kappa I + vA - \eta Q\}]$$

Let's think about how to put ADM to use by

$$S = \sum_{i=0}^{\infty} S_i \quad , \quad E = \sum_{i=0}^{\infty} E_i \ , \ I = \sum_{i=0}^{\infty} I_i$$

$$A = \sum_{i=0}^{\infty} A_i \ , R = \sum_{i=0}^{\infty} R_i \quad , \quad Q = \sum_{i=0}^{\infty} Q_i$$

The nonlinearities SI , SA AND SQ can be written as

$$SI = \sum_{i=0}^{\infty} Y_i \quad , Y_i = \sum_{j=0}^{i} S_j \, I_{i-j} \quad , AS = \sum_{i=0}^{\infty} X_i \quad , \ X_i = \sum_{j=0}^{i} A_j \, S_{i-j} \ ,$$

$$SQ = \sum_{i=0}^{\infty} Z_i \quad , \quad Z_i = \sum_{j=0}^{i} S_j \, Q_{i-j}$$

$$\sum_{i=0}^{\infty} S_i = w_1 + S^{-1}\left[u^{\varepsilon} S\left\{\Delta - \lambda \sum_{i=0}^{\infty} S_i - \frac{\alpha \sum_{i=0}^{\infty} S_i\left(\sum_{i=0}^{\infty} I_i + \beta \sum_{i=0}^{\infty} A_i\right)}{N} - \gamma \sum_{i=0}^{\infty} Z_i\right\}\right]$$

$$\sum_{i=0}^{\infty} E_i = w_2 + S^{-1}\left[u^{\varepsilon} S\left\{\frac{\alpha \sum_{i=0}^{\infty} S_i\left(\sum_{i=0}^{\infty} I_i + \beta \sum_{i=0}^{\infty} A_i\right)}{N} + \gamma \sum_{i=0}^{\infty} Z_i - ((1-\phi)\delta - \phi\mu - \lambda) \sum_{i=0}^{\infty} E_i\right\}\right]$$

$$\sum_{i=0}^{\infty} I_i = w_3 + S^{-1}[u^{\varepsilon} S\{(\delta - \phi\delta) \sum_{i=0}^{\infty} E_i - (\sigma + \lambda) \sum_{i=0}^{\infty} I_i\}]$$

$$\sum_{i=0}^{\infty} A_i = w_4 + S^{-1}[u^{\varepsilon} S\{\phi\mu \sum_{i=0}^{\infty} E_i - (\rho + \lambda) \sum_{i=0}^{\infty} A_i\}]$$

$$\sum_{i=0}^{\infty} R_i = w_5 + S^{-1}[u^{\varepsilon} S\{\sigma \sum_{i=0}^{\infty} I_i + \rho \sum_{i=0}^{\infty} A_i - \lambda \sum_{i=0}^{\infty} R_i\}]$$

$$\sum_{i=0}^{\infty} Q_i = w_6 + S^{-1}[u^{\varepsilon} S\{\kappa \sum_{i=0}^{\infty} I_i + v \sum_{i=0}^{\infty} A_i - \eta \sum_{i=0}^{\infty} Q_i\}]$$

$$S_0 = w_1 \ , \ E_0 = w_2 \ , I_0 = w_3 \ , A_0 = w_4 \ , R_0 = w_5 \ , Q_0 = w_6$$

$$S_{n+1} = S^{-1}\left[u^{\varepsilon} S\left\{\Delta - \lambda S_n - \frac{\alpha S_n I_n + \alpha S_n \beta A_n}{N} - \gamma Z_n\right\}\right]$$

$$S_{n+1} = S^{-1}\left[u^{\varepsilon}S\left\{\Delta - \ \lambda S_n - \frac{\alpha Y_n + \alpha\beta X_n}{N} - \gamma Z_n\right\}\right]$$

$$E_{n+1} = S^{-1}\left[u^{\varepsilon}S\left\{\frac{\alpha S_n I_n + \alpha S_n \beta A_n}{N} + \ \gamma Z_n - ((1-\phi)\delta - \phi\mu - \lambda)E_n\right\}\right]$$

$$E_{n+1} = S^{-1}\left[u^{\varepsilon}S\left\{\frac{\alpha Y_n + \alpha\beta X_n}{N} + \ \gamma Z_n - ((1-\phi)\delta - \phi\mu - \lambda)E_n\right\}\right]$$

$$I_{n+1} = S^{-1}[u^{\varepsilon}S\{(\delta - \phi\delta)E_n - (\sigma + \lambda)I_n\}]$$

$$A_{n+1} = S^{-1}[u^{\varepsilon}S\{\phi\mu E_n - (\rho + \lambda)A_n\}]$$

$$R_{n+1} = S^{-1}[u^{\varepsilon}S\{\sigma I_n + \rho A_n - \lambda R_n\}]$$

$$Q_{n+1} = S^{-1}[u^{\varepsilon}S\{\kappa I_n + \upsilon A_n - \eta Q_n\}]$$

$$S_1 = S^{-1}\left[u^{\varepsilon}S\left\{\Delta - \ \lambda S_0 - \frac{\alpha Y_0 + \alpha\beta X_0}{N} - \gamma Z_0\right\}\right]$$

$$= \ S^{-1}\left[\left\{\Delta - \ \lambda S_0 - \frac{\alpha Y_0 + \alpha\beta X_0}{N} - \gamma Z_0\right\}u^{\varepsilon}\right] \quad , \quad u^{\varepsilon} = \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$S_1 \ = \ \left[\left\{\Delta - \ \lambda S_0 - \frac{\alpha Y_0 + \alpha\beta X_0}{N} - \gamma Z_0\right\}\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$S_1 \ = \ \left[\left\{\Delta - \ \lambda S_0 - \frac{\alpha S_0 I_0 + \alpha S_0 \beta A_0}{N} - \gamma S_0 Q_0\right\}\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$S_1 = W_1 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$E_1 = S^{-1}\left[\left\{\frac{\alpha Y_n + \alpha\beta X_n}{N} + \ \gamma Z_n - ((1-\phi)\delta - \phi\mu - \lambda)E_n\right\}u^{\varepsilon}\right]$$

$$u^{\varepsilon} = \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$E_1 = \left[\left\{\frac{\alpha Y_0 + \alpha\beta X_0}{N} + \gamma Z_0 - ((1-\phi)\delta - \phi\mu - \lambda)E_0\right\}\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$E_1 = W_2 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$I_1 = S^{-1}[u^{\varepsilon} S\{(\delta - \phi\delta)E_0 - (\sigma+\lambda)I_0\}]$$

$$I_1 = [\{(\delta - \phi\delta)E_0 - (\sigma+\lambda)I_0\}u^{\varepsilon}]$$

$$u^{\varepsilon} = \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$I_1 = \left[\{(\delta - \phi\delta)E_0 - (\sigma+\lambda)I_0\}\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$I_1 = W_3 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$A_1 = S^{-1}[u^{\varepsilon} S\{\phi\mu E_0 - (\rho+\lambda)A_0\}]$$

$$A_1 = [\{\phi\mu E_0 - (\rho+\lambda)A_0\}u^{\varepsilon}]$$

$$u^{\varepsilon} = \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$A_1 = \left[\{\phi\mu E_0 - (\rho+\lambda)A_0\}\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$A_1 = W_4 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$R_1 = S^{-1}[u^{\varepsilon} S\{\sigma I_n + \rho A_n - \lambda R_n\}]$$

$$R_1 = [\{\sigma I_0 + \rho A_0 - \lambda R_0\} u^{\varepsilon}]$$

$$u^{\varepsilon} = \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$R_1 = \left[\{\sigma I_0 + \rho A_0 - \lambda R_0\} \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$R_1 = W_5 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$Q_1 = S^{-1}[u^{\varepsilon} S\{\kappa I_0 + v A_0 - \eta Q_0\}]$$

$$Q_1 = [\{\kappa I_0 + v A_0 - \eta Q_0\} u^{\varepsilon}]$$

$$u^{\varepsilon} = \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$Q_1 = \left[\{\kappa I_0 + v A_0 - \eta Q_0\} \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right]$$

$$Q_1 = W_6 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$S_2 = S^{-1}\left[u^{\varepsilon} S\left\{- \lambda S_1 - \frac{\alpha Y_1 + \alpha\beta X_1}{N} - \gamma Z_1\right\}\right]$$

$$S_2 = S^{-1}\left[u^{\varepsilon} S\left\{- \lambda w_1 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} - \frac{\alpha(S_0 w_3 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + w_1 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} I_0) + \alpha\beta(A_0 w_1 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + w_4 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} S_0)}{N} - \right.\right.$$

$$\left.\left.\gamma(S_0 w_6 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + w_1 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} Q_0)\right\}\right]$$

$$S_2 = S^{-1}\left[u^{\varepsilon}S\left\{-\lambda w_1 u^{\varepsilon} - \frac{\alpha(S_0 w_3 u^{\varepsilon} + w_1 u^{\varepsilon} I_0) + \alpha\beta(A_0 w_1 u^{\varepsilon} + w_4 u^{\varepsilon} S_0)}{N} - \gamma(S_0 w_6 u^{\varepsilon} + w_1 u^{\varepsilon} Q_0)\right\}\right]$$

$$S_2 = \left[\left\{\Delta - \lambda w_1 u^{2\varepsilon} - \frac{\alpha(S_0 w_3 u^{2\varepsilon} + w_1 u^{2\varepsilon} I_0) + \alpha\beta(A_0 w_1 u^{2\varepsilon} + w_4 u^{2\varepsilon} S_0)}{N} - \gamma(S_0 w_6 u^{2\varepsilon} + w_1 u^{2\varepsilon} Q_0)\right\}\right]$$

$$S_2 = \left[\left\{-\lambda w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \frac{\alpha(S_0 w_3 + w_1 I_0)\frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + \alpha\beta(A_0 w_1 + w_4 S_0)\frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}}{N} - \gamma(S_0 w_6 + w_1 Q_0)\frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\right\}\right]$$

$$E_2 = S^{-1}\left[u^{\varepsilon}\, S\left\{\frac{\alpha Y_1 + \alpha\beta X_1}{N} + \gamma Z_1 - ((1-\phi)\delta - \phi\mu - \lambda)E_1\right\}\right]$$

$$E_2 = S^{-1}\left[u^{\varepsilon}\, S\left\{\frac{\alpha Y_1 + \alpha\beta X_1}{N} + \gamma Z_1 - ((1-\phi)\delta - \phi\mu - \lambda)E_1\right\}\right]$$

$$E_2 = S^{-1}\left[u^{\varepsilon}\, S\left\{\frac{\alpha}{N} S_0 w_3 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{\alpha}{N} w_1 I_0 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{\alpha\beta}{N} A_0 w_1 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{\alpha\beta}{N} w_4 S_0 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \gamma S_0 w_6 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \gamma w_1 Q_0 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} - (\delta - \phi\delta + \phi\mu + \lambda) w_2 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right\}\right]$$

$$E_2 = S^{-1}\left[u^{\varepsilon}\, S\left\{\frac{\alpha}{N} S_0 w_3 u^{\varepsilon} + \frac{\alpha}{N} w_1 I_0 u^{\varepsilon} + \frac{\alpha\beta}{N} A_0 w_1 u^{\varepsilon} + \frac{\alpha\beta}{N} w_4 S_0 u^{\varepsilon} + \gamma S_0 w_6 u^{\varepsilon} + \gamma w_1 Q_0 u^{\varepsilon} - (\delta - \phi\delta + \phi\mu + \lambda) w_2 u^{\varepsilon}\right\}\right]$$

$$E_2 = \left[\left\{\frac{\alpha}{N} S_0 w_3 u^{2\varepsilon} + \frac{\alpha}{N} w_1 I_0 u^{2\varepsilon} + \frac{\alpha\beta}{N} A_0 w_1 u^{2\varepsilon} + \frac{\alpha\beta}{N} w_4 S_0 u^{2\varepsilon} + \gamma S_0 w_6 u^{2\varepsilon} + \gamma w_1 Q_0 u^{2\varepsilon} - (\delta - \phi\delta + \phi\mu + \lambda) w_2 u^{2\varepsilon}\right\}\right]$$

$$E_2 = \left[\left\{\frac{\alpha}{N} S_0 w_3 + \frac{\alpha}{N} w_1 I_0 + \frac{\alpha\beta}{N} A_0 w_1 + \frac{\alpha\beta}{N} w_4 S_0 + \gamma S_0 w_6 + \gamma w_1 Q_0 - (\delta - \phi\delta + \phi\mu + \lambda) w_2\right\}\right] \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$I_2 = S^{-1}[u^\varepsilon S\{(\delta - \phi\delta)E_1 - (\sigma + \lambda) I_1\}]$$

$$I_2 = S^{-1}\left[u^\varepsilon S\left\{(\delta - \phi\delta) w_2 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)} - (\sigma + \lambda) w_3 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)}\right\}\right]$$

$$I_2 = S^{-1}[u^\varepsilon \{(\delta - \phi\delta) w_2 u^\varepsilon - (\sigma + \lambda) w_3 u^\varepsilon\}]$$

$$I_2 = S^{-1}[\{(\delta - \phi\delta) w_2 u^{2\varepsilon} - (\sigma + \lambda) w_3 u^{2\varepsilon}\}]$$

$$I_2 = [\{(\delta - \phi\delta) w_2 - (\sigma + \lambda) w_3\}] \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$A_2 = S^{-1}[u^\varepsilon S\{\phi\mu E_1 - (\rho + \lambda) A_1\}]$$

$$A_2 = S^{-1}\left[u^\varepsilon S\left\{\phi\mu w_2 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)} - (\rho + \lambda) w_4 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)}\right\}\right]$$

$$A_2 = [\{\phi\mu w_2 - (\rho + \lambda) w_4\}] \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$R_2 = S^{-1}[u^\varepsilon S\{\sigma I_1 + \rho A_1 - \lambda R_1\}]$$

$$R_2 = S^{-1}\left[u^\varepsilon S\left\{\sigma w_3 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)} + \rho w_4 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)} - \lambda w_5 \frac{t^\varepsilon}{\Gamma(\varepsilon+1)}\right\}\right]$$

$$R_2 = [\{\sigma w_3 + \rho w_4 - \lambda w_5\}] \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon + 1)}$$

$$Q_2 = S^{-1}[u^\varepsilon S\{\kappa I_1 + v A_1 - \eta Q_1\}]$$

$$Q_2 = S^{-1}\left[u^{\varepsilon} S\left\{\kappa w_3 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + v w_4 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} - \eta w_6 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}\right\}\right]$$

$$Q_2 = [\{\kappa w_3 + v w_4 - \eta w_6\}] \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$S_3 = S^{-1}\left[u^{\varepsilon} S\left\{-\lambda S_2 - \frac{\alpha Y_2 + \alpha\beta X_2}{N} - \gamma Z_2\right\}\right]$$

$$S_3 = S^{-1}\left[u^{\varepsilon} S\left\{-\lambda S_2 - \frac{\alpha(S_0 I_2 + S_1 I_1 + S_2 I_0) + \alpha\beta(A_0 S_2 + A_1 S_1 + A_2 S_0)}{N} - \gamma(S_0 Q_2 + S_1 Q_1 + S_2 Q_0)\right\}\right]$$

$$S_3 = S^{-1}\left[u^{\varepsilon} S\left\{-\lambda w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \frac{\alpha(S_0 w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + S_2 I_0)}{N} + \frac{\alpha\beta(A_0 w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} S_0)}{N} - \gamma(S_0 w_6 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} w_6 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} Q_0)\right\}\right]$$

$$S_3 = S^{-1}\left[u^{\varepsilon} S\left\{-\lambda w_1 u^{2\varepsilon} - \frac{\alpha(S_0 w_3 u^{2\varepsilon} + w_3 u^{2\varepsilon} w_1 u^{2\varepsilon} + S_2 w_3 u^{2\varepsilon})}{N} + \frac{\alpha\beta(A_0 w_1 u^{2\varepsilon} + w_4 u^{2\varepsilon} w_1 u^{2\varepsilon} + w_4 u^{2\varepsilon} S_0)}{N} - \gamma(S_0 w_6 u^{2\varepsilon} + w_1 u^{2\varepsilon} w_6 u^{2\varepsilon} + w_1 u^{2\varepsilon} Q_0)\right\}\right]$$

$$S_3 = \left[\left\{-\lambda w_1 u^{3\varepsilon} - \frac{\alpha(S_0 w_3 u^{3\varepsilon} + w_3 u^{3\varepsilon} w_3 u^{3\varepsilon} + w_1 u^{3\varepsilon} w_3 u^{3\varepsilon})}{N} + \frac{\alpha\beta(A_0 w_1 u^{3\varepsilon} + w_4 u^{3\varepsilon} w_1 u^{3\varepsilon} + w_4 u^{3\varepsilon} S_0)}{N} - \gamma(S_0 w_6 u^{3\varepsilon} + w_1 u^{3\varepsilon} w_6 u^{3\varepsilon} + w_1 u^{3\varepsilon} Q_0)\right\}\right]$$

$$S_3 = \left[\left\{- \lambda S_2 - \frac{\alpha(S_0 w_3 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + w_1 w_3 \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} + w_1 I_0 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)})}{N} + \right.\right.$$

$$\frac{\alpha\beta(A_0 w_1 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + w_4 w_1 \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} + w_4 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} S_0)}{N} - \gamma(S_0 w_6 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + w_1 w_6 \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} +$$

$$\left.\left. w_1 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} Q_0)\right\}\right]$$

$$E_3 = S^{-1}\left[u^{\varepsilon}\, S\left\{\frac{\alpha Y_2 + \alpha\beta X_2}{N} + \gamma Z_2 - ((1-\phi)\delta - \phi\mu - \lambda)E_2\right\}\right]$$

$$E_3 = S^{-1}\left[u^{\varepsilon}\, S\left\{\frac{\alpha(S_0 I_2 + S_1 I_1 + S_2 I_0) + \alpha\beta(A_0 S_2 + A_1 S_1 + A_2 S_0}{N} + \gamma S_0 Q_2 + S_1 Q_1 + S_2 Q_0 - \right.\right.$$

$$\left.\left.((1-\phi)\delta - \phi\mu - \lambda)E_2\right\}\right]$$

$$E_3 = S^{-1}\left[u^{\varepsilon}\, S\left\{\begin{array}{c} \frac{\alpha\left(S_0 w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} I_0\right)}{N} \\ \frac{\alpha\beta\left(A_0 w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} S_0\right)}{N} \\ + \gamma(S_0 w_6 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} w_6 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + w_1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} Q_0) \\ -((1-\phi)\delta - \phi\mu - \lambda) w_2 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} \end{array}\right\}\right]$$

$$E_3 = S^{-1}\left[u^{\varepsilon}\, S\left\{+ \begin{array}{c} \frac{\alpha\left(S_0 w_3 u^{2\varepsilon} + w_1 u^{2\varepsilon} w_3 u^{2\varepsilon} + w_1 u^{2\varepsilon} I_0\right)}{N} \\ \frac{\alpha\beta\left(A_0 w_1 u^{2\varepsilon} + w_4 u^{2\varepsilon} w_1 u^{2\varepsilon} + w_4 u^{2\varepsilon} S_0\right)}{N} \\ + \gamma(S_0 w_6 u^{2\varepsilon} + w_1 u^{2\varepsilon} w_6 u^{2\varepsilon} + w_1 u^{2\varepsilon} Q_0) \\ -((1-\phi)\delta - \phi\mu - \lambda) w_2 u^{2\varepsilon} \end{array}\right\}\right]$$

$$E_3 = \left[\left\{+ \begin{array}{c} \frac{\alpha\left(S_0 w_3 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + w_1 w_3 \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} + w_1 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} I_0\right)}{N} \\ \frac{\alpha\beta\left(A_0 w_1 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + w_4 w_1 \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} + w_4 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} S_0\right)}{N} \\ + \gamma(S_0 w_6 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + w_1 w_6 \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} + w_1 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} Q_0) \\ -((1-\phi)\delta - \phi\mu - \lambda) w_2 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} \end{array}\right\}\right]$$

$$I_3 = S^{-1}[u^{\varepsilon} S\{(\delta - \phi\delta)E_2 - (\sigma + \lambda)I_2\}]$$

$$I_3 = S^{-1}\left[u^{\varepsilon} S\left\{(\delta - \phi\delta)w_2 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - (\sigma + \lambda)w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\right\}\right]$$

$$I_3 = S^{-1}[u^{\varepsilon} S\{(\delta - \phi\delta)w_3 u^{2\varepsilon} - (\sigma + \lambda)w_3 u^{2\varepsilon}\}]$$

$$I_3 = [\{(\delta - \phi\delta)w_2 u^{3\varepsilon} - (\sigma + \lambda)w_3 u^{3\varepsilon}\}]$$

$$I_3 = [\{(\delta - \phi\delta)w_2 - (\sigma + \lambda)w_3\}] \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$A_3 = S^{-1}[u^{\varepsilon}\, S\{\phi\mu E_2 - (\rho + \lambda)\, A_2\}]$$

$$A_3 = S^{-1}\left[u^{\varepsilon}\, S\left\{\phi\mu w_2 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - (\rho + \lambda)\, w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\right\}\right]$$

$$A_3 = S^{-1}[u^{\varepsilon}\, S\{\phi\mu w_2 u^{2\varepsilon} - (\rho + \lambda)\, w_4 u^{2\varepsilon}\}]$$

$$A_3 = [\{\phi\mu w_2 u^{3\varepsilon} - (\rho + \lambda)\, w_4 u^{3\varepsilon}\}]$$

$$A_3 = \left[\left\{\phi\mu w_2 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} - (\rho + \lambda)\, w_4 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}\right\}\right]$$

$$R_3 = S^{-1}[u^{\varepsilon}\, S\{\sigma I_2 + \rho A_2 - \lambda R_2\}]$$

$$R_3 = S^{-1}\left[u^{\varepsilon}\, S\left\{\sigma w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + \rho w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \lambda w_5 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\right\}\right]$$

$$R_3 = S^{-1}[u^{\varepsilon}\, S\{\sigma w_3 u^{2\varepsilon} + \rho w_4 u^{2\varepsilon} - \lambda w_5 u^{2\varepsilon}\}]$$

$$R_3 = [\{\sigma w_3 u^{3\varepsilon} + \rho w_4 u^{3\varepsilon} - \lambda w_5 u^{3\varepsilon}\}]$$

$$R_3 = \left[\left\{\sigma w_3 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + \rho w_4 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} - \lambda w_5 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}\right\}\right]$$

$$Q_3 = S^{-1}[u^{\varepsilon} S\{\kappa I_2 + v A_2 - \eta Q_2\}]$$

$$Q_3 = S^{-1}\left[u^{\varepsilon} S\left\{\kappa w_3 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + v w_4 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \eta w_6 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\right\}\right]$$

$$Q_3 = S^{-1}[u^{\varepsilon} S\{\kappa w_3 u^{2\varepsilon} + v w_4 u^{2\varepsilon} - \eta w_6 u^{2\varepsilon}\}]$$

$$Q_3 = [\{\kappa w_3 u^{3\varepsilon} + v w_4 u^{3\varepsilon} - \eta w_6 u^{3\varepsilon}\}]$$

$$Q_3 = \left[\left\{\kappa w_3 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + v w_4 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} - \eta w_6 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}\right\}\right]$$

Here, we take three terms of the infinite series of S, E, I, A, R, and using Q as a rough estimate. In the following we display the result of the mathematical model.

$$\begin{cases} s(t) = \sum_{i=0}^{\infty} S_i(t) \approx S_0(t) + S_1(t) + S_2(t) + S_3(t) \\ E(t) = \sum_{i=0}^{\infty} E_i(t) \approx E_0(t) + E_1(t) + E_2(t) + E_3(t) \\ I(t) = \sum_{i=0}^{\infty} I_i(t) \approx I_0(t) + I_1(t) + I_2(t) + I_3(t) \\ A(t) = \sum_{i=0}^{\infty} A_i(t) \approx A_0(t) + A_1(t) + A_2(t) + A_3(t) \\ R(t) = \sum_{i=0}^{\infty} R_i(t) \approx R_0(t) + R_1(t) + R_2(t) + R_3(t) \\ Q(t) = \sum_{i=0}^{\infty} Q_i(t) \approx Q_0(t) + Q_1(t) + Q_2(t) + Q_3(t) \end{cases}$$

3.5 Numerical Simulation

Here, we make some adjustments to the starting points $S(0) = 1350900000$, $E(0) = 1724266$, $I(0) = 745$, $A(0) = 413$, $R(0) = 66$, $Q(0) = 10000$, $N = 1352600000$. We take $\varepsilon = 1$ for results closer to reality.

$$S_0 = 1350900000$$

$$S_1 = -164212.6 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$S_2 = 677 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$S_3 = -1262 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$s(t) = \sum_{i=0}^{\infty} S_i(t) \approx S_0(t) + S_1(t) + S_2(t) + S_3(t)$$

$$\approx 1350900000 - 164212.6 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + 677 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - 1262 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$E_0 = 1724266$$

$$E_1 = 212809 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$E_2 = 5599 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$E_3 = 1102 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$E(t) = \sum_{i=0}^{\infty} E_i(t) \approx E_0(t) + E_1(t) + E_2(t) + E_3(t)$$

$$\approx 1724266 + 212809 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + 5599 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + 1102 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$I_0 = 745$$

$$I_1 = 21824 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$I_2 = 543 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$I_3 = 17 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$I(t) = \sum_{i=0}^{\infty} I_i(t) \approx I_0(t) + I_1(t) + I_2(t) + I_3(t)$$

$$\approx 745 + 212809 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + 5599 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + 1102 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$A_0 = 413$$

$$A_1 = 70686 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$A_2 = -51315 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$A_3 = 43847.6 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$A(t) = \sum_{i=0}^{\infty} A_i(t) \approx A_0(t) + A_1(t) + A_2(t) + A_3(t)$$

$$\approx 413 + 70686 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} - 51315 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + 43847.6 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$R_0 = 66$$

$$R_1 = 418 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$R_2 = 62050 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$R_3 = -43570 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$R(t) = \sum_{i=0}^{\infty} R_i(t) \approx R_0(t) + R_1(t) + R_2(t)$$

$$\approx 66 + 418 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + 62050 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - 43570 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$Q_0 = 10000$$

$$Q_1 = -99.29 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}$$

$$Q_2 = 80.1 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$

$$Q_3 = -51.2 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

$$Q(t) = \sum_{i=0}^{\infty} Q_i(t) \approx Q_0(t) + Q_1(t) + Q_2(t)$$

$$\approx \ 10000 - 99.29 \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + 80.17 \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - 51.89 \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}$$

Numerical simulations of the exposed population S(t), the exposed population E(t), the affected population I (t), the asymptotic affected population A (t), the recovered population R(t), and the reservoir population Q will be put through their paces (t) Figures 3.5, 3.6, 3.7, 3.8, 3.9, and 3.10; Tables 3.2, 3.3, 3.4, 3.5, 3.6, 3.7.

Table 3.2 Numerical simulations of S(t) susceptible population

t	susceptible population S(t)
	$\varepsilon = 1$
0	1.3509×10^{9}
50	1.3172×10^{9}
100	1.1275×10^{9}
150	6.2400×10^{8}
200	3.5106×10^{8}

Table 3.3 Numerical simulations of E(t) exposed people

t	Exposed population E(t)
	$\varepsilon = 1$
0	1.7242$\times 10^{6}$
50	4.2321$\times 10^{7}$
100	2.2973$\times 10^{8}$
150	7.1650$\times 10^{8}$
200	1.3130$\times 10^{9}$

Table 3.4 Numerical simulations of Affected persons I (t)

t	Infected population I(t)
	$\varepsilon = 1$
0	745
50	2.1311×10^{6}
100	7.7564×10^{6}
150	1.9001×10^{7}
200	3.7892$\times 10^{7}$

Table 3.5 Numerical simulations asymptotic affected persons A (t)

t	Infected population A asymptotically (t)
	$\varepsilon = 1$
0	413
50	8.5286$\times 10^{8}$
100	7.0583$\times 10^{9}$
150	2.3530$\times 10^{10}$
200	5.7454$\times 10^{10}$

Table 3.6 Numerical simulations of recovered population R(t)

t	Recovered population R(t)
	$\varepsilon = 1$
0	**66**
20	4.5660×10^{7}
40	4.1506×10^{8}
60	1.4568×10^{9}
80	3.5193×10^{9}

Table 3.7 Numerical simulations of Reservoir population Q(t)

t	Reservoir population Q(t)
	$\varepsilon = 1$
0	10000
1.5	9912.5
3.0	9045
4.5	6512
6.0	5176

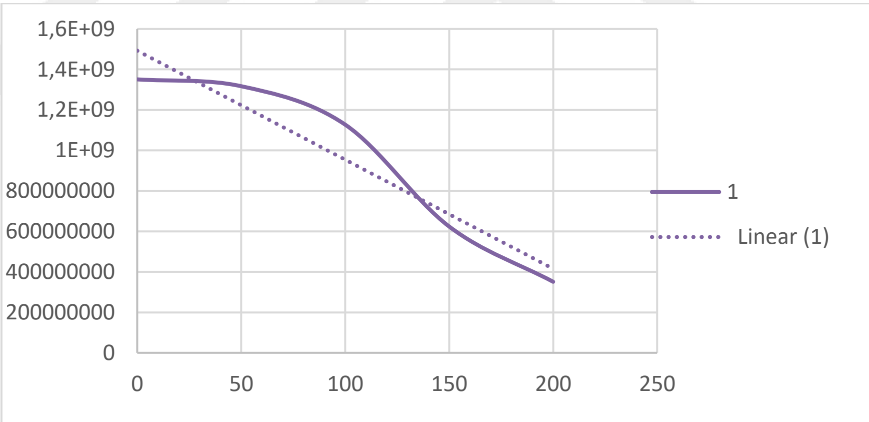

Figure 3.5 Dynamics of susceptible population

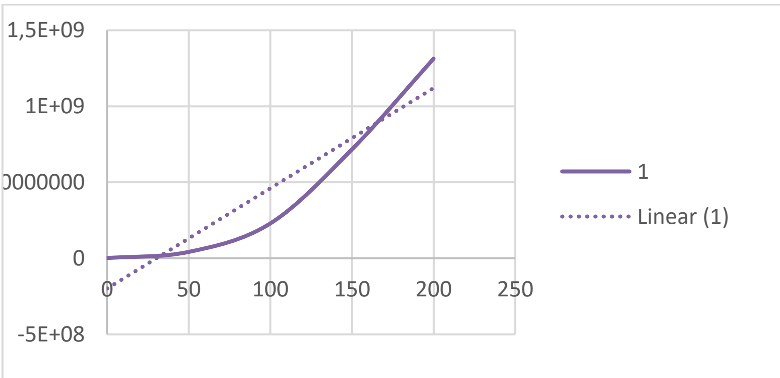

Figure 3.6 Dynamics of exposed people

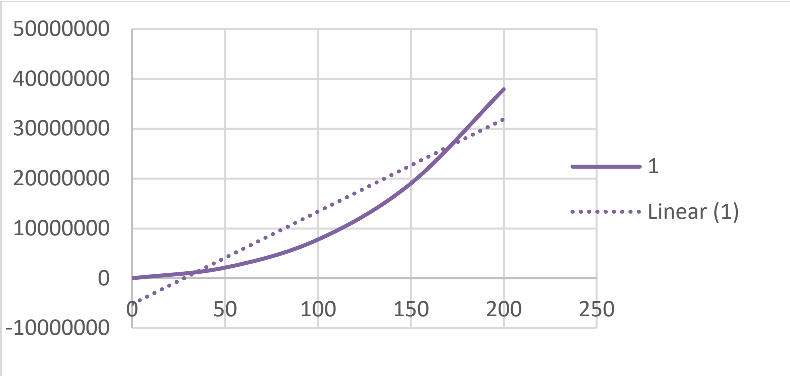

Figure 3.7 Dynamics of affected persons

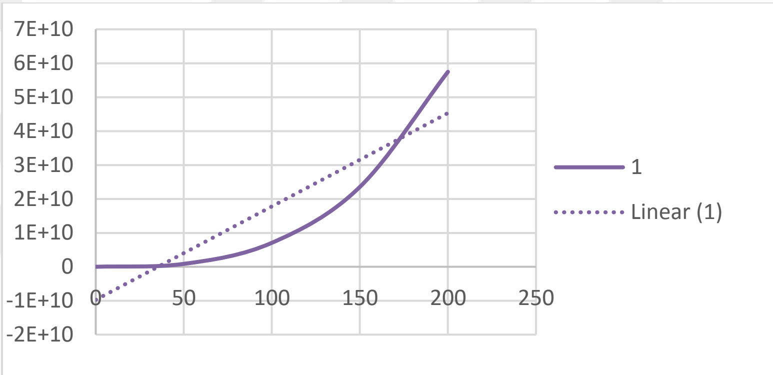

Figure 3.8 Dynamics of asymptotic affected persons

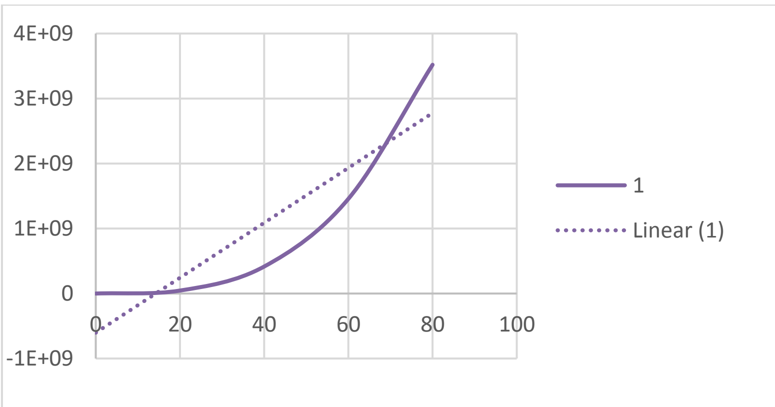

Figure 3.9 Dynamics of recovered population

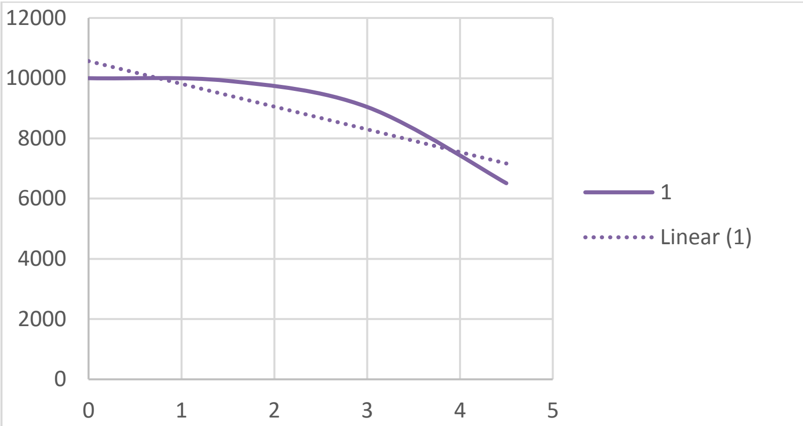

Figure 3.10 Dynamics of Reservoir population

4. CONCLUSIONS AND RECOMMENDATION

Conclusions and recommendations for further research are presented in this section.

- We found the equilibrium point, tested the stability, and according to Hartmann's theory, they have the same behavior. We then solved the modified model by the Sumudu transformation method. The results obtained graphically illustrated.

- We noticed that the values of $S(t)$, $E(t)$, $I(t)$, $A(t)$, $R(t)$, $Q(t)$ are closer to reality when it is a $\varepsilon = 1$

- The fractional models are better than the classical models with natural order derivatives.

- $S(t), E(t), I(t), A(t), R(t), Q(t)$ close values agree with medical measurement.

- In the future, the concerned fractional COVID-19 model may also be examined using additional cutting-edge numerical techniques with new findings and conclusions in the context of numerical simulations.

REFERENCES

Ahmad, M., Jiang, J., Zada, A., Shah, S. O. and Xu, J. 2020. Analysis of coupled system of implicit fractional differential equations involving Katugampola–Caputo fractional derivative. Complexity, 2020: 11 pages.

Barreira, L. and Valls, C. 2013. Dynamical systems. An introduction. Translated from the 2012 Portuguese original. Universitext. Springer, London, x+209 pp.

Belgacem, F. B. M. and Karaballi, A. A. 2006. Sumudu transform fundamental properties investigations and applications. International Journal of Stochastic Analysis, 2006: 1-23

Boyce, W. E. and DiPrima, R.C. 1965. Elementary differential equations and boundary value problems. John Wiley & Sons, Inc., New York-London-Sydney, xi+485 pp.

Burton, T. A. 2005. Volterra integral and differential equations. Second edition. Mathematics in Science and Engineering, 202. Elsevier B. V., Amsterdam, 2005. x+353 pp

Caputo, M. and Fabrizio, M. 2015. A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation & Applications, 1(2): 73-85.

Chen, T. M., Rui, J., Wang, Q. P., Zhao, Z. Y., Cui, J. A. and Yin, L. 2020. A mathematical model for simulating the phase-based transmissibility of a novel coronavirus. Infectious diseases of poverty, 9(1): 1-8.

Cohen, A. M. 2007. Numerical methods for Laplace transform inversion (Vol. 5). Springer Science & Business Media, New York, xiv+251 pp.

Daftardar-Gejji, V. 2014. Fractional Calculus: Theory and Applications. Narosa Publishing House, Mumbai, India, 232 pages.

Darzi, R., Mohammadzade, B., Mousavi, S. and Beheshti, R. 2013. Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation. J. Math. Comput. Sci, *6*(1): 79-84.

Das, S. 2020. Kindergarten of Fractional Calculus, Cambridge Scholars Publishing, Newcastle, UK, 533 pages.

Derrick, W. and Grossman, S. I. 1976. Elementary Differential Equations With Applications. Addison-Wesley Publishing Company, Philippines, 468-489.

Diethelm, K. 2010. The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type. Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, viii+247 pp.

Ferguson, N., Laydon, D., Nedjati Gilani, G., Imai, N., Ainslie, K., Baguelin, M., Bhatia, S., Boonyasiri, A., Cucunuba Perez, Z., Cuomo-Dannenburg, G., Dighe, A., Dorigatti, I., Fu, H., Gaythorpe, K., Green, W., Hamlet, A., Hinsley, W., Okell, L., Van Elsland, S., Thompson, H., Verity, R., Volz, E., Wang, H., Wang, Y., Walker, P., Walters, C., Winskill, P., Whittaker, C., Donnelly, C., Riley, S. and Ghani, A. 2020. Report 9: Impact of non-pharmaceutical interventions (NPIs) to reduce COVID-19 mortality and healthcare demand. DOI: 10.25561/77482

Ge, F., Chen, Y. and Kou, C. 2018. Regional analysis of time-fractional diffusion processes. Springer, Berlin, Germany, xix+250 pp.

Guckenheimer, J. and Holmes, P. 1983. Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Applied Mathematical Sciences, 42. Springer-Verlag, New York, xvi+453 pp.

Heymans, N. and Podlubny, I. 2006. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. Rheologica Acta, 45(5):765-771.

Kazem, S. 2013. Exact solution of some linear fractional differential equations by Laplace transform. International Journal of nonlinear science, 16(1): 3-11.

Khan, M. A. and Atangana, A. 2020. Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative. Alexandria Engineering Journal, 59(4): 2379-2389.

Latifi, S. 2020. 17th International Conference on Information Technology-New Generations (ITNG 2020). Springer International Publishing, Switzerland AG, xv+742 pages

Lichae, B. H., Biazar, J. and Ayati, Z. 2019. The fractional differential model of HIV-1 infection of CD4+ T-cells with description of the effect of antiviral drug treatment. Computational and mathematical methods in medicine, 2019:12 pages.

Losada, J. and Nieto, J. J. 2015. Properties of a new fractional derivative without singular kernel. Progr. Fract. Differ. Appl, 1(2): 87-92.

May, R. M. 2001. Stability and complexity in model ecosystems. In Stability and Complexity in Model Ecosystems. Princeton university press, New Jersey, 301 pages.

Mirevski, S. P., Boyadjiev, L. and Scherer, R. 2007. On the Riemann–Liouville fractional calculus, g-Jacobi functions and F-Gauss functions. Applied Mathematics and Computation, 187(1): 315-325.

Organization, W. H. 2020. Coronavirus disease 2019 (COVID-19): situation report—37. https://www.who.int/docs/default-source/coronaviruse/situation-reports/20200226-sitrep-37-covid-19.pdf?sfvrsn=6126c0a4_2

Owolabi, K. M. and Atangana, A. 2019. Numerical methods for fractional differentiation. Springer Series in Computational Mathematics, 54. Springer, Singapore, xvi+328 pp.

Podlubny, I. 1998. Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, xxiv+340 pp.

Rahman, M. 2007. Integral equations and their applications. WIT Press, Southampton, xiv+356 pp.

Roberts, A. J. 2009. Normal form of stochastic or deterministic multiscale differential equations. Technical report, http://www. maths. adelaide. edu. au/anthony. roberts/sdenf. html.

Shaikh, A. S., Shaikh, I. N. and Nisar, K. S. 2020. A mathematical model of COVID-19 using fractional derivative: outbreak in India with dynamics of transmission and control. Advances in Difference Equations, 2020(1) 1-19.

Srivastava, H. M. and Choi, J. 2012. Zeta and q-Zeta functions and associated series and integrals. Elsevier, Inc., Amsterdam, xvi+657 pp.

Tuluce Demiray, S., Bulut, H. and Belgacem, F. B. M. 2015. Sumudu transform method for analytical solutions of fractional type ordinary differential equations. Mathematical Problems in Engineering, 2015: 6 pages.

Xue, D. 2017. Fractional-order control systems. Fundamentals and numerical implementations. De Gruyter, Berlin, xv+372 pp.

CURRICULUM VITAE

Personal Information

Name and Surname : Alyaa Mhmood Faraj FARAJ

Education

MSc	Çankırı Karatekin University Graduate School of Natural and Applied Sciences Department of Mathematics	2019-Present
Undergraduate	Baghdad University Faculty of Sciences Department of Mathematics	2003-2008