PATTERN-AVOIDING PERMUTATIONS: THE CASE OF LENGTH THREE, FOUR, AND FIVE

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

PATTERN-AVOIDING PERMUTATIONS: THE CASE OF LENGTH THREE, FOUR, AND FIVE

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A shorter permutation of length k is said to appear as a pattern in a longer permutation of length n if the longer permutation has a subsequence of length k that is order isomorphic to the shorter one. Otherwise, the longer permutation avoids the shorter one as a pattern. We use $S_n(\tau)$ to denote the set of permutations of length n that avoid pattern τ. Pattern avoidance induces an equivalence relation on the pattern set S_k. For $\rho, \tau \in S_k$, we define the equivalence relation as follows: $\rho \sim_W \tau$ if and only if $|S_n(\rho)| = |S_n(\tau)|$ for all $n \geq 1$. The equivalence classes of this relation are called Wilf classes. The main questions are determining the Wilf classes of S_k and enumerating each class. We first study the Wilf classification and enumeration of each class for S_3 and S_4. We then present some new numerical results regarding the Wilf classification of pairs of patterns of length five. We define a Wilf class as small if it contains only one pair and big if it contains more than one pair. We show that there are at least 968 small Wilf classes and at most 13 big Wilf classes.

Keywords: pattern avoiding permutations, generating functions, Catalan numbers, Fibonacci numbers, Dyck paths, trees.

ÖZET

MOTİF İÇERMEYEN PERMÜTASYONLAR: UZUNLUĞU ÜÇ, DÖRT VE BEŞ OLAN DURUMLAR

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Uzunluğu n olan bir permütasyon, kendisinden daha kısa k uzunluğundaki bir permütasyonla aynı göreceli sıralamaya sahip bir altdizi içeriyorsa, bu permütasyonu motif olarak içerir. Aksi takdirde, bu permütasyon bu motifi içermez. Uzunluğu n olan permütasyonlar kümesi S_n ile, uzunluğu n olan ve τ motifini içermeyen permütasyonlar kümesi $S_n(\tau)$ ile gösterilir. Motif içermeme, motif kümesi S_k üzerinde bir denklik bağıntısı tanımlar. $\rho, \tau \in S_k$ iken her $n \geq 1$ için $|S_n(\rho)| = |S_n(\tau)|$ doğruysa ρ ve τ aynı Wilf sınıfındadır denir ve $\rho \sim_W \tau$ ile gösterilir. Bu denklik sınıflarına Wilf sınıfları denir. Başlıca sorular S_k'nın Wilf sınıflarını belirlemek ve her sınıftaki eleman sayısını bulmaktır. Öncelikle S_3 ve S_4'ün Wilf sınıflarının belirlenmesine ve Wilf sınıflarının sayımına yönelik çalışmaları inceliyoruz. Daha sonra, uzunluğu beş olan ikili motiflerin Wilf sınıflandırmasına ilişkin bazı yeni sayısal sonuçlar sunuyoruz. Eğer bir Wilf sınıfında birden fazla motif ikilisi varsa buna büyük Wilf sınıfı, sadece bir tane motif ikilisi varsa küçük Wilf sınıfı denir. Uzunluğu beş olan ikili motifler için en az 968 küçük Wilf sınıfı ve en fazla 13 büyük Wilf sınıfı olduğunu gösteriyoruz.

Anahtar sözcükler: motif içermeyen permütasyonlar, üretici fonksiyonlar, Katalan sayıları, Fibonacci sayıları, Dyck yolları, ağaçlar.

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Chapter 1

Introduction

1.1 Permutations and patterns

A permutation of length n is a one-to-one and onto function from the set $[n] = \{1, 2, 3, \ldots, n\}$ to itself.

Let $\sigma : [n] \to [n]$ be a permutation. We can represent permutations in two-line notation:

$$\begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_i & \cdots & \sigma_n \end{pmatrix}$$

We usually omit the first row and denote the permutation as $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$. We can see that σ_i is the image of i under σ where $1 \leq i \leq n$.

We denote the set of all length n permutations with S_n. For example, $S_1 = \{1\}$, $S_2 = \{12, 21\}$, $S_3 = \{123, 132, 213, 231, 312, 321\}$.

A pattern is simply a permutation in S_k. Let $\tau \in S_k$ and $\sigma \in S_n$. We say that σ contains τ as a pattern if a subsequence $\sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_k}$ of length k in σ has the same relative order as the pattern τ. That is,

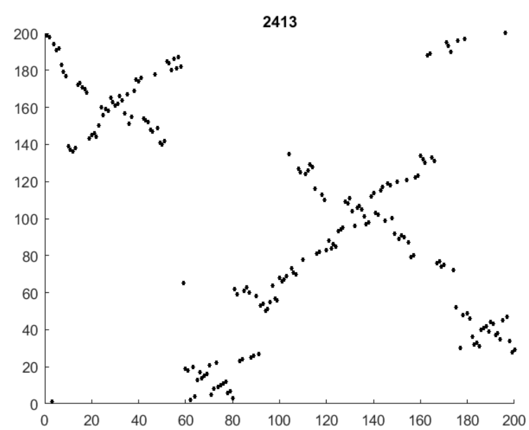
$$\sigma_{i_a} < \sigma_{i_b} \text{ if and only if } \tau_a < \tau_b$$

for all $1 \leq a, b \leq k$.

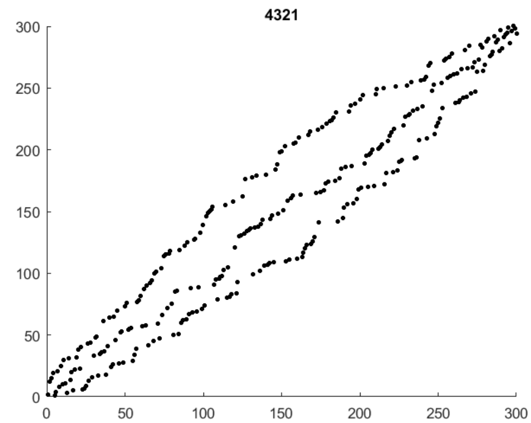
For example, let's look at $\sigma = 1763452 \in S_7$. We see that it contains the pattern 123 by the subsequence 134: **1**76**34**52 or the subsequence 345: 176**345**2.

If σ does not contain τ, we say that σ avoids the pattern τ. In the previous example, σ avoids the pattern 213. We use $S_n(\tau)$ to denote the set of all length n permutations that avoid τ. For example $S_4(132) = \{1234, 2134, 2314, 2341, 3124, 3241, 3214, 3412, 3421, 4123, 4231, 4213, 4312, 4321\}$.

We can use the plot of the permutations to visualize them, which helps us to understand the structures of pattern-restricted permutations. We put a node at every (i, σ_i) on the $xy-$plane to graphically represent a permutation σ. We have generated the following plots using the Markov Chain Monte Carlo method developed by Liu and Madras [1].



Plot of a permutation $\sigma \in S_{200}(2413)$



Plot of a permutation $\sigma \in S_{300}(4321)$

One of the fundamental research questions is to enumerate $S_n(\tau)$ for any given pattern $\tau \in S_k$. In the following chapters, we will see that it is crucial to understand the structures of permutations in $S_n(\tau)$ to enumerate them.

1.2 Wilf equivalence on S_k

Pattern avoidance induces an equivalence relation on S_k. Let $\rho, \tau \in S_k$,

$$\rho \sim_W \tau \text{ if and only if } |S_n(\rho)| = |S_n(\tau)| \text{ for all } n \geq 1.$$

We say that ρ and τ are in the same Wilf class if $\rho \sim_W \tau$. There are two main questions:

- How many Wilf classes does $(S_k, \sim_W)$ have?
- For a pattern τ, how can we enumerate $S_n(\tau)$ for all $n \geq 1$?

For $(S_2, \sim_W)$, the answer is trivial because there is only one permutation of length n that avoids 12, which is the permutation $n(n-1)\cdots 21$. Therefore $|S_n(12)| = 1$ for all $n \geq 1$. Similarly, we have $|S_n(21)| = 1$ for all $n \geq 1$. Therefore there is only one Wilf class for this case.

For the case $(S_k, \sim_W)$ with $k > 2$, the answer is not as straightforward as in the case of $k = 2$. We utilize trivial bijections as a helpful tool to determine the number of Wilf classes in $(S_k, \sim_W)$.

Trivial bijections can determine if certain patterns are in the same Wilf class, and hence reduce the potential Wilf classes. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$.

- The complement of σ is $\sigma^c = \sigma_1^c\sigma_2^c \ldots \sigma_n^c$ where $\sigma_i^c = n + 1 - \sigma_i$ for $i \in [n]$.
- The reverse of σ is $\sigma^r = \sigma_n\sigma_{n-1} \ldots \sigma_1$.
- The inverse of σ is $\sigma^i = \sigma_1^i\sigma_2^i \ldots \sigma_n^i$ where $\sigma_k^i = l$ if and only if $\sigma_l = k$.

Let $\tau \in S_k$ and $\sigma \in S_n$. It follows from the bijections that

- σ avoids τ if and only if σ^c avoids τ^c.
- σ avoids τ if and only if σ^r avoids τ^r.
- σ avoids τ if and only if σ^i avoids τ^i.

These claims will be proven in the second chapter. Therefore, for all $n \geq 1$, we have

$$|S_n(\tau)| = |S_n(\tau^c)| = |S_n(\tau^r)| = |S_n(\tau^i)|$$

and

$$|S_n(\tau^{rc})| = |S_n(\tau^{ri})| = |S_n(\tau^{ci})| = |S_n(\tau^{rci})|.$$

We can use these results to reduce the potential Wilf classes in $(S_k, \sim_W)$. However, the trivial bijections alone are insufficient to determine the exact number of Wilf classes. Non-trivial bijections are necessary to show that two patterns are in the same Wilf class. Nonetheless, establishing non-trivial bijections can be highly challenging.

In the case of $(S_3, \sim_W)$, after applying trivial bijections, we observe that there are potentially two Wilf classes because $|S_n(123)| = |S_n(321)|$ and $|S_n(132)| =$

$|S_n(213)| = |S_n(312)| = |S_n(231)|$ for all $n \geq 1$. In 1969, Knuth [2] demonstrated that $S_n(132)$ is enumerated by Catalan numbers. Rotem [3] showed that Catalan numbers also enumerate $S_n(321)$. Hence there is only one Wilf class in $(S_3, \sim_W)$. Simion and Schmidt [4] established a direct bijection between $S_n(123)$ and $S_n(132)$. Further details and proof of these facts are presented in the second chapter.

The case of $(S_4, \sim_W)$ is more complex. When we apply trivial bijections, we are left with seven potential Wilf classes: 1243, 1432, 1324, 2143, 2413, 1234, and 1342. Stankova [5] demonstrated that $|S_n(1342)| = |S_n(2413)|$ for all $n \geq 1$. Backelin, West, and Xin [6] presented non-trivial bijections implying that $|S_n(1234)| = |S_n(1243)| = |S_n(2143)| = |S_n(1432)|$ for all $n \geq 1$. Hence, the number of potential Wilf classes reduces to three, represented by 1234, 1324, and 1342. West [7] provided numerical evidence for the following counts:

- $|S_n(1342)|$ is counted as 1, 2, 6, 23, 103, 512, 2740, 15485, ...
- $|S_n(1234)|$ is counted as 1, 2, 6, 23, 103, 513, 2761, 15767, ...
- $|S_n(1324)|$ is counted as 1, 2, 6, 23, 103, 513, 2762, 15793, ...

Therefore there are exactly three Wilf classes in $(S_4, \sim_W)$. In 1990, Gessel [8] proved that for all $n \geq 1$,

$$|S_n(1234)| = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

In 1997, Bona [9] proved that for all $n \geq 1$,

$$|S_n(1342)| = (-1)^{n-1} \frac{7n^2 - 3n - 2}{2} + 3 \sum_{i=2}^{n} (-1)^{i-1} 2^{i+1} \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2}.$$

The enumeration of $S_n(1324)$ is still an important open problem in the field. In 2018, Conway et al. [10] found a recursive algorithm and calculated $|S_n(1324)|$ up to $n = 50$. The first twelve terms are as follows:

$$1, 2, 6, 23, 103, 513, 2.762, 15.793, 94.776, 591.950, 3.824.112, 25.431.452, \ldots$$

For $(S_5, \sim_W)$, there are 16 Wilf classes [11]. We also know the exact number of elements in $S_n(\tau)$ up to $n = 25$ calculated by Clisby et al. [12]. For any $\tau \in S_5$, $|S_n(\tau)|$ from $n = 1$ to $n = 6$ is $1, 2, 6, 24, 119, 694$. The rest of the sequences are as follows:

Representative pattern τ	$\lvert S_n(\tau)\rvert$ from $n = 7$ to $n = 13$
25314	4578, 33184, 258757, 2136978, 18478134, 165857600, 1535336290
31524	4579, 33216, 259401, 2147525, 18632512, 167969934, 1563027614
35214	4579, 33218, 259483, 2149558, 18672277, 168648090, 1573625606
43251	4581, 33283, 260805, 2171393, 18994464, 173094540, 1632480259
34215	4581, 33285, 260886, 2173374, 19032746, 173741467, 1642533692
53124	4580, 33252, 260202, 2161837, 18858720, 171285237, 1609282391
32541	4581, 33284, 260847, 2172454, 19015582, 173461305, 1638327423
35124	4580, 33249, 260092, 2159381, 18815124, 170605392, 1599499163
31245	4581, 33286, 260927, 2174398, 19053058, 174094868, 1648198050
42351	4580, 33252, 260204, 2161930, 18861307, 171341565, 1610345257
42315	4581, 33287, 260967, 2175379, 19072271, 174426353, 1653484169
12345	4582, 33324, 261808, 2190688, 19318688, 178108704, 1705985883
35241	4580, 33254, 260285, 2163930, 18900534, 172016256, 1621031261
53241	4580, 33256, 260370, 2166120, 18945144, 172810050, 1633997788
53421	4582, 33325, 261853, 2191902, 19344408, 178582940, 1713999264
52341	4582, 33325, 261863, 2192390, 19358590, 178904675, 1720317763

We know the generating function only for $|S_n(12345)|$ obtained by Bostan et al. [13]. It is given by

$$F(x) = x(1-16x)^{-\frac{1}{3}}(1+2x)^{-\frac{2}{3}} \cdot {}_2F_1\left(\frac{1}{6}, \frac{2}{3}, 1, -\frac{108x^2}{(1-16x)(1+2x)^2}\right)^2$$

where ${}_2F_1(a, b, c, H)$ is the hypergeometric function.

1.3 The growth rate of $|S_n(\tau)|$ for any $\tau \in S_k$

In 1980, Stanley and Wilf conjectured that for any pattern $\tau \in S_k$, there exists a constant c_τ such that

$$|S_n(\tau)| \leq c_\tau^n \text{ for all } n \geq 1.$$

Arratia (1999) showed that this conjecture is equivalent to the existence of the following limit [14],

$$\mathrm{gr}(\tau) = \lim_{n\to\infty} \sqrt[n]{|S_n(\tau)|}$$

This limit is called the growth rate or the Wilf constant of the pattern τ. Marcus and Tardos [15] proved this conjecture in 2003. We will present their proof in the fourth chapter.

We use the notation

$$a_n \sim b_n \text{ if and only if } \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

It follows from the singularity analysis of the generating functions that [16]

$$|S_n(132)| \sim \frac{4^n}{\sqrt{n^3\pi}},$$

$$|S_n(1234)| \sim \frac{81\sqrt{3}\cdot 9^n}{16\pi n^4},$$

and

$$|S_n(1342)| \sim \frac{64\cdot 8^n}{243\sqrt{\pi} n^{\frac{5}{2}}}.$$

Therefore, $\mathrm{gr}(132) = 4, \mathrm{gr}(1234) = 9$, and $\mathrm{gr}(1342) = 8$.

Even though an explicit formula for $|S_n(1324)|$ is unknown, there are some results on the growth rate of 1324. The best-known bounds are recently given by Bevan et al. [17] as $10.271 \leq \mathrm{gr}(1324) \leq 13.5$.

1.4 Wilf equivalence on the subsets of S_k

For a set of patterns T, we set

$$S_n(T) = \bigcap_{\tau \in T} S_n(\tau).$$

Let $\mathcal{P}_m^k = \{T \subseteq S_k \mid |T| = m\}$. For any $T_1, T_2 \in \mathcal{P}_m^k$, we define the equivalence relation

$$T_1 \sim_W T_2 \text{ if and only if } |S_n(T_1)| = |S_n(T_2)| \text{ for all } n \geq 1.$$

We know the number of Wilf classes of subsets of S_3 and their exact enumeration due to Simion and Schmidt [4]. Let $T \in \mathcal{P}_m^3$.

m	Number of subsets	Number of Wilf classes	Representative	$\lvert S_n(T)\rvert$
1	6	1	132	c_n
2	15	3	123,132	2^{n-1}
			123,321	0 if $n \geq 5$
			132,321	$\binom{n}{2} + 1$
3	20	3	123,132,213	f_{n+1}
			123,132,231	n
			123,132,321	0 if $n \geq 5$
4	15	2	$\{123, 321\} \subset T$	0 if $n \geq 5$
			$\{123, 321\} \not\subset T$	2 if $n \geq 2$
5	6	2	$S_3 - \{123\}$	1 if $n \geq 3$
			$S_3 - \{132\}$	0 if $n \geq 4$
6	1	1	S_n	0 if $n \geq 3$

Note that c_n is the n^{th} Catalan number and f_n is the n^{th} Fibonacci number. All of the work above will be discussed in the second chapter. There are extensive results for the subsets of S_4. We look at $(\mathcal{P}_m^4, \sim_W)$.

For $m = 2$, we know the number of Wilf classes due to Kremer and Shiu (2003) [18] and Le (2005) [19]. There are 38 Wilf classes, where every generating function is known except one. See [20] for the complete list of generating functions.

For $m = 3$, there are 242 Wilf classes. A series of papers by Mansour, Callan, and Shattuck (2017-2019) completed the enumeration of those Wilf classes [21, 22, 23]. Generating functions of 241 of them are known.

Mansour (2020) proved there are 1100 Wilf classes for $m = 4$ and 3441 Wilf classes for $m = 5$. He found the generating function of all of them [24, 25].

For $5 \leq m \leq 24$, we also know the exact number of Wilf classes in $(\mathcal{P}^4_m, \sim_W)$ thanks to Mansour and Schork (2016-2017) [26, 27, 28]. We list the number of Wilf classes for any subset of S_4.

m	$\binom{24}{m}$	Number of potential Wilf classes after trivial bijections	Number of Wilf classes
1	24	7	3
2	276	56	38
3	2,024	317	242
4	10,626	1,524	1,100
5	42,504	5,733	3,441
6	134,596	17,728	8,438
7	346,104	44,767	15,392
8	735,471	94,427	19,002
9	1,307,504	166,766	16,293
10	1,961,256	249,624	10,624
11	2,496,144	316,950	5,857
12	2,704,156	343,424	3,044
13	2,496,144	316,950	1,546
14	1,961,256	249,624	786
15	1,307,504	166,766	393
16	735,471	94,427	198

17	346,104	44,767	105
18	134,596	17,728	55
19	42,504	5,733	28
20	10,626	1,524	14
21	2,024	317	8
22	276	56	4
23	24	7	2
24	1	1	1

It is natural to ask whether the same counting sequence enumerates two pattern classes with different lengths. Two patterns with distinct lengths are called unbalanced Wilf equivalent if the same counting sequence enumerates them. Burstein and Pantone (2015) showed that [29]

$$|S_n(1324, 3416725)| = |S_n(1234)| \text{ for all } n \geq 1$$

and

$$|S_n(2143, 3142, 246135)| = |S_n(2413, 3142)| \text{ for all } n \geq 1.$$

In 2022, Albert et al. [30] introduced a general approach to compute the enumerations of Wilf classes in $(S_k, \sim_W)$ or $(\mathcal{P}^k_m, \sim_W)$. The website PermPAL [31] provides the proof trees and generating function systems to enumerate Wilf classes of chosen S_k and $\mathcal{P}^k_m$.

1.4.1 Some results on $(\mathcal{P}^5_2, \sim_W)$

In this section, we will summarize our new numerical results. Let $T \in \mathcal{P}^5_2$. Since $|S_5| = 120$, there are $\binom{120}{2} = 7140$ subsets with two elements. Applying trivial bijections to these 7140 pattern subsets resulted in 1012 potential Wilf classes. For each 1012 pair of patterns of length five, we enumerate $S_n(\tau, \rho)$ where $\tau, \rho \in S_5$ for $n = 1, 2, \ldots, 9$. We observe the following:

- There are 968 counting sequences corresponding to one pair.
- There are 13 counting sequences that correspond to more than one pair.

If a counting sequence corresponds to more than one pair, we classify it as a big Wilf class. Otherwise, it is classified as a small Wilf class. Based on our numerical works, there are at least 968 small Wilf classes. Additionally, there are at most 13 big Wilf classes. Let's proceed by listing the potential big Wilf classes.

	Representative sets	$\lvert S_n(T)\rvert$ up to $n = 9$ or more
1	25143, 24153 15342, 15324 35241, 34251 23514, 23415	1, 2, 6, 24, 118, 672, 4.254, 29.116, 211.464, ...
2	31524, 51324 25134, 23154	1, 2, 6, 24, 118, 668, 4.156, 27.576, 191.510, ...
3	43512, 34512 12345, 12354 45321, 35421 53412, 45312 21354, 12354 35412, 54312 45312, 45213 32145, 32154 12534, 12543 13245, 31245 12354, 12435 21453, 21534 23145, 31245 24531, 15432	This sequence was also calculated by Kitaev in 2020, see A224295 on OEIS [32]. Kitaev and Gao conjectured [33] a formula for partially ordered patterns that have length 4 or 5. 1, 2, 6, 24, 118, 672, 4.256, 29.176, 212.586, ...
4	21453, 21345 12453, 12345	1, 2, 6, 24, 118, 671, 4.231, 28.771, 207.133, ...
5	24531, 14532 42135, 42153	

	35241, 35214 53214, 53241	1, 2, 6, 24, 118, 672, 4.254, 29.116, 211.466, ...
6	24135, 31524 54213, 35214	1, 2, 6, 24, 118, 670, 4.201, 28.226, 199.260, ...
7	32145, 21345 21354, 32154	1, 2, 6, 24, 118, 671, 4.233, 28.838, 208.484, ...
8	21543, 12345 43215, 23451 54321, 51234 45123, 15432	1, 2, 6, 24, 118, 668, 4.124, 26.728, 177.736, ...
9	41352, 41325 35412, 35121	1, 2, 6, 24, 118, 672, 4.252, 29.056, 210.342, 1.591.808, 12.475.444, ...
10	54231, 54123 45231, 45123	1, 2, 6, 24, 118, 671, 4.231, 28.770, 207.094, 1.558.207, 12.145.725, ...
11	54231, 54321 12354, 13254	1, 2, 6, 24, 118, 672, 4.258, 29.241, 213.865, ...
12	12534, 23154 21354, 32145	1, 2, 6, 24, 118, 669, 4.175, 27.821, 194.301,...
13	54123, 54321 45123, 45321	1, 2, 6, 24, 118, 668, 4160, 27.680, 193.158, ...

Enumerating $S_n(\tau, \rho)$ for all the pairs listed above with $n \geq 10$ is computationally demanding. Note that two distinct pair in the same potential big Wilf class may have different counting sequences for larger n values. These numerical results provide valuable insights into our future research on the Wilf classification of pattern pairs of length five.

We also have the following observation.

Lemma 1.1. Let $T = \{54321, 51234\}$ and $I_n(T) = \{\sigma \in S_n(T) \mid \sigma_1 = n\}$, then we have $|I_n(T)| = 0$ for $n \geq 11$.

Proof. Let $\sigma \in I_n(T)$. We see that $\sigma = n\sigma_2\sigma_3 \ldots \sigma_{n-1}\sigma_n$. Let $\sigma_R =$

$\sigma_2\sigma_3 \ldots \sigma_{n-1}\sigma_n$. It implies that $\sigma_R \in S_{n-1}(1234, 4321)$. We know that an integer sequence with length $k^2 + 1$ or larger has an increasing or a decreasing subsequence with length $k+1$, by the Proposition 2.18. Therefore we see that for $k = 3$, any σ_R that has length ten or more will contain 1234 or 4321. Therefore $|I_n(T)| = 0$ for $n \geq 11$. $\square$

For the rest of the thesis, we mainly use the following books of M. Bona: Combinatorics of Permutations [34] and Handbook of Enumerative Combinatorics [35].

Chapter 2

Patterns of Length Three

2.1 Preliminaries

The previous chapter delved into permutation, pattern avoidance, and Wilf class definitions. In this chapter, we will explore examples of pattern avoidance. We study the proofs that only one Wilf class is present in $(S_3, \sim_W)$. To prepare for this, we will first recall the definition of pattern avoidance for a more thorough understanding.

Definition 2.1. Let $\tau \in S_k$ and $\sigma \in S_n$. σ contains τ as a pattern if a subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ of length k in σ has the same relative order as the pattern τ. That is,

$$\sigma_{i_a} < \sigma_{i_b} \text{ if and only if } \tau_a < \tau_b$$

for all $1 \leq a, b \leq k$.

Example 2.1. Let $\sigma = 15243 \in S_5$. σ contains the pattern $\tau = 132$ with different subsequences: **152**43, **1**5**243**, 15**243**. It also contains the pattern 123, **1**5**24**3, **1**5**2**4**3**.

Definition 2.2. A permutation σ avoids the pattern τ if σ does not contain τ.

Example 2.2. $\sigma = 3271654 \in S_7$ avoids 123. Therefore $3271654 \in S_7(123)$.

Lemma 2.1. Let $\rho, \tau \in S_k$.

$$\rho \sim_W \tau \text{ if and only if } |S_n(\rho)| = |S_n(\tau)| \text{ for all } n \geq 1.$$

defines an equivalence relation.

This relation is called the Wilf equivalence relation. Two patterns in the same equivalence class are called Wilf equivalent.

Proof. First we see that $|S_n(\rho)| = |S_n(\rho)|$ for all n, therefore $\sim_W$ is reflexive. If $|S_n(\rho)| = |S_n(\tau)|$, then $|S_n(\tau)| = |S_n(\rho)|$ for all n. Thus $\sim_W$ is symmetric. If $|S_n(\rho)| = |S_n(\tau)|$ and $|S_n(\tau)| = |S_n(\alpha)|$, then $|S_n(\rho)| = |S_n(\alpha)|$ for all n. Therefore $\sim_W$ is transitive. □

Next, we will recall the definitions of trivial bijections.

Definition 2.3. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$.

- The complement of σ is $\sigma^c = \sigma_1^c\sigma_2^c \dots \sigma_n^c$ where $\sigma_i^c = n + 1 - \sigma_i$ for $i \in [n]$.
- The reverse of σ is $\sigma^r = \sigma_n\sigma_{n-1} \dots \sigma_1$.
- The inverse of σ is $\sigma^i = \sigma_1^i\sigma_2^i \dots \sigma_n^i$ where $\sigma_k^i = l$ if and only if $\sigma_l = k$.

Example 2.3. Let $\sigma = 4123$. We have $\sigma^c = 1432$, $\sigma^r = 3214$, and $\sigma^i = 2341$.

Proposition 2.1. Let $\sigma \in S_n$ and $\tau \in S_k$.

- σ avoids τ if and only if σ^c avoids τ^c.
- σ avoids τ if and only if σ^r avoids τ^r.
- σ avoids τ if and only if σ^i avoids τ^i.

Proof. Suppose σ avoids τ, but σ^c contains τ^c. This implies that there exists a subsequence in σ^c with the same relative order as τ^c. Taking the complement of σ^c gives us $(\sigma^c)^c = \sigma$, and the subsequence will have the same relative order as $(\tau^c)^c = \tau$. This contradicts the assumption that σ avoids τ. Similarly, if σ^c avoids τ^c but σ contains τ, it means that σ has a subsequence with the same relative order as τ. By taking the complement of σ, the corresponding subsequence of σ will have the same relative order as τ^c. Therefore, σ^c contains τ^c, which contradicts the initial assumption. The same argument can be applied to the remaining statements in the proposition. □

Example 2.4. Let $\sigma = 34125 \in S_5(132)$. We see that $\sigma^c = 32541 \in S_5(312)$, $\sigma^r = 52143 \in S_5(231)$, $\sigma^i = 34125 \in S_5(132)$.

Therefore, we can conclude that

$$|S_n(\tau)| = |S_n(\tau^c)| = |S_n(\tau^r)| = |S_n(\tau^i)|$$

and

$$|S_n(\tau^{rc})| = |S_n(\tau^{ri})| = |S_n(\tau^{ci})| = |S_n(\tau^{rci})|.$$

Next, we will see the application of these results on S_3.

Recall that $S_3 = \{123, 132, 231, 213, 312, 321\}$ and $|S_3| = 6$. We have the following:

$$123 \overset{i,r,c}{\longleftrightarrow} 321 \qquad \begin{tikzcd} 132 \arrow[r, leftrightarrow, "r"] \arrow[d, leftrightarrow, "c"] & 231 \arrow[d, leftrightarrow, "c"] \arrow[dl, leftrightarrow, "i"] \\ 312 \arrow[r, leftrightarrow, "r"] & 213 \end{tikzcd}$$

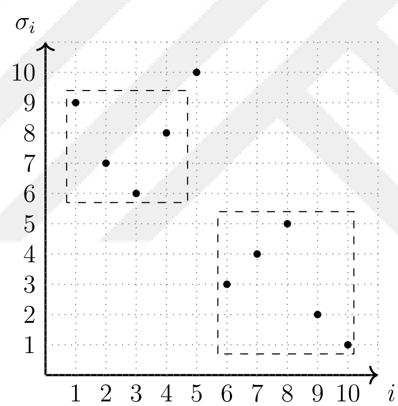
Thus, we have $|S_n(123)| = |S_n(321)|$ and $|S_n(132)| = |S_n(231)| = |S_n(312)| = |S_n(213)|$ for all $n \geq 1$.

Therefore, there are at most two Wilf classes for $(S_3, \sim_W)$. Simion and Schmidt [4] introduced the first nontrivial bijection between $S_n(132)$ and $S_n(123)$, thereby concluding the Wilf classification of $(S_3, \sim_W)$. The next section will discuss the enumeration of $S_n(132)$.

2.2 Enumeration of $S_n(132)$

We want to determine the sequence $|S_n(132)|$ for all $n \geq 1$. We will use the structure of such permutations and generating function method. For a brief introduction to generating functions, see the Appendix. First, we will understand the structure of permutations that avoid 132.

Example 2.5. Let $\sigma = 97681053412 \in S_{10}(132)$. We can see the plot of σ as follows:



We see that $\sigma_5 = 10$ is the largest entry. Every entry on the left of 10 is larger than every entry on the right of 10.

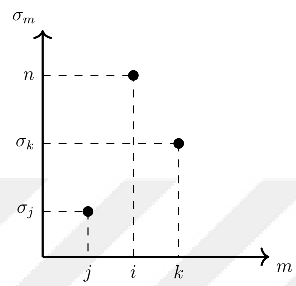
Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ and $\sigma_i = n$. We define σ_L as the permutation of length $i-1$ with the same relative order as the string $\sigma_1\sigma_2\ldots\sigma_{i-1}$. Similarly, we define σ_R as the permutation of length $n-i$ that has the same relative order with the string $\sigma_{i+1}\sigma_{i+2}\ldots\sigma_n$.

Proposition 2.2. Let $\sigma \in S_n$. $\sigma \in S_n(132)$ if and only if it satisfies the following conditions. Suppose that $\sigma_i = n$, then

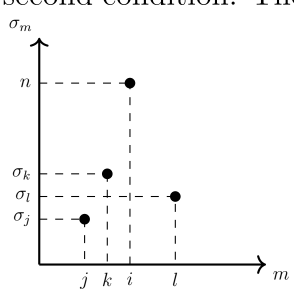
1. $\sigma_k < \sigma_j$ for every $1 \leq j \leq i < k \leq n$.

2. σ_L and σ_R avoid the pattern 132.

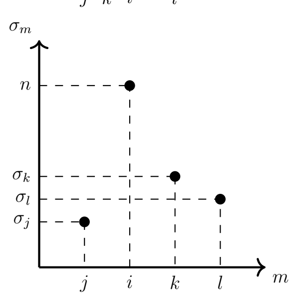
Proof. Assume that $\sigma \in S_n(132)$. We will use contradiction, assuming that there exist two entries $\sigma_j < \sigma_k$ where $1 \leq j < i < k \leq n$. We look at the following picture:



We can see that $\sigma_j\sigma_i\sigma_k$ forms a 132 pattern. Therefore the first condition must be satisfied. Since σ avoids 132, the second condition immediately follows. Conversely, assume that the two conditions are satisfied. We use contradiction again. Suppose that $\sigma \notin S_n(132)$. Let $\sigma_j\sigma_k\sigma_l$ be a 132 pattern in σ. $\sigma_j\sigma_k\sigma_l$ can not be entirely on the left of n. Similarly, it can not be entirely on the right of n by the second condition. There are two possible cases. We look at the following picture:



In this case, only σ_l is on the right of n. It contradicts the first condition since there exists an entry σ_j such that $\sigma_j < \sigma_l$ where $1 \leq j < i < l \leq n$.



In the following case, only σ_j is on the left of n. Similarly, it contradicts the first condition since an entry σ_j exists, such as $\sigma_j < \sigma_l$ where $1 \leq j < i < l \leq n$.

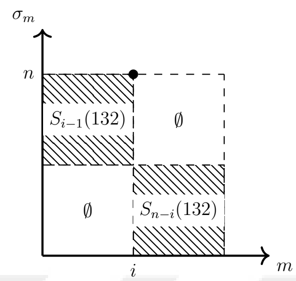
Therefore, if the conditions are satisfied then $\sigma \in S_n(132)$. $\square$

We will use the previous proposition to enumerate $S_n(132)$.

Theorem 2.1. [2] For all $n \geq 1$,

$$|S_n(132)| = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Let $C_n = |S_n(132)|$.



Suppose $\sigma_i = n$ for some $1 \leq i \leq n$. Recall that we must have $\sigma_k < \sigma_j$ for any $j < i < k$ otherwise $\sigma_j n \sigma_k$ would form the pattern 132. By the previous proposition, σ_L must form 132−avoiding permutation. We have C_{i-1} many of them. Similarly, σ_R must form a 132−avoiding permutation. We have C_{n-i} many permutations. We will have the sum for $i = 1$ up to $i = n$, since $\sigma_i = n$ where $i \in [n]$. Therefore, we get $C_0 = 1$ and

$$C_n = \sum_{i=1}^{n} C_{i-1} C_{n-i}.$$

We know that this is the recurrence relation satisfied by the Catalan numbers. Therefore, based on the calculations in Proposition A.2, we have,

$$|S_n(132)| = \frac{1}{n+1}\binom{2n}{n}.$$

□

2.3 Three bijections

First bijection is between $S_n(132)$ and $S_n(123)$. We use the left-to-right minima of the permutations in $S_n(132)$ and $S_n(123)$. The second bijection is between $S_n(132)$ and Dyck paths of length n. Here, we use the property in Proposition 2.2. There is another bijection between Dyck paths and $S_n(321)$. We know that the Catalan numbers enumerate Dyck paths. Therefore these bijections also show that Catalan numbers enumerate $S_n(\tau)$ where $\tau \in S_3$.

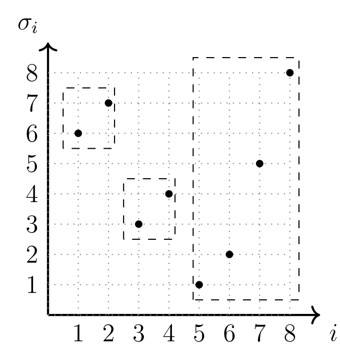
2.3.1 A bijection between $S_n(132)$ and $S_n(123)$

In 1985, Simion and Schmidt [4] introduced a direct bijection between $S_n(132)$ and $S_n(123)$. As previously demonstrated in Section 2.1, it was established that there could be at most two Wilf classes in $(S_3, \sim_W)$. These classes are represented by the patterns 132 and 123. Hence, this bijection confirms the existence of only one Wilf class in $(S_3, \sim_W)$.

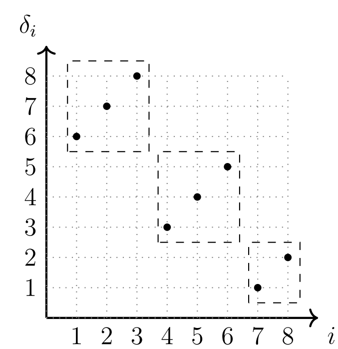
Definition 2.4. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ be a permutation. An entry σ_i is called a left-to-right minimum of σ if $\sigma_i < \sigma_k$ for all $k < i$.

A left-to-right minimum is an entry in the permutation that is smaller than all the entries to their left. It is worth noting that the first entry of a permutation is always a left-to-right minimum. Permutations that avoid pattern 132 exhibits a distinct structure in their arrangement of non-left-to-right minima. To illustrate this, let's consider the following examples.

Let $\sigma = 67341258 \in S_8(132)$. Left-to-right minima are $\sigma_1 = 6$, $\sigma_3 = 3$ and $\sigma_5 = 1$. We look at the plot of σ:



Let $\delta = 67834512 \in S_8(132)$. Left-to-right minima are $\delta_1 = 6$, $\delta_4 = 3$ and $\delta_7 = 1$. We see the plot as follows:



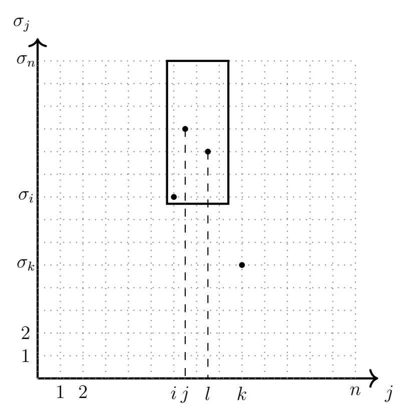
Both permutations have the same set of left-to-right minima, but the position of the left-to-right minimum differs. We see that non-left-to-right entries between two left-to-right minima are always in increasing order. We will now examine the structure of permutations that satisfy the conditions outlined in the following proposition.

Proposition 2.3. Let $\sigma \in S_n$. Assume that $i < k$ and σ_i, σ_k are left-to-right minima such that there is no left-to-right minimum between them.
$\sigma \in S_n(132)$ if and only if the following conditions are satisfied:

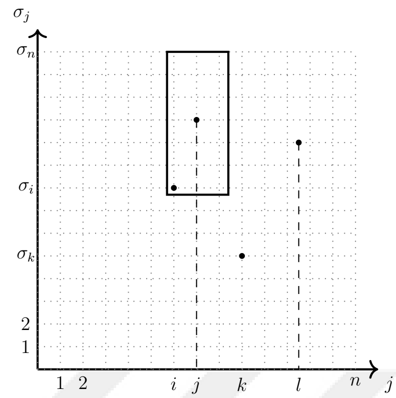
1. If there are more than one entry between σ_i and σ_k, they are in increasing order.

2. Entries between σ_i and σ_k are the smallest entries that are larger than σ_i, which also does not occur before σ_i.

First, we will show that any $\sigma \in S_n(132)$ satisfies the above conditions. We look at the first condition. We will prove it by contradiction. Let σ_i and σ_k be left-to-right minima such that there is no left-to-right minimum between them. Assume that there are two entries σ_j and σ_l such that $i < j < l < k$ and $\sigma_l < \sigma_j$. We see the following picture:

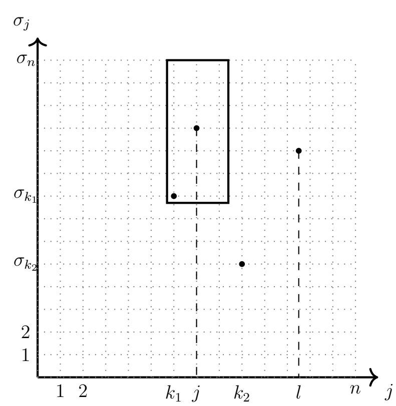


σ_j and σ_l are not left-to-right minima. Therefore both are larger than σ_i. It implies that $\sigma_i\sigma_j\sigma_l$ forms a 132 pattern, which is a contradiction. Thus the first condition must be satisfied. Next, we will discuss the second condition. Again, we will prove it by contradiction. Let σ_i and σ_k be left-to-right minima such that there is no left-to-right minimum between them. Assume that the second condition is not satisfied. It implies that the entries between σ_i and σ_k are not the smallest entries that did not occur before σ_i. We look at the following picture:



It implies that there exist σ_l and σ_j such that $\sigma_i < \sigma_l < \sigma_j$ where $i < j < k < l$. Both are always larger than σ_i. Therefore the pattern $\sigma_i \sigma_j \sigma_l$ is the same as the 132 pattern, which is a contradiction. Therefore the second condition must be satisfied.

Conversely, assume that the conditions are satisfied. We will use contradiction. Assume that $\sigma \notin S_n(132)$. It means that there is a pattern $\sigma_i \sigma_j \sigma_l$ where $i < j < l$ and $\sigma_i < \sigma_l < \sigma_j$. We see that σ_l and σ_j are not left-to-right minima since they are larger than σ_i. They can not be between the same left-to-right minima since it would contradict the first condition.



Let σ_{k_1} and σ_{k_2} be left-to-right minima such that $k_1 < j < k_2 < l$. We see that $\sigma_{k_1} < \sigma_l$ and $\sigma_l < \sigma_j$. It implies that the entries between σ_{k_1} and σ_{k_2} are not the smallest entries that are also larger than σ_{k_1}. It contradicts the second condition. Therefore if the conditions are satisfied, $\sigma \in S_n(132)$.

Next, we will show that if two permutations in $S_n(132)$ have the same entries as their left-to-right minima in the same places, they are the same permutations.

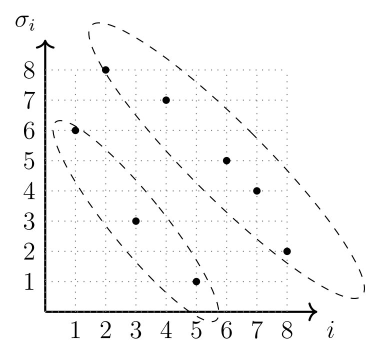
Let $\sigma \in S_n$, we define $E_\sigma = \{\sigma_i \mid \sigma_i \text{ is a left-to-right minimum}\}$ and $I_\sigma = \{i \mid \sigma_i \text{ is a left to right minimum}\}$.

Proposition 2.4. If $\sigma, \delta \in S_n(132)$ where $E_\sigma = E_\delta$ and $I_\sigma = I_\delta$, then $\sigma = \delta$.

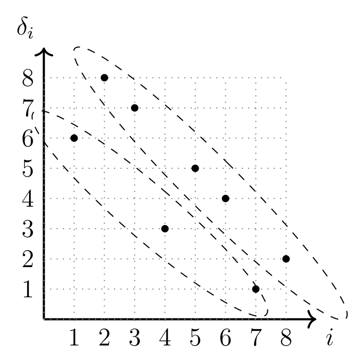
Proof. Since $E_\sigma = E_\delta$ and $I_\sigma = I_\delta$, the left-to-right minima of σ and δ are the same and in the same places. We start from the first left-to-right minimum of σ. Until the next left-to-right minimum, entries also satisfy the conditions of Proposition 2.3. δ also has the same number of elements between the next left-to-right minimum, since $I_\sigma = I_\delta$. Therefore between them, they have the same entries by the Proposition 2.3. We repeat the same process until the last entry. It implies that $\sigma = \delta$. □

There is a similar structure for the permutations that avoid 123. First, we will look at the following examples.

Let $\sigma = 68371542 \in S_8(123)$. Left-to-right minima are $\sigma_1 = 6$, $\sigma_3 = 3$ and $\sigma_5 = 1$. We look at the plot of σ:



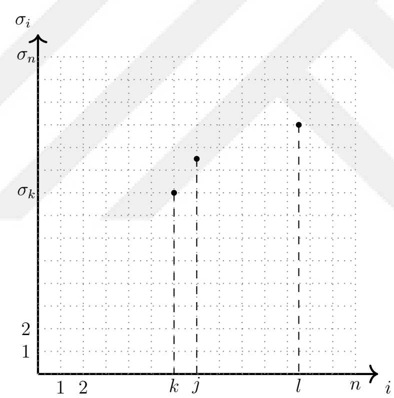
Let $\delta = 68735412 \in S_8(123)$. Left-to-right minima are $\delta_1 = 6$, $\delta_4 = 3$ and $\delta_7 = 1$. We see the plot as follows:



We consider two permutations that avoid 123. We can observe that while the placement of entries may differ, the non-left-to-right minima are arranged in decreasing order.

Proposition 2.5. Let $\sigma \in S_n$. $\sigma \in S_n(123)$ if and only if the non-left-to-right minima entries are in decreasing order.

Proof. Assume that $\sigma \in S_n(123)$. We will use contradiction. Let σ_j, σ_l be two entries of σ that are not left-to-right minima where $1 < j < l \leq n$ and $\sigma_j < \sigma_l$. σ_j is not a left-to-right minimum, therefore there is an entry $\sigma_k < \sigma_j$ such that $k < j$. σ_j and σ_l are larger than σ_k. Therefore there is a subsequence $\sigma_k < \sigma_j < \sigma_l$, which forms a 123 pattern. It is a contradiction.



Conversely, assume that non-left-to-right minima are in decreasing order. We will use contradiction, suppose that $\sigma \notin S_n(123)$. It means that there is a subsequence in σ, $\sigma_k < \sigma_j < \sigma_l$ where $k < j < l$. We see that σ_j and σ_l can not be left-to-right minima since they are larger than σ_k. It implies that two non-left-to-right minima are in increasing order. It is a contradiction. Therefore $\sigma \in S_n(123)$. $\square$

Similar to permutations in $S_n(132)$, if two permutations in $S_n(123)$ have identical entries as their left-to-right minima in the same positions, they are the same permutation.

Proposition 2.6. Let $\sigma, \delta \in S_n(123)$. If $E_\sigma = E_\delta$ and $I_\sigma = I_\delta$, then $\sigma = \delta$.

Proof. If $E_\sigma = E_\delta$, then $[n]/E_\sigma = [n]/E_\delta$. We showed that there is exactly one possibility for the array of them by Proposition 2.5. We also know that $I_\sigma = I_\delta$, therefore $[n]/I_\sigma = [n]/I_\delta$. It implies that the places where the remaining entries will be the same. Therefore $\sigma = \delta$. $\square$

The structure of permutations in $S_n(132)$ and $S_n(123)$ has been explained. We will now employ this structure to establish the bijection between these sets.

Theorem 2.2. There exists a bijection between $S_n(123)$ and $S_n(132)$.

Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n(123)$. We will define

$$\alpha : S_n(123) \longrightarrow S_n(132)$$

as

1 Fix the left-to-right minima of σ in their places.

2 We scan left-to-right minima of σ from left to right. After every left-to-right minimum, put the smallest non-left-to-right minimum larger than the current left-to-right minimum that has not been used.

Let's consider the following example.

Example 2.6. $398276154 \in S_9(123)$. Our left to right minima are $3, 2, 1$. We fix them, and we have the following procedure.

- 3 _ _ 2 _ _ 1 _ _ Remaining entries $= \{9, 8, 7, 6, 5, 4\}$,
- 3 **4** _ 2 _ _ 1 _ _. Remaining entries $= \{9, 8, 7, 6, 5\}$,
- 3 4 **5** 2 _ _ 1 _ _. Remaining entries $= \{9, 8, 7, 6\}$
- 3 4 5 2 **6** _ 1 _ _. Remaining entries $= \{9, 8, 7\}$
- 3 4 5 2 6 **7** 1 8 9.

We get $345267189 \in S_9(132)$.

Let $\theta = \theta_1\theta_2\ldots\theta_n \in S_n(132)$. We will define

$$\beta : S_n(132) \longrightarrow S_n(123)$$

as

1 Fix the left-to-right minima of θ in their places.

2 Write all the non-left-to-right minimum entries of θ in decreasing order for the remaining places.

Example 2.7. $875649123 \in S_9(132)$. We fix the left to right minima.

- 8 7 5 _ 4 _ 1 _ _ Remaining entries $= \{9, 6, 3, 2\}$
- 8 7 5 **9** 4 _ 1 _ _ Remaining entries $= \{6, 3, 2\}$
- 8 7 5 9 4 **6** 1 _ _ Remaining entries $= \{3, 2\}$
- 8 7 5 9 4 6 1 **3** 2

and we have $875946132 \in S_9(123)$.

We will use the following proposition to show that α and β do not create a new left-to-right minimum.

Proposition 2.7. Let $\sigma \in S_n$. Assume that $\sigma_i \in E_\sigma$, then $i - 1 \leq n - \sigma_i$.

Proof. We know that the number of entries that is larger than σ_i is $(n - \sigma_i)$. We also know that there are $i - 1$ entries on the left of σ_i. Since σ_i is a left-to-right minimum, it is smaller than every entry on its left. Therefore the number of entries on the left of σ_i, should be smaller or equal to $n - \sigma_i$. Therefore $i - 1 \leq n - \sigma_i$. □

Proposition 2.8. α and β do not create a new left-to-right minimum.

Proof. Let $\sigma \in S_n(123)$. We will show that $\alpha(\sigma)$ has the same left-to-right minima as σ. We will use contradiction. Assume that α created a new left-to-right minimum in $\alpha(\sigma)$. Let $\sigma_k \in E_{\alpha(\sigma)}$ be the first left-to-right minimum such that $\sigma_k \notin E_\sigma$. Assume that $\sigma_i \in E_\sigma$ is the closest left-to-right minimum such that $i < k$. We look at the entries between σ_i and σ_k. Since $\sigma_k < \sigma_i$, all of the $n - \sigma_i$ entries larger than σ_i have been used. It implies that $i - 1 \geq n - \sigma_i$. The previous proposition shows that only $i - 1 = n - \sigma_i$ is possible. It implies that $\sigma_i = n - (i - 1)$. In that case $\sigma = n(n-1)(n-2)\ldots 21$. We see that every entry is a left-to-right minimum. Therefore α fixes every entry in such a case. We can say that $E_{\alpha(\sigma)} \subseteq E_\sigma$. Note that every left-to-right minimum of σ is still a left-to-right minimum of $\alpha(\sigma)$ since we always put larger entries after every left-to-right minimum. Therefore we can say that $E_\sigma \subseteq E_{\alpha(\sigma)}$. We can conclude that $E_\sigma = E_{\alpha(\sigma)}$ and since we fix them in their places, $I_\sigma = I_{\alpha(\sigma)}$.

We can use the same arguments for β. Therefore β does not create a new left-to-right minimum as well. We can say that for $\theta \in S_n(123)$, we have $E_\theta = E_{\beta(\theta)}$ and $I_\theta = I_{\beta(\theta)}$. □

Proposition 2.9. For every $\sigma \in S_n(123)$, $\alpha(\sigma) \in S_n(132)$. Similarly, for every $\theta \in S_n(132)$, $\beta(\theta) \in S_n(123)$.

Proof. We have shown that $E_\sigma = E_{\alpha(\sigma)}$ and $I_\sigma = I_{\alpha(\sigma)}$. α puts the smallest possible entries in increasing order between two left-to-right minima. They are also larger than the current left-to-right minimum. Therefore by Proposition 2.3, $\alpha(\sigma) \in S_n(132)$.

Similarly, we know that $E_\theta = E_{\beta(\theta)}$ and $I_\theta = I_{\beta(\theta)}$. β puts the non-left-to-right minima in the remaining places decreasingly. Therefore by Proposition 2.5, $\beta(\theta) \in S_n(123)$. □

Lemma 2.2. α and β are inverses of each other.

Proof. Let $\sigma \in S_n(123)$. We know that $E_\sigma = E_{\alpha(\sigma)}$ by the Proposition 2.8. Again by the same proposition, we have $E_\sigma = E_{\alpha(\sigma)} = E_{\beta(\alpha(\sigma))}$ and $I_\sigma = I_{\alpha(\sigma)} = I_{\beta(\alpha(\sigma))}$.

We have that $\alpha(\sigma) \in S_n(132)$ and $\beta(\alpha(\sigma)) \in S_n(123)$ by Proposition 2.9. Finally, by the Proposition 2.6, we have $\sigma = \beta(\alpha(\sigma))$.

Let $\theta \in S_n(132)$. We know that $E_\theta = E_{\beta(\theta)}$ by the Proposition 2.8. Again by the same proposition, we have $E_\theta = E_{\beta(\theta)} = E_{\alpha(\beta(\theta))}$ and $I_\theta = I_{\beta(\theta)} = I_{\alpha(\beta(\theta))}$. We have that $\beta(\theta) \in S_n(123)$ and $\alpha(\beta(\theta)) \in S_n(132)$ by Proposition 2.9. Finally, by the Proposition 2.4, we have $\theta = \alpha(\beta(\theta))$. $\square$

2.3.2 A bijection between $S_n(132)$ and Dyck paths

Knuth introduced this bijection [2] in 1969.

Recall that a Dyck path of length n is a sequence of steps (1,1) and (1,-1) that starts from $(0,0)$ and ends at $(2n,0)$ where it never goes below $x-$axis. We denote the step (1,1) by u and (1,-1) by d. Note that $\mathcal{D}_n$ denotes the set of Dyck paths of length n.

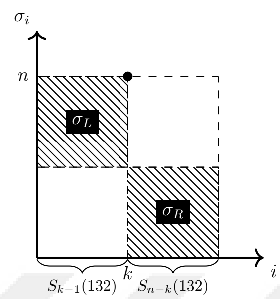
Recall from Proposition A.2 that Dyck paths are enumerated with Catalan numbers. Similarly, 132-avoiding permutations are also enumerated with Catalan numbers, as seen in Section 2.2. Therefore we know that

$$|\mathcal{D}_n| = |S_n(132)|$$

for all $n \geq 1$. It indicates that there is a bijection between these two sets. We will use the structure of the elements in $\mathcal{D}_n$ and $S_n(132)$.

Let $\sigma \in S_n(132)$ and $\sigma_k = n$. Here, we slightly change the notation we defined at the beginning of Section 2.2. We do it to understand the bijection better. Recall that if $\sigma \in S_n(132)$, then $\sigma_r < \sigma_l$ for any $1 \leq l < k < r \leq n$. Therefore the string $\sigma_{k+1}\sigma_{k+2}\ldots\sigma_n$ has the smallest $n-k$ entries. We see that $\sigma_R = \sigma_{k+1}\sigma_{k+2}\ldots\sigma_n \in S_{n-k}(132)$. We know that the string $\sigma_L = \sigma_1\sigma_2\ldots\sigma_{k-1}$ does not define a permutation. Let σ'_L be the permutation obtained by subtracting $n-k$ from every entry of σ_L. $\sigma'_L \in S_{k-1}(132)$ and it has the same relative order as σ_L.

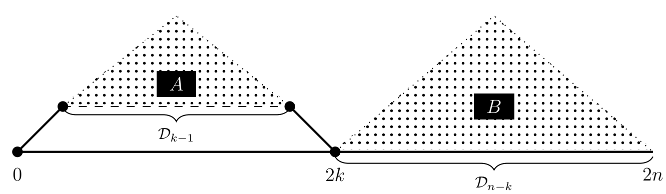
We can see the plot of σ as follows:



We can say that

$$\sigma = \sigma_L n \sigma_R \text{ where } \sigma_L' \in S_{k-1}(132) \text{ and } \sigma_R \in S_{n-k}(132).$$

Let $D \in \mathcal{D}_n$ and $(2k, 0)$ be the coordinate where D touches the $x-$axis for the first time.



We can write

$$D = uAdB \text{ where } A \in \mathcal{D}_{k-1} \text{ and } B \in \mathcal{D}_{n-k}.$$

We can see that Dyck paths and permutations that avoid 132-pattern have the same structure. Therefore if we define a function where $f(A) = \sigma_L$ and $f(B) = \sigma_R$, it will be a bijection.

Theorem 2.3. There exists a bijection between $S_n(132)$ and $\mathcal{D}_n$.

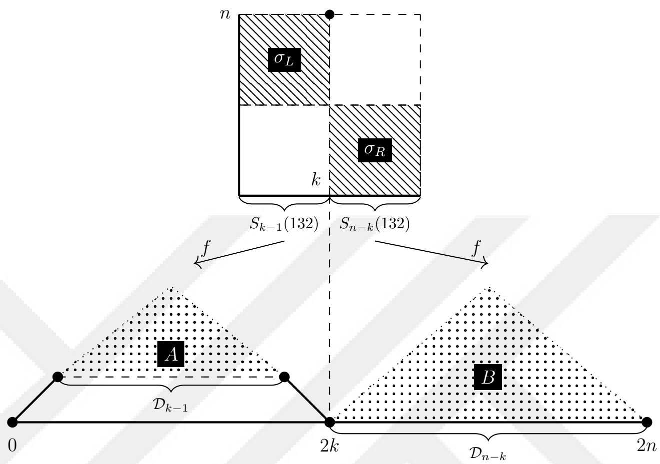
We will define

$$f : S_n(132) \longrightarrow \mathcal{D}_n$$

recursively,

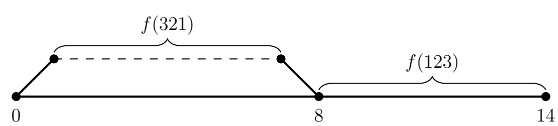
$$f(\sigma) = uf(\sigma_L')df(\sigma_R).$$

and let E be an empty string of entries, we define $f(E) = e$ where e is the empty string of steps. We can see the above map as follows:

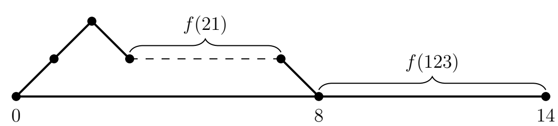


Example 2.8. Let $\sigma = 6547123 \in S_7(132)$.

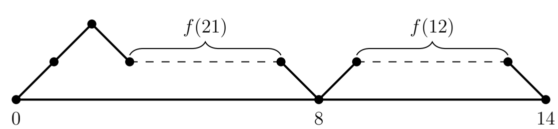
We have $\sigma_L = 654$ and $\sigma_R = 123$, the length of σ_R is 3. Thus $\sigma'_L = 321$. We see that $f(\sigma) = uf(321)df(123)$. By definition, $f(321)$ and $f(123)$ both have 6 steps. Therefore $f(\sigma)$ touches $x-$axis for the first time at the point $(8, 0)$.



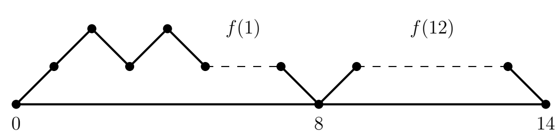
We want to find $f(321)$. We see that $\sigma_L = E$. Therefore $f(\sigma'_L) = e$ and $f(321) = uf(\sigma'_L)df(\sigma_R) = udf(21)$.



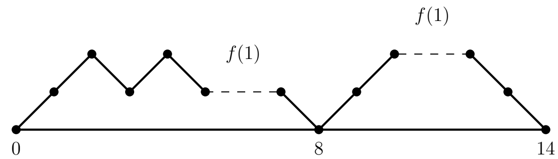
We want to find $f(123)$. We see that $\sigma_R = E$. Therefore $f(\sigma_R) = e$, and $f(123) = uf(\sigma'_L)df(\sigma_R) = uf(12)d$.



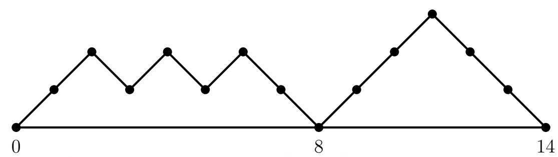
We want to find $f(21)$. We see that $\sigma_L = E$. Therefore $f(\sigma'_L) = e$ and $f(21) = uf(\sigma'_L)df(\sigma_R) = udf(1)$.



We want to find $f(12)$. We see that $\sigma_R = E$ and $f(\sigma_R) = e$. Therefore $f(12) = uf(\sigma'_L)df(\sigma_R) = uf(1)d$.



We want to find $f(1)$. We see that $\sigma_R = E$ and $\sigma_L = E$. Therefore $f(\sigma_R) = f(\sigma'_L) = e$ and $f(1) = uf(\sigma'_L)df(\sigma_R) = ud$.

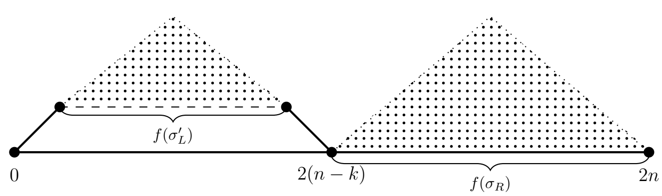


Lemma 2.3. For every $\sigma \in S_n(132)$, $f(\sigma) \in \mathcal{D}_n$.

Proof. We will use induction on n. Let $n = 1$, then $\sigma = 1$, we have

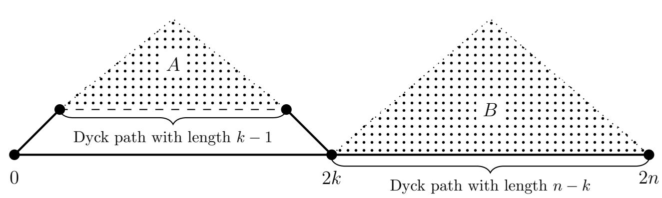
$$f(\sigma) = f(1) = uf(\sigma'_L)df(\sigma_R) = uf(E)df(E) = uede = ud.$$

Assume that if $\sigma \in S_k(132)$ where $k < n$, then $f(\sigma) \in \mathcal{D}_k$. Let $\sigma = \sigma_L n \sigma_R \in S_n(132)$. We know that $f(\sigma) = uf(\sigma'_L)df(\sigma_R)$. If σ_R has length k, then σ'_L has the length $n - k - 1$. Therefore $f(\sigma'_L) \in \mathcal{D}_{n-1-k}$ and $f(\sigma_R) \in \mathcal{D}_k$. We have the following picture:



We can see that $f(\sigma)$ is a Dyck path of length n. □

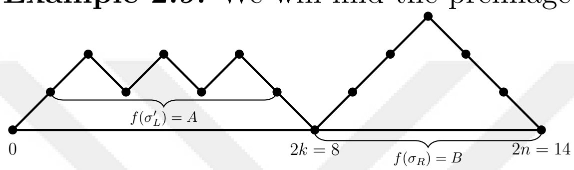
We want to show that f is surjective, therefore we will use the following process. Let $D \in \mathcal{D}_n$ and $(2k, 0)$ be the coordinate where D touches the x axis for the first time.



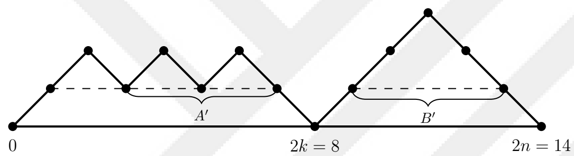
We see that $D = uAdB$. There should be $\sigma^1 \in S_{k-1}(132)$ and $\sigma^2 \in S_{n-k}(132)$ such that $f(\sigma^1) = A$ and $f(\sigma^2) = B$. It is a recursive process. Assume that such permutations exist. Let $\sigma^{1'}$ be the partial permutation where we add $n-k$ to every entry. Then,

$$f(\sigma) = f(\sigma^{1'} n \sigma^2) = uAdB = D.$$

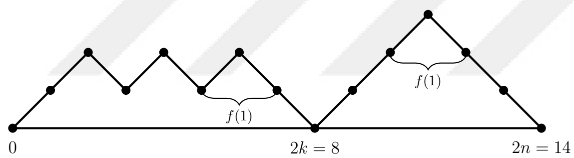
Example 2.9. We will find the preimage of the following Dyck path $D \in \mathcal{D}_7$.



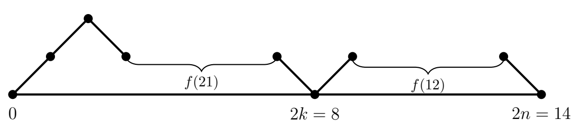
We see that $D = uAdB$ where $A = ududud$ and $B = uuuddd$



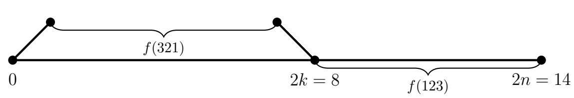
We see that $A = f(\sigma'_L) = uEdA'$ and $B = f(\sigma_R) = uB'dE$.



Note that $f(1) = ud$. We see that $A' = uf(E)df(1)$ and $B' = uf(1)df(E)$.



We want to find preimages of A' and B'. We can see that $f(21) = uf(E)df(1) = A'$ and $f(12) = uf(1)df(E) = B'$.



$A = f(\sigma'_L) = uEdf(21) = f(321)$, and $B = f(\sigma_R) = uf(12)dE = f(123)$.

Since $D = uAdB = uf(321)df(123)$, by the process again, we add $|\sigma_R| = |123| = 3$ to 321, hence $\sigma_L = 654$. Thus preimage of D is $\sigma = 6547123$.

Lemma 2.4. f is bijective.

Proof. First, we want to show that f is surjective. We will use induction on n. Assume that for every $A \in \mathcal{D}_k$ there exist $\sigma \in S_k(132)$ such that $f(\sigma) = A$ where $k < n$. Note that $f(1) = ud$. Let $D \in \mathcal{D}_n$, we want to find $\sigma \in S_n(132)$ such that $f(\sigma) = D$. Assume that D touches the $x-$axis for the first time at $(2k, 0)$. We can write $D = uAdB$ such that $A \in \mathcal{D}_{k-1}$ and $B \in \mathcal{D}_{n-k}$. By our assumption, there exist $\sigma^1 \in S_{k-1}(132)$ and $\sigma^2 \in S_{n-k}(132)$ such that $f(\sigma^1) = A$ and $f(\sigma^2) = B$. Let $\sigma^{1'}$ be the permutation where we add $n-k$ to every entry of σ^1. We claim that $\sigma = \sigma^{1'} n \sigma^2$ where $f(\sigma) = D$. By the Proposition 2.2, $\sigma \in S_n(132)$. We can also see that $f(\sigma) = D$ by the definition of f. Therefore f is surjective.

We will use induction on n to show that f is also injective. Assume that f is injective for $\sigma \in S_k(132)$ where $k < n$. We see that $f(1) = ud$. Let $D \in \mathcal{D}_n$ and $(2k, 0)$ be the first coordinate where D touches the $x-$axis. $D = uAdB$ where $A \in \mathcal{D}_{k-1}$ and $B \in \mathcal{D}_{n-k}$. Since we assume injection, we can say that $f(\rho_L) = A$ and $f(\rho_R) = B$. We know that $\rho_L \in S_{k-1}(132)$ and $\rho_R \in S_{n-k}(132)$. We know that $f(\rho) = D$ where $\rho = \rho'_L n \rho_R$. Since both ρ_L and ρ_R are unique, then ρ is also unique. Therefore f is injective.

$\square$

2.3.3 A bijection between $S_n(321)$ and Dyck paths

Rotem has proved the next bijection [3]. Rotem uses the property that any permutation in $S_n(321)$ can be partitioned into two increasing subsequences. First, we show the bijection between $S_n(321)$ and ballot sequences. Second, we discuss the bijection between ballot sequences and Dyck paths. We can see the structure as follows:

$$S_n(321) \longleftrightarrow \text{Ballot sequences} \longleftrightarrow \text{Dyck paths}$$

2.3.3.1 Bijection between $S_n(321)$ and ballot sequences of length n

There is also a certain structure for the permutations in $S_n(321)$. It will allow us to define the bijection between $S_n(321)$ and ballot sequences of length n.

Definition 2.5. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ be permutation. If $\sigma_k < \sigma_i$ for every $k < i$, then σ_i is called left-to-right maximum. Note that the first entry is always a left-to-right maximum.

It means that an entry is a left-to-right maximum if it is larger than every entry on its left.

Example 2.10. Let $\sigma = 435621 \in S_6$. We see that the entries $\sigma_1 = 4$, $\sigma_3 = 5$, and $\sigma_4 = 6$ are left-to-right maxima.

Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ be permutation. We define $M_\sigma = \{\sigma_i \mid \sigma_i \text{ is a left-to-right maximum}\}$ and $N_\sigma = \{\sigma_i \mid \sigma_i \text{ is not a left-to-right maximum}\}$.

Proposition 2.10. Let $\sigma \in S_n$. $\sigma \in S_n(321)$ if and only if elements in N_σ appear in increasing order in σ.

Proof. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n(321)$. We will show that entries in N_σ are in increasing order. We will use contradiction. Assume that there are two entries in N_σ in decreasing order. It means that there exist $\sigma_j, \sigma_k \in N_\sigma$ and $\sigma_k < \sigma_j$ where $j < k$. Since σ_j is not a left-to-right maximum, there is at least one entry σ_i such that $\sigma_j < \sigma_i$ where $i < j$. We have the subsequence $\sigma_i\sigma_j\sigma_k$, forming a 321 pattern. It is a contradiction. Therefore every entry in N_σ should be in increasing order. Conversely, we will show that if N_σ is in increasing order, then $\sigma \in S_n(321)$. We will prove the contrapositive argument. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \notin S_n(321)$, then there is a subsequence $\sigma_k < \sigma_j < \sigma_i$ where $i < j < k$. We see that $\sigma_j, \sigma_k \in N_\sigma$ since there is an entry on their left that is larger than both. Therefore there are two elements in N_σ in decreasing order. Thus if every element in N_σ increasing order, then $\sigma \in S_n(321)$. $\square$

Next, we will define a ballot sequence.

Definition 2.6. Let $b = b_1b_2b_3 \ldots b_n$ be a sequence of non-negative integers. b is called a ballot sequence of length n if it satisfies the following conditions:

1. $b_1 \leq b_2 \leq \cdots \leq b_n$

2. $0 \leq b_i \leq i-1$ where $i = 1, 2, \ldots, n$.

We denote the set of all length n ballot sequences with B_n. We see that $B_1 = \{0\}$, $B_2 = \{00, 01\}$, and $B_3 = \{000, 001, 002, 011, 012\}$.

Example 2.11. Note that 001224 is a ballot sequence, but 01132 is not a ballot sequence since it does not satisfy the first condition.

Let $\sigma \in S_n(321)$. The idea is to fix the non-left-to-right maxima of σ in their places. Non-left-to-right maxima form an increasing subsequence. It will allow us to construct a bijection between $S_n(321)$ and B_n.

We define

$$B : S_n(321) \longrightarrow B_n$$

$$B(\sigma_1\sigma_2 \ldots \sigma_n) = b_1 b_2 \ldots b_n$$

as

$$b_i = \begin{cases} 0 & \text{if } i = 1, \\ b_{i-1} & \text{if } \sigma_i \in M_\sigma, \\ \sigma_i & \text{if } \sigma_i \in N_\sigma. \end{cases}$$

Example 2.12. Let $\sigma = 2513476 \in S_7(321)$. We want to find $B(\sigma)$. Note that $M_\sigma = \{2, 5, 7\}$, then
$\sigma_1 = 2, b_1 = 0$,
$\sigma_2 = 5, b_2 = 0$ since $5 \in M_\sigma$,
$\sigma_3 = 1, b_3 = 1$ since $1 \in N_\sigma$,
$\sigma_4 = 3, b_4 = 3$ since $3 \in N_\sigma$,
$\sigma_5 = 4, b_5 = 4$ since $4 \in N_\sigma$,
$\sigma_6 = 7, b_6 = 4$ since $7 \in M_\sigma$,
$\sigma_7 = 6, b_7 = 6$ since $6 \in N_\sigma$.
To understand the process better, we look at the following table. Non-left-to-right maxima are written in bold.

i	1	2	3	4	5	6	7
σ_i	2	5	**1**	**3**	**4**	7	**6**
b_i	0	0	**1**	**3**	**4**	4	**6**

We have $B(\sigma) = 0013446 \in B_7$.

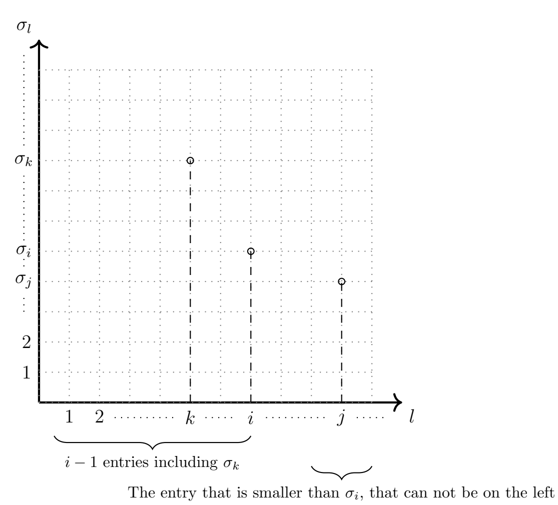
Lemma 2.5. For every $\sigma \in S_n(321)$, $B(\sigma) \in B_n$.

Proof. Let $\sigma \in S_n(321)$. There are two conditions that $B(\sigma)$ should satisfy so that $B(\sigma) \in B_n$.

First condition is to satisfy $b_1 \leq b_2 \leq \ldots \leq b_n$. We will use induction on n. First, we want to show that $b_1 \leq b_2$. For any $B(\sigma)$, we have $b_1 = 0$. We see that if $\sigma_2 \in N_\sigma$, then $b_2 = \sigma_2 > 0$. Otherwise, $\sigma \in M_\sigma$ and $b_2 = 0$. Thus $b_1 \leq b_2$ is proved. Next, we assume that $b_1 \leq b_2 \leq \ldots \leq b_{i-1}$. We will look at two cases.

- If $\sigma_i \in M_\sigma$, then we have $b_i = b_{i-1}$. It satisfies the first condition.
- If $\sigma_i \in N_\sigma$, then $b_i = \sigma_i$. We know that the elements in N_σ are in increasing order. Therefore $b_k < b_i$ for every $\sigma_k \in N_\sigma$ where $k < i$. Again, it satisfies the first condition.

Second condition is to satisfy $0 \leq b_i \leq i - 1$ for every $1 \leq i \leq n$. First we will check the case when $\sigma_i \in N_\sigma$. We will prove it by contradiction. Assume that $b_i = \sigma_i \geq i$. Since $\sigma_i \in N_\sigma$, there exist an entry σ_k such that $\sigma_i < \sigma_k$ where $k < i$.



We know that there are at least $i-1$ entries that is smaller than σ_i by our assumption. There are $i-1$ entries before σ_i, and one of the entries is σ_k. Therefore there exists an entry $\sigma_j < \sigma_i$ where $i < j$ since there are only $i-2$ entries on the left of σ_i that can be smaller than σ_i. We see that $\sigma_j < \sigma_i < \sigma_k$ where $k < i < j$. It is a contradiction since the 321 pattern is formed. Therefore we can say that $0 \leq b_i \leq i-1$ when σ_i is not a left-to-right maximum.

Second case is when σ_i is a left to right maximum. By the definition of B, $b_i = b_{i-1}$. It means that $b_i = b_k$ such that $\sigma_k \in N_\sigma$ and $k < i$. We know that $0 \leq b_k \leq k-1$. It implies that $0 \leq b_k = b_i \leq k-1 \leq i-1$. Therefore the second condition is also satisfied. $\square$

We can see that the b_i is determined by the entries in N_σ. Therefore the inverse of B will use that property. We define

$$T : B_n \longrightarrow S_n(321)$$

$$T(b_1 b_2 \dots b_n) = \sigma_1 \sigma_2 \dots \sigma_n$$

as

1. If a positive number occurs for the first time, fix it in its index.
2. Put the remaining entries in the remaining places in increasing order.

Example 2.13. Consider the ballot sequence $b = 0013446$. We want to find $T(b)$. Fixing positive integers in their places when they first occur, we have

i	1	2	3	4	5	6	7
b_i	0	0	1	3	4	4	6
First step			1	3	4		6

The remaining entries are $2, 5, 7$. If we place them according to T, we have

i	1	2	3	4	5	6	7
b_i	0	0	1	3	4	4	6
First step			1	3	4		6
Second step	2	5	1	3	4	7	6

$T(b) = 2513476$.

Proposition 2.11. For every $b \in B_n$, $T(b) \in S_n(321)$.

Proof. We will use contradiction, suppose that $b \in B_n$ and $T(b) \notin S_n(321)$. Therefore there exists a subsequence $\sigma_k < \sigma_j < \sigma_i$ where $i < j < k$. Since b is a ballot sequence, fixed entries in the first step are increasing. Thus at least two entries of the subsequence should be inserted in the second step. However, the second step also puts the entries increasingly, therefore it is a contradiction. $\square$

We will use the following proposition to show that T and B are inverses of each other.

Proposition 2.12. Let $b \in B_n$ and $T(b_1 b_2 \ldots b_n) = \sigma_1 \sigma_2 \ldots \sigma_n$. Assume that $1 \leq k \leq n$, then $b_k = \sigma_k$ if and only if $\sigma_k \in N_\sigma$.

Proof. Assume that $b_k = \sigma_k = j$. Therefore by the definition of ballot sequence, $\sigma_k = j = b_k \leq k - 1$ There are $k - 1$ elements on the left of σ_k. Note that $j - 1 \leq k - 2$. We have equality $j - 1 = k - 2$ at the largest case. It implies at least one entry on the left of σ_k larger than $j = \sigma_k$. Thus $j = \sigma_k \in N_\sigma$. Conversely, assume that $\sigma_i \in N_\sigma$. We will use contradiction, suppose that $\sigma_i \neq b_i$. It means that σ_i is inserted in the second step of T. We know that $\sigma_i \in N_\sigma$. Therefore there exist an entry σ_j such that $\sigma_i < \sigma_j$ and $j < i$. σ_j is inserted in the first step since both steps put entries in increasing order. Therefore $\sigma_j \in N_\sigma$, by the first part of the proof. We see that both $\sigma_j, \sigma_i \in N_\sigma$ are in increasing order. It contradicts with $T(b) \in S_n(321)$ by the Proposition 2.10. Therefore if $\sigma_k \in N_\sigma$ then $b_k = \sigma_k$. $\square$

Theorem 2.4. There exists a bijection between $S_n(321)$ and B_n.

Proof. We will show that B and T are inverses of each other. Assume that $\sigma \in S_n(321)$. We have $B(\sigma) = b_1 b_2 \ldots b_n$. For $T(B(\sigma))$, by the definition of B, $b_i \neq b_{i-1}$ for $\sigma_i \in N_\sigma$. Also M_σ appears in increasing order, thus $T(B(\sigma)) = \sigma$. Conversely, let b be a ballot sequence and $T(b) = \sigma$. Checking $B(T(b))$, B fixes

N_σ, and by the Proposition 2.12, elements of N_σ should be back in their places. Thus $B(T(b)) = b$. □

2.3.3.2 A bijection between B_n and $\mathcal{D}_n$

There are two ways to define a Dyck path. The details of the definitions are discussed in Appendix. We will use the other definition that we used for the second bijection. Note that a lattice path with the steps $(0,1)$, $(1,0)$ where it never goes above the $x = y$ diagonal also defines a Dyck path. We denote the step $(1,0)$ with r and $(0,1)$ with u.

We denote the set of Dyck paths with length n as $\mathcal{D}_n$.

We know that by the definition of ballot sequence, $b_i \geq 0$ and $0 \leq b_i \leq i-1$. We can see $0 \leq b_{i-1} \leq i-2$. It is possible that $b_i = i-1$ and $b_{i-1} = 0$, and $b_i = b_{i-1} = i-2$. It implies that $0 \leq b_i - b_{i-1} \leq i-1$. Note that by definition, there can be at most i times u steps until the line $x = i$ in a Dyck path. We will use these properties to construct a bijection between B_n and $\mathcal{D}_n$.

Theorem 2.5. There exists a bijection between B_n and $\mathcal{D}_n$.

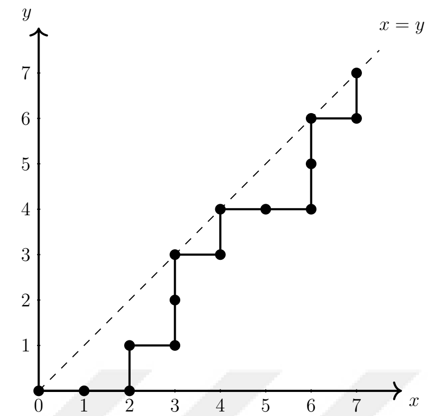
We denote the k many consecutive u steps with $u^{(k)}$. Note that $u^{(0)}$ is an empty string. Let $b_1 b_2 \ldots b_n \in B_n$. Define

$$f : B_n \longrightarrow \mathcal{D}_n$$

as for every b_i where $1 \leq i \leq n-1$, we will have a $ru^{(b_{i+1}-b_i)}$. In the end, we append $ru^{(n-b_n)}$. That is

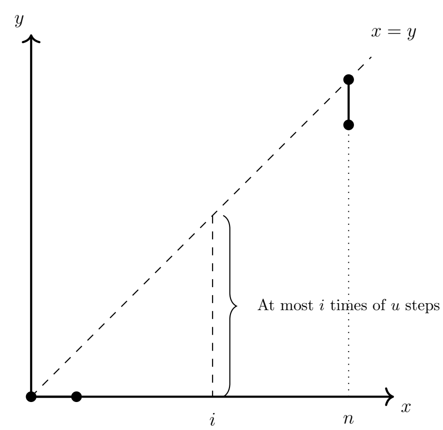
$$f(b_1 b_2 \ldots b_n) = ru^{(b_2-b_1)} ru^{(b_3-b_2)} \ldots ru^{(b_{i+1}-b_i)} \ldots ru^{(n-b_n)}$$

Example 2.14. Let $\sigma = 2513476 \in S_7(321)$, we already showed that $B(\sigma) = 0013446 \in B_7(321)$.
We have $f(0013446) = ru^{(0-0)} ru^{(1-0)} ru^{(3-1)} ru^{(4-3)} ru^{(4-4)} ru^{(6-4)} ru^{(7-6)} = rruruururruuru$.



Lemma 2.6. For every $b \in B_n$, $f(b) \in \mathcal{D}_n$.

Proof. Let $f(b_1 b_2 \ldots b_n) = ru^{(b_2-b_1)} ru^{(b_3-b_2)} \ldots ru^{(b_{i+1}-b_i)} \ldots ru^{(n-b_n)}$. We will check the Dyck path's number of u steps up to the line $x = i$. There must be at least $i+1$ times of u steps to pass the $x = y$ line.



We will show that there are at most i times of u steps up to $x = i$. We have at most $(b_2 - b_1) + (b_3 - b_2) + \ldots + (b_i - b_{i-1}) + (b_{i+1} - b_i) = b_{i+1} - b_1$ times of u steps by definition of f. We know that $b_1 = 0$ for any $b \in B_n$. Therefore, at most b_{i+1} times of u steps exist. By the definition of ballot sequence, we have $0 \leq b_{i+1} \leq i$. Thus it never passes the $x = y$ line for any $1 \leq i \leq n$. We will have n times r steps since for every $1 \leq i \leq n$, we put one of r. There are $(b_2-b_1)+(b_3-b_2)+\ldots+(b_i-b_{i-1})+(b_{i+1}-b_i)+\ldots+(b_n-b_{n-1})+(n-b_n) = n-b_1 = n$

many u steps. Therefore the path will end at (n, n) where it never goes above the $x = y$ line. Thus $f(\sigma) \in \mathcal{D}_n$. $\square$

If there is a $b \in B_n$ for every $A \in \mathcal{D}_n$ such that $f(b) = A$, then f is surjective. Note that we can represent $A = r_1 u^{(k_1)} r_2 u^{(k_2)} \ldots r_i u^{(k_i)} \ldots r_n u^{(k_n)}$ where r_i denotes the ith occurence of the step r. We can find $b = b_1 b_2 \ldots b_n$ such that $f(b) = A$ by defining $b_n = n - k_n$ and $b_i = b_{i+1} - k_i$.

Example 2.15. Let $A = rruruururrruuru \in \mathcal{D}_7$. We see that $A = r_1 u^{(0)} r_2 u^{(1)} r_3 u^{(2)} r_4 u^{(1)} r_5 u^{(0)} r_6 u^{(2)} r_7 u^{(1)}$. Therefore $b_7 = 7 - 1 = 6$, then $b_6 = 6 - 2 = 4$, $b_5 = 4 - 0 = 4$, $b_4 = 4 - 1 = 3$, $b_3 = 3 - 2 = 1$, $b_2 = 1 - 1 = 0$, $b_1 = 0 - 0 = 0$. We have $f(0013446) = A$.

Lemma 2.7. f is a bijection.

Proof. We want to show that there exists $b = b_1 b_2 \ldots b_n \in B_n$ such that $f(b) = A$ for every $A \in \mathcal{D}_n$. Let $A = r_1 u^{(k_1)} r_2 u^{(k_2)} \ldots r_i u^{(k_i)} \ldots r_n u^{(k_n)}$. First, we will show that the definition satisfies $0 \leq b_i \leq i - 1$. We see that

$$b_n = n - k_n$$
$$b_{n-1} = b_n - k_{n-1}$$
$$\ldots$$
$$b_{i+1} = b_{i+2} - k_{i+1}$$
$$b_i = b_{i+1} - k_i$$

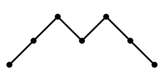
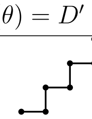
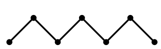
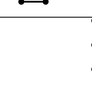
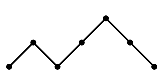
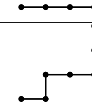
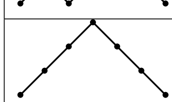
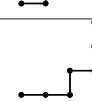
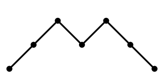
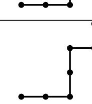
Therefore $b_i = n - (k_i + k_{i+1} + \ldots + k_n)$. We know that $(k_i + k_{i+1} + \ldots + k_n)$ is the number of u steps after the line $x = i$. We know there are at most i times of u steps before the $x = i$ line. Therefore $(k_i + k_{i+1} + \ldots + k_n) \leq n - i$. We also know that k_n is at least one since Dyck paths always end with the step u. Therefore $b_i \leq n - (n - i + 1)$. We can claim that $0 \leq b_i \leq i - 1$. Recall that we defined $b_i = b_{i+1} - k_i$, thus it is clear that $b_1 \leq b_2 \leq \ldots \leq b_n$. Therefore f is surjective. Assume that $b, b' \in B_n$ and $f(b) = f(b') = A \in \mathcal{D}_n$ where $A = r_1 u^{(k_1)} r_2 u^{(k_2)} \ldots r_i u^{(k_i)} \ldots r_n u^{(k_n)}$. By the definition of f, $b_n = b'_n = n - k_n$ and $b_i = b'_i = b_{i+1} - k_i$. Therefore for every $i \in [n]$, $b_i = b'_i$. Thus $b = b'$ and f is injective. $\square$

2.3.4 Overview of the bijections

We can see the structure of the bijections as follows:

$$
\begin{tikzcd}
S_n(132) \arrow[r, leftrightarrow, "\text{First bijection}"] \arrow[d, leftrightarrow, "\text{Second bijection}"'] & S_n(123) \arrow[d, leftrightarrow, "\text{Reverse of the permutations}"] \\
\text{Dyck paths} \arrow[r, leftrightarrow, "\text{Third bijection}"'] & S_n(321)
\end{tikzcd}
$$

Let f_2 denote the inverse of the second bijection and f_3 denote the function $f(B(\sigma))$ in the third bijection. We know that $C_3 = 5$, and we can see the process of the bijections.

$D \in \mathcal{D}_3$	$f_2(D) = \sigma \in S_3(132)$	$\beta(\sigma) = \rho \in S_3(123)$	$\rho^r = \theta \in S_3(321)$	$f_3(\theta) = D' \in \mathcal{D}_3$
	213	213	312	
	321	321	123	
	312	312	213	
	123	132	231	
	231	231	132	

2.4 Subsets of S_3

The proofs presented in this section are rooted in the research conducted by Simion and Schmidt [4]. While the previous sections focused on permutations that avoid a single pattern, our attention now turns to permutations that avoid multiple patterns from the set S_3. The proofs in this section revolve around the positions of the element n in permutations that avoid the chosen set of patterns. To enhance readability, we introduce the following notations.

Recall that for a set of patterns T,

$$S_n(T) = \bigcap_{\tau \in T} S_n(\tau),$$

$$\mathcal{P}^k_m = \{T \subseteq S_k \mid |T| = m\},$$

and if $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ and $\sigma_k = n$, then σ_L is the permutation of length $k-1$ with the same relative order as the string $\sigma_1\sigma_2\ldots\sigma_{k-1}$. Similarly, σ_R is the permutation of length $n-k$ that has the same relative order with the string $\sigma_{k+1}\sigma_{k+2}\ldots\sigma_n$.

2.4.1 Wilf classes of $\mathcal{P}^3_2 = \{T \subset S_3 \mid |T| = 2\}$

We will look at the subsets with two elements in S_3. We will list all $\binom{6}{2} = 15$ subsets.

We have the following cases for $T \in \mathcal{P}^3_2$.

1. $|S_n(123, 132)| = |S_n(123, 213)| = |S_n(231, 321)| = |S_n(312, 321)|$
2. $|S_n(132, 213)| = |S_n(231, 312)|$
3. $|S_n(132, 231)| = |S_n(213, 312)| = |S_n(132, 312)| = |S_n(231, 213)|$

4. $|S_n(132,321)| = |S_n(231,123)| = |S_n(312,123)| = |S_n(213,321)|$

5. $|S_n(123,321)|$

By Proposition 2.1, we have the following equalities.

1. $\{123,132\}^r = \{321,231\}$

   $\{321,231\}^c = \{123,213\}$

   $\{123,213\}^r = \{321,312\}$

2. $\{132,213\}^r = \{231,312\}$

3. $\{132,231\}^c = \{312,213\}$

   $\{312,213\}^i = \{231,213\}$

   $\{231,213\}^r = \{132,312\}$

4. $\{132,321\}^r = \{231,123\}$

   $\{132,321\}^c = \{312,123\}$

   $\{231,123\}^c = \{213,321\}$

Therefore, we will check the 5 chosen sets to understand the Wilf classes of $(\mathcal{P}_2^3, \sim_W)$.

Proposition 2.13. Let $T_1 = \{123, 132\}$, then $|S_n(T_1)| = 2^{n-1}$ for all $n \geq 1$.

Proof. Let $\sigma \in S_n(T_1)$. Assume that $\sigma_k = n$, then we have two cases.

1. $k = n$, then $\sigma_L = (n-1)(n-2)\dots 1$, since σ avoids 123. Therefore σ is determined.

2. $k < n$, then entries in σ_L are decreasing order, since σ avoids 123. σ_L also have the $k-1$ largest entries, since σ avoids 132. Therefore σ_L is determined. We also have $\sigma_R \in S_{n-k}(T_1)$.

For $k = n$, we have one permutation. For every other k, we have $\sigma_R \in S_{n-k}(T_1)$. Therefore we have $|S_{n-k}(T_1)|$ many permutations for every $k < n$. Thus

$$|S_n(T_1)| = 1 + \sum_{k=1}^{n-1} |S_{n-k}(T_1)|.$$

We will use induction to prove our claim. Assume that $|S_k(T_1)| = 2^{k-1}$ is true for all $k < n$. Note that $|S_{n-k}(T_1)| = 2^{n-k-1}$. Therefore

$$|S_n(T_1)| = 1 + \sum_{k=1}^{n-1} 2^{n-k-1}.$$

We change the variables by $n - k = t$, we have the sum

$$|S_n(T_1)| = 1 + \sum_{t=1}^{n-1} 2^{t-1} = 1 + \sum_{t=0}^{n-2} 2^{t}$$

We use the finite geometric series formula,

$$|S_n(T_1)| = 1 + \frac{(1 - 2^{n-1})}{1 - 2} = 1 - 1 + 2^{n-1}$$

and

$$|S_n(T_1)| = 2^{n-1}.$$

□

Proposition 2.14. Let $T_2 = \{132, 213\}$, then $|S_n(T_2)| = 2^{n-1}$ for all $n \geq 1$.

Proof. Let $\sigma \in S_n(T_2)$. Assume that $\sigma_k = n$, then we have two cases.

1. $k = n$, then $\sigma_L = 12\ldots(n-1)$, since σ avoids 213. Therefore σ is determined.

2. $k < n$, then the entries of σ_L are in increasing order, since σ avoids 213. σ also avoids 132, therefore σ_L has the largest $k - 1$ entries. Thus σ_L is determined. We also have $\sigma_R \in S_{n-k}(T_2)$.

It implies that

$$|S_n(T_2)| = 1 + \sum_{k=1}^{n-1} |S_{n-k}(T_2)|.$$

Therefore, we have the same result as the previous proof. It implies that $|S_n(T_2)| = 2^{n-1}$. $\square$

Proposition 2.15. Let $T_3 = \{132, 231\}$, then $|S_n(T_3)| = 2^{n-1}$ for all $n \geq 1$.

Proof. Let $\sigma \in S_n(T_3)$. Assume that $\sigma_k = n$, then we have two cases.

1. $1 < k < n$ is not possible, since σ avoids 132 and 231.
2. $k = 1$, then $\sigma_R \in S_{n-1}(T_3)$.
3. $k = n$, then $\sigma_L \in S_{n-1}(T_3)$.

Therefore
$$|S_n(T_3)| = |S_{n-1}(T_3)| + |S_{n-1}(T_3)| = 2|S_{n-1}(T_3)|.$$
By iteration, we have
$$|S_n(T_3)| = 2^k |S_{n-k}(T_3)|.$$
It implies that $|S_n(T_3)| = 2^{n-1}$. $\square$

Proposition 2.16. Let $T_4 = \{132, 321\}$, then $|S_n(T_4)| = \binom{n}{2} + 1$ for all $n \geq 1$.

Proof. Let $\sigma \in S_n(T_4)$. Assume that $\sigma_k = n$, then we have two cases.

1. $k = n$, then $\sigma_L \in S_{n-1}(T_4)$.
2. $k < n$, then σ_L has the largest $k - 1$ entries, to avoid 132. They are in increasing order to avoid 321. Therefore σ_L is determined. Similarly, σ_R has the remaining $n - k$ entries in increasing order. Thus σ_R is also determined.

Thus in the case of $k \neq n$, there is one permutation for every k. Therefore there are $n - 1$ permutations except for the case $k = n$. We have the sum

$|S_n(T_4)| = |S_{n-1}(T_4)| + n - 1$. We will use induction, assume that for $k < n$, $|S_k(T_4)| = \binom{k}{2} + 1$ holds. Note that $|S_{n-1}(T_4)| = \binom{n-1}{2} + 1$. We see that

$$\begin{aligned} |S_n(T_4)| &= \binom{n-1}{2} + 1 + n - 1 = \binom{n-1}{2} + n = \frac{(n-1)(n-2)}{2} + n \\ &= \frac{n^2 - 3n + 2n + 2}{2} = \frac{n^2 - n}{2} + 1 = \frac{n(n-1)}{2} + 1. \end{aligned}$$

We have $|S_n(T_4)| = \binom{n}{2} + 1$. □

The following propositions are used to enumerate permutations that avoid 123 and 321.

Proposition 2.17. Let there be $n^2 + 1$ balls and n boxes. If we distribute all of the balls to the boxes, there is at least one box with $n + 1$ or more balls.

Proof. Assume that there is no box with $n + 1$ balls. It means that there are at most n balls in a box. Therefore the total number of balls is less than or equal to n^2. This contradicts the fact that we have $n^2 + 1$ balls. □

Proposition 2.18. [36] Let $\alpha_1, \alpha_2, \ldots, \alpha_{n^2+1}$ be a sequence where $\alpha_i \in [n^2 + 1]$ and if $\alpha_i = \alpha_j$ then $i = j$. There exists a subsequence with length $n + 1$ which is decreasing or increasing.

Proof. Assume that neither increasing or decreasing subsequence that has length $n + 1$ exists. Define $f : [n^2 + 1] \longrightarrow [n]$ where $f(\alpha_i)$ is the length of longest increasing subsequence that ends with α_i. By the Proposition 2.17, there exist $k \in [n]$ such that

$$f(\alpha_{i_1}) = f(\alpha_{i_2}) = \ldots f(\alpha_{i_{n+1}}) = k$$

Let $1 \leq j \leq n$ then $\alpha_{i_j} > \alpha_{i_{j+1}}$. Otherwise, there would be an increasing subsequence that has length $k + 1$ that ends with $\alpha_{i_{j+1}}$. Hence

$$\alpha_{i_1} > \alpha_{i_2} > \cdots > \alpha_{i_{n+1}}$$

which is a decreasing subsequence that has length $n+1$. It is a contradiction. □

Proposition 2.19. Let $T_5 = \{123, 321\}$, then

$$|S_n(T_5)| = \begin{cases} n & \text{if } n = 1,2 \\ 4 & \text{if } n = 3,4 \\ 0 & \text{if } n \geq 5. \end{cases}$$

Proof. The cases are clear up to $n = 3$. For $n = 4$, only the following permutations avoid T_5 : 2143, 3142, 2413, and 3412. A permutation of length 5 or more has a decreasing or increasing subsequence of length 3. Therefore $|S_n(T_5)| = 0$ for every $n \geq 5$. □

Therefore we see that $|S_n(T_1)| = |S_n(T_2)| = |S_n(T_3)|$ for all $n \geq 1$.

Representative	$\lvert S_n(T)\rvert$
123,132	2^{n-1}
132,321	$\binom{n}{2} + 1$
123,321	0 if $n \geq 5$

There are 3 Wilf classes in $(\mathcal{P}_2^3, \sim_W)$.

2.4.2 Wilf classes of $\mathcal{P}_3^3 = \{T \subset S_3 \mid |T| = 3\}$

We will look at the subsets with three elements in S_3. We will list all $\binom{6}{3} = 20$ subsets.

We have the following cases for $T \in \mathcal{P}_3^3$.

1. $|S_n(123, 132, 213)| = |S_n(231, 312, 321)|$
2. $|S_n(123, 132, 231)| = |S_n(132, 231, 321)| =$
   $|S_n(213, 312, 321)| = |S_n(123, 213, 312)|$

3. $|S_n(213,231,312)| = |S_n(132,231,312)| =$
   $|S_n(132,213,231)| = |S_n(132,213,312)|$

4. $|S_n(123,132,312)| = |S_n(132,312,321)| =$
   $|S_n(213,231,321)| = |S_n(123,213,231)|$

5. $|S_n(123,231,312)| = |S_n(132,213,321)|$

6. $|S_n(123,321,213)| = |S_n(123,321,312)| =$
   $|S_n(123,321,231)| = |S_n(123,321,132)|$

By Proposition 2.1, we have the following equalities.

1. $\{123,132,213\}^c = \{231,312,321\}$

2. $\{123,132,231\}^r = \{132,231,321\}$

   $\{123,132,231\}^c = \{213,312,321\}$

   $\{132,231,321\}^c = \{123,213,312\}$

3. $\{213,231,312\}^c = \{132,213,231\}$

   $\{132,213,231\}^r = \{132,213,312\}$

   $\{132,213,312\}^c = \{132,231,312\}$

4. $\{123,132,312\}^c = \{132,312,321\}$

   $\{123,132,312\}^r = \{213,231,321\}$

   $\{213,231,321\}^c = \{123,213,231\}$

5. $\{123,231,312\}^r = \{132,213,321\}$

6. $\{123,321,213\}^c = \{123,321,231\}$

   $\{123,132,312\}^r = \{123,321,312\}$

   $\{123,321,231\}^r = \{123,321,132\}$

Therefore we only look at the six sets for the Wilf classes in $(\mathcal{P}_3^3, \sim_W)$.

Definition 2.7. n^{th} Fibonacci number f_n is defined as

$$f_n = f_{n-1} + f_{n-2}$$

where $f_0 = 0, f_1 = 1$ and $n \geq 2$. We can see the sequence as

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots$$

Proposition 2.20. Let $T_1 = \{123, 132, 213\}$, then $|S_n(T_1)| = f_{n+1}$ for $n \geq 1$ where f_n is the Fibonacci sequence starting with $f_0 = 0, f_1 = 1$.

Proof. Let $\sigma \in S_n(T_1)$. Assume that $\sigma_k = n$. We only have the cases $k = 1$ or $k = 2$. Otherwise, we will have the pattern 213 or 123.

1. $k = 1$, then $\sigma_R \in S_{n-1}(T_1)$.
2. $k = 2$, then $\sigma_L = \sigma_1 = n - 1$, since σ avoids 132. Thus $\sigma_R \in S_{n-2}(T_1)$.

Therefore

$$|S_n(T_1)| = |S_{n-1}(T_1)| + |S_{n-2}(T_1)|.$$

It is the recurrence relation satisfied by the Fibonacci sequence, thus $|S_n(T_1)| = f_{n+1}$. □

Proposition 2.21. Let $T_2 = \{123, 132, 231\}$, then $|S_n(T_2)| = n$ for every $n \geq 1$.

Proof. Let $\sigma \in S_n(T_2)$ and $\sigma_k = n$. It is either $k = 1$ or $k = n$. Otherwise, pattern 132 or 231 is formed.

1. $k = n$, then $\sigma_L = (n-1)(n-2)\ldots 21$, since σ avoids 123. Thus σ is determined.
2. $k = 1$, then $\sigma_R \in S_{n-1}(T_2)$.

Therefore we have $|S_n(T_2)| = 1 + |S_{n-1}(T_2)|$. By iteration, we have $|S_n(T_2)| = 1 + n - 1 = n$. □

Proposition 2.22. Let $T_3 = \{213, 231, 312\}$, then $|S_n(T_3)| = n$ for every $n \geq 1$.

Proof. Let $\sigma \in S_n(T_3)$. Assume that $\sigma_k = n$, then we have three cases.

1. $k = 1$, then $\sigma_R = (n-1)(n-2)\ldots 21$, since σ avoids 312.
2. $k = n$, the $\sigma_L = 12\ldots(n-2)(n-1)$, since σ avoids 213.
3. $1 < k < n$, the σ_L has smallest $k-1$ entries, since σ avoids 231. They are in increasing order since σ avoids 213. It follows that σ_R has the remaining entries. Entries of σ_R are in decreasing order since σ avoids 312. Therefore σ is determined.

We showed that for every k where $1 \leq k \leq n$, there is only one possible permutation that is in $S_n(T_3)$. Thus $|S_n(T_3)| = n$. $\square$

Proposition 2.23. Let $T_4 = \{123, 132, 312\}$, then $|S_n(T_4)| = n$ for every $n \geq 1$.

Proof. Let $\sigma \in S_n(T_4)$. Assume that $\sigma_k = n$, then we have three cases.

1. $k = 1$, then $\sigma_R = (n-1)(n-2)\ldots 21$, since σ avoids 312.
2. $k = n$, then $\sigma_L = (n-1)(n-2)\ldots 21$, since σ avoids 123.
3. $1 < k < n$, then σ_L has the largest $k-1$ entries, since σ avoids 132. They should be in decreasing order since σ avoids 123. Similarly, σ_R has the smallest $n-k$ entries. They should be in decreasing order since σ avoids 312. Therefore σ is determined.

We showed that for every k where $1 \leq k \leq n$, there is only one possible permutation that is in $S_n(T_4)$. Thus $|S_n(T_4)| = n$. $\square$

Proposition 2.24. Let $T_5 = \{123, 231, 312\}$, then $|S_n(T_5)| = n$ for every $n \geq 1$.

Proof. Let $\sigma \in S_n(T_5)$. Assume that $\sigma_k = n$, then we have three cases.

1. $k = 1$, then $\sigma_R = (n-1)(n-2)\ldots 21$, since σ avoids 312.

2. $k = n$, then $\sigma_L = (n-1)(n-2)\ldots 21$, since σ avoids 123.

3. $1 < k < n$, then σ_L has the smallest $k-1$ entries, since σ avoids 231. They are in decreasing order since σ avoids 123. Similarly, σ_R has the remaining $n-k$ entries. They are in decreasing order since σ avoids 312. Therefore σ is determined.

We showed that for every $\sigma_i = n$ where $1 \leq i \leq n$, there is only one possible permutation that is in $S_n(T_5)$. Thus $|S_n(T_5)| = n$. □

Proposition 2.25. Let $T_6 = \{123, 321, 213\}$ then

$$|S_n(T_6)| = \begin{cases} n & \text{if } n \leq 3 \\ 1 & \text{if } n = 4, \\ 0 & \text{if } n \geq 5. \end{cases}$$

Proof. For $n \leq 3$ our case is clear. For $n = 4$, only 3412 avoids the patterns in T_6. For $n \geq 5$, we use Proposition 2.18. Any permutation that has a length larger than 5, will have a subsequence of length three which is decreasing or increasing. □

We see that $|S_n(T_2)| = |S_n(T_3)| = |S_n(T_4)| = |S_n(T_5)|$ for all $n \geq 1$.

Representative	$\lvert S_n(T)\rvert$
123, 132, 213	f_{n+1}
123, 132, 231	n
123, 321, 213	0 if $n \geq 5$

There are 3 Wilf classes in $(\mathcal{P}_3^3, \sim_W)$.

2.4.3 Wilf classes of $\mathcal{P}_m^3 = \{T \subset S_3 \mid |T| = m \geq 4\}$

We will look at the subsets with four elements in S_3. We will list all $\binom{6}{4} = 15$ subsets.

We have the following cases for $T \in \mathcal{P}_4^3$.

1. $|S_n(123, 132, 213, 231)| = |S_n(132, 231, 312, 321)| = |S_n(231, 213, 321, 312)| = |S_n(123, 132, 213, 312)| = |S_n(213, 231, 312, 321)|$
2. $|S_n(123, 213, 231, 312)| = |S_n(123, 132, 231, 312)| = |S_n(132, 213, 231, 321)| = |S_n(132, 213, 312, 321)|$
3. $|S_n(123, 321, 231, 312)| = |S_n(123, 321, 132, 213)|$
4. $|S_n(123, 321, 132, 312)| = |S_n(123, 321, 132, 231)| = |S_n(123, 321, 213, 312)| = |S_n(123, 321, 213, 231)|$

By Proposition 2.1, we have the following equalities.

1. $\{123, 132, 213, 231\}^r = \{132, 231, 312, 321\}$

   $\{123, 132, 213, 231\}^c = \{231, 213, 321, 312\}$

   $\{123, 132, 213, 231\}^i = \{123, 132, 213, 312\}$

   $\{123, 132, 213, 312\}^c = \{213, 231, 312, 321\}$

2. $\{132, 213, 231, 321\}^r = \{123, 132, 231, 312\}$

   $\{123, 132, 231, 312\}^c = \{132, 213, 312, 321\}$

   $\{132, 213, 312, 321\}^r = \{123, 213, 231, 312\}$

3. $\{123, 321, 231, 312\}^r = \{123, 321, 132, 213\}$

4. $\{123, 321, 132, 312\}^r = \{123, 321, 213, 231\}$

   $\{123, 321, 132, 312\}^c = \{123, 321, 213, 312\}$

   $\{123, 321, 132, 312\}^i = \{123, 321, 132, 231\}$

Therefore we only look at the four subsets for the Wilf classes in $(\mathcal{P}_4^3, \sim_W)$.

Proposition 2.26. Let $T_1 = \{123, 132, 213, 231\}$, then $|S_n(T_1)| = 2$ for every $n \geq 2$.

Proof. Let $\sigma \in S_n(T_1)$. Assume that $\sigma_k = n$, we have three cases.

1. $1 < k < n-1$ is not possible, since σ avoids 231 and 132. $k = n$ is also not possible, since σ avoids 123 and 213.
2. $k = 1$ and $\sigma_{n-1} = 1$. It implies $\sigma_n = 2$, since σ avoids 213. We have $\sigma = n(n-2)(n-1)\ldots 4312$. Entries before 1 are in decreasing order since σ avoids 231. Therefore σ is determined.
3. $k = 1$ and $\sigma_n = 1$, the remaining entries are in decreasing order, since σ avoids 231. It implies that $\sigma = n(n-2)(n-1)\ldots 4321$. Therefore σ is determined.

Note that 1 can not be positioned in any other place since it would create the pattern 132 or 123. Therefore there are only two permutations in $S_n(T_1)$ for every $n \geq 2$. $\square$

Proposition 2.27. Let $T_2 = \{123, 213, 231, 312\}$, then $|S_n(T_2)| = 2$ for every $n \geq 2$.

Proof. Let $\sigma \in S_n(T_2)$. Assume that $\sigma_k = 1$, then we have three cases.

1. $2 < k \leq n$ is not possible, since σ avoids 213 and 123.
2. $k = 2$, then $\sigma_L = \sigma_1 = 1$, since σ avoids 231. $\sigma_R = (n-1)(n-2)\ldots 2$, since σ avoids 312. Therefore σ is determined.
3. $k = 1$, then $\sigma_R = (n-1)(n-2)\ldots 21$, since σ avoids 312. Therefore σ is determined.

Therefore there are only two permutations in $S_n(T_2)$ for every $n \geq 2$. $\square$

Proposition 2.28. Let $T_3 = \{123, 321, 231, 312\}$, then

$$|S_n(T_3)| = \begin{cases} n & \text{if } n \leq 2 \\ 2 & \text{if } n = 3 \\ 1 & \text{if } n = 4 \\ 0 & \text{if } n \geq 5 \end{cases}$$

Proof. Let $\sigma \in S_n(T_3)$. For $n = 4$, only 2143 avoids T_3. For $n \geq 5$, by Proposition 2.18, there is not any permutation in $S_n(T_3)$. $\square$

Proposition 2.29. Let $T_4 = \{123, 321, 132, 312\}$, then

$$|S_n(T_4)| = \begin{cases} n & \text{if } n \leq 2, \\ 2 & \text{if } n = 3, \\ 0 & \text{if } n \geq 4 \end{cases}$$

Proof. For $n = 4$, there is no permutation that avoids T_4. We know that for $n \geq 5$, again by Proposition 2.18 we do not have any permutations. Therefore there is not any permutation in $S_n(T_4)$ where $n \geq 4$. $\square$

Therefore we see that $|S_n(T_1)| = |S_n(T_2)|$ and $|S_n(T_3)| = |S_n(T_4)|$ for all $n \geq 5$.

Representative	$\lvert S_n(T)\rvert$
123, 132, 213, 231	2 if $n \geq 2$
123, 321, 231, 312	0 if $n \geq 5$

We can see it as two Wilf classes in $(\mathcal{P}_4^3, \sim_W)$.

We will look at the subsets with five elements in S_3. We will list all $\binom{6}{5} = 6$ subsets.

We have the following cases for $T \in \mathcal{P}_5^3$.

1. $|S_n(132, 231, 213, 312, 321)| = |S_n(123, 132, 213, 231, 312)|$
2. $|S_n(123, 231, 213, 312, 321)| = |S_n(123, 132, 213, 312, 321)| = |S_n(123, 132, 231, 312, 321)|$
3. $|S_n(123, 132, 213, 231, 321)|$

By Proposition 2.1, we have the following equalities.

1. $\{132, 231, 213, 312, 321\}^r = \{123, 132, 213, 231, 312\}$
2. $\{123, 231, 213, 312, 321\}^r = \{123, 132, 213, 312, 321\}$
   $\{123, 132, 213, 312, 321\}^i = \{123, 132, 231, 312, 321\}$

It is enough to look at three subsets for the Wilf classes in $(\mathcal{P}_5^3, \sim_W)$.

Proposition 2.30. Let $T_1 = \{132, 231, 213, 312, 321\}$, then

$$|S_n(T_1)| = \begin{cases} n & \text{if } n \leq 2 \\ 1 & \text{if } n \geq 3 \end{cases}$$

Proof. Let $\sigma \in S_n(T_1)$. Assume that $\sigma_k = n$, then we have two cases.

1. $1 < k < n$ is not possible, since σ avoids 132 and 231. $k = 1$ is also not possible, since σ avoids 312 and 321.
2. $k = n$, then $\sigma_L = 12\ldots(n-2)(n-1)$, since σ avoids 213. Thus σ is determined.

Therefore there is only one permutation in $S_n(T_1)$ for every $n \geq 3$. $\square$

Proposition 2.31. Let $T_2 = \{123, 213, 231, 312, 321\}$, then

$$|S_n(T_2)| = \begin{cases} n & \text{if } n \leq 2 \\ 1 & \text{if } n = 3 \\ 0 & \text{if } n \geq 4 \end{cases}$$

Proof. For $n = 4$, there are no permutations that avoid T_2 in the previous cases where T has four elements. We know that for $n \geq 5$, by Proposition 2.18, we do not have any permutations in $S_n(T_2)$ where $n \geq 5$. $\square$

Proposition 2.32. Let $T_3 = \{123, 132, 213, 231, 321\}$, then

$$|S_n(T_3)| = \begin{cases} n & \text{if } n \leq 2 \\ 1 & \text{if } n = 3 \\ 0 & \text{if } n \geq 4 \end{cases}$$

Proof. For $n = 4$, there are no permutations that avoid T_3 in the previous cases where T has four elements. We know that for $n \geq 5$, by Proposition 2.18, we do not have any permutations in $S_n(T_3)$ where $n \geq 5$. $\square$

Therefore we see that $|S_n(T_2)| = |S_n(T_3)|$ for all $n \geq 5$.

Representative	$\lvert S_n(T)\rvert$
$S_3 - 123$	1 if $n \geq 3$
$S_3 - 132$	0 if $n \geq 4$

We can see it as two Wilf classes in $(\mathcal{P}_5^3, \sim_W)$.

Therefore, we have the following table:

m	Number of subsets	Wilf class number	Representative	$\lvert S_n(T)\rvert$
1	6	1	132	c_n
2	15	3	123,132	2^{n-1}
			123,321	0 if $n \geq 5$
			132,321	$\binom{n}{2} + 1$
3	20	3	123,132,213	f_{n+1}
			123,132,231	n
			123,132,321	0 if $n \geq 5$

4	15	2	$123, 321 \subset T$	0 if $n \geq 5$
			$123, 321 \not\subset T$	2 if $n \geq 2$
5	6	2	S_3 - 123	1 if $n \geq 3$
			S_3 - 132	0 if $n \geq 4$
6	1	1	S_n	0 if $n \geq 3$

Chapter 3

Patterns of Length Four

3.1 Wilf classes of $(S_4, \sim_W)$

In this section, we will study the Wilf classes of $(S_4, \sim_W)$. Let's first consider trivial bijections: the complement, inverse, and reverse.

$$P_1 = \begin{tikzcd} 1243 \arrow[loop left, "i"] \arrow[r, leftrightarrow, "r"] \arrow[d, leftrightarrow, "c"] & 3421 \arrow[d, leftrightarrow, "c"] \\ 4312 \arrow[r, leftrightarrow, "r"] \arrow[ur, leftrightarrow, "i"'] & 2134 \arrow[loop right, "i"] \end{tikzcd} \qquad P_2 = \begin{tikzcd} 1432 \arrow[loop left, "i"] \arrow[r, leftrightarrow, "r"] \arrow[d, leftrightarrow, "c"] & 2341 \arrow[d, leftrightarrow, "c"] \\ 4123 \arrow[r, leftrightarrow, "r"] \arrow[ur, leftrightarrow, "i"'] & 3214 \arrow[loop right, "i"] \end{tikzcd}$$

$$P_3 = \begin{tikzcd} 1324 \arrow[loop left, "i"] \arrow[r, leftrightarrow, "r\ c"] & 4231 \arrow[loop right, "i"] \end{tikzcd}$$

$$P_4 = \begin{tikzcd} 2143 \arrow[loop left, "i"] \arrow[r, leftrightarrow, "r\ c"] & 3412 \arrow[loop right, "i"] \end{tikzcd}$$

$$P_5 = 2413 \xleftrightarrow{i\ r\ c} 3142$$

$$P_6 = \begin{tikzcd} 1234 \arrow[loop left, "i"] \arrow[r, leftrightarrow, "r\ c"] & 4321 \arrow[loop right, "i"] \end{tikzcd}$$

$$
P_7 = \begin{tikzcd}
4213 \arrow[d, leftrightarrow, "c"] \arrow[ddd, leftrightarrow, bend right, "r"'] \arrow[dddrr, leftrightarrow, bend left, "i"] & & \\
1342 \arrow[r, leftrightarrow, "i"] \arrow[d, leftrightarrow, "r"] & 1423 \arrow[d, leftrightarrow, "c"] \arrow[ddr, leftrightarrow, bend left, "r"] & \\
2431 \arrow[r, leftrightarrow, "i"] \arrow[d, leftrightarrow, "c"] & 4132 \arrow[d, leftrightarrow, "r"] & \\
3124 \arrow[r, leftrightarrow, "i"] & 2314 \arrow[r, leftrightarrow, "c"] & 3241
\end{tikzcd}
$$

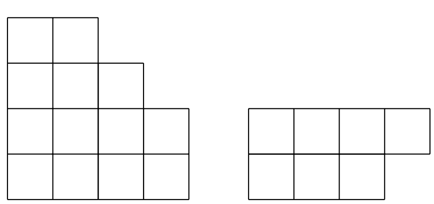
We are left with seven potential Wilf classes. Next, we will discuss the non-trivial bijections between the potential Wilf classes.

3.1.1 Shape-Wilf equivalence and its implications on $(S_4, \sim_W)$

This subsection is based mostly on the work of Babson and West [37]. Their approach utilizes the concepts of shape-Wilf equivalence and the direct sum of permutations to establish Wilf equivalence between specific patterns. To provide a foundation for our discussion, we introduce the following definitions that outline the concept of shape-Wilf equivalence.

Definition 3.1. A Ferrers board is a left justified array of cells in which the number of cells in each row is at least the number of cells in the row above. We say that a Ferrers board has the shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where λ_i is the number of cells in the i^{th} row and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Note that we order the rows from bottom to top, meaning that most bottom row is the first row.

Example 3.1. We look at the following examples:

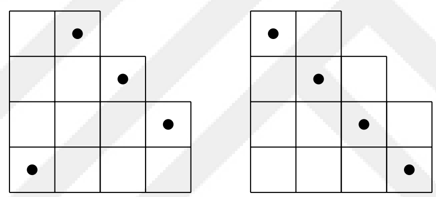


The first figure is a Ferrers board. It has the shape (4, 4, 3, 2). The second figure is not a Ferrers board.

If a Ferrers board has a designated set of cells, called rooks, so each row and column contains precisely one rook, then it is called a full rook placement.

If the underlying Ferrers of a full rook placement is a square, then it is called square full rook placement.

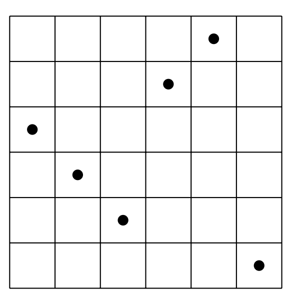
Example 3.2. We look at the following examples.



We see that there can be different full rook placements for the same Ferrers board.

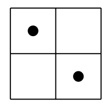
Remark. Let $\sigma \in S_n$. σ corresponds to a square full rook placement with n rows and n columns with the rooks in the cells (σ_i, i) for every i.

Example 3.3. Let $\sigma = 432561 \in S_6$. We see that the corresponding full rook placement is as follows:

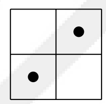


Definition 3.2. Let R and S be two full rook placements. We say that R is contained in S if R can be obtained by deleting rows and columns in S.
A full rook placement contains the permutation σ if it contains the square full rook placement corresponding to σ. Otherwise, it avoids σ.

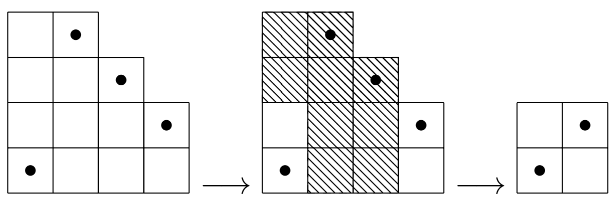
Example 3.4. Let $\tau_1 = 21 \in S_2$ and $\tau_2 = 12 \in S_2$. First, we will find the corresponding full rook placements of them.
For τ_1, we should put rooks on $(2,1)$ and $(1,2)$.



For τ_2, we should put rooks on $(1,1)$ and $(2,2)$.



Next, we look at the following full rook placement.



We see that this full rook placement contains τ_2, if we delete the fourth and third row and the second and third column. It avoids τ_1 since it does not contain the corresponding square full rook placement.

Definition 3.3. We say that the permutations σ and γ are shape-Wilf equivalent if any given shape λ, the number of σ avoiding full rook placements of shape λ is equal to γ avoiding full rook placements of shape λ.

Note that shape-Wilf equivalence implies Wilf equivalence by considering only square shapes.

Theorem 3.1. The permutations 12 and 21 are shape-Wilf equivalent.

Theorem 3.2. The permutations 123 and 321 are shape-Wilf equivalent.

The theorems mentioned, Theorem 3.1 and Theorem 3.2, were proven by Babson and West in 2000 [37]. We will omit the details of the proofs. The key idea behind Theorem 3.1 is to establish a direct bijection for each full rook placement. As for Theorem 3.2, Babson and West used an induction on full rook placements.

Theorem 3.3. The permutations $12\ldots(n-1)n$ and $n(n-1)\ldots 21$ are shape-Wilf equivalent for every $n \geq 4$.

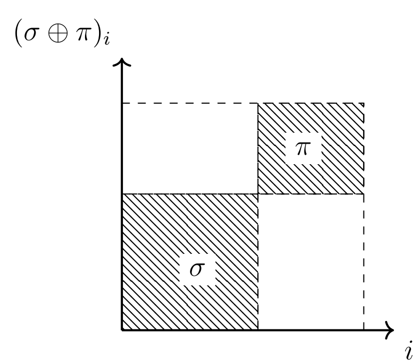
Backelin, West, and Xin [6] formulated a theorem that provides a generalization of both Theorem 3.2 and Theorem 3.1. For brevity, we will omit the details of the proof.

Moving forward, we will introduce the concept of the sum of permutations. It is highly valuable in reducing the number of potential Wilf classes.

Definition 3.4. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_k \in S_k$ and $\pi = \pi_1\pi_2\ldots\pi_\ell \in S_\ell$. We define the i^{th} entry of the direct sum of σ and π by

$$(\sigma \oplus \pi)_i = \begin{cases} \sigma_i & \text{if } i \in \{1, 2, \ldots, k\}, \\ \pi_{i-k} + k & \text{if } i \in \{k+1, k+2, \ldots, k+\ell\} \end{cases}$$

We can see how the structure of the direct sum of the permutations works by the following plot:



Theorem 3.4. Let σ and γ be shape-Wilf equivalent patterns. For any permutation δ, $\sigma \oplus \delta$ and $\gamma \oplus \delta$ are also shape-Wilf equivalent.

This theorem also has been proved by Babson and West [37]. We will omit the proof. The idea in the proof is first to fix a $\sigma \oplus \delta$ avoiding full rook placement and its shape. First, find the copy of δ in the right corner of the placement if it exists. We can cover the remaining placements, and they build a bijection to $\gamma \oplus \delta$ avoiding full rook placement with the same shape. It allows us to find bijections between $S_n(\tau)$ and $S_n(\rho)$ where $\rho, \tau \in S_4$. We can see the bijections in the following table:

Representative of the symmetry class	Chosen permutation from the class	Written as the direct sum of permutations	Equivalence	Corresponding permutation
1234	1234	$12 \oplus 12$	$123 \oplus 1$	1234
1243	2134	$21 \oplus 12$	$12 \oplus 12$	1234
2143	2143	$21 \oplus 21$	$12 \oplus 21$	1243
1432	3214	$321 \oplus 1$	$123 \oplus 1$	1234

We see that $|S_n(1234)| = |S_n(1243)| = |S_n(2143)| = |S_n(1432)|$ for any $n \geq 1$. Thus we are left with 4 potential Wilf classes: 1234, 1342, 1324, and 2413.

Theorem 3.5. There exists a bijection between $S_n(4132)$ and $S_n(3142)$.

Stankova has proved this theorem [5]. She uses the generating tree method. By applying the trivial bijections, we see that $|S_n(3142)| = |S_n(2413)|$ and $|S_n(4132)| = |S_n(1342)|$. Therefore, based on Stankova's proof, we can conclude that $|S_n(3142)| = |S_n(4132)| = |S_n(2413)| = |S_n(1342)|$. As a result, we are left with three potential Wilf classes: 1342, 1234, and 1324. Numerical evidence computed by Julian West [7], shows that, starting by $n = 1$,

- $|S_n(1342)|$ is counted as 1, 2, 6, 23, 103, 512, 2740, 15485, ...
- $|S_n(1234)|$ is counted as 1, 2, 6, 23, 103, 513, 2761, 15767, ...
- $|S_n(1324)|$ is counted as 1, 2, 6, 23, 103, 513, 2762, 15793, ...

Next, we will discuss the enumeration of $|S_n(1342)|$.

3.2 Enumeration of $|S_n(1342)|$

A significant breakthrough after Gessel's result [8] for $|S_n(1234)|$ was achieved by Bona in 1997. He computed the generating function and the enumerating formula corresponding to $S_n(1342)$ [9]. It is the second non-trivial enumeration result in the field for a pattern of length more than three. The crucial idea behind the proof depends on a bijection between a specific subset of $S_n(1342)$ and $\beta(0,1)$-trees. We will give the formal definitions in the following pages. It is known that the number of $\beta(0,1)$-trees with n vertices is given by [38]:

$$3 \cdot 2^{n-1} \cdot \frac{(2n)!}{n!(n+2)!}.$$

The permutations in that specific subset of $S_n(1342)$ considered in the proof are called indecomposable and all other 1342-avoiding permutations can be obtained by concatenations of such permutations. Let's introduce their definition.

Definition 3.5. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ be a permutation. σ is called decomposable if there exists $k \in [n-1]$ such that for all $i \leq k < j$, we have $\sigma_j < \sigma_i$.

In other words, σ is decomposable if it can be cut into two parts so that every entry before the cut is larger than every entry after the cut.

Example 3.5. Let $\sigma = 564123 \in S_6$. We see that it is decomposable since for all $i \leq 3 < k$, we have $\sigma_k < \sigma_i$.

Note that a permutation may have more than one decomposition. If we look at the example above, we can see that $56|4123$ and $564|123$.

Let σ be a permutation. If σ is not decomposable, it is called indecomposable.

Example 3.6. Let $\sigma = 156423 \in S_6$. We see that it is indecomposable since the first entry is $\sigma_1 = 1$, and it can not be larger than any entry.

We use $D_n(\tau)$ to denote the set of all indecomposable permutations of length n that avoid the pattern τ.

The crucial bijection in Bona's proof shows that $|D_n(1342)| = |\beta_n(0,1)|$ for all $n \geq 1$.

Tutte calculated the generating function corresponding to $\beta_n(0,1)$ [39] and obtained the number of $\beta_n(0,1)$-trees [38].

Let $F(x) = \sum_{n=0}^{\infty} |\beta_n(0,1)| x^n$. We know that

$$F(x) = \frac{8x^2 + 12x - 1 + (1-8x)^{3/2}}{32x}.$$

Let $H(x) = \sum_{n=0}^{\infty} |S_n(1342)| x^n$.

Since any $\sigma \in S_n(1342)$ is a concatenation of indecomposable 1342-avoiding permutations, we get

$$H(x) = \sum_{i \geq 0} F^i(x) = \frac{((-8x^2 + 20x + 1) + (1-8x)^{\frac{3}{2}})32x}{64x(x^3 + 3x^2 + 3x + 1)}$$

Therefore

$$H(x) = \sum_{n=0}^{\infty} |S_n(1342)| x^n = \frac{-8x^2 + 20x + 1 + (1-8x)^{\frac{3}{2}}}{2(x+1)^3}.$$

This implies that

$$|S_n(1342)| = (-1)^{n-1}\frac{7n^2 - 3n - 2}{2} + 3\sum_{i=2}^{n} (-1)^{i-1} 2^{i+1} \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2}.$$

We will now provide the details of the above arguments. As preliminary tools, in the following section, we discuss the relevant aspects of graph theory and introduce $\beta(0,1)$-trees. The graph theoretic part draws upon the content of Stanley's book, "Enumerative Combinatorics" [40].

3.2.1 Preliminaries

Definition 3.6. A finite graph G is a triple $G = (V, E, \phi)$ where V is a finite set of vertices, E is a finite set of edges, ϕ is a function that assigns to each edge

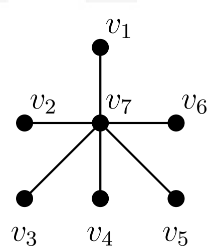
$e \in E$ to a 2-element multi-set of vertices.

In this chapter, we will exclusively work with finite graphs. Thus, whenever we refer to a "graph" it should be understood to mean a finite graph.

Definition 3.7. Let $G = (V, E, \phi)$ be a graph. Assume that $\phi(e) = \{v_1, v_2\}$, then we can see e as a joining of the vertices v_1 and v_2. If $v_1 = v_2$, then e is called a loop.

Definition 3.8. A graph $G = (V, E, \phi)$ is called simple if ϕ is injective and G has no loops.

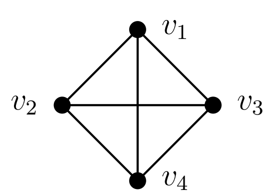
Example 3.7. These are some examples of a simple graph. Let $G_1 = (V_1, E_1, \phi_1)$ where $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E_1 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $\phi(e_i) = \{v_i, v_7\}$ for $i = 1, 2, \ldots, 6$. We can see G_1 as follows:



Let $G_2 = (V_2, E_2, \phi_2)$ where $V_2 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

$$\phi(e_i) = \begin{cases} v_i, v_{i+1} & \text{if } i = 1, 2, 3 \\ v_4, v_2 & \text{if } i = 4 \\ v_4, v_1 & \text{if } i = 5 \\ v_3, v_1 & \text{if } i = 6 \end{cases}$$

We can see G_2 as follows:



Definition 3.9. Let $G = (V, E, \phi)$ be a graph. $v \in V$ and $e \in E$ are incident if $v \in \phi(e)$. It means that e joins two vertices.

Definition 3.10. Let $G = (V, E, \phi)$ be a finite graph and $u, v \in V$. A walk from v to u is a sequence $v_0 e_1 v_1 e_2 v_2 \ldots e_n v_n$ such that $v_0 = v$, $v_n = u$, each $e_i \in E$ and $v_i \in V$, and any two consecutive terms are incident.

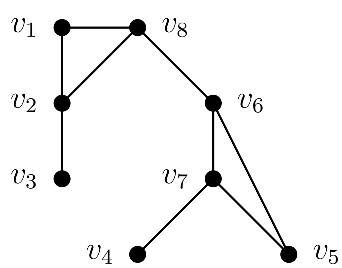
A walk is called a path if v_i's are distinct. Note that it also means that e_i's are distinct.

A walk is called a cycle if it has distinct v_i's except $v_0 = v_n$.

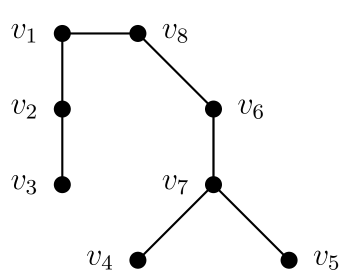
Definition 3.11. A graph G is called connected if any two distinct vertices are joined by a walk.

Definition 3.12. [41] Let G be a connected simple graph. G is called a tree if it does not contain a cycle.

Example 3.8. We see that there are cycles in the following graph because some of the edges have a path to themselves. Therefore it is not a tree.



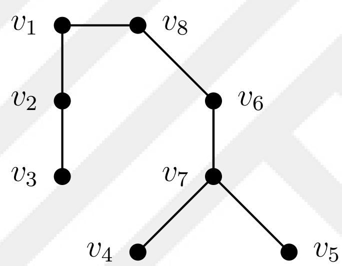
If we remove the edge between v_2 and v_8, and the one between v_6 and v_5, new graph will be a tree.



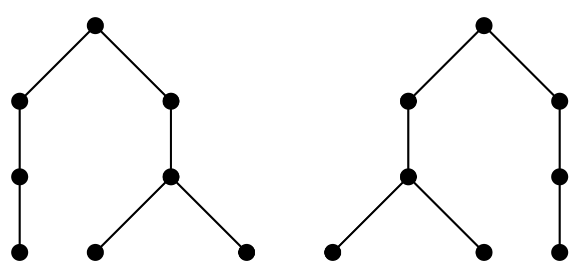
Definition 3.13. [42] Let G be a tree with the vertex g and $w \in V$ such that $w \neq g$. Assume that $v_0e_1v_1e_2 \ldots e_kv_ke_{k+1}v_{k+1}$ is a unique path where $v_0 = g$ and $v_{k+1} = w$. We call v_k the parent of w and call w a child of v_k. A vertex with no children is a leaf. All other vertices are internal vertices of G.

Definition 3.14. A tree G is called a plane tree if there is one vertex called the root and children of the root are ordered.

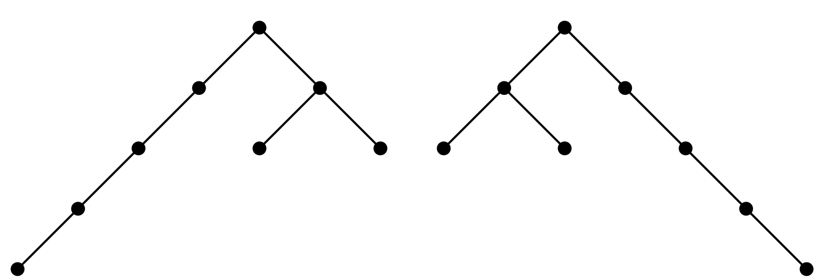
Example 3.9. Let G be the following tree:



If we choose v_8 as the root, we have the following plane trees:



If we choose v_6 as the root, we have the following plane trees:



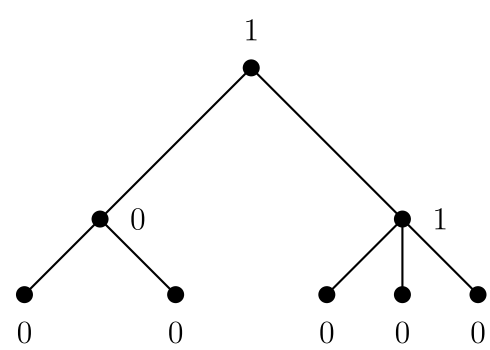
For the rest of this chapter, we will use the left to right order for the plane trees.

Definition 3.15. Let G be a plane tree. G with nonnegative labels $\ell(v)$ on every $v \in V$ is called a $\beta(0,1)$-tree if it satisfies the following conditions.

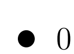
- If v is a leaf, then
$$\ell(v) = 0$$
- If v is the root and $v_1, v_2, \ldots v_k$ are its children, then
$$\ell(v) = \sum_{i=1}^{k} \ell(v_k)$$
- If v is an internal vertex that has the children $v_1, v_2, \ldots v_k$, then the inequality
$$\ell(v) \leq \sum_{i=1}^{k} \ell(v_k) + 1$$
holds.

We denote the set of $\beta(0,1)$-trees that has n vertices by $\beta_n(0,1)$

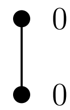
Example 3.10. Following plane tree is a $\beta(0,1)$-tree.



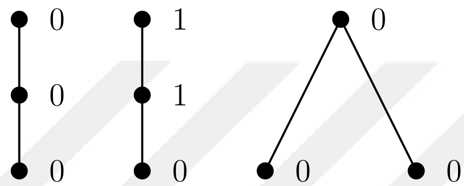
We will list all the $\beta(0,1)$-trees that has 1, 2, 3 and 4 vertices. For $n = 1$,



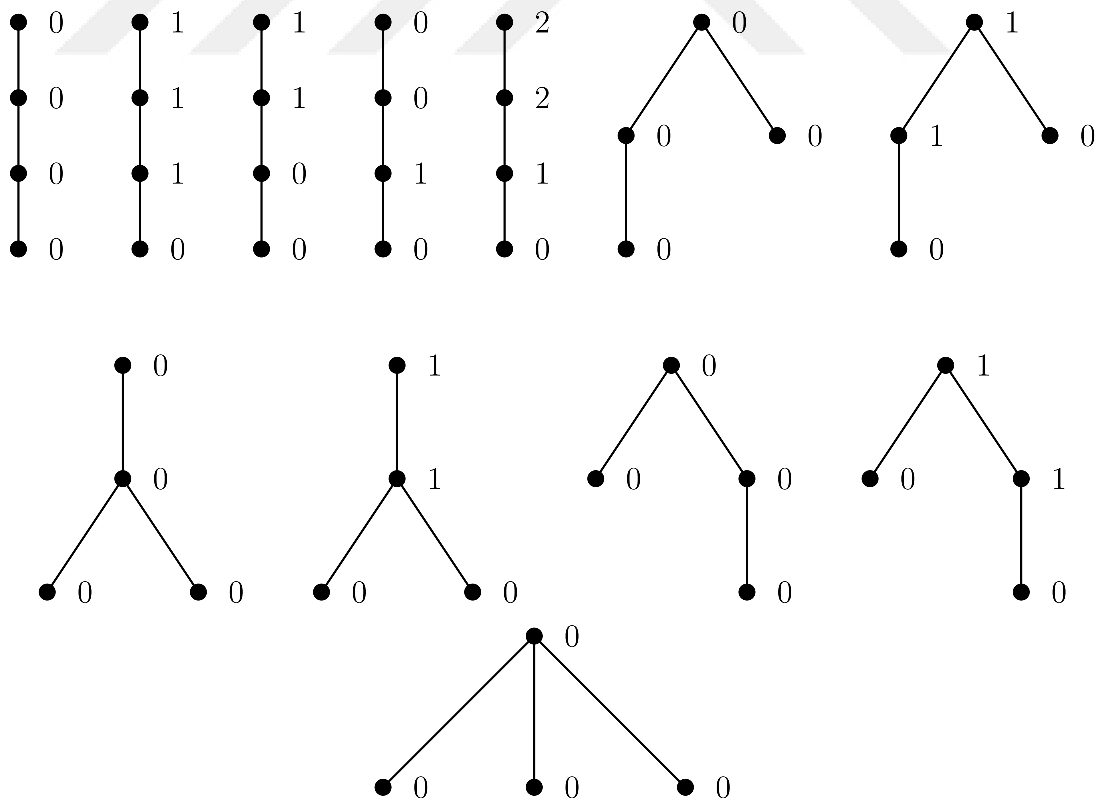
For $n = 2$,



For $n = 3$,



For $n = 4$,

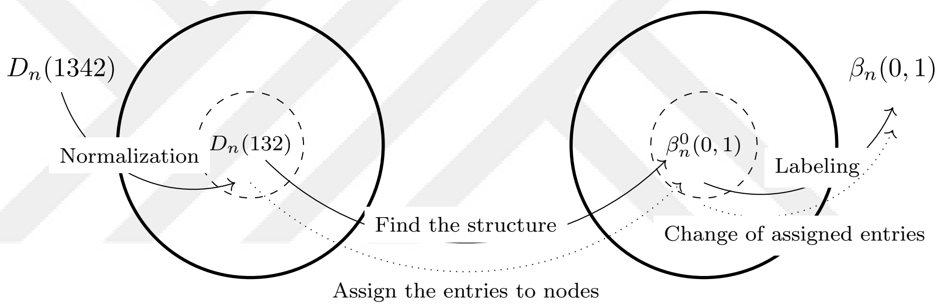


3.2.2 Bijection between $D_n(1342)$ and $\beta_n(0,1)$-trees

Bona's proof is based on a bijection between indecomposable permutations of length n that avoid 1342 and the set of $\beta(0,1)$-trees with n vertices. Recall that we denote the set of $\beta(0,1)$-trees that has n vertices with $\beta_n(0,1)$.

Theorem 3.6. There exists a bijection $F : D_n(1342) \to \beta_n(0,1)$.

First, we will discuss the structure of the proof.



The diagram above illustrates the multiple steps used to define F where for $\sigma \in D_n(1342)$ $F(\sigma) \in \beta_n(0,1)$. Observe that a $\beta(0,1)$-tree consists of two main components: its shape as a tree and the labels of its vertices. Bona's bijection F tells us how to determine these components for a permutation in $D_n(1342)$.

- First, we use the following steps to find the structure.
  1. The proof uses a function called the normalization function, which maps from $D_n(1342)$ to $D_n(132)$.
  2. It is known that there is a bijection between $D_n(132)$ and plane trees. The proof uses this to find the structure of $F(\sigma) \in \beta_n(0,1)$.
- The second step is to find the labels of the vertices. This procedure is highly technical, so we will illustrate it with examples.

After defining the function F from $D_n(1342)$ to $\beta_n(0,1)$, the proof verifies that $F^{-1} : \beta_n(0,1) \to D_n(1342)$ exists. Since we know the exact number of $\beta_n(0,1)$-trees, we can conclude that

$$|D_n(1342)| = 3 \cdot 2^{n-1} \cdot \frac{(2n)!}{n!(n+2)!}.$$

3.2.2.1 Normalization and the bijection g

The following part focuses on the shape of $F(\sigma) \in \beta_n(0,1)$.

Let $\sigma \in D_n(1342)$. We will exploit the same algorithm used in the definition of α in Subsection 2.3.1.

Definition 3.16. Consider the function $N : D_n(1342) \to D_n(132)$ defined as follows. Let $\sigma \in D_n(1342)$, then

1. Fix the left-to-right minima of σ,
2. After every left-to-right minimum of σ, we put the smallest entry that is also larger than the current left-to-right minimum.

$N(\sigma)$ is called the normalization of σ. Note that $N(\sigma) \in S_n(132)$ by the Proposition 2.8.

Example 3.11. Let $\sigma = 658732194 \in D_9(1342)$, then $N(\sigma) = 657832149 \in D_9(132)$.

Lemma 3.1. For every $\sigma \in D_n(1342)$, $N(\sigma) \in D_n(132)$.

Proof. We know that $N(\sigma)$ and $\sigma \in D_n(1342)$ have the same left-to-right minima in the same places, by the Proposition 2.8. We will show that left-to-right minima's positions decide whether the permutation is decomposable. Let $\sigma = \sigma_1 \dots \sigma_n \in S_n$. We will look into two cases, first assuming that $a \in [n]$ exists, such that $\sigma_{n-a+1} = a$ is a left-to-right minimum.

$$\begin{array}{cccccc} 1 & 2 & \cdots\cdots & n-a+1 & \cdots\cdots & n \end{array}$$

$$\underbrace{\boxed{\sigma_1 \quad \sigma_2 \quad \cdots\cdots \quad a}}_{n-a \text{ entries that is larger than } a}\ \underbrace{\cdots\cdots \quad \sigma_n}_{a-1 \text{ entries after } a}$$

There are $n-a$ entries before a, and a is smaller than all since it is a left-to-right minimum. Therefore on the right of a, there are $a-1$ entries. The number of entries that is smaller than a is $a-1$. All of them should be on the right of a. Otherwise, a can not be a left-to-right minimum. Therefore, we can cut before σ_{n-a+2}. It implies that σ is decomposable.

Second case is when there is no index $n-a+1 \in [n]$ such that $\sigma_{n-a+1} = a$ is a left-to-right minimum. Let $\sigma_{n-m+1} = k$ be a left-to-right minimum. We know that there are $n-m$ entries before k. We also know that, in general, there are $n-k$ entries that are larger than k. Since $\sigma_{n-m+1} = k$ is a left-to-right minimum, we should have $n-m < n-k$. Note that we have the first case if $k = m$. It implies that $k < m$.

$$\begin{array}{cccccccc} 1 & 2 & \cdots\cdots & i & \cdots\cdots & n-m+1 & \cdots\cdots & n \end{array}$$

$$\underbrace{\sigma_1 \quad \sigma_2 \quad \cdots\cdots \quad m \quad \cdots\cdots}_{n-m+1 \text{ entries that is larger than } k} \quad k \quad \underbrace{\cdots\cdots \quad \sigma_n}_{m-1 \text{ entries after } k}$$

We know that there are $k-1$ entries smaller than k. They should be on the right of $\sigma_{n-m+2} = k$. There are $m-1$ entries after k. We also know that $m-1 > k-1$ means that there is an entry after k larger than k. Thus the permutation is not decomposable. $\square$

Next lemma is important to understand the bijection between $D_n(132)$ and plane trees.

Lemma 3.2. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in D_n(132)$, then $\sigma_n = n$.

Proof. Proof by contradiction. Assume that $\sigma \in D_n(132)$ and $\sigma_n \neq n$. Suppose that $\sigma_i = n$ where $1 \leq i < n$. Since it avoids 132, every entry before n must be greater than every entry on the right of it. Thus it is decomposable by cutting after n. This is a contradiction. $\square$

Corollary 3.1. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in D_n(132)$. It implies that $\sigma' = \sigma_1\sigma_2\ldots\sigma_{n-1} \in S_{n-1}(132)$.

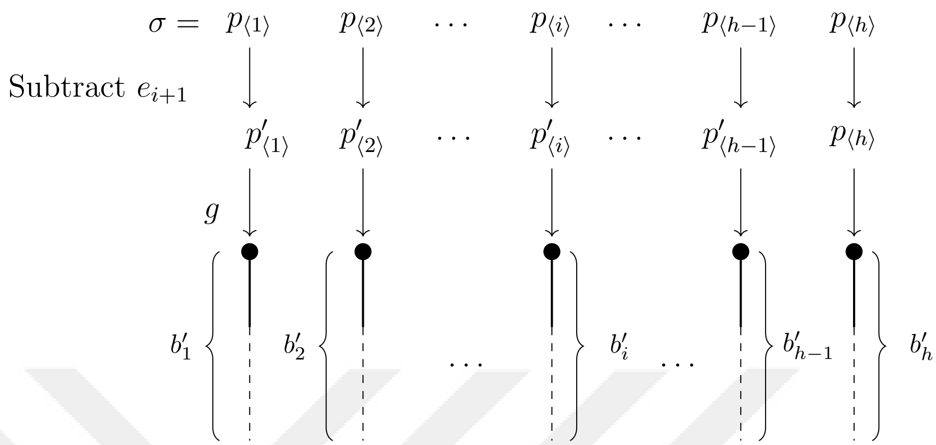
Example 3.12. Let $\sigma = 34512678 \in D_8(132)$ and $\delta = 65734128 \in D_8(132)$. We can see that $\sigma' = 3451267 \in D_7(132)$ since it is indecomposable. $\delta' = 6573412 \in S_7(132)$ is decomposable since we can cut after 7.

We use $\beta_n^0(0,1)$ to denote the set of all $\beta_n(0,1)$-trees with labels being 0. Note that $\beta_n^0(0,1)$ is the same set with the plane trees with n vertices. We see that $N(\sigma) \in D_n(132)$. The function g will determine the shape of the tree corresponding to the given permutation in $D_n(1342)$.

Proposition 3.1. There exists a bijection g between $D_n(132)$ and $\beta_n^0(0,1)$.

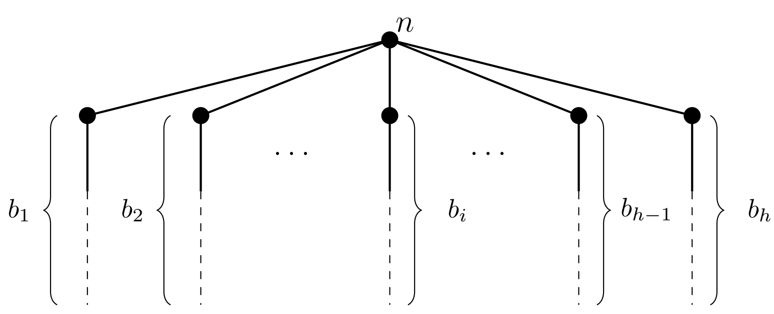
The bijection $g : D_n(132) \longrightarrow \beta_n^0(0,1)$ is defined recursively. Note that the image of the permutation 1 is a single vertex. Let $\sigma \in D_n(132)$. We know that $\sigma_n = n$, we first put a single vertex that is associated with n. This vertex is the root of our plane tree. We look at $\sigma' \in S_{n-1}(132)$. There are two cases, σ' decomposable or indecomposable. The first case is when σ' is decomposable. Cut σ' into strings $\sigma' = p_{\langle 1\rangle}p_{\langle 2\rangle}\ldots p_{\langle h\rangle}$. Here, $p_{\langle i\rangle}$'s are partial permutations such that for any $i < j$, every entry of $p_{\langle i\rangle}$ is larger than every entry of $p_{\langle j\rangle}$. There are h subtrees of the root. Next, we will find the plane tree that corresponds to $p_{\langle i\rangle}$.

Let e_i be the largest entry in the string $p_{\langle i\rangle}$. Let $p'_{\langle i\rangle}$ be the permutation where we subtract e_{i+1} from every entry of $p_{\langle i\rangle}$ for every $1 \leq i < h$. Note that $p'_{\langle i\rangle} \in D_k(132)$ for some $k < n$. Therefore, we can recursively find the assigned entries and the structure of $b'_i = g(p'_{\langle i\rangle})$. We can see the process in the following diagram.

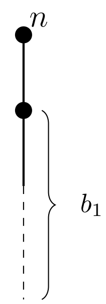


Next, we want to find b_i. The plane tree b_i is the same as the b'_i. The only change is with assigned entries of vertices. We add e_{i+1} to every assigned entry of b'_i. We repeat it for every b'_i for $1 \leq i < h$. Since $p_{\langle h \rangle}$ already defines a permutation, we keep the assigned entries and the plane tree. Therefore we find the assigned entries of every subtree.

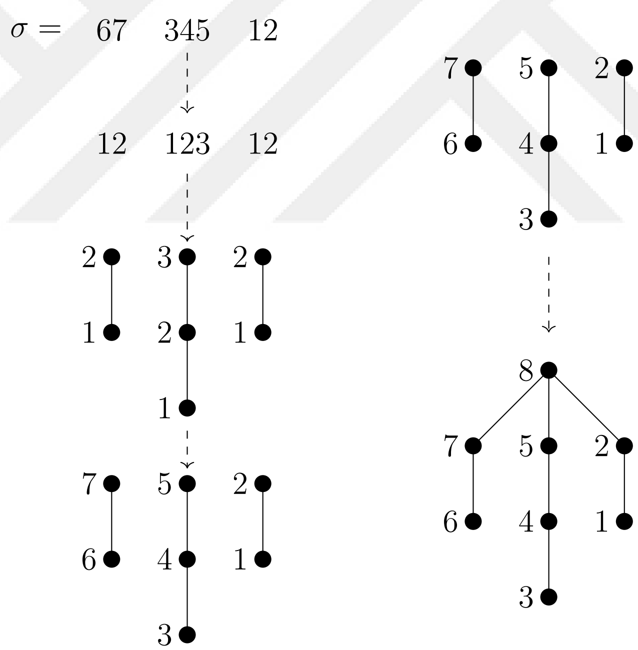
We obtain $g(\sigma)$ by connecting all h plane trees to the common root where b_i is the ith subtree. We already assigned n to the common root. Thus $g(\sigma)$ is found, we can see it as follows:



The second case when $\sigma' \in D_{n-1}(132)$. We say the root has only one subtree b_1 where $g(\sigma') = b_1$. We find b_1 recursively. We can see the $g(\sigma)$ as following:



Example 3.13. Let $\sigma = 67345128 \in D_8(132)$. We see that $\sigma' = 6734512$ and it is decomposable. We see that $\sigma' = 67|345|12$. We follow the steps of the first case, then



Next, will define

$$g^{-1} : \beta_n^0(0,1) \longrightarrow D_n(132)$$

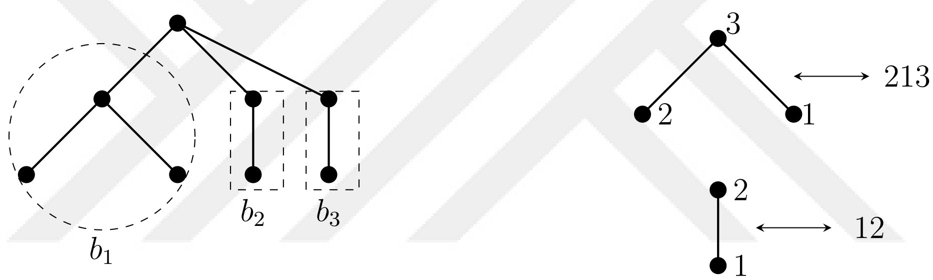
Let T be a $\beta(0,1)$-tree with all zero labels with root q. Assume that q has k subtrees. Let the subtrees be $b_1, b_2, \ldots, b_k$ and they have $n_1, n_2, \ldots, n_k$ vertices from left to right. We will define g^{-1} recursively. First note that single vertex $\beta(0,1)$-tree corresponds to the permutation 1.

1. Each b_i corresponds to an indecomposable 132-avoiding permutation with length n_i.

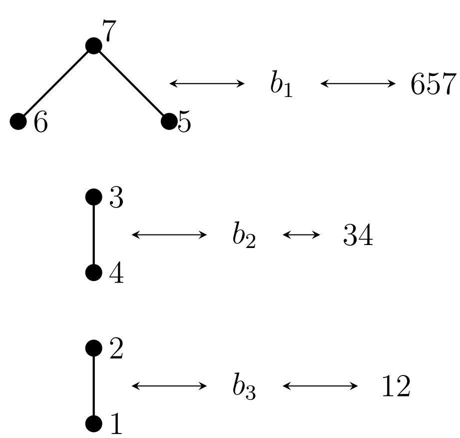
2. Add $\sum_{j=i+1}^{t} n_j$ to all entries of the permutation p_i associated with b_i.

3. Concatenate all the permutations and add n to the end of σ that is associated with T.

Example 3.14. Note that in the second step, we find the number of vertices in the subtrees that are on the right of the chosen subtree. We add this number to the corresponding permutation of the chosen subtree. It allows us to find a valid permutation in $D_n(132)$. Let T be the following tree. We see that b_2 and b_3 are the same trees.



By the definition, we do not add anything to the permutation that corresponds to b_3. We add $\sum_{i=3}^{3} n_i = n_3 = 2$ to the entries of the corresponding permutation of b_2. We add $\sum_{i=2}^{3} n_i = n_2 + n_3 = 2 + 2 = 4$ to the entries of the corresponding permutation of b_1. Therefore we see that



We concatenate all the permutations and add $n = 8$ to the end, we see that $\sigma = 65734128 \in D_8(132)$.

Note that g^{-1} is the direct inverse of g, therefore g is a bijection.

3.2.2.2 Labeling and change of assigned entries

Next, we will label the vertices of the corresponding plane tree of $\sigma \in D_n(1342)$.

Definition 3.17. The entry σ_i beats the entry σ_j if there exists an entry σ_h where $h < i < j$ such that $\sigma_h \sigma_i \sigma_j$ forms a 132 pattern.

Moreover σ_i reaches σ_k if there is a subsequence $\sigma_i, \sigma_{i+a_1}, \sigma_{i+a_2}, \ldots \sigma_{i+a_t}, \sigma_k$ of entries where $i < i + a_1 < i + a_2 < \cdots < i + a_t < k$ so that each entry in the subsequence beats the next one. When σ_i beats σ_j it also reaches σ_j.

Let v_i be the vertex of $\beta(0,1)$-tree T, that is associated with σ_i.

Label of v_i is the number of descendants of v_i (including itself) such that at least there is one index i for the descendant v_j such that σ_j reaches σ_k where $k > i$.

It means that we are looking for the number of descendants of v_i that reach an entry that is after σ_i, which will be the label of v_i.

Example 3.15. Let $\sigma = 5716243 \in D_7(1342)$. We see the subsequence 576, therefore 7 beats 6. We also see the subsequence 162, therefore 6 beats 2. We can say that 7 reaches 2.

Let $\sigma \in D_n(1342)$. The mapping

$$F : D_n(1342) \to \beta_n(0,1)$$

is defined as follows:

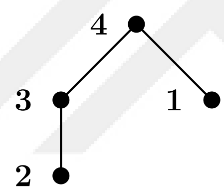
- First, determine $N(\sigma) \in D_n(132)$ using Definition 3.16.
- Second, obtain $g(N(\sigma)) \in \beta_n^0(0,1)$ using Proposition 3.1.

- Third, update the assigned entries of vertices in $g(N(\sigma))$ based on the function $((\sigma) \circ ((N(\sigma))^{-1})$.

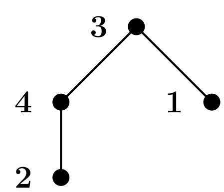
- Fourth, apply the labeling defined in Definition 3.17.

We will see the process with the next examples. Assigned entries are shown in bold text.

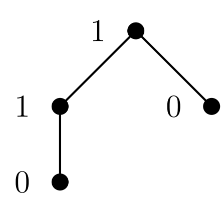
Example 3.16. Let $\sigma = 2413 \in D_4(1342)$.
First step: $N(\sigma) = 2314$.
Second step: We see that $g(N(\sigma)) = g(2314)$ is



Third step: $\sigma(N(\sigma))^{-1} = [2413][3124] = 1243$. It means that 3 goes to 4 in the tree, 4 goes to 3.

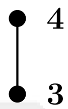


Fourth step: For the labeling, we check the 132 patterns in σ. There is only one of them, which is 243. It means that 4 reaches 3. Thus $\ell(4) = 1$. By the root conditions we have the following tree:

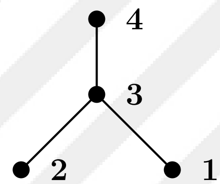


Example 3.17. Let $\sigma = 2143 \in D_4(1342)$.
First step: We see that $N(\sigma) = 2134$.
Second step: We want to find $g(N(\sigma))$.

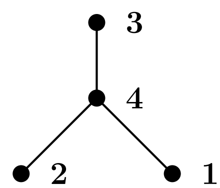
$\bullet$ **4** $N(\sigma)' = 213$. It is indecomposable .



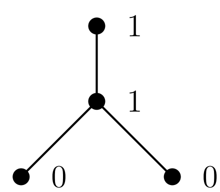
$N(\sigma)'' = 21$. It is decomposable. We have the tree



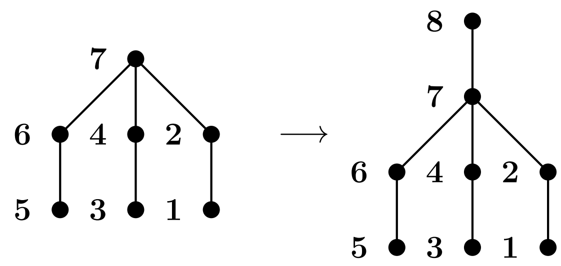
Third step: $\sigma(N(\sigma))^{-1} = [2143][2134] = 1243$. Thus we change the places of 3 and 4.



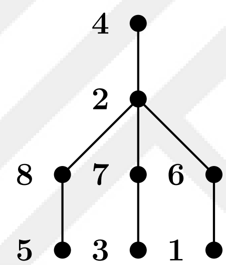
Fourth step: Only 132 pattern formed is 243 for thus 4 reaches 3. $\ell(4) = 1$.



Example 3.18. Let $\sigma = 58371624 \in D_8(1342)$.
First step: Observe that $N(\sigma) = 56341278$ and $(N(\sigma))^{-1} = 56341278$.
Second step: We want to find $g(N(\sigma))$. $N(\sigma)' = 5634127$ and it is indecomposable. **7** $\bullet$ $N(\sigma)' = 563412$ and it is decomposable where $p_{\langle 1 \rangle} = 56, p_{\langle 2 \rangle} = 34, p_{\langle 3 \rangle} = 12$. Meaning that we have three branches and every $p_{\langle i \rangle}$ is indecomposable, thus every branch is a single path.



Third step: Proceed to $\sigma(N(\sigma))^{-1} = [58371624][56341278] = 16375824$. We change the associated entries according to this bijection.



Fourth step:

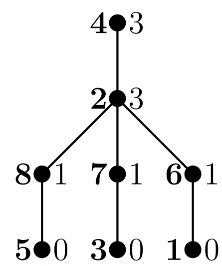
Label of 8: 8 only has one descendant which is 5. 5 can not reach anywhere since there is no entry on its left. There is the subsequence 586, meaning 8 reaches to 6. Thus $\ell(8) = 1$.

Label of 7: There is only one descendant which is 3. 3 does not reach anyhwere since both 1 and 2 on its right. 132 formed with 374. Thus 7 reaches 4, $\ell(7) = 1$.

Label of 6: Similarly, by the subsequence 162 and since 1 can not reach anywhere, $\ell(6) = 1$.

Label of 2: We see that 2 has 6 descendants, we already discussed that 5, 3, and 1 can not reach anywhere. We look at 8, 7, and 6. There is only one entry after 2, which is 4. 8 reaches 4 since 8 beats 6 by pattern 586, and 6 beats 4 by pattern 364. 7 beats 4 by the pattern 374. We also showed that 6 beats 4. 2 can not reach 4. There are three descendants of 2 that reach 4. $\ell(2) = 3$.

Label of 4, 5, 3, 1: By the root condition, $\ell(4) = 3$. $\ell(5) = \ell(3) = \ell(1) = 0$ since they are leaves. We have $F(58371624)$ as following:



3.2.2.3 Special case of F

There exists a special case for the function F concerning a subset of $D_n(1342)$. Let us define $I_n(1342) = \{\sigma \in D_n(1342) \mid \sigma_1 = 1\}$. $F(\sigma)$ for $\sigma \in I_n(1342)$ has a characteristic. It follows from the fact that there is only one left-to-right minimum in σ. Consequently, all permutations within this subset with the same length share the same normalization.

Proposition 3.2. There exists a bijection between $I_n(1342)$ and the set of $\beta_n(0,1)$-trees with one single path.

We will follow the steps of F.

<u>First step:</u> We want to understand the normalization of $\sigma \in I_n(1342)$. We see that $N(\sigma) = 12\ldots(n-1)n$ for every $\sigma \in I_n(1342)$.

<u>Second step:</u> We find $g(N(\sigma))$. We look at $g(N(\sigma)) = g(12\ldots(n-1)n)$, and it will be a single path plane tree with n vertices, by the definition of g. Note that g also assigns i to ith vertex from the bottom.

<u>Third step:</u> We use $((\sigma) \circ ((N(\sigma))^{-1})$ and change the assigned entries. We see that $N(\sigma)$ is the identity function, therefore $((\sigma) \circ ((N(\sigma))^{-1}) = \sigma$. We assign σ_i to ith vertex from the bottom.

<u>Fourth step:</u> We want find the labeling. By the third step, we know that every descendant of σ_i is σ_k where $k < i$. We also know that $\sigma_1 = 1$. Therefore if we

look at the number of σ_k that is larger than at least an entry on the right of σ_i, we find the number of descendants that reaches an entry after σ_i.

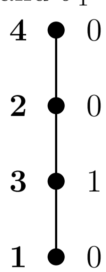
Therefore we can define

$$f : I_n(1342) \longrightarrow \text{Single path } \beta_n(0,1)\text{-trees}$$

as follows:

1. Assign the entry σ_i to the ith vertex from the bottom.
2. Find the label of vertices as follows. Let v_i be the ith vertex from the bottom:
   - $i < n$, then
   $$\ell(v_i) = |\{j \leq i \mid \sigma_j > \sigma_s \text{ for at least one } s > i\}|$$
   - $i = n$, then
   $$\ell(v_i) = \ell(n-1).$$

Example 3.19. Let $\sigma = 1324 \in I_4(1342)$. It is indecomposable, it avoids 1342 and $\sigma_1 = 1$. Assigned entries are shown in bold text.



$\ell(1) = 0$ since 1 is smaller than any entry. $\ell(2) = 1$ since $\sigma_2 = 3 > 2$. $l(3) = 0$ since nothing on the left of $\sigma_3 = 2$ including itself is not larger than 4. $\ell(4) = 0$ by definition.

Next, we will examine the inverse of this particular case. Let's define the mapping

$$f^{-1} : \text{Single path } \beta_n(0,1)\text{-trees} \longrightarrow I_n(1342)$$

as follows:

1. The vertex assigned to n marks the beginning of a positive sequence, as it is larger than any other entry.

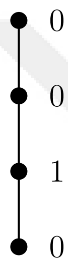
2. If there is no such vertex, the last vertex is assigned to n.

3. After locating n, decrease the labels of all subsequent vertices by 1.

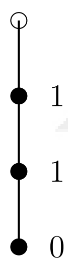
4. Adjust the label of the last vertex to satisfy the root condition.

5. Repeat the process recursively.

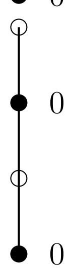
Example 3.20. We look at the following tree.



There is no positive sequence thus $\sigma_4 = 4$. Deleting the last node, we have the following new tree.



In order to satisfy the root condition, the root label is changed. The new tree has 3 vertices, the second vertex starts a positive sequence. Thus $\sigma_2 = 3$



By definition, the following label is decremented by 1. The new tree has 2 vertices. There is no positive sequence. It follows that $\sigma_3 = 2$ and $\sigma_1 = 1$.

Thus we have the permutation $\sigma = 1324 \in I_4(1342)$.

3.2.2.4 Definition of F^{-1}

We want to show that

$$F^{-1} : \beta_n(0,1) \longrightarrow D_n(1342)$$

exists. We will use induction, assume that every $T' \in \beta_{n-1}(0,1)$ has a unique $F^{-1}(T') \in D_{n-1}(1342)$. Let $T \in \beta_n(0,1)$, v_1 be the leftmost leaf of T and x be the parent of v_1. We show the uniqueness of $F^{-1}(T) \in D_n(1342)$ for any $T \in \beta_n(0,1)$ in four cases. We can see them as follows:

	Children of x	Label of x	Labels of ancestors of x
Case 1	More than v_1	Positive or zero	Positive or zero
Case 2	Only v_1	Zero	Positive or zero
Case 3	Only v_1	Positive	One or more of them have zero
Case 4	Only v_1	Positive	All of them have positive

Note that g^{-1} takes v_1 to σ_1 since $N(\sigma)$ does not change the first entry.

1. v_1 is not the only child of x.
    - Delete v_1, we still have a valid tree T' since $\ell(v_1) = 0$.
    - T' is a valid tree with a unique preimage $F(\sigma') = T'$.
    - Compare the entries of σ' with σ_1. Add 1 to every entry that is larger than σ_1 or equal σ_1.
    - We put σ_1 in the first entry; we have the preimage of T.
2. v_1 is the only child of x and $\ell(x) = 0$.
    - Delete v_1, we still a valid tree T'.
    - Proceed as in the first case.
3. v_1 is the only child of x, $\ell(x) \neq 0$, and an ancestor of x has label 0.
    - Let u be the closest ancestor with label 0.
    - u is not the root. Otherwise, it should have all children with the label 0.
    - Remove the subtree T_u that has u as its root.
    - Let T' be T where T_u is removed. It has a unique preimage by induction.
    - There is one possibility of a set of entries from σ assigned to T_u. Let A be the assigned entries of T_u by $N(\sigma)$ and B the assigned entries of T_u by σ. If $A \neq B$, then there is a 132 pattern that starts in T_u and ends outside of T_u. It is a contradiction since $\ell(u) = 0$.

- T_u is a valid $\beta(0,1)$-tree if we change the label of u to sum of its children. It has a unique preimage σ^u by induction.
- Thus, we showed that T has a unique preimage.

4. v_1 is the only child of x, $\ell(x) \neq 0$ and no ancestor of x has 0 label. If n is not associated with the root vertex, then all of its ancestors and itself have a positive label, and n is the leftmost entry with such a property. These claims need proof. We will present them after we describe this case.
   - x is associated with n. It means that n is the second entry of the preimage σ.
   - We connect v_1 with the parent of x and decrease the label by 1 of ancestors of x. It means that we delete x.
   - The new tree T' has a unique preimage σ' by the induction.
   - Insert n to the second position of σ'. We get the preimage of T. It does not create a 1342 pattern since there is only one entry before n, and $\sigma' \in D_n(1342)$.

We use the following lemmas for the fourth case.

Lemma 3.3. Let $\sigma \in D_n(1342)$. Assume that $\sigma_n \neq n$. It means that n is not associated with the root. Let v_n be the vertex that is associated with n. Each ancestor of v_n has a positive label under F.

Proof. We want to show that n reaches an entry after any of the ancestors. Let $a_1 > \ldots > a_r > a_{r+1} > \ldots > a_m = 1$ be the left-to-right minima of σ. Assume that n is located between a_r and a_{r+1}.

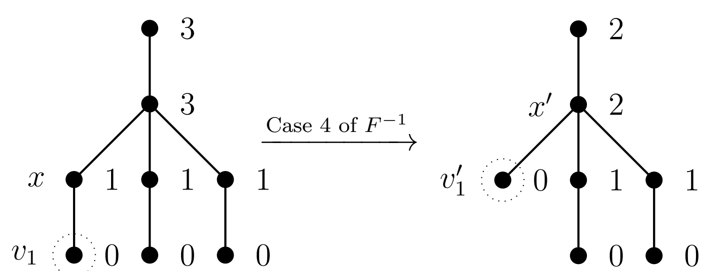
$$a_r \cdots\cdots\; n \;\cdots\cdots a_{r+1} \cdots\cdots\cdots\cdots a_{r_1} \cdots\; y_1 \cdots a_{r_1+1} \cdots\cdots\; y_2$$

n beats everything between itself and a_{r+1}. Here, a_r plays the role of 1 since the mentioned entries are larger than a_r. We claim that n beats an entry after a_{r+1}. Assume that n does not beat an entry after a_{r+1}. It means that every entry larger than a_{r+1} is on the left of itself. It would imply that σ is decomposable. Therefore it is a contradiction. Let y_1 be in between the left-to-right minima a_{r_1} and a_{r_1+1}. Similarly, y_1 beats an entry after a_{r_1+1}. Eventually, we bypass all the left-to-right minima and get to the chosen ancestor's right. n reaches an entry on the right of chosen left-to-right minimum. Therefore every ancestor of n has a positive label. $\square$

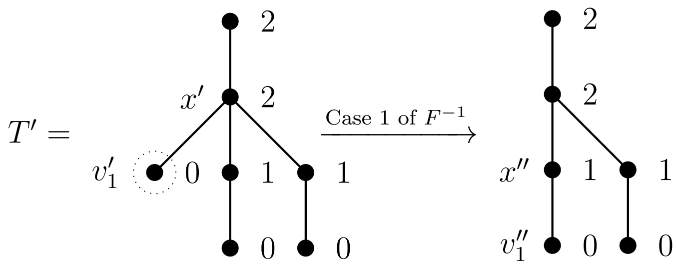
Lemma 3.4. Let $\sigma \in D_n(1342)$ and $\sigma_n \neq n$. n is the leftmost entry of σ, which has the property that each of its ancestors, including itself, has a positive label.

Proof. Assume that σ_k and n have the mentioned property, and σ_k is on the left of n. If more entries have the same property, choose the rightmost one. If σ_k beats an entry on the right of n, then we have the subsequence $x\sigma_k ny$, which forms a 1342 pattern. It is a contradiction. Therefore σ_k does not beat such element y. Since σ_k has the property, it should reach an entry after n. Thus it should beat an entry v between σ_k and n. Let v beat the element w with the subsequence tvw. We know that w is after n; therefore, $tvnw$ forms a 1342 pattern. It is a contradiction. Therefore n is the leftmost entry with such property. $\square$

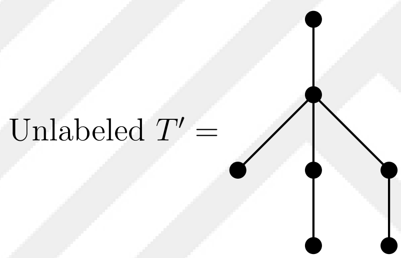
Example 3.21. Let T be the following tree:



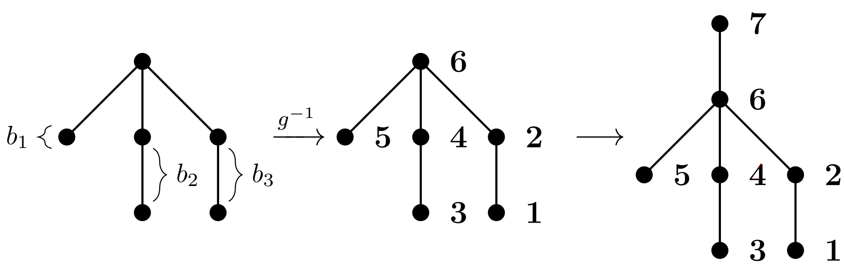
We see that v_1 is the only child of x, $\ell(x) \neq 0$, and all of the ancestors of x have a positive label. It is the case 4, thus for $F(\sigma) = T$, $\sigma_2 = 8$ since T has 8 vertices. We will follow it by deleting x and decreasing all of the ancestors' labels by 1.



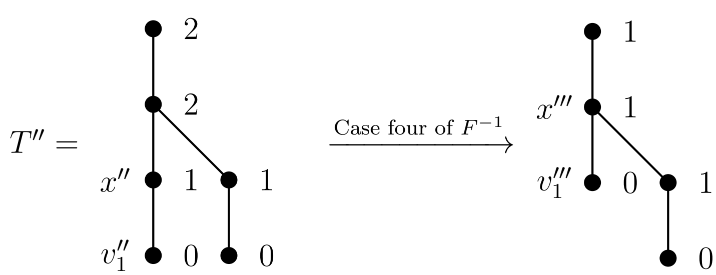
We see that v_1' is not the only child of x'. It is the case 1 of the F^{-1}. We look at the unlabeled version of T'.



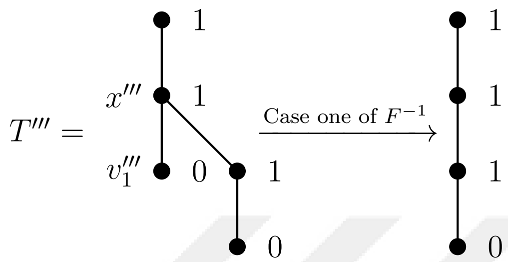
We find the leftmost entry as follows:



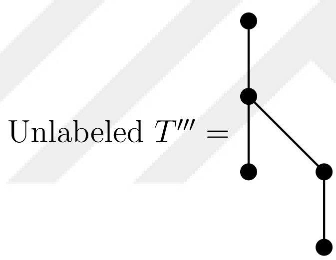
Thus for $F(\sigma') = T'$, $\sigma_1' = 5$. Since T' is also the case two, we delete the v_1'. We have



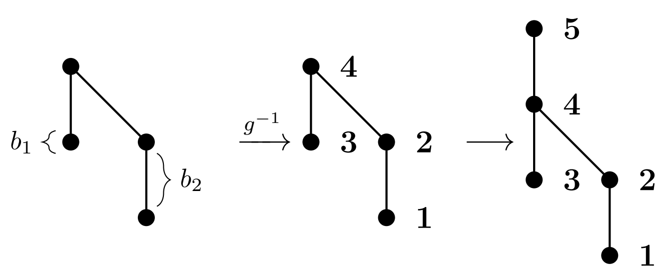
Only child of x'' is v_1'', hence by case 4, we should delete x'', decrease the label of ancestors by 1. x'' is associated by 6 by case 4, meaning for $F(\sigma'') = T''$, $\sigma_2'' = 6$.



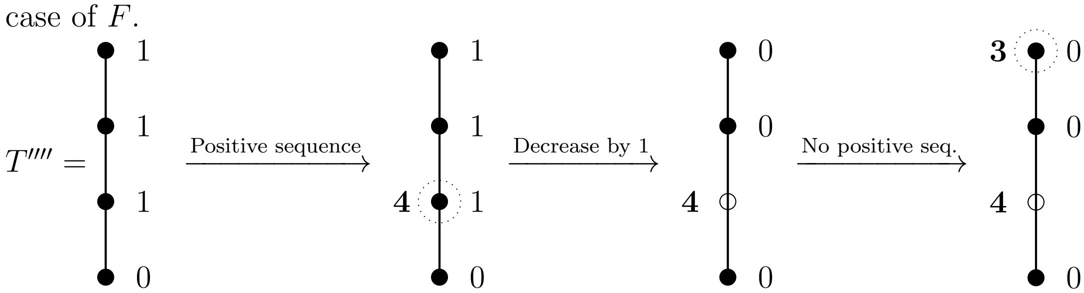
v_1''' is not the only child of x'''. Thus by case 1, we should find the leftmost entry.

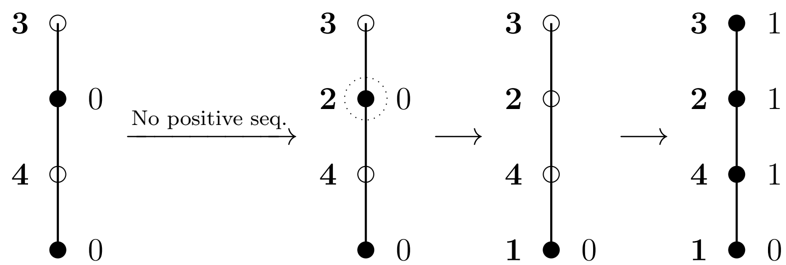


We find the preimage as follows:



Thus for $F(\sigma''') = T'''$, $\sigma_1''' = 3$. Since T''' has the case 1, we delete v_1'''. We have a single path $\beta(0,1)-$tree. We will proceed according to f, which is the special case of F.





Hence we conclude that $F(1423) = T''''$. Now we will proceed to T'''. $T'''' \xrightarrow{\text{Case 1}} T'''$ and $\sigma_1''' = 3$. Adding 1 to 3 and 4 in 1423, we have $\sigma''' = 31524$. We observe that $T''' \xrightarrow{\text{Case 4}} T''$ and $\sigma_2'' = 6$, we have that $\sigma'' = 361524$. We see that $T'' \xrightarrow{\text{Case 1}} T'$ and find that $\sigma_1' = 5$. We add 1 to the appropriate entries of σ'', then $\sigma' = 5371624$. Finally, $T' \xrightarrow{\text{Case 4}} T$, meaning we plug 8 to the second entry, therefore $\sigma = 58371624 \in D_8(1342)$.

Chapter 4

Stanley-Wilf Conjecture

Let $\tau \in S_k$. Stanley-Wilf's conjecture claims that $|S_n(\tau)|$ has an exponential upper bound. Klazar and Marcus [43] showed that the Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. Füredi-Hajnal conjecture has been proved by Marcus and Tardos [15]. First, we will introduce some definitions to understand the Füredi-Hajnal conjecture. Second, we show that the Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture.

4.1 Preliminaries

Definition 4.1. Let A be a matrix. If A has only 0 and 1 as its entries, then it is called a 0-1 matrix.

Definition 4.2. Let P be a $k \times k$ 0-1 matrix. If P has exactly one 1-entry in every row and column, then it is called a permutation matrix.

Example 4.1. Let

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that P_2 is not a permutation matrix since the first row and column have two 1-entry.

Definition 4.3. Let A be a matrix. A submatrix of A is a matrix formed by deleting some rows and columns of A.

Example 4.2. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

If we delete the first row and third column, we get the following submatrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Definition 4.4. Let A and P be 0-1 matrices with the size $m \times n$ and $k \times l$, respectively. A contains P if A has a $k \times l$ submatrix D so that $D_{ij} = 1$ whenever $P_{ij} = 1$ for all i, j. If A does not contain P, then A avoids P.

Note that only the positions of 1-entries are important. Let P be a $k \times k$ permutation matrix. We denote the set of $n \times n$ 0-1 matrices that avoids P as $\mathcal{M}_n(P)$.

Example 4.3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

If we delete the second row and the second column of A, then we arrive at the following submatrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Since this submatrix has the 1-entry in the same positions with P, it implies that A contains P.

Definition 4.5. Let P be a $k \times k$ permutation matrix and $A \in \mathcal{M}_n(P)$. We define $\mathrm{num}_1(A)$ as the number of 1 entries in A and $f(n, P) = \max\limits_{A \in \mathcal{M}_n(P)} \mathrm{num}_1(A)$.

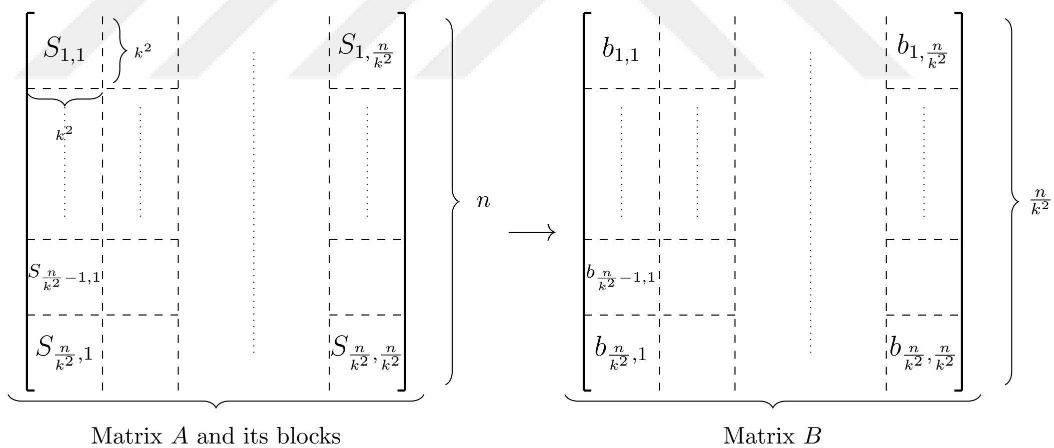
4.2 Füredi-Hajnal conjecture

Füredi-Hajnal conjecture claims that there exists a constant c_P that depends only on $k \times k$ permutation matrix P such that

$$f(n, P) \leq c_P \cdot n \text{ for all } n \geq 1$$

The proof uses induction. First, let P be a $k \times k$ permutation matrix and $A \in \mathcal{M}_n(P)$. Assume that k^2 divides n.

We define S_{ij} to be square submatrix of A consisting of the entries $a_{i'j'}$ where $i' \in [k^2(i-1)+1, k^2(i-1)+k^2]$ and $j' \in [k^2(j-1)+1, k^2(j-1)+k^2]$. It means dividing A into $k^2 \times k^2$ blocks. We can see the matrix as follows:



Let $B = (b_{ij})$ be the $\frac{n}{k^2} \times \frac{n}{k^2}$ 0-1 matrix. We define $b_{ij} = 0$ if and only if all the entries of S_{ij} is 0. Otherwise we define $b_{ij} = 1$.

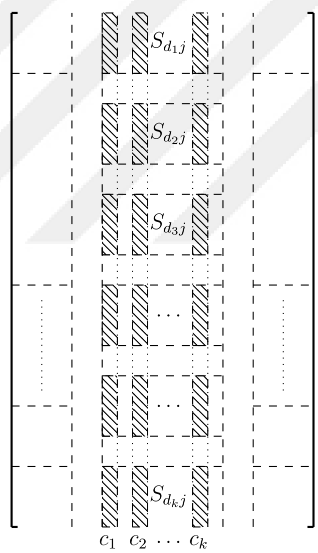
Lemma 4.1. The matrix B avoids P.

Proof. Assume that B contains P. Consider the entries of B representing P. We choose an arbitrary 1-entry from the corresponding block for every 1-entry in P. It would imply that A contains P, which is a contradiction. □

Definition 4.6. We say that a block of A (with the size $k^2 \times k^2$) is wide if it contains 1-entry in at least k different columns. Similarly, a block of A is tall if it contains 1-entry in at least k different rows.

Lemma 4.2. Consider the set of column blocks of A, the set $C_j = \{S_{ij} : i = 1, 2, \ldots, \frac{n}{k^2}\}$. The number of blocks in C_j that is wide is less than $k\binom{k^2}{k}$.

Proof. First, note that we fix the columns of the blocks we choose. We look at the blocks that correspond to jth column of matrix B. We know that there are k^2 columns. We choose k columns from k^2 columns, there are $\binom{k^2}{k}$ options. Assume that there are $k\binom{k^2}{k}$ wide blocks in C_j. By the pigeonhole principle, k blocks in C_j have $1-$entry in the same columns. Suppose we order the columns from left to right as $c_1, c_2, \ldots, c_k$. We will describe the matrix.



Let $S_{d_1j}, \ldots, S_{d_kj}$ be those blocks where $1 \leq d_1 < d_2 < \ldots < d_k \leq \frac{n}{k^2}$. As seen in the picture, for every 1-entry in P, we can choose the needed column accordingly. It is possible since there are k possibilities for every position. Therefore we can find a submatrix of A that corresponds to P. It is a contradiction. $\square$

We can use the same argument for the row blocks. The number of tall blocks in the set of row blocks $R_i = \{S_{ij} : j = 1, 2, \ldots, \frac{n}{k^2}\}$ is less than $k\binom{k^2}{k}$.

Lemma 4.3. Let P be a $k \times k$ permutation matrix and assume that n is divisible by k^2. Then we have

$$f(n, P) \leq (k-1)^2 f\left(\frac{n}{k^2}, P\right) + 2k^3\binom{k^2}{k}n.$$

Proof. Let T_1 be the set of wide blocks, T_2 be the set of tall blocks, and T_3 be the set of nonempty blocks that are neither wide nor tall. Note that number of sets of column or row blocks is $\frac{n}{k^2}$. Therefore

$$|T_1| \leq \frac{n}{k^2} k \binom{k^2}{k},$$

$$|T_2| \leq \frac{n}{k^2} k \binom{k^2}{k},$$

and

$$|T_3| \leq f\Big(\frac{n}{k^2}, P\Big)$$

If a block in T_3 is nonempty, it has at most $(k-1)$ entries in every row and column. It implies it has at most $(k-1)^2$ many $1-$entry. Note that wide or tall blocks have at most k^4 many $1-$entry. Thus

$$f(n,P) \leq k^4 \Big(2\frac{n}{k^2} k \binom{k^2}{k}\Big) + (k-1)^2 f\Big(\frac{n}{k^2}, P\Big)$$
$$f(n,P) \leq 2k^3 \binom{k^2}{k} n + (k-1)^2 f\Big(\frac{n}{k^2}, P\Big).$$

□

Theorem 4.1. Let P be a $k \times k$ permutation matrix. We have

$$f(n,P) \leq 2k^4 \binom{k^2}{k} n$$

for all $n \geq 1$.

Proof. We will use induction on n. Let n' be the largest integer less than n divisible by k^2. By the previous lemma, we have

$$f(n,P) \leq f(n',P) + 2k^2 n$$
$$f(n,P) \leq (k-1)^2 f\Big(\frac{n'}{k^2}, P\Big) + 2k^3 \binom{k^2}{k} n' + 2k^2 n$$

By induction,

$$f(n,P) \leq (k-1)^2 \Big[2k^4 \binom{k^2}{k} \frac{n}{k^2}\Big] + 2k^3 \binom{k^2}{k} n' + 2k^2 n$$

Since $n' \leq n$,

$$f(n,P) \leq 2k^2\binom{k^2}{k}n((k-1)^2+k-1)$$
$$f(n,P) \leq 2k^4\binom{k^2}{k}n.$$

$\square$

4.3 Proof of Stanley-Wilf conjecture

The Stanley-Wilf conjecture claims that for any $\tau \in S_k$, there exists a constant c_τ that only depends on τ such that

$$|S_n(\tau)| \leq c_\tau^n \text{ for all } n \geq 1.$$

We will define the permutation matrix corresponding to $\sigma \in S_n$.

Definition 4.7. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$. We define the matrix $B_\sigma = (b_{ij})_{1\leq i,j\leq n}$ as follows:

$$b_{ij} = \begin{cases} 1 & \text{if } \sigma_i = j \\ 0 & \text{otherwise} \end{cases}$$

We call B_σ corresponding matrix of σ. Note that B_σ is always a permutation matrix since every element of $[n]$ occur once.

Example 4.4. Let $\sigma^1 = 132 \in S_3$ and $\sigma^2 = 1342 \in S_4$. We can see that

$$B_{\sigma^1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$B_{\sigma^2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Let B_τ be the corresponding matrix of τ. It is clear that

$$|S_n(\tau)| \leq |\mathcal{M}_n(B_\tau)|$$

since not every 0-1 matrix does not correspond to a permutation that avoids B_τ. We know from Füredi-Hajnal conjecture that

$$f(n, B_\tau) \leq 2k^4 \binom{k^2}{k} n.$$

Let $A \in \mathcal{M}_n(B_\tau)$. We will cut A into 2×2 blocks. We replace the block with 0 if all of its entries are 0. Otherwise, we place it with 1. We have a $\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil$ matrix. This new matrix also avoids B_τ. Therefore

$$f\Big(\Big\lceil \frac{n}{2} \Big\rceil, B_\tau\Big) \leq 2k^4 \binom{k^2}{k} \Big\lceil \frac{n}{2} \Big\rceil.$$

We know that all the 1-entries come from a nonzero block. There are $2^4 - 1 = 15$ many nonzero blocks since they have the size 2×2. Thus

$$|\mathcal{M}_n(B_\tau)| \leq 15^{2k^4 \binom{k^2}{k} \lceil \frac{n}{2} \rceil} \cdot |\mathcal{M}_{\lceil \frac{n}{2} \rceil}(B_\tau)|.$$

Therefore by iteration, we have,

$$|S_n(\tau)| \leq |\mathcal{M}_n(B_\tau)| \leq (15^{2k^4 \binom{k^2}{k}})^n.$$

Appendix A

Combinatorics background

We use $\binom{n}{k}$ to denote the number of k element subsets of an n element set. We know that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Note that $\binom{n}{0} = 1$.

A.1 Generalized binomial theorem

First, we show the binomial theorem, which is defined for positive integers. For the generating function method, we need the definition for any real number. The generalized binomial theorem is defined for any real number, therefore we use to explain it in the next part.

Proposition A.1. (Pascal's identity) Let $n, k \in \mathbb{Z}^+$, then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof. Assume that $k \leq n$.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(n-k)!k!}$$

$$= \frac{n!k}{k(k-1)!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)(n-k)!}$$
$$= \frac{(n-k+1+k)n!}{(n-k+1)!k!} = \frac{(n+1)n!}{(n+1-k)!k!} = \binom{n+1}{k}.$$

□

Theorem A.1. (Binomial theorem) Let $n \in \mathbb{Z}^+$ and $x, y \in \mathbb{R}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof. The proof is by induction on n. For the $n = 1$ case we have,

$$(x+y) = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k$$
$$= \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x + y$$

Assume that the theorem holds for n.

Note that

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + y^n\right)$$

$$= x^{n+1} + \left[1 + \binom{n}{1}\right]x^n y + \left[\binom{n}{1} + \binom{n}{2}\right]x^{n-1}y^2 + \ldots$$
$$\ldots + \left[\binom{n}{k-1} + \binom{n}{k}\right]x^{n-k+1}y^k + \ldots + y^{n+1}$$

Summing up by the previous proposition,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Therefore it implies that

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k.$$

□

Definition A.1. Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. We define
$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))}{n!}$$

Theorem A.2. (Generalized binomial theorem)
$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$
where α is any real number.

Proof. We will use the Maclaurin series, we know that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Let $f(x) = (1+x)^\alpha$, then
$$f^{(n)}(x) = \alpha(\alpha-1)\dots(\alpha-(n-1))(1+x)^{\alpha-n}$$
and it implies that
$$f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-(n-1))(1)^{\alpha-n} = \alpha(\alpha-1)\dots(\alpha-(n-1)).$$

Therefore
$$f(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-(n-1))}{n!} x^n = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

□

A.2 Catalan numbers

Generating function method is usually used to find the enumeration of the set of permutations that avoids a pattern. We will first see how it is used in general cases. Later, we will show how to determine the generating function of Catalan numbers.

Definition A.2. The generating function $G(x)$ of $\{a_n\}_{n\geq 0}$ is defined to be the formal power series

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example A.1. We want to use the generation function method to find the number of subsets with r element from a set with n elements. For any element, we have $(1+x)$ since the element is included or not included. Therefore by binomial theorem,

$$G(x) = (1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r.$$

Definition A.3. The n^{th} Catalan number is defined as

$$c_n = \sum_{i=1}^{n} c_{i-1} c_{n-i}$$

where $c_0 = 1$.

We will find the exact formula of Catalan numbers by the generating function method.

Proposition A.2. The Catalan numbers are given by

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. We want to compute

$$C(x) = \sum_{n=0}^{\infty} c_n x^n.$$

From the previous recurrence for c_n, we have

$$C(x) = 1 + \sum_{n=1}^{\infty} c_n x^n = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} c_{k-1} c_{n-k} \right) x^n$$
$$C(x) = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} c_{k-1} c_{n-k} \right) x^n$$

By the Cauchy product formula for the series,

$$C^2(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_n c_{n-k} \right) x^n$$
$$C(x) = 1 + xC^2(x)$$
$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Since $\lim_{x\to 0} C(x) = 1$,

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Using Theorem A.2,

$$\begin{aligned}
C(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) &= \frac{1}{2x}\left(1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n \right) \\
&= \frac{1}{2x}\left(1 - 1 - \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n \right) = \frac{1}{2x}\left(\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-1)^{n+1} 2^{2n} x^n \right) \\
&= \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-1)^{n+1} 2^{2n-1} x^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2}) \dots (\frac{-2(n-1)+1}{2})}{n!} (-1)^{n+1} 2^{2n-1} x^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{1.3.5 \dots [2(n-1)-1]}{n! 2^n} (-1)^{n-1} (-1)^{n+1} 2^{2n-1} x^{n-1}
\end{aligned}$$

Changing variables, we get

$$\sum_{m=0}^{\infty} \frac{1.3.5 \dots (2m-1)}{(m+1)!} 2^m x^m = \sum_{m=0}^{\infty} \frac{1}{(m+1)} . \frac{1.3.5 \dots (2m-1) 2^m}{m!} x^m =$$
$$\sum_{m=0}^{\infty} \frac{1}{(m+1)} . \frac{1.3.5 \dots (2m-1) 2^m m!}{m! m!}$$

$$= \sum_{m=0}^{\infty} \frac{1}{(m+1)} . \frac{1.3.5\ldots(2m-1)2^m(1.2.3\ldots m)}{m!m!} x^m$$

$$= \sum_{m=0}^{\infty} \frac{1}{(m+1)} . \frac{1.3.5\ldots(2m-1)(2.4.6\ldots 2m)}{m!m!} x^m$$

$$= \sum_{m=0}^{\infty} \frac{1}{(m+1)} . \frac{1.2.3.4\ldots(2m-1)(2m)}{m!m!} x^m = \sum_{m=0}^{\infty} \frac{1}{(m+1)} . \frac{2m!}{m!m!} x^m$$

Thus,

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

$\square$

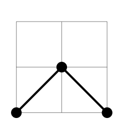
A.3 Dyck paths

Definition A.4. A Dyck path is a lattice path from (0,0) to (n, n) with the steps $(1, 0)$ and $(0, 1)$, which does not go above the diagonal line $y = x$.

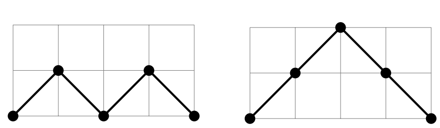
Equivalently, a Dyck path of length n is a sequence of steps (1,1) and (1,-1) that starts from $(0, 0)$ and ends at $(2n, 0)$ where it never goes below the $x-$axis.

We will mostly use the second definition.

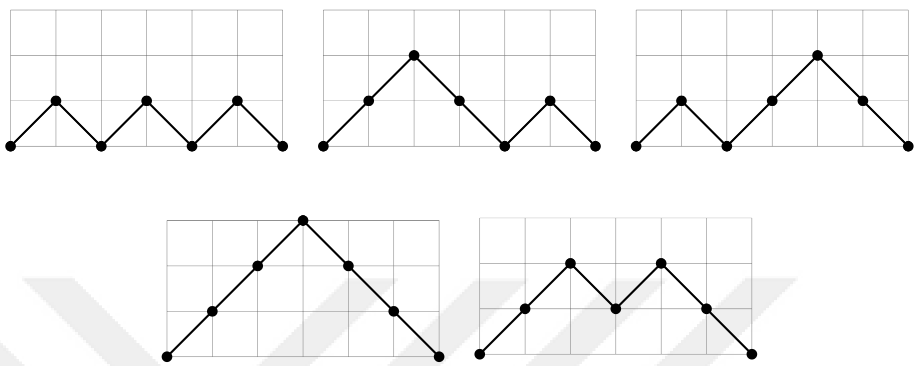
Example A.2. Let $n = 1$, then it ends at $(2, 0)$. There is one Dyck path.



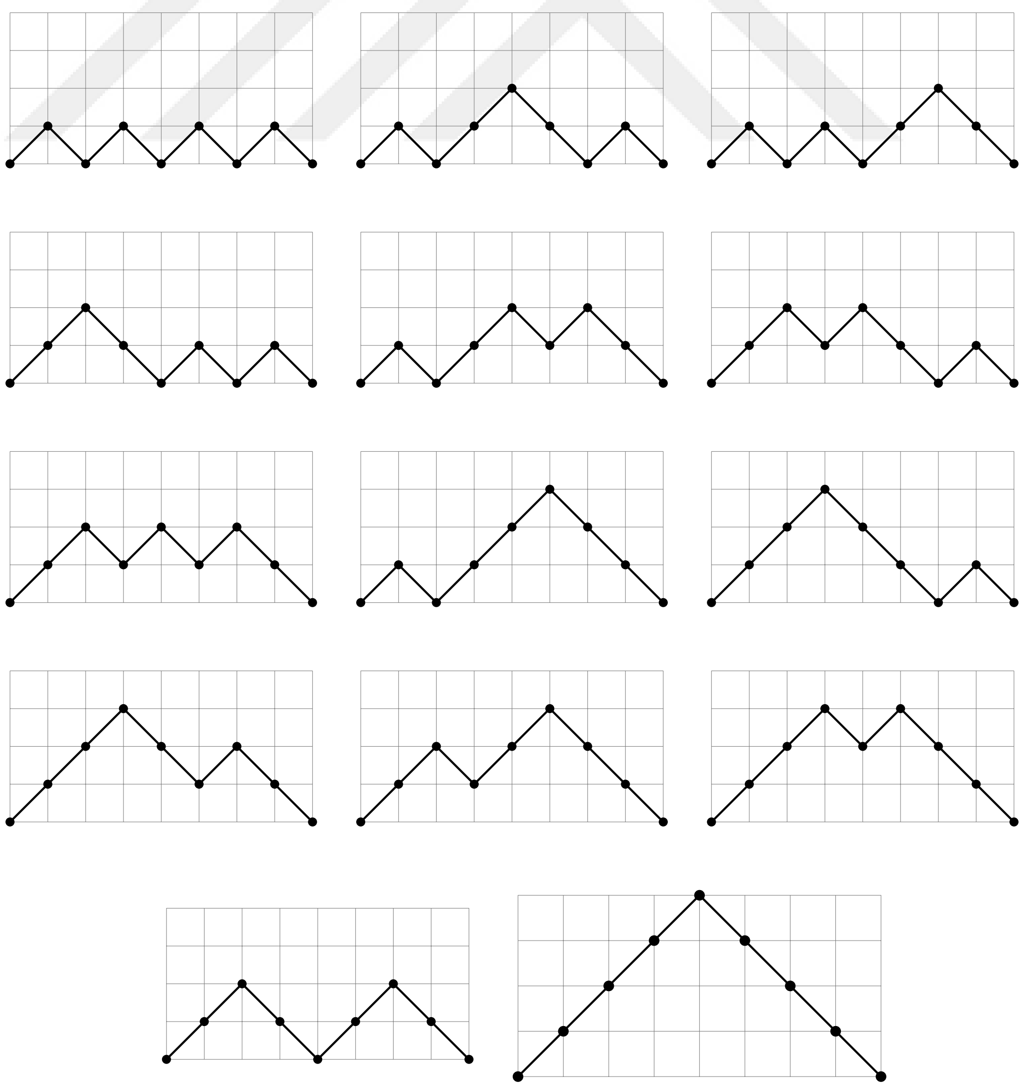
Let $n = 2$, then it ends at $(4, 0)$. There are two such Dyck paths.



Let $n = 3$, then it ends at $(6, 0)$. There are 5 of them.



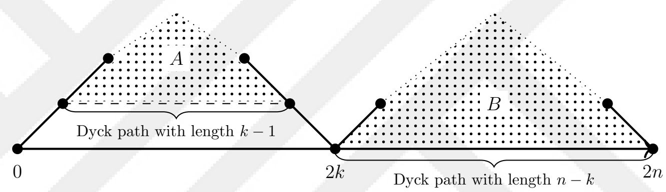
Let $n = 4$, it ends at $(8, 0)$. There are 14 of such paths.



Recall that n^{th} Catalan number is equal to $\frac{1}{n+1}\binom{2n}{n}$ where $c_0 = 1$. There are many distinct combinatorial objects enumerated by Catalan numbers. For an extensive list, see [44]. First, we will show that the Dyck paths are enumerated by Catalan numbers. Let $\mathcal{D}_n$ denote the set of Dyck paths with $2n$ steps.

Proposition A.3. The number of elements in the set $\mathcal{D}_n$ is equal to c_n.

Proof. Let $D \in \mathcal{D}_n$. Assume that $(2k, 0)$ is the coordinate where D touches the $x-$axis for the first time.



We see that $D = uAdB$, $A \in \mathcal{D}_{k-1}$ and $B \in \mathcal{D}_{n-k}$. We want to count the number of Dyck paths. We see that D can touch the $x-$axis on $2i$ where $i \in [n]$. Therefore for any i, corresponding $A \in \mathcal{D}_{i-1}$ and $B \in \mathcal{D}_{n-i}$. Therefore we conclude that

$$|\mathcal{D}_n| = \sum_{i=1}^{n} |\mathcal{D}_{i-1}||\mathcal{D}_{n-i}|$$

which is the same recursion satisfied by Catalan numbers. $\square$

Next, we will show the following bijections:

$$\text{Dyck paths} \longleftrightarrow \text{Parentheses} \longleftrightarrow \text{Plane trees}$$

Note that Dyck paths have $2n$ steps, parentheses have n pairs, and plane trees have $n+1$ vertices. See the subsection 3.2.1 for the definition of a plane tree.

Definition A.5. n pairs of parentheses is an ordering of $2n$ parentheses. It consists of n times opening parentheses, denoted as (, and n times closing parentheses, denoted as). It always starts with (and ends with). We denote the set of n pairs of parentheses as $\mathcal{P}_{|n|}$.

Proposition A.4. A bijection exists between length n Dyck paths and n pairs of parentheses.

Proof. Define

$$g : \mathcal{D}_n \longrightarrow \mathcal{P}_{|n|}$$

as $g(u) = ($ and $g(d) =)$. g is well-defined since there are n down and n up steps, and it always starts with the up step and ends with the down step. g is surjective since for any $\mathcal{P} \in \mathcal{P}_{|n|}$, we can find a Dyck path in $\mathcal{D}_n$ by mapping (to u and) to d. It is also injective since every Dyck path has a different ordering of the steps u and d. Therefore the preimage of a pair of parentheses is unique. Thus g is bijective. $\square$

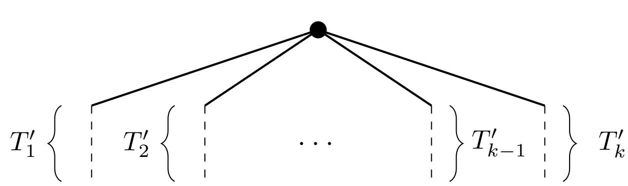
Proposition A.5. There exists a bijection between n pairs of parentheses and plane trees with $n + 1$ vertices.

We denote the set of plane trees with n vertices as T_n. We will define

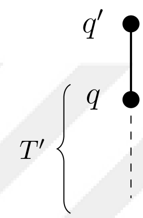
$$f : \mathcal{P}_{|n|} \longrightarrow T_{n+1}$$

in two cases. The first one is when $\mathcal{P} \in \mathcal{P}_{|n|}$ can be written as a joint of some pairs of parentheses. The second case is when $\mathcal{P}$ can not be written in that form. It is defined recursively.

- Note that $f(()) =$ 
- Assume that $\mathcal{P} \in \mathcal{P}_n$ and $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2 \ldots \mathcal{P}_k$ where $\mathcal{P}_i \in \mathcal{P}_{|i|}$. $f(\mathcal{P})$ is defined as follows: If $f(\mathcal{P}_i) = T_i$, then $f(\mathcal{P})$ is the tree where we glue the T_i's on their roots, from left to right. We denote T_i except its root as T_i'. We see the following picture:

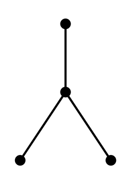


- Assume that $\mathcal{P}$ can not be written as a joint of some pairs of parentheses. We delete the last and first parenthesis of $\mathcal{P}$ and denote it as $\mathcal{P}'$. It will be valid pair of parentheses since it can not be jointly written. Suppose that $f(\mathcal{P}') = T'$, then we define $f(\mathcal{P})$ as adding a new node above the current root and drawing a new edge between them. It can be seen as the following:

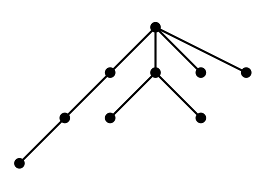


We will show that f is well-defined. We denote the number of vertices in the tree T as $|T|$. In the first case, $\sum_{i=1}^{k} |T_i'| = n + k$. We delete the root of every tree, we have n vertices. We glue them on a common root, we end up with a new tree that has $n+1$ vertices. In the second case of f, $\mathcal{P}'$ has $n-1$ pairs, meaning that T' has n vertices. Therefore the new tree has $n+1$ vertices. Thus our claim is true.

Example A.3. Let $\mathcal{P} = ((()))(()())()()$, we see that $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4$ where $\mathcal{P}_1 = ((())), \mathcal{P}_2 = (()())$ and $\mathcal{P}_3 = \mathcal{P}_4 = ()$. Note that $\mathcal{P}$ has 8 pairs and

$f(\mathcal{P}_1) =$  $f(\mathcal{P}_2) =$  $f(\mathcal{P}_3) = f(\mathcal{P}_4) =$ 

Therefore

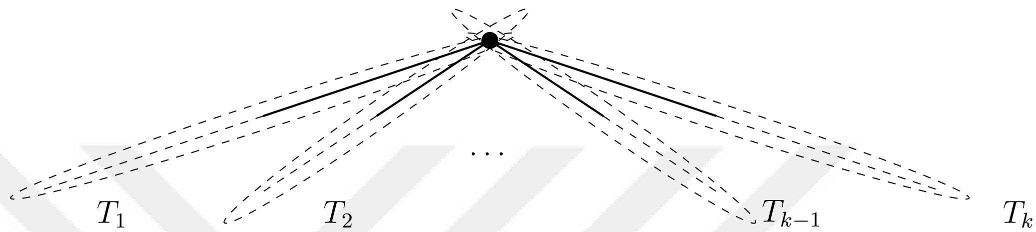
$f(\mathcal{P}) =$ 

See that $f(\mathcal{P})$ has 9 vertices.

Similarly, define

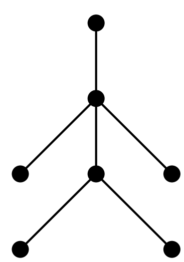
$$f^{-1} : T_{n+1} \longrightarrow \mathcal{P}_{|n|}$$

where  goes to (). Let T be a plane tree with $n+1$ vertices with k subtrees where $k \geq 2$. We look at the subtrees of T;

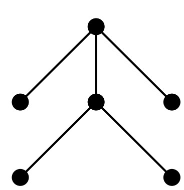


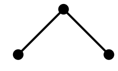
Let T_i be the plane tree such that we connect ith subtree of T to a new root. We define f^{-1} recursively. Assume that the number of vertices in T_i is $n_i + 1$ and $f^{-1}(T_i) = \mathcal{P}_i$ where $\mathcal{P}_i \in \mathcal{P}_{|n_i|}$. We define $f^{-1}(T) = \mathcal{P}_1\mathcal{P}_2 \dots \mathcal{P}_k$. It is well-defined since $\sum_{i=1}^{k} n_i = n$, then $f^{-1}(T) \in \mathcal{P}_{|n|}$. If $k = 1$, let T' be the subtree that we get by deleting the root of T, then $f^{-1}(T) = (\ f^{-1}(T')\ )$.

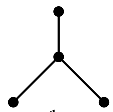
Example A.4. We will find the corresponding parentheses for the following plane tree T:



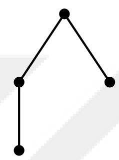
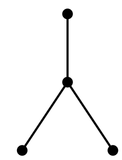
Since the root has one branch, we will delete the root and look at their preimages.

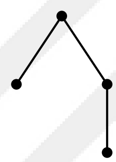
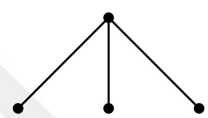


We know that  goes to (). Now we look at , its image under f^{-1} is

$()()$. Therefore preimage of  is $(()())$. Thus $\mathcal{P}_1 = ()$, $\mathcal{P}_2 = (()())$ and $\mathcal{P}_3 = ()$. By the definition of f^{-1}, parentheses corresponding to T is $(()(()())())$.

Example A.5. Let $n = 3$. We have the following pairs of parentheses and their matching plane trees.

$((()))$ $\longleftrightarrow$  , $(())()$ $\longleftrightarrow$  , $(()())$ $\longleftrightarrow$ 

$()(())$ $\longleftrightarrow$  , $()()()$ $\longleftrightarrow$ 

Therefore we can conclude that

$$|\mathcal{D}_n| = |\mathcal{P}_{|n|}| = |T_{n+1}| = c_n = \sum_{i=1}^{n} c_{i-1}c_{n-i} = \frac{1}{n+1}\binom{2n}{n}.$$

A.4 Functions and sets

We use bijections throughout the second and third sections. We review some definitions and important results.

Definition A.6. A function f is injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ where $x_1, x_2 \in X$. A function f is surjective if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. A function $f : X \longrightarrow Y$ is a bijection if f is injective and surjective.

Definition A.7. Let $f : X \to Y$ be a function. If there exists a function $g : Y \to X$ such that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$, then f is called invertible. The function g is called the inverse of f, and denoted as f^{-1}.

Proposition A.6. Let $f : X \to Y$. f is a bijection if and only if f^{-1} exists.

Next, we give an example of the bijective method, where we prove a well-known fact.

Proposition A.7. There are 2^n subsets of a set with n elements.

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set with n elements, $S \subseteq X$ and $B = \{b_1 b_2 \dots b_n \mid b_i \in \{0, 1\} \text{ for } 1 \leq i \leq n\}$. Define

$$f : S \longrightarrow B$$

as $f(S) = b_1 b_2 \dots b_n$ where

$$b_i = \begin{cases} 1 & \text{if } x_i \in S \\ 0 & \text{if } x_i \notin S. \end{cases}$$

We see that $|B| = 2^n$. f is a bijection, note that it is surjective since we can find a subset S whenever $b \in B$. It is also injective since $f^{-1}(b) = S$ is unique. Therefore our claim is implied. $\square$

Definition A.8. Let $a, b, c \in \mathbb{R}$ where $c \notin \mathbb{Z}^{\leq 0}$. The hypergeometric function is defined for $|H| < 1$, by the formal power series

$${}_2F_1(a, b, c, H) = \sum_{n=0}^{\infty} \frac{(a)_n (b_n)}{(c)_n} \cdot \frac{H^n}{n!}$$

where $(q)_n$ is defined by

$$(q)_n = \begin{cases} 1 & \text{if } n = 0 \\ q(q+1)\dots(q+n-1) & \text{if } n > 0. \end{cases}$$

A.5 Notations

- $[n] = \{1, 2, 3, \dots, n\}$

- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$
- $\mathbb{Z}^{\geq n} = \{n, n+1, n+2, n+3, \ldots\}$
- $\mathbb{Z}^{\leq n} = \{\ldots, n-3, n-2, n-1, n\}$

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