

NONLINEAR MODELING OF NON-NEWTONIAN HYDRODYNAMIC LUBRICATION

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By
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Nonlinear modeling of non-Newtonian hydrodynamic lubrication

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We certify that we have read this thesis and that in our opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Master of Science.



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ABSTRACT

NONLINEAR MODELING OF NON-NEWTONIAN HYDRODYNAMIC LUBRICATION

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Lubrication is essential to improve the performance of sliding surfaces. Power transmission in mechanical and biological systems rely on proper lubrication to minimize wear and energy losses. However, most practical applications involve conditions that cause or require the lubricant to exhibit non-Newtonian behavior, namely shear thinning and viscoelasticity. In this study a novel non-linear 1D Reynolds equation is proposed for modeling shear thinning and a 1D viscoelastic Reynolds equation is proposed to model viscoelastic effects. The models are compared with the direct numerical simulation (DNS) of thin films for different geometries. The results are in good qualitative and quantitative agreement indicating the simplified models are valid. The pressure presents strong variations as lubricant elasticity becomes significant, but stagnates as the polymer relaxation time becomes slow compared to the characteristic flow time. The net film pressure is shown to be a superposition of a Newtonian and viscoelastic component. The viscoelastic pressure varies as contact geometry changes. Surfaces with constant slope (plane slider) exhibit a constant decrease in film pressure whereas parabolic surfaces can enhance pressure for low relaxation times.

Keywords: non-Newtonian lubrication, shear thinning, viscoelasticity.

ÖZET

NEWTON OLMAYAN HİDRODİNAMİK YAĞLAMANIN DOĞRUSAL OLMAYAN MODELLEMESİ

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Yüzeylerin kayma performansını artırmak için yağlama çok önemlidir. Mekanik ve biyolojik sistemlerde güç iletimi verimi, aşınma ve enerji kayıplarını en aza indirmek için uygun yağlamaya dayanır. Fakat pratik uygulamaların çoğu, kayganlaştırıcının Newtonyen olmayan davranış sergilemesine ya da kayma incelenmesi ve viskoelastisite göstermesine neden olan koşulları içermektedir. Bu çalışmada, kayma incelenmesini ve viskoelastik etkileri modellemek için bir boyutlu viskoelastik Reynolds denklemi önerilmiştir. Model, farklı geometriler için ince filmlerin doğrudan sayısal simülasyonu (DNS) ile karşılaştırılmıştır. Sonuçlar sadeleştirilmiş modellerin niteliksel ve niceliksel yönden DNS sonuçları ile uyumlu olduğunu göstermektedir. Yağlayıcı madde elastisitesi önem kazandıkça basınçta yüksek değişimler meydana gelmekte, polimer relaksasyon zamanı karakteristik akış zamanına göre yavaş kaldıkça ise basınçta değişim gözlenmemektedir. Net film basıncının bir Newton ve viskoelastik bileşenin bir üst düşümü olduğu gösterilmiştir. Viskoelastik basınç, temas geometrisi değiştikçe değişmektedir. Sabit eğimli yüzeyler (düz sürgü), film basıncında sabit bir düşüş sergilerken, parabolik yüzeyler düşük gevşeme süreleri için basıncı artırabilir.

Anahtar sözcükler: Newtonyen olmayan, kesme inceltme, viskoelastisite.

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Contents

1	Introduction	1
1.1	Motivations	1
1.2	Piezoviscosity	4
1.3	Shear thinning	4
1.3.1	Polymer additives and shear stability	6
1.4	Viscoelasticity	8
1.5	Constitutive Relations	11
1.5.1	Generalized Maxwell	11
1.6	Aims and objectives	12
1.6.1	Aims	12
1.6.2	Objectives	13
2	Models and methodology	14
2.1	Reynolds equation	14

2.2	Generalized Reynolds equation	17
2.2.1	Shear thinning lubricants	17
2.2.2	Multigrade lubricants	20
2.3	Viscoelastic Reynolds equation	26
2.3.1	Linearized viscoelastic Reynolds equation	32
2.4	Bearing number	33
3	Numerical Methods	35
3.1	Non-linear Reynolds equation	36
3.2	Generalized Reynolds	39
3.2.1	Discretization	39
3.2.2	Convergence scheme	41
3.2.3	Viscoelastic Reynolds equation	43
3.3	Direct Numerical Simulation	44
3.3.1	Multigrid	44
3.3.2	Pressure implicit split operator (PISO)	45
3.3.3	Viscoelasticity	46
4	Results and Discussion	48
4.1	Surface geometry and Assumptions	48

4.1.1	Geometry	48
4.1.2	Assumptions	50
4.2	Model validation	51
4.2.1	Mesh quality	53
4.3	Load and coefficient of friction	54
4.4	Modified non-linear Reynolds equation	59
4.5	One-dimensional generalized Reynolds	64
4.6	Viscoelasticity	67
4.6.1	Viscoelastic pressure	72
4.7	Log conformation representation	79
5	Conclusion and perspectives	82
A		92
A.1	1D generalized Reynolds equation	92
A.2	Viscoelastic pressure	93
A.3	Linearized viscoelastic Reynolds	96
A.4	Log conformation representation	97

List of Figures

1.1	A comparison of the Ree-Eyring and Bair-Winer shear thinning models. Yield stress $\tau_0 = 5$ MPa and ambient viscosity $\eta_0 = 1$ Pa $\cdot$ s	5
4.1	Surface profiles adopted for the analysis of all models and non-Newtonian effects in this study.	50
4.2	Pressure profiles for the ridge (see figure 4.1) [1]. (a) Newtonian fluid $p_0 = 0.15 \times 10^9$ Pa, (b) Newtonian fluid $p_0 = 0.16 \times 10^9$ Pa and (c) Non-Newtonian fluid $p_0 = 0.16 \times 10^9$ Pa.	52
4.3	Maximum pressure (a, c, e) and average pressure (b, d, f) versus Λ for varying mesh sizes for the plane slider (a, b), the parabolic slider (c, d) and the ridge (e, f).	55
4.4	Maximum allowable load L (a, c, e) and coefficient of friction f_{tot} (b, d, f) versus Λ for the plane slider (a, b), the parabolic slider (c, d) and the ridge (e, f).	56
4.5	Pressure variation along the ridge, see figure (4.1), for $\Lambda = 150$. $\eta_0 = 0.01$ Pa $\cdot$ s, $p_0 = 0.15 \times 10^9$ GPa	58
4.6	A comparison of the viscosity terms appearing in equation (4.6). μ assumes no variation along y whereas, in μ_{eq} the dependency on y is accounted.	61

4.7	(a) Pressure profiles obtained using the nonlinear Reynolds with viscosity updated at different locations along y . (b) Pressure profile using the effective viscosity and strain rate at the mid-plane. $\eta_0 = 0.01\text{Pa.s}$, $p_0 = 1 \times 10^5 \text{ Pa}$, $\Lambda = 5000$ and $\tau_0 = 5 \times 10^5\text{Pa}$	61
4.8	Maximum allowable load L (a, c, e) and coefficient of friction f_{tot} (b, d, f) versus Λ using the modified viscosity (see equation 4.7) for the plane slider (a, b), the parabolic slider (c, d) and the ridge (e, f).	63
4.9	A plot of CPU time in seconds against increasing mesh size using the non-linear Reynolds.	64
4.10	A comparison of pressure profiles for the parabolic slider (see figure 4.1) at $\Lambda = 1500$ with the multigrade formulation at $\beta = 0.9$	64
4.11	A comparison of the pressure profiles for the plane, parabola and ridge (see figure 4.1). The plane slider at $De = 0.2$, $\beta = 0.2$ (a), the parabolic slider at $De = 0.2$, $\beta = 0.5$ and (c) the ridge at $De = 0.5$, $\beta = 0.2$	68
4.12	Pressure profiles for increasing De for the plane slider at $\beta = 0.2$. (a) $De = 0.5$, (b) $De = 1$, (c) $De = 1.5$, (d) $De = 2$, (e) $De = 2.5$ and (f) $De = 3$	70
4.13	Pressure profiles at increasing De for the parabolic slider at $\beta = 0.2$. (a) $De = 0.2$, (b) $De = 0.5$, (c) $De = 0.8$, (d) $De = 1$, (e) $De = 1.2$ and (f) $De = 1.5$	71
4.14	Pressure profiles for increasing De for the ridge at $\beta = 0.2$. (a) $De = 0.5$, (b) $De = 1$, (c) $De = 1.5$, (d) $De = 2$, (e) $De = 2.5$ and (f) $De = 3$	73

4.15	Polymer shear stress at the top surface ($y = h$) (a), at the bottom surface $y = 0$ (b), viscoelastic pressure p_{ve} (c) and total pressure p_{tot} (d) for the plane slider.	74
4.16	Polymer shear stress at the top surface ($y = h$) (a), at the bottom surface $y = 0$ (b), viscoelastic pressure p_{ve} (c) and total pressure p_{tot} (d) for the parabolic slider.	75
4.17	Polymer shear stress at the top surface ($y = h$) (a), at the bottom surface ($y = 0$) (b), viscoelastic pressure p_{ve} (c) and total pressure p_{tot} (d) for the ridge.	76
4.18	Total pressure for $De = 0.2$ and $\beta = 0.2$ (a) and polymer shear stress at the top surface ($y = h$) (b) for the parabolic slider using equation (4.20) as the constitutive relation. Mesh size is 128×64 for DNS.	78
4.19	Total pressure for $De = 0.2$ and $\beta = 0.5$ (a) and polymer shear stress at the top surface ($y = h$) (b) for the parabolic slider using equation (4.20) as the constitutive relation. Mesh size is 128×64 for DNS.	78
4.20	Numerical stability analysis of the cartoon model given by equation (4.24).	81

List of Tables

4.1	Pressure boundary conditions for the three surfaces (see figure 4.1).	50
4.2	Modeling parameters used in figure 4.2	51
4.3	A list of abbreviations for the models used in this study.	55
4.4	Values for the constants appearing in the Bair-Winer Model (see equation 1.3)	62
4.5	Boundary conditions for the linearized viscoelastic model (LVE) .	67
4.6	Boundary and initial conditions for the viscoelastic model (VE) .	67
4.7	Parameters for the one dimensional model problem, equation (4.24).	81

Chapter 1

Introduction

1.1 Motivations

Lubrication of mechanical components is an essential requirement in applications that involve the risk of surface-to-surface contact. Power transmission via relative motion of two surfaces is common to many industries such as the automotive and power sector. In these industries, lubrication serves to mitigate the losses due to friction. It is not just the energetic losses but mechanical wear as a result of excessive stresses generated at the surfaces that must be minimized. Tribological contacts account for 23% of energy losses globally [2].

Lubrication, however, is a very broad term. It encompasses several mechanisms and modes that are very different. Naturally, lubricant chemistry and operating conditions also influence the lubricating characteristics. Therefore, a proper design needs to factor the environmental effects, type of contact, loading conditions and material properties. Finally, the machine element that demands well lubricated operating modes dictates the selection of lubricant and lubrication regime.

A certain class of mechanical components termed as bearings are frequently

used to reduce wear or contact between sliding surfaces. The central idea is to have a replaceable or repairable component that undergoes wear in place of the primary element, say a shaft. If the operating mode of the bearing relies on direct solid-to-solid contact, the lubrication regime is said to be boundary lubrication [3][4]. The term boundary lubrication was coined approximately one century ago [5]. This mode may also be present during a slow-speed startup of rotating shafts or due to high-temperature thinning of a lubricant. Under normal operating conditions, it is assumed that the surfaces are fully separated in fluid film bearings but localized contact can be established for those reasons. In fact lubricants are added to form a chemical layer on the solid surfaces to aid sliding motion. The chemistry of these layers is important in reducing friction but the main mode is due to a 'plowing' action as the asperities come into contact [6]. Therefore, a pure boundary or hydrodynamic mode may not be the true physical picture.

Hydrodynamic lubrication is a specific term that refers to the action of a thin fluid layer that separates the surfaces in relative motion. This thin lubricant layer is maintained purely due to the fast motion of the, otherwise contacting, surfaces. It is fairly obvious that if sliding action is maintained and the surfaces are not close to each other, then wear will be significantly minimized. Hence, hydrodynamic modes offer the lowest values of traction coefficients. Intuitively, liquid is immediately displaced as a solid surface applies some force but if the fluid is sufficiently viscous, it will offer some resistance to its displacement, thus resulting in a high pressure that sustains the film.

Lubricant type has a great impact on the film thickness. A more viscous lubricant will generate higher pressures and is able to support load. Initially, early modeling efforts assumed a Newtonian fluid as the lubricant in calculations for film thickness and load carrying capacity. Since under low operating pressures, load is small enough to ensure Newtonian behavior [7].

Under heavily loaded conditions, which are more typical of elastohydrodynamic lubrication, the solid surfaces deform under immense film pressure leading

to strain rates that induce highly non-Newtonian conditions. Under these pressures, the lubricant also undergoes solidification or glass-transition and very high viscosities are observed [8]. In addition to the nonlinear rheological properties (thickening due to pressure, temperature dependence etc.), viscoelastic effects become important. Therefore, a departure from Newtonian behavior is expected under these conditions. Two primary non-Newtonian effects emerge, firstly the decrease of viscosity on the strain rate termed as shear thinning and the elasticity of the lubricant due to addition of high molecular weight polymers.

The following sections cover the literature relating to non-linear and non-Newtonian effects in detail with references. A brief review of studies is provided to stimulate the discussion pertaining to numerical modeling of thin film lubrication. A detailed mathematical overview is given in section 2.

Direct numerical simulations of lubricant films are often performed to test the validity of numerous assumptions of Reynolds equations. Although done in different contexts, DNS confirms the findings of Reynolds equation when its assumptions are fully satisfied. The first, as mentioned is that of neglected momentum advection or inertial effects. If the Reynolds number (Re) is small, a Stokes-flow assumption is easily invoked [9], and in some cases not strictly relating to thin films, the assumption itself reduces the mathematical problem to that given by lubrication theory [10]. Unless, Re is not very large, say in excess of 2000, inertial effects can be neglected [11]. Although in certain cases fluid inertia offers an increase in pressure as opposed to a Stokes assumption due to an increase in pressure as a result of cavitation [12].

Thin-film assumptions are suitable for slowly varying geometries but sharp protrusions or pockets for load optimizations can require a direct numerical simulation to capture very sharp or near infinite gradients close to the pocket entry or exit [13]. Therefore, a direct numerical simulation cannot be ruled out entirely. It is important to phase out scenarios where complicated models fail to provide an accurate pressure distribution or load capacity. A detailed review provided in [14] focuses modifications made to Reynolds equation and DNS applied to lubrication in textured surfaces.

1.2 Piezoviscosity

The pressure dependence of the fluid lubricant is described as piezoviscosity. It refers to the increase in viscosity as pressure increases [15]. The pressure response of a liquid, in general, on viscosity may not always be positive, but sufficiently large pressures have an increasing trend that is not linear [16]. A detailed study suggests that as molecular structure changes, the intertwining nature of neighboring molecules has an influence on the viscosity [17]. In this study, Roelands model, equation (1.1), is used to account for the piezoviscous effect [18] shown below,

$$\eta = \eta_0 \exp \left((\ln \eta_0 + 9.67) \left(-1 + \left(1 + \frac{p}{p_0} \right)^z \right) \right), \quad (1.1)$$

where η_0 is the viscosity at atmospheric pressure, p_0 and z are empirical constants.

Roelands model has the advantage of accurately predicting solidification of the lubricant due to very high pressures. Additionally, it offers a better fit to experimental data and is thus widely popular in numerical analysis of lubricating films.

1.3 Shear thinning

A dominant non-Newtonian effect is the dependency of viscosity on the shear stress or strain rate. Unlike a Newtonian fluid where viscosity remains constant for any applied strain rate, it has been observed that lubricants undergo significant decrease in viscosity as a result of the high strain rates typical of hydrodynamic or elastohydrodynamic contacts. Typically, power-law type fluids are used to model the effects of viscosity loss in hydrodynamic lubrication [19]. However, these fluids are termed as generalized Newtonian fluids and the functional form degrades to the Newtonian viscosity when the shear modulus of the lubricant is very large. In order to isolate the effects of elasticity, it is assumed that **(i)** the effect of shear thinning is simply the decrease in viscosity due to strain rate and **(ii)** a nonlinear relationship that satisfies this behavior is valid [20][21]. Experimentally,

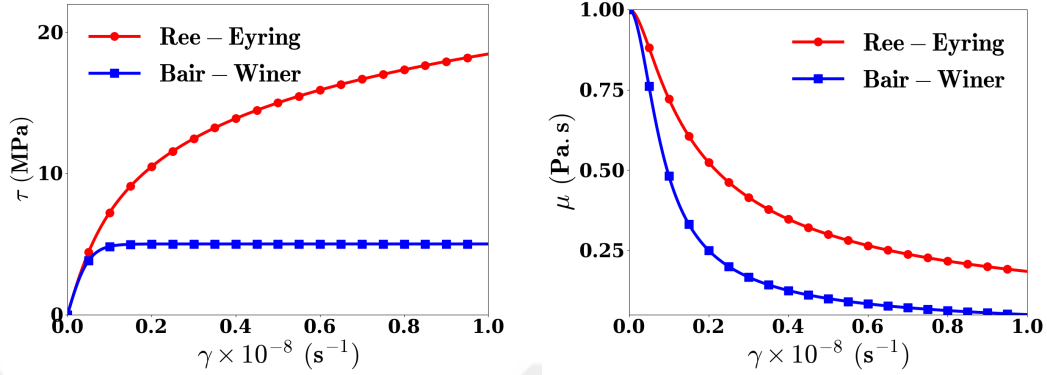


Figure 1.1: A comparison of the Ree-Eyring and Bair-Winer shear thinning models. Yield stress $\tau_0 = 5$ MPa and ambient viscosity $\eta_0 = 1$ Pa · s

the exact nature of this non-linear relation can be argued and constructed to match experimental observation. In this study, the nonlinear function is what remains after elasticity is ignored from the viscoelastic relation.

The first formulation that gained popularity in hydrodynamic lubrication at high strain rates is the Ree-Eyring relation [22]. While it is not strictly derived from molecular theory, its empirical form has the qualitative aspects expected due to molecular motion,

$$\mu = \frac{\tau_0}{\gamma} \sinh\left(\frac{\tau}{\tau_0}\right). \quad (1.2)$$

The proposed form has been used extensively, since its inception, in studying the growth of pressure in lubricated contacts. Experimental evidence suggested a significant thinning of the lubricating film at high sliding speeds or strain rates. This was inconsistent with the typical Newtonian model which maintains a strain independent viscosity resulting in no decrease in film thickness due to increase in shear but rather a decrease in viscosity due to thermal effects.

The Ree-Eyring model predicts an ever increasing shear stress with strain rate even after the yield point is crossed which was shown to be inconsistent with experimental data for very high strain rate [23], see figure 1.1.

The model proposed by Bair and Winer [24] alleviates this problem by introducing pressure and temperature dependent fluid properties of shear modulus G

and yield stress τ_0 which are sufficient to describe the behavior under high strain rates and agree well with experimental data. The initial proposed logarithmic form of the function was later modified into a symmetric tangent hyperbolic function [25].

The full argument against the use of a Ree-Eyring type model has been presented in literature previously and goes beyond the scope of this study. The more recent Bair-Winer model is adopted in this study since there is no evidence against its validity as an empirical model. Additionally, it is built upon constants that are now accepted as properties of the fluid, namely the yield stress τ_0 and limiting shear modulus G ,

$$\mu = \frac{\tau_0}{\dot{\gamma}} \tanh\left(\frac{\eta_0 \dot{\gamma}}{\tau_0}\right), \quad (1.3)$$

where the equivalent shear stress is given by $\tau = \mu \dot{\gamma}$.

An important point to note is that the shear stress and strain rate appearing in (1.3) is a scalar. The formulation is purely empirical, hence the strain rate is taken to be the magnitude of the strain tensor. The strain tensor is simply twice the deformation tensor given by $\underline{\underline{D}}$. Although this relation is valid for high pressures and strain rates, the Ree-Eyring model is still a popular choice in many EHL studies. From a mathematical perspective, the decrease in viscosity due to shear stress is less in the case of equation (1.2) as opposed to equation (1.3).

1.3.1 Polymer additives and shear stability

Lubricants are multipurpose fluids which perform a critical role in preventing wear and energy losses due to surface contact. Thus the primary attribute for a lubricant, from a mechanical perspective, is the load carrying capacity. This is linked to the net viscosity of the oil at the given operating temperature. The sensitivity of the oil to thermal effects must therefore be controlled in order to maximize or minimize load. If a lubricant base oil is too viscous, the ability to flow will be hindered, but a low viscosity will result in loss of load carrying capacity. Hence, a lubricant base oil is enhanced with polymers that are inactive

at low temperatures and undergo small changes in viscosity at high temperatures. This ensures a sufficiently viscous mixture that can carry the load.

The viscosity of a polymer is related to its molecular weight. These high molecular weight polymers are long polymer chains that acquire the shape of a random coil based on the solubility of the solvent and ambient conditions. At high enough shear rates, a marked decrease in viscosity can be observed. There are essentially two main underlying reasons. The first mode is a temporary loss and is simply due to the alignment of the polymer chains with the direction of shear. In a given flow field, the polymer chains simply tend to align themselves, *reversibly*, with the direction of flow which minimizes the resistance they offer as a result of neighboring chains being entangled [26]. The second is a more permanent and harmful mode in which the polymer coils break down into smaller components in a scission process. These smaller fragments possess roughly half the molecular weight and hence are less viscous than their larger counterpart [27]. The decrease in viscosity of the lubricant due to chain scission can be severe. Under significant stresses the net mixture can be exhibiting only a Newtonian viscosity. This break down of the coil is responsible for a sharp loss in viscosity at high shearing rates. Continual degradation of the coiled polymers due to applied strain tends to reduce their effectiveness to that of linear polymers [28]. For this reason, the more shear stable polymer additives are ones with low molecular weight [29]. This does not limit their design traits however.

The architecture of a polymer molecule (size, shape, molecular density etc.) has a considerable influence on its rheological characteristics. A high molecular weight polymer with a high branched structure offers a good increase in viscosity but is relatively more shear stable than linear polymers [30]. This indicates that the distribution of molecular weight is a key factor. A highly branched, termed hyper-branching, polymer can possess good thickening ability and shear stability [31].

It is interesting to note that the base stock also has an influence on the shear stability index of the polymer. The composition, viscosity and ability to integrate with the polymer additive impact the stability of the resulting mixture

[32]. Although the degradation of larger polymers appears unavoidable, it still offers better performance than a monograde lubricant despite the viscosity loss [33]. The allowable load is not the only factor but the friction induced due to a high viscosity is also of concern, especially for automotive applications.

The base solvent of the lubricant does not undergo such a change as it already comprises of a polymer of low molecular weight. Hence it can be assumed to be shear stable. This assumption is based on the eventual saturation of viscosity loss at high strain rates beyond which no further polymer breakdown occurs. A generalized Reynolds equation is derived on the basis of this assumption.

1.4 Viscoelasticity

Fluids in lubrication are known to exhibit both Newtonian and non-Newtonian behavior depending on the application. A Newtonian lubricant maintains an almost shear independent viscosity with variations occurring purely due to the change in pressure or temperature. Non-Newtonian effects arise when the viscosity of the lubricant varies with the flow conditions independent of the temperature or pressure. A particular type of non-Newtonian effect that is termed as viscoelasticity is due to elasticity of the lubricant measured by the shear modulus G . This is arguably related to the molecular weight of the polymeric lubricant and can also vary with temperature and pressure [34]. In simple terms, the time history of the shear rate is insignificant for fluids in ordinary conditions. As the lubricant gets more solid-like due to the addition of heavy polymers or if the flow conditions require that the fluid should adapt its shear stress very quickly the time history of the shear stress is no longer constant and can influence the deformation of the fluid. Hence, this effect is measured by the ability of the fluid to adjust to the surrounding flow conditions (relaxation time) while the characteristic flow time over which it must adapt is application sensitive.

The ratio of relaxation time to characteristic flow time is referred to as Deborah number. Relaxation times for lubricants are in the order of 10^{-4} to 10^{-6} seconds.

While this may seem small, the characteristic time of flow for most lubrication applications is of comparable magnitude. Consider a bearing sliding velocity of $0.25m/s$ and film thickness of $25\mu m$. In this case the characteristic time (λ) is 10^{-4} s. Clearly, the stress field in this case will be significantly different as the fluid has little time to relax. Conversely if the relaxation time is very large, then the fluid will exhibit purely viscous behavior [35].

Modeling viscoelastic effects in the context of lubrication can be traced over the last few decades. The first attempt to incorporate a uniform model that describes the strain-rate dependent viscosity and elasticity of the lubricant was achieved in the context of elastohydrodynamic lubrication in 1976 [36] [37]. Although its mathematical construction arises from the more general linear Maxwell model which describes the fluid as a combination of a linear elastic element and a linear viscous element. Consider the total displacement of the fluid element to be the sum of the displacement due to viscous interactions and an elastic response. A crude but simple approach to its derivation is given as,

$$\gamma_{tot} = \gamma_e + \gamma_v,$$

where, γ_e is the displacement due to fluid elasticity and γ_v is displacement due to fluid viscosity

$$\begin{aligned}\frac{\partial \gamma_{tot}}{\partial t} &= \frac{\partial}{\partial t}(\gamma_e + \gamma_v), \\ \dot{\gamma}_{tot} &= \frac{\partial \gamma_e}{\partial t} + \frac{\partial \gamma_v}{\partial t}, \\ 2\underline{\underline{D}} &= \frac{\partial(\frac{1}{G}\underline{\underline{\tau}})}{\partial t} + \dot{\gamma}_v, \\ 2\underline{\underline{D}} &= \frac{1}{G} \frac{\partial \underline{\underline{\tau}}}{\partial t} + \frac{\underline{\underline{\tau}}}{\eta}.\end{aligned}\tag{1.4}$$

The total deformation rate of the flow field is given by the deformation gradient tensor $\underline{\underline{D}}$ which is expressed in terms of the velocity gradient tensor $\underline{\underline{L}}$ as $\underline{\underline{D}} = \frac{1}{2}(\underline{\underline{L}} + \underline{\underline{L}}^T)$. Rearranging (1.4) into the more familiar form adopted in literature gives the linear-Maxwell model,

$$\underline{\underline{\tau}} + \frac{\eta}{G} \frac{\partial \underline{\underline{\tau}}}{\partial t} = 2\eta \underline{\underline{D}},\tag{1.5}$$

where the ratio of viscosity η_p and shear modulus G is the fluid relaxation time λ .

The time rate of change operator appearing in the constitutive relation is due to a finite elastic modulus or relaxation time. Its treatment has differed since deformation theory is described differently for small or large strains. In the case of small strains, it has been shown mathematically that the time rate of change operator $\frac{\partial \underline{\tau}}{\partial t}$ can be accurately represented by the material or substantive derivative [37], i.e.,

$$\frac{\partial \underline{\tau}}{\partial t} = \frac{d \underline{\tau}}{dt} + (u \cdot \nabla) \underline{\tau}, \quad (1.6)$$

The linear Maxwell type fluid is modified to include a nonlinear shear thinning effect which is meant to account for the strain dependency of the viscosity. Thus the model is parameterized by the elastic shear modulus and a limiting yield stress [38]. The nonlinear viscosity function can be the sine hyperbolic as suggested by Ree-Eyring or tangent hyperbolic as given by Bair-Winer [24]. In either case, the inclusion of a nonlinear viscous element alters the form of the constitutive relation to the following,

$$\frac{\underline{\tau}}{\mu} + \frac{1}{G} \frac{d \underline{\tau}}{dt} = 2 \underline{D}. \quad (1.7)$$

Under the assumption of a thin film, where the ratio of film height to film length is small, axial strains and deformations remain small enough to be neglected. The normal stresses rise proportionally to these strain and hence are neglected in the case of thin film lubrication. The same phenomena is observed for viscoelastic lubricants [39] and terms involving the stretch are discarded prior to the analysis [40]. A major problem with viscoelastic effects is the distinction between shearthinning and elastic response of polymer solutions. In this study a constant viscosity lubricant with a large elastic component is assumed. It is possible to design such fluids by adding small concentrations of polymers to a viscous solvent [41].

1.5 Constitutive Relations

Other large scale applications where the thin-film assumption does not hold can exhibit viscoelasticity. Several models have been proposed over the past decades. The validity of each model depends on the application and assumptions implicit within the model. In this study a the focus is on the Oldroyd-B model.

1.5.1 Generalized Maxwell

The simplest approach towards modeling viscoelasticity is by approximating the fluid to consist of an elastic and viscous component connected in series. The elastic component can be modeling as a spring where the shear-strain is directly proportional to the shear-stress. This viscous element, modelled as a dashpot, consists of the generalized newtonian fluid approximation where the viscosity if a function of the shear-strain. The total strain rate is then simply the sum of elastic and viscous strain rate (Note the connection in series). Mathematically, it is written as

$$\underline{\underline{\tau}} + \lambda \frac{d\underline{\underline{\tau}}}{dt} = -2\eta \underline{\underline{D}}, \quad (1.8)$$

where, $\underline{\underline{D}}$ is the deformation tensor and λ is the relaxation time in seconds.

A note regarding the time derivative ($\frac{\delta \tau}{\delta t}$) appearing in the equation. In the strictest sense, it should take into account the rotation as well as translation of the reference frame in the direction of particle flow. For example, if we consider the derivative to be simply the time derivative in the Lagrangian system, then the model fails to account for the change in properties due to particle flux. If the derivative is approximated by the total or substantive derivative, then the situation improves slightly. However, the total derivative only accounts for changes along the streamline of the particles and not due to rotation. Hence, several relations are possible for the expansion of the derivative appearing in equation 1.8. The choice, depends on the conditions and accuracy required.

The Maxwell relation is a linear viscoelastic constitutive relation. It can be

modified to include non-linear viscosity models by replacing the viscous strain-rate to be a function of the shear-stress [36]. Note that it is obtained under the assumption of an incompressible lubricant [42]. Hence, several shear-thinning or piezoviscous models can be easily incorporated into the visco-elastic relation.

1.6 Aims and objectives

The thesis is organized as follows. In Chapter 2 a detailed introduction is given for the models present in literature that are accepted and those derived in this study. Chapter 3 discusses the numerical algorithms used to solve the models and results are presented in Chapter 4.

1.6.1 Aims

The aims of this work are given as follows,

- Develop simplified computational models of two distinct non-Newtonian phenomena that arise in lubricants under heavily loaded conditions.
- Determine the suitability and accuracy of the simplified models in predicting load carrying capacity and coefficient of friction.
- Compare and contrast existing techniques for modeling shear thinning (inelastic non-Newtonian) and viscoelasticity (elastic isoviscous) lubricants in hydrodynamic lubrication.
- Develop a code for the direct numerical simulation of the Navier-Stokes equations using the **OpenFOAM** C++ library [43]. implement viscoelastic, piezoviscous and shearthinning effects.
- Compare results obtained from simplified model with a direct numerical simulation of the Navier-Stokes equations to assess validity of models and assumptions.

- Develop simple 1D models for analyzing viscoelastic effects as an alternative to more complex 2D approaches. Analyze and assess the limitations of the 1D model by comparing results with the direct numerical simulation of Navier-Stokes.
- Assess the ability of proposed models to accurately predict load carrying capacity and coefficient of friction in lubricated contacts.

1.6.2 Objectives

To reach the aims highlighted in the previous section, the following approach is adopted in this study.

- Analyze the steps in deriving existing Reynolds type equations.
- Derive a Reynolds equation for non-Newtonian lubricants considering the split stress tensor approach.
- Discuss advantages and disadvantages of derived models with respect to accuracy, computational effort and simplicity.

Chapter 2

Models and methodology

Numerous models have been proposed in literature to describe non-Newtonian effects in thin lubricating films. The goal is to obtain the pressure distribution in the film which ultimately governs the load carrying capacity of the bearing. In this section a brief description of models presented and discussed in literature is provided and a more detailed derivation and discussion is offered for models derived in this study.

2.1 Reynolds equation

The classical mathematical form proposed by Osborne Reynolds in 1886 [44] describing the pressure of a thin lubricating film is known as Reynolds equation. A lubricating film formed in a closed channel generates a pressure that depends on the relative speed of the sliding surfaces and viscosity of the fluid. The Reynolds equation can be derived quite easily and has been discussed thoroughly in literature, see [45], or within the general context of low Reynolds number flows through closed channels [46]. Therefore, a very brief overview is given below to enable a comparison with more detailed formulations for non-Newtonian lubricants.

The central idea behind elementary lubrication theory is to retain terms in the

mass and momentum conservation laws for fluids that are of the order of ϵ , where $\epsilon = \frac{h}{L}$. Terms that are smaller than that scale are discarded. This results in the following system of equations given below,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \quad (2.1)$$

$$-\frac{dp}{dx} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) = 0. \quad (2.2)$$

The set of equations (2.1) to (2.2) are complete in describing the pressure and velocity distribution in a thin lubricant film. Notice that if the viscosity μ is only a function of x then, the system can be solved to give the classical form of Reynolds equation,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_0 \frac{(U_h - U_0)}{2} \frac{dh}{dx}. \quad (2.3)$$

Equation (2.3) is obtained for an incompressible, isoviscous lubricant bounded between two surfaces in relative motion ($U_h - U_0$). In certain mechanical systems, such as a journal, roller or thrust type bearings, the top surface is rotating or moving at a given speed (U_h). The bottom surface spanning the length of the contact is assumed to be flat. Hence, the film thickness, represented by h is the gap between the top and bottom surfaces. Therefore, Reynolds equation (2.3) predicts the pressure distribution in a lubricant provided the film thickness is known.

Note that the viscosity of the lubricant is assumed to be constant. It is well known that fluid properties vary with temperature and pressure. Generally, the flow rate of the lubricant is sufficient to carry the heat generated due to viscous dissipation, thus allowing isothermal conditions to prevail. However, the high pressures and strain rates can induce nonlinear effects in the fluid. In this case, Reynolds equation takes a slightly different form given by equation (2.4),

$$\frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) = \frac{U}{2} \frac{dh}{dx}, \quad (2.4)$$

where μ is the nonlinear viscosity that is a function of the pressure and shear strain rate i.e. $\mu = f(p, \dot{\gamma})$.

A nonlinear form of Reynolds equation is obtained that can accurately account for the viscosity depending on pressure and to some extent strain-rate. If μ is a function of x only, then equation (2.4) provides accurate results as long as the normal strain $\frac{\partial u}{\partial x}$ is small [45]. In certain cases, the dependency on pressure is significant ($\frac{d\mu}{dx} \frac{\partial u}{\partial x} \neq 0$) and further corrections may be required [47].

In typical hydrodynamic lubrication, pressures are not large and shear-thinning may be of secondary importance. The use of the non-linear variant of Reynolds equation is very common in such cases, for example studying forces experienced by cam-followers [48]. It allows coupling other dynamical equations to solve for the pressure in a purely one dimensional model. Other applications include studying surface texturing with simple Newtonian lubricants [49].

The problem arises when non-Newtonian lubricants are considered. The strain rate dependent nature of synthetic polymers additives causes a decrease in viscosity due to the high shear stresses prevalent within the lubricating contact. This requires that the Reynolds equation should be derived in conjunction with either an empirical formulation for the strain rate dependent viscosity or a constitutive relation of the shear stress. Following this approach a generalized Reynolds equation can be obtained for viscosity varying along the vertical axis y . This was first achieved by Dowson in 1962 [50] and a good discussion was presented for various cases.

In a similar manner, the energy equation can also be coupled for thermal problem where the temperature can vary along the film thickness and as a result, the density and viscosity vary along the film as well [51]. Similarly, for other constitutive laws for the polymer shear stress, such as fluids exhibiting power law behavior in sinovial joints, an exact formulation can be obtained [52]. The use of a generalized Reynolds equation is becoming frequent in studying lubricating contacts since it allows any functional formulation for viscosity. The problem however, can become increasingly complex if an analytical solution is sought but is still solvable using numerical methods.

2.2 Generalized Reynolds equation

2.2.1 Shear thinning lubricants

Naturally, lubricants do not maintain an isoviscous or Newtonian state in most real world applications. Consider a strain rate dependent viscosity obeying the Bair-Winer empirical relation with a yield stress τ_0 and a dynamic viscosity η_0 at ambient pressure and temperature. The starting point in deriving a generalized Reynolds equation is to approximate the shear stress due to a non-Newtonian lubricant as the product of an equivalent viscosity function and the strain rate. The general relation between the shear stress and strain rate is then assumed to be [51],

$$\tau = \mu^*(p, \dot{\gamma})\dot{\gamma}, \quad (2.5)$$

where $\dot{\gamma} = \frac{\partial u}{\partial y}$ is the dominant component of the strain rate tensor under the thin film approximation and τ is the shear stress component along xy . This notation is assumed throughout this section, unless stated otherwise.

Note that (2.5) does not imply a Newtonian type interaction between the shear stress and strain rate. Since $\frac{d\tau}{d\dot{\gamma}} \neq \mu^*$, the above relationship describes a generalized Newtonian fluid. The choice for μ^* is mostly empirical in lubrication simulations. The equivalent viscosity μ^* is represented by μ in the proceeding derivation. The momentum conservation for a thin lubricating film is reformulated in terms of the shear stress as,

$$\frac{dp}{dx} = \frac{\partial \tau}{\partial y}, \quad (2.6)$$

substituting the relation for the polymer shear stress, τ into the momentum equation and integrating once along the y axis,

$$\frac{dp}{dx}y + c_1 = \tau, \quad (2.7)$$

to ensure symbolic consistency with the formulation of a generalized Reynolds equation [53], the shear stress is substituted as,

$$\tau = \frac{1}{F} \frac{\partial u}{\partial y},$$

where F is the inverse function of the nonlinear equivalent viscosity μ .

In terms of F , equation (2.7) is integrated again along y after multiplying throughout by F to obtain,

$$\frac{dp}{dx} \int_0^y F y dy + c_1 \int_0^y F dy + c_2 = u, \quad (2.8)$$

applying the boundary conditions, the constants of integration are obtained in terms of F ,

$$c_2 = 0$$

$$c_1 = \frac{1}{\int_0^h F dy} \left(U - \frac{dp}{dx} \int_0^h F y dy \right),$$

substituting the expressions for the constants c_1 and c_2 back into equation (2.8), a final expression for the velocity field is obtained in terms of F and p .

$$u = \frac{dp}{dx} \left(\int_0^y F y dy - \frac{\int_0^h F y dy}{\int_0^h F dy} \int_0^y F dy \right) + U \frac{\int_0^y F dy}{\int_0^h F dy}, \quad (2.9)$$

where U is the upper surface velocity.

Note that it is simple to retrieve the Newtonian velocity field from equation (2.9) by simply setting $F = \frac{1}{\eta}$. The governing equation for the pressure field is obtained by using the continuity equation and applying Leibnitz theorem as shown below,

$$\frac{\partial}{\partial x} \int_0^h u dy - u|_h \frac{dh}{dx} + \int_0^h \frac{\partial v}{\partial y} dy = 0,$$

applying the boundary conditions for v and

$$\begin{array}{lll} y = 0 & u = 0 & v = 0 \\ y = h & u = U & v = U \frac{dh}{dx} \end{array}$$

simplifying the expression resulting from Leibnitz theorem to obtain,

$$\frac{\partial}{\partial x} \int_0^h u dy = 0$$

substituting for the velocity, u , and integrating along the film height h , a governing equation for pressure is obtained,

$$\frac{d}{dx} \left(m_2 \frac{dp}{dx} \right) = U \frac{d}{dx} \left(\frac{m_1}{f_0} \right), \quad (2.10)$$

$$\begin{aligned}
m_2 &= \int_0^h \int_0^y F y dy dy' - \frac{\int_0^h F y dy}{\int_0^h F dy} \int_0^h \int_0^y F dy dy', \\
m_1 &= \int_0^h \int_0^y F dy dy', \\
f_0 &= \int_0^h F dy.
\end{aligned}$$

An expression for the strain rate is required to formulate a closed set of equations that can be solved (given a valid constitutive relation) to obtain the pressure, velocity and shear stress. Differentiating the velocity field given by equation (2.9), u , with respect to y , the strain rate, $\frac{\partial u}{\partial y}$, is obtained in terms of p and F ,

$$\frac{\partial u}{\partial y} = F \frac{dp}{dx} \left(y - \frac{\int_0^h F y dy}{\int_0^h F dy} \right) + U \frac{F}{\int_0^h F dy}, \quad (2.11)$$

finally, the Bair-Winer model is used to include the shear thinning effect,

$$\mu = \frac{\tau_0}{\dot{\gamma}} \tanh\left(\frac{\eta \dot{\gamma}}{\tau_0}\right), \quad (2.12)$$

where η is the Newtonian viscosity that depends on pressure and temperature, τ_0 is the lubricant yield stress and $\dot{\gamma} = \frac{\partial u}{\partial y}$.

The set of equation (2.10) to (2.12) are solved iteratively to obtain the pressure distribution for any surface profile subject to the assumptions of no-slip and negligible lubricant elasticity. While, this approach has gained popularity, it is important to mention that it is not the only generalized form available in literature [52]. However, other approaches generally assume some power law as the constitutive relation for shear stress. Similarly, a Reynolds equation considering only the Ree-Eyring constitutive relation can be derived as an alternative but fails to incorporate other rheological laws [54].

In principle the generalized Reynolds equation can be coupled with the energy equation for a thermal problem without any change. The assumption of a non-Newtonian lubricant simply implies that the viscosity is a two-dimensional function and needs to be defined at every point in the flow field. This implies the variation of viscosity with y . In this study the focus is primarily on isothermal problems.

The generalized Reynolds is unfortunately an integral differential equation that requires evaluating the integrals at each iteration when computing the numerical solution. For large meshes, the problem can become computationally intensive. Additionally, the nonlinearity in the problem due to dependency of viscosity on pressure and strain rate can require large under relaxation of the solution at each iteration.

2.2.2 Multigrade lubricants

Lubricants, especially engine oils that are expected to operate in hydrodynamic regimes are comprised of a base stock of low molecular weight and a high molecular weight polymer that functions as a viscosity index improver (VII). The base stock itself can be a low molecular weight polymer since the vast majority of synthetic lubricants are essentially polymeric in their composition. These so called multi-grade oils are intrinsically non-Newtonian. However, the primary base stock may not exhibit non-Newtonian behavior since it has a high shear stability index. The response to a high shear strain is gradual for low molecular weight polymers. The total shear stress can then be considered as a sum of the Newtonian base stock and a non-Newtonian polymer additive.

A generalized Reynolds equation for such a lubricant mixture can be similarly obtained without much effort. A key difference arises in attempting to obtain a purely one dimensional description. The derivation starts with the same set of equations (2.1) and (2.2) but is modified to include a Newtonian stress field,

$$\begin{aligned} \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} &= 0, \\ \frac{dp}{dx} &= \left. \frac{\partial \tau}{\partial y} \right|_{\text{solvent}} + \left. \frac{\partial \tau}{\partial y} \right|_{\text{polymer}}. \end{aligned} \quad (2.13)$$

The solvent stress is Newtonian while that of the polymer additive is non-Newtonian. Note that the contributions of each stress component in equation (2.13) appear within the definition of the shear stress component, (see below). Under the assumptions of a thin lubricating film, as in Reynolds equation, only

the xy component of the stress tensor is considered important,

$$\tau_s = \eta_s \frac{\partial u}{\partial y},$$

$$\tau_p = \eta_p \frac{\partial u}{\partial y},$$

where the subscript s and p denote solvent and polymer respectively. Note that the solvent viscosity is constant or a function of pressure in case of piezoviscous effects. The low molecular weight of the solvent will imply a small viscosity at ambient pressure and temperature hence, even piezoviscous effects may not be significant. The polymer viscosity is given by the Bair-Winer model (1.3) and is also subject to piezoviscous effects. The momentum equation (2.14) is integrated across the film i.e. along y ,

$$\frac{dp}{dx} = \eta_s \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \quad (2.14)$$

$$\frac{dp}{dx} + c_1 = \eta_s \frac{\partial u}{\partial y} + \tau, \quad (2.15)$$

Equation (2.15) is further integrated along y but without substituting for the polymer stress, τ . Instead it is kept as an unknown quantity at this point, otherwise integral expressions will appear in the differential equation and the result will not offer any simplification similar to the generalized Reynolds equation (2.10). Integrating (2.15) along y for the velocity field u ,

$$\frac{dp}{dx} \frac{y^2}{2} + c_1 y + c_2 = \eta_s u + \int_0^y \tau dy, \quad (2.16)$$

applying the no-slip boundary conditions to determine the constants of integration c_1 and c_2 in terms of the shear stress, τ , and pressure, p ,

$$\begin{aligned} c_2 &= 0, \\ c_1 &= \left(-\frac{dp}{dx} \frac{h}{2} + \eta_s \frac{U}{h} + \frac{1}{h} \int_0^h \tau dy \right), \end{aligned} \quad (2.17)$$

the velocity field, u , is obtained by substituting the expression for c_1 and c_2 in terms of p and τ into equation (2.16),

$$u = \frac{1}{2\eta_s} \frac{dp}{dx} (y^2 - yh) + U \frac{y}{h} + \frac{1}{\eta_s} \left(\frac{y}{h} \int_0^h \tau dy - \int_0^y \tau dy \right). \quad (2.18)$$

Equation (2.18) is the velocity field expressed in terms of the unknown pressure and stress fields. Comparing with equation (2.9), the form is much simpler. Additionally, it is written as the summation of **(i)** a pressure driven (Poiseuille), **(ii)** boundary driven (Couette) and a **(iii)** non-Newtonian flow field. The governing equation for pressure is obtained in the conventional manner, see equation (2.10).

$$\frac{\partial}{\partial x} \int_0^h u dy = 0,$$

$$\frac{d}{dx} \left(\frac{h^3}{12\eta_s} \frac{dp}{dx} \right) = \frac{U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{1}{\eta_s} \left(\int_0^h \frac{y}{h} \int_0^h \tau dy dy' - \int_0^h \int_0^y \tau dy dy' \right) \right). \quad (2.19)$$

If piezoviscosity for the solvent is taken into account, the final term on the right hand side of equation (2.19) must retain the solvent viscosity inside the differential. However, the base stock is a very low molecular weight polymer to prevent high viscosity at very low temperatures, which may be the case in many applications for example cold engine startup, the solvent is assumed to be isoviscous. Therefore the pressure equation is given as,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_s \frac{U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\int_0^h \frac{y}{h} \int_0^h \tau dy dy' - \int_0^h \int_0^y \tau dy dy' \right). \quad (2.20)$$

As noted in the derivation of the generalized Reynolds equation for a mono-grade lubricant, an expression for the strain rate is required to close the system of equations,

$$\dot{\gamma} = \frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\frac{1}{2\eta_s} \frac{dp}{dx} (y^2 - yh) + U \frac{y}{h} + \frac{1}{\eta_s} \left(\frac{y}{h} \int_0^h \tau - \int_0^y \tau \right) \right),$$

note the appearance of an average polymer stress term, $\tau_{avg} = \frac{1}{h} \int_0^h \tau dy$ that is independent of y .

$$\frac{\partial u}{\partial y} = \frac{1}{2\eta_s} \frac{dp}{dx} (2y - h) + \frac{U}{h} + \frac{1}{\eta_s} \left(\frac{1}{h} \int_0^h \tau dy - \tau \right). \quad (2.21)$$

The set of equations (2.20), (2.21) and (1.3) constitute a closed system of equations for determining the pressure, velocity and shear stress under the assumptions of a thin film. However, they are still bound in the form of an integral

differential equation. Although the required number of integrations are considerably less, it can still be further reduced into a purely one dimensional problem by noting a linear functional form of the shear stress from equation (2.14).

The derivative of the shear stress along y must result in a function that depends only on x . The left hand side of the momentum equation is solely a function of x and therefore $\frac{\partial \tau_s}{\partial y}$ and $\frac{\partial \tau_p}{\partial y}$ must also satisfy this constraint [45]. The polymer shear stress along y is expressed simply as a linear function in terms of the shear stress at the top τ_h and bottom τ_0 surfaces,

$$\tau = \tau_0 + (\tau_h - \tau_0) \frac{y}{h}.$$

Substituting the linear relation in the equations for pressure, velocity and shear stress itself to obtain a reduced set of one dimensional equations. The average shear stress is given by equation (2.22),

$$\frac{1}{h} \int_0^h \tau = \frac{1}{h} \int_0^h (\tau_0 + (\tau_h - \tau_0) \frac{y}{h}) dy,$$

simplifying to obtain,

$$\frac{1}{h} \int_0^h \tau = \frac{1}{2}(\tau_0 + \tau_h), \quad (2.22)$$

and replacing the average shear stress, given by equation (2.22), in the expression for strain rate, equation (2.21), evaluated at the top surface ($y = h$) and at the bottom ($y = 0$) surface respectively,

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{-h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{\eta_s} \left(\frac{1}{2}(\tau_0 + \tau_h) - \tau_0 \right), \quad (2.23)$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=h} = \frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{\eta_s} \left(\frac{1}{2}(\tau_0 + \tau_h) - \tau_h \right). \quad (2.24)$$

The integrals appearing in governing equation (2.20) for pressure are simplified similarly,

$$\int_0^h \frac{y}{h} \int_0^h \tau = \int_0^h \frac{y}{h} \int_0^h (\tau_0 + (\tau_h - \tau_0) \frac{y}{h}) dy dy,$$

$$\int_0^h \frac{y}{h} \int_0^h \tau = \frac{h^2}{4} (\tau_h + \tau_0), \quad (2.25)$$

$$\begin{aligned} \int_0^h \int_0^y \tau &= \int_0^h \int_0^y (\tau_0 + (\tau_h - \tau_0) \frac{y}{h}) dy dy, \\ \int_0^h \int_0^y \tau &= \frac{h^2}{6} (\tau_h + 2\tau_0), \end{aligned} \quad (2.26)$$

taking the difference of the integral expressions,

$$\begin{aligned} \int_0^h \frac{y}{h} \int_0^h \tau - \int_0^h \int_0^y \tau &= \frac{1}{4} (\tau_h + \tau_0) - \frac{1}{6} (\tau_h + 2\tau_0), \\ \int_0^h \frac{y}{h} \int_0^h \tau - \int_0^h \int_0^y \tau &= \frac{h^2}{12} (\tau_h - \tau_0), \end{aligned}$$

substituting the simplified expressions for the integrals of the polymer stress into equation (2.20), the one dimensional form of the generalized Reynolds equation for polymer enhanced lubricants is,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_s \frac{U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{h^2}{12} (\tau_h - \tau_0) \right). \quad (2.27)$$

The shear stresses at the surfaces τ_0 and τ_h can be determined using the Bair-Winer model (1.3),

$$\mu|_{y=0} = \frac{\tau_L}{\frac{\partial u}{\partial y}|_{y=0}} \tanh \left(\frac{\eta_p \frac{\partial u}{\partial y}|_{y=0}}{\tau_L} \right) \quad (2.28)$$

$$\mu|_{y=h} = \frac{\tau_L}{\frac{\partial u}{\partial y}|_{y=h}} \tanh \left(\frac{\eta_p \frac{\partial u}{\partial y}|_{y=h}}{\tau_L} \right) \quad (2.29)$$

where μ is the polymer viscosity,

$$\tau_0 = \mu|_{y=0} \frac{\partial u}{\partial y} \Big|_{y=0}, \quad (2.30)$$

$$\tau_h = \mu|_{y=h} \frac{\partial u}{\partial y} \Big|_{y=h}. \quad (2.31)$$

The system of equations (2.23) to (2.31) completes the one dimensional generalized Reynolds equation for polymer enhanced non-Newtonian lubricants. The resulting problem set comprises a system of 6 equations. Piezoviscous thickening

and shear thinning affect the polymer additive and are described by Roelands model (1.1) and Bair-Winer model (1.3) respectively.

It is straight forward to extend the implementation towards more than one type of polymer by simply taking the sum of the shear stresses for each polymer component. Expressed as a summation over the components of the oil, the momentum equation can be written for N components as,

$$-\frac{dp}{dx} + \frac{\partial \tau_s}{\partial y} + \sum_{n=1}^N \frac{\partial \tau_{pn}}{\partial y} = 0, \quad (2.32)$$

The classical two dimensional form is also derived in accordance with the procedure adopted in literature [50]. The derivation presented in this study considers an inelastic non-Newtonian fluid as a lubricant. The lubricant is assumed to be incompressible and under isothermal operating conditions. The resulting form is an integral differential equation which incorporates the variation of the fluid properties along *and* across the film. Thus offering a powerful alternative to the direct numerical simulation of lubricating films. It is fair to point out that all forms of the generalized Reynolds equation are subject to the thin film approximation. Inaccurate results will be obtained if the film thickness approaches the film length.

A close inspection of the governing equations for pressure in the one and two dimensional versions of the generalized Reynolds equation will show that they differ significantly in functional form. This difference arises due to the absence of a Newtonian solvent shear stress which manifests as a second derivative in the momentum equation (2.14). The two dimensional generalized Reynolds then requires a substitution of the polymer stress if any useful form is to be derived. In the case where the lubricant undergoes significant thinning and piezoviscous thickening the Newtonian component of the lubricant mixture is entirely inactive and the polymer provides the increase in film pressure which is the expected outcome. The models converge to the same solution in the limit of a high polymer viscosity.

Interestingly, the one dimensional version requires more iterations to converge

but solves very quickly for a large mesh size if the polymer concentration is small. However, it possesses a glaring singularity that arises as the solvent viscosity vanishes. As long as the solvent viscosity is non-zero a numerical solution is attainable given the proper relaxation factors for shear stress and pressure. The nonlinear coupling however, is more severe in the one dimensional case as the system comprises of *six* coupled equations, which allows errors to propagate and increase in the iterative scheme resulting in divergence of the solution. It may not offer a replacement for the two dimensional equation in the limiting case where the lubricant comprises of a non-Newtonian fluid only, i.e. $\eta_s = 0$ and $\eta_p > 0$. It is more efficient to solve the two dimensional version in this limiting case.

For large to moderate bearing numbers, the solvent contribution to the stress and pressure fields becomes negligible and any finite solvent viscosity does not influence the solution. Hence, numerically the two models will return the same solution.

2.3 Viscoelastic Reynolds equation

The generalized Reynolds equation [55] can be extended to polymer solutions where the base stock or lubricant is assumed to be newtonian and iso-viscous. The base lubricant is enhanced with polymer additives that not only serve as viscosity index improvers (VIIs) but owing to their high molecular weight, induce a significant viscoelastic effect. The strength of this elastic effect is measured by the Deborah (De) or Wiessenberg number (Wi).

$$De = \lambda \frac{U}{L}$$

where λ is the polymer relaxation time, U is the maximum surface velocity and L is the characteristic length.

In hydrodynamic lubrication, this will be assumed as the length of the contact zone. However, for applications not pertaining to lubrication, a more appropriate choice is the diameter or gap of the closed channel.

Reynolds originally proposed a one dimensional model for the film pressure by discarding terms in the momentum equations that were not of the order of the film thickness. The idea that the film pressure remains one dimensional does not imply that equations for shear stress or velocity will remain one dimensional. To illustrate this point, consider the constitutive relation for shear stress and considering only the material derivative of the shear stress,

$$\underline{\underline{\tau}} + \lambda \frac{d\underline{\underline{\tau}}}{dt} = 2\eta_p \underline{\underline{D}},$$

where,

$$\frac{d\underline{\underline{\tau}}}{dt} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + u \frac{\partial \underline{\underline{\tau}}}{\partial x} + v \frac{\partial \underline{\underline{\tau}}}{\partial y}, \quad (2.33)$$

and

$$\underline{\underline{D}} = \frac{1}{2}(\underline{\underline{L}} + \underline{\underline{L}}^T).$$

The shear stress varies along x and y . Additionally, the velocity vector field components u and v along the x and y directions also appear. The problem definition is inherently two dimensional in the polymer shear stress unless the second term in the convected derivative, equation (2.33), is dropped by enforcing the assumption of a one dimensional model. However, in the approach outlined below, a fully one dimensional model for an isoviscous elastic lubricant can be developed by carefully considering the boundary conditions and the reduced Navier-Stokes equations.

The viscoelastic problem is formulated similar to the generalized Reynolds equation. Non-dimensionalizing and neglecting the second and higher order terms ($O(\epsilon^2)$, where $\epsilon = \frac{h}{L}$) the following set of equations are obtained,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.34)$$

$$\frac{dp}{dx} = \eta_s \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau_{xy,p}}{\partial y}, \quad (2.35)$$

$$\tau_{xy,p} + \lambda \left(\frac{\partial \tau_{xy,p}}{\partial t} + u \frac{\partial \tau_{xy,p}}{\partial x} + v \frac{\partial \tau_{xy,p}}{\partial y} \right) = \eta_p \frac{\partial u}{\partial y}, \quad (2.36)$$

where η_s is the solvent viscosity and the subscript p denotes the polymer. A viscoelastic Reynolds equation can be obtained by following the approach adopted

for multigrade lubricants (see section 2.2.2). However, the polymer stress term in the momentum equation (2.35) is not replaced with equation (2.36). A simpler model is obtained by allowing the shear stress to remain as an unknown field. Integrating the momentum equation (2.35) twice with respect to y we obtain,

$$\frac{dp}{dx} \frac{y^2}{2\eta_s} + \frac{c_1}{\eta_s} y + \frac{c_2}{\eta_s} = u + \int_0^y \frac{1}{\eta_s} \tau_p dy, \quad (2.37)$$

Note in (2.37) that the polymer shear stress is an unknown and an integral operator is appearing in the differential equation. The two constants of integration c_1 and c_2 can be determined using the boundary conditions,

$$y = 0, u = 0, v = 0, \quad (2.38)$$

$$y = h, u = U, v = V = U \frac{dh}{dx}, \quad (2.39)$$

Applying the boundary conditions, the values of c_1 and c_2 are as follows,

$$c_2 = 0,$$

$$c_1 = \eta_s \left(-\frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{h\eta_s} \int_0^h \tau_p dy \right),$$

substituting the constants c_1 and c_2 back into equation (2.37) gives an expression for u in terms of the pressure gradient and polymer stress

$$u = \frac{1}{2\eta_s} \frac{dp}{dx} (y^2 - yh) + U \frac{y}{h} + \frac{1}{\eta_s} \left(\frac{y}{h} \int_0^h \tau_p dy - \int_0^y \tau_p dy \right). \quad (2.40)$$

The final expression for the velocity field, u , given by equation (2.40) contains contributions from the pressure gradient, boundary motion and elasticity of the polymer. If the polymer stress goes to zero, the term inside the parenthesis will be zero. The equation for pressure is obtained by using the continuity equation (2.34). Applying Leibnitz theorem to the continuity equation prior to the substitution gives,

$$\frac{\partial}{\partial x} \int_0^h u dy - U \frac{dh}{dx} + \int_0^h \frac{\partial v}{\partial y} dy = 0,$$

applying the boundary conditions and performing the necessary integration along y , the equation for pressure is obtained in terms of the polymer stress,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \frac{U\eta_s}{2} \frac{dh}{dx} + \frac{\partial}{\partial x} \left(\int_0^h \frac{y}{h} \int_0^h \tau_p dy dy' - \int_0^h \int_0^y \tau_p dy dy' \right). \quad (2.41)$$

Equation (2.41) governs the film pressure for viscoelastic lubricants with polymer additives. The integral operators in the differential equations are along the direction of the film thickness and hence will render the problem two dimensional. In fact, numerically, the integrations along y can also be costly for very large meshes. To overcome this issue and express the problem purely along one dimension, x a convenient expression for shear stress along y is required.

The momentum equation (2.35) has the gradient of pressure $\frac{dp}{dx}$ on the left hand side and the $\frac{d\tau_p}{dy}$ on the right hand side. Assuming the pressure does not vary along y , consistent with the assumption of a thin film, the functions appearing on the left and right hand side of (2.35) must be purely functions of x . This directly implies that the shear stress is linear along y [45]. Hence, assuming a linear profile for the shear stress in terms of the stress at the lower surface $\tau_{p,0}$ and upper surface $\tau_{p,h}$ the polymer stress is expressed as (ignoring the subscript p for clarity),

$$\tau = \tau_0 + (\tau_h - \tau_0)\frac{y}{h}. \quad (2.42)$$

The terms appearing as integrals in the equations for pressure and shear stress can now be explicitly obtained using the linear relationship (2.42). The constitutive equation contains the strain rate $\gamma_{xy} = \frac{\partial u}{\partial y}$. The expression for the strain rate, equation (2.43), is easily obtained by differentiating equation (2.40) with respect to y ,

$$\frac{\partial u}{\partial y} = \frac{1}{2\eta_s} \frac{dp}{dx}(2y - h) + \frac{U}{h} + \frac{1}{\eta_s} \left(\frac{1}{h} \int_0^h \tau_p dy - \tau_p \right). \quad (2.43)$$

A set of one dimensional equations can now be obtained by evaluating the strain rate in equation (2.43) and polymer shear stress in equation (2.42) at the bottom and top surfaces. The linear expression for the polymer stress is used to simplify the integrals. (Note that the integral term $(\frac{1}{h} \int_0^h \tau_p dy)$ appearing in the expression for the strain rate is the average stress along y). Substituting equation (2.42) into equation (2.43), the strain rate at the top and at the bottom surface is given by equations (2.45) and (2.44) respectively as shown below,

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = -\frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{\eta_s} \left(\tau_{avg} - \tau_p \Big|_{y=0} \right), \quad (2.44)$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=h} = \frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{\eta_s} \left(\tau_{avg} - \tau_p \Big|_{y=h} \right). \quad (2.45)$$

The constitutive relation at the bottom and top surfaces $y = 0$, $y = h$ respectively, is given as

$$\tau \Big|_{y=0} + \lambda \left(\frac{\partial \tau}{\partial t} \Big|_{y=0} + u \Big|_{y=0} \frac{\partial \tau}{\partial x} \Big|_{y=0} + v \Big|_{y=0} \frac{\partial \tau}{\partial y} \Big|_{y=0} \right) = \eta_p \frac{\partial u}{\partial y} \Big|_{y=0}, \quad (2.46)$$

$$\tau \Big|_{y=h} + \lambda \left(\frac{\partial \tau}{\partial t} \Big|_{y=h} + u \Big|_{y=h} \frac{\partial \tau}{\partial x} \Big|_{y=h} + v \Big|_{y=h} \frac{\partial \tau}{\partial y} \Big|_{y=h} \right) = \eta_p \frac{\partial u}{\partial y} \Big|_{y=h}. \quad (2.47)$$

The following simplifications are performed for each of the terms appearing in the above expressions (making efficient use of the boundary conditions). Beginning from the left hand side of (2.46) and (2.47),

$$\tau \Big|_{y=0} = \tau_0,$$

$$\tau \Big|_{y=h} = \tau_h,$$

where τ_0 and τ_h are the polymer shear stresses at the bottom and upper surfaces.

The time rate of change on the shear stress is given below,

$$\begin{aligned} \frac{\partial \tau}{\partial t} &= \frac{\partial}{\partial t} \left(\tau_0 + (\tau_h - \tau_0) \frac{y}{h} \right), \\ \frac{\partial \tau}{\partial t} &= \frac{\partial \tau_0}{\partial t} + \left(\frac{\partial \tau_h}{\partial t} - \frac{\partial \tau_0}{\partial t} \right) \frac{y}{h} - \frac{y}{h^2} \frac{\partial h}{\partial t} (\tau_h - \tau_0), \end{aligned}$$

where $\frac{\partial h}{\partial t}$ is the squeezing term. In the absence of any squeeze motion,

$$\frac{\partial h}{\partial t} = 0,$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial \tau_0}{\partial t} + \left(\frac{\partial \tau_h}{\partial t} - \frac{\partial \tau_0}{\partial t} \right) \frac{y}{h},$$

$$u \Big|_{y=0} \frac{\partial \tau}{\partial x} \Big|_{y=0} = 0,$$

$$v \Big|_{y=0} \frac{\partial \tau}{\partial y} \Big|_{y=0} = 0.$$

The convected derivative of the shear stress given by $u \frac{\partial \tau}{\partial x} + v \frac{\partial \tau}{\partial y}$, is simplified by evaluating the terms at the top surface ($y = h$) and at the bottom surface ($y = 0$) as shown below,

$$\left(u \frac{\partial \tau}{\partial x} \right) \Big|_{y=h} = u \Big|_{y=h} \frac{\partial \tau}{\partial x} \Big|_{y=h},$$

$$u \Big|_{y=h} \frac{\partial \tau}{\partial x} \Big|_{y=h} = U \frac{\partial \tau_h}{\partial x} - \frac{U}{h} \frac{dh}{dx} (\tau_h - \tau_0),$$

similarly the second term in the convected derivative $v \frac{\partial}{\partial y}$ is given as,

$$\begin{aligned} \left(v \frac{\partial \tau}{\partial y} \right) \Big|_{y=h} &= v \Big|_{y=h} \frac{\partial \tau}{\partial y} \Big|_{y=h} \\ v \Big|_{y=h} \frac{\partial \tau}{\partial y} \Big|_{y=h} &= \frac{U}{h} \frac{dh}{dx} (\tau_h - \tau_0) \end{aligned}$$

the final expressions for the polymer shear stress at $y = 0$ and $y = h$ are given as,

$$\tau_0 + \lambda \frac{\partial \tau_0}{\partial t}, = \eta_p \frac{\partial u}{\partial y} \Big|_{y=0}, \quad (2.48)$$

$$\tau_h + \lambda \frac{\partial \tau_h}{\partial t} + \lambda U \frac{\partial \tau_h}{\partial x} = \eta_p \frac{\partial u}{\partial y} \Big|_{y=h}, \quad (2.49)$$

the double integral expressions appearing in the pressure equation (2.41) are evaluated similarly as,

$$\int_0^h \frac{y}{h} \int_0^h \tau_p dy dy' = \frac{h^2}{4} (\tau_h + \tau_0),$$

$$\int_0^h \int_0^y \tau_p dy dy' = \frac{h^2}{6} (\tau_h + 2\tau_0),$$

substituting into equation (2.41) to obtain a form for a non-Newtonian Reynolds equation,

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) = \frac{U \eta_s}{2} \frac{dh}{dx} + \frac{\partial}{\partial x} \left(\frac{h^2}{12} (\tau_h - \tau_0) \right). \quad (2.50)$$

The non-Newtonian effects considered in this section may not manifest separately in practical situations. Lubricant shear thinning and elasticity have been treated separately, to isolate the mechanism behind these effects. Note that equations (2.27) and (2.50) are identical. It appears that the non-Newtonian effects appear purely as a variation in the non-Newtonian shear stress at the top and bottom surfaces. A more detailed analysis is presented in section 4.

2.3.1 Linearized viscoelastic Reynolds equation

Linearization techniques are often used to simplify expressions that may not yield an analytical solution otherwise. If the scalar equations of shear stress and momentum are taken as the initial problem set, a Reynolds equation cannot be derived simply because of terms that contain a nonlinear coupling of the variables.

Prior to linearizing, a set of equations satisfying the thin-film assumption must be obtained in accordance with lubrication theory. The thin film approximation is applied by denoting a parameter $\epsilon = \frac{h}{L}$ and non-dimensionalizing or scaling the momentum, continuity and stress transport equations. In this manner, terms may contain ϵ or higher powers of ϵ . Since ϵ is very small, all higher order terms involving ϵ are truncated and the conservation equations scaled to include the terms only on the order of the film thickness [35]. The final set of equations are given below,

$$\frac{dp}{dx} = (1 - \beta) \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \quad (2.51)$$

$$\tau_{xx} + \beta De \left(u \frac{\partial \tau_{xx}}{\partial x} + v \frac{\partial \tau_{xx}}{\partial y} - 2 \frac{\partial u}{\partial y} \tau_{xy} - 2 \frac{\partial u}{\partial x} \tau_{xx} \right) = 0, \quad (2.52)$$

$$\tau_{xy} + \beta De \left(u \frac{\partial \tau_{xy}}{\partial x} + v \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial v}{\partial x} \tau_{xx} - \frac{\partial u}{\partial y} \tau_{yy} \right) = \beta \frac{\partial u}{\partial y}, \quad (2.53)$$

$$\tau_{yy} + \beta De \left(u \frac{\partial \tau_{yy}}{\partial x} + v \frac{\partial \tau_{yy}}{\partial y} - 2 \frac{\partial v}{\partial y} \tau_{yy} - 2 \frac{\partial v}{\partial x} \tau_{xy} \right) = 2\beta \frac{\partial v}{\partial y}. \quad (2.54)$$

An analytical solution can be obtained by removing the nonlinear coupling via an expansion of the variables in terms of De [35, 56, 57, 58],

$$\begin{aligned} p &= p^0 + De p^1 + De^2 p^2 \dots, \\ u &= u^0 + De u^1 + De^2 u^2 \dots, \\ v &= v^0 + De v^1 + De^2 v^2 \dots, \\ \tau_{xx} &= \tau_{xx}^0 + De \tau_{xx}^1 + De^2 \tau_{xx}^2 \dots, \\ \tau_{xy} &= \tau_{xy}^0 + De \tau_{xy}^1 + De^2 \tau_{xy}^2 \dots, \\ \tau_{yy} &= \tau_{yy}^0 + De \tau_{yy}^1 + De^2 \tau_{yy}^2 \dots, \end{aligned}$$

where the superscripts 0, 1 and 2 denote the zeroth, first and second order terms in the expansion. Naturally for a linear model the second order terms are truncated in the scalar momentum and shear stress equations.

The definition of De is not strictly binding in such analyses but since it has been assumed that polymer additives induce viscoelastic effects, a small difference between the original work by [35] is introduced because of the presence of a finite solvent concentration. Hence, the De implies only a viscoelastic polymer component. Take $De = \lambda \frac{U}{L}$ where λ is the relaxation time of the polymer additive and linearize the scalar equations (2.51) to (2.54) with an expansion in De . Note the appearance of the polymer concentration β due to the polymer viscosity η_p appearing in the constitutive relation for the polymer shear stress. Linearizing the scalar equations in increasing powers of De and truncating all terms containing De^2 , the linearized viscoelastic Reynolds equation for the Deborah order pressure p^1 is given as

$$\begin{aligned} \frac{d}{dx} \left(\frac{h^3}{12} \frac{dp^1}{dx} \right) = & \beta^2 \left(\frac{d^2 h}{dx^2} \left(-\frac{1}{3} + 1.5 \frac{h_m}{h} - 1.5 \frac{h_m^2}{h^2} \right) \right. \\ & \left. + \frac{1}{h} \left(\frac{dh}{dx} \right)^2 \left(-1.5 \frac{h_m}{h} + 3 \frac{h_m^2}{h^2} \right) \right). \end{aligned} \quad (2.55)$$

2.4 Bearing number

To establish a fair comparison between the results obtained for each geometry, a non dimensional number is used to describe the operating pressure regime. The bearing number describes the ratio between viscous and pressure forces and appears in the context of journal bearings. It can be obtained for any arbitrary surface by considering the non-dimensional Reynolds equation by allowing a small modification to the process. Conventionally, the pressure is non dimensionalized using the flow parameters. However, to force the appearance of the bearing number, the pressure is simply non-dimensionalized by the ambient pressure in the hydrodynamic zone. Substituting the following non-dimensional variables, $x^* = \frac{x}{L}$, $h^* = \frac{h}{h}$, $p^* = \frac{p}{p_0}$ where, L is the length, p_0 is the ambient pressure,

μ is the viscosity at ambient pressure and $\hat{h}$ is the mean surface height.

$$\frac{d}{dx^*}(h^{*3} \frac{dp^*}{dx^*}) = \frac{6U\mu L}{\hat{h}^2 p_0} \frac{dh^*}{dx^*} \quad (2.56)$$

where, $\hat{h} = \frac{\int_0^L h dx}{L}$ is the mean surface height. For a plane slider it is simply the average of the maximum and minimum height.

The physical parameters appearing on the right hand side of equation (2.56) are collectively termed as the bearing number which is expressed as $\Lambda = \frac{U\mu L}{\hat{h}^2 p_0}$. This dimensionless number is the ratio between the viscous bearing forces and pressure forces close to the point of contact. It provides a measure of the operating pressures in the lubricant film. Note that the same bearing number may not always produce the same load or pressure profile due to nonlinear effects such as piezoviscosity and shear thinning.

Chapter 3

Numerical Methods

The following section is a detailed explanation on the numerical solution of the classical Reynolds equation with non-linear effects and generalized Reynolds equation presented in the preceding chapters. It is assumed that the theory relating to the equations and the assumptions involved in their derivations are already known to the reader from the previous section. Nonetheless, a brief explanation is added to establish relevance with the numerical procedure where it is required.

The discretization of all governing equations is accomplished via finite difference schemes. The algebraic system of equations arising from the discretization are given for pressure, velocity and shear stress. Finally, a numerical algorithm is presented for the efficient solution of the system of algebraic equations.

The classical Reynolds equation in 1D offers a good starting point for estimating the pressure profile for incompressible Newtonian lubricants or fluids. From a numerical standpoint it is a straight forward linear problem that does not require much computational effort for simplistic geometries. Interestingly, the solution for the velocity is completely decoupled from the pressure. The resulting velocity contours for u and v can be obtained once the pressure is known. Mathematically it is a second-order linear differential equation and given a known function of the surfaces in terms of the spatial variable x , the exact solution can also be

determined. The general form of the 1D Reynolds equation is given as

$$\frac{d}{dx}\left(\frac{\rho h^3}{12\mu} \frac{dp}{dx}\right) = \frac{(U_2 - U_1)}{2} \frac{d}{dx}(\rho h), \quad (3.1)$$

where, U_2 and U_1 are the top and bottom surface velocities respectively. Assuming, incompressibility and an isoviscous Newtonian fluid equation 3.1 can be simplified to give,

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = 6U_0\mu_0 \frac{dh}{dx}, \quad (3.2)$$

where U_0 and μ_0 are the relative surface velocity and ambient viscosity respectively

Note that ambient properties in lubrication need not be defined at atmospheric pressure. The solution to equation 3.2 is straightforward and involves solving a linear differential equation. Since the scope of this study focuses primarily on non-linear and non-Newtonian effects, the numerical routine defined for the non-linear version of equation 3.2 will prove sufficient for the linear case as well.

3.1 Non-linear Reynolds equation

In its classical form, the Reynolds equation, equation (2.3) can include compressible effects, if it is derived using the steady-state compressible continuity equation. The additional assumption of an isoviscous fluid may not be valid for large pressures in the order of GPa as explained in the earlier sections (see, section 1 and 1.3). However, it may still be evaluated numerically by retaining the viscosity term as in equation 3.1.

The numerical solution of the non-linear form first requires a discretization over a given grid. An arbitrary variable ζ is substituted to simplify the process. It does not possess any physical significance but aides in the computational effort to follow. Let ζ be defined as $\zeta = \frac{\rho h^3}{\mu}$. Replacing the inner term in the parenthesis of equation (4.6), the 1D form of the Reynolds equation is given below,

$$\frac{d}{dx}\left(\zeta \frac{dp}{dx}\right) = 6U \frac{d(\rho h)}{dx}, \quad (3.3)$$

while it may be tempting to expand the derivative and discretize the resulting equation as shown below,

$$\frac{d\zeta}{dx} \frac{dp}{dx} + \zeta \frac{d^2p}{dx^2} = 6U \left(h \frac{d\rho}{dx} + \rho \frac{dh}{dx} \right), \quad (3.4)$$

however, it is to be noted that equation 3.4 requires greater computational effort and its finite difference discretization is not shown here. A more numerically convenient and compact form [59] appears upon discretization using finite differences if ζ is retained within the differential. Equation (3.3) is discretized using 2nd order finite-difference approximation for both 2nd and 1st order derivatives. For the first term appearing on the left-hand-side of equation (3.3),

$$\frac{d}{dx} \left(\zeta \frac{dp}{dx} \right) = \frac{1}{\Delta x^2} (\zeta_{i+1/2} p_{i+1} + (-\zeta_{i+1/2} - \zeta_{i-1/2}) p_i + \zeta_{i-1/2} p_{i-1}), \quad (3.5)$$

where $\zeta_{i+1/2} = \frac{1}{2}(\zeta_{i+1} + \zeta_i)$ and $\zeta_{i-1/2} = \frac{1}{2}(\zeta_i + \zeta_{i-1})$.

The grid spacing Δx is assumed to be constant. Since a finite difference form has been adopted, all the values of the pressure, density and viscosity exist at the node points. The half-nodal values of ζ must be evaluated numerically but can also be obtained directly from the given functions of density, viscosity and surface height or film thickness. The right hand side of equation (3.4) is discretized in a similar fashion using a second order central difference approximation,

$$6U \frac{d(\rho h)}{dx} = \frac{6U}{2\Delta x} [(\rho h)_{i+1} - (\rho h)_{i-1}]. \quad (3.6)$$

Collecting the terms involving the discretized pressure nodes p_i , p_{i+1} and p_{i-1} on the left hand side and placing the remaining terms, not containing any pressure node, on the right. Equations 3.5 and 3.6 are re-written as,

$$w_{i-1} p_{i-1} + e_{i+1} p_{i+1} + c_i p_i = f_i, \quad (3.7)$$

where the coefficients are defined as,

$$\begin{aligned} w_i &= \frac{1}{2\Delta x^2} (\zeta_i + \zeta_{i-1}), \\ e_i &= \frac{1}{2\Delta x^2} (\zeta_{i+1} + \zeta_i), \\ c_i &= \frac{1}{2\Delta x^2} (-\zeta_{i+1} - 2\zeta_i - \zeta_{i-1}), \\ f_i &= \frac{6U_0}{2\Delta x} (\rho h)_{i+1} - (\rho h)_{i-1}. \end{aligned}$$

The algebraic equation 3.7 for a particular internal nodal point i can be written for all nodal points $i = 2$ to $i = N-1$. The resulting system of algebraic equations can be written in terms of a matrix vector equation as $\mathbf{A}p = f$ where A is the coefficient matrix and f is the right hand side vector. For the case where ζ does not depend on pressure (Newtonian, isoviscous and incompressible), the coefficient matrix A and the right hand side vector are linear.

The equations for the boundaries ($i = 1$ to $i = N$) are obtained by discretizing the boundary conditions. In this study, we use pure Dirchlet conditions for the pressure at $x = 0, L$. Hence, the equations are, $p_1 = C_1$ and $p_2 = C_2$ where C_1 and C_2 are the pressures at the boundaries. In most cases, they will simply be equal to the prevailing ambient pressure. The linear system of equations can be solved without any iterative procedures making use of the fact that the coefficient matrix is sparse and tri-diagonal. The upper diagonal is the vector formed by e_i and lower diagonal is the vector formed by w_i in equation (3.7).

For the non-linear case, i.e $\mathbf{A} = A(p)$ and $f = f(p)$, a linearization is required. Stationary methods present the simplest form of linearization by assuming an initial value of pressure and evaluating the coefficient matrix and forcing terms at a given iteration point. This form is typical of fixed-point methods. Equation (3.7) is then recast into a non-linear form by noting the dependency of the coefficients on pressure,

$$w_i^k(p)p_{i-1}^l + c_i^k(p)p_i^l + e_i^k(p)p_{i+1}^l = f_i^k(p). \quad (3.8)$$

The indices k and l represent the choice of pressure at the current or next iteration level. If $k = l$ then the iterations are implicit and difficult (but not impossible) to solve. Since the coefficients cannot be expressly resolved and unknown pressure variables isolated from the remaining terms. The algebraic system of equations (3.8) require a linearization for an efficient solution. The simplest linearization is to consider $l = k + 1$. Then the numerical approach is a simple form of a Fixed point iteration. The algorithm is given below,

1. Initialize all coefficients with an initial guess for pressure. $p^k = p_0$.

2. Solve the system of linearized equations ($\mathbf{A}(p^k)p = f$) using a fast exact algorithm. Note that the coefficient matrix A is tri-diagonal and diagonally dominant, we can employ Thomas-Algorithm for its exact solution.
3. Initialize the coefficients w, c, e and forcing term f at the new pressure p^{k+1} .
4. Repeat 2 - 3 until convergence criteria is fulfilled.

3.2 Generalized Reynolds

The algorithm to solve the generalized Reynolds is similar to the method proposed earlier. Instead of solving a single differential equation at each iterative step denoted by k . A system of equations is solved until a suitable convergence criteria is fulfilled.

3.2.1 Discretization

The system of equations are discretized using finite difference approximations to the numerical derivatives and integrals. The two dimensional form of the generalized Reynolds requires an evaluation of the integral quantities that are then differentiated. The system of equations are shown below,

$$\frac{\partial u}{\partial y} = F \frac{dp}{dx} \left(y - \frac{\int_0^h F y dy}{\int_0^h F dy} \right) + U \frac{F}{\int_0^h F dy}, \quad (3.9)$$

$$u = \frac{dp}{dx} \left(\int_0^y F y dy - \frac{\int_0^h F y dy}{\int_0^h F dy} \int_0^y F dy \right) + U \frac{\int_0^y F dy}{\int_0^h F dy}, \quad (3.10)$$

$$\frac{d}{dx} \left(m_2 \frac{dp}{dx} \right) = U \frac{d}{dx} \left(\frac{m_1}{f_0} \right). \quad (3.11)$$

Note that in order to calculate the viscosity from (1.3), the strain rate $\frac{\partial u}{\partial y}$ is required which can be obtained either from equation (2.9) or (2.11).

The integral operators are evaluated numerically using a second order trapezoidal rule as shown below,

$$\begin{aligned}\mathbf{I}_y^{y+\Delta y} &= \int_0^y F_x = \int_y^{y+\Delta y} F_x, \\ \mathbf{I}_y^{y+\Delta y} &= \frac{\Delta y}{2}(F_{x,y} + F_{x,y+\Delta y}),\end{aligned}\tag{3.12}$$

similarly for the definite integral bounded between $y = 0$ and $y = h$,

$$\begin{aligned}\mathbf{I}_0^h &= \mathbf{I}_0^{y_0} + \mathbf{I}_{y_0}^{y_0+\Delta y} + \dots + \mathbf{I}_{h-\Delta y}^h, \\ \mathbf{I}_0^h &= \sum_{j=1}^N \frac{\Delta y}{2}(F_{x,y_j} + F_{x,y_{j+1}}), \\ \mathbf{I}_0^h &= \Delta y \left(\sum_{j=1}^N \frac{1}{2}(F_{x,y_j} + F_{x,y_{j+1}}) \right) \\ \mathbf{I}_0^h &= \Delta y \left(\sum_{j=1}^N F_{x,y_{j+\frac{1}{2}}} \right),\end{aligned}\tag{3.13}$$

for the integration of the quantity Fy between the relevant intervals, the same formulation (3.12) and (3.13) is used. Simply multiplying F with y and setting the new quantity as the input to the primitive function $\mathbf{I}$ is sufficient. If a suitably large number of points are used, no loss in accuracy is observed.

Once the integrals have been evaluated for every point along the x axis. The differential operators are discretized using a second order central difference scheme as shown previously. The rest of the equations are algebraic and their discrete values are simply obtained by plugging into the relevant equations as shown below,

$$\dot{\gamma}_{xy} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} \Big|_{ij}\tag{3.14}$$

$$\dot{\gamma}_{ij} = \frac{1}{2\Delta y}(u_{ij+1} - u_{ij-1}),$$

$$\begin{aligned}\frac{1}{F_{x,y}} &= \frac{\tau_0}{\dot{\gamma}_{x,y}} \tanh\left(\frac{\eta_x \dot{\gamma}_{x,y}}{\tau_0}\right), \\ \frac{1}{F_{ij}} &= \frac{\tau_0}{\dot{\gamma}_{ij}} \tanh\left(\frac{\eta_i \dot{\gamma}_{ij}}{\tau_0}\right).\end{aligned}\tag{3.15}$$

3.2.2 Convergence scheme

The previous fixed point iteration used to solve the discretized set of equations is also suitable for the generalized Reynolds equation but only for small values of the Λ (see section 2.4 for the definition of Λ) given a small enough under relaxation factor. For large values of Λ the piezoviscous effect results in a large change in viscosity every subsequent iteration k . The numerical damping effect due to shear thinning is insufficient.

An alternative to a more robust yet complicated nonlinear solution method such as Newton-Raphson is to implement a Runge-Kutta type marching scheme in the iterative space. Consider a general form for an under relaxation of the pressure solution at a given iteration k ,

$$p^{k+1} = \omega(p^* - p^k) + p^k, \quad (3.16)$$

where ω is an under relaxation parameter. The update equation for p can be re-written in the form,

$$\frac{dp}{dk} = (p^* - p^k) = f(p). \quad (3.17)$$

Equation (3.17) represents a marching scheme of first order in an iterative space. If an forward Euler type process is applied, the resulting update equation is that of the ordinary under relaxation above. However, for more stable updates, one can use a Runge-Kutta type marching scheme that is more accurate but more importantly has a larger stability region. Marching schemes are typically used for physical variables such as time t or space x hence the accuracy of the scheme matters more than its stability for a given step size. In an iterative process, the aim is to take larger steps to achieve convergence faster without divergence.

A typical 4th order accurate Runge-Kutta scheme is given as,

$$\begin{aligned}
\frac{dp}{dk} &= f(p), \\
k_1 &= \omega(f(p^k) - p^k), \\
k_2 &= \omega(f(p^k + \frac{k_1}{2}) - p^k), \\
k_3 &= \omega(f(p^k + \frac{k_2}{2}) - p^k), \\
k_4 &= \omega(f(p^k + k_3) - p^k), \\
p^{k+1} &= p^k + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4),
\end{aligned} \tag{3.18}$$

where ω is the size of the update or under relaxation factor. Interestingly, when a RK4 type scheme is applied to solve the system of equations (2.9) to (2.10) the iterative process can be *over-relaxed* but more iterations are required in this case. This simply implies that the region for stability is much larger in an RK4 scheme. In this study, the RK4 type scheme was applied globally to u and p as described below,

$$\begin{aligned}
\frac{\partial p}{\partial k} &= f(p, u), \\
\frac{\partial u}{\partial k} &= g(p, u), \\
k_{p1} &= \omega(f(p^k, u^k) - p^k), \\
k_{u1} &= \omega(g(p^k, u^k) - u^k), \\
k_{p2} &= \omega(f(p^k + \frac{k_{p1}}{2}, u^k + \frac{k_{u1}}{2}) - p^k), \\
k_{u2} &= \omega(g(p^k + \frac{k_{p1}}{2}, u^k + \frac{k_{u1}}{2}) - u^k), \\
k_{p3} &= \omega(f(p^k + \frac{k_{p2}}{2}, u^k + \frac{k_{u2}}{2}) - p^k), \\
k_{u3} &= \omega(g(p^k + \frac{k_{p2}}{2}, u^k + \frac{k_{u2}}{2}) - u^k), \\
k_{p4} &= \omega(f(p^k + k_{p3}, u^k + k_{u3}) - p^k), \\
k_{u4} &= \omega(g(p^k + k_{p3}, u^k + k_{u3}) - u^k), \\
p^{k+1} &= p^k + \frac{1}{6}(k_{p1} + 2(k_{p2} + k_{p3}) + k_{p4}), \\
u^{k+1} &= u^k + \frac{1}{6}(k_{u1} + 2(k_{u2} + k_{u3}) + k_{u4}),
\end{aligned}$$

The choice of ω for the pressure and velocity update can differ. A suitable choice depends on the mesh size and bearing number. As pressure has an exponential effect on the viscosity, it is easy for a fixed point iterative scheme to diverge despite a numerical damping effect due to shear thinning. It is observed while solving the generalized Reynolds for monograde and especially multigrade lubricants, that divergence is extremely likely if the choice of ω is not satisfactory. The RK4 scheme allows for a larger margin in selecting the relaxation factor.

3.2.3 Viscoelastic Reynolds equation

The generalized Reynolds for a viscoelastic lubricant described by a linear Maxwell type fluid requires the solution of an advection type equation for the shear stress. The differencing schemes for the pressure equation is not described as it is given in the previous sections. The polymer shear stress evolves according to the equation below,

$$\tau_h + \lambda \frac{\partial \tau_h}{\partial t} + \lambda U \frac{\partial \tau_h}{\partial x} = \eta_p \gamma_h. \quad (3.19)$$

The stress transport equation (3.19) is a straight forward linear advection equation with source and sink terms. To improve stability, the spatial derivative is discretized using a first order backward finite difference scheme and the time derivative is discretized using a backward Euler scheme. The term containing the shear stress is kept implicit as shown below,

$$\begin{aligned} \tau_{h,i}^{t+1} + \lambda \frac{\tau_{h,i}^{t+1} - \tau_{h,i}^t}{\Delta t} + \lambda U \frac{\tau_{h,i}^t - \tau_{h,i-1}^t}{\Delta x} &= \eta_p \gamma_{h,i}, \\ \tau_{h,i}^{t+1} \Delta t + \lambda (\tau_{h,i}^{t+1} - \tau_{h,i}^t) + \frac{\lambda U \Delta t}{\Delta x} (\tau_{h,i}^t - \tau_{h,i-1}^t) &= \eta_p \Delta t \gamma_{h,i}, \\ \tau_{h,i}^{t+1} (\Delta t + \lambda) &= \lambda \tau_{h,i}^t - \frac{\lambda U \Delta t}{\Delta x} (\tau_{h,i}^t - \tau_{h,i-1}^t) + \eta_p \Delta t \gamma_{h,i}, \\ \tau_{h,i}^{t+1} &= \frac{1}{\Delta t + \lambda} \left(\lambda \tau_{h,i}^t - \frac{\lambda U \Delta t}{\Delta x} (\tau_{h,i}^t - \tau_{h,i-1}^t) + \eta_p \Delta t \gamma_{h,i} \right), \end{aligned} \quad (3.20)$$

Similarly for the shear stress at the bottom surface it is possible to write,

$$\tau_{0,i}^{t+1} = \frac{1}{\Delta t + \lambda} \left(\lambda \tau_{0,i}^t + \eta_p \Delta t \gamma_{0,i} \right), \quad (3.21)$$

Equations (3.20) and (3.21) are first order accurate in time and space. The disadvantage is offset by choosing small time steps and a large number of nodes, since the implementation is very simple in any programming language. Vectorized languages such as Python or Matlab offer efficient one line implementations. In this study, all models excluding the direct numerical simulation were programmed using the Python 3 programming language.

3.3 Direct Numerical Simulation

The DNS of Navier-Stokes is performed using an in-house code written using the OpenFOAM [43] C++ library. A detailed description of the numerical sub-routines is not within the scope of this work but a brief comment with some references is provided for interested readers.

The Navier-Stokes system of equations for non-Newtonian fluid are written in terms of the unknown viscosity which is heuristically updated at each iterative step (or time step) until convergence is achieved. Algorithmically, it is identical to the process described in the previous sections,

$$\nabla \cdot u = 0, \tag{3.22}$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho_0} \nabla p + \nabla \cdot (\nu \nabla u). \tag{3.23}$$

3.3.1 Multigrid

Iterative solution methods such as the Jacobi or Gauss-Seidel are generally avoided for the solution of a very large matrix-vector system because of the number of iterations required to converge to an acceptable solution. Successive over-relaxation of the Gauss-Seidel method improves the convergence rate but is still quite slow within the context of direct numerical simulations. A significant improvement in convergence rate is obtained by using the multi-level methods [60]. The general principle involves solving the problem on a hierarchy of grids.

An approximate solution is obtained using a few iterations of Gauss-Seidel on a given grid and the residual is transferred to a coarser grid, (referred to as restriction), where the problem is represented in terms of the error and residual. The solution on the finest grid is then updated with an improved guess for the error interpolated from the coarser grid.

An example for a two-grid cycle is as follows.

1. Obtain an approximate solution using a few Gauss-Seidel iterations on a given grid defined by a grid spacing of h . Denoting the new approximate solution by v^h and the residual by $r^h = f^h - A^h v^h$.
2. Restrict the approximate solution on the coarser grid denoted by twice the grid spacing of the finer mesh, i.e $2h$. Perform a few Gauss-Seidel operations on $A^{2h} e^{2h} = r^{2h}$.
3. Interpolate the error, e^{2h} , to the fine grid and update the previous solution on that grid, i.e. $v^{2h} = v^{2h} + e^{2h}$

The above is a simple 2 grid v-cycle in multigrid terminology. For more details involving complicated cycling schemes such as the full multigrid and algebraic multigrid, the reader is referred to [60].

3.3.2 Pressure implicit split operator (PISO)

Equations (3.22) and (3.23) are solved for the velocity using a multigrid solver [61]. Using an initial guess for the pressure field, the velocity equation is discretized using a second-order accurate scheme with a biasing factor. Once, the velocity field is obtained, the pressure field is calculated by the well-established PISO algorithm [62] which decouples the pressure from velocity. A simple example is given below,

Discretizing all the terms in the momentum equation except the gradient of the scalar pressure field, p ,

$$Au = -\nabla p + r, \quad (3.24)$$

$$u = A^{-1}(-\nabla p) + A^{-1}r, \quad (3.25)$$

substituting equation (3.25) into the continuity equation for incompressible fluids,

$$\nabla \cdot u = 0, \quad (3.26)$$

$$\nabla \cdot (A^{-1}(-\nabla p)) + \nabla \cdot (A^{-1}r) = 0, \quad (3.27)$$

a second order equation for pressure is obtained,

$$\nabla \cdot (A^{-1}(\nabla p)) = \nabla \cdot (A^{-1}r), \quad (3.28)$$

note the similarity of PISO with that of obtaining Reynolds equation from the continuity equation. The values of u and p are staggered on the grid with u being solved at the cell center and p interpolated to the faces for implementation in (3.28).

Additionally the stress term given in equation (3.23) is expanded before being implemented, $\nabla \cdot (\nu \nabla u) = \nu \nabla^2 u + \nabla \nu \nabla u$. Then the viscosity is updated for piezoviscous thickening and shear thinning via equations (1.1) and (1.3) respectively.

3.3.3 Viscoelasticity

Implementation of viscoelastic effects is not too difficult in a transient setting. Additional to the Navier-Stokes system of equations (3.22) and (3.23), the constitutive relation for the polymer stress is solved (1.5). Note also that the momentum equation is modified to include the non-Newtonian polymer stress. The subscript p denotes a polymer and the shear stress field τ is the non-Newtonian polymer shear stress. The equations are given below,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho_0} \nabla p + \nu_s \nabla^2 u + \frac{1}{\rho_0} \nabla \cdot \bar{\bar{\tau}}, \quad (3.29)$$

$$\bar{\bar{\tau}} + \lambda \left(\frac{D\bar{\bar{\tau}}}{Dt} \right) = \bar{\bar{L}}\bar{\bar{\tau}} + \bar{\bar{\tau}}\bar{\bar{L}}^T + 2\eta_p \bar{\bar{D}}, \quad (3.30)$$

where, L is the gradient of velocity and $\frac{D}{Dt}$ is the material derivative operator.

Equation (3.30) is solved via a second order upwinding for the advective term appearing in the material derivative. A first order Euler scheme is used for the time derivatives appearing in the momentum and constitutive relation. The solution is stopped when a steady state tolerance is satisfied for the stress and pressure.

The only difference in the DNS algorithm due to a constitutive relation for stress is that an additional equation for the unknown stress field must be solved before implementing PISO for the pressure and velocity fields.

Chapter 4

Results and Discussion

The following chapter presents the major findings of this study. The results obtained from the models presented in chapter 2 are presented in detail. First, surface geometries and assumptions used in the analysis are given. Followed by a discussion on the models derived in section 2.

4.1 Surface geometry and Assumptions

4.1.1 Geometry

Generally, contact geometries have been defined in literature for typical bearings such as thrust pads, rollers or journal bearings. The surfaces are chosen to help in delineating certain effects that arise in viscoelastic lubricants described later. As an example, the plane slider always results in a decrease in load carrying capacity while the parabola does exhibit positive load for certain values of De . Regardless, the load is dependent on the geometry. Three typical surfaces are given below,

1. A plane slider contact used to approximate a thrust-pad. It is defined as a straight line governed by two parameters, the slope and maximum or

minimum height. See equation (4.1). Additionally, the second derivative of the surface is zero i.e. $\frac{d^2h}{dx^2} = 0$.

$$H = h_{max} + \frac{h_{min} - h_{max}}{L}x, \quad (4.1)$$

2. A parabolic slider contact which can approximate the zone of hydrodynamic lubrication in rollers or journals, see equation (4.2). The second derivative is constant and depends on the length, maximum and minimum height,

$$\frac{d^2h}{dx^2} = \frac{h_{max} - h_{min}}{l^2}$$

$$H = \frac{h_{max} - h_{min}}{0.5L^2}x^2 + \frac{h_{min} - h_{max}}{0.5L}x + h_{max}, \quad (4.2)$$

3. Lubricated surfaces can be engineered or textured to provide improvement in sliding properties (reduced friction coefficient) or simply an enhancement in load as shown previously. However, very rarely are real surfaces considered to be smooth. Sharp protrusions and deformations may exist on the primary surface in the form of roughness or load enhancing textures that need to be resolved with a finer mesh. A simpler approach, initially, is to consider a single unit cell of the protrusion (4.1) in isolation. Assuming a reasonable value for the ambient pressure depending on the location, and purpose of the protrusion. A detailed analysis on the validity of employing a Reynolds type assumption has already been performed [1]. It has been proven that as long as the fundamental assumption of a thin film is not violated, even very small surface deformities can be successfully modeled by the Reynolds equation. A ridge type surface is used to approximate a single unit of deformation or surface roughness. Note that the function is infinitely differentiable,

$$H = A \times 10^{(-10(\frac{x-x_c}{\lambda})^2)} \cos\left(\frac{2\pi(x-x_c)}{\lambda}\right), \quad (4.3)$$

$$A = h_{max} - h_{min}.$$

The results obtained are subject to the following boundary conditions unless stated otherwise.

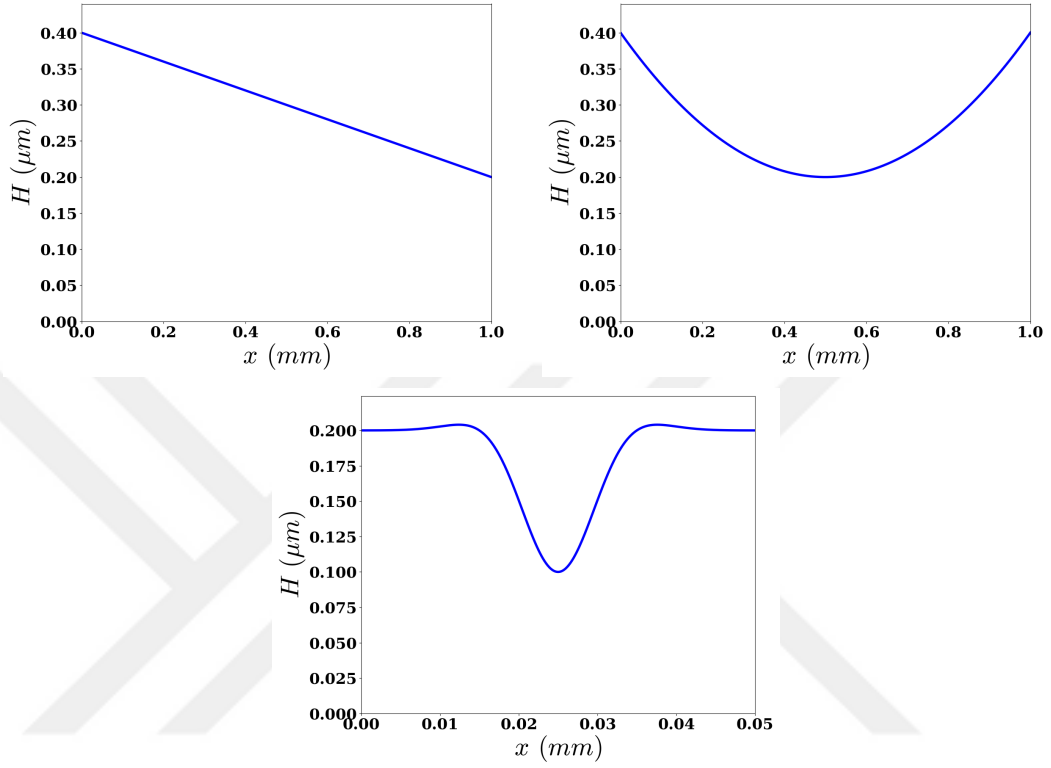


Figure 4.1: Surface profiles adopted for the analysis of all models and non-Newtonian effects in this study.

4.1.2 Assumptions

The primary purpose of this study is to investigate load variations due to two main non-Newtonian effects arising in polymer enhanced lubricants or multigrade oils. The models derived in this study or those adopted from literature contain several flow and rheological parameters surface speed U , ambient pressure p_0 , viscosity η_0 etc. The pressure distribution and consequently the load and friction coefficients vary if any parameter changes. It is therefore expedient to examine

Surface	$p(x = 0)$ Pa	$p(x = L)$ Pa
Plane slider	1×10^5	1×10^5
Parabolic slider	1×10^5	1×10^5
Ridge	0.15×10^9	0.15×10^9

Table 4.1: Pressure boundary conditions for the three surfaces (see figure 4.1).

Parameter	Value
η_0	0.1 Pa·s
ρ_0	870 $\frac{kg}{m^3}$
τ_0	7×10^6 Pa
p_0	0.15×10^9 Pa
U	0.1 $\frac{m}{s}$

Table 4.2: Modeling parameters used in figure 4.2

variations in these quantities against suitable non-dimensional numbers, such as the slide to roll ratio S or film thickness ratio. In this study, dependence on the bearing number is chosen since it includes effects from most flow and rheological parameters [3]. The results for a purely inelastic non-Newtonian interaction are presented under certain assumptions.

- Firstly, a viscoelastic lubricant is assumed in the models that follow.
- The nonlinear viscous component of the lubricant is approximated by the Bair-Winer model.
- This analysis is a precursor to a more detailed EHL study at high bearing numbers using viscoelastic lubricants. Hence, the film thicknesses assumed for the plane and parabolic slider are not commonly encountered without a deformation of the surface. Hence, the strain rates are not typical within the context of a purely hydrodynamic regime. A detailed explanation is given in the passages that follow.

4.2 Model validation

It is necessary to establish the validity of models presented in this study. Results are compared with those found in [1]. The rheological properties are given in Table 4.2.

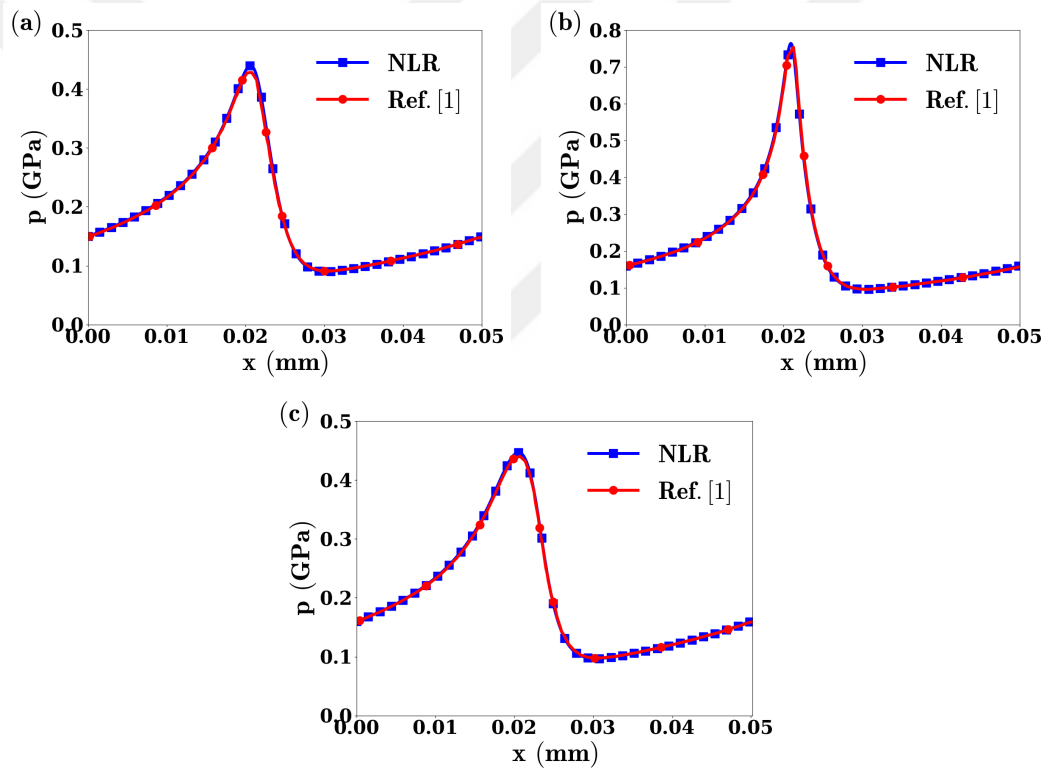


Figure 4.2: Pressure profiles for the ridge (see figure 4.1) [1]. (a) Newtonian fluid $p_0 = 0.15 \times 10^9$ Pa, (b) Newtonian fluid $p_0 = 0.16 \times 10^9$ Pa and (c) Non-Newtonian fluid $p_0 = 0.16 \times 10^9$ Pa.

In this study, the Bair-Winer constitutive relation is used to model shearthinning. However, it is uncommon in literature due to the popularity of the Ree-Eyring constitutive relation, despite its validity. Figure 4.2 is produced using the nonlinear Reynolds equation (see equation (4.5)) model with a Ree-Eyring constitutive relation. This is the only case in which the Ree-Eyring model is used. The central idea is to validate the governing equation rather than a particular empirical model. It is implied that if the governing equation is suitable for modeling non-Newtonian effects, then substituting a different constitutive law is trivial since the equations are solved in a decoupled manner. In the original work, a compressible, non-Newtonian lubricant was assumed using the Dowson Higginson model given below,

$$\rho = \rho_0 \frac{c_1 + c_2 p}{c_1 + p}, \quad (4.4)$$

where $c_1 = 5.9 \times 10^9$ and $c_2 = 1.34$ and $\rho_0 = 870 \frac{kg}{m^3}$ is the ambient or reference density.

The results presented in this study do not include compressibility for several reasons. Primarily because it is not the goal to study the effects of cavitation and the viscoelastic models derived in this study assume an incompressible fluid. Hence, compressibility is not considered for the analysis of non-Newtonian effects in this study. All inelastic models presented are in good agreement with the non-linear Reynolds under the appropriate assumptions.

4.2.1 Mesh quality

Numerical solutions are only reliable if the values obtained are consistent across different mesh sizes. This would show that a maximum accuracy for the set of values in question has been reached and further refinement of the mesh does not influence the results. For low bearing numbers, a coarse mesh is sufficient to deliver accurate results for allowable load, maximum and average pressure. However, for low to moderate values of bearing number (this varies based on the geometry) a fine mesh is required, even for simple geometries such as the plane

slider 4.1. Figure 4.3 compares the average and maximum pressure for against the bearing number obtained from the generalized Reynolds equation.

The type of surface clearly has an impact on the mesh quality, this is not surprising. High values of Λ also require a finer mesh to resolve the pressure distribution. Note that a finer mesh is required to properly construct smooth surfaces such as the slider and ridge but also to capture the variation in viscosity at high Λ . Based on the results from figure 4.3 the mesh quality for the direct numerical simulation is selected as the final value appearing in figure 4.3.

4.3 Load and coefficient of friction

Figure 4.4 shows variation of load and coefficient of friction for the three typical surfaces. The range of bearing numbers was arbitrarily chosen to capture an increasing and decreasing trend for each metric. The exact value of the load depends on the surface and lubricant rheology (yield stress, ambient viscosity, ambient density etc), hence the trends are more useful than the specific values. Following is a discussion of the important aspects observed.

The dependence of the **(i)** load and **(ii)** coefficient of friction against Λ (see section 2.4 for the definition of Λ) are not new to this study for the case of hydrodynamic lubrication [3]. However, the inclusion of nonlinear effects at a high bearing number do not exist in literature for the surfaces examined. It should be evident that a small bearing number is indicative of a nearly Newtonian fluid but a Newtonian profile may not necessarily have a low bearing number. As Λ increases the non linear effects start to dominate. For simplicity, the models are referred to as,

For small Λ all models are in good agreement. At certain Λ the departure from a Newtonian assumption becomes significant and nonlinear effects dominate. At moderate values of Λ piezoviscous thickening is dominant and a steady increase in pressure is observed. As the shear stress in the contact approach the yield stress

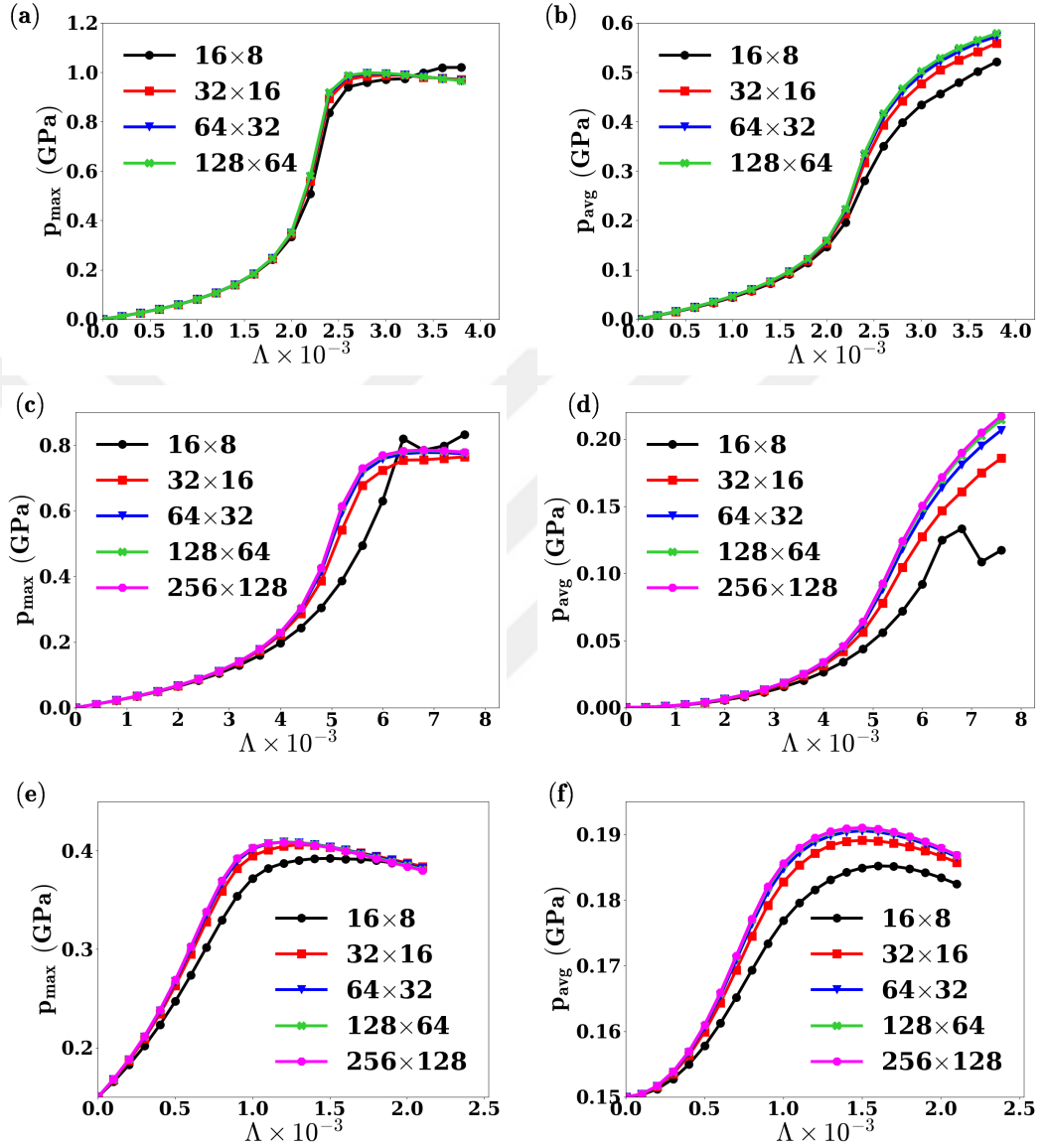


Figure 4.3: Maximum pressure (a, c, e) and average pressure (b, d, f) versus Λ for varying mesh sizes for the plane slider (a, b), the parabolic slider (c, d) and the ridge (e, f).

RE	Reynolds equation (2.3)
LVE	Linearized Viscoelastic Reynolds equation (2.55)
NLR	Non-linear Reynolds equation (4.6)
GR	Generalized Reynolds equation (2.10)
DNS	Direct numerical simulation

Table 4.3: A list of abbreviations for the models used in this study.

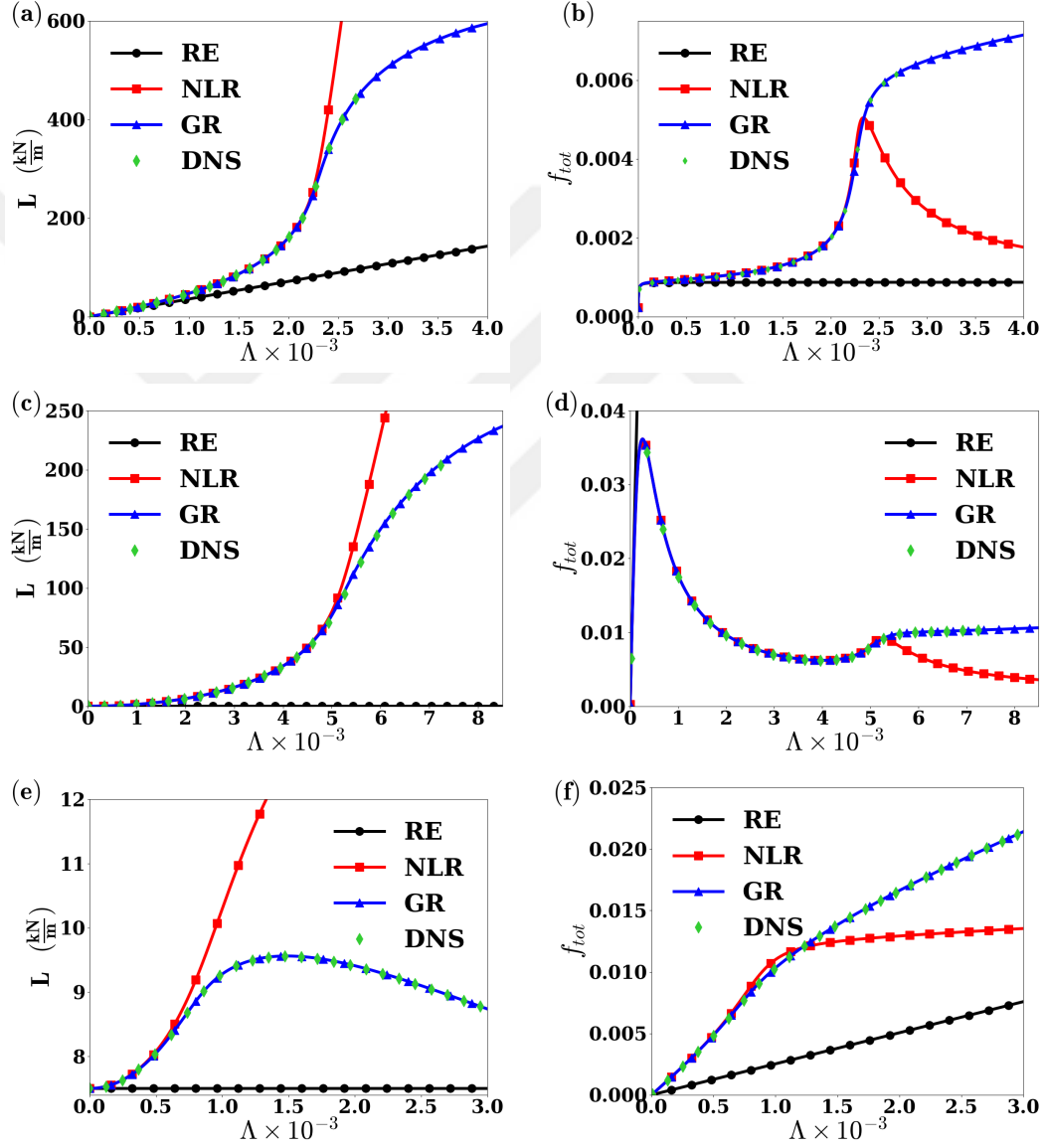


Figure 4.4: Maximum allowable load L (a, c, e) and coefficient of friction f_{tot} (b, d, f) versus Λ for the plane slider (a, b), the parabolic slider (c, d) and the ridge (e, f).

of the lubricant, the increase in load is damped and either decreases (ridge) or approaches a plateau. For the purpose of comparison, the plane and parabolic slider are assumed to be rigid and subject to similar pressures. Although an exact physical scenario is not the focus of this study, these profiles can be assumed as micro-textures on a deformed surface, for example dimples and inclined pockets etc. Compared to the ridge deformity, the parabolic surface is more efficient as it minimizes friction and increases load. The most reliable non-Newtonian models are the generalized Reynolds and DNS of the Navier-Stokes equations.

At large enough bearing numbers, the load calculated from the more reliable generalized Reynolds or DNS shows a stagnation in the maximum allowable load. There is no significant gain in load due to very high strain rates. Therefore, for an optimal choice of the operating bearing number, the variation in friction coefficient is also calculated. The friction coefficient continues to rise in the region where load has stagnated indicating that frictional losses are increasing without any useful gain in load. s

A ridge significantly impedes fluid flow resulting in a large increase in shear stress. Hence, a high stress leads to significant thinning of the lubricant, reducing load. Figure 4.4 shows low value for the coefficient of friction that rises as bearing number is increased. Generally surface roughness is problematic in boundary lubrication where direct contact of the solid boundaries causes a dramatic increase in friction.

The nonlinear Reynolds is valid upto $\Lambda < 700$ the non linear Reynolds provides excellent agreement with the solutions from generalized Reynolds and DNS. A typical value of the bearing number from literature for this type of surface moving at $1m/s$ and atmospheric pressure is typically $\Lambda < 500$. Using the exact values in [1], $\Lambda = 150$ but since the study also included the effects of compressibility, the exact results differ.

As Λ further increases, the ridge allows an increase in pressure beyond which a continuous decrease is observed. The argument for an incompressible fluid in

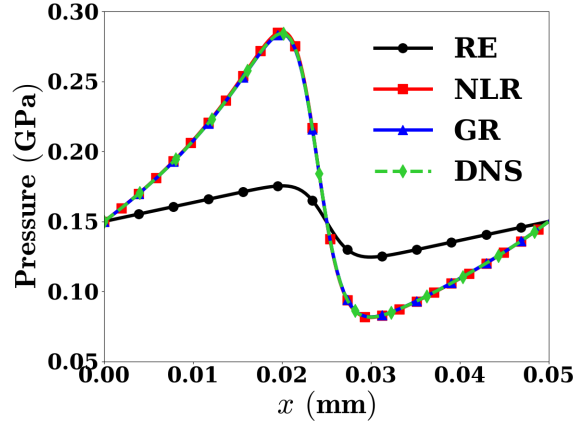


Figure 4.5: Pressure variation along the ridge, see figure (4.1), for $\Lambda = 150$. $\eta_0 = 0.01 \text{ Pa}\cdot\text{s}$, $p_0 = 0.15 \times 10^9 \text{ GPa}$

the context of such extreme pressures, close to 1 GPa starts to breakdown. However, compressibility of the lubricant serves to increase the pressure, although not significantly, for all Λ . It is similar in effect to the piezoviscous thickening but is not exponential. Therefore, even if the lubricant was assumed to be compressible in this case, the pressure maximum would not be affected as it is limited by the value of the yield stress.

Eventual increase in Λ will lead to a decline in load and shift the pressure towards an almost Newtonian response at very high bearing numbers. To give some perspective, under the same geometric parameters of the ridge (4.3), $\Lambda = 3000$ implies a surface moving at $U = 20 \text{ m/s}$.

Figure 4.5 shows gradual flattening of the pressure along the ridge at high Λ . For other choices of the lubricant rheology and geometric parameters, the exact values of the pressure might differ but eventual decrease in the maximum will occur at some Λ . This decline is due to the existence of a yield stress beyond which there is no gain in traction. If the maximum value of the shear stress is fixed, and strain rate increases as Λ increases then significant thinning of the lubricant will occur. This loss in viscosity can be permanent since beyond the yield point polymers coils are ruptured.

4.4 Modified non-linear Reynolds equation

Numerically, a non-linear Reynolds equation is perhaps the most favorable approach. It is a stationary iterative scheme in which the non-linear relations for viscosity are updated at each iteration until a suitable convergence criteria is met [63]. The Reynolds equation in either linear or non-linear form is a one dimensional equation whereas the strain rate varies significantly along and across the contact,

$$\frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) = \frac{U}{2} \frac{dh}{dx}. \quad (4.5)$$

Equation (4.5) restricts the viscosity to be a 1D function varying along x or the contact length only i.e. $\mu \neq F(x, y)$. However, the lubricant viscosity varies strongly along y as strain rates increase beyond a critical value ($\dot{\gamma}_c \approx \frac{\tau_0}{\eta_0}$). Hence, the functional approximation of the viscosity due to shear thinning is no longer one dimensional i.e. $\mu = f(\dot{\gamma}, p)$. The strain rate is a two dimensional function that depends on x and y . Since the strain rate varies along y , it can be argued as to what set of values along the film thickness is suitable. For example, it can be assumed that the strain rate is largely due to a Couette flow, $\gamma = \frac{U}{h}$ [20]. The strain rate for figure 4.4 was evaluated at the mid-plane which is equivalent to averaging the strain rate at the top and bottom surface. In fact, this is equivalent to assuming a Couette flow since the Poiseuille interactions will cancel out exactly,

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{1}{2\mu} \frac{dp}{dx} (2y - h) + \frac{U}{h}, \\ \left. \frac{\partial u}{\partial y} \right|_{average} &= \frac{\left. \frac{\partial u}{\partial y} \right|_{y=0} + \left. \frac{\partial u}{\partial y} \right|_{y=h}}{2} = \frac{U}{h}. \end{aligned}$$

Nevertheless, the simplistic form of equation (4.5) is appealing in studies that focus on different aspects of lubrication, such as cavitation [63]. At low Λ , the form of equation (4.6) is accurate enough to predict the pressure. However, at high Λ , significant overestimation of pressure will occur if an update of viscosity is performed without any modifications. A simple solution to the problem is proposed below but first it is necessary to identify the reason for a sharp loss

in accuracy. Expanding the derivative appearing on the right hand side of the momentum equation,

$$\begin{aligned}\frac{dp}{dx} &= \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right), \\ \frac{dp}{dx} &= \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial \mu}{\partial y} \frac{\partial u}{\partial y},\end{aligned}$$

Applying the chain rule to expand $\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial \gamma} \frac{\partial \gamma}{\partial y}$

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial \mu}{\partial \gamma} \frac{\partial \gamma}{\partial y} \frac{\partial u}{\partial y},$$

where $\dot{\gamma}$ is given simply by,

$$\begin{aligned}\dot{\gamma} &= \frac{\partial u}{\partial y}, \\ \frac{dp}{dx} &= \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial \mu}{\partial \gamma} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y}, \\ \frac{dp}{dx} &= \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial \mu}{\partial \gamma} \frac{\partial u}{\partial y}, \\ \frac{dp}{dx} &= \frac{\partial^2 u}{\partial y^2} \left(\mu + \frac{\partial \mu}{\partial \gamma} \frac{\partial u}{\partial y} \right).\end{aligned}\tag{4.6}$$

Equation (4.6) now contains contributions to the pressure arising from a strain dependent viscosity function. Interestingly, there is no need to proceed further and re-derive a Reynolds equation with this new expression. Denoting an equivalent viscosity that depends on both x and y ,

$$\mu_{eq} = \mu + \frac{\partial \mu}{\partial \gamma} \frac{\partial u}{\partial y}.\tag{4.7}$$

The difference due to the absence of the second term in (4.7) is shown in figure 4.6. Note that μ by itself implies a 1D viscosity either due to piezoviscosity or the variation in the strain rate along x .

Incorporating the equivalent viscosity into equation (4.5) and comparing the solution with a more robust two dimensional generalized Reynolds shows that accuracy is greatly improved. Figure 4.4 is produced with the strain rate in

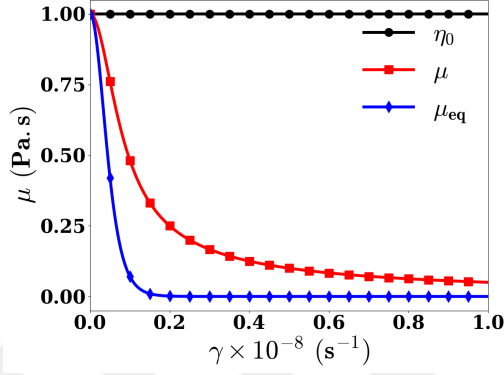


Figure 4.6: A comparison of the viscosity terms appearing in equation (4.6). μ assumes no variation along y whereas, in μ_{eq} the dependency on y is accounted.

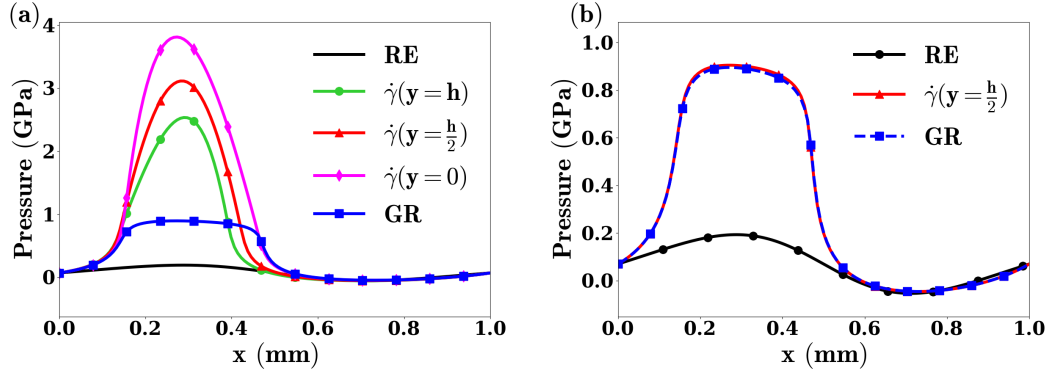


Figure 4.7: **(a)** Pressure profiles obtained using the nonlinear Reynolds with viscosity updated at different locations along y . **(b)** Pressure profile using the effective viscosity and strain rate at the mid-plane. $\eta_0 = 0.01\text{Pa.s}$, $p_0 = 1 \times 10^5\text{ Pa}$, $\Lambda = 5000$ and $\tau_0 = 5 \times 10^5\text{ Pa}$.

Parameter	Value
Yield Stress (τ_0)	$7 \times 10^6 \text{ Pa}$
Dynamic Viscosity η_0	$1 \times 10^{-2} \text{ Pa.s}$

Table 4.4: Values for the constants appearing in the Bair-Winer Model (see equation 1.3)

equation (4.7) equal to the mid-plane i.e. $\frac{\partial u}{\partial y} = \frac{U}{h}$. The yield stress was reduced to $5 \times 10^5 \text{ Pa}$ in order that shear thinning remains dominant at low Λ . Interestingly, any other choice for the strain rate did not result in accurate pressure profiles. However, this is probably due to an increase in non-linearity that renders the expression (4.7) as inappropriate for use in an iterative scheme.

Implementing the modified expression for viscosity, equation (4.7), produces substantially better results, shown in figure 4.8, for the heuristic non-linear approach. The simpler geometries are in excellent agreement, even for very large Λ . The more complex ridge may not agree quantitatively but demonstrates identical trends.

Figure 4.4 shows that a simple 1D model can accurately predict the pressure profile at moderate to large values of Λ . In fact, numerically the advantage is significant in terms of computation time and mesh quality. For complex surfaces such as the ridge, a fine mesh is a basic requirement for simple Newtonian lubricants. Introducing non-linear functions further increases the burden on mesh quality. In the case of two dimensional models, refining mesh quality along one axis alone will be insufficient. Thus implementing equation 4.5 with the viscosity correction can offer a computationally appealing alternative (see figure 4.4).

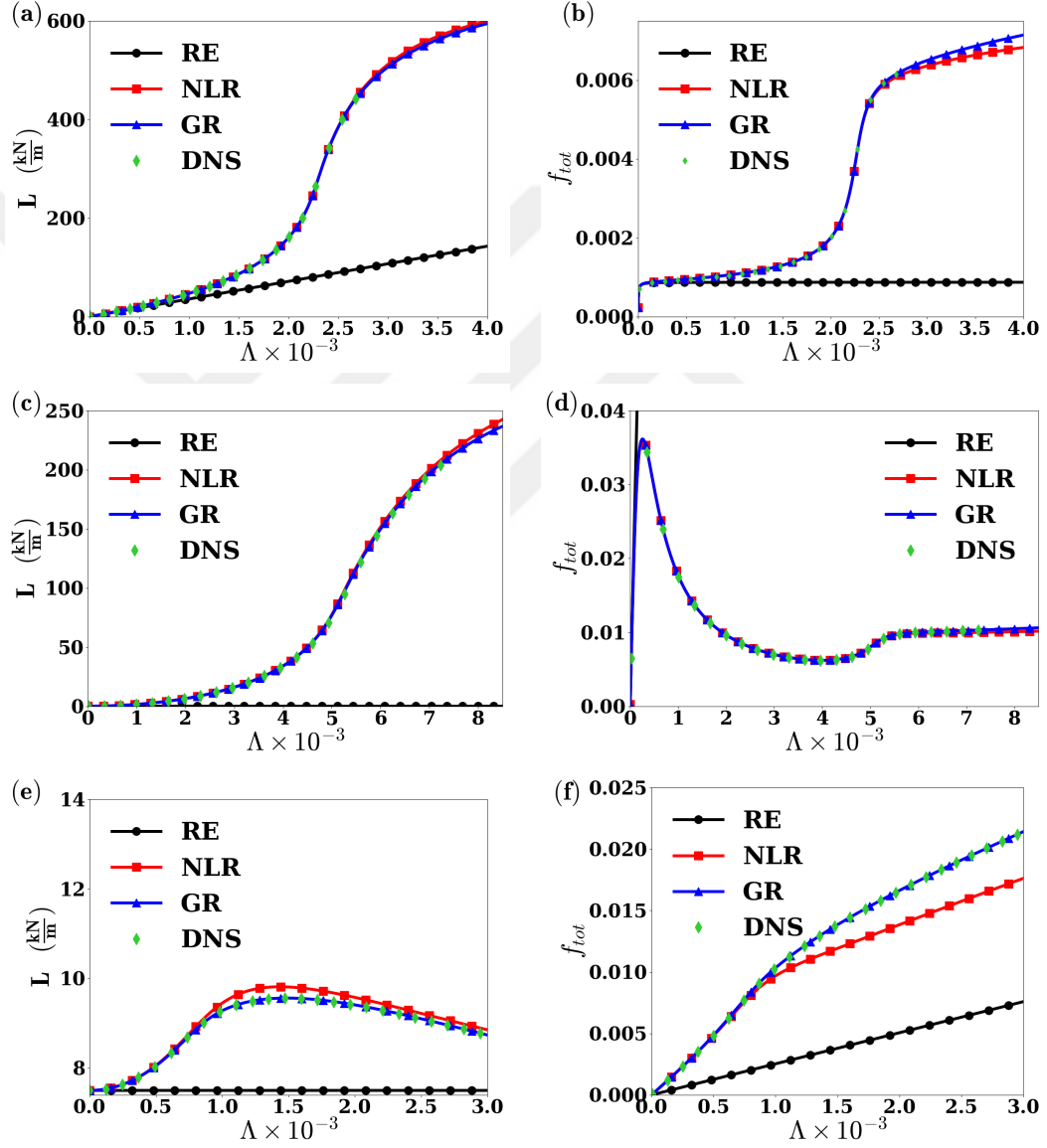


Figure 4.8: Maximum allowable load L (a, c, e) and coefficient of friction f_{tot} (b, d, f) versus Λ using the modified viscosity (see equation 4.7) for the plane slider (a, b), the parabolic slider (c, d) and the ridge (e, f).

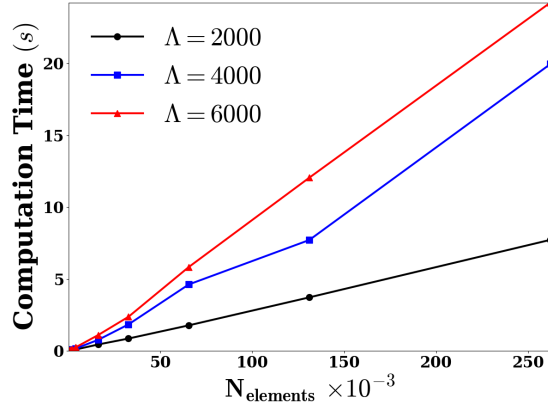


Figure 4.9: A plot of CPU time in seconds against increasing mesh size using the non-linear Reynolds.

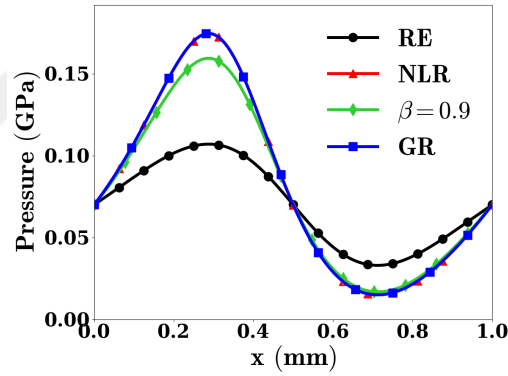


Figure 4.10: A comparison of pressure profiles for the parabolic slider (see figure 4.1) at $\Lambda = 1500$ with the multigrade formulation at $\beta = 0.9$

4.5 One-dimensional generalized Reynolds

The generalized form of Reynolds equation considers density and viscosity variation along the film thickness. In the original form found in literature (see section 2), the resulting governing equation is an integral differential equation. In this study, a 1D variant is derived (see section 2) that arises when the polymer concentration is finite (i.e. $\beta < 1$) and the base solvent is not exhibiting non-Newtonian effects. However, the 1D form results in a system of equations that are numerically less appealing at high polymer concentrations. It would seem that less iterative effort is required to solve the equations only if $\beta < 0.7$ and a reduced mesh size allows for faster computation.

Figure 4.5 shows that in the limit of $\beta = 1$, the multigrade formulation given by equation (2.20) will approach the result given by generalized Reynolds for a monograde non-Newtonian lubricant. However, as Λ increases, the limit on β for is restrictive since the iterative effort required to obtain a solution rises considerably. In fact, it is almost impossible to obtain a solution unless very coarse meshes are used. However, this limitation can be overcome by using sophisticated numerical techniques, such as Newtons multigrid or simply a non-linear multigrid cycle. Further numerical investigation is not the purpose of this study and hence is not considered here.

It is interesting that equations (2.20) and (2.50) are identical in terms of the unknowns. Source terms in the equations depend on the shear stress at the surfaces only. It may not yield to an appealing analytical form but it can be used to understand the dependency of pressure on the rheological variables.

Additionally, it is trivial to show that all generalized models, derived here or otherwise reduce to the classical form of Reynolds equation in the limit of a Newtonian lubricant. The system of equations for multigrade lubricants, can be integrated to give the pressure in terms of the solvent and polymer viscosity. Details of the formulation are given in the appendix and only the final form is given below,

$$\begin{aligned}
p = p_0 + & \left(\int_0^x \frac{3U}{h^2} (2\eta_s + f) dx - \frac{\int_0^l \frac{3U}{h^2} (2\eta_s + f) dx}{\int_0^l \frac{1}{h^3} (2\eta_s + f) dx} \int_0^x \frac{1}{h^3} (2\eta_s + f) dx \right) \\
& + \int_0^x \frac{U}{h^2} g dx - \frac{\int_0^l \frac{U}{h^2} g dx}{\int_0^l \frac{(2\eta_s + f)}{h^3} dx} \int_0^x \frac{1}{h^3} (2\eta_s + f) dx \\
& - (p_0 - p_l) \frac{\int_0^x \frac{2\eta_s + f}{h^3}}{\int_0^l \frac{2\eta_s + f}{h^3}}, \tag{4.8}
\end{aligned}$$

where f and g are given as,

$$f = \mu_{Top} + \mu_{Bot} \tag{4.9}$$

$$g = \mu_{Top} - \mu_{Bot}. \tag{4.10}$$

Equation (4.8) expresses the pressure in terms of the unknown viscosity at the top and bottom surface. The arbitrary function f and g exist because of the

variation in viscosity along the film thickness. Note that if the viscosity does not vary along y , then $g = 0$ and the second term vanishes. The left-over terms are simply the Newtonian pressure of the lubricant subject to piezoviscous thickening. The last term does however show that different pressures at the entrance and exit influence the pressure profile in a non-linear manner.

Note that the one dimensional model, although not numerically appealing (for the numerical schemes used in this study) provides an explanation of the dependence of pressure on fluid flow properties. Equation (2.27) shows that non-Newtonian effects emerge as a difference in the shear stress at the top and bottom surfaces,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_s \frac{U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{h^2}{12} (\tau_h - \tau_0) \right).$$

In deriving this final form for the pressure, no functional form for the shear stress was chosen. It implies that this equation can be coupled with any equation that describes the growth of stress. If the Newtonian stress fields are substituted, the classical Reynolds equation is easily retrieved. Thus, equation (2.27) can be coupled with arbitrary shear stress profiles and the pressure evolution can be studied against several different shear stress models.

Therefore, any effort that hopes to deliver accurate results must factor in variation of lubricant viscosity at the surfaces. For typical hydrodynamic applications, pressures are low enough to neglect large variations in viscosity. However, it is the lubricant rheology and specifically, the yield stress that dictates onset of non-Newtonian effects. Therefore, careful consideration of viscosity variation is required.

The generalized form of Reynolds equation (2.13) and (2.10) are a good alternative to a DNS of Navier-Stokes equations. Clearly, for the load and friction coefficient the results are in excellent agreement. Pressure profiles are accurately predicted by the generalized Reynolds for nonlinear lubricants at very high Λ . It is well known that DNS require significant computational effort and finer meshes in general which is not a limitation of the generalized Reynolds.

Surface	$p(x)$ Pa	
Plane slider	$p^0(x = 0)$	1×10^5
	$p^0(x = l)$	1×10^5
	$p^1(x = 0)$	0
	$p^1(x = l)$	0
Parabolic slider	$p^0(x = 0)$	1×10^5
	$p^0(x = l)$	1×10^5
	$p^1(x = 0)$	0
	$p^1(x = l)$	0
Ridge	$p^0(x = 0)$	0.15×10^9
	$p^0(x = l)$	0.15×10^9
	$p^1(x = 0)$	0
	$p^1(x = l)$	0

Table 4.5: Boundary conditions for the linearized viscoelastic model (LVE)

$\tau_p(x, t)$ Pa	$u(x, t = 0) \frac{m}{s}$
$\frac{d\tau_p}{dx}(x = 0, t) = 0$	$u(x, t = 0) = \frac{U}{h}$
$\tau_p(x, t = 0) = 0$	

Table 4.6: Boundary and initial conditions for the viscoelastic model (VE)

Does it provide an absolute alternative to DNS?. Violation of the thin-film assumption will cause a breakdown of all generalized Reynolds type equations. However, it can serve to provide a good initial guess for the DNS and significantly improve computational time. In fact, the relaxation parameters at large Λ were taken to be an order of magnitude less than that for the generalized Reynolds.

4.6 Viscoelasticity

Viscoelastic response is modeled using three approaches. In increasing order of complexity, they are the linearized Reynolds **LVE**, the viscoelastic Reynolds equation **GR** and finally the direct numerical simulation of Navier-Stokes **DNS** as before. The boundary and initial conditions are given in tables 4.6,

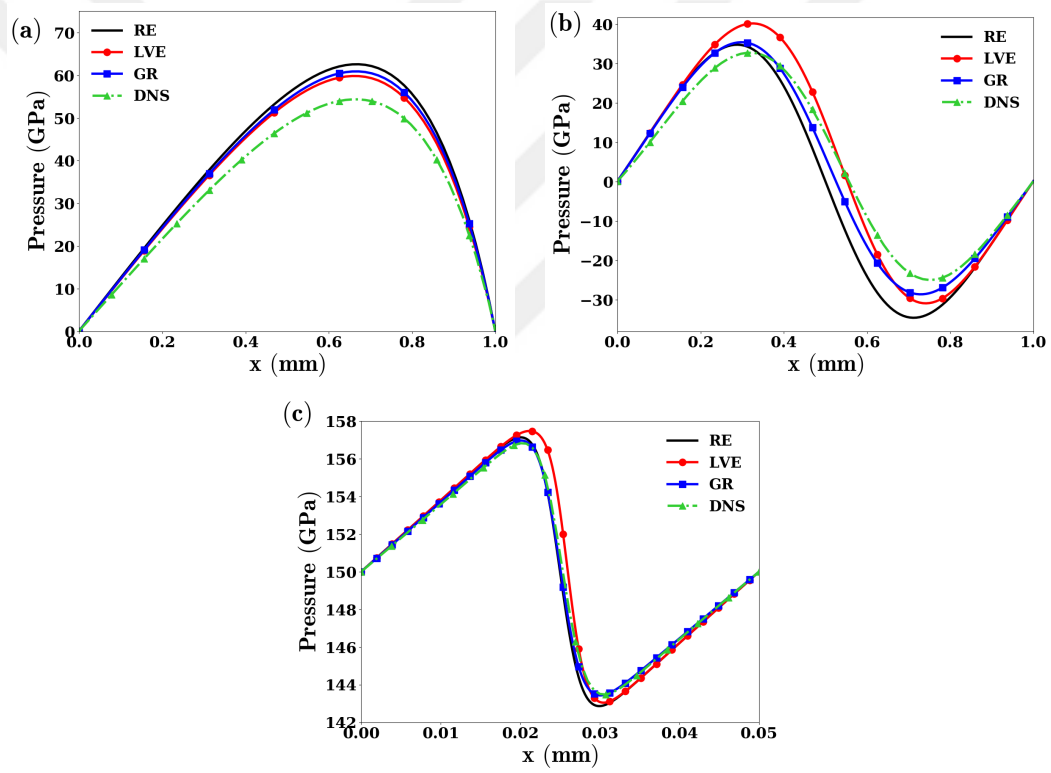


Figure 4.11: A comparison of the pressure profiles for the plane, parabola and ridge (see figure 4.1). The plane slider at $De = 0.2$, $\beta = 0.2$ (a), the parabolic slider at $De = 0.2$, $\beta = 0.5$ and (c) the ridge at $De = 0.5$, $\beta = 0.2$

Figure 4.6 depicts pressure profiles that are in good qualitative agreement for low De . The linearized Reynolds is strictly valid for low De [35] and equation (2.55) contains an explicit dependency on the surface profile, specifically on the first and second derivative of the surface height h . It shows no dependency on higher order derivatives of the surface. Therefore, the plane slider serves as a good test case for the validity of the range for De over which LVE is valid.

The second derivative of surface height is zero for the plane slider. Therefore the viscoelastic response will be due to the constant negative slope only. The deviation is not significant for a low De implying that higher order effects ($O(De^2)$) are related to the 'smoothness' of the surface profile. In the absence of the higher order surface derivatives, it is safe to use a linearized model provided De is kept low (≤ 0.5). Hence, for low De viscoelastic Reynolds is in good quantitative agreement. However, the linear model simply scales the viscoelastic pressure (p^1) and leads to erroneous profiles (negative pressures) at moderate values of De .

Similarly, for the parabola and ridge, a monotonic increase in pressure is seen as De increases. Again the LVE overestimates the increase in pressure. Note that unlike the plane slider, the overestimation is more significant. Therefore, the range of low De that will give good agreement between the models is even lower for surface profiles that exhibit some form of smoothness. That is if the second and higher order derivatives are non-zero, then the deviation from a linearized model is larger.

The LVE model is not valid for moderate to large values of De . Hence, GR and DNS are used to study effects at increasing order of De for the three surfaces. Figures 4.12 (a) to (f) show the total pressure profiles for the plane slider. As De increases, the pressure continues to drop until reaching a minimum. Any further increase in De does not induce any significant changes in the pressure. Qualitatively the GR and DNS are in good agreement, in so far as the trends are concerned. This indicates that all key physical mechanisms are captured by the GR model and that second order effects ($O(De^2)$) are significant at moderate De numbers and are responsible for the eventual stagnation of pressure change.

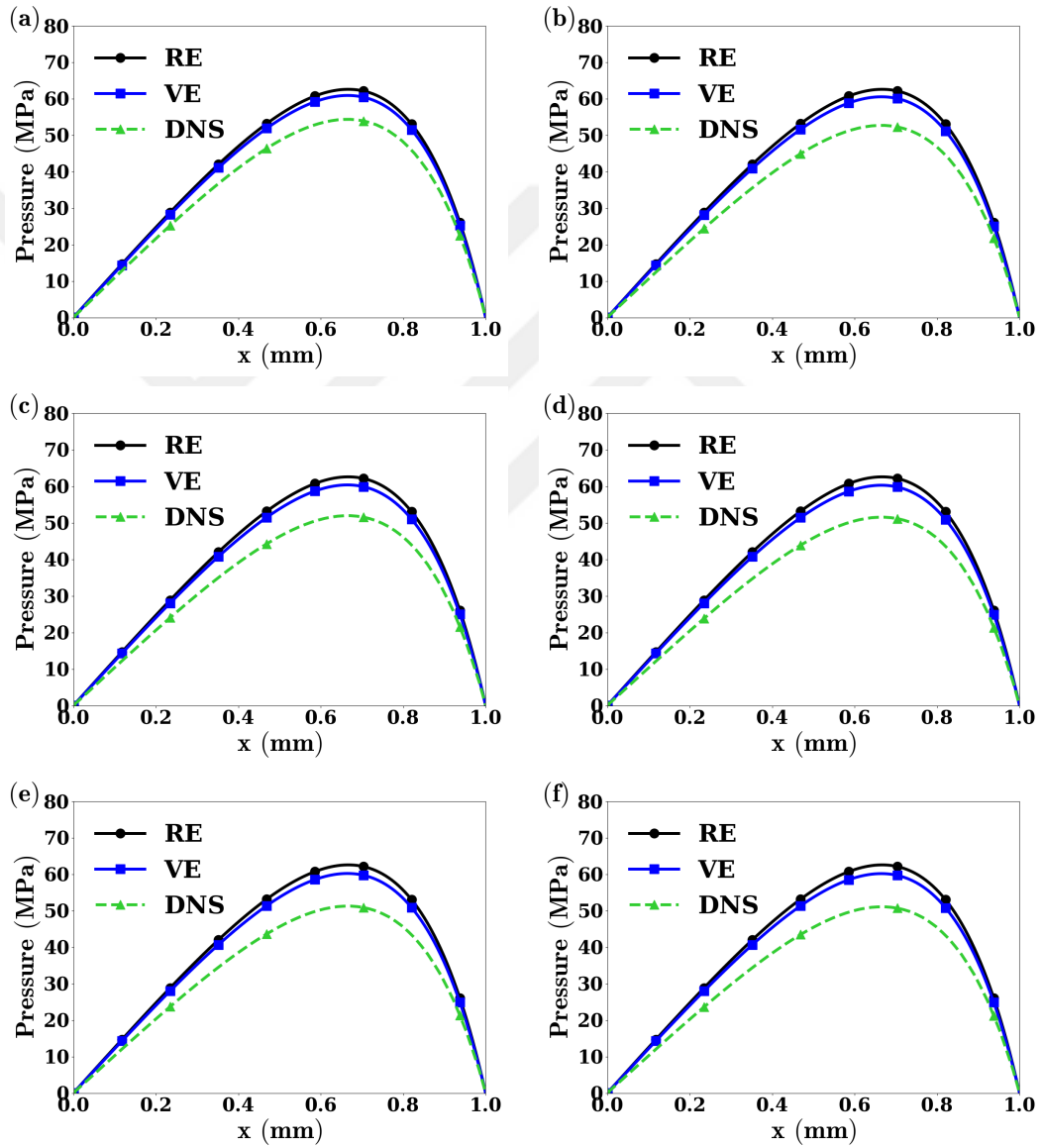


Figure 4.12: Pressure profiles for increasing De for the plane slider at $\beta = 0.2$. (a) $De = 0.5$, (b) $De = 1$, (c) $De = 1.5$, (d) $De = 2$, (e) $De = 2.5$ and (f) $De = 3$.

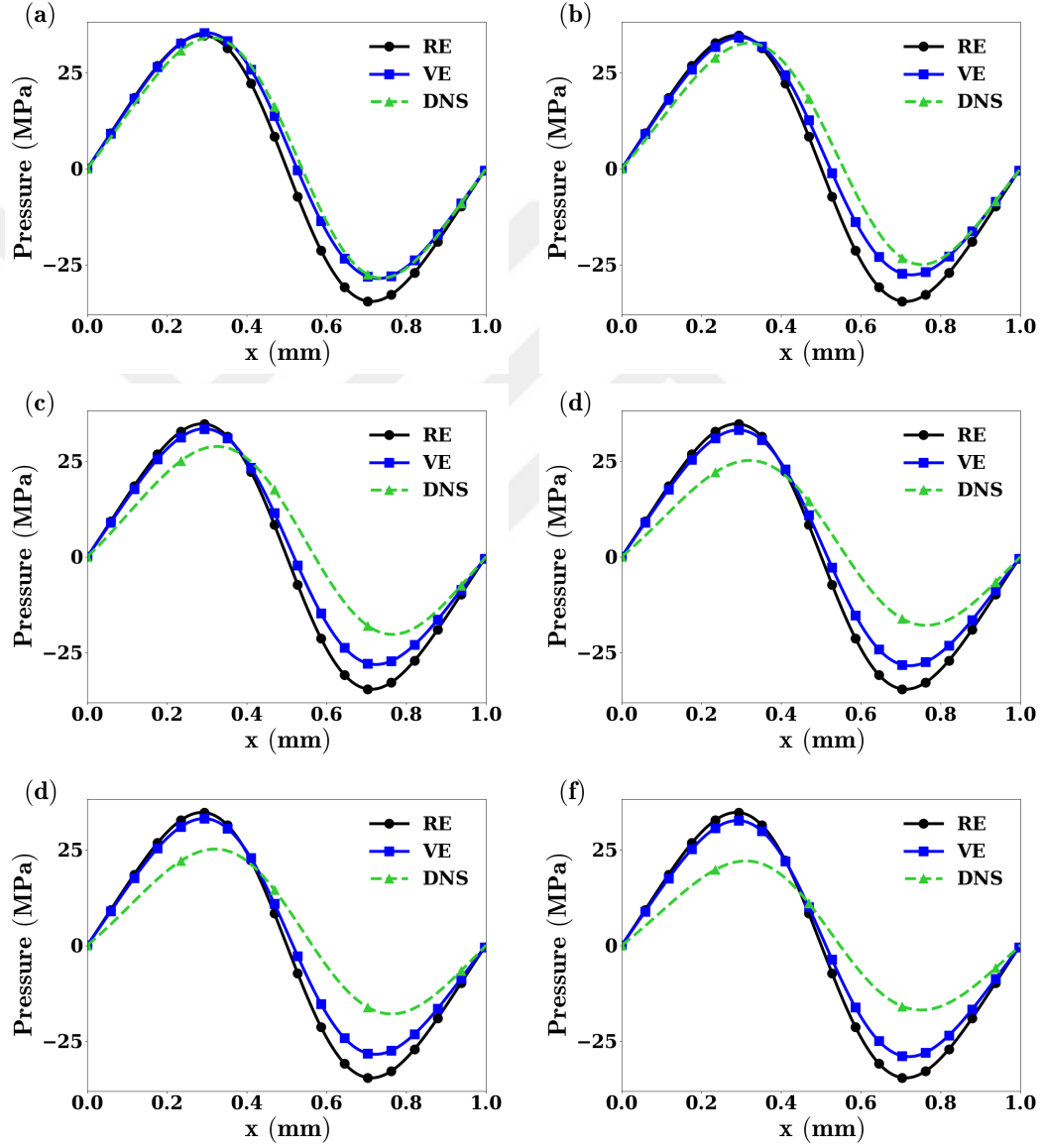


Figure 4.13: Pressure profiles at increasing De for the parabolic slider at $\beta = 0.2$. (a) $De = 0.2$, (b) $De = 0.5$, (c) $De = 0.8$, (d) $De = 1$, (e) $De = 1.2$ and (f) $De = 1.5$

The viscoelastic response of the parabolic slider, figure 4.13 (a) to (f), is interesting. In this case, as De increases the net pressure is increased until reaching a maximum ($De \approx 0.3$) and then declines steadily as De reaches large values. Hence, the stress due to polymer additives can contribute to an increase in pressure.

The ridge does not exhibit any significant viscoelastic response at any value of De . A minor increase in pressure is observed but it is negligible.

4.6.1 Viscoelastic pressure

To better understand the viscoelastic response, the pressure due to the polymers must be decoupled from the total pressure. Equation (4.16) shows that the net pressure in the film is a superposition of the Newtonian pressure (p_s) due to the base stock, a Newtonian equivalent pressure of the polymer (p_p) and a viscoelastic polymer pressure (p_{ve}). Considering the system of equations for a linear viscoelastic lubricant,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \frac{\eta_s U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{h^2}{12} (\tau_h - \tau_0) \right), \quad (4.11)$$

$$\tau_h = \eta_p \gamma_h - \lambda U \frac{d\tau_h}{dx}, \quad (4.12)$$

$$\tau_0 = \eta_p \gamma_0, \quad (4.13)$$

$$\gamma_h = \frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} - \frac{1}{2\eta_s} (\tau_h - \tau_0), \quad (4.14)$$

$$\gamma_0 = -\frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{2\eta_s} (\tau_h - \tau_0), \quad (4.15)$$

The system of equations can be simplified to show that the final pressure is given as a function of the geometry and polymer shear stress at the sliding surface,

$$p = p_0 + p_s + p_p + p_{ve}, \quad (4.16)$$

$$p_s = 6U\eta_s \int_0^x h^{-2} - 6U\eta_s h_m \int_0^x h^{-3}, \quad (4.17)$$

$$p_p = 6U\eta_p \int_0^x h^{-2} - 6U\eta_p h_m \int_0^x h^{-3}, \quad (4.18)$$

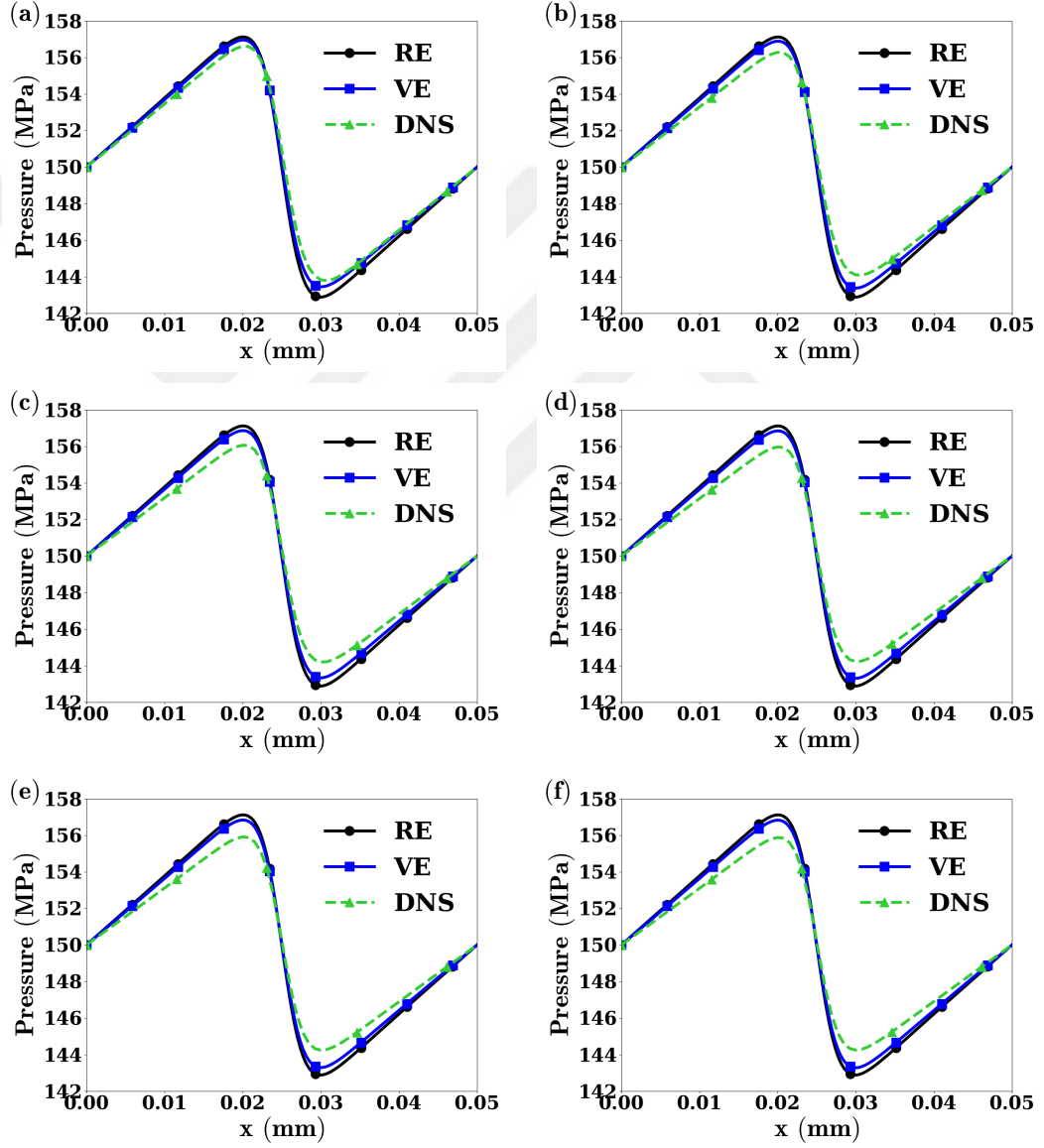


Figure 4.14: Pressure profiles for increasing De for the ridge at $\beta = 0.2$. (a) $De = 0.5$, (b) $De = 1$, (c) $De = 1.5$, (d) $De = 2$, (e) $De = 2.5$ and (f) $De = 3$.

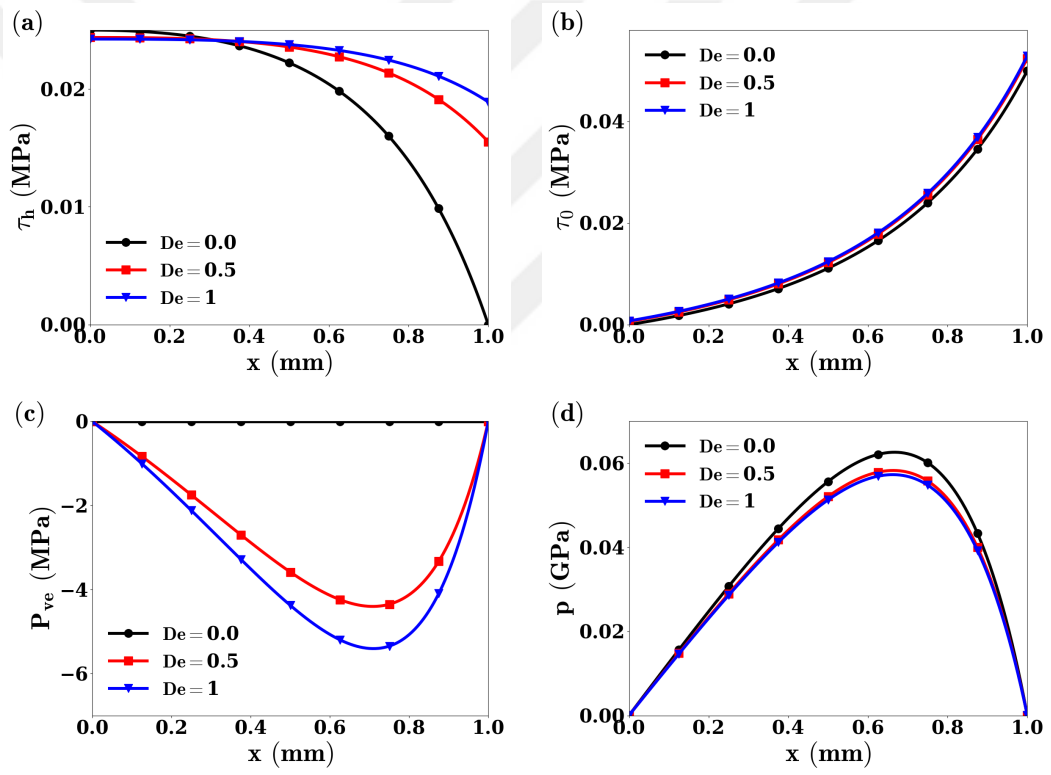


Figure 4.15: Polymer shear stress at the top surface ($y = h$) (a), at the bottom surface $y = 0$ (b), viscoelastic pressure p_{ve} (c) and total pressure p_{tot} (d) for the plane slider.

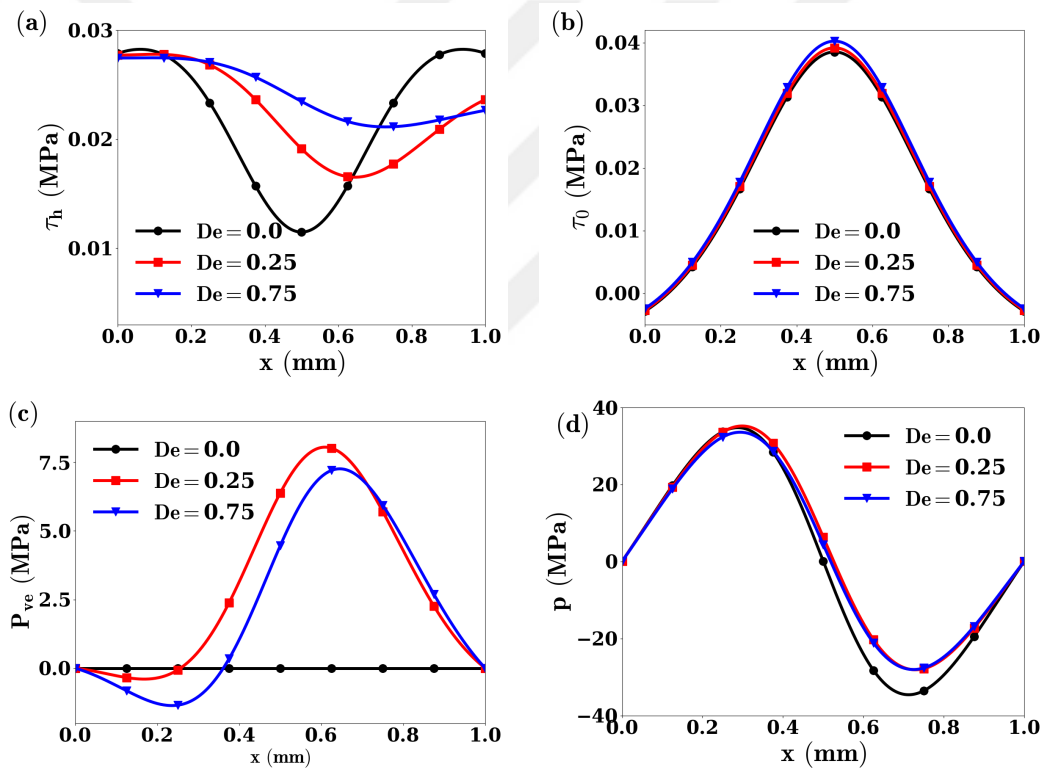


Figure 4.16: Polymer shear stress at the top surface ($y = h$) (a), at the bottom surface $y = 0$ (b), viscoelastic pressure p_{ve} (c) and total pressure p_{tot} (d) for the parabolic slider.

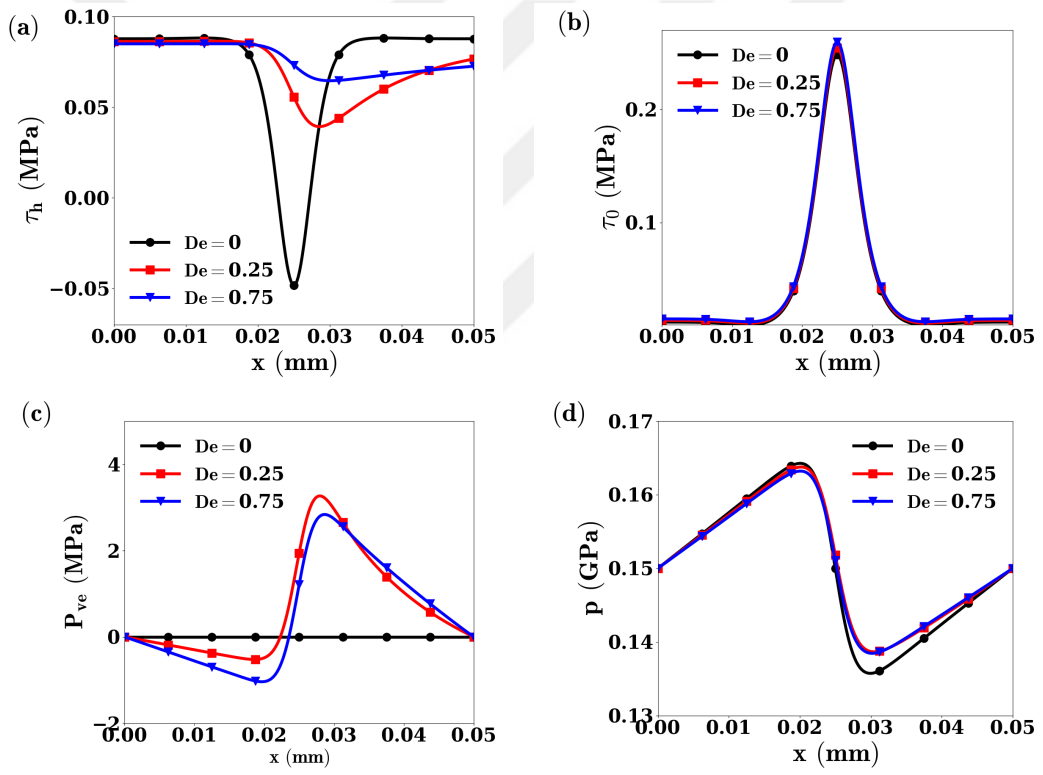


Figure 4.17: Polymer shear stress at the top surface ($y = h$) (a), at the bottom surface ($y = 0$) (b), viscoelastic pressure p_{ve} (c) and total pressure p_{tot} (d) for the ridge.

$$p_{ve} = \lambda U \left(\frac{\int_0^x h^{-3}}{\int_0^l h^{-3}} q - q \right), \quad (4.19)$$

where q is a transformation given as,

$$\frac{d\tau_h}{dx} = h \frac{dq}{dx},$$

Equations (4.16) to (4.19) are purely meant to analyze the viscoelastic pressure distribution *given* a shear stress profile. It serves to qualitatively explain the pressure increase or decrease for different geometries. Figures 4.12, 4.16 and 4.17 show the effect of viscoelasticity as for different surface profiles.

Firstly, all surfaces eventually reach a stagnation De beyond which no significant change in pressure takes place. A polymer under the Oldroyd-B formulation is allowed to stretch infinitely but its contribution to pressure eventually subsides as De increases further. This stagnation of the change in pressure is expected due to the mathematical form of the stress transport i.e. equation (3.20). The polymer stress evolves at the surfaces due to a balance between the advective transport and the net generation or loss. The net velocity at which shear stress is transported or advected out of the domain at $y = h$ increases linearly as $U_{adv} = \lambda U$. While the rate of stress generation decreases as $\approx \lambda^{-1}$. Hence, the polymer shear stress profile at $y = h$ continues to flatten (see figures 4.12, 4.16 and 4.17), leading to a decrease in the stress gradient. Note also that the absence of an advective component in the shear stress at the bottom surface ensures that the stress profile remains Newtonian.

The effect of the polymer concentration, β , is to linearly increase the viscoelastic effect. Since the polymer viscosity remains constant, dependent of neither strain nor thermophysical conditions, this is an expected result.

The validity of the proposed 1D GR viscoelastic model still needs to be addressed. If the reduced momentum equation (2.35) which forms the basis of the GR model is examined, it is obvious that inertial effects and the normal stress components, i.e. xx and yy , are ignored since they are not of the order of the film thickness. Secondly, the polymer stress constitutive relation factors only the

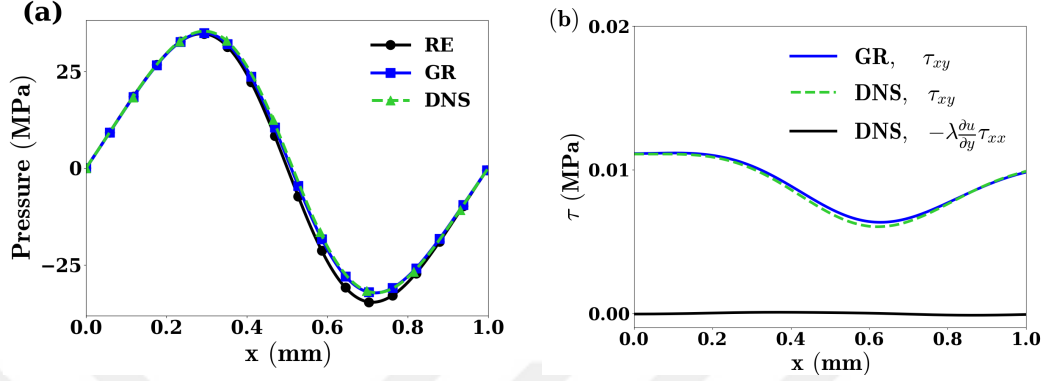


Figure 4.18: Total pressure for $De = 0.2$ and $\beta = 0.2$ (a) and polymer shear stress at the top surface ($y = h$) (b) for the parabolic slider using equation (4.20) as the constitutive relation. Mesh size is 128×64 for DNS.

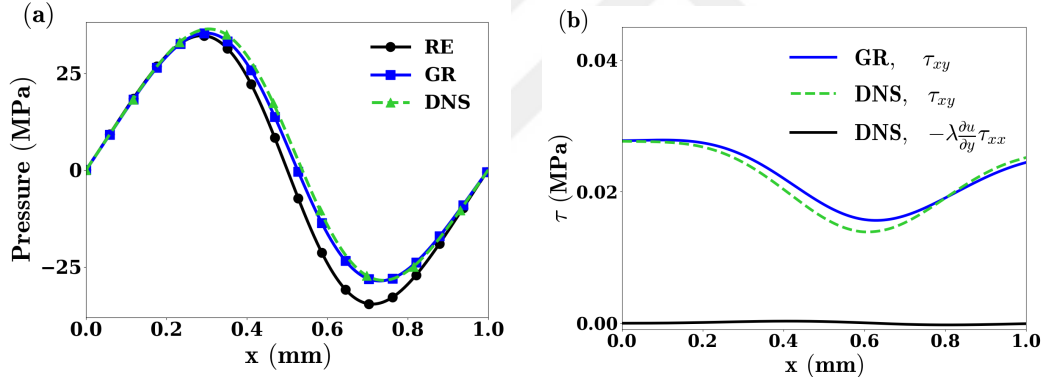


Figure 4.19: Total pressure for $De = 0.2$ and $\beta = 0.5$ (a) and polymer shear stress at the top surface ($y = h$) (b) for the parabolic slider using equation (4.20) as the constitutive relation. Mesh size is 128×64 for DNS.

material derivative of the shear stress component.

A DNS of the Navier-Stokes is performed on the reduced form of the Oldroyd-B equation (4.20), to compare the differences in the pressure profiles from GR and DNS under the same set of assumptions,

$$\underline{\underline{\tau}} + \lambda \frac{D\underline{\underline{\tau}}}{Dt} = 2\eta_p \underline{\underline{D}}. \quad (4.20)$$

Figures 4.18 and 4.19 show a small deviation between pressure and shear stress profiles along the contact. The remaining minor difference is due to an insufficient mesh quality for the DNS. For a refined mesh the computational time increases

dramatically, due to the limitation on a stable CFL number. Since, lubricating films in general deal with large flow velocities and relatively small film heights, the restriction on the CFL in ordinary Newtonian conditions can be severe.

4.7 Log conformation representation

The discrepancy in the results of figures 4.12 to 4.14 are due to a coarse mesh size of 64×32 . Further refinement in this study was limited due to instabilities arising from the high Deborah number. To overcome this issue, a log-transformation of the conformation tensor has been proposed in literature [64][65]. The stress constitutive equation can also be expressed in the form of the conformation tensor, $\underline{\underline{c}}$, defined as,

$$\underline{\underline{\tau}} = G(\underline{\underline{c}} - \underline{\underline{I}}), \quad (4.21)$$

where, G is the shear modulus of the fluid or polymer and $\underline{\underline{I}}$ is the identity tensor.

$$\underline{\underline{c}} + \frac{d\underline{\underline{c}}}{dt} = -\frac{1}{\lambda}(\underline{\underline{c}} - \underline{\underline{I}}), \quad (4.22)$$

where λ is the relaxation time and $\frac{d\underline{\underline{c}}}{dt}$ is the upper-convected rate or Oldroyd rate, see Appendix A.4 for details.

The current formulation for Oldroyd-B is coupled with the momentum equation and solved systematically in a transient or steady-state analyses. However, for large De , any attempt to solve either equation (4.21) using standard discretization techniques fails irrespective of an implicit or explicit formulation [66][67]. The numerical instability affecting the advective constitutive relation was identified as a restriction on the grid spacing [64].

The solution to the numerical instability is termed the log-conformation approach and involves taking the logarithm of the conformation tensor and reformulating equation (4.22) [64]. The purpose of the logarithm is to ensure that the conformation matrix remains positive definite. Therefore any spurious numerical growth due to numerical errors or instabilities are damped considerably and

a solution can still be obtained. Note that this does not remove the numerical problem but merely improves the stability criteria for the conformation tensor. In general, it is also possible to construct other relationships that ensure c to be positive definite [68].

Equation (4.22) offers no advantage in terms of numerical stability [65] as previously stated. It is beneficial to get rid of the velocity gradient tensor by decomposing it as follows [64],

$$L = \Omega + B + Nc^{-1},$$

where Ω and N are skew-symmetric tensors and B is traceless ($tr[B] = 0$) and symmetric positive definite. Substituting for the logarithm of c , the resulting constitutive equation in terms of s is given as,

$$\frac{\partial s}{\partial t} + (u \cdot \nabla)s - \Omega s - s\Omega^T - 2Bs = -\frac{1}{\lambda}(I - s^{-1}). \quad (4.23)$$

Stability analysis of the constitutive relations in their tensor form is more involved. A 1D, simplified model is used to numerically show the difference in stability. The approach is based on a cartoon-model [66]. A 1D linear advection equation is considered with a source term and two governing parameters a and b ,

$$\frac{\partial c}{\partial t} + a \frac{\partial c}{\partial x} = bc.$$

Discretization using a standard first order upwind scheme for the spatial derivative will show that the criteria for stability restricts the size of the stencil, i.e $\Delta x < \frac{a}{b}$. Now, considering a log transformation $s = \log c$, the same one-dimensional model is given as,

$$\frac{\partial s}{\partial t} + a \frac{\partial s}{\partial x} = b. \quad (4.24)$$

The most notable, immediate feature of the transformation is that the source term no longer depends on c or s . It is beneficial to inspect the stability with respect to the transformation and numerical schemes. The application of the LCR will be performed as future work as the numerical procedure goes beyond the scope of this study.

a	b	Δx	Δt	$\frac{d}{dx}$	scheme
1	10	0.125	1e-3		upwind
1	0.5	0.015625	1e-4		central

Table 4.7: Parameters for the one dimensional model problem, equation (4.24).

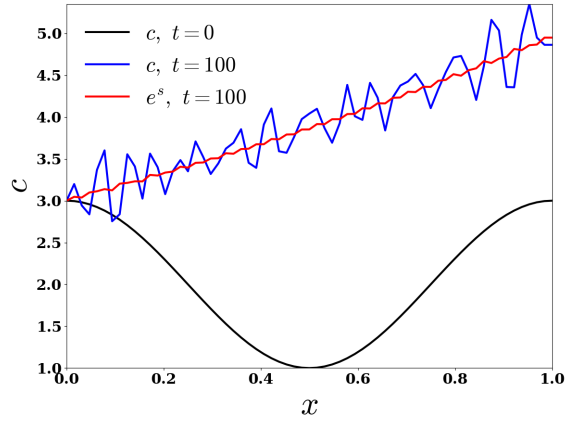


Figure 4.20: Numerical stability analysis of the cartoon model given by equation (4.24).

Chapter 5

Conclusion and perspectives

In this study several models are used for obtaining the film pressure in hydrodynamically loaded contacts using non-Newtonian lubricants. Two main non-Newtonian effects are examined, namely shear thinning and viscoelasticity. Three surface profiles are used in accordance with literature to compare the effect of surface properties on pressure.

A novel 1D model is derived for both non-Newtonian effects that factors a Newtonian solvent and non-Newtonian polymer additive. It is shown that numerical simulations of hydrodynamically lubricated contacts using non-Newtonian fluids of complex rheology is possible to different degrees of accuracy. The capacity to solve non-Newtonian rheology via simple numerical algorithms is possible.

The load and friction coefficient are obtained versus the bearing number for different surface profiles. The parabolic slider minimizes the friction coefficient as Λ increases. Whereas the plane slider and ridge continually exhibit an increase in friction.

In all the cases examined, non linear effects dominate beyond a certain critical Λ . Conditions leading to large Λ can be modeled using the modified non-linear Reynolds which gives a good estimation of load and coefficient of friction. It is not very accurate for modeling deformations where the variation in strain rate

along the film thickness can be large. Hence, the simple correction is insufficient but nonetheless provides close estimates of the load and coefficient of friction.

A viscoelastic Reynolds equation is derived which gives excellent qualitative agreement with the direct numerical simulation of Navier-Stokes. Linear profiles present a monotonic decrease in pressure until saturation but curved surfaces such as a parabola benefit from viscoelastic effects for small values of De . Linearized approaches lead to inaccurate results at moderate to high De for all geometries considered in this study.

The 1D viscoelastic Reynolds can be used to show that the total film pressure is a sum of the Newtonian solvent pressure and a viscoelastic polymer pressure that depends only on the surface shear stress. For the plane slider, this polymer pressure is always negative. The parabolic slider does provide an increase in pressure for small De . The ridge remains largely unaffected with a negligible increase in pressure for all De . Regardless of the surface profile, as De increases, all the change in pressure becomes very small approaching some minimum value.

Hence, it can be concluded that viscoelastic effects are significant in lubrication analyses and can be modeled using simple approaches depending on the level of accuracy desired.

The modified non-linear Reynolds equation can be coupled with solid deformation models to solve EHL problems. Prior to this study, the full generalized Reynolds equation must be solved to accurately account for shear thinning which can be computationally expensive. Owing to the 1D form of the non-linear Reynolds equation, it can allow more stable and computationally inexpensive solution routines. Additionally adding the energy equation and compressible effects is straight forward.

The 1D viscoelastic Reynolds equation derived in this study can be extended to account for the missing rotational operator which causes a discrepancy in results compared with the full DNS. Adding the normal components of the stress by following the procedure outlined in section 2 will yield simplified 1D expressions.

The numerical instability at higher mesh qualities in the DNS of viscoelastic lubricants can be overcome by applying the log-conformation transformation, see section 4.7 for details. Simulations involving fine meshes require large computation times which can be overcome by this transformation. The only accurate model that can capture viscoelastic effects is direct numerical simulation. However, the constitutive relation requires advanced numerical techniques such as the log-conformation representation of the conformation tensor $\bar{\bar{c}}$ to obtain results at high De and fine meshes.

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Appendix A

A.1 1D generalized Reynolds equation

To obtain the Newtonian Reynolds equation from the one dimensional generalized Reynolds equation,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_s \frac{U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{h^2}{12} (\tau_h - \tau_0) \right), \quad (\text{A.1})$$

where τ_h and τ_0 are given as,

$$\tau_h = \eta_p \frac{\partial u}{\partial y} \Big|_{y=h}, \quad (\text{A.2})$$

$$\tau_0 = \eta_p \frac{\partial u}{\partial y} \Big|_{y=0}, \quad (\text{A.3})$$

substitute the expression for the surface shear stresses due to inactive polymers into the expression for strain rate,

$$\eta_p \frac{\partial u}{\partial y} \Big|_{y=h} = \frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} + \frac{\eta_p}{2\eta_s} (\tau_0 - \tau_h), \quad (\text{A.4})$$

$$\eta_p \frac{\partial u}{\partial y} \Big|_{y=0} = -\frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} + \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.5})$$

simplifying to obtain an expression for $\tau_h - \tau_0$,

$$\tau_h = \frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} - \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.6})$$

$$\tau_0 = -\frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} + \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.7})$$

$$(\tau_h - \tau_0) = \frac{h\eta_p}{\eta_0} \frac{dp}{dx}, \quad (\text{A.8})$$

substitute the expression into the generalized Reynolds equation and simplify and note that $\eta_0 = \eta_s + \eta_p$,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_s \frac{U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{h^2}{12} \left(\frac{h\eta_p}{\eta_0} \frac{dp}{dx} \right) \right), \quad (\text{A.9})$$

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \eta_0 \frac{U}{2} \frac{dh}{dx}, \quad (\text{A.10})$$

hence, the Newtonian Reynolds equation is retrieved.

A.2 Viscoelastic pressure

The decomposition of film pressure for an isoviscous viscoelastic lubricant into a Newtonian and viscoelastic component P_{ve} is given in this section. The system of equations governing viscoelastic hydrodynamic lubrication is given as,

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \frac{\eta_s U}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\frac{h^2}{12} (\tau_h - \tau_0) \right), \quad (\text{A.11})$$

$$\tau_h + \lambda U \frac{d\tau_h}{dx} = \eta_p \gamma_h, \quad (\text{A.12})$$

$$\tau_0 = \eta_p \gamma_0, \quad (\text{A.13})$$

$$\gamma_h = \frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} - \frac{1}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.14})$$

$$\gamma_0 = -\frac{h}{2\eta_s} \frac{dp}{dx} + \frac{U}{h} + \frac{1}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.15})$$

simplifying the system to obtain the quantity $\tau_h - \tau_0$,

$$\eta_p \gamma_h = \frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} - \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.16})$$

$$\tau_h + \lambda U \frac{d\tau_h}{dx} = \frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} - \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.17})$$

From (A.16) and (A.17)

$$\tau_h = \lambda U \frac{d\tau_h}{dx} \frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} - \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.18})$$

and similarly for τ_0

$$\tau_0 = -\frac{h\eta_p}{2\eta_s} \frac{dp}{dx} + \frac{\eta_p U}{h} + \frac{\eta_p}{2\eta_s} (\tau_h - \tau_0), \quad (\text{A.19})$$

after some calculations,

$$(\tau_h - \tau_0) = \frac{h\eta_p}{\eta_0} \frac{dp}{dx} - \lambda U \frac{\eta_s}{\eta_0} \frac{d\tau_h}{dx}, \quad (\text{A.20})$$

substitute (A.20) into (A.11)

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \frac{\eta_0 U}{2} \frac{dh}{dx} - \frac{d}{dx} \left(\lambda U h^2 \frac{d\tau_h}{dx} \right), \quad (\text{A.21})$$

note that the forcing term in the RHS of equation (A.21) is now decomposed into a Newtonian and Viscoelastic portion. Integrating (A.21) along x ,

$$\frac{h^3}{12} \frac{dp}{dx} = \frac{\eta_0 U}{2} h - \lambda U h^2 \frac{d\tau_h}{dx} + c_1, \quad (\text{A.22})$$

to proceed further, a variable substitution is required for τ_h ,

$$\frac{d\tau_h}{dx} = h \frac{dq}{dx}, \quad (\text{A.23})$$

substituting $h \frac{dq}{dx}$ into (A.22)

$$\frac{h^3}{12} \left(\frac{dp}{dx} + \lambda U \frac{dq}{dx} \right) = \frac{\eta_0 U}{2} h + c_1, \quad (\text{A.24})$$

integrating and simplifying,

$$\frac{d}{dx} (p + \lambda U q) = \frac{6U\eta_0}{h^2} + \frac{c_1}{h^3}, \quad (\text{A.25})$$

Let $p' = p + \lambda U q$ be a modified pressure

$$\frac{dp'}{dx} = \frac{6U\eta_0}{h^2} + \frac{c_1}{h^3}, \quad (\text{A.26})$$

$$p' = 6U\eta_0 \int_0^x \frac{1}{h^2} + c_1 \int_0^x \frac{1}{h^3} + c_2, \quad (\text{A.27})$$

$$c_2 = p'_0, \quad (\text{A.28})$$

For c_1

$$p'(x = l) = p'_l, \quad (\text{A.29})$$

$$c_1 = \frac{1}{\int_0^l h^{-3}} (p'_l - p'_0) - 6U\eta_0 h_m, \quad (\text{A.30})$$

$$h_m = \frac{\int_0^l h^{-2}}{\int_0^l h^{-3}}, \quad (\text{A.31})$$

$$p' = 6U\eta_0 \int_0^x h^{-2} + \frac{\int_0^x h^{-3}}{\int_0^l h^{-3}} (p'_l - p'_0) - 6U\eta_0 h_m \int_0^x h^{-3} + p'_0, \quad (\text{A.32})$$

$$p' = p_0 + p_N + \frac{\int_0^x h^{-3}}{\int_0^l h^{-3}} \lambda U q, \quad (\text{A.33})$$

$$p = p_0 + p_N + \frac{\int_0^x h^{-3}}{\int_0^l h^{-3}} \lambda U q - \lambda U q, \quad (\text{A.34})$$

the final form of the total film pressure is given as,

$$p = p_0 + p_N + \lambda U \left(\frac{\int_0^x h^{-3}}{\int_0^l h^{-3}} q - q \right), \quad (\text{A.35})$$

where the viscoelastic pressure denoted by p_{ve} is given by the last term on the right hand side of equation (A.35),

$$p_{ve} = \lambda U \left(\frac{\int_0^x h^{-3}}{\int_0^l h^{-3}} q - q \right), \quad (\text{A.36})$$

since $q_0 = 0$ note that $p'_0 = p_0$,

$$\frac{d\tau_h}{dx} = h \frac{dq}{dx}, \quad (\text{A.37})$$

$$\tau_h = hq - \int_0^x \frac{dh}{dx} q + k, \quad (\text{A.38})$$

where,

$$k = \tau_h(x = 0). \quad (\text{A.39})$$

A.3 Linearized viscoelastic Reynolds

Lets apply an asymptotic expansion to all the field variables in increasing order of De to the scalar equations of momentum, continuity and shear stress,

$$\frac{dp}{dx} = (1 - \beta) \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \quad (\text{A.40})$$

$$\tau_{xx} + \beta De \left(u \frac{\partial \tau_{xx}}{\partial x} + v \frac{\partial \tau_{xx}}{\partial y} - 2 \frac{\partial u}{\partial y} \tau_{xy} - 2 \frac{\partial u}{\partial x} \tau_{xx} \right) = 0, \quad (\text{A.41})$$

$$\tau_{xy} + \beta De \left(u \frac{\partial \tau_{xy}}{\partial x} + v \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial u}{\partial y} \tau_{xx} - \frac{\partial v}{\partial x} \tau_{xx} \right) = \beta \frac{\partial u}{\partial y}, \quad (\text{A.42})$$

$$\tau_{yy} + \beta De \left(u \frac{\partial \tau_{yy}}{\partial x} + v \frac{\partial \tau_{yy}}{\partial y} - 2 \frac{\partial v}{\partial y} \tau_{yy} - 2 \frac{\partial v}{\partial x} \tau_{xy} \right) = 2\beta \frac{\partial v}{\partial y}, \quad (\text{A.43})$$

expanding the variables and collecting the terms denoted by superscript 0 as the leading order solution,

$$\frac{dp^0}{dx} = (1 - \beta) \frac{\partial^2 u^0}{\partial y^2} + \frac{\partial \tau_{xx}^0}{\partial x} + \frac{\partial \tau_{xy}^0}{\partial y}, \quad (\text{A.44})$$

$$\tau_{xx}^0 = 0, \quad (\text{A.45})$$

$$\tau_{xy}^0 = \beta \frac{\partial u^0}{\partial y}, \quad (\text{A.46})$$

$$\tau_{yy}^0 = 2\beta \frac{\partial v^0}{\partial y}, \quad (\text{A.47})$$

similarly for 1st order terms in De denoted by a superscript 1 and discarding terms containing De^2 and higher,

$$\frac{dp^1}{dx} = (1 - \beta) \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial \tau_{xx}^1}{\partial x} + \frac{\partial \tau_{xy}^1}{\partial y}, \quad (\text{A.48})$$

$$\tau_{xx}^1 = 2\beta \frac{\partial u^0}{\partial y} \tau_{xy}^0, \quad (\text{A.49})$$

$$\tau_{xy}^1 = \beta \frac{\partial u^1}{\partial y} - \beta \left(u^0 \frac{\partial \tau_{xy}^0}{\partial x} - v^0 \frac{\partial \tau_{xy}^0}{\partial y} + \frac{\partial u^0}{\partial y} \tau_{yy}^0 \right), \quad (\text{A.50})$$

when the first order terms are substituted using the leading order expressions, the final form obtained is the same as given in [56],

$$\frac{dp^1}{dx} = \frac{\partial^2 u^1}{\partial y^2} + \beta^2 \left(- \frac{\partial}{\partial y} \left(u^0 \frac{\partial}{\partial x} \left\{ \frac{\partial u^0}{\partial y} \right\} \right), - \frac{\partial}{\partial y} \left(v^0 \frac{\partial^2 v^0}{\partial y^2} \right) + 2 \frac{\partial}{\partial y} \left(\frac{\partial v^0}{\partial y} \frac{\partial u^0}{\partial y} \right) \right), \quad (\text{A.51})$$

the term on the RHS of equation (A.51) can be integrated along y independently of the polymer concentration β to give the De -order pressure distribution.

A.4 Log conformation representation

To obtain the constitutive relation, equation (4.22), substitute the definition for c , equation (4.21) into equation (1.5) and expand the terms,

$$G(\underline{\underline{c}} - \underline{\underline{I}}) + \lambda \left(G \frac{\partial(\underline{\underline{c}} - \underline{\underline{I}})}{\partial t} + G(u \cdot \nabla)(\underline{\underline{c}} - \underline{\underline{I}}) - \underline{\underline{L}}(\underline{\underline{c}} - \underline{\underline{I}}) - (\underline{\underline{c}} - \underline{\underline{I}})\underline{\underline{L}}^T \right) = 2\eta \underline{\underline{D}},$$

substituting the deformation tensor $\underline{\underline{D}}$ with its definition in terms of the velocity gradient tensor $\underline{\underline{L}}$ and noting that $\lambda = \frac{\eta}{G}$

$$\underline{\underline{D}} = \frac{1}{2} \left(\underline{\underline{L}} + \underline{\underline{L}}^T \right),$$

$$(\underline{\underline{c}} - \underline{\underline{I}}) + \lambda \left(\frac{\partial(\underline{\underline{c}} - \underline{\underline{I}})}{\partial t} + (u \cdot \nabla)\underline{\underline{c}} - \underline{\underline{L}}\underline{\underline{c}} - \underline{\underline{c}}\underline{\underline{L}}^T \right) + \lambda(\underline{\underline{L}} + \underline{\underline{L}}^T) = \lambda(\underline{\underline{L}} + \underline{\underline{L}}^T),$$

rearranging and simplifying the velocity gradient terms to obtain the Oldroyd-B model for viscoelasticity.

$$\frac{\partial(\underline{\underline{c}} - \underline{\underline{I}})}{\partial t} + (u \cdot \nabla)\underline{\underline{c}} - \underline{\underline{L}}\underline{\underline{c}} - \underline{\underline{c}}\underline{\underline{L}}^T = -\frac{1}{\lambda}(\underline{\underline{c}} - \underline{\underline{I}}).$$