GENERIC INITIAL IDEALS OF MODULAR POLYNOMIAL INVARIANTS

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

GENERIC INITIAL IDEALS OF MODULAR POLYNOMIAL INVARIANTS

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July 2020

We study the generic initial ideals (gin) of certain ideals that arise in modular invariant theory. For all the cases where an explicit generating set is known, we calculate the generic initial ideal of the Hilbert ideal of a cyclic group of prime order for all monomial orders. We also clarify gin for the Klein four group and note that its Hilbert ideals are Borel fixed with certain orderings of the variables. In all the situations we consider, there is a monomial order such that the gin of the Hilbert ideal is equal to its initial ideal. Along the way we show that gin respects a permutation of the variables in the monomial order.

Keywords: Generic initial ideals, modular polynomial invariants.

ÖZET

MODÜLER POLİNOM DEĞİŞMEZLERİNİN GENERİK BAŞ TERİM İDEALLERİ

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Biz modüler değişmez teorisinde ortaya çıkan ideallerin generik baş terim ideallerini çalışıyoruz. Üretici kümesi bilinen tüm durumlar için, eleman sayısı asal olan devirli grubun Hilbert idealinin generik baş terim idealini tüm monom sıralamaları için hesaplıyoruz. Klein 4-grubunun Hilbert idealinin değişkenlerin belli sıralaması ile birlikte Borel sabit olduğunu not ederek Klein 4-grubunun generik baş terim idealini de açıklığa kavuşturuyoruz. Düşündüğümüz tüm durumlarda bir monom sıralaması vardır öyle ki Hilbert idealinin generik baş terim ideali kendi baş terim idealine eşittir. Bu esnada generik baş terim idealinin monom sıralamanın içindeki değişkenlerin yer değiştirmesine saygı duyduğunu gösteriyoruz.

Anahtar sözcükler: Generik baş terim idealleri, modüler polinom değişmezleri.

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Contents

Chapter 1

Introduction

The aim of this thesis is to present rigorously a computation of generic initial ideals that arise in modular invariant theory. In order to accomplish this job, we add preliminary results concerning generic intial ideals and its basic properties, Borel fixed ideals and its connection with generic initial ideals and modular polynomial invariants.

Studying the initial monomials of elements of an ideal gives an important information about the algebraic and combinatoric properties of the ideal. The results of initial term computations depend on the choice of coordinates. We eliminate this dependency by using a generic change of basis and coordinates. After the change of coordinates, the initial ideal is coordinate-independent. This coordinate-independent ideal is called generic initial ideal. In other words, for a homogeneous ideal I in a polynomial ring, its generic initial ideal $gin_>(I)$ with respect to a term order $>$ is the ideal of initial monomials after a generic change of coordinates.

The generic initial ideal encodes much information on the combinatorial, geometrical and homological properties of I and the associated variety and plays a significant role in commutative algebra as well as in algebraic geometry. We exemplify this subject by considering Hartshorne's proof of the connectedness of

Hilbert schemes. Another supportive example is the fact that generic initial ideals are used to bound the invariants of projective varieties. Describing $gin_>(I)$ is a very difficult task in general and there are relatively few classes of ideals for which generic initial ideals are explicitly computed. For further information, see [1].

In this thesis, we study generic initial ideals that arise in invariant theory. We consider a finite dimensional module V of a group G over an infinite field F. There is an induced action on the symmetric algebra $F[V] := S(V^*)$ on V^*. This is a polynomial algebra $F[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ is a basis for V^*. A classical object is the ring of invariants

$$F[V]^G := \{f \in F[V] \mid \sigma(f) = f \text{ for all } \sigma \in G\}$$

which is a graded subalgebra of $F[V]$. The ideal in $F[V]$ generated by homogeneous invariants of positive degree is the Hilbert ideal of V and we denote it by $H(V)$.

When the characteristic of the field divides the order of the group, i.e., V is a modular module, the invariant ring is more complicated and difficult to obtain. Invariants are not known in general even in the simplest modular situation when G is a cyclic group of prime order. For this group, we consider the cases where an explicit generating set is known for the Hilbert ideal, and we compute the generic initial ideals of these Hilbert ideals for all orders. It turns out that, with the upper triangular ordering of the variables, gin is equal to the initial ideal of the Hilbert ideal in these cases.

We also consider the Klein four group and note that its Hilbert ideals are Borel fixed with certain orderings of the variables. In all the situations we consider, we observe that it is possible to select a monomial order such that the gin of the Hilbert ideal is equal to its initial ideal.

In the following, we briefly describe the contents of this thesis.

In Chapter 2, we closely follow [2], [3], and [1]. We discuss the results on generic initial ideals and then we give the definition of Borel fixed ideals. After the notion of Borel fixedness, we focus on its connections with generic initial ideals. Chapter 3 is dedicated to basic properties and definitions of invariant rings by following [4] and [5].

In Chapter 4, we present a proof of some results concerning how gin changes when the variables are permuted. More specifically, gin respects a permutation of the variables in the monomial order. For the details, see [6].

The main goal of the thesis is to compute generic initial ideals that appear in invariant theory. With the help of Chapter 4, we fulfill this task in Chapter 5. Case-by-case analysis and computations of the gin of the modular Hilbert ideals of a cyclic group of prime order are done in Chapter 5. Note that an explicit generating set of ideals that arise in modular invariant theory is known in the article [7]. Relying on this fact, we deal with the computations of the gin of the Hilbert ideals.

In the final chapter, we note that any Hilbert ideal of the Klein four group in characteristic two is Borel fixed with the right choice of the monomial order and therefore gin is equal to the Hilbert ideal itself. We feel that these findings support Conjecture 6.0.4 which states that for a given module over characteristic p of a p-group there is always a choice of basis for the module and a monomial order such that gin of the Hilbert ideal is equal to its initial ideal. For further details, see [6] and [8].

Chapter 2

Generic initial ideals

We review the definition and basic properties of generic initial ideals. We closely follow [2], [3], and [1]. Let S denote the polynomial ring $F[x_1, x_2, ..., x_n]$. We fix a monomial order $>$ on S which satisfies $x_1 > x_2 > \cdots > x_n$. To define generic initial ideals, we need to explain some preliminaries such as initial ideals, Zariski open sets, linear actions, and exterior powers.

Biggest monomial of a polynomial is called initial monomial. For a homogeneous ideal I, the initial ideal of it is the ideal generated by the initial monomial of all polynomials in I. It is denoted by $In_>(I)$. It provides an important information about algebraic and combinatorial properties of the ideal.

Let F be a field. A subset of F^k is called Zariski closed if it is the set of common zeros of a collection of polynomials. We say $U \in F^k$ is Zariski open if the complement of U is Zariski closed. In this chapter, we assume F is infinite since if it is finite, then any subset is Zariski closed, equivalently it is Zariski open as well.

Lemma 2.0.1. *Non-empty Zariski open sets have a non-empty intersection.*

Proof. Let $V_1, V_2, .., V_m$ be non-empty Zariski open sets. It is sufficient to prove for m=2. Let assume the complement of V_1 is the set of common zeros of $f_1, f_2, ..f_s$

and the complement of V_2 is the set of common zeros of $h_1, h_2, ..., h_r$. For $a \in V_1$ and $b \in V_2$, there exist f_i and h_j such that $f_i(a) \neq 0$, $h_j(b) \neq 0$. Since F is infinite, it follows that $\exists c \in F^k$ satisfying $f_i h_j(c) \neq 0$. Thus, $c \in V_1 \cap V_2$. $\square$

Remark 2.0.2. Any Zariski open set is a dense subset of F^k by 2.0.1.

Note that the general linear group $GL_n(F)$ is the group of all invertible $n \times n$ matrices. For $\alpha = (\alpha_{ij}) \in GL_n(F)$, the action of the general linear group on S is defined by

$$x_j \to \alpha_{1j}x_1 + \alpha_{2j}x_2 + \cdots + \alpha_{nj}x_n \text{ for } 1 \leq j \leq n.$$

It is a linear, degree preserving isomorphism on S. This type of isomorphism is called a linear isomorphism or linear automorphism.

Let $\alpha = (\alpha_{ij})$ be an element in $GL_n(F)$. We know that α is an invertible matrix by the definition of $GL_n(F)$. Thus, we have that the determinant of α is not zero and we get the following remark.

Remark 2.0.3. Let $M_n(F)$ be the set of all n-by-n matrices. The general linear group $GL_n(F)$ is a Zariski open subset of $M_n(F)$.

Before giving the definition of generic initial ideals, we need to give the concept of exterior power since it is necessary to show that the existence of generic initial ideals.

Definition 2.0.4. Let M be R-module and $M^{\otimes r}$ denotes the tensor product of M with itself r times. Then, rth exterior power of M, $\bigwedge^r(M)$ is the $M^{\otimes r}/I_r$ where I_r is spanned by $m_1 \otimes m_2 \otimes \cdots \otimes m_r$ such that $m_i = m_j$ for some $i \neq j$. The coset of $m_1 \otimes m_2 \otimes \cdots \otimes m_r$ is called the wedge product and denoted by $m_1 \wedge m_2 \wedge \cdots \wedge m_r$.

Let S_d denote the d-th homogeneous component of S and we consider the t-th exterior power $\wedge^t S_d$ of S_d. Recall that an element $m_1 \wedge m_2 \wedge \cdots \wedge m_t$, where m_i

is a monomial of degree d with $m_1 > m_2 > \cdots > m_t$, is called a standard exterior monomial of $\wedge^t S_d$ with respect to $>$.

One can order standard exterior monomials lexicographically: If $m_1 \wedge m_2 \wedge \cdots \wedge m_t$ and $w_1 \wedge w_2 \wedge \cdots \wedge w_t$ are two standard exterior monomials with respect to $>$, then we set

$$m_1 \wedge m_2 \wedge \cdots \wedge m_t > w_1 \wedge w_2 \wedge \cdots \wedge w_t$$

if $m_i > w_i$ for the smallest index i with $m_i \neq w_i$.

Theorem 2.0.5 (**Eisenbud, 1995**). *For any homogeneous ideal I, there exists a Zariski open set $U \in GL_n(F)$ such that $In_>(gI)$ is constant for all $g \in U$.*

Definition 2.0.6. Assume the convention of the previous theorem. Then, the constant ideal $In_>(gI)$ is called the generic initial ideal of I and notation is $gin_>(I)$.

Recall that $In_>(I)$ is depend on the choice of coordinates but after a generic change of the coordinates, the resulting initial ideal is coordinate independent. We call it by generic initial ideal.

Proof of Theorem 2.0.5 . Let $f_1, f_2, \cdots f_t$ be a F-basis for I_d and $m_1 \wedge \cdots \wedge m_t$ be the biggest standard exterior monomial appearing in the support of $g(f_1) \wedge \cdots \wedge g(f_t)$ for all $g \in GL_n(F)$. We consider $g(f_1) \wedge \cdots \wedge g(f_t)$ as a linear combination of standard exterior monomials. $C(g)$ denotes the coefficient of $m_1 \wedge \cdots \wedge m_t$ in $g(f_1) \wedge \cdots \wedge g(f_t)$.

We define U_d as the set of all $g \in GL_n(F)$ such that $C(g) \neq 0$. With this definition, observe that U_d is a Zariski open subset of the general linear group. We write J_d for the initial ideal $In_>(gI_d)$ where $g \in U_d$. Note that J_d is well defined because it is independent of the choice of $g \in U_d$.

We will show that $J = \bigoplus J_d$ is an ideal. By 2.0.1, there is an element $g \in U_d \cap U_{d+1}$. For this g, we have $In_>(gI_d) = J_d$ and $In_>(gI_{d+1}) = J_{d+1}$. This implies $S_1 J_d \subset J_{d+1}$ and the assertion follows.

Assume that the generators of J is of at most degree h. Next, we will show that $U = \bigcap_{d=1}^{h} U_d$ has the desired property and we say that J is the generic initial ideal of I.

Take an element $g \in \bigcap_{d=1}^{h} U_d$ and we have $J_d = In_>(gI_d)$ for all $d \leq h$. Hence, we see that $In_>(gI) \supset J$. Since the dimensions of J_d, I_d and gI_d are same for all d, it follows that $In_>(gI) = J$ as required.

□

Note that we fix the ordering of the variables as $x_1 > x_2 > \cdots > x_n$ in the beginning of this chapter.

Definition 2.0.7. The Borel subgroup contains all upper triangular matrices in $GL_n(F)$, that is,

$$\mathcal{B} = \{g = (g_{ij}) \in GL_n(F) : g_{ij} = 0 \text{ for } i > j\}.$$

In the definition of the Borel subgroup, we assume $x_1 > x_2 > \cdots > x_n$. If we change the ordering of the variables, the Borel subgroup is not same.

For example, if we change the ordering of the variables as $x_1 < x_2 < \cdots < x_n$, then the Borel subgroup contains all lower triangular matrices in $GL_n(F)$ and it is called the non-standart Borel subgroup accommodating the new ordering.

Definition 2.0.8. If $gI = I$ for all $g \in \mathcal{B}$, we say that I is Borel-fixed. In other words, I is fixed under the action of Borel subgroup $\mathcal{B}$.

After the definition of Borel-fixed ideal, we have the following proposition.

Proposition 2.0.9. *Let $I \in S$ be a graded Borel fixed ideal. Then, I is a monomial ideal.*

Proof. Let $f \neq 0$ be a homogeneous polynomial in I and m be a monomial appearing in f. Our aim is to show the existence of a homogeneous polynomial $g \in I$ with $supp(g) = supp(f) \setminus \{m\}$.

Suppose that $f = a_m m + \sum_{u \neq m} a_u u$ and α is a diagonal matrix with diagonal $b_1, b_2, \cdots, b_n$. By applying α to f, we get

$$\alpha(f) = a_m m(b_1, b_2, \cdots, b_n)m + \sum_{u \neq m} a_u u(b_1, b_2, \cdots, b_n)u.$$

Recall that we assume the field F is infinite in the beginning of this chapter. Since F is infinite, it follows that it is possible to choose $b_1, b_2, \cdots, b_n$ such that $m(b_1, b_2, \cdots, b_n) \neq u(b_1, b_2, \cdots, b_n)$ for $u \neq m$. By taking $g = m(b_1, b_2, \cdots, b_n)f - \alpha(f)$, we have $supp(g) = supp(f) \setminus \{m\}$.

It remains to show $g \in I$. By Borel-fixedness of I, we know $\alpha(f) \in I$. Thus, $g = m(b_1, b_2, \cdots, b_n)f - \alpha(f)$ belongs to I.

□

We have an important property for generic initial ideals as follows (for the proof, see [9]):

Theorem 2.0.10 (Bayer-Stillman, 1987). *For a homogeneous ideal I, $gin_{>}(I)$ is a Borel-fixed ideal.*

Remark 2.0.11. Note that the generic initial ideal can be obtained by at least one element of $\mathcal{B}$(for details, see [3, 15.18]).

Next, we give the definition of strongly stable ideals.

Definition 2.0.12. Let I be a monomial ideal and $x_1 > \cdots > x_n$. Then, we say that I is strongly stable if $x_i(m/x_j)$ lies in I when m is a monomial in I divisible by x_j and $i < j$.

It is enough to check this property for the monomial generators.

Proposition 2.0.13 (**Herzog-Hibi, 2011**). *Strongly stable ideals are Borel-fixed.*

Proof. Let I be a strongly stable ideal. It is easy to see that upper elementary matrices and diagonal matrices generate the Borel subgroup $\mathcal{B}$. Since I is fixed under the action of diagonal matrices, it suffices to show the proposition for the action of upper elementary matrices.

Take an upper elementary matrix $\alpha \in GL_n(F)$ whose action is given by $\alpha(x_k) = x_k$ for $j \neq k$ and $\alpha(x_j) = ex_i + x_j$ for $1 \leq i < j$.

Let $m = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial in I. A direct computation of $\alpha(m)$ gives

$$\alpha(m) = m + a_j e x_i (m/x_j) + \cdots .$$

By the property of a strongly stable ideal, we get $\alpha(m) \in I$ and the proposition follows.

$\square$

Remark 2.0.14. The reverse implication of 2.0.13 does not hold in general but if char $F = 0$, we also have the reverse implication. Thus, we say that generic initial ideals are strongly stable if the characteristic of F is zero.

Conca proved the following proposition.

Proposition 2.0.15 (**Conca, 2005**). *For a homogeneous ideal I, $gin_>(I) = I$ if and only if I is a Borel-fixed ideal.*

Proof. If $gin_>(I) = I$, I is Borel-fixed since we know gin is Borel-fixed by 2.0.10. Assume I is Borel-fixed. Then, take $g \in GL_n(F)$ such that $g = ab$ where $a \in GL_n(F)$ is lower triangular and $b \in GL_n(F)$ is upper triangular and $gin_>(I) = In_>(gI)$.

By the assumptions, we may get the assertion as follows:

$$gin_>(I) = In_>(abI) = In_>(bI) = In_>(I) = I.$$

The last equation follows since I is a Borel-fixed ideal (so it is a monomial ideal). Details can be found in [10, 1.8].

$\square$

By the previous proposition and 2.0.10, we have the following corollary.

Corollary 2.0.16. *Assume the notation of this chapter. Then,*

$$gin_>(gin_>(I)) = gin_>(I).$$

Chapter 3

Invariant rings

We give briefly the basic definitions and notations of invariant rings. We closely follow [4] and [5].

Let G be a group and V be a finite dimensional vector space over a field F. We give the definition of a finite dimensional representation of G over F.

Definition 3.0.1. We say that ρ is a finite dimensional representation of G over F if $\rho\colon G \to GL(V)$ is a group homomorphism.

After this definition, we focus on the modular representations because we study the certain ideals appearing in modular invariant theory.

Definition 3.0.2. If the order of G is divisible by the characteristic of F, the representation of G is called a modular representation.

For $g \in G$, $v \in V$, and ρ is a representation of G ; we send

$$(g, v) \text{ to } \rho(g)(v).$$

The map defines an action on V. From this action, we have an action of G on the dual space of V.

Recall that the vector space dual of V is the set of all linear functions from V to F and it is denoted by V^*. We get an action on V^* as follows

$$(g(f))(v) := f(g^{-1}(v))$$

where $g \in G$, $v \in V$ and $f \in V^*$. Note that the new representation is called the dual representation and it is obtained from algebra automorphisms on the symmetric algebra $S(V^*)$.

Definition 3.0.3. The ring of invariants is defined as

$$S(V^*)^G := \{ f \in S(V^*) \mid g.f = f \text{ for all } g \in G\}.$$

After the definition of the ring of invariants, we are ready to define the Hilbert ideal.

Definition 3.0.4. The ideal in $S(V^*)$ generated by the homogeneous invariants of positive degree is the Hilbert ideal of V and the notation is $H(V)$.

A main problem in invariant rings is to find the generating set of $S(V^*)^G$. For instance, we have the following generating set for invariants of symmetric group S_n.

Example 3.0.5. Consider the action of the symmetric group S_n on F^n. Recall that S_n acts on F^n by

$$\sigma(x_1, \cdots, x_n) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)}), \quad \sigma \in S_n.$$

The invariant ring of S_n is generated by elementary symmetric polynomials $f_1, \cdots, f_n$ where

$$
\begin{aligned}
f_1 &= x_1 + x_2 + \cdots + x_n \\
f_2 &= \sum_{i<j} x_i x_j \\
&\vdots \\
f_n &= x_1 x_2 \cdots x_n.
\end{aligned}
$$

Describing the generating set of invariant rings is a difficult problem in general even in the modular situation when G is a cyclic group of prime order. Note that the cyclic group of prime order is one of the most basic groups in invariant theory.

We study shortly the representation theory of cyclic groups which we will use later in the following chapters to understand the invariants of cyclic groups. Let $G = C_p$ denote the cyclic group of prime order p and assume that the characteristic of F is p. We begin with a finite dimensional indecomposable C_p-module, namely V. In other words, we have an indecomposable representation of C_p in characteristic p

$$\rho : C_p \to GL(V).$$

Assume g is a generator of C_p. Recall that $g^p = 1$ and $\lambda^p - 1 = (\lambda - 1)^p = 0$ for an eigenvalue λ. Since the characteristic of the field is p it follows that $\lambda = 1$ is the only eigenvalue of g lies in F. Thus, we can choose a basis $\{e_1, \cdots, e_n\}$ of V such that the image of the generator g is in Jordan form.

We focus on the C_p action on V^* instead of V because invariant rings consider the dual vector space. As we did in the previous paragraph, we may choose an upper triangular basis $\{x_1, \cdots, x_n\}$ of V^* with $g(x_1) = x_1$ and $g(x_k) = x_k + x_{k-1}$ for $2 \leq k \leq n$. Hence, a representation of the cyclic group in the field of order p is determined by the Jordan canonical form of $\rho(g)$.

Definition 3.0.6. V_n is the indecomposable C_p-module of dimension n for $1 \leq n \leq p$.

After this definition, observe that we have the following chain of inclusions.

Remark 3.0.7. We have $V_1 \subset V_2 \subset \cdots \subset V_p$.

Chapter 4

Generic Initial Ideals and Change of Order

We assume that $>$ is a fix monomial order on $F[x_1, \ldots, x_n]$ which satisfies $x_1 > x_2 > ... > x_n$ to define generic initial ideals in Chapter 1. A natural question that arises is whether gin is permuted in the same way when we permute the variables $x_1, \cdots, x_n$. The results of permutation of generic initial ideals appear in [6].

Let S denote the polynomial ring $F[x_1, \ldots, x_n]$. We fix a term order $>$ on the set of monomials in S. Let π be a permutation of $\{1, 2, \ldots, n\}$ and $>_\pi$ denote the term order such that

$$\pi(M_1) >_\pi \pi(M_2) \text{ if and only if } M_1 > M_2.$$

Recall that we order standard exterior monomials lexicographically. Standard exterior monomials with respect to the new order are defined and ordered similarly.

In other words, If $m_1 \wedge m_2 \wedge \cdots \wedge m_t$ and $w_1 \wedge w_2 \wedge \cdots \wedge w_t$ are two standard exterior monomials with respect to $>_\pi$, then we set

$$m_1 \wedge m_2 \wedge \cdots \wedge m_t >_\pi w_1 \wedge w_2 \wedge \cdots \wedge w_t$$

if $m_i >_\pi w_i$ for the smallest index i with $m_i \neq w_i$.

Let $\alpha \in GL_n(F)$ and consider the polynomial ring in the extended set of variables $R = F[x_1, \ldots, x_n, \alpha_{i,j} \mid 1 \leq i, j \leq n]$. We extend π to R by saying $\pi(\alpha_{ij}) = \alpha_{\pi(i)j}$.

Lemma 4.0.1. *Let M be a monomial in $\alpha(f)$ with coefficient $c \in F[\alpha_{ij}]$. Then the coefficient of $\pi(M)$ in $\alpha(f)$ is $\pi(c)$ where $f \in S$ and $\alpha \in GL_n(F)$.*

Proof. We show $\pi(\alpha(f)) = \alpha(f)$ for all $f \in S$. It is enough to show the claim for only variables since we know both π and α are ring homomorphisms. Observe that

$$\pi(\alpha(x_j)) = \pi(\sum_{1 \leq i \leq n} \alpha_{ij} x_i) = \sum_{1 \leq i \leq n} \alpha_{\pi(i)j} x_{\pi(i)} = \alpha(x_j).$$

Thus, we have $\pi(\alpha(f)) = \alpha(f)$ for all $f \in S$. Let $c_k \in F[\alpha_{ij}]$ and M_k is a monomial in S. We say $\alpha(f) = \sum c_k M_k$. Then, we have

$$\alpha(f) = \pi(\alpha(f)) = \sum \pi(c_k)\pi(M_k).$$

Therefore, we obtain $\sum c_k M_k = \sum \pi(c_k)\pi(M_k)$ and the assertion of the lemma follows. $\square$

After the lemma, we get the following theorem.

Theorem 4.0.2. *For a homogeneous ideal I, we have*

$$\mathrm{gin}_{>_\pi}(I) = \pi(\mathrm{gin}_{>}(I)).$$

Proof. Let I_d be a homogeneous component of I with a basis $f_1, \ldots, f_t$. Assume that $m_{j_1}, \ldots, m_{j_t}$ are monomials in $\alpha(f_1), \ldots, \alpha(f_t)$ with coefficients $c_1, \ldots, c_t \in F[\alpha_{ij}]$, respectively.

Suppose that $c_1 m_{j_1} \wedge \cdots \wedge c_t m_{j_t}$ is a $c \in F[\alpha_{ij}]$ multiple of the standard exterior monomial $m_1 \wedge \cdots \wedge m_t$ (with respect to $>$) in $\wedge^t S_d$. Then, we have $\pi(m_{j_1}), \ldots, \pi(m_{j_t)}$ appear in $\alpha(f_1), \ldots, \alpha(f_t)$ with coefficients $\pi(c_1), \ldots, \pi(c_t) \in F[\alpha_{ij}]$, respectively by the previous lemma 4.0.1.

Note that the ranking of the monomials in $>$ is preserved in $>_\pi$ after applying π. Hence, the coefficient of the standard exterior monomial $\pi(m_1) \wedge \cdots \wedge \pi(m_t)$ (with respect to $>_\pi$) in $\pi(c_1)\pi(m_{j_1}) \wedge \cdots \wedge \pi(c_t)\pi(m_{j_t})$ is $\pi(c)$.

Then, we get

$$\begin{aligned}\alpha(f_1) \wedge \cdots \wedge \alpha(f_t) &= \sum_{m_1 > \cdots > m_t} c(m_1, \ldots, m_t)(m_1 \wedge \cdots \wedge m_t) \\ &= \sum_{\pi(m_1) >_\pi \cdots >_\pi \pi(m_t)} \pi(c(m_1, \ldots, m_t))(\pi(m_1) \wedge \cdots \wedge \pi(m_t)).\end{aligned}$$

Since π is a permutation of variables in $F[\alpha_{ij}]$, $c(m_1, \ldots, m_t)$ is the zero polynomial if and only if $\pi(c(m_1, \ldots, m_t))$ is the zero polynomial. Thus, we have that if $w_1 \wedge \cdots \wedge w_t$ is the largest exterior monomial (with respect to $>$) with the property that there is $\alpha \in GL_n(F)$ with $\mathrm{In}_>(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = w_1 \wedge \cdots \wedge w_t$, then $\pi(w_1) \wedge \cdots \wedge \pi(w_t)$ is the largest exterior monomial (with respect to $>_\pi$) such that there exists $\alpha \in GL_n(F)$ with $\mathrm{In}_{>_\pi}(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = \pi(w_1) \wedge \cdots \wedge \pi(w_t)$.

Since $(\mathrm{gin}_> I)_d$ and $(\mathrm{gin}_{>_\pi} I)_d$ are generated by $w_1, \ldots, w_t$ and $\pi(w_1), \ldots, \pi(w_t)$ (see [2, 4.1.4, 4.1.5]), respectively and d is arbitrary, the theorem follows, that is,

$$\mathrm{gin}_{>_\pi}(I) = \pi(\mathrm{gin}_>(I)).$$

$\square$

After this theorem, we have the following remark.

Remark 4.0.3. If gin satisfies some property with respect to $>$, then it also satisfies this property with respect to $>_\pi$ when we permute the variables.

Note that there are $n!$ permutations of the variables. Thus, we get the following corollary.

Corollary 4.0.4. *The number of generic initial ideals of a homogeneous ideal I is divisible by $n!$.*

Remark 4.0.5. Recall that the standart Borel subgroup contains all upper triangular matrices when we fix the ordering of the variables as $x_1 > x_2 > \cdots > x_n$. If we permute the variables, then we need to consider the non-standart Borel subgroup determined by the permutation instead of the standart Borel subgroup.

Chapter 5

Results

We focus on the generic initial ideals of the ideals that appear in modular invariant theory. Let G be a cyclic group of prime order p over a field F with characteristic p. Recall that we have p indecomposable G-modules $V_1, \cdots, V_p$. We consider the Hilbert ideal of these modules. For some cases, we know an explicit generating set of the Hilbert ideals by [7]. For these cases, we compute the generic initial ideals. The results appear in [6].

5.0.1 The monomial cases: $mV_2 \oplus lV_3$ and V_4

We investigate the generic initial ideal of the Hilbert ideal of $mV_2 \oplus lV_3$ and V_4. By [7], we have

$$H(mV_2 \oplus lV_3) = \langle x_i, y_i^p | i = 1, ..., m\rangle \cup \langle x_i, y_i y_j, z_i^p | i = m+1, ..., m+l; i \leq j\rangle$$

and

$$H(V_4) = \langle x_1, x_2^2, x_2 x_3^{p-3}, x_3^{p-1}, x_4^p\rangle.$$

Note that $gin_<(I) = I$ for a Borel-fixed ideal I with respect to a monomial order $<$. Thus, if we prove that $H(mV_2 \oplus lV_3)$ and $H(V_4)$ are Borel-fixed with respect to some order, we find the generic initial ideals of them.

Recall that mV_2 is an ideal in $F[x_1, ...x_m, y_1, ..., y_m]$ and $\mathcal{B}_{2m}$ is the Borel subgroup of $GL_{2m}(F)$ and lV_3 is an ideal in $F[x_1, ...x_l, y_1, ..., y_l, z_1, ..., z_l]$ and $\mathcal{B}_{3l}$ is the Borel subgroup of $GL_{3l}(F)$.

Similarly, $mV_2 \oplus lV_3$ is an ideal in the polynomial ring containing $2m + 3l$ variables and the Borel subgroup $\mathcal{B}$ of $GL_{2m}(F) \times GL_{3l}(F)$ is $\mathcal{B}_{2m} \times \mathcal{B}_{3l}$.

Theorem 5.0.1. *Let $<$ be a monomial order which satisfies $x_1 > ... > x_{m+l} > y_1 > ... > y_{m+l} > z_{m+1} > ... > z_{m+l}$. $H(mV_2 \oplus lV_3)$ is a Borel-fixed ideal.*

Proof. All invertible diagonal matrices and upper elementary matrices generate the Borel subgroup $\mathcal{B}$ and a monomial ideal keeps fixed under the action of a diagonal matrix. Hence, it is enough to show $H(mV_2 \oplus lV_3)$ is fixed under the action of upper elementary matrices.

Take the upper elementary matrix $\alpha = [\alpha_{ij}]$ sending x_i to $x_i + cx_k$ with $c \in F$, $k < i$. Since all x_i's lie inside $H(mV_2 \oplus lV_3)$, it follows that

$$\alpha(H(mV_2 \oplus lV_3)) = H(mV_2 \oplus lV_3).$$

For the upper elementary matrix α sending y_i to $y_i + cx_k$ or z_i to $z_i + cx_k$, $H(mV_2 \oplus lV_3)$ keeps fixed because all x_i's belong to $H(mV_2 \oplus lV_3)$.

We consider the upper elementary matrix α sending y_i to $y_i + cy_k$ with $k < i$. If $i \leq m$, then

$$\alpha(y_i^p) = y_i^p + c^p y_k^p \in H(mV_2 \oplus lV_3).$$

If $i > m$ and $i < j$, we have

$$\alpha(y_i y_j) = (y_i + cy_k)y_j = y_i y_j + cy_k y_j \in H(mV_2 \oplus lV_3).$$

Observe that $i = j > m$ yields that

$$\alpha(y_i^2) = y_i^2 + 2cy_i y_k + c^2 y_k^2 \in H(mV_2 \oplus lV_3).$$

Note that if $i > m$, we should take $k > m$ by definition of the Borel subgroup of $GL_{2m}(F) \times GL_{3l}(F)$.

For the upper elementary matrix α sending z_i to $z_i + cz_j$ or z_i to $z_i + cy_j$, we get

$$\alpha(z_i^p) = z_i^p + c^p z_j^p \in H(mV_2 \oplus lV_3)$$

and

$$\alpha(z_i^p) = z_i^p + c^p y_j^p \in H(mV_2 \oplus lV_3).$$

Notice that we take $m < j < i$ because of definition of the Borel subgroup of $GL_{2m}(F) \times GL_{3l}F)$.

So, $H(mV_2 \oplus lV_3)$ is fixed for all upper elementary matrices, which gives $H(mV_2 \oplus lV_3)$ is a Borel-fixed ideal. □

By 2.0.15, we get the following corollary.

Corollary 5.0.2. *$gin_<(H(mV_2 \oplus lV_3)) = H(mV_2 \oplus lV_3)$ for $<$ as given in Theorem 5.0.1.*

With the help of Chapter 4, we have the following corollary.

Corollary 5.0.3. *There are $(2m + 3l)!$ generic initial ideals of $H(mV_2 \oplus lV_3)$ by Theorem 4.0.2.*

After computing the generic initial ideal of $H(mV_2 \oplus lV_3)$, we study the generic initial ideal of $H(V_4)$. Firstly, we show that $H(V_4)$ is a Borel-fixed ideal.

Theorem 5.0.4. *Let $S = K[x_1, ..., x_4]$ be a polynomial ring and $<$ be a monomial order satisfying $x_1 > ... > x_4$. $H(V_4)$ is a Borel-fixed ideal.*

Proof. We only consider the action of upper elementary matrices as in the proof of Theorem 5.0.1.

We neglect the upper elementary matrices sending x_i to $x_i + cx_1$ for $i = 2, 3, 4$ since x_1 belongs to $H(V_4)$.

For the upper elementary matrix α with $x_3 \to x_3 + cx_2$ where $c \in F$, we have

$$\alpha(x_2 x_3^{p-3}) = x_2(x_3 + cx_2)^{p-3} \in H(V_4)$$

and

$$\alpha(x_3^{p-1}) = (x_3 + cx_2)^{p-1} \in H(V_4).$$

Note that

$$x_2^2 \text{ and } x_2 x_3^{p-3} \in H(V_4).$$

For the upper elementary matrix α with $x_4 \to x_4 + cx_2$ where $c \in F$, we obtain

$$\alpha(x_4^p) = x_4^p + c^p x_2^p \in H(V_4).$$

For the upper elementary matrix α with $x_4 \to x_4 + cx_3$ where $c \in F$, we get

$$\alpha(x_4^p) = x_4^p + c^p x_3^p \in H(V_4).$$

We show $H(V_4)$ is fixed for all upper elementary matrices and this implies it is a Borel-fixed ideal.

□

For Borel-fixed ideals, we calculate the generic initial ideal by Conca's proposition. Thus, we have the following corollary.

Corollary 5.0.5. *Let $<$ be as in Theorem 5.0.4. We obtain $gin_<(H(V_4)) = H(V_4)$ by Proposition 2.0.15.*

We compute the generic initial ideal of $H(V_4)$ for a monomial order $<$ satisfying $x_1 > ... > x_4$. If we change the ordering of the variables, we use the results of the previous chapter.

By the content of the previous chapter, we get the number of generic initial ideals of $H(V_4)$.

Corollary 5.0.6. *There are $4!$ generic initial ideals of $H(V_4)$ by Theorem 4.0.2.*

For this subsection, we deal with the generic initial ideal of the monomial cases $H(mV_2) \oplus H(lV_3)$ and $H(V_4)$.

5.0.2 The non-monomial case: V_5 for $p > 5$

In the previous subsection, we have the Borel-fixed ideals $H(mV_2) \oplus H(lV_3)$ and $H(V_4)$. By using Conca's proposition(Proposition 2.0.15), we calculate the generic initial ideals of these Hilbert ideals. However, for this subsection, the Hilbert ideal of V_5 is non-monomial, that is, it can't be a Borel-fixed ideal. Hence, Conca's proposition is not applicable.

Let $S = K[x_1, x_2, ..., x_5]$ be a polynomial ring in 5 variables and $p > 5$. In [7], it is shown that $H(V_5)$ is generated by

$$(x_1, x_2^2, x_3^2 - 2x_2x_4 - x_2x_3, x_2x_3x_4, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p).$$

Let $\mathcal{B}_5$ denote the Borel subgroup of $GL_5(F)$. We calculate $\alpha(H(V_5))$ for all $\alpha \in \mathcal{B}_5$. Note that the generating set of $\alpha(H(V_5))$ is coming from applying α to the generating set of $H(V_5)$.

Lemma 5.0.7. *For any $\alpha = [\alpha_{ij}] \in \mathcal{B}_5$, we have*

$$\alpha(H(V_5)) = \langle x_1, x_2^2, x_3^2 + Cx_2x_3 + Dx_2x_4, x_2x_3x_4, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\rangle \quad (5.1)$$

where $C = \alpha_{33}^{-1}(2\alpha_{23}\alpha_{33} - 2\alpha_{22}\alpha_{34} - \alpha_{22}\alpha_{33})$ and $D = \alpha_{33}^{-1}(-2\alpha_{22}\alpha_{44})$.

Proof. Lets take $\alpha = [\alpha_{ij}] \in \mathcal{B}_5$,that is, $\alpha_{ij} = 0$ for $i > j$ and $\alpha_{ii} \neq 0$ for $i = 1, 2, ..., 5$. Applying α to x_1, we get x_1 belongs to generating set of $\alpha(H(V_5))$ since $\alpha(x_1) = \alpha_{11}x_1$ and set $I := (x_1)$.

We have

$$\alpha(x_2^2) \equiv \alpha_{22}^2 x_2^2 \mod I.$$

So, x_2^2 is a generator of $\alpha(H(V_5))$ corresponding to $\alpha(x_2^2)$ and set $I := I \cup (x_2^2)$.

In $\mod I$, we know that $\alpha(x_3^2 - 2x_2x_4 - x_2x_3)$ is equivalent to

$$\alpha_{33}x_3^2 + (2\alpha_{23}\alpha_{33} - 2\alpha_{22}\alpha_{34} - \alpha_{22}\alpha_{33})x_2x_3 - 2\alpha_{22}\alpha_{44}x_2x_4.$$

This gives $x_3^2 + Cx_2x_3 + Dx_2x_4$ is a generator of $\alpha(H(V_5))$ corresponding to $\alpha(x_3^2 - 2x_2x_4 - x_2x_3)$ where

$$C = \alpha_{33}^{-1}(2\alpha_{23}\alpha_{33} - 2\alpha_{22}\alpha_{34} - \alpha_{22}\alpha_{33}) \text{ and } D = \alpha_{33}^{-1}(-2\alpha_{22}\alpha_{44}).$$

Observe that

$$x_2x_3^2 = x_2(x_3^2 + Cx_2x_3 + Dx_2x_4) - (x_2^2)(Cx_3 + Dx_4) \in \alpha(H(V_5)).$$

Then, we set $I := I \cup (x_2x_3^2)$. Applying α to $x_2x_3x_4$, we get

$$\alpha(x_2x_3x_4) \equiv \alpha_{22}\alpha_{33}\alpha_{44}x_2x_3x_4 \mod I.$$

It shows that $x_2x_3x_4$ is a generator of $\alpha(H(V_5))$ corresponding to $\alpha(x_2x_3x_4)$.

This implies both

$$x_3^3 = x_3(x_3^2 + Cx_2x_3 + Dx_2x_4) - Cx_2x_3^2 - Dx_2x_3x_4 \in \alpha(H(V_5))$$

and

$$x_4x_3^2 + Dx_2x_4^2 = x_4(x_3^2 + Cx_2x_3 + Dx_2x_4) - Cx_2x_3x_4 \in \alpha(H(V_5)).$$

Setting $I := I \cup (x_2x_3x_4, x_3^3)$ and applying α to $x_2x_4^{p-4}$, we obtain

$$\begin{aligned}\alpha(x_2x_4^{p-4}) &\equiv \alpha_{22}x_2(\alpha_{24}x_2 + \alpha_{34}x_3 + \alpha_{44}x_4)^{p-4} \\ &\equiv \alpha_{22}x_2(\alpha_{34}x_3 + \alpha_{44}x_4)^{p-4} \\ &\equiv \alpha_{22}\alpha_{44}^{p-4}x_2(x_4)^{p-4} \mod I.\end{aligned}$$

Hence, $x_2x_4^{p-4}$ is a generator of $\alpha(H(V_5))$ corresponding to $\alpha(x_2x_4^{p-4})$.

Notice that

$$x_3^2x_4^{p-5} = x_4^{p-6}(x_4x_3^2 + Dx_2x_4^2) - D(x_2x_4^{p-4}) \in \alpha(H(V_5))$$

and set

$$I := I \cup (x_2x_4^{p-4}, x_3^2x_4^{p-5}).$$

Applying α to $x_3x_4^{p-3}$, we have

$$\begin{aligned}\alpha(x_3x_4^{p-3}) &\equiv (\alpha_{23}x_2+\alpha_{33}x_3)(\alpha_{24}x_2+\alpha_{34}x_3+\alpha_{44}x_4)^{p-3}\\ &\equiv \alpha_{33}x_3(\alpha_{24}x_2+\alpha_{34}x_3+\alpha_{44}x_4)^{p-3}\\ &\equiv \alpha_{33}x_3(\alpha_{34}x_3+\alpha_{44}x_4)^{p-3}\\ &\equiv \alpha_{33}\alpha_{44}^{p-3}x_3x_4^{p-3} \mod I.\end{aligned}$$

So, $x_3x_4^{p-3}$ is a generator of $\alpha(H(V_5))$ coming from $\alpha(x_3x_4^{p-3})$.

We set $I := I \cup (x_3x_4^{p-3})$ and get

$$\begin{aligned}\alpha(x_4^{p-1}) &\equiv (\alpha_{24}x_2+\alpha_{34}x_3+\alpha_{44}x_4)^{p-1}\\ &\equiv (\alpha_{34}x_3+\alpha_{44}x_4)^{p-1} \equiv (\alpha_{44}x_4)^{p-1} \mod I.\end{aligned}$$

This shows x_4^{p-1} is a generator of $\alpha(H(V_5))$ coming from $\alpha(x_4^{p-1})$. Now, we set $I := I \cup (x_4^{p-1})$.

We apply α to x_5^p to obtain

$$\alpha(x_5^p) = (\alpha_{25}x_2+\alpha_{35}x_3+\alpha_{45}x_4+\alpha_{55}x_5)^p \equiv \alpha_{55}^p x_5^p \mod I.$$

It gives x_5^p is a generator of $\alpha(H(V_5))$ following from $\alpha(x_5^p)$. □

We calculate $In_<(\alpha(H(V_5)))$ for all monomial orders $<$ satisfying $x_1 > x_2 > x_3 > x_4 > x_5$. The initial ideal of $\alpha(H(V_5))$ depends on whether C is 0 or not. Note that D is always non-zero since $\alpha_{ii} \neq 0$ for all $i = 1, 2, ..., 5$.

Lemma 5.0.8. *We fix a monomial order $<$ with $x_1 > x_2 > x_3 > x_4 > x_5$. If $C = 0$ and $x_2x_4 < x_3^2$,*

$$In_<(\alpha(H(V_5))) = \langle x_1, x_2^2, x_3^2, x_2x_3x_4, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\rangle.$$

If $C = 0$ and $x_3^2 < x_2x_4$,

$$In_<(\alpha(H(V_5))) = \langle x_1, x_2^2, x_2x_4, x_2x_3^2, x_3^3, x_3^2x_4^{p-5}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\rangle.$$

Proof. We calculate S-polynomials of the generators and apply Buchberger's criterion, see [11]. Observe that the S-polynomial of f and g reduces to 0 with respect to generators of $\alpha(H(V_5))$ if f and g are monomials or if $in_<(f)$ and $in_<(g)$ are relatively prime.

Then, we only need to check the following S-polynomials:

$$S(x_3^2 + Dx_2x_4, x_2x_3x_4) \text{ and } S(x_3^2 + Dx_2x_4, x_3x_4^{p-3}) \text{ for } x_2x_4 < x_3^2.$$

We always consider reducing to zero with respect to generators of $\alpha(H(V_5))$.

If $x_2x_4 < x_3^2$, then

$$S(x_3^2 + Dx_2x_4, x_2x_3x_4) = Dx_2^2x_4^2 \text{ reduces to } 0.$$

Also we have

$$S(x_3^2 + Dx_2x_4, x_3x_4^{p-3}) = Dx_2x_4^{p-2} = Dx_4^2(x_2x_4^{p-4}) \text{ reduces to } 0.$$

So, the set of generators of $\alpha(H(V_5))$ is a Gröbner basis of $\alpha(H(V_5))$ and

$$In_<(\alpha(V_5)) = (x_1, x_2^2, x_3^2, x_2x_3x_4, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p).$$

If $x_3^2 < x_2x_4$, then we obtain

(i) $S(x_3^2 + Dx_2x_4, x_2^2) = x_2x_3^2$ which is a non-zero remainder ;

(ii) $S(x_3^2 + Dx_2x_4, x_2x_3x_4) = x_3^3$ which is a non-zero remainder ;

(iii) $S(x_3^2 + Dx_2x_4, x_2x_4^{p-4}) = x_3^2x_4^{p-5}$ which is a non-zero remainder ;

(iv) $S(x_3^2 + Dx_2x_4, x_3x_4^{p-3}) = x_3^3x_4^{p-4}$ reduces to 0;

(v) $S(x_3^2 + Dx_2x_4, x_4^{p-1}) = x_3^2x_4^{p-2}$ reduces to 0.

Note that item (v) reduces to 0 because the remainder $x_3^2x_4^{p-5}$ will be added to the generating set according to the Buchberger's algorithm.

We add all the non-zero remainders to the generating set of $\alpha(H(V_5))$ and we compute S-polynomials of the non-zero remainders and $x_3^2 + Dx_2x_4$. Since $gcd(x_2x_4, x_3^3) = 1$, it follows that there is no need to calculate $S(x_3^2 + Dx_2x_4, x_3^3)$.

We need to calculate following two S-polynomials

$$S(x_3^2 + Dx_2x_4, x_2x_3^2) = x_3^4 \text{ reduces to } 0$$

and

$$S(x_3^2 + Dx_2x_4, x_3^2x_4^{p-5}) = x_3^4x_4^{p-6} \text{ reduces to } 0.$$

Hence, we get

$$In_<(\alpha(H(V_5))) = (x_1, x_2^2, x_2x_4, x_2x_3^2, x_3^3, x_3^2x_4^{p-5}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p).$$

□

Recall that gin is a Borel fixed ideal by Theorem 2.0.10. By this theorem and the previous lemma, we have the following remark.

Remark 5.0.9. Observe that $In_<(\alpha(H(V_5)))$ is not a Borel-fixed ideal if $C = 0$. Since $gin_<(H(V_5))$ is a Borel-fixed ideal it follows that $gin_<(H(V_5)) \neq In_<(\alpha(H(V_5)))$ for an element $\alpha \in \mathcal{B}_5$ giving $C = 0$.

Proof. If $C = 0$ and $x_2x_4 < x_3^2$, then $x_3 \to x_3 + x_2$ gives

$$x_3^2 + 2x_2x_3 + x_2^2 \notin In_<(\alpha(H(V_5))).$$

If $C = 0$ and $x_3^2 < x_2x_4$, then $x_4 \to x_4 + x_3$ gives

$$x_2x_4 + x_2x_3 \notin In_<(\alpha(H(V_5))).$$

□

Alternatively, if $C = 0$ and $x_2x_4 < x_3^2$, then $In_<(\alpha(H(V_5)))_2$ is not a strongly-stable ideal since

$$x_2x_3 \notin In_<(\alpha(H(V_5)))_2 \text{ while } x_3^2 \in In_<(\alpha(H(V_5)))_2.$$

If $C = 0$ and $x_3^2 < x_2x_4$, then $In_<(\alpha(H(V_5)))_2$ is not a strongly-stable ideal since

$$x_2x_3 \notin In_<(\alpha(H(V_5)))_2 \text{ while } x_2x_4 \in In_<(\alpha(H(V_5)))_2.$$

By [2, 4.2.4], if the largest exponent of monomial generators of an ideal is less than $char(F) = p$, then we have that Borel-fixedness is equivalent to strongly stability.

Note that the largest exponent of monomial generators of $In_<(\alpha(H(V_5)))_2$ is 2 which is smaller than $char(F) = p$. Thus, $In_<(\alpha(H(V_5)))_2$ is not a Borel-fixed ideal and it implies that $In_<(\alpha(H(V_5)))$ is not Borel-fixed.

The Remark 5.0.9 shows that the generic initial ideal of $H(V_5)$ can not be obtained by an element α satisfying $C = 0$.

Lemma 5.0.10. *Let $<$ be a monomial order that satisfies $x_1 > x_2 > x_3 > x_4 > x_5$ and assume $C \neq 0$. If $x_2x_4 < x_3^2$,*

$$In_<(\alpha(H(V_5))) = \langle x_1, x_2^2, x_2x_3, x_3^3, x_3^2x_4, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\rangle.$$

If $x_3^2 < x_2x_4$,

$$In_<(\alpha(H(V_5))) = \langle x_1, x_2^2, x_2x_3, x_3^3, x_2x_4^2, x_3^2x_4^{p-5}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\rangle.$$

Proof. Firstly, we consider the case $x_2x_4 < x_3^2$. We have that $S(x_2^2, x_3^2 + Cx_2x_3 + Dx_2x_4) = x_2x_3^2 + Dx_2^2x_4 = C^{-1}x_3(x_3^2 + Cx_2x_3 + Dx_2x_4) - C^{-1}Dx_2x_3x_4 + Dx_2^2x_4 - C^{-1}x_3^3$.

This shows x_3^3 is a non-zero remainder so we add it to the generating set of $\alpha(H(V_5))$. Note that

$$S(x_3^2 + Cx_2x_3 + Dx_2x_4, x_3^3) = x_3^4 + Dx_2x_3^2x_4 \text{ reduces to } 0.$$

Notice that

$$S(x_3^2 + Cx_2x_3 + Dx_2x_4, x_2x_3x_4) = x_3^2x_4 + Dx_2x_4^2$$

is a non-zero remainder and add it to the generating set. For this case, we know

$$In_<(x_3^2x_4 + Dx_2x_4^2) = x_3^2x_4.$$

So, we check the S-polynomial

$$S(x_3^2 + Cx_2x_3 + Dx_2x_4, x_3^2x_4 + Dx_2x_4^2) = x_3^3x_4 + Dx_2x_3x_4^2 - CDx_2^2x_4^2$$

which reduces to 0.

Additonally, we compute the following S-polynomials to get a Gröbner basis.

$$S(x_3^2 + Cx_2x_3 + Dx_2x_4, x_2x_4^{p-4}) = x_3^2x_4^{p-4} + Dx_2x_4^{p-3} = x_4^{p-5}(x_3^2x_4 + Dx_2x_4^2)$$

is reducing to 0.

$$S(x_3^2 + Cx_2x_3 + Dx_2x_4, x_3x_4^{p-3}) = x_3^2x_4^{p-3} + Dx_2x_4^{p-2} = x_4^{p-4}(x_3^2x_4 + Dx_2x_4^2)$$

is reducing to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_3^3) = Dx_2x_3x_4^2$$

reduces to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_2x_3x_4) = Dx_2^2x_4^2$$

reduces to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_2x_4^{p-4}) = Dx_2^2x_4^{p-3}$$

is reducing to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_3x_4^{p-3}) = Dx_2x_4^{p-2} = Dx_4^2(x_2x_4^{p-4})$$

reduces to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_4^{p-1}) = Dx_2x_4^p$$

is reducing to 0.

After these computations, we get

$$\{x_1, x_2^2, x_3^2 + Cx_2x_3 + Dx_2x_4, x_3^3, x_3^2x_4 + Dx_2x_4^2, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\}$$

is a Gröbner basis and we have the intended initial ideal according to the Gröbner basis.

Now, we consider the case $x_3^2 < x_2x_4$. In this case, only difference comes from

$$In_<(x_3^2x_4 + Dx_2x_4^2) = x_2x_4^2.$$

Thus, we focus on the S-polynomials including $x_3^2x_4 + Dx_2x_4^2$. We compute the following S-polynomials.

$$S(x_3^2x_4 + Dx_2x_4^2, x_2^2) = x_3^2x_4x_2$$

is reducing to 0 because it is divisible by $x_2x_3x_4$.

$$\begin{aligned} S(x_3^2x_4 + Dx_2x_4^2, x_3^2 + Cx_2x_3 + Dx_2x_4) &= Cx_3^3x_4 - Dx_4^2x_3^2 - D^2x_2x_4^3 \\ &= Cx_3^3x_4 - Dx_4(x_3^2x_4 + Dx_2x_4^2) \end{aligned}$$

is reducing to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_2x_3x_4) = x_3^3x_4$$

reduces to 0.

$$S(x_3^2x_4 + Dx_2x_4^2, x_2x_4^{p-4}) = x_3^2x_4^{p-5}$$

is a non-zero remainder and we need to check

$$S(x_3^2 + Cx_2x_3 + Dx_2x_4, x_3^2x_4^{p-5}) = x_3^3x_4^{p-5} + Dx_2x_3x_4^{p-4}$$

which reduces to 0. For the non-zero remainder, we also compute

$$S(x_3^2x_4 + Dx_2x_4^2, x_3^2x_4^{p-5}) = x_3^4x_4^{p-6}$$

reducing to 0.

Both

$$S(x_3^2x_4 + Dx_2x_4^2, x_3x_4^{p-3}) = x_3^3x_4^{p-4}$$

and

$$S(x_3^2x_4 + Dx_2x_4^2, x_4^{p-1}) = x_3^2x_4^{p-2} = x_4^3(x_3^2x_4^{p-5})$$

reduce to 0 with respect to generators of $\alpha(H(V_5))$.

By all calculations, we obtain

$$\{x_1, x_2^2, x_3^2 + Cx_2x_3 + Dx_2x_4, x_3^3, x_3^2x_4 + Dx_2x_4^2, x_3^2x_4^{p-5}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p\}$$

is a Gröbner basis and it gives the desired initial ideal.

$\square$

Note that there exist an element $\alpha \in \mathcal{B}_5$ such that $gin_<(H(V_5)) = In_<(\alpha(H(V_5)))$, see Remark 2.0.11. We have $gin_<(H(V_5)) \neq In_<(\alpha(H(V_5)))$ for an element $\alpha \in \mathcal{B}_5$ giving $C = 0$ by Remark 5.0.9. Thus, we obtain $gin_<(H(V_5)) = In_<(\alpha(H(V_5)))$ for $\alpha \in \mathcal{B}_5$ giving $C \neq 0$. Recall that $In_<(\alpha(H(V_5)))$ is independent of $\alpha \in \mathcal{B}_5$ if $C \neq 0$ by Lemma 5.0.10.

By Lemma 5.0.7, Lemma 5.0.8, Remark 5.0.9 and Lemma 5.0.10, we have the following theorem. This theorem clarifies the situation for the gin of $H(V_5)$ when we fix the ordering of the variables as $x_1 > x_2 > ... > x_5$.

Theorem 5.0.11. *If $x_2x_4 < x_3^2$,*

$$gin_<(H(V_5)) = \langle x_1, x_2^2, x_2x_3, x_3^3, x_3^2x_4, x_2x_4^{p-4}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p \rangle.$$

If $x_3^2 < x_2x_4$,

$$gin_<(H(V_5)) = \langle x_1, x_2^2, x_2x_3, x_3^3, x_2x_4^2, x_3^2x_4^{p-5}, x_3x_4^{p-3}, x_4^{p-1}, x_5^p \rangle.$$

Let e be identity element of the Borel subgroup $\mathcal{B}_5$. For e, we know $C = -1$ and so we get following corollary.

Corollary 5.0.12. *$gin_<(H(V_5)) = In_<(H(V_5))$ for all monomial orders when we fix ordering of variables as $x_1 > x_2 > ... > x_5$ and the non-empty Zariski open set U contains*

$$\{\alpha = [\alpha_{ij}] \in \mathcal{B}_5 : \alpha_{33}^{-1}(2\alpha_{23}\alpha_{33} - 2\alpha_{22}\alpha_{34} - \alpha_{22}\alpha_{33}) \neq 0\}.$$

Corollary 5.0.13. *There are $2(5!)$ generic initial ideals of $H(V_5)$ for $p > 5$ by Theorem 4.0.2 and Theorem 5.0.11.*

5.0.3 The non-monomial case: V_5 for $p = 5$

We compute the generic initial ideal of $H(V_5)$ for $p > 5$. In this subsection, we investigate the gin of $H(V_5)$ for $p = 5$.

For $p = 5$, It is shown that

$$N = \{x_1, x_2^2, x_3^2 - 2x_2x_4 - x_2x_3, x_2x_3x_4, 2x_2x_4^2 + x_3x_4^2, x_2x_4^3, x_4^4, x_5^5\}$$

is a reduced Gröbner basis for $H(V_5)$ by [7].

Lemma 5.0.14. *For any $\alpha = (\alpha_{ij}) \in \mathcal{B}_5$ and $p = 5$, we obtain $\alpha(H(V_5))$ is generated by*

$$N' := \{x_1, x_2^2, x_3^2 + Cx_2x_3 + Dx_2x_4, x_2x_3x_4, C_0x_2x_4^2 + D_0x_3x_4^2, x_2x_4^3, x_4^4, x_5^5\}$$

where $C = \alpha_{33}^{-2}(2\alpha_{23}\alpha_{33} - 2\alpha_{22}\alpha_{34} - \alpha_{22}\alpha_{33})$, $D = \alpha_{33}^{-2}(-2\alpha_{22}\alpha_{44})$, $C_0 = (4\alpha_{33}^{-1}\alpha_{22}\alpha_{34} + 2\alpha_{22} + \alpha_{23})\alpha_{44}^2$ and $D_0 = \alpha_{33}\alpha_{44}^2$.

Proof. We denote the generators of $H(V_5)$ in N with f_i for $1 \leq i \leq 8$ with $f_1 = x_1$ and $f_8 = x_5^5$. Let J_i denote the ideal generated by $\alpha(f_1), \ldots, \alpha(f_i)$. Since α is a ring homomorphism, $\alpha(H(V_5))$ is generated by $\alpha(f_1), \ldots, \alpha(f_8)$ and so we have $\alpha(H(V_5)) = J_8$. We also denote $x_3^2 + Cx_2x_3 + Dx_2x_4$ with f_3' and $C_0x_2x_4^2 + D_0x_3x_4^2$ with f_5'.

Note that, since α sends x_1 to a multiple of x_1 and x_2 to a linear combination of x_1 and x_2 we have that $J_2 = (x_1, x_2^2)$. On the other hand, direct computation gives

$$\alpha(x_3^2 - 2x_4x_2 - x_2x_3) \equiv \alpha_{33}^2 f_3' \mod J_2.$$

Thus, we have $J_3 = (x_1, x_2^2, f_3')$. Note that

$$x_2x_3^2 = x_2f_3' - x_2^2(Cx_3 + Dx_4) \in J_3.$$

Therefore, since $x_1, x_2^2 \in J_3$ as well we have

$$\alpha(f_4) = \alpha(x_2x_3x_4) \equiv \alpha_{22}\alpha_{33}\alpha_{44}x_2x_3x_4 \mod J_3.$$

It follows that $J_4 = (x_1, x_2^2, f_3', x_2x_3x_4)$. Since $x_3^3 \in J_4$, we get

$$\alpha(2x_2x_4^2 + x_3x_4^2) \equiv C_0x_2x_4^2 + D_0x_3x_4^2 = f_5' \mod J_4.$$

It follows that $J_5 = (x_1, x_2^2, f_3', x_2x_3x_4, f_5')$.

We finish the proof by showing that $\alpha(f_i)$ is a scalar multiple of f_i modulo J_{i-1} for $6 \le i \le 8$. This gives $J_i = (f_i) + J_{i-1}$ for $6 \le i \le 8$ and hence $J_8 = \alpha(H(V_5))$ is generated by N'.

We have

$$\alpha(f_6) = \alpha(x_2x_4^3) \equiv \alpha_{22}x_2(\alpha_{24}x_2 + \alpha_{34}x_3 + \alpha_{44}x_4)^3 \equiv \alpha_{22}\alpha_{44}^3x_2x_4^3 \mod J_5,$$

where the first equivalence uses $x_1 \in J_5$ and the second equivalence uses that $x_2^2, x_2x_3^2, x_2x_3x_4 \in J_5$. To compute $\alpha(f_7)$, we note the identities

$$x_3^3 = x_3f_3' - Cx_2x_3^2 - Dx_2x_3x_4$$

and

$$D_0x_3^2x_4^2 = x_3f_5' - C_0x_2x_3x_4^2$$

and

$$D_0x_3x_4^3 = x_4f_5' - C_0x_2x_4^3$$

which gives that $x_3^3, x_3^2x_4^2, x_3x_4^3 \in J_6$. For f_7, we have

$$\begin{aligned} \alpha(x_4^4) &= (\alpha_{14}x_1 + \alpha_{24}x_2 + \alpha_{34}x_3 + \alpha_{44}x_4)^4 \\ &\equiv (\alpha_{34}x_3 + \alpha_{44}x_4)^4 \equiv (\alpha_{44}x_4)^4 \mod J_6, \end{aligned}$$

where the first equivalence uses that $x_1, x_2^2, x_2x_3x_4, x_2x_3^2, x_2x_4^3 \in J_6$ and the second one uses that $x_3x_4^3, x_3^2x_4^2, x_3^3 \in J_6$. Finally

$$\alpha(f_8) \equiv \alpha_{55}^5x_5^5 \mod J_7$$

because $x_i^5 \in J_7$ for $1 \le i \le 4$ and the assertion follows.

$\square$

Note that the sets N and N' differ by two polynomials only. For the simplicity of notation, we set

$$f_3 = x_3^2 + Cx_2x_3 + Dx_2x_4 \text{ and } f_5 = C_0x_2x_4^2 + D_0x_3x_4^2$$

and so that $\alpha(H(V_5))$ is generated by f_i for $1 \le i \le 8$.

Fix a term order $>$ with $x_1 > \cdots > x_5$. We set

$$A^{'} = \{x_1, x_2^2, x_2x_3, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^3, x_4^4, x_5^5\}.$$

We compute the Gröbner basis for $\alpha(H(V_5))$ for a special class of C and C_0.

Lemma 5.0.15. *Let $\alpha = (\alpha_{ij}) \in \mathcal{B}_5$ such that $C \neq 0, C_0 \neq 0$. Then $\mathrm{In}_>(\alpha(H(V_5)))$ is generated by $A^{'}$.*

Proof. The reduction of the S polynomial $S(f_2, f_3)$ of f_3 with f_2 via f_2, f_3, f_4 is x_3^3 and the $S(f_3, f_4)$ is $x_3^2x_4 + Dx_2x_4^2$ and the $S(f_5, f_6)$ is $x_3x_4^3$. We denote x_3^3, $x_3^2x_4 + Dx_2x_4^2$ and $x_3x_4^3$ by f_9, f_{10}, f_{11}, respectively.

Firstly, we consider the case $x_3^2 > x_2x_4$. Note then the set $A^{'}$ consists of $\mathrm{In}_>(f_i)$ for $1 \leq i \leq 11$, $i \neq 4, 6$ ($Cf_4 = x_4f_3 - f_{10}$ and $\mathrm{In}_>(f_4)$ is divisible by $\mathrm{In}_>(f_3)$; $C_0f_6 = x_4f_5 - D_0f_{11}$ and $\mathrm{In}_>(f_6)$ is divisible by $\mathrm{In}_>(f_5)$). Therefore, it remains to show that this set of polynomials satisfy the Buchberger criterion. That is, the S-polynomial of any pair of polynomials f_i, f_j with $1 \leq i, j \leq 11$ and $i, j \neq 4, 6$ reduce to zero. Since the S-polynomial of two monomials are zero, it suffices to consider the S-polynomials involving f_3 or f_5 or f_{10}.

We go through the pairs and write the polynomials in the order they appear in the polynomial division: $S(f_2, f_3)$ reduces to zero via f_2, f_3, f_9, f_{10} and $S(f_2, f_5)$ reduces to zero via f_3, f_{10}. The S-polynomial $S(f_3, f_5)$ reduces to zero via f_5, f_{10}, f_{11}.

The S-polynomials $S(f_5, f_7)$, $S(f_7, f_{10})$ and $S(f_{10}, f_{11})$ reduce to zero at one step each via f_7. $S(f_3, f_9)$ reduces to zero via f_3, f_9, f_{10} and $S(f_3, f_{10})$ reduces to zero via f_2, f_3, f_9, f_{10}.

The S-polynomial $S(f_3, f_{11})$ reduces to zero via f_7, f_{11} and $S(f_5, f_{10})$ reduces to zero via f_2, f_9. The reduction of $S(f_5, f_{11})$ is zero via f_{11}. Finally, $S(f_9, f_{10})$ reduces to zero via f_3, f_{10}.

We have considered all pairs whose initial terms are not relatively prime. So, the proof for this case is complete because the S-polynomial of two polynomials that have relatively prime initial terms reduces to zero.

Now we consider the case $x_2x_4 > x_3^2$. Define

$$f_{12} = DD_0x_3x_4^2 - C_0x_3^2x_4 = Df_5 - C_0f_{10}.$$

From this equality and $\mathrm{In}_>(f_5) = \mathrm{In}_>(f_{10})$, we get that the set A' consists of $\mathrm{In}_>(f_i)$ for $1 \leq i \leq 12$, $i \neq 4, 6, 10$. Thus, it remains to show that this set of polynomials satisfy the Buchberger criterion.

We only need to check the S-polynomials involving f_{12}. Both S-polynomials $S(f_3, f_{12})$ and $S(f_5, f_{12})$ reduce to zero via f_3, f_9, f_{10}. The S-polynomials $S(f_7, f_{12})$ and $S(f_{11}, f_{12})$ reduce to zero at one step each via f_7. Finally, the S-polynomial $S(f_9, f_{12})$ reduces to zero via f_5, f_{11}, f_{12}.

□

By the previous lemmas and Theorem 4.0.2, we get the following theorem.

Theorem 5.0.16. *For $p = 5$, there are $5!$ generic initial ideals of $H(V_5)$. Each of them is generated by $\Pi(A')$ where Π is a permutation of the variables in $F[V_5]$.*

Proof. Let $\alpha = (\alpha_{ij}) \in \mathcal{B}_5$ such that $C = 0, C_0 \neq 0$. Then, we get that the monic generators of $\mathrm{In}_>(\alpha(H(V_5)))$ of degree at most two are x_1, x_2^2 and x_3^2 or x_2x_4. Note that if $C = 0$, we have

$$x_2x_3 \notin \mathrm{In}_>(\alpha(H(V_5))).$$

Therefore, $\mathrm{In}_>(\alpha(H(V_5)))$ fails to be Borel fixed because either x_3^2 or x_2x_4 lies in $\mathrm{In}_>(\alpha(H(V_5)))$. For $C = 0$, it follows that

$$\mathrm{gin}_>(H(V_5)) \neq \mathrm{In}_>(\alpha(H(V_5))).$$

Moreover, If $C \neq 0$ and $C_0 = 0$, then we have

$$x_3x_4^2 \in \mathrm{In}_>(\alpha(H(V_5))).$$

However, we know either

$$x_3^2x_4 \notin \mathrm{In}_>(\alpha(H(V_5)))$$

or

$$x_2x_4^2 \notin \mathrm{In}_>(\alpha(H(V_5))).$$

Hence, $\mathrm{In}_>(\alpha(H(V_5)))$ is not Borel fixed.

On the other hand, by the previous lemma, all other $\alpha \in \mathcal{B}_5$ generates the same initial ideal $\mathrm{In}_>(\alpha(H(V_5)))$ and so $\mathrm{gin}_>(H(V_5)) = \mathrm{In}_>(\alpha(H(V_5)))$ for α satisfying $C \neq 0$ and $C_0 \neq 0$. The final claim of the theorem follows from Theorem 4.0.2. $\square$

Chapter 6

Further Study

The most fundamendal groups appearing in modular invariant theory are cyclic groups and the Klein four group. So far we study cyclic groups and now we consider the Klein four group.

Let G be the Klein four group over a field F with characteristic 2 and W denotes a G-module over F. Assume that W does not contain the regular representation as a summand. In [8], it is found the generators of $H(W)$.

From [8, Proposition 16, 17], we know that there is a basis for dual space W^* such that $H(W)$ is generated by first, second and forth powers of these basis elements.

This result follows from the following two propositions.

Proposition 6.0.1 (Elmer-Sezer, 2018). *Let W denote a G-module over F containing $k+1$ indecomposable summands. There is a basis $\{x_0, x_1, \ldots, x_n\}$ of W^* in which $x_0, x_1, \ldots, x_k$ are terminal variables such that $F[W]_G$ is free as a module over its subalgebra $\mathcal{T}$ generated by the images $X_0, X_1, \ldots, X_k$ of the terminal variables.*

Moreover, we have

$$\mathcal{T} \cong F[t_0, \ldots, t_k]/(t_0^{a_0}, \ldots, t_k^{a_k}),$$

where $t_0, \ldots, t_k$ are independent variables, and for each i, we have $a_i = 2$ or 4.

Proposition 6.0.2 (Elmer-Sezer, 2018). *Let W be a FG-module such that W does not contain the regular representation as a summand. Then, there exists a basis of W^* such that $H(W)$ is generated by powers of basis elements.*

After these two propositions, we note that $F[W] = F[x_1, \ldots, x_n]$ is a polynomial ring in these basis elements and $H(W)$ is generated by $x_i^{a_i}$, where $a_i \in \{1, 2, 4\}$ for $1 \leq i \leq n$.

Then, we show that the Hilbert ideal of W is Borel fixed with the suitable ordering of the variables in the monomial order. Note that this implies the generic initial ideal of $H(W)$ is equal to itself by Proposition 2.0.15.

Proposition 6.0.3. *Assume the notation of the previous paragraphs. Then, there is choice for a basis for W^* and a ranking of variables in $F[W]$ such that $H(W)$ is Borel fixed for all monomial orders compatible with this ranking. In particular, we get*

$$\mathrm{gin}_>(H(W)) = H(W) \text{ for all such orders.}$$

Proof. Let $>$ be a monomial order such that $x_i > x_j$ whenever $a_i < a_j$. Since we are in characteristic two and a_i is a 2-th power, a member of the non-standard Borel subgroup sends $x_i^{a_i}$ to some combination of a_i-th powers of variables of higher or equal rank. By the definition of $>$, this combination belongs to $H(W)$, so $H(W)$ is a Borel fixed ideal. Thus, by [10, 1.8], we have

$$\mathrm{gin}_>(H(W)) = H(W).$$

□

Reviewing the all cases we have studied so far, we notice that

$$\mathrm{gin}_>(H(V_5)) = \mathrm{In}_>(H(V_5)).$$

Together with the previous proposition on the Klein four group, we consider that this equality is satisfied for a more general situation and state the following conjecture.

Conjecture 6.0.4. Let G be a p-group and V be a G-module over a field F of characteristic p. Then, there is a choice of a basis for V^* and a monomial order $>$ on the monomials in $F[V]$ such that

$$\mathrm{gin}_>(H(V)) = \mathrm{In}_>(H(V)).$$

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