MONOMIAL GOTZMANN SETS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

MONOMIAL GOTZMANN SETS

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A homogeneous set of monomials in a quotient of the polynomial ring $S := F[x_1, \ldots, x_n]$ is called Gotzmann if the size of this set grows minimally when multiplied with the variables. We note that Gotzmann sets in the quotient $R := F[x_1, \ldots, x_n]/(x_1^a)$ arise from certain Gotzmann sets in S. Then we partition the monomials in a Gotzmann set in S with respect to the multiplicity of x_i and obtain bounds on the size of a component in the partition depending on neighboring components. We show that if the growth of the size of a component is larger than the size of a neighboring component, then this component is a multiple of a Gotzmann set in $F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n]$. We also adopt some properties of the minimal growth of the Hilbert function in S to R.

Keywords: Gotzmann sets, Macaulay-Lex rings, The Hilbert functions.

ÖZET

MONOM GOTZMANN KÜMELERİ

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$S := F[x_1, \ldots, x_n]$ bir polinom halkası olsun. S'in bir bölümünde olan homojen bir monom kümesi alalım. Eğer bu kümeyi tüm değişkenler ile çarptığımızda bu kümenin boyutundaki artış minimum ise bu kümeye Gotzmann kümesi denir. İlk olarak $R := F[x_1, \ldots, x_n]/(x_1^a)$ bölümünde yer alan Gotzmann kümelerinin S'te bulunan belli Gotzmann kümelerinden geldiğini göreceğiz. Daha sonra S'teki Gotzmann kümelerini x_i çarpanlarına göre bölümleyip komşu bileşenlere bağlı olarak bölümlerin boyutlarını sınırlandıracağız. Eğer bir bileşenin boyutundaki artış komşu bileşenin boyutundan büyük ise bu bileşenin $F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n]$'deki bir Gotzmann kümesinin bir çarpanı olduğunu göstereceğiz. Son olarak da S'teki Hilbert fonksiyonlarının en düşük artma özelliklerinden bazılarını R'ye uyarlayacağız.

Anahtar sözcükler: Gotzmann kümeleri, Macaulay-Lex halkaları, Hilbert fonksiyonları.

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Contents

Chapter 1

Introduction

In this thesis the main subject is Gotzmann monomial ideals in a polynomial ring. A homogeneous set of monomials in a quotient of the polynomial ring $S := F[x_1, \ldots, x_n]$ is called Gotzmann if the size of this sets grow minimally when multiplied with the variables. We begin by giving some background information on monomial ideals. Then we will introduce the Hilbert functions and give a classical theorem on minimal growth of the Hilbert functions. After studying these topics we will be able to start studying Gotzmann monomial sets in a polynomial ring.

1.1 Background Information on Monomial Ideals

Let $S = F[x_1, \ldots, x_n]$ be a polynomial ring over a field F with $\deg(x_i) = 1$ for $1 \leq i \leq n$.

Definition 1.1.1. A **monomial** in S is a product $x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ for a vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ of nonnegative integers. An ideal $I \subseteq S$ is called a **monomial ideal** if it is generated by monomials.

Recall that a partial order on a set P is a relation $\leq$ on P such that, for all $x, y, z \in P$ one has

(i) $x \leq x$ (reflexivity);

(ii) If $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry);

(iii) If $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

A total order on a set P is a partial order $\leq$ on P such that, for any two elements x and y belonging to P, one has either $x \leq y$ or $y \leq x$.

By a monomial order on S we mean a total order on S satisfying

(i) $1 < x^a$ for all $x^a \in S$ with $x^a \neq 1$

(ii) If $x^a < x^b$ then $x^a x^c < x^b x^c$ for all $x^c \in S$.

Note that if $n \geq 2$ then there are infinitely many monomial orders on S and for $n = 1$ there is a unique monomial order on S.

Example 1.1.2. (a) Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be vectors belonging to $\mathbb{N}^n$. We define the total order $<_{lex}$ on S by setting $x^a <_{lex} x^b$ if either

(i) $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or

(ii) $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the leftmost nonzero component of the vector $a - b$ is negative.

It follows that $<_{lex}$ is a monomial order on S, which is called the **lexicographic order** on S induced by the ordering $x_1 > x_2 > \ldots > x_n$.

(b) Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be vectors belonging to $\mathbb{N}^n$. We define the total order $<_{rev}$ on S by setting $x^a <_{rev} x^b$ if either

(i) $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or

(ii) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the rightmost nonzero component of the vector $a - b$ is positive.

It follows that $<_{rev}$ is a monomial order on S, which is called the **reverse lexicographic order** on S induced by the ordering $x_1 > x_2 > \ldots > x_n$.

(c) Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be vectors belonging to $\mathbb{N}^n$. We define the total order $<_{purelex}$ on S by setting $x^a <_{purelex} x^b$ if the leftmost nonzero component of the vector $a - b$ is negative. It follows that $<_{purelex}$ is a monomial order on S, which is called the **pure lexicographic order** on S induced by the ordering $x_1 > x_2 > \ldots > x_n$.

In this thesis we will use *lexicographic* order on S with $x_1 > x_2 > \ldots > x_n$.

Definition 1.1.3. A set M of monomials in S is called **lexsegment** if for monomials $m \in M$ and $v \in S$ we have: If $\deg m = \deg v$ and $v > m$, then $v \in M$. A monomial ideal I is called lexsegment if the set of monomials in I is lexsegment.

Example 1.1.4. The set $A := \{x_1^2, x_1x_2, x_2^2\}$ is lexsegment in $F[x_1, x_2]$.
Note that a lexsegment set in a polynomial ring may no longer be a lexsegment set in a polynomial ring extension. For example the set A is not lexsegment in $F[x_1, x_2, x_3]$.

Definition 1.1.5. For a set of monomials M in the homogeneous component S_t of degree t in S, let $lex_S(M)$ denote the lexsegment set of $|M|$ monomials in S_t.

1.2 The Hilbert Function

Definition 1.2.1. Let M be a $\mathbb{N}^n$**-graded** S-module and $M = \oplus_{a\in\mathbb{N}^n} M_a$. Then the **Hilbert function** $H(M, -) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ of M is the numerical function defined by

$$H(M, a) = dim_F M_a.$$

If I is a homogeneous ideal in S the Hilbert function is defined by

$$H(I, t) = dim_F I_t$$

where I_t is the homogeneous component of degree t of I.

If the vector space dimension $dim_F(M_a)$ is finite for all $a \in \mathbb{N}^n$ then the formal power series

$$\mathcal{F}(M, x) = \sum_{a \in \mathbb{N}^n} dim_F(M_a) x^a$$

is the $\mathbb{N}^n$-**graded Hilbert series** of M. Setting $x_i = t$ for all i yields the **Hilbert series** $\mathcal{F}(M; t, \ldots, t)$. Similarly if I is a homogeneous ideal in S , then the Hilbert series of I is defined by

$$\mathcal{F}(I, x) = \sum_{t \in \mathbb{N}} dim_F(I_t) x^t$$

Example 1.2.2. A basic example is when $S = F[x]$. Since the dimension of each homogeneous component is 1, the Hilbert function $H(S, t) = 1$ for all $t \in \mathbb{N}$. Hence we have the Hilbert series of S;

$$\mathcal{F}(S, t) = 1 + t + t^2 + \ldots = \frac{1}{1 - t}.$$

Now let's take $S = F[x, y]$. This time each homogeneous component of degree t of S contains $t + 1$ monomials, namely $x^t, x^{t-1}y, \ldots, xy^{t-1}, y^t$. Hence $H(S, t) = t + 1$ for all $t \in \mathbb{N}$. Hence we have the Hilbert series of S;

$$\mathcal{F}(S, t) = \sum_{i \in \mathbb{N}} (i + 1) t^i = \frac{1}{(1 - t)^2}.$$

If we take $S = F[x_1, \ldots, x_n]$ we will get

$$\mathcal{F}(S, t) = \frac{1}{(1 - t)^n}.$$

Note that in each cases Hilbert series of S is a rational function. Actually Miller and Sturmfels [11, C4] show that the Hilbert series of every monomial quotient I of S can be expressed as a rational function as

$$\mathcal{F}(I, x) = \frac{\mathcal{K}(I, x)}{(1 - x_1) \ldots (1 - x_n)},$$

where $\mathcal{K}(I, x)$ is called **K-polynomial** of I.

What are the possible Hilbert functions of homogeneous ideals in S? This question was answered by Macaulay [7], who showed that for every homogeneous ideal there exists a lexsegment ideal with the same Hilbert function. Lexsegment ideals are highly structured: they are defined combinatorially and it is easy to derive inequalities characterizing their possible Hilbert functions.

Gotzmann's persistence theorems [4] determine the growth of the Hilbert function of a homogeneous ideal. Before we give Gotzmann's persistence theorem let's give some definitions.

Definition 1.2.3. Let n and h be positive integers.

$$h = \binom{h(n)+n}{n} + \binom{h(n-1)+n-1}{n-1} + \cdots + \binom{h(i)+i}{i}$$

where $h(n) \geq h(n-1) \geq \ldots \geq h(i) \geq 0$, $i \geq 1$ is called the *n-th binomial representation* of h.

Theorem 1.2.4. *For given positive integers n and h this representation exists and it is unique.*

Proof. We choose maximal $h(n)$ such that $h \geq \binom{h(n)+n}{n}$. If the equality holds then we are done. Otherwise let $h' = h - \binom{h(n)+n}{n}$. Then $h' > 0$ and by using induction on h, and since $h' < h$ we may assume

$$h' = \binom{h'(n-1)+n-1}{n-1} + \cdots + \binom{h'(i)+i}{i}$$

where $h'(n-1) \geq h'(n-2) \geq \ldots \geq h'(i) \geq 0$, $i \geq 1$. Therefore

$$h = \binom{h(n)+n}{n} + \binom{h'(n-1)+n-1}{n-1} + \cdots + \binom{h'(i)+i}{i}.$$

Take $h(k) = h'(k)$ for $i \leq k \leq n-1$. It remains to show $h(n) \geq h'(n-1)$. Since $\binom{h(n)+n+1}{n} > h$ it follows

$$\binom{h(n)+n}{n-1} = \binom{h(n)+n+1}{n} - \binom{h(n)+n}{n} > h' > \binom{h'(n-1)+n-1}{n-1}$$

hence $h(n)+n > h'(n-1)+n-1$, i.e., $h(n) \geq h'(n-1) = h(n-1)$. This proves the existence of binomial representation.

Next we show if

$$h = \binom{h(n)+n}{n} + \binom{h(n-1)+n-1}{n-1} + \cdots + \binom{h(i)+i}{i}$$

with $h(n) \geq h(n-1) \geq \ldots \geq h(i) \geq 0$ then $h(n)$ is the largest integer such that $h \geq \binom{h(n)+n}{n}$. We prove this by induction on h. The assertion is trivial for $h = 1$. Suppose $h > 1$ and that $h \geq \binom{h(n)+n+1}{n}$. Then

$$h' = \sum_{j=1}^{n-1} \binom{h(j)+j}{j} \geq \binom{h(n)+n+1}{n} - \binom{h(n)+n}{n} = \binom{h(n)+n}{n-1} \geq \binom{h(n-1)+n}{n-1}$$

contradicting the induction hypothesis.

Now since the first summand in the representation of h is uniquely determined, and since by induction on the length of the representation we may assume that the representation of h' is unique we conclude that also the representation of h is unique. $\square$

Notation 1.2.1. If

$$h = \binom{h(n)+n}{n} + \binom{h(n-1)+n-1}{n-1} + ... + \binom{h(i)+i}{i}$$

is the n-th binomial representation of h, then we define

$$h^{\langle n \rangle} = \binom{h(n)+n+1}{n} + \binom{h(n-1)+n}{n-1} + ... + \binom{h(i)+i+1}{i}.$$

Remark 1.2.5. Let $h \geq k$ and n be positive integers, then $h^{<n>} \geq k^{<n>}$.

Now we are ready to give the following theorems;

Theorem 1.2.6 (Minimal growth of the Hilbert function)**.** *Let I be a homogeneous ideal in S. Then one has*

$$H(I, d+1) \geq H(I, d)^{<n-1>}.$$

This theorem is proved by F. Macaulay. We refer to reader to [2, 4.2] for further information.

Theorem 1.2.7 (Gotzmann's Persistence Theorem, [4])**.** *Let I be a homogeneous ideal of S generated in degree $\leq d$. If $H(I, d+1) = H(I, d)^{<n-1>}$ then*

$$H(I, k+1) = H(I, k)^{<n-1>}$$

for all $k \geq d$.

In [13] Murai gives a combinatorial proof of this theorem using binomial representations. He derives some properties of these representations which provide information on the growth of the Hilbert functions.

For a set of monomials M, $S_1 \cdot M$ denotes the set of monomials of the form um, where u is a variable and $m \in M$. By a classical theorem of Macaulay [6, C4] we have

$$|(S_1 \cdot lex_S(M))| \leq |(S_1 \cdot M)|. \tag{1.1}$$

One course of research inspired by Macaulay's theorem is the study of the homogeneous ideals I such that every Hilbert function in S/I is obtained by a lexsegment ideal in S/I. Such quotients are called Macaulay-Lex rings. A well-known example is $S/(x_1^{a_1}, \ldots, x_n^{a_n})$ with $a_1 \leq \cdots \leq a_n \leq \infty$ and $x_i^{\infty} = 0$ which is due to Clements and Lindström [3]. These rings have important applications in combinatorics and algebraic geometry. For a good account of these matters and basic properties of Macaulay-Lex rings we direct the reader to Mermin and Peeva [9], [10]. Some recently discovered classes of Macaulay-Lex rings can be found in Mermin and Murai [8].

Chapter 2

Monomial Gotzmann Ideals

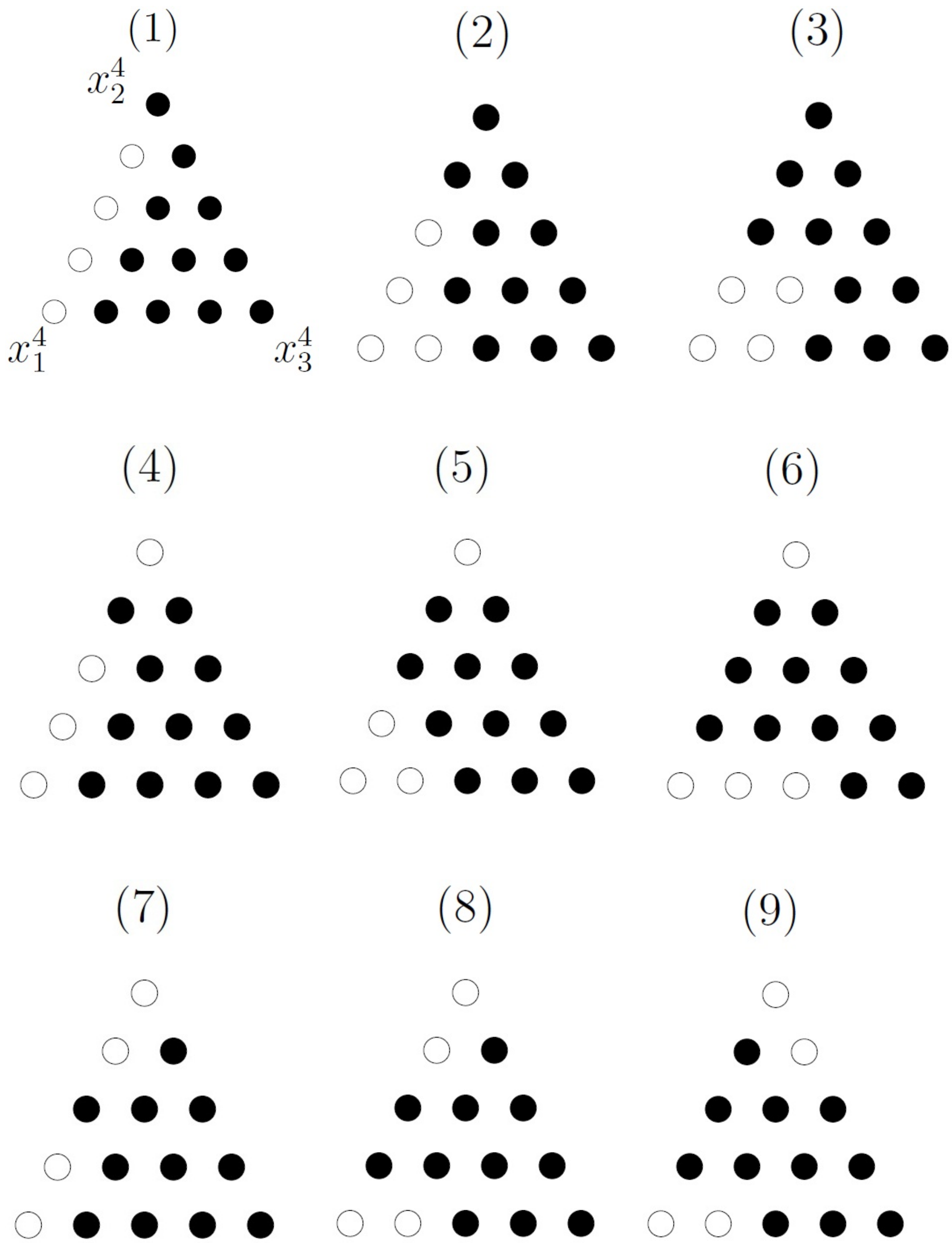
2.1 Preliminaries

Monomial sets in S whose sizes grow minimally in the sense of Macaulay's inequality have also attracted attention:

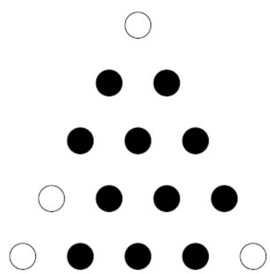
Definition 2.1.1. A homogeneous set M of monomials is called Gotzmann if $|(S_1 \cdot lex_S(M)| = |(S_1 \cdot M)|$ and a monomial ideal I is Gotzmann if the set of monomials in I_t is a Gotzmann set for all t.

In [14], Gotzmann ideals in S that are generated by at most n homogeneous polynomials are classified in terms of their Hilbert functions. In [12] Murai finds all integers j such that every Gotzmann set of size j in S is lexsegment up to a permutation. He also classifies all Gotzmann sets for $n \leq 3$.

Example 2.1.2. According to Murai's classification there are ten Gotzmann sets of degree 4 in $F[x_1, x_2, x_3]$. Let's show them on pictures. The monomial x_1^4 is in the lower left corner, x_3^4 is in the lower right corner and x_2^4 is at the top. The black dots denote the monomials in the set and the empty circles denote the monomials that are not in the set. For example figure (1) means $x_1^3x_3$, $x_1^2x_3^2$, $x_1x_3^3$, x_3^4, $x_1^2x_2x_3$, $x_1x_2x_3^2$, $x_2x_3^3$, $x_1x_2^2x_3$, $x_2^3x_3$, and x_2^4 are in the set and x_1^4, $x_1^3x_2$, $x_1^2x_2^2$ and $x_1x_2^3$ are missing. In the pictures below it is classified all Gotzmann sets in $F[x_1, x_2, x_3]$ up to permutation.



(10)



According to Murai's classification all empty circles must be at corner and each connected component of emptysets look like the Young diagram Also the number of empty circles must be less than the degree of the elements in the set.

Gotzmann persistence theorem states that if M is a Gotzmann set in S, then $S_1 \cdot M$ is also a Gotzmann set, see [4]. In [13] Murai gives a combinatorial proof of this theorem using binomial representations. Among other related works, Aramova, Herzog and Hibi obtains Macaulay's and Gotzmann's theorems for exterior algebras, [1]. More recently, Hoefel shows that the only edge ideals that are Gotzmann are the ones that arise from star graphs, see [5]. Also some results on generation of lexsegment and Gotzmann ideals by invariant monomials can be found in [16].

2.2 Monomial Gotzmann Sets in R

In this section we study the Gotzmann sets and the minimal growth of the Hilbert function in the Macaulay-Lex quotient $R := F[x_1, \ldots, x_n]/(x_1^a)$, where a is a positive integer. A set M of monomials in R can also be considered as a set of monomials in S and by $R_1 \cdot M$ we mean the set of monomials in $S_1 \cdot M$ that are not zero in R. We show that Gotzmann sets in R arise from certain Gotzmann sets in S: When a Gotzmann set in R_t with $t \geq a$ is added to the set of monomials in S_t that are divisible by x_1^a, one gets a Gotzmann set in S_t. Then we partition the monomials in a Gotzmann set in S with respect to the multiplicity of x_i

and show that if the growth of the size of a component is larger than the size of a neighboring component, then this component is a multiple of a Gotzmann set in $F[x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n]$. Otherwise we obtain lower bounds on the size of the component in terms of sizes of neighboring components. We also note down adoptions of some properties concerning the minimal growth of the Hilbert function in S to R. These results can also be found in [15].

We begin with the definition of Gotzmann sets in R.

Definition 2.2.1. A set M of monomials in R_t is Gotzmann if $|(R_1 \cdot lex_R(M))| = |(R_1 \cdot M)|$, where R_t is homogeneous component of degree t of R and $lex_R(M)$ denotes the lexsegment set of monomials in R_t that has the same size as M.

For a homogeneous lexsegment set L in S with $|L| = d$, the size of $S_1 \cdot L$ was computed by Macaulay. This number is very closely related to the n-th binomial representation of d and is denoted by $d^{<n-1>}$. In contrast to the situation in S, for the homogeneous lexsegment set $L \subseteq R_t$ of size d, the size of the set $R_1 \cdot L$ depends also on t. We let $d_{n,t}$ denote this size. In the sequel when we talk about $d_{n,t}$ we will always assume that d is smaller than the number of monomials in R_t because otherwise $d_{n,t}$ is not defined. Notice that we have $d_{n,t} = d^{<n-1>}$ for $t < a - 1$.

Definition 2.2.2. For a non-negative integer i, let S_t^i and R_t^i denote the set of monomials in S_t and R_t respectively that are divisible by x_1^i but not by x_1^{i+1}. For a set of monomials M in R_t, let M^i denote the set $R_t^i \cap M$. Similarly, if M is in S_t, then M^i denotes $S_t^i \cap M$. Also let $I(M)$ denote the smallest integer such that $M^{I(M)} \neq \emptyset$.

Set $S' = F[x_2, \dots, x_n]$ and let $S'_1 \cdot M$ denote the set of monomials of the form $x_i m$, where $2 \leq i \leq n$ and $m \in M$. For a monomial $u \in R$ and a monomial set M in R we let $u \cdot M$ denote the set of monomials in R that are of the form um with $m \in M$. We also let M^i/x_1 denote the set of monomials m in S' such that $mx_1^i \in M^i$. We start by noting down a result of Murai [13, 1.5] that is very useful for our purposes.

Lemma 2.2.3. *Let b_1, b_2, n be positive integers. Then*

$$b_1^{<n>} + b_2^{<n>} > (b_1 + b_2)^{<n>}.$$

The following lemmas squeeze $d_{n,t}$ between $d^{<n-2>}$ and $d^{<n-1>}$.

Lemma 2.2.4. *Let $t \geq a-1$. Then $d_{n,t} \geq d^{<n-2>}$.*

Proof. Let L be the lexsegment size of d in R_t with $t \geq a-1$ and let j denote $min(L)$. Since L is lexsegment, we have $L^i = R_t^i$ for $j < i \leq a-1$ giving $x_1 \cdot L^i \subseteq S_1' \cdot L^{i+1}$ for $j \leq i < a-1$. Moreover $x_1 \cdot L^{a-1}$ is empty and so we get

$$R_1 \cdot L = \bigsqcup_{j \leq i \leq a-1} S_1' \cdot L^i.$$

Note that L^i/x_1 is a lexsegment set in S'. Therefore $|S_1' \cdot L^i| = |S_1' \cdot (L^i/x_1)|$ and $|S_1' \cdot (L^i/x_1)| = |L^i|^{<n-2>}$. It follows that

$$d_{n,t} = |R_1 \cdot L| = \sum_{j \leq i \leq a-1} |L^i|^{<n-2>}.$$

From this identity and Lemma 2.2.3 we obtain $d_{n,t} \geq d^{<n-2>}$, as desired. □

Lemma 2.2.5. *Let M be a set of monomials in R_t with $t \geq a$. Let B denote the set of monomials in S_t that are divisible by x_1^a. We have the disjoint union*

$$S_1 \cdot (B \sqcup M) = (S_1 \cdot B) \sqcup ((S_1 \cdot M) \cap R).$$

Therefore $d_{n,t} = (d + |B|)^{<n-1>} - |B|^{<n-1>}$. In particular, $d_{n,t} < d^{<n-1>}$.

Proof. Since $t \geq a$, B is non-empty. Note also that B is a lexsegment set in S because x_1 is the highest ranked variable. Since no monomial in R is divisible by x_1^a the sets $S_1 \cdot B$ and $(S_1 \cdot M) \cap R$ are disjoint and we clearly have $S_1 \cdot (B \cup M) \supseteq (S_1 \cdot B) \sqcup ((S_1 \cdot M) \cap R)$. Conversely, let m be a monomial in $S_1 \cdot (B \sqcup M)$. We may take $m \in (S_1 \cdot M) \setminus R$. Then $m = x_1^a m'$ for some monomial m' that is not divisible by x_1. Since the degree of m is at least $a+1$, m' is divisible by one of the variables, say x_i for some $2 \leq i \leq n$. Then $m = x_i(x_1^a m'/x_i) \in S_1 \cdot B$. Secondly putting a lexsegment set L for M in this formula yields $d_{n,t} = (d + |B|)^{<n-1>} - |B|^{<n-1>}$ because $R_1 \cdot L = (S_1 \cdot L) \cap R$ and $L \sqcup B$ is lexsegment in S_t. It also follows that $d_{n,t} < d^{<n-1>}$ by Lemma 2.2.3. □

Since $R_1 \cdot M = (S_1 \cdot M) \cap R$, the previous lemma yields the following theorem.

Theorem 2.2.6. *Let M be a set of monomials in R_t. Then we have:*

(1) If $t \geq a$, then M is Gotzmann in R_t if and only if $B \sqcup M$ is Gotzmann in S_t.

(2) If $t = a-1$, then M is Gotzmann in R_t if and only if M is Gotzmann in S_t and $x_1^{a-1} \in M$.

(3) If $t < a-1$, then M is Gotzmann in R_t if and only if M is Gotzmann in S_t.

Proof. Let L denote the lexsegment set in R_t of the same size as M with $t \geq a$. Then Lemma 2.2.5 implies that $|R_1 \cdot L| = |R_1 \cdot M|$ if and only if $|S_1 \cdot (B \sqcup M)| = |S_1 \cdot (B \sqcup L)|$. Hence the first statement of the proposition follows because $B \sqcup L$ is lexsegment in S_t.

For $t = a-1$, we have $d_{n,a-1} = d^{<n-1>} - 1$. Let $M \in S_{a-1}$ be a set of monomials with $x_1^{a-1} \notin M$. Then $R_1 \cdot M = S_1 \cdot M$ and so $|R_1 \cdot M| \geq d^{<n-1>} > d_{n,a-1}$. Conversely if $x_1^{a-1} \in M$, then $|R_1 \cdot M| = |S_1 \cdot M| - 1$. Hence the second assertion of the theorem follows.

Finally, the last statement follows easily because for $t < a-1$ we have $R_1 \cdot M = S_1 \cdot M$ and lexsegment sets in R_t and S_t are the same. $\square$

Remark 2.2.7. This theorem does not generalize to all Macaulay-Lex quotients. Consider the set of monomials $A := \{x_1^3x_2, x_1^3x_3, x_1x_2^3, x_2^3x_3\}$ whose size grows minimally in $F[x_1, x_2, x_3]/(x_1^4, x_2^4)$. But the set $A \cup \{x_2^4\}$ is not Gotzmann in $F[x_1, x_2, x_3]/(x_1^4)$. Furthermore $A \cup \{x_1^4, x_2^4\}$ is not Gotzmann in $F[x_1, x_2, x_3]$.

2.3 A Combinatorial Result on Deleting a Variable

We know prove our second result which is about Gotzmann sets in S. Let M be a Gotzmann set in S_t. We show that M^i is a product of x_1^i with a Gotzmann set in S' if $|M^i|^{<n-2>} \geq |M^{i-1}|$. Otherwise we provide lower bounds on $|M^i|$ depending on the sizes of neighboring components. We need the following lemma:

Lemma 2.3.1. *Let M be a Gotzmann set of monomials in S_t with $t \geq 0$. For $0 \leq i \leq t$ set $d_i = |M^i|$. For $0 \leq i \leq t+1$ we have*

$$|(S_1 \cdot M)^i| = \max\{d_i^{<n-2>}, d_{i-1}\}.$$

Proof. For a set of monomials K in S_t and a monomial $u \in S$, let $u \cdot K$ denote the set of monomials uk, where $k \in K$. Note that a monomial in $(S_1 \cdot K)^i$ is either product of a variable in S' with a monomial in K^i or a product of x_1 with a monomial in K^{i-1}. It follows that $(S_1 \cdot K)^i = S_1' \cdot K^i \cup x_1 \cdot K^{i-1}$. We also have $S_1' \cdot K^i = S_1' \cdot (x_1^i \cdot (K^i/x_1)) = x_1^i \cdot (S_1' \cdot (K^i/x_1))$. Applying this to the set M we get that the size of the set $S_1' \cdot M^i$ is equal to the size of $S_1' \cdot (M^i/x_1)$ which is at least $d_i^{<n-2>}$ by Macaulay's theorem. Meanwhile the size of the set $x_1 \cdot M_{i-1}$ is d_{i-1}. It follows for all $0 \leq i \leq t+1$ that

$$|(S_1 \cdot M)^i| \geq \max\{d_i^{<n-2>}, d_{i-1}\}. \tag{2.1}$$

Define

$$T = \bigsqcup_{0 \leq i \leq t} x_1^i \cdot (lex_{S'}(M^i/x_1)).$$

Notice that we have $|T^i| = d_i$ for $0 \leq i \leq t$. We compute $|(S_1 \cdot T)^i|$ for $0 \leq i \leq t+1$ as follows. From the first paragraph of the proof we have $(S_1 \cdot T)^i = S_1' \cdot T^i \cup x_1 \cdot T^{i-1}$ and that $S_1' \cdot T^i = x_1^i \cdot (S_1' \cdot (T^i/x_1))$. But T^i/x_1 is a homogeneous lexsegment set by construction and so $S_1' \cdot (T^i/x_1)$ is also a lexsegment set in S'_{t-i+1}, see [2, 4.2.5.]. Hence $|S'_1 \cdot T^i| = d_i^{<n-2>}$. On the other hand $|x_1 \cdot T^{i-1}| = d_{i-1}$. Moreover, since T^{i-1}/x_1 is a lexsegment set in S'_{t-i+1}, the identity $x_1 \cdot T^{i-1} = x_1^i \cdot (T^{i-1}/x_1)$ gives that $x_1 \cdot T^{i-1}$ is obtained by multiplying each element in a homogeneous lexsegment set in S' with x_1^i. Since $S'_1 \cdot T^i$ is also obtained by multiplying the lexsegment set $S'_1 \cdot (T^i/x_1)$ with x_1^i we have either $S'_1 \cdot T^i \subseteq x_1 \cdot T^{i-1}$ or $S'_1 \cdot T^i \supseteq x_1 \cdot T^{i-1}$. Hence $(S_1 \cdot T)^i = S'_1 \cdot T^i$ if $d_i^{<n-2>} \geq d_{i-1}$ and $(S_1 \cdot T)^i = x_1 \cdot T^{i-1}$ otherwise. Moreover, $|(S_1 \cdot T)^i| = \max\{d_i^{<n-2>}, d_{i-1}\}$. Since the size of M has the minimal possible growth, from Inequality 2.1 we get $|(S_1 \cdot M)^i| = \max\{d_i^{<n-2>}, d_{i-1}\}$ as desired. □

We remark that the statement of the following theorem (and the previous lemma) stays true if we permute the variables and write the assertion with respect to another variable. It is also instructive to compare this with [12, 2.1].

Theorem 2.3.2. *Assume the notation of the previous lemma. if $d_i^{<n-2>} \geq d_{i-1}$, then M^i/x_1 is Gotzmann in S'. Moreover, if $d_i^{<n-2>} < d_{i-1}$, then we have either $(d_i+1)^{<n-2>} > d_{i-1} - 1$ or $d_i + 1 > d_{i+1}^{<n-2>}$.*

Proof. Assume that $d_i^{<n-2>} \geq d_{i-1}$ for some $0 \leq i \leq t+1$. Then from the first statement we have $|(S_1 \cdot M)^i| = d_i^{<n-2>}$. But $S'_1 \cdot M^i$ is a subset of $(S_1 \cdot M)^i$ and $|S'_1 \cdot M^i| = |x_1^i \cdot (S'_1 \cdot (M^i/x_1))| = |S'_1 \cdot (M^i/x_1)| \geq d_i^{<n-2>}$. It follows that $|S'_1 \cdot (M^i/x_1)| = d_i^{<n-2>}$and so M^i/x_1 is Gotzmann.

We now prove the final assertion of the proposition. Assume that there exists an integer $1 \leq q \leq t$ such that $d_q^{<n-2>} < d_{q-1}$. By way of contradiction assume further that $(d_q+1)^{<n-2>} \leq d_{q-1} - 1$ and $d_q + 1 \leq d_{q+1}^{<n-2>}$. We obtain a contradiction by constructing a set W in S_t whose size grows strictly less than the size of M. Let w_{q-1} be the minimal monomial in T^{q-1}. Notice also that $d_q^{<n-2>} < d_{q-1}$ implies that $S_t^q \setminus T^q \neq \emptyset$ and let w_q be the monomial that is maximal among the monomials in $S_t^q \setminus T^q$. Define

$$W = \Big(\bigsqcup_{0 \leq i \leq t,\ i \neq q-1, q} T^i \Big) \sqcup (T^{q-1} \setminus \{w_{q-1}\}) \sqcup (T^q \cup \{w_q\}).$$

Notice that by construction W^i/x_1 is a lexsegment set in S' for all $0 \leq i \leq t$. Therefore, just as we saw for T, we have $|(S_1 \cdot W)^i| = \max\{|W^i|^{<n-2>}, |W^{i-1}|\}$. We also have $|W^i| = d_i$ for $i \neq q-1, q$, and $|W^{q-1}| = d_{q-1} - 1$ and $|W^q| = d_q + 1$. It follows that $|(S_1 \cdot T)^i| = |(S_1 \cdot W)^i|$ for all $i \neq q-1, q, q+1$. We finish the proof by showing that

$$\sum_{q-1 \leq i \leq q+1} |(S_1 \cdot W)^i| < \sum_{q-1 \leq i \leq q+1} |(S_1 \cdot T)^i|.$$

We have $|(S_1 \cdot W)^{q-1}| = \max\{(d_{q-1} - 1)^{<n-2>}, d_{q-2}\} \leq \max\{(d_{q-1})^{<n-2>}, d_{q-2}\} = |(S_1 \cdot T)^{q-1}|$. Notice also that $|(S_1 \cdot W)^q| = \max\{(d_q + 1)^{<n-2>}, d_{q-1} - 1\} = d_{q-1} - 1 < d_{q-1} = \max\{d_q^{<n-2>}, d_{q-1}\} = |(S_1 \cdot T)^q|$. Finally, $|(S_1 \cdot W)^{q+1}| = \max\{d_{q+1}^{<n-2>}, d_q + 1\} = d_{q+1}^{<n-2>} = |(S_1 \cdot T)^{q+1}|$.

□

2.4 Generalization of Some Properties of the Hilbert Function in S to R

We generalize some properties of the minimal growth of the Hilbert function in S to R. Firstly, we show that $d_{n,t}$ is increasing in the first parameter and decreasing in the second parameter.

Proposition 2.4.1. *Let d, n, t be positive integers. Then the following statements hold:*

1. $d_{n+1,t} > d_{n,t}$.
2. $d_{n,t+1} \leq d_{n,t}$.

Moreover, for $n \geq 3$ we have $d_{n,t'} = d^{<n-2>}$ for t' sufficiently large.

Proof. Since $d_{n,t} = d^{<n-1>}$ for $t < a-1$, the first statement is precisely [13, 1.7] for $t < a-1$. Let L be the lexsegment set of size d in R_t. For $t = a-1$, we have $R_1 \cdot L = (S_1 \cdot L) \setminus \{x_1^a\}$. So $d_{n,a-1} = d^{<n-1>} - 1$ and the first statement again follows from [13, 1.7]. For $t \geq a$ by Lemma 2.2.5 we have $d_{n,t} < d^{<n-1>}$. On the other hand,$d_{n+1,t} \geq d^{<n-1>}$ by Lemma 2.2.4. This establishes the first statement.

Since $d_{n,t} = d^{<n-1>}$ for $t \leq a-2$, the second statement holds trivially for $t < a-2$. Moreover, we have $d_{n,a-1} = d^{<n-1>} - 1$ from the previous paragraph and so $d_{n,a-1} < d_{n,a-2}$ as well. Also we eliminate the case $n = 2$ because $d_{2,t} = d$ for $t \geq a-1$. So we assume that $t \geq a-1$ and $n > 2$. Note that $n > 2$ implies that $|R_t^i| < |R_{t+1}^i|$ and $|R_t^i| < |R_t^{i-1}|$ for $t \geq a-1$ and $i \leq a-1$. Let L_1 and L_2 be two lexsegment sets of equal sizes in R_t and R_{t+1} respectively. The rest of the proof of the second statement is devoted to showing $|R_1 \cdot L_1| \geq |R_1 \cdot L_2|$. Set $j_1 = I(L_1)$ and $j_2 = I(L_2)$. Since L_1 and L_2 are lexsegment sets, we have $L_1^i = R_t^i$ for $j_1 < i \leq a-1$ and $L_2^i = R_{t+1}^i$ for $j_2 < i \leq a-1$. We also have $|L_1^{j_1}| \leq |R_t^{j_1}|$ and $|L_2^{j_2}| \leq |R_{t+1}^{j_2}|$. But since $|R_t^i| < |R_{t+1}^i|$, we have $|L_1^i| < |L_2^i|$ for all i that is strictly bigger than both j_1 and j_2. Therefore $j_1 \leq j_2$ because the sizes of L_1 and L_2 are the same. Also note that $|R_t^i| = |R_{t+1}^{i+1}|$ and so $|L_1^i| = |L_2^{i+1}|$ for $\max\{j_1, j_2 - 1\} < i < a-1$. We claim that $j_1 + 1 \geq j_2$. Otherwise we obtain a contradiction as follows. We have $\max\{j_1, j_2-1\} = j_2 - 1 \neq j_1$ and $|L_1^{j_2-1}| = |R_t^{j_2-1}| = |R_{t+1}^{j_2}| \geq |L_2^{j_2}|$. Therefore

$$|L_2| = \sum_{j_2 \leq i \leq a-1} |L_2^i| \leq \sum_{j_2-1 \leq i \leq a-2} |L_1^i| < \sum_{j_2-1 \leq i \leq a-1} |L_1^i| < \sum_{j_1 \leq i \leq a-1} |L_1^i| = |L_1|.$$

Thus we have either $j_1 = j_2$ or $j_1 + 1 = j_2$. We handle these cases separately.

We first assume that $j_1 = j_2$. Set $j = j_1$. If $j = a-1$, then by Lemma 2.2.4 we have $|R_1 \cdot L_1| = |L_1^{a-1}|^{<n-2>} = |L_1|^{<n-2>}$ and similarly we get $|R_1 \cdot L_2| = |L_1|^{<n-2>}$ giving the desired inequality. So assume that $j < a-1$. Since $|L_1| = |L_2|$ and $|L_1^i| = |L_2^{i+1}|$ for $j+1 < i < a-1$, we have $|L_1^{a-1}| + |L_1^j| = |L_2^j| + |L_2^{j+1}|$. Moreover, by Lemma 2.2.4, we have

$$|R_1 \cdot L_1| - |R_1 \cdot L_2| = |L_1^{a-1}|^{<n-2>} + |L_1^j|^{<n-2>} - |L_2^j|^{<n-2>} - |L_2^{j+1}|^{<n-2>}.$$

If $|L_1^j| = |L_2^{j+1}|$, then $|L_1^{a-1}| = |L_2^j|$ as well and the right hand side of the equation above is zero giving $|R_1 \cdot L_1| = |R_1 \cdot L_2|$. If $|L_1^j| \neq |L_2^{j+1}|$, then we necessarily have $|L_1^j| < |L_2^{j+1}|$ because L_1^j is a subset of R_t^j whose size is equal to the size of $R_{t+1}^{j+1} = L_2^{j+1}$. Moreover $|L_1^{a-1}| = |R_t^{a-1}| < |R_{t+1}^{a-1}| \leq |R_{t+1}^{j+1}|$ because $j+1 \leq a-1$ and so $|L_1^{a-1}| < |L_2^{j+1}|$ as well. Now we get that $|L_2^j| < |L_2^{j+1}|$ because both $|L_1^j|$ and $|L_1^{a-1}|$ is strictly smaller than $|L_2^{j+1}|$ and $|L_1^{a-1}|+|L_1^j| = |L_2^j|+|L_2^{j+1}|$. But $|L_2^{j+1}|$ is the number of all monomials of degree $t-j$ in $n-1$ variables. Hence [13, 1.6] applies and we get

$$|L_1^{a-1}|^{<n-2>} + |L_1^j|^{<n-2>} - |L_2^j|^{<n-2>} - |L_2^{j+1}|^{<n-2>} \geq 0.$$

Equivalently, $|R_1 \cdot L_1| \geq |R_1 \cdot L_2|$ as desired.

If $j_1 + 1 = j_2$, then from $|L_1^i| = |L_2^{i+1}|$ for $j_1 < i < a-1$ and Lemma 2.2.4 we have $|L_1^{a-1}| + |L_1^{j_1}| = |L_2^{j_1+1}|$ and

$$|R_1 \cdot L_1| - |R_1 \cdot L_2| = |L_1^{a-1}|^{<n-2>} + |L_1^{j_1}|^{<n-2>} - |L_2^{j_1+1}|^{<n-2>}.$$

So we get that $|R_1 \cdot L_1| - |R_1 \cdot L_2| > 0$ from [13, 1.5].

Finally, fix an integer d. Then for a sufficiently large integer t', the number of all monomials of degree $t' - a - 1$ in S' is bigger than d ($n > 2$ is essential here). Now let L be the lexsegment set of size d in $R_{t'}$. Then we have $L^{a-1} = L$ and therefore Lemma 2.2.4 gives $d_{n,t'} = d^{<n-2>}$ as desired. □

Let L_1 and L_2 be two lexsegment sets in R_t with sizes b and c respectively with $b \geq c$. Let j_1 and j_2 denote $I(L_1)$ and $I(L_2)$. Note that we have $j_1 \leq j_2$. Define

$$t' = \begin{cases} t - j_2 \text{ if } j_1 = j_2 \text{ and } j_1 \neq a-1 \text{ and } x_1^{j_1} x_n^{t-j_1} \notin (L_1 \cup L_2); \\ t + 1 - j_2 \text{ otherwise.} \end{cases}$$

We finish by noting down an adoption of [13, 1.5] for the ring R.

Proposition 2.4.2. *Assume the notation of the previous paragraph. For $n \geq 3$ we have*

$$b_{n,t} + c_{n,t} > (b+c)_{n,t+t'}.$$

Proof. Note that for $t < a-1$, the statement of the lemma follows from [13, 1.5] because then $b_{n,t} = b^{<n-1>}$, $c_{n,t} = c^{<n-1>}$ and $(b+c)_{n,t+t'} \leq (b+c)^{<n-1>}$. So we assume $t \geq a-1$. To prove the lemma it is enough to show that there exist monomials m_1 and m_2 of degree t' in S' such that $m_1 \cdot L_1$ and $m_2 \cdot L_2$ are disjoint and that $R_1 \cdot (m_1 \cdot L_1)$ and $R_1 \cdot (m_2 \cdot L_2)$ have non-empty intersection. Because then

$$(b+c)_{n,t+t'} \leq |R_1 \cdot \big(m_1 \cdot L_1 \sqcup m_2 \cdot L_2\big)| < |R_1 \cdot (m_1 \cdot L_1)| + |R_1 \cdot (m_2 \cdot L_2)| = b_{n,t} + c_{n,t}.$$

In the following we use the fact that if a minimal element in a set is of higher rank than the maximal element in another set then these two sets do not intersect.

We handle the case $j_1 = j_2 = a-1$ separately and the proof for this case essentially carries over from [13, 1.5]. Let $x_1^{a-1} w$ be the minimal element in L_1, where w is a monomial in S'. Consider $x_2^{t'} \cdot L_1$ and $w x_n \cdot L_2$. The minimal element of $x_2^{t'} \cdot L_1$ is $x_1^{a-1} x_2^{t'} w$ and the maximal element of $w x_n \cdot L_2$ is $x_1^{a-1} x_2^{t-a+1} w x_n$. Then $x_1^{a-1} x_2^{t'} w > x_1^{a-1} x_2^{t-a+1} w x_n$ since $t' = t - a + 2$. On the other hand $x_1^{a-1} x_2^{t'} w x_n$ lie in both $R_1 \cdot (x_2^{t'} \cdot L_1)$ and $R_1 \cdot (w x_n \cdot L_2)$.

It turns out that for all the remaining cases one can choose $m_1 = x_2^{t'}$ and $m_2 = x_n^{t'}$. We first show $x_2^{t'} \cdot L_1 \cap x_n^{t'} \cdot L_2 = \emptyset$ and to this end it suffices to show that $x_2^{t'} \cdot L_1^i \cap x_n^{t'} \cdot L_2^i = \emptyset$ for $i \leq a-1$.

Since $L_1^i = R_t^i$ for $i > j_1$, the minimal element in $x_2^{t'} \cdot L_1^i$ is $x_1^i x_2^{t'} x_n^{t-i}$ for $i > j_1$. Similarly, $L_2^i = R_t^i$ for $i > j_2$, so the maximal element in $x_n^{t'} \cdot L_2^i$ is $x_1^i x_2^{t-i} x_n^{t'}$. But $t' \geq t - i$ for $i > j_2$ and so we have $x_1^i x_2^{t'} x_n^{t-i} > x_1^i x_2^{t-i} x_n^{t'}$ for $j_2 < i$. Also $L_2^i = \emptyset$ for $i < j_2$ and therefore it only remains to show $x_2^{t'} \cdot L_1^{j_2} \cap x_n^{t'} \cdot L_2^{j_2} = \emptyset$. Let u and v denote the maximal monomial in $x_n^{t'} \cdot L_2^{j_2}$ and minimal monomial in $x_2^{t'} \cdot L_1^{j_2}$, respectively. Note that $u = x_1^{j_2} x_2^{t-j_2} x_n^{t'}$. We consider different cases and show that $v > u$ in each case. First assume that $j_2 > j_1$. Then we have $L_1^{j_2} = R_t^{j_2}$ and $t' = t + 1 - j_2$. It follows that $v = x_1^{j_2} x_2^{t'} x_n^{t-j_2} = x_1^{j_2} x_2^{t+1-j_2} x_n^{t-j_2} > u$. Now assume that $j_1 = j_2$ and $x_1^{j_2} x_n^{t-j_2} \in (L_1 \cup L_2)$. Then $t' = t + 1 - j_2$. Also, since $|L_1| \geq |L_2|$ and $x_1^{j_2} x_n^{t-j_2}$ is the minimal element in $R_t^{j_2}$, we have $x_1^{j_2} x_n^{t-j_2} \in L_1^{j_2}$ and so $v = x_1^{j_2} x_2^{t'} x_n^{t-j_2} = x_1^{j_2} x_2^{t+1-j_2} x_n^{t-j_2} > u$. Finally, if $j_1 = j_2$ and $x_1^{j_2} x_n^{t-j_2} \notin (L_1 \cup L_2)$, then the minimal monomial in $L_1^{j_2}$ is bigger than $x_1^{j_2} x_n^{t-j_2}$ and so $v > x_1^{j_2} x_2^{t'} x_n^{t-j_2} = x_1^{j_2} x_2^{t-j_2} x_n^{t-j_2} = u$.

We now show that $R_1 \cdot (x_2^{t'} \cdot L_1)$ and $R_1 \cdot (x_n^{t'} \cdot L_2)$ have non-empty intersection. If $j_1 < j_2$, then as we saw above $L_1^{j_2} = R_t^{j_2}$ and so $x_1^{j_2} x_n^{t-j_2}$ is the minimal element in $L_1^{j_2}$ and hence $x_1^{j_2} x_2^{t+1-j_2} x_n^{t-j_2+1} \in R_1 \cdot (x_2^{t'} \cdot L_1)$. But since $x_1^{j_2} x_2^{t-j_2}$ is the maximal element in $L_2^{j_2}$, $x_1^{j_2} x_2^{t-j_2+1} x_n^{t+1-j_2} \in R_1 \cdot (x_n^{t'} \cdot L_2)$ as well. For the case $j_1 = j_2$ first assume that $x_1^{j_2} x_n^{t-j_2} \in (L_1 \cup L_2)$. We have computed the maximal monomial u in $x_n^{t'} \cdot L_2^{j_2}$ and the minimal monomial v in $x_2^{t'} \cdot L_1^{j_2}$ for this case in the previous paragraph. Notice that we have $vx_n = ux_2$ giving $R_1 \cdot (x_2^{t'} \cdot L_1) \cap R_1 \cdot (x_n^{t'} \cdot L_2) \neq \emptyset$. Finally, assume $j_1 = j_2 < a - 1$ and $x_1^{j_1} x_n^{t-j_1} \notin (L_1 \cup L_2)$. Then $L_1^{j_2+1} = L_2^{j_2+1} = R_t^{j_2+1}$ and so the minimal element in $x_2^{t'} \cdot L_1^{j_2+1}$ is $x_1^{j_2+1} x_2^{t-j_2} x_n^{t-1-j_2}$ giving $x_1^{j_2+1} x_2^{t-j_2} x_n^{t-j_2} \in R_1 \cdot (x_2^{t'} \cdot L_1)$. But this monomial is in $R_1 \cdot (x_n^{t'} \cdot L_2)$ as well because the maximal element in $x_n^{t'} \cdot L_2^{j_2+1}$ is $x_1^{j_2+1} x_2^{t-1-j_2} x_n^{t-j_2}$. $\square$

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