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NIJENHUIS TENSOR OF ALMOST COMPLEX STRUCTURE

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

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In the work complex structures are considered, we construct two bases: RA-basis and A-basis. We also define an almost complex structure and its associated G-structure. We define locally the Nijenhuis tensor of an almost complex structure, prove that this tensor is defined globally, and obtain an explicit expression for this tensor.

2022, 46 pages

Keywords: Associated G-Structure, Complex Structure, Almost Complex Structure, Nijenhuis Tensor.

ÖZET

NEREDEYSE KARMAŞIK YAPILI NIJENHUIS TENSÖRÜ

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Çalışmada karmaşık yapılar göz önünde bulundurulur, iki temel oluştururuz: RA-temeli ve A-temeli. Ayrıca, neredeyse karmaşık bir yapı ve onunla ilişkili G-yapısını tanımlıyoruz. Neredeyse karmaşık bir yapının Nijenhuis tensörünü yerel olarak tanımlıyoruz, bu tensörün global olarak tanımlandığını kanıtlıyoruz ve bu tensör için açık bir ifade elde ediyoruz.

2022, 46 sayfa

Anahtar Kelimeler: İlişkili G Yapısı, Karmaşık Yapı, Neredeyse Karmaşık Yapı, Nijenhuis Tensörü.

PREFACE AND ACKNOWLEDGEMENTS

In the name of God the Merciful, and thanks to God, the Lord of the Worlds, and peace and prayers be upon the master of the prophets and messengers, Muhammad peace be upon him. After that, I thank God almighty, who helped me to write this letter, and I dedicate my thanks to my Advisor (Assoc. Prof. Dr. Ufuk ÖZTÜRK) for his impact on the completion of my letter as well as my thanks to my Co-Advisor (Asst. Prof. Dr. Ali Abdulmajeed SHIHAB) to give me a lot of time, support and good guidance, and I ask God to reward them with the best reward, and I offer my thanks and gratitude to my great family and my dear wife for their support to me.

Ahmed Thamer ABD ALI
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CONTENTS

LIST OF SYMBOLS

$C^{\infty}(\mathbf{M})$	Algebra of smooth functions on $\mathbf{M}$
$\otimes$	Tensor product
$\wedge$	Exterior product
$\mathbb{C}$	Complex numbers
$\mathbf{X}(\mathbf{M})$	Module of vector fields
$\boldsymbol{\nabla}$	Kozal's operator
$\boldsymbol{\theta}$	Connection form

LIST OF ABBREVIATIONS

DimV	Dimension of alinear space V
A-basis	Adapted Basis
RA-basis	Real adapted basis

1. INTRODUCTION

Complex manifolds are one of the most remarkable mathematical objects studied intensively both in differential geometry and in algebraic geometry, the theory of Lie groups and homogeneous spaces, topology, the theory of differential operators, and mathematical physics. The almost complex structure was first considered by Ehresmann and Hopf in 1940. Complex and almost complex manifolds are introductory to Kahlerian manifolds. Kähler manifolds were first defined in 1933 (Kähler 1933). Their significance for algebraic geometry became clear after the publication of Hodge's work, the results of which were subsequently combined in the book (Al-Dulaimi *et al.* 2022). These works basically determined the direction of research on Kahlerian manifolds for many years. The flow of research into the geometry and topology of Kahlerian manifolds continues unabated even in our time.

Our work is devoted to the study of the Nijenhuis tensor of almost complex manifolds and is structured as follows. In Section 2, we consider complex structures, the complexification of a linear space, and introduce the concept of linearity extension. We give the definition of a complex structure, build two bases: RA-basis and A-basis. In Section 3 we define an almost complex structure and its adjoint G-structure. In Section 4, we define the Nijenhuis tensor of an almost complex structure, prove that this tensor is defined globally, and obtain an explicit expression for this tensor.

2. COMPLEX STRUCTURES

2.1 Tensor Products of Modules

Let A and B be modules over a commutative associative ring K with identity. Consider a free Abelian group $A \circ B$ whose set of generators is the set of all symbols of the form $a \circ b; a \in A, b \in B$. Its elements are all formal (finite) sums of such symbols, i.e. entries of the form $a_1 \circ b_1 + \cdots + a_N \circ b_N; N \in \boldsymbol{N}$. Consider its subgroup $S \subset A \circ B$ generated by elements of the form $(a' + a'') \circ b - a' \circ b - a'' \circ b, a \circ (b' + b'') - a \circ b' - a \circ b'', (\alpha a) \circ b - a \circ (\alpha b);\ \alpha \in \boldsymbol{K}$. Consider an Abelian group $A \otimes B = A \circ B/S$. Its elements are finite formal sums of symbols of the form $a \otimes b = (a \circ b) + S$. It naturally introduces the structure of a K-module with the outer composition operation $\alpha(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} (\alpha a_i) \otimes b_i \big(= \sum_{i=1}^{n} b_i \otimes (\alpha a_i)\big)$. This K-module is called the tensor product of K-models A and B (Kobayashi and Nomizu 1969).

Remark 2.1 If A and B have a module structure over some ring $\boldsymbol{K}_1$, then $A \otimes B$ obviously also has a natural $\boldsymbol{K}_1$-module structure. In particular, if A is an algebra, then $A \otimes B$ has the natural A-module structure.

2.2 Complexification of Linear Space

Let, in particular, $A = \boldsymbol{C}$ be the field of complex numbers and $B = V$ be an R-linear space. Then the C-linear space (C-module) $\boldsymbol{C} \otimes V$ is denoted by $V^{\boldsymbol{C}}$ and is called the complexification of the linear space V. Its elements are entries of the form $\sum_{k=1}^{N} z_k \otimes X_k$; $z_k \in \boldsymbol{C}, X_k \in V$, N is an arbitrary natural number. The sum of two such elements $\sum_{k=1}^{N_1} z_k \otimes X_k$ and $\sum_{p=1}^{N_2} z_p \otimes X_p$is $\sum_{k=1}^{N_1} z_k \otimes X_k + \sum_{p=1}^{N_2} z_p \otimes X_p$, and the product of the element $\sum_{k=1}^{N_1} z_k \otimes X_k$ by the complex number $z \in \boldsymbol{C}$ is written as $\sum_{k=1}^{N_1} (z z_k) \otimes X_k$.

In a C-linear space $V^{\boldsymbol{C}}$, a mapping $\tau: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$ is canonically defined, acting by the formula $\tau\big(\sum_{k=1}^{N_1} z_k \otimes X_k\big) = \sum_{k=1}^{N_1} \overline{z_k} \otimes X_k$, where $z \to \bar{z}$ is the usual complex conjugation in the field of complex numbers. It is directly verified that τ is an involutive anti-automorphism of a C-linear space $V^{\boldsymbol{C}}$, i.e., bijection with the properties

$$1)\ \tau^2 = id;\ 2)\ \tau(X+Y) = \tau(X) + \tau(Y);\ 3)\ \tau(zX) = \bar{z}\tau(X); z \in \boldsymbol{C}, X, Y \in V. \qquad (2.1)$$

It is called the complex conjugation operator.

Note that V naturally admits an embedding of j in $V^{\boldsymbol{C}}$ by means of the identification $X \equiv j(X) = 1 \otimes X; X \in V$. At the same time, $\tau(X) \equiv \tau(1 \otimes X) = 1 \otimes X \equiv X$. Moreover, $\tau\left(\sum_{k=1}^{N_1} z_k \otimes X_k\right) = \sum_{k=1}^{N_1} z_k \otimes X_k \Leftrightarrow z_k = \overline{z_k}(k = 1, \dots, N) \Leftrightarrow z_k = x_k \in \boldsymbol{R}$, and hence $\sum_{k=1}^{N_1} z_k \otimes X_k = \sum_{k=1}^{N_1} x_k (1 \otimes X_k) \equiv \sum_{k=1}^{N_1} x_k X_k \in V$. This proves

Proposition 2.1 Let $Y = \sum_{k=1}^{N_1} z_k \otimes X_k \in V^{\boldsymbol{C}}$. Then, taking into account the accepted identification, $Y \in V \Leftrightarrow \tau(Y) = Y$.

Note that if V is an n-dimensional R-linear space, then $V^{\boldsymbol{C}}$ is an n-dimensional C-linear space. Moreover, if $b = \{e_1, \dots, e_n\}$ is a basis of the R-linear space V, then under the above canonical identification of the elements $e_k \in V$ with the elements $1 \otimes e_k \in V^{\boldsymbol{C}}$, the set b is also a basis of the C-linear space $V^{\boldsymbol{C}}$. This easily follows from a more general fact of independent interest (Efimov and Rozendorn 1975):

Proposition 2.2 Let $\{e_1, \dots, e_n\}$ be a basis of a real linear space V, $\{\varepsilon_1, \dots, \varepsilon_n\}$ be a basis of an R-linear space W. Then $\{e_i \otimes \varepsilon_a | i = 1, \dots, n; a = 1, \dots, m\}$ is a basis of the R-linear space $V \otimes W$.

Indeed, this Proposition implies that the elements $\{1 \otimes e_k, \sqrt{-1} \otimes e_k | k = 1, \dots, n\}$ form a basis of the R-linear space $V^{\boldsymbol{C}}$, whence it easily follows that the first n elements of this basis form a basis for $V^{\boldsymbol{C}}$ as a C-linear space.

Any operator $f: V \to V$ canonically defines a C-linear mapping $f^{\boldsymbol{C}} = id \otimes f: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$ by the formula $f^{\boldsymbol{C}}(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} z_k \otimes f(X_k)$.

Obviously, in view of the indicated identification, $f^{\boldsymbol{C}}\big|_N = f$, which is why the mapping $f^{\boldsymbol{C}}$ is called the linear extension of the operator f.

Proposition 2.3 A C-linear operator $F: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$ is a linear extension of some R-linear operator $f: V \to V$ if and only if $\tau \circ F = F \circ \tau$.

Proof. Indeed, if $F = f^{\boldsymbol{C}}$, then $\tau \circ F(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} \overline{z_k} \otimes f(X_k) = F \circ \tau(\sum_{k=1}^{N} z_k \otimes X_k)$, which implies that $\tau \circ F = F \circ \tau$. Conversely, if this relation holds, then $\tau \circ F(1 \otimes X) = F \circ \tau(1 \otimes X) = F(1 \otimes X), X \in V$, and, by virtue of Proposition 2.1, the restriction $f = F|_V$ of the operator F on V by the formula $1 \otimes f(X) = F(1 \otimes X)$. Obviously, in this case $F(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} F(z_k \otimes X_k) = \sum_{k=1}^{N} z_k F(1 \otimes X_k) = \sum_{k=1}^{N} z_k \big(1 \otimes f(X_k)\big) = \sum_{k=1}^{N} z_k \otimes f(X_k) = f^{\boldsymbol{C}}(\sum_{k=1}^{N} z_k \otimes X_k)$, and hence $F = f^{\boldsymbol{C}}$.

The following assertion is proved in exactly the same way:

Proposition 2.4 An r-aryC-linear mapping $T: V^{\boldsymbol{C}} \times \ldots \times V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$ is a linear extension of an r-aryR-linear mapping $T: V \times \ldots \times V \to V$ if and only if $\tau \circ T(X_1, \ldots, X_r) = T(\tau X_1, \ldots, \tau X_r);\ X_1, \ldots, X_r \in V^{\boldsymbol{C}}$.

One can give another definition of complexification that is equivalent to the one given. Let V be an R-linear space. Let us introduce the following operations in the set $V \times V$:

1) Addition. If $X_1 = (A_1, B_1), X_2 = (A_2, B_2)$ are elements from $V \times V$, then the pair $(A_1 + A_2, B_1 + B_2)$ is called their sum and denoted by $X_1 + X_2$.

2) Multiplication by a complex number. If $X = (A, B) \in V \times V, z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}$, then we set $zX = (\alpha a - \beta b, \alpha b + \beta a)$. We call an element zX the product of a complex number z and an element X.

It is directly verified that this introduces in the set $V \times V$ the structure of a C-linear space $\tilde{V}^{\boldsymbol{C}}$, which is naturally isomorphic to the C-linear space $V^{\boldsymbol{C}}$. The natural isomorphism $\varphi: V^{\boldsymbol{C}} \to \tilde{V}^{\boldsymbol{C}}$ assigns to the element $\sum_{k=1}^{N}(\alpha_k + \sqrt{-1}\beta_k)X_k \in V^{\boldsymbol{C}}$ the pair $(A, B) \in \tilde{V}^{\boldsymbol{C}}$, where $A = \sum_{k=1}^{N} \alpha_k X_k, B = \sum_{k=1}^{N} \beta_k X_k$. Under this isomorphism, the above-described embedding $j: V \subset V^{\boldsymbol{C}}$ corresponds to the embedding $\tilde{j}: V \subset \tilde{V}^{\boldsymbol{C}}$ defined by the formula $\tilde{j}(X) = (X, 0); X \in V$, the operator τ of complex conjugation corresponds to the operator $\tilde{\tau}: \tilde{V}^{\boldsymbol{C}} \to \tilde{V}^{\boldsymbol{C}}$ defined by the formula $\tilde{\tau}(X, Y) = (X, -Y); X, Y \in V$, and the C-linear operator $f^{\boldsymbol{C}} = id \otimes f$ is C-linear operator $\tilde{j}^{\boldsymbol{C}}: \tilde{V}^{\boldsymbol{C}} \subset \tilde{V}^{\boldsymbol{C}}$ defined by $\tilde{f}^{\boldsymbol{C}}(X, Y) = (fX, fY); X, Y \in V$.

2.3 Complex Structures

Let V be a complex linear space. It can, in particular, be considered as a real linear space $V^{\boldsymbol{R}}$ (Called the reification of the space V) in which an R-linear endomorphism $J_0: V^{\boldsymbol{R}} \to V^{\boldsymbol{R}}$ is given, defined by the formula $J_0(X) = \sqrt{-1}X; X \in V^{\boldsymbol{R}}$. This endomorphism allows us to completely restore the structure of a complex linear space to V. Namely, if $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in V$, then $zX = \alpha X + \beta J_0(X)$. Moreover, it is obvious that the endomorphism J_0 is anti-involutive, i.e., $J_0^2 = -id$.

Let V be a real linear space.

Definition 2.1 A complex structure in V is an endomorphism $J: V \to V$ such that $J^2 = -id$. In other words, a complex structure is an anti-involutive automorphism of a real linear space.

The fixation of a complex structure in V canonically determines in V the structure of a complex linear space (i.e. a C-module). Indeed, if $X \in V, z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}$, then we set

$$zX = \alpha X + \beta(JX). \tag{2.2}$$

It is directly verified that in this case all 8 axioms of a C-linear space are satisfied, which we will denote by the same symbol V. Obviously, V as an R-linear space is its reification $V^{\boldsymbol{R}}$.

Let the dimension $dim_{\boldsymbol{C}}V$ of a linear space V as a complex space be equal to n, and let $\{e_1, \dots, e_n\}$ be the basis of this space. Let $X \in V$. Then $X = z^k e_k$, where $z^k = \alpha^k + \sqrt{-1}\beta^k \in \boldsymbol{C}; k = 1, \dots, n$. Taking into account (1.2), $X = \alpha^k e_k + \beta^k(Je_k)$, i.e. any vector $X \in V^{\boldsymbol{R}}$ is represented as a linear combination of vectors $e_1, \dots, e_n, Je_1, \dots, Je_n$. On the other hand, let $\lambda^k e_k + \mu^k Je_k = 0, \lambda^k, \mu^k \in \boldsymbol{R}$. Then, due to (1.2), $\left(\lambda^k + \sqrt{-1}\mu^k\right)e_k = 0$, and due to C-linear independence of the vectors $e_1, \dots, e_n, \lambda^k + \sqrt{-1}\mu^k = 0$, which means that $\lambda^k = \mu^k = 0; k = 1, \dots, n$. Therefore, the vectors $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ form a basis of the space V as an R-linear space (that is, a basis of the space $V^{\boldsymbol{R}}$). Such a basis is called a real-adapted complex structure, in short, an RA-basis.

Remark 2.2 Obviously, any complex structure is defined in the RA-basis by a matrix of the form

$$\left(J_j^i\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{2.3}$$

As a simple but important corollary, we get the following

Proposition 2.5 A finite-dimensional real linear space admits a complex structure if and only if it is even-dimensional.

Proof. Since the RA-basis contains an even number of vectors, the space that admits a complex structure is necessarily even-dimensional. Conversely, let V be a 2n-dimensional real linear space. We fix an arbitrary basis $\{e_1, \dots, e_{2n}\}$ in it. Then the endomorphism J of the space V, defined in this basis by the matrix (2.3), is obviously a complex structure.

Let J be a complex structure in an R-linear space V. Consider the endomorphism $\sigma: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$ defined by the formula $\sigma = \frac{1}{2}\left(id - \sqrt{-1}J^{\boldsymbol{C}}\right)$. Obviously, $\sigma^2 = \sigma$, i.e. σ is a projector. The projector $\bar{\sigma}$ complementary to it is defined by the formula $\bar{\sigma} = \frac{1}{2}\left(id + \sqrt{-1}J^{\boldsymbol{C}}\right)$. In what follows, with a liberty of speech, we will denote the endomorphism $J^{\boldsymbol{C}}$ simply by J. Note that $J \circ \sigma = \frac{1}{2}\left(J + \sqrt{-1}id\right) = \frac{\sqrt{-1}}{2}\left(id - \sqrt{-1}J\right) = \sqrt{-1}\sigma$, and hence $Im\sigma \subset D_J^{\sqrt{-1}}$. (Here and in what follows, we denote by D_F^{λ} the proper subspace of the endomorphism F corresponding to the eigenvalue λ).

Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sigma X = \frac{1}{2}\left(X - \sqrt{-1}JX\right) = \frac{1}{2}(2X) = X$, in particular, $X \in Im\sigma$. Thus Im$Im\sigma = D_J^{\sqrt{-1}}$. Similarly, Im $Im\bar{\sigma} = D_J^{-\sqrt{-1}}$. Since $\mathcal{X}^{\boldsymbol{C}}(M) = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, we get:

Theorem 2.1 The linear space $V^{\boldsymbol{C}}$ decomposes into a direct sum of eigenspaces of the endomorphism J corresponding to the eigenvalues $\sqrt{-1}$and $-\sqrt{-1}$, i.e. $V^{\boldsymbol{C}} = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, and the endomorphisms σ and $\bar{\sigma}$ are projections onto the subspaces $D_J^{\sqrt{-1}}$ and $D_J^{-\sqrt{-1}}$, respectively.

Moreover, it is true

Theorem 2.2 The specification of a complex structure on an R-linear space V is equivalent to splitting $V^{\boldsymbol{C}}$ into a direct sum of two complex conjugate subspaces serving as proper subspaces of this complex structure.

Proof. Necessity follows from Theorem 2.1. Let now $V^{\boldsymbol{C}} = D \oplus \tau D$. Then $\forall X \in V^{\boldsymbol{C}} \Longrightarrow X = X_1 + X_2;\ X_1 \in D, X_2 \in \tau D$. We construct an endomorphism $\mathcal{J}: V^{\boldsymbol{C}} \longrightarrow V^{\boldsymbol{C}}$ by setting $\mathcal{J}(X) = \sqrt{-1}(X_1 - X_2)$. Obviously, $\tau(X) = \tau(X_1) + \tau(X_2)$, moreover, $\tau(X_1) \in \tau D, \tau(X_2) \in D$. Therefore $(\mathcal{J} \circ \tau)(X) = \sqrt{-1}\big(\tau(X_2) - \tau(X_1)\big)$. On the other hand, due to the antilinearity of the operator τ, $(\tau \circ \mathcal{J})(X) = -\sqrt{-1}\big(\tau(X_1) - \tau(X_2)\big) = \sqrt{-1}(\tau X_2 - \tau X_1)$. Thus, $\mathcal{J} \circ \tau = \tau \circ \mathcal{J}$. By Proposition 2.3 $\mathcal{J} = J^{\boldsymbol{C}}$ for some R-linear endomorphism $J: V \to V$. Obviously, $\mathcal{J}^2 = -id$, in particular, $J^2 = -id$, i.e. J is a complex structure on V. If $X \in D$, then $X = X_1$, and hence $\mathcal{J}(X) = \sqrt{-1}X_1 = \sqrt{-1}X$. Therefore, $D \subset D_J^{\sqrt{-1}}$. Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sqrt{-1}(X_1 - X_2) = \mathcal{J}(X) = \sqrt{-1}X = \sqrt{-1}(X_1 + X_2)$, whence $X_2 = 0$, and hence $X \in D$. Therefore, $D_J^{\sqrt{-1}} \subset D$, i.e. $D_J^{\sqrt{-1}} = D$. Similarly, $D_J^{-\sqrt{-1}} = \tau D$.

Lemma 2.1 In the introduced notation, 1) $\tau \circ \sigma = \bar{\sigma} \circ \tau$, 2) $\tau \circ \bar{\sigma} = \sigma \circ \tau$.

Proof. Taking into account the antilinearity of the mapping τ and using Proposition 2.3, we have: $\tau \circ \sigma(X) = \frac{1}{2}\tau\big(X - \sqrt{-1}JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}\tau \circ JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}J \circ \tau X\big) = \sigma \circ \tau(X); X \in V^{\boldsymbol{C}}$.

These condrelation is proved similarly.

Theorem 2.3 The maps $\sigma|_V: V \to D_J^{\sqrt{-1}}$ and $\bar{\sigma}|_V: V \to D_J^{-\sqrt{-1}}$ are, respectively, an isomorphism and an anti-isomorphism of C-linear spaces.

Proof. The additivity of the mappings $\sigma|_V$ and $\bar{\sigma}|_V$ is obvious. Let now $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in V$. As already seen, $\sigma \circ J = J \circ \sigma = \sqrt{-1}\sigma, \bar{\sigma} \circ J = J \circ \bar{\sigma} = -\sqrt{-1}\sigma$. Therefore $\sigma(zX) = \sigma(\alpha X + \beta JX) = \alpha\sigma X + \beta\sqrt{-1}\sigma X = z(\sigma X)$. Similarly, $\bar{\sigma}(zX) = \bar{z}(\bar{\sigma}X)$, and thus the mappings $\sigma|_V$ and $\bar{\sigma}|_V$ are, respectively, a homomorphism and an antihomomorphism of C-linear spaces.

Let $\exists\, X \in V$and $\sigma X = 0$. Applying the operator τ to both parts of this identity, taking into account Lemma 2.1 and Proposition 2.1, we obtain that $\bar{\sigma}X = 0$, and hence $X = \sigma X + \bar{\sigma}X = 0$. Therefore, $\ker\sigma|_V = \{0\}$. Similarly, $\ker\bar{\sigma}|_V = \{0\}$, i.e. σ and $\bar{\sigma}$ are monomorphism and antimonomorphism, respectively.

Let, finally, $X \in D_J^{\sqrt{-1}}$. Consider the vector $Y = X + \tau X$. By Proposition 2.1, $Y \in V$. On the other hand, since $X \in Im\sigma = \ker\bar{\sigma}$, in view of Lemma 2.1 we have $\sigma Y = \sigma X + (\tau \circ \sigma)X = X + (\tau \circ \bar{\sigma})X = X$. Similarly, if $X \in D_J^{-\sqrt{-1}}$, then $\bar{\sigma}Y = X$, and thus $\sigma|_V$ and $\bar{\sigma}|_V$ are an epimorphism and an anti-epimorphism, respectively.

Let, in particular, V be a real R-linear space, $\dim V = 2n$, and let $b = \{e_1, \dots, e_{2n}\}$ be its basis as a C-module. Consider a system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$, where $\varepsilon_a = \sigma(e_a), \varepsilon_{\hat{a}} = \bar{\sigma}(e_a); a = 1, \dots, n$. By virtue of Theorem 2.3, the vectors $\{\varepsilon_1, \dots, \varepsilon_n\}$ form the basis of the C-linear space $D_J^{\sqrt{-1}}$, and the vectors $\{\varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$ form the basis of the C-linear space $D_J^{-\sqrt{-1}}$ and, by virtue of Lemma 2.1 and Proposition 2.1, $\tau\varepsilon_a = (\tau \circ \sigma)e_a = (\bar{\sigma} \circ \tau)e_a = \bar{\sigma}e_a = \varepsilon_{\hat{a}}$.

Moreover, by virtue of Theorem 2.1, the system of vectors b_A forms a basis of the space $V^{\boldsymbol{C}}$, characterized by the fact that the endomorphism matrix J in this basis has the form

$$(J_j^i) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \qquad (2.4)$$

Such a basis is called an adapted complex structure, in short, an A-basis.

Conclusion. Fixing the complex structure J in the $2n$-dimensional real linear space $V^{\boldsymbol{R}}$ induces the assignment of the structure of the n-dimensional complex linear space Vin $V^{\boldsymbol{R}}$. Each basis $b = \{e_1, \dots, e_{2n}\}$ of the space V canonically induces two bases:

1) RA-basis $b_{RA} = \{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ of the space $V^{\boldsymbol{R}}$. Of course, taking into account the canonical identification $X \equiv 1 \otimes X$, this basis can also be considered as a basis of the C-linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

2) A-basis $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$ of the C-linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

Let now $b = \{e_1, \dots, e_n\}$ and $\tilde{b} = \{\tilde{e}_1, \dots, \hat{e}_n\}$ be two bases of the space V, $C = C_{b\tilde{b}} = (c_b^a)$ be the transition matrix from the basis b to basis $\tilde{b}$, $C = A + \sqrt{-1}B$, where $A = (\alpha_b^a)$ and $B = (\beta_b^a)$ are the real and imaginary parts of the matrix C, respectively. Since, taking into account (2.1), $\tilde{e}_a = c_a^b e_b = \alpha_a^b e_b + \beta_a^b (Je_b)$, we have: $J\tilde{e}_a = \alpha_a^b Je_b + \beta_a^b J^2 e_b = -\beta_a^b e_b + \alpha_a^b Je_b$, which means

$$C_{b_{RA}\tilde{b}_{RA}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{2.5}$$

Further, $\tilde{\varepsilon}_a = \sigma(\tilde{e}_a) = \sigma(c_a^b e_b) = c_a^b \sigma(e_b) = c_a^b \varepsilon_b$; $\tilde{\varepsilon}_{\hat{a}} = \bar{\sigma}(\tilde{e}_a) = \bar{\sigma}(c_a^b e_b) = \bar{c}_a^b \bar{\sigma}(e_b) = \bar{c}_a^b \varepsilon_{\hat{b}}$, and hence

$$C_{b_a\tilde{b}_A} = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}. \tag{2.6}$$

Obviously, both matrices $C_{b_{RA}\tilde{b}_{RA}}$ and $C_{b_a\tilde{b}_A}$ can be treated as matrices of the same endomorphism of the linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$, namely, the endomorphism $f^{\boldsymbol{C}}$, where f is the endomorphism space $V^{\boldsymbol{R}}$, considered as an endomorphism of the space V, taking the basis b to the basis $\tilde{b}$. In particular,

$$\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix} = |\det C|^2. \tag{2.7}$$

An important consequence of this relation is

Proposition 2.6 The fixation of a complex structure in an R-linear space $V^{\boldsymbol{R}}$ canonically determines the orientation of this space. It consists of bases oriented in the same way as any RA-basis.

3. ALMOST COMPLEX STRUCTURES AND THE ASSOCIATED G-STRUCTURE

Definition 3.1 An almost complex structure on a manifold M is an anti-involutive endomorphism of a module $\mathcal{X}(M)$, i.e., $C^\infty(M)$-linear map $J: \mathcal{X}(M) \to \mathcal{X}(M)$ such that $J^2 = -id$. The endomorphism J is also called a structural endomorphism. A manifold on which an almost complex structure is fixed is called an almost complex manifold. A diffeomorphism $f: M_1 \to M_2$ of an almost complex manifold (M_1, J_1) onto an almost complex manifold (M_2, J_2) is called a holomorphic diffeomorphism if $f_* \circ J_1 = J_2 \circ f_*$.

It is obvious that an almost complex structure can be considered as a complex structure of the module $\mathcal{X}(M)$ considered as an R-linear space. As we have seen, the structure of a C-linear space is naturally induced on this linear space, and hence the structure of a $\boldsymbol{C} \otimes C^\infty(M)$-module, i.e., module over the ring of smooth complex-valued functions on the manifold M. The smoothness of such a function is understood as the smoothness of its real and imaginary parts. For a better understanding of this structure, it is convenient to use the alternative definition of $\boldsymbol{C} \otimes C^\infty(M)$ as a complexification of the R-linear space $C^\infty(M)$, according to $\boldsymbol{C} \otimes C^\infty(M) = C^\infty(M) \times C^\infty(M)$. If $(f, g) = f + \sqrt{-1}g \in \boldsymbol{C} \otimes C^\infty(M), X \in \mathcal{X}(M)$, then by definition $\left(f + \sqrt{-1}g\right)X = fX + g(JX)$.

Let J be an almost complex structure on a manifold M. It induces complex structures $J_m: T_m(M) \to T_m(M)$ at every point $m \in M$. In view of Proposition 2.5, the space $T_m(M)$, and hence the manifold M itself, are even-dimensional. Let $\dim M = 2n$. The number n is called the complex dimension of the manifold M.

Theorem 3.1 Defining an almost complex structure on a smooth manifold M^{2n} is equivalent to defining a G-structure on this manifold with the structure group $G = GL^R(n, \boldsymbol{C})$.

Proof. Let J be an almost complex structure on a manifold M. Then, at each point $m \in M$, a family $\mathcal{R}_m$ of frames of the space $T_m(M)$, considered as an n-dimensional C-linear space, is defined. It follows from the definition of the frame that the group $GL(n,\boldsymbol{C})$ acts freely and transitively in each such family.

Lemma 3.1 In some neighborhood U of an arbitrary point $m \in M$, one can construct a family of vector fields $\{e_1^0, \dots, e_n^0\}$ on U that form a basis of the module $\mathcal{X}(U)$ as a $\boldsymbol{C} \otimes C^\infty(U)$-module.

Proof. We fix some RA-frame $p = \{\xi_1, \dots, \xi_n, J_m\xi_1, \dots, J_m\xi_n\}$ at the point m. As we know, the system of vectors ξ_k can be extended to the system of vector fields $e_k^0 (k = 1, \dots, n)$ on M. In this case, the system of vectors $J_m\xi_k$ can be extended to the system of vector fields Je_k^0. Since the linear independence of the vectors of the frame p is equivalent to the inequality zero of the determinant of the transition matrix from the natural basis at the point m to the basic part of the frame p, this property is preserved in some neighborhood U of the point m and for some vector fields $\{e_1^0, \dots, e_n^0, Je_1^0, \dots, Je_n^0\}$. But then, obviously, the system $\{e_1^0, \dots, e_n^0\}$ of vector fields on U will be $\boldsymbol{C} \otimes C^\infty(U)$-linearly independent, and hence forms a basis of the $\boldsymbol{C} \otimes C^\infty(U)$-module $\mathcal{X}(U)$.

We continue the proof of the theorem. Denote $\mathcal{R} = \bigcup_{m \in M} \mathcal{R}_m$, and introduce the natural projection $\pi: \mathcal{R} \to M$ that maps the frame $p \in \mathcal{R}$ to its vertex. Now we can construct the mapping $F_U: \pi^{-1}(U) \to GL(n, \boldsymbol{C})$ by setting $F_U(p) = g$, where g is the transition matrix from the frame $(m, e_1^0|_m, \dots, e_n^0|_m)$ to the frame p. It is easy to see that the quadruple $B_J(M) = \big(\mathcal{R}, M, \pi, G = GL(n, \boldsymbol{C})\big)$ forms a principal bundle. This principal bundle can be regarded as a G-structure with respect to the monomorphism$\big(\tilde{f}, \tilde{\rho}\big)$ of the principal bundle $B_J(M)$ into the principal bundle $B(M)$, where $\tilde{f}: \mathcal{R} \to BM$ is the mapping that associates the frame $(m, e_1, \dots, e_n)$ of the space $T_m(M)$ as a C-module corresponding to the RA-frame, and $\tilde{\rho}: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{R})$ is the canonical monomorphism of Lie groups corresponding to the matrix $C = A + \sqrt{-1}B \in GL(n, \boldsymbol{C})$ the matrix $\tilde{\rho}(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, whose image is the Lie group $GL^{\boldsymbol{R}}(n, \boldsymbol{C})$.

Conversely, let $(\mathcal{R}, M, \pi, GL^R(n, \boldsymbol{C}))$ be a G-structure of this type on M. Let J_0 be the standard complex structure in the space $\boldsymbol{R}^{2n}$ given by a matrix of the form (2.3). Let $m \in M$ be an arbitrary point. In the space $T_m(M)$, we define an endomorphism J_m by the formula $J_m = p \circ J_0 \circ p^{-1}; p \in \pi^{-1}(m)$. Obviously, $J_m^2 = -id$, i.e. J_m is a complex structure on $T_m(M)$. Let us show that it is well defined in the sense of being independent of the choice of the element $p \in \pi^{-1}(m)$. Indeed, if $\tilde{p} \in \pi^{-1}(m)$ is another such element, then $\exists h \in GL^R(n, \boldsymbol{C})$ and $\tilde{p} = ph$. Therefore, $\tilde{p} \circ J_0 \circ \tilde{p}^{-1} = (ph) \circ J_0 \circ (ph)^{-1} = p \circ (h \circ J_0 \circ h^{-1}) \circ p^{-1} = p \circ J_0 \circ p^{-1} = J_m$, since the group $GL^R(n, \boldsymbol{C})$ is obviously the invariance group of the endomorphism J_0, i.e., $hJ_0 = J_0 h; h \in GL^{\boldsymbol{R}}(n, \boldsymbol{C})$, which is verified directly.

We show that it is defined correctly in the sense of independence from the choice of the element $p \in \pi^{-1}(m)$. In fact, if $\tilde{p} \in \pi^{-1}(m)$ is another such element, then $\exists h \in GL^R(n, \boldsymbol{C})$ and $\tilde{p} = ph$. Therefore, $\tilde{p} \circ J_0 \circ \tilde{p}^{-1} = (ph) \circ J_0 \circ (ph)^{-1} = p \circ (h \circ J_0 \circ h^{-1}) \circ p^{-1} = p \circ J_0 \circ p^{-1} = J_m$ as the group $GL^R(n, \boldsymbol{C})$ is clearly the invariance group of the endomorphism J_0, т.е. $hJ_0 = J_0 h; h \in GL^{\boldsymbol{R}}(n, \boldsymbol{C})$, what is checked directly.

Let us show that the family of tensors $J = \{J_m | m \in M\}$ defines a tensor field on the manifold M. To do this, it suffices to prove that for any admissible chart (U, φ) on M, the functions $m \to J_j^i(m) = dx^i\left(J_m\left(\left.\frac{\partial}{\partial x^j}\right|_m\right)\right), m \in M$, are smooth on U. Let us fix a local section $s: U \to \mathcal{R}$ of the space of the G-structure. Then, by construction, in the RA-frame $\sigma(m)$ (and the dual coreframe) we have: $\left(e^i\left(J_m(e_j)\right)\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \left((J_0)_j^i\right)$. The smoothness of the section is expressed in the fact that the components of the matrix C of the transition from the natural basis of the module $\mathcal{X}(U)$ to the RA-basis $\sigma(U)$ of this module, and hence the components of the inverse matrix $\tilde{C}$, are smooth functions. Therefore

$m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right) = C_k^i(m)e^k\left(J_m\left(\tilde{C}_j^r(m)e_r\right)\right) =$

$C_k^i(m)\tilde{C}_j^r(m)e^k\left(J_m(e_r)\right) = C_k^i(m)\tilde{C}_j^r(m)(J_0)_r^k$ are smooth functions on U. Thus, J is an almost complex structure. Obviously, the family of RA-frames generated by it coincides with the space of the G-structure.

Theorem 3.2 Every almost complex manifold is even-dimensional and orientable.

Proof. The even-dimensionality of an almost complex manifold (M^{2n},J) follows from Proposition 2.5 applied to any linear space of the form $T_m(M); m \in M$. By definition, the orientability of this manifold means the existence of a differential $2n$-form τ that nowhere vanishes on it. But its existence is obvious for any neighborhood of local triviality of the bundle $B_J(M)$. Indeed, if U is such a neighborhood, $s: U \to \mathcal{R}$ is a section of this bundle over it, given by vector fields $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$, then, by Proposition 2.6, it suffices to put $\tau_U = e_1 \wedge \dots \wedge e_n \wedge J^*e_1 \wedge \dots \wedge J^*e_n$, where $J^*(u)(X) = u(JX); X \in \mathcal{X}(U), u \in \mathcal{X}^*(U)$. Now let $\{\psi_\alpha\}_{\alpha \in A}$ be a partition of unity subject to the covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of the manifold M by local triviality domains of the bundle $B_J(M)$. Due to the paracompactness of the manifold, this covering can be considered locally finite without loss of generality. Let τ_α be the $2n$-form constructed for the domain U_α; $\alpha \in A$. Then $\tau = \sum_{\alpha \in A} \psi_\alpha \tau_\alpha$ is a well-defined $2n$-form on M. Indeed, due to the local finiteness of U in some neighborhood U of each point $m \in M$, the form $\tau|_U$ is the sum of at most a finite number of smooth forms $(\tau_\alpha)|_U$, which means that τ is a nowhere vanishing $2n$-form on M.

Remark 3.1 The even-dimensionality and orientability of a manifold are thus necessary conditions for the existence of an almost complex structure on this manifold. However, these conditions are not sufficient. For example, a well-known deep result of a topological nature is the assertion that a $2n$-dimensional sphere S^{2n} admits an almost complex structure if and only if either $n = 1$ or $n = 3$ (Samelson, 1940). Therefore, for example, a 4-dimensional sphere, being, as is well known, an even-dimensional orientable manifold, does not admit an almost complex structure. The question of finding necessary and sufficient conditions for the existence of an almost complex structure on a smooth manifold is still open.

Remark 3.2 Along with the principal bundle of BM frames over a smooth manifold M^n, we can consider a more extensive principal bundle of complex frames over M, which we denote by $B^{\boldsymbol{C}}(M) = \left(B^{\boldsymbol{C}}M, M, \pi, GL(n, \boldsymbol{C})\right)$, where $B^{\boldsymbol{C}}M$ is the union of all frames of the spaces $\left(T_m(M)\right)^{\boldsymbol{C}}; m \in M$. The corresponding justifications do not differ in any way from the corresponding justifications for the main bundle of the BM. This principal bundle plays a particularly important role for almost complex manifolds (M^{2n}, J), since it allows us, along with the G-structure $B_J(M)$ constructed above, to consider one more defining G-structure defined by the monomorphism (f, ρ) of the principal bundle $B_J(M)$ into the principal bundle $B^{\boldsymbol{C}}M$, where $f: \mathcal{R} \to B^{\boldsymbol{C}}M$is a mapping that associates the frame $(m, e_1, \dots, e_n)$ of the space $T_m(M)$ as a C-module with the corresponding A-frame, and $\rho: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{C})$ is the canonical Lie group monomorphism that assigns to the matrix $C \in GL(n, \boldsymbol{C})$ the matrix $\rho(C) = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix} \in GL(2n, \boldsymbol{C})$. As above, it is proved that specifying such a G-structure is equivalent to specifying the original almost complex structure. This G-structure will be especially important for our subsequent considerations; we call it a G-structure attached to an almost complex structure.

4. NIJENHUIS TENSOR

4.1 Local Definition

Let (M, J) be an almost complex manifold, $dimM = 2n$. Let us agree that in what follows, unless otherwise stated, the indices $i, j, k, \ldots$ range from 1 to $2n$, the indices $a, b, c. d, \ldots$ range from 1 to n. and denote $\hat{a} = a + n$. Let (U, φ) be a local map on a manifold M. According to the Main Theorem of tensor analysis, specifying an almost complex structure J on a manifold M induces specifying on the total space BM a bundle of frames over M of a system of functions $\{J_j^i\}$ satisfying in the coordinate neighborhood $W = \pi^{-1}(U) \subset BM$ to a system of differential equations of the form

$$\triangle J_j^i \equiv dJ_j^i - J_k^i \omega_j^k + J_j^k \omega_k^i = J_{jk}^i \omega^k. \tag{4.1}$$

We differentiate these relations externally:

$$\left(\triangle J_{jk}^i + J_r^i \omega_{jk}^r - J_j^r \omega_{rk}^i\right) \wedge \omega^k = 0.$$

According to Cartan's lemma,

$$\triangle J_{jk}^i + J_r^i \omega_{jk}^r - J_j^r \omega_{rk}^i = J_{jkr}^i \omega^r \tag{4.2}$$

For suitable functions J_{jkr}^i. Since, by virtue of the Corollary to the Generalized Cartan Lemma, $\omega_{jk}^i \equiv \omega_{kj}^i mod\ (\omega^1, \ldots, \omega^n)$, then alternating both parts of (4.2) by indices j and k, we get:

$$\triangle \left(J_{jk}^i - J_{kj}^i\right) = -J_k^r \omega_{rj}^i + J_k^r \omega_{rj}^i + A_{jkr}^i \omega^r$$

For suitable functions A^i_{jkr}. Multiply both sides of this ratio by J^j_s and sum over j from 1 to $2n$:

$$J^j_s \triangle \left(J^i_{jk} - J^i_{kj}\right) = -J^j_s J^r_k \omega^i_{rj} + J^j_s J^r_k \omega^i_{rj} + J^j_s A^i_{jkr} \omega^r$$

Or, taking into account (4.1),

$$\triangle \left\{J^j_s \left(J^i_{jk} - J^i_{kj}\right)\right\} = -\omega^i_{sk} - J^j_s J^r_k \omega^i_{jr} + B^i_{skr} \omega^r$$

For suitable functions B^i_{skr}. Alternating with respect to the indices s and k, we finally obtain:

$$\triangle \left\{J^j_s \left(J^i_{jk} - J^i_{kj}\right) - J^j_k \left(J^i_{js} - J^i_{sj}\right)\right\} = C^i_{skr} \omega^r$$

For suitable C^i_{skr} functions. By virtue of the Main Theorem of Tensor Analysis, the set of functions

$$N^i_{sk} = \frac{1}{4}\left\{J^j_s \left(J^i_{jk} - J^i_{kj}\right) - J^j_k \left(J^i_{js} - J^i_{sj}\right)\right\}. \tag{4.3}$$

Defines in some neighborhood U of an arbitrary point of the manifold M a tensor of type (2,1), called the Nijenhuis tensor of the almost complex structure J. Our immediate task is to prove that this tensor is in fact defined globally on M and to find an explicit expression for it. To achieve this, we first of all note that, taking into account (1.4), relations (4.1) on the space of the associated G-structure can be written in the form

$$1)\ \omega^{\hat{a}}_b = -\frac{1}{2} J^{\hat{a}}_{bk} \omega^k;\ \ 2)\ \omega^a_{\hat{b}} = \frac{1}{2} J^a_{\hat{b}k} \omega^k;\ \ 3)\ J^a_{bk} = 0;\ \ 4)\ J^{\hat{a}}_{\hat{b}k} = 0. \tag{4.4}$$

Indeed, putting $i = a, j = b$ in (4.1), we obtain, taking into account (1.4):

$$dJ_b^{\hat{a}} - J_{\hat{c}}^{\hat{a}}\omega_b^{\hat{c}} + J_b^c\omega_c^{\hat{a}} = J_{bk}^{\hat{a}}\omega^k,$$

Whence, taking into account (1.4), we obtain that $2\sqrt{-1}\omega_b^{\hat{a}} = J_{bk}^{\hat{a}}\omega^k$, which means that $\omega_b^{\hat{a}} = -\frac{\sqrt{-1}}{2}J_{bk}^{\hat{a}}\omega^k$. The rest of the relations are proved similarly.

Taking into account (3.41-2), we obtain that the first group of structural equations of an almost complex manifold on the space of the associated G-structure can be written in the form

$$1)\ d\omega^a = -\omega_b^a \wedge \omega^b + \frac{\sqrt{-1}}{2}J_{\hat{b}c}^a\omega^{\hat{b}} \wedge \omega^c + \frac{\sqrt{-1}}{2}J_{[\hat{b}\hat{c}]}^a\omega^{\hat{b}} \wedge \omega^{\hat{c}};$$

$$2)\ d\omega^{\hat{a}} = -\omega_{\hat{b}}^{\hat{a}} \wedge \omega^{\hat{b}} - \frac{\sqrt{-1}}{2}J_{b\hat{c}}^{\hat{a}}\omega^b \wedge \omega^{\hat{c}} - \frac{\sqrt{-1}}{2}J_{[bc]}^{\hat{a}}\omega^b \wedge \omega^c. \qquad (4.5)$$

Further, taking into account (3.3), (1.4) and (3.4:3-4), we calculate the components of the Nijenhuis tensor in the A-frame (i.e., on the space of the associated G-structure):

$$N_{\hat{b}\hat{c}}^a = \sqrt{-1}J_{[\hat{b}\hat{c}]}^a;\ \ N_{bc}^{\hat{a}} = -\sqrt{-1}J_{[bc]}^{\hat{a}}. \qquad (4.6)$$

The other components of this tensor are zero.

Taking into account (4.6), structural equations (4.5) can be written in the form

$$1)\ d\omega^a = -\omega_b^a \wedge \omega^b + \frac{1}{2}N_{\hat{b}\hat{c}}^a\omega^{\hat{b}} \wedge \omega^{\hat{c}} + \frac{\sqrt{-1}}{2}J_{\hat{b}c}^a\omega^{\hat{b}} \wedge \omega^c;$$

$$2)\ d\omega^{\hat{a}} = -\omega_{\hat{b}}^{\hat{a}} \wedge \omega^{\hat{b}} - \frac{\sqrt{-1}}{2}J_{b\hat{c}}^{\hat{a}}\omega^b \wedge \omega^{\hat{c}} - \frac{1}{2}N_{bc}^{\hat{a}}\omega^b \wedge \omega^c. \qquad (4.7)$$

From these relations follows the following important

Theorem 4.1 The eigendistributions of an almost complex structure J on a smooth manifold M^{2n} are involutive if and only if its Nijenhuis tensor vanishes identically.

Proof. This immediately follows from the fact that the eigendistributions $D_J^{\sqrt{-1}}$ and $D_J^{-\sqrt{-1}}$ are given, respectively, by the Pfaffian systems $s^*(\omega^{\hat{a}}) = 0$ and $s^*(\omega^a) = 0$ $(a = 1, \ldots, n)$, (Where s is the local section of the associatedG-structure) due to the definition of the A-frame and also relations (4.7).

4.2 Almost Complex Connections

Definition 4.1 An almost complex connection on an almost complex manifold (M, J) is a connection in the principal bundle $B_J(M)$.

By virtue of the existence of a canonical monomorphism

$$(\tilde{f}, \tilde{\rho}): B_J(M) \to B(M)$$

An almost complex connection induces a path connection on M, and by virtue of the existence of a canonical monomorphism

$(f, \rho): B_J(M) \to B^C(M)$,

Which is an isomorphism onto the image, the associated G-structure, the almost complex connection induces connections in these principal bundles. All these four connections can be identified. In particular, a connection in the principal bundle $B^C(M)$ as an almost complex connection is characterized by the fact that its connection form θ takes values in the Lie algebra of the structure group of the associated G-structure, and hence its components satisfy the relations

$$1)\ \overline{\theta_b^a} = \theta_{\hat{b}}^{\hat{a}};\ \ 2)\ \theta_b^{\hat{a}} = 0;\ \ 3)\ \theta_{\hat{b}}^a = 0. \tag{4.8}$$

Theorem 4.2 A linear connection ∇ on an almost complex manifold (M,J) is an almost complex connection if and only if $\nabla J = 0$.

Proof. Let ∇ be an arbitrary connection on M and θ its form on $B^C(M)$. Then, as above, the identities

$$\nabla J_j^i \equiv dJ_j^i - J_k^i \theta_j^k + J_j^k \theta_k^i = J_{j,k}^i \omega^k.$$

As above, it is shown that on the space of the associated G-structure these identities take the form:

$$1)\ \theta_b^{\hat{a}} = -\tfrac{1}{2} J_{\hat{b},k}^{\hat{a}} \omega^k;\ \ 2)\ \theta_{\hat{b}}^a = \tfrac{1}{2} J_{\hat{b},k}^a \omega^k;\ \ 3)\ J_{b,k}^a = 0;\ \ 4)\ J_{\hat{b},k}^{\hat{a}} = 0. \qquad (4.9)$$

Relations (4.6) for the nonzero components of the Nijenhuis tensor then take the form:

$$N_{\hat{b}\hat{c}}^a = \sqrt{-1} J_{[\hat{b},\hat{c}]}^a;\ \ N_{bc}^{\hat{a}} = -\sqrt{-1} J_{[b,c]}^{\hat{a}}. \qquad (4.10)$$

Let now ∇ be an almost complex connection. Then, due to (4.8), (4.9:1-2) and due to the linear independence of the basic forms, we have: $\nabla J_{\hat{b},k}^a = 0, \nabla J_{b,k}^{\hat{a}} = 0$. In view of (3.9:3-4), $\nabla J_{j,k}^i = 0$, i.e. $\nabla J = 0$.

Conversely, let $\nabla J = 0$. Then, taking into account (3.9:1-2), we obtain that $\theta_{\hat{b}}^a = 0, \theta_b^{\hat{a}} = 0$. On the other hand, since the connection ∇ is real, it follows that $\overline{\theta_b^a} = \theta_{\hat{b}}^{\hat{a}}$. Consequently, θ is a connection form with values in the Lie algebra of the structure group of the associated G-structure, and due to the isomorphism (f,ρ) of principal bundles, it can be identified with an almost complex connection.

Theorem 4.3 Every torsion-free linear connection on an almost complex manifold induces an almost complex connection whose torsion tensor coincides with the Nijenhuis tensor.

Proof. Let ∇ be an arbitrary torsion-free path connection on an almost complex manifold (M, J), and let θ be the form of this connection. By virtue of relations (4.9) and (4.10), its first group of structural equations on the space of the associated G-structure takes the form:

$$1)\ d\omega^a = \left(-\theta^a_b \wedge \omega^b + \frac{\sqrt{-1}}{2} J^a_{\hat{c},b}\omega^{\hat{c}}\right) \wedge \omega^b + \frac{1}{2} N^a_{\hat{b}\hat{c}}\omega^{\hat{b}} \wedge \omega^{\hat{c}};$$

$$2)\ d\omega^{\hat{a}} = -\left(\theta^{\hat{a}}_{\hat{b}} + -\frac{\sqrt{-1}}{2} J^{\hat{a}}_{c,\hat{b}}\omega^c\right) \wedge \omega^{\hat{b}} + \frac{1}{2} N^{\hat{a}}_{bc}\omega^b \wedge \omega^c. \tag{4.11}$$

Let us introduce the 1-form ζ with values in the complex complete matrix Lie algebra of order $2n$ and its components

$$1)\ \zeta^a_b = \theta^a_b - \frac{\sqrt{-1}}{2} J^a_{\hat{c},b}\omega^{\hat{c}};\ 2)\ \zeta^a_{\hat{b}} = 0;\ 3)\ \zeta^{\hat{a}}_{\hat{b}} = \theta^{\hat{a}}_{\hat{b}} + \frac{\sqrt{-1}}{2} J^{\hat{a}}_{c,\hat{b}}\omega^c;\ 4)\ \zeta^{\hat{a}}_b = 0. \tag{4.12}$$

Obviously, this form takes values in the Lie algebra of the structure group of the associated G-structure. In addition, externally differentiating (4.12:1) on the space of the associated G-structure, taking into account (4.9), we obtain:

$$d\zeta^a_b = d\theta^a_b - \frac{\sqrt{-1}}{2} dJ^a_{\hat{c},b} \wedge \omega^{\hat{c}} - \frac{\sqrt{-1}}{2} J^a_{\hat{c},b} d\omega^{\hat{c}} =$$

$$= -\theta^a_c \wedge \theta^c_b - \theta^a_{\hat{c}} \wedge \theta^{\hat{c}}_b + \frac{1}{2} R^a_{bij}\omega^i \wedge \omega^j$$

$$+ \frac{\sqrt{-1}}{2}\left\{\left(J^a_{\hat{h},b}\theta^{\hat{h}}_{\hat{c}} + J^a_{\hat{c},h}\theta^h_b + J^a_{\hat{c},\hat{h}}\theta^{\hat{h}}_b - J^h_{\hat{c},b}\theta^a_h + J^a_{\hat{c},\hat{b}}\omega^k\right) \wedge \omega^{\hat{c}}\right.$$

$$\left. - J^a_{\hat{c},b}\left(\theta^{\hat{c}}_h \wedge \omega^h + \theta^{\hat{c}}_{\hat{h}} \wedge \omega^{\hat{h}}\right)\right\} \equiv$$

$$\equiv \left(\zeta_c^a - \frac{\sqrt{-1}}{2} J_{\hat{h},c}^a \omega^{\hat{h}}\right) \wedge \left(\zeta_b^c - \frac{\sqrt{-1}}{2} J_{\hat{g},b}^c \omega^{\hat{g}}\right) + \frac{\sqrt{-1}}{2}\left(J_{\hat{h},b}^a \theta_{\hat{c}}^{\hat{h}} + J_{\hat{c},h}^a \theta_b^h - J_{\hat{c},b}^h \theta_h^a\right) \wedge \omega^{\hat{c}} - J_{\hat{c},b}^a \theta_{\hat{h}}^{\hat{c}} \wedge \omega^{\hat{h}} \equiv -\zeta_c^a \wedge \zeta_b^c = -\zeta_k^a \wedge \zeta_b^k \left(mod \omega^i \wedge \omega^j\right).$$

Here the symbol " $\equiv$ " is understood as an equality up to terms of the form $f\omega^i \wedge \omega^j; f \in C^\infty(B_J^C M)$. Thus, $d\zeta_b^a \equiv -\zeta_k^a \wedge \zeta_b^k\left(mod\omega^i \wedge \omega^j\right)$. Similarly, $d\zeta_{\hat{b}}^{\hat{a}} \equiv -\zeta_k^{\hat{a}} \wedge \zeta_{\hat{b}}^k\left(mod\omega^i \wedge \omega^j\right)$. Finally, taking into account (4.12),

$$d\zeta_{\hat{b}}^a = 0 - \zeta_c^a \wedge \zeta_{\hat{b}}^c - \zeta_{\hat{c}}^a \wedge \zeta_{\hat{b}}^{\hat{c}} = -\zeta_k^a \wedge \zeta_{\hat{b}}^k.$$

Similarly, $d\zeta_b^{\hat{a}} = 0 = -\zeta_k^{\hat{a}} \wedge \zeta_b^k$, and thus

$$d\zeta_j^i \equiv -\zeta_k^i \wedge \zeta_j^k\left(mod\omega^i \wedge \omega^j\right).$$

By the Cartan-Laptev theorem, ζ is a connection form. What already. It was noted that this form takes values in the Lie algebra of the structure group of the associated G-structure, and hence is an almost complex connection. It follows from (4.11) and (4.12) that the torsion tensor S of this connection coincides with the Nijenhuis tensor.

Definition 4.2 An almost complex connection whose torsion tensor coincides with the Nijenhuis tensor is said to be semi-canonical.

4.3 Global Definition

We are now ready to prove that the Nijenhuis tensor of an almost complex manifold is defined globally and obtain an explicit formula for calculating it. First of all, we recall that on the space of the associated G-structure

$$1)\ N_{ab}^c = 0;\ \ 2)\ N_{\hat{a}b}^c = 0;\ \ 3)\ N_{a\hat{b}}^c = 0; \tag{4.13}$$

And complex conjugate formulas (in short, f.c.s.).

Proposition 4.1 Relations (4.13) are equivalent to the following identities:

1) $\sigma N(\sigma X, \sigma Y) = 0$; 2) $\sigma N(\bar{\sigma} X, \sigma Y) = 0$ 3) $\sigma N(\sigma X, \bar{\sigma} Y) = 0$;

And f.c.s., respectively.

Proof. Note that for $p = (m, \varepsilon_1, \ldots, \varepsilon_{\hat{n}}) \in B_J^C M$, $N_{ab}^c(p) = 0 \iff N(\varepsilon_a, \varepsilon_b)^c \varepsilon_c = 0 \iff \sigma\left(N\big(\sigma(e_a), \sigma(e_b)\big)\right) = 0$, and since the vectors $\{e_1, \ldots, e_n\}$ form the basis of the space $T_m(M)$ as a C-module, then $\sigma\left(N\big(\sigma(e_a), \sigma(e_b)\big)\right) = 0 \iff \sigma N(\sigma X, \sigma Y) = 0$.

The rest of the assertions are proved similarly.

Proposition 4.2 Let ∇ be an almost complex connection on an almost complex manifold (M, J). Then

1) $\nabla_X \circ \sigma = \sigma \circ \nabla_X$; 2) $\nabla_X \circ \bar{\sigma} = \bar{\sigma} \circ \nabla_X; X \in \mathcal{X}(M)$.

Proof. By condition $\nabla_X J = 0$. That's why

$$2\nabla_X(\sigma Y) = \nabla_X Y - \sqrt{-1}\nabla_X(JY) = \nabla_X Y - \sqrt{-1}\{\nabla_X(J)Y + J\nabla_X Y\} = \big(id - \sqrt{-1}J\big)\nabla_X Y = 2\sigma(\nabla_X Y).$$

The second relation is proved similarly.

In view of these assertions, if S is the torsion tensor of the semi-canonical connection ∇, then

$N(X,Y) = (\sigma + \bar{\sigma})N\big((\sigma + \bar{\sigma})X, (\sigma + \bar{\sigma})Y\big) = \sigma N(\bar{\sigma}X, \bar{\sigma}Y) + \bar{\sigma}N(\sigma X, \sigma Y) = \sigma S(\bar{\sigma}X, \bar{\sigma}Y) + \bar{\sigma}S(\sigma X, \sigma Y) = \sigma\{\nabla_{\bar{\sigma}X}(\bar{\sigma}Y) - \nabla_{\bar{\sigma}Y}(\bar{\sigma}X) - [\bar{\sigma}X, \bar{\sigma}Y]\} + \bar{\sigma}\{\nabla_{\sigma X}(\sigma Y) - \nabla_{\sigma Y}(\sigma X) - [\sigma X, \sigma Y]\} = -\sigma[\bar{\sigma}X, \bar{\sigma}Y] - \bar{\sigma}[\sigma X, \sigma Y] = -\frac{1}{8}\big(id - \sqrt{-1}J\big)\big([X - \sqrt{-1}JX, Y - \sqrt{-1}JY]\big) - \frac{1}{8}\big(id + \sqrt{-1}J\big)\big([X - \sqrt{-1}JX, Y - \sqrt{-1}JY]\big) = \frac{1}{4}\{-[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]\}.$

This proves

Theorem 4.4 The Nijenhuis tensor of an almost complex structure J is calculated by the formula

$$N(X,Y) = \frac{1}{4}\{-[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]\}.$$

5. INTEGRABILITY OF ALMOST COMPLEX STRUCTURES

5.1 Complex Manifolds

In this section, we will give an example of the use of the Neuenhuis tensor in the geometry of almost complex structures.

Let $\boldsymbol{C}^m$ and $\boldsymbol{C}^n$ be complex arithmetic space sof dimensions m and n, respectively,$and\ let\ U \subset \boldsymbol{C}^m\ and\ V \subset \boldsymbol{C}^n$be open subsets, or feilds, in them. A map $f: V \longrightarrow Vis$ called holomorphic if the functions ωi=$\omega i^i = \omega^i\left(z^{z\,1}, \ldots, zm^{,zj=xj}\right), z^j = x^j + \sqrt{-1}y^j, \omega^+ - 1\ yj, \omega i = ui + -1\ u^i + \sqrt{-1}v^{vi}, i = 1, \ldots, n, j = 1, \ldots, n$, those defining the map f are holomorphic, i.e. they decompose into a convergent power series in the vicinity of each point of their domains of definition, or, equivalently, their real and imaginary parts are smooth functions, and the Cauchy-Riemann conditions $\frac{\partial u^i}{\partial x^j} = \frac{\partial v^i}{\partial y^j}; \frac{\partial u^i}{\partial y^j} = -\frac{\partial v^i}{\partial x^j}$.

A holomorphic map f is called a biholomorphismof U on V if f is bijective and f^{-1} is holomorphic. If such a map exists, the domains U and V are called biholomorphic.

Let M be a Hausdorff topological space with a countable base (i.e. Satisfying its second countability axiom). (An open) complex map (In short, a c-map) of dimension n in M is a pair (U, φ), where $U \subset M$ is an open subset, called the map area, $and\ \varphi$ is its homeomorphism to an open subset in $\boldsymbol{C}^n$, called the mapping map. Two maps (U, φ) and (V, ψ) are called holomorphically connected if $\psi \circ \varphi^{-1}: \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ is a biholomorphism of fields of complex Euclidean spaces.

The map (U, φ) is also called the local coordinate system on M, and the set of numbers $(z^1, \ldots, z^n) = \varphi(p)$ is called the local coordinates of the point p ∈ U. A family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of n-dimensional complex maps on M is called a (complex) atlas if:

1. All maps of the family are pairwise holomorphic connected,

2.$\bigcup_{\alpha\in A} U_\alpha = M$.

Definition 5.1 An atlas on M is called maximal if every map on M that is holomorphically connected to any map of the atlas belongs to this atlas. A maximal atlas on M is called a complex analytic structure on M. The space M on which the complex-analytic structure is fixed is called a complex manifold. The number n is called the complex dimension of the manifold (or structure), and is denoted by $\dim_{\mathbf{C}}$M..

Thus, a complex analytic structure on a-manifold M is a family $\mathfrak{C} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of open maps of a given dimension on M, such that

1. (CM1) $M = \bigcup_{\alpha\in A} U_\alpha$;

2. (CM2) $\forall \alpha, \beta \in A \implies \varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ – holomorphic mapping regions of Euclidean spaces;

3. (CM3) $\mathfrak{C}$is the maximum family for which (CMl) and (CM2) hold.

As in the real case, it is quite obvious that any complex atlas of hand M can be extended to a complex-analytical structure, and in the only way, by adding to it all the maps holomorphically related to any hag of this atlas. Two complex-analytic structures on M are called equivalentif any pair of their maps is holomorphically connected.

Examples of complex manifolds are the n-dimensional complex linear space V and any of its domains, as well as the complex projective space $\boldsymbol{CP}(V)$. In the first of these examples, a complex atlas is defined by a single map defined by specifying an arbitrary basis and mapping to each vector $X \in V$ a set of n complex numbers that are the coordinates of this в vector in this basis, and the maps given by different bases are obviously biholomorphic related by the relations $w^i = C^i_j z^j$, where C is the matrix of the transition from the second basis to the first. In the second example, the complex structure is defined by the complex maps constructed in this example.

Note that every n-dimensional c-map can be considered as $a\ 2n$-dimensional map of a smooth structure called the reification of a given c-map, and according to the definition, a holomorphic connection of two c-maps implies a smooth connection of their reifications. Hence, every n-dimensional complex manifold can be considered as $a\ 2n$-dimensional real manifold, called the reification of this complex manifold.

5.2 Canonical Almost Complex Structure and Integrability

Let M be $a\ 2n$-dimensional smooth manifold, $m \in M$ is an arbitrary point, (U, φ) is a c-map on M with coordinates $\left\{z^a = x^a + \sqrt{-1}x^{\hat{a}} | a = 1, \dots, n\right\}$. We define an endomorphism I_m of the space $T_m(M)$ by the formula

$$I_m = r_\varphi^{-1} \circ I_O \circ r_\varphi, \tag{5.1}$$

Where $r_\varphi : T_m(M) \to \boldsymbol{R}^{2n}$ is a reference map , I_O is a standard complex structure on the reification $\boldsymbol{R}^{2n}$ of the space $\boldsymbol{C}^n$, corresponding to multiplication in $\boldsymbol{C}^n$ by $\sqrt{-1}$ and given by a matrix of the form $\left(J^i_j\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Obviously, $I_m^2 = -id$, i.e. I_m is the –complex structure. It is also obvious that the family

$I = \{I_m | m \in U\}$ of endomorphisms defines a smooth tensor field of type (1,1) on U, since the matrix of its components is in the natural basis

$$(I_j^i) = dx^i\left(I\left(\frac{\partial}{\partial x^j}\right)\right) = dx^i\left(r_\varphi^{-1} \circ I_O \circ r_\varphi\left(\frac{\partial}{\partial x^j}\right)\right) = \varepsilon^i\left(I_O(\varepsilon_j)\right) = \left((I_O)_j^i\right) =$$

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{5.2}$$

It consists of constants and, therefore, smooth functions. Here $\{\varepsilon_a, \varepsilon_{\hat{a}} = I_O(\varepsilon_a)\}$– real-adapted complex structure $of the I_O$ basis of the space $\boldsymbol{R}^{2n}$, corresponding to the standard basis $\{\varepsilon_a\}$ space $\boldsymbol{C}^n$, $\{\varepsilon^i\}$ is the dual cobasis. Let us call the constructed almost complex structure I on the domain U the canonical almost complex structure of this c-map.

Note 5.1 Obviously, the relations (5.2) are equivalent to the relations

$$I\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial x^{\hat{a}}}, I\left(\frac{\partial}{\partial x^{\hat{a}}}\right) = -\frac{\partial}{\partial x^a}; \;\; a = 1, \dots, n.$$

Case 5.1 Canonical almost complex structures of two c-maps coincide at the intersection of the domains of these maps if and only if these c-maps are holomorphically connected.

Proof. Let (U, φ), and (V, ψ) be two c-charts on M with coordinates $\{z^a = x^a + \sqrt{-1}x^{\hat{a}} | a = 1, \dots, n\}$ and $\{w^a = u^a + \sqrt{-1}u^{\hat{a}} | a = 1, \dots, n\}$, respectively, and I and $J$are their coordinates. Canonical almost, complex structures. By virtue of (5.1),

$$I|_{U \cap V} = J_{U \cap V} \iff \varphi^{-1} \circ I_O \circ \varphi = \psi^{-1} \circ I_O \circ \psi \iff (\psi \circ \varphi^{-1}) \circ I_O = I_O \circ (\psi \circ \varphi^{-1}). \tag{5.3}$$

Given that the endomorphisms I_OIo and $\zeta = \varphi^{\varphi^{-1}}$ in the natural real-adapted basis are given, respectively, by the matrices

$$\left((I_O)^i_j\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix};\ \left(\zeta^i_j\right) = \left(\frac{\partial u^i}{\partial x^j}\right) = \begin{pmatrix} \frac{\partial u^a}{\partial x^b} & \frac{\partial u^a}{\partial x^{\hat{b}}} \\ \frac{\partial u^{\hat{a}}}{\partial x^b} & \frac{\partial u^{\hat{a}}}{\partial x^{\hat{b}}} \end{pmatrix},$$

Conditions (5.3) can be rewritten in the form

$$\frac{\partial u^a}{\partial x^b} = \frac{\partial u^{\hat{a}}}{\partial x^{\hat{b}}};\ \frac{\partial u^{\hat{a}}}{\partial x^b} = -\frac{\partial u^a}{\partial x^{\hat{b}}},$$

That is, in the form of classical Cauchy-Riemann equations expressing the holomorphic conditions of the functions $w^a = w^a(z^1, \ldots, z^n)$; $a = 1, \ldots, n$, definingtheir map $\psi \circ \varphi^{-1}$. Due to the equality of charts, the inverse chart is also holomorphic, which means that the charts are holomorphic, and the converse is also true.

A proven statement immediately follows:

Theorem 5.1 An almost complex structure is defined internally on the reification of every complex manifold.

Proof. We fix on it a c-chart (U, φ) with coordinates $\left\{z^a = x^a + \sqrt{-1}x^{\hat{a}} | a = 1, \ldots, n\right\}$. Then the vector fields $\left\{\frac{\partial}{\partial x^a}; \frac{\partial}{\partial x^{\hat{a}}} / \ a|1, \ldots, n\right\}$ form the basis of the module $\mathcal{X}(U^{\boldsymbol{R}})$ of the reification of the manifold U (the natural basis of this module). We define an endomorphism I of this module by requiring that in the natural basis it is given by the matrix

$$\left(I^i_j\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

In other words, let

$$I\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial x^{\hat{a}}}, I\left(\frac{\partial}{\partial x^{\hat{a}}}\right) = -\frac{\partial}{\partial x^a};\ \ a = 1, \ldots, n. \tag{5.4}$$

Obviously, $I^2 = -id$, i.e. Iis an almost complex structure on $the\ U^{\boldsymbol{R}}$ manifold. Let's show that it does not depend on the map selection. Let (V, ψ) be another c- chart on M with coordinates $\{w^a = u^a + \sqrt{-1}u^{\hat{a}} | a = 1, \dots, n\}$. According to the definition of a complex-analytic structure, the chart

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

Is a holomorphic map of regions of space $\boldsymbol{C}^n$, and according to the equations Cauchy-Riemann,

$$\frac{\partial u^a}{\partial x^b} = \frac{\partial u^{\hat{a}}}{\partial x^{\hat{b}}}; \ \frac{\partial u^{\hat{a}}}{\partial x^b} = -\frac{\partial u^a}{\partial x^{\hat{b}}}. \tag{5.5}$$

The new map corresponds to the endomorphism $\tilde{I}$, which operates according to the formulas

$$\tilde{I}\left(\frac{\partial}{\partial u^a}\right) = \frac{\partial}{\partial u^{\hat{a}}}, \tilde{I}\left(\frac{\partial}{\partial u^{\hat{a}}}\right) = -\frac{\partial}{\partial u^a}; \ \ a = 1, \dots, n. \tag{5.6}$$

Considering the narrowing of these endomorphisms to the domain $U \cap V$, taking into account (5.6), we obtain:

$$\tilde{I}\left(\frac{\partial}{\partial x^a}\right) = \tilde{I}\left(\frac{\partial u^b}{\partial x^a}\frac{\partial}{\partial u^b} + \frac{\partial u^{\hat{b}}}{\partial x^a}\frac{\partial}{\partial u^{\hat{b}}}\right) = \frac{\partial u^b}{\partial x^a}\tilde{I}\left(\frac{\partial}{\partial u^b}\right) + \frac{\partial u^{\hat{b}}}{\partial x^a}\tilde{I}\left(\frac{\partial}{\partial u^{\hat{b}}}\right) = \frac{\partial u^b}{\partial x^a}\frac{\partial}{\partial u^{\hat{b}}} + \frac{\partial u^{\hat{b}}}{\partial x^a}\frac{\partial}{\partial u^b},$$

And, taking into account (5.4) and (5.6),

$$\tilde{I}\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial u^{\hat{b}}}{\partial x^{\hat{a}}}\frac{\partial}{\partial u^{\hat{b}}} + \frac{\partial u^b}{\partial x^{\hat{a}}}\frac{\partial}{\partial u^b} = \frac{\partial}{\partial x^{\hat{a}}} = I\left(\frac{\partial}{\partial x^a}\right),$$

That is,

$\tilde{I}\left(\frac{\partial}{\partial x^{a}}\right) = I\left(\frac{\partial}{\partial x^{a}}\right)$.

Similarly, we prove that:

$\tilde{I}\left(\frac{\partial}{\partial x^{\hat{a}}}\right) = I\left(\frac{\partial}{\partial x^{\hat{a}}}\right)$.

Thus, the values of both endomorphisms coincide on the basis vectors, and therefore, $I_{Iu\cap V} = \tilde{I}_{Iu\cap V.}$. Thus, it is built on fields . A c- chart is an almost complex structure on a manifold $M^{\boldsymbol{R}}$ defined. It is global and does not depend on the chart selection, i.e. it is defined internally.

Definition 5.2 An almost complex structure I that is internally attached to the reification of a complex manifold is called the canonical almost complex structure of this manifold.

Definition 5.3 An almost complex structure J on a smooth manifoldis called integrableif it is a canonical almost complex structure of some complex-analytic structure of this manifold.

Note that according to (5.2), natural bases of reification of c- charts of a manifold M are real-adapted bases. The corresponding A-bases $\{\varepsilon_{a}, \varepsilon_{\hat{a}}\}$ are formed by vector fields

$\varepsilon_{a} = \sigma\left(\frac{\partial}{\partial x^{a}}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^{a}} - \sqrt{-1}I\frac{\partial}{\partial x^{a}}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^{a}} - \sqrt{-1}\frac{\partial}{\partial x^{\hat{a}}}\right) = \frac{1}{2}\frac{\partial}{\partial z^{a}}$; $\varepsilon_{\hat{a}} = \bar{\sigma}\left(\frac{\partial}{\partial x^{a}}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^{a}} + \sqrt{-1}I\frac{\partial}{\partial x^{a}}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^{a}} + \sqrt{-1}\frac{\partial}{\partial x^{\hat{a}}}\right) = \frac{1}{2}\frac{\partial}{\partial z^{\hat{a}}}$.

Their dual bases are formed by 1-forms

$dz^{a} = dx^{a} + \sqrt{-1}dx^{\hat{a}}$и $dz^{\hat{a}} = d\bar{z}^{a} = dx^{a} - \sqrt{-1}dx^{\hat{a}}$.

In fact,

$$dz^a\left(\frac{\partial}{\partial z^b}\right)=\frac{1}{2}\left(dx^a+\sqrt{-1}dx^{\hat{a}}\right)\left(\frac{\partial}{\partial x^b}-\sqrt{-1}\frac{\partial}{\partial x^{\hat{b}}}\right)=\frac{1}{2}(\delta_b^a-0+0+\delta_b^a)=\delta_b^a \quad ,$$
$$dz^a\left(\frac{\partial}{\partial z^{\hat{b}}}\right)=\frac{1}{2}\left(dx^a+\sqrt{-1}dx^{\hat{a}}\right)\left(\frac{\partial}{\partial x^b}+\sqrt{-1}\frac{\partial}{\partial x^{\hat{b}}}\right)=\frac{1}{2}(\delta_b^a+0+0-\delta_b^a)=0.$$

Similarly, it is verified that $dzaz^{\hat{a}}\left(\frac{\partial}{\partial z^b}\right)\partial\partial zb \;=\; 0, dza\partial^{\hat{\partial}}\left(\frac{\partial}{\partial z^{\widehat{zb=\delta ba}}}\right)=\delta_{\hat{b}}^{\hat{a}}$, i.e.

$$dz^i\left(\frac{\partial}{\partial z^j}\right)=\delta_j^i.$$

Since the vector fields $\{\varepsilon_a\}$ and $\{\varepsilon_{\hat{a}}\}$ form a local basis of their own distributions $D_I^{\sqrt{-1}}$ and $D_I^{-\sqrt{1}}$ of endomorphism I, respectively, the systems of 1-forms $\{dz^{\hat{a}}\}$ and $\{dz^a\}$ form Pfaff systems of these distributions, respectively. Obviously, these Pfaff systems, and hence their corresponding distributions, are completely integrable, by virtue of Theorem 4.1, the Neuenhuis tensor N_I identically equal to Pool.

Conversely, let (M,J) be a $2n$-dimensional almost complex manifold, and the Neuenhuis tensor N_I is identically equal to the pool. By Theorem 4.1, this is equivalent to the complete integrability of the eigendistributions $D_J^{\sqrt{-1}}$ and $D_J^{-\sqrt{-1}}$ endomorphism J. Let $\{\omega^{\hat{a}}\}$ and $\{\omega^a\}$– be the Pfaff systems of these distributions, respectively, $a=1,\dots,n$. Note that these Pfaff systems are complex conjugate. Consider the first of these systems. Due to its complete integrability, its forms locally satisfy the structural equations

$$d\omega^a=\omega_b^a\wedge\omega^b. \tag{5.7}$$

Let (U,φ) be a local chart of the manifold with coordinates $\{x^1,\dots,x^{2n}\}$, in the domain of which these equations are defined. Then the map φ is a diffeomorphism of the domain U to the domain $V=\varphi(U)\subset\boldsymbol{R}^{2n}$, and hence defines the map $\varphi_*:\mathcal{T}(U)\longrightarrow T(V)$. Applying this map to a contractionm of the forms ω^a on U, we obtain a system of forms $\{\theta^a\}$ on V satisfying, by virtue of (5.7), the equations $d\theta^a = \theta^a_b \wedge \theta^b$, where $\theta^a_b = \varphi_*(\omega^a_b)$. If the manifold M is real-analytic, it follows from the classical Frobenius theorem (extended to complex forms) that there exists a system $z^a = z^a(x^1, \ldots, x^{2n})$of real-analytic complex-valued functions on Vsuch that

$$\theta^a = A^a_b dz^b; a = 1, \ldots, n, \tag{5.8}$$

Where $\{A^a_b\}$ is a matrix of complex-valued real-analytic functions that is non degenerate at each point of the domain of definition. Next, let $z^a = u^a + \sqrt{-1}u^{\hat{a}}$. Since the forms $\{dz^a\}$ are complexly linearly independent, the forms $\{du^a, du^{\hat{a}}\}$, and their real and imaginary parts will be linearly independent in the real sense, hence $det\left(\frac{\partial u^i}{\partial x^j}\right) \neq 0$ at each point of the domain of definition, which means that the system of functions $\{u^i = u^i \circ \varphi(p)/\, p \in M\}$ defines valid maps on a manifold M, and the system of complex-valued functions $\{z^a = z^a \circ \varphi(p) | p \in M\}$ defines c-charts of this manifold. Let Ibe a canonical almost complex structure corresponding to such a c-chart. As we have seen, the forms $\{d(z^a \circ \varphi)\}$ form the Pfaff system $of the distribution D_I^{--\sqrt{1}}$. On the other hand, the forms ω^a form the Pfaff system $D_J^{-\sqrt{-1}}$. By virtue of (5.8),

$$\omega^a = \varphi^*(\theta^a) = \varphi^*(A^a_b)\varphi^*(dz^b) = \varphi^*(A^a_b)d\varphi^*(z^b) = (A^a_b \circ \varphi)d(z^b \circ \varphi)$$

This means that these Pfaff systems are equivalent, which means that$D_I^{-\sqrt{1}} = D_J^{-\sqrt{1}}$. But the distribution $D_I^{-\sqrt{1}}$ is locally spanned by vector fields $\left\{\frac{\partial}{\partial z^{\hat{a}}}\right\}$, which means $that\, J\left(\frac{\partial}{\partial z^{\hat{a}}}\right) = -\sqrt{1}\frac{\partial}{\partial z^{\hat{a}}}$, i.e.

$$J\left(\frac{\partial}{\partial x^a}\right) + \sqrt{-1}J\left(\frac{\partial}{\partial x^{\hat{a}}}\right) = -\sqrt{-1}\frac{\partial}{\partial x^a} + \frac{\partial}{\partial x^{\hat{a}}},$$

From where, subject to Remark 5.1,

$$J\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial x^{\hat{a}}} = I\left(\frac{\partial}{\partial x^a}\right), J\left(\frac{\partial}{\partial x^{\hat{a}}}\right) = -\frac{\partial}{\partial x^a} = I\left(\frac{\partial}{\partial x^{\hat{a}}}\right).$$

Hence, the endomorphisms I and J coincide on the map region. In particular, the canonical almost complex structures generated by the constructed c-charts coincide at the intersection of map domains and, according to Proposition 5.1, these c-charts are holomorphically connected. Since their domains cover M, they form an atlas, which means that they define a complex-analytic structure on a manifold Mwhose canonical almost complex structure coincides with J.Thus proved

Theorem 5.2 The Neuenhuis tensor of an integrable almost complex structure on a smooth manifold is zero. If the manifold is real-analytic, then the integrability of an almost complex structure on it is equivalent to the equality of its Neuenhuis tensor to zero.

Note 5.2 Newlander and Nirenberg [6] proved that the assumptionие of real analyticity of a manifold is insignificant and, thus, an almost complex structure on a smooth manifold is integrable if and only if its Neuenhuis tensor is zero.

6. TYPE CONSTANCY OF ALMOST HERMITIAN MANIFOLDS

In this section, we give another example of the use of the Neuenhuis tensor in the geometry of almost Hermitian manifolds.

Let M^{2n} be a smooth manifold, $\mathcal{X}(M)$ be a module of smooth vector fields on M, ∇ be a Riemannian connection on M, $and\ d$ be an external differentiation operator. All manifolds, tensor fields, etc. are assumed to be smooth objects of class C^{∞}.

Definition 6.1 An almost Hermitian structure on a manifold M^{2n} is a pair (J, g), where J is an almost complex structure on M, $and\ g = \langle\cdot,\cdot\rangle$ is a Riemannian metric on M.At the same time

$$\langle JX, JY\rangle = \langle X, Y\rangle;\ \forall X, Y \in \mathcal{X}(M).$$

An endomorphism J is called a structural endomorphism. A manifold on which an almost Hermitian structure is fixed is called an almost Hermitian manifold.

It is well known that specifying an almost Hermitian structure on M is equivalent to specifying a G-structure on M with the structure group $U(n)$, whose space elements are benchmarks consisting of pairwise conjugate eigenvectors of the structural endomorphism J and unitary in the natural Hermitian metric of complexification of the corresponding tangent space (Such benchmarks are called A-benchmarks) [6]. This G-structure is called an attached structure. We assume that the indices $i, j, k, \ldots$run through the values from 1 to $2n$, and the indices a, b, c, d, f, g, h run through the values from 1 to n. Let $\hat{a} = a + n$. As usual, the indices enclosed in square (respectively round) brackets imply alternation (respectively symmetrization).It is easy to verify that on the space of the adjoint G-structure, the components of tensors g and J are given by matrices

$$\left(g_{ij}\right) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \ \left(J_j^i\right) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \tag{6.1}$$

Where I_n is a unit matrix of order n.

Let (M, J, g) be an almost Hermitian manifold. $Letdenote$ the form of Riemannian connectivity ∇ on M. Taking into account relations (4.9), on the space of the attached G-structure

$$1)\ \theta_{\hat{b}}^{a} = \frac{\sqrt{-1}}{2} J_{\hat{b},c}^{a}\omega^{c} + \frac{\sqrt{-1}}{2} J_{\hat{b},\hat{c}}^{a}\omega^{\hat{c}};\ \ 2)\ \theta_{b}^{\hat{a}} = -\frac{\sqrt{-1}}{2} J_{b,c}^{\hat{a}}\omega^{c} - \frac{\sqrt{-1}}{2} J_{b,\hat{c}}^{\hat{a}}\omega^{\hat{c}};\ \ 3)\ J_{b,k}^{a} = 0;\ \ 4)\ J_{\hat{b},k}^{\hat{a}} = 0. \tag{6.2}$$

In addition, by virtue of the relation $\nabla g = 0$, $dg_{ij} - g_{kj}\theta_i^k - g_{ik}\theta_j^k = 0$. On the space of the attached G-structure, these relations, by virtue of (6.1:1), will take the form:

$$1)\ \theta_{b}^{\hat{a}} + \theta_{a}^{\hat{b}} = 0;\ \ 2)\ \theta_{\hat{b}}^{a} + \theta_{\hat{a}}^{b} = 0;\ \ 3)\ \theta_{b}^{a} + \theta_{\hat{a}}^{\hat{b}} = 0. \tag{6.3}$$

Taking these relations into account, the first group of structural equations of Riemannian connectivity on the space of the attached G-structure is written in the form:

$$\begin{cases} 1)\ d\omega^{a} = -\theta_{b}^{a} \wedge \omega^{b} + B^{ab}{}_{c}\omega^{c} \wedge \omega_{b} + B^{abc}\omega_{b} \wedge \omega_{c}; \\ 2)\ d\omega_{a} = \theta_{a}^{b} \wedge \omega_{b} + B_{ab}{}^{c}\omega_{c} \wedge \omega^{b} + B_{abc}\omega^{b} \wedge \omega^{c}. \end{cases} \tag{6.4}$$

Where

$$\omega_{a} = \omega^{\hat{a}};\ B^{ab}{}_{c} = -\frac{\sqrt{-1}}{2} J_{\hat{b},c}^{a};\ B^{abc} = \frac{\sqrt{-1}}{2} J_{[\hat{b},\hat{c}]}^{a};\ B_{ab}{}^{c} = \frac{\sqrt{-1}}{2} J_{b,\hat{c}}^{\hat{a}};\ B_{abc} = -\frac{\sqrt{-1}}{2} J_{[b,c]}^{\hat{a}};\ \overline{B^{ab}{}_{c}} = B_{ab}{}^{c};\ \overline{B^{abc}} = B_{abc}. \tag{6.5}$$

Equations (6.4) are called the first group of structural equations of almost Hermitian structure.

According to Theorem 4.4, the Neuenhuis tensor of a structural endomorphism J has the form Nx,$Y(X,Y) = \frac{1}{4}\{-[X,Y] + [JX,JY] - [JJX,Y] - [Jx,JY]\}$. It is easy to see that this tensor has the properties:

$$N(X,Y) = -N(Y,X);\ N(JX,Y) = N(X,JY) = -JN(X,Y);\ \forall X,Y \in \mathcal{X}(M). \tag{6.6}$$

A direct calculation with the identity $[X,Y] = \nabla_X Y - \nabla_Y X$ shows that

$$N(X,Y) = \frac{1}{4}\{\nabla_{JX}(J)Y - J\nabla_X(J)Y - \nabla_{JY}(J)X + J\nabla_Y(J)X\}. \tag{6.7}$$

Taking into account (6.5), we obtain from this that on the space of the adjoint G-structure, the components of the tensor N are determined by the equalities

$$N^a_{\hat{b}\hat{c}} = B^{abc}, N^{\hat{a}}_{bc} = B_{abc}. \tag{6.8}$$

The remaining components of the tensor N are zero.

Definition 6.2 An almost Hermitian variety (M,J,g) is called a variety of constant type c if

$$\forall X,Y \in \mathcal{X}(M): \langle\langle X,Y\rangle\rangle = 0 \implies \|N(X,Y)\|^2 = c\|X\|^2\|Y\|^2, \tag{6.9}$$

Where

$$\langle\langle X,Y\rangle\rangle = \langle X,Y\rangle + \sqrt{-1}\,\langle X,JY\rangle; X,Y \in \mathcal{X}(M). \tag{6.10}$$

Let us analyze definition 6.1. Taking into account (6.1: 1), we have $\langle X,Y\rangle = g_{ij}X^iY^j = g_{a\hat{b}}X^aY^{\hat{b}} + g_{\hat{a}b}X^{\hat{a}}Y^b = X^aY_a + X_aY^a$. According to (6.1) we have $\langle X,JY\rangle = g_{ij}X^iJ_k^jY^k = g_{a\hat{b}}X^aJ_{\hat{c}}^{\hat{b}}Y^{\hat{c}} + g_{\hat{a}b}X^{\hat{a}}J_c^bY^c = -\sqrt{-1}X^aY_a + \sqrt{-1}X_aY^a$.

Hence, $\langle\langle X,Y\rangle\rangle = \langle X,Y\rangle + \sqrt{-1}\,\langle X,JY\rangle = 0$ is equivalent to performing equalities on the space of the attached G-structure:

$X^aY_a + X_aY^a = 0$; $X^aY_a - X_aY^a = 0$,

Which are executed if and only if $X^aY_a = 0$.

Consider the equality $\|N(X,Y)\|^2 = c\|Nx,Y2 =\|^2\|Y\|^2$ on the spaceof the attached G-structure. On the one hand, by calculation (6.8) $\|N(X,Y)\|^2 = \langle N(X,Y),N(X,Y)\rangle Nx,Y2 = NX,Y,Nx,Y = 2NX,YANX,YA =$
$2\big(N(X,Y)\big)^a\big(N(X,Y)\big) \quad = 2\cdot B^{abc}X_bY_c\cdot B_{adh}X^dY^h$.

On the other hand, $c\|X\|^2\|Y\|^2CX2Y2 = c\times 2X^aX_a\times 2Y^dY_d = 4(X^aX_a)(Y^dY_d)$, so $B^{abc}X_bY_c\times B_{adh}X^dY^h = 2c(X^aX_a)(Y^dY_d)$. Consider a second-order Kronecker delta $\delta_{bc}^{ad} = \delta_b^a\delta_c^d - \delta_b^d\delta_c^a$, then $\delta_{bc}^{ad}X^bY^cX_aX_d = \big(\delta_b^a\delta_c^d - \delta_b^d\delta_c^a\big)X^bY^cX_aX_d = X^bY^cX_aX_d - X^bY^cX_aX_d = (X^aX_a)(Y^dY_d) - (X^dX_d)(Y^aY_a)$. Therefore, condition (6.9) on the space of an attached G-structure can be written as:

$\forall X,Y\in\mathcal{X}(M)$: $X^aY_a = 0 \implies B^{had}B_{hbc}X_aY_dX^bY^c = 2c\delta_{bc}^{ad}X_aY_dX^bY^c$.

We introduce the tensor $\mathcal{B}(X,Y,Z,W) = \big(B^{had}B_{hbc} - 2c\delta_{bc}^{ad}\big)X^bY^cZ_aW_d$; $\forall X,Y,Z,W\in\mathcal{X}(M.)$.

Theorem 6.1 The tensor $\mathcal{B}(X,Y,Z,W) = \big(B^{had}B_{hbc} - 2c\delta_{bc}^{ad}\big)X^bY^cZ_aW_d$ has the properties:

1) $\mathcal{B}(JX,Y,Z,W)=\mathcal{B}(X,JY,Z,W)=\sqrt{-1}\mathcal{B}(X,Y,Z,W)$;

2) $\mathcal{B}(X,Y,JZ,W)=\mathcal{B}(X,Y,Z,JW)=-\sqrt{-1}\mathcal{B}(X,Y,Z,W)$;

3) $\overline{\mathcal{B}(X,Y,Z,W)}=\mathcal{B}(Z,W,X,Y)$;

4) $\mathcal{B}(X,Y,Z,W)=-\mathcal{B}(Y,X,Z,W)$;

5) $\mathcal{B}(X,Y,Z,W)=-\mathcal{B}(X,Y,W,Z)$;

6) $\mathcal{B}(X,Y,X,Y)=0$, if $X^aY_a=0$.

Proof. Let 's prove the property

1 $\mathcal{B}(JX,Y,Z,W)=\left(B^{had}B_{hbc}-2c\delta_{bc}^{ad}\right)(JX)^bY^cZ_aW_d=\left(B^{had}B_{hbc}-2c\delta_{bc}^{ad}\right)J_g^bX^gY^cZ_aW_d=\sqrt{-1}\left(B^{had}B_{hbc}-2c\delta_{bc}^{ad}\right)X^bY^cZ_aW_d=\sqrt{-1}\mathcal{B}(X,Y,Z,W)$.

Similarly, we prove complex linearity by the second argument.

Let 's prove the property

2): $\mathcal{B}(X,Y,JZ,W)=\left(B^{had}B_{hbc}-2c\delta_{bc}^{ad}\right)X^bY^c(JZ)_aW_d=\left(B^{had}B_{hbc}-2c\delta_{bc}^{ad}\right)X^bY^cJ_a^gZ_gW_d=-\sqrt{-1}\left(B^{had}B_{hbc}-2c\delta_{bc}^{ad}\right)X^bY^cZ_aW_d=\sqrt{-1}\mathcal{B}(X,Y,Z,W)$.

Similarly, the proof of complex antilinearity is performed with respect to the fourth argument.

Let 's prove the property

3): $\overline{\mathcal{B}(X,Y,Z,W)} = \overline{\left(B^{had}B_{hbc} - 2c\delta_{bc}^{ad}\right)X^bY^cZ_aW_d} = \left(B_{had}B^{hbc} - 2c\delta_{ad}^{bc}\right)X_bY_cZ^aW^d = \left(B^{hbc}B_{had} - 2c\delta_{ad}^{bc}\right)Z^aW^dX_bY_c = \mathcal{B}(Z,W,X,Y)$.

For the property

4) We have: $\mathcal{B}(X,Y,Z,W) = \left(B^{had}B_{hbc} - 2c\delta_{bc}^{ad}\right)X^bY^cZ_aW_d = -\left(B^{had}B_{hcb} - 2c\delta_{cb}^{ad}\right)X^bY^cZ_aW_d \underbrace{=}_{b\leftrightarrow c} -\left(B^{had}B_{hbc} - 2c\delta_{bc}^{ad}\right)Y^bX^cZ_aW_d = -\mathcal{B}(Y,X,Z,W)$.

Similarly, we prove property 5).

Property 6) follows directly from condition (6.9).

Theorem 6.2 An almost Hermitian manifold is a manifold of constant type c if and only if on the space of the adjoint G-structure

$$B^{had}B_{hbc} = 2c\delta_{bc}^{ad}, \tag{6.11}$$

Where $\delta_{bc}^{ad} = \delta_b^a\delta_c^d - \delta_b^d\delta_c^a$ is the second-order Kronecker delta.

Proof. Let an almost Hermitian manifold (M,J,g) be a manifold of constant type C.Then the equality holds

$$\mathcal{B}(X,Y,X,Y) = 0, \text{ if } X^aY_a = 0. \tag{6.12}$$

Let us perform an involutional polarization of identity (6.12) with respect to X[10]. First, we replace Xна $withX + Zand$, get Bx $\mathcal{B}(,Y,X,Y) + \mathcal{B}(+Bz),\mathcal{Y},(X,Y + Bx,Y,Z,Y) + \mathcal{B}z(,Y,Z,Y) = 0$, from which, taking into account (6.12), we get

$$\mathcal{B}(Z,Y,X,Y) + \mathcal{B}(X,Y,Z,Y) = 0. \tag{6.13}$$

In (4.13), we replace $X \to JX$, then, taking into account Theorem 6.1, we get

$-\sqrt{1}\mathcal{B}(Z,Y,X,Y) + \sqrt{-1}\mathcal{B}(X,Y,Z,Y) = 0$. Add to the obtained equality the identity (6.13) multiplied by $\sqrt{-1}$, and we get

$$\mathcal{B}(X,Y,Z,Y) = 0. \tag{6.14}$$

By performing an involutional polarization of identity (6.14) with respect to Y, we obtain

$$\mathcal{B}(X,Y,Z,W) = 0;\ \forall X,Y,Z,W \in \mathcal{X}(M).$$

It follows that the tensor $\mathcal{B} \equiv 0$, i.e. $B^{had}B_{hbc} = 2c\delta^{ad}_{bc}$.

Theorem 6.3 An almost Hermitian manifold of constant type zero coincides with the class of Hermitian manifolds.

Proof. Let an almost Hermitian manifold (M,J,g) be a manifold of constant type zero. Then the identity (6.11) takes the form $B^{had}B_{hbc} = 0$. We collapse the last equality first by indices a and b, then by indices c and d, then we get $0 = B^{abc}B_{abc} = \sum_{abc}|B_{abc}|^2 \iff B_{abc} = 0$. Taking into account the obtained equality, the system (6.4) will take the form:

$$\begin{cases} 1)\ d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b; \\ \ 2)\ d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b. \end{cases} \tag{6.15}$$

The resulting system of equations tells us that the manifold is Hermitian.

We obtain a number of corollaries from Theorem 6.3.

Corollary 6.1 A quasi-Koehler manifold of type zero (i.e., the nearly Koehler and almost Koehler manifold) is a Koehler manifold.

Corollary 6.2 G_{G1}-and G_{G1}-manifolds of constant type zero are Hermitian manifolds.

7. CONCLUSIONS AND RECOMMENDATION

In this thises, we give a definition of a complex structure on a linear space and prove that a finite-dimensional real space admits a complex structure if and only if it is even-dimensional. It is shown that the fixation of the complex structure J in the $2n$-dimensional real linear space $V^{\boldsymbol{R}}$ induces the specification of the structure of the n-dimensional complex linear space V in $V^{\boldsymbol{R}}$. Each basis $b = \{e_1, \dots, e_{2n}\}$ of the space V canonically induces two bases:

1) RA-basis $b_{RA} = \{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ of the space $V^{\boldsymbol{R}}$.

2) A-basis $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$ of the C-linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

A G-structure attached to an almost complex structure is constructed.

We have given a local definition of the Nijenhuis tensor of an almost complex structure J. We have proved that this tensor is in fact defined globally on the manifold M, and we have found an explicit expression for it.

We give two examples of the use of the Neuenhuis tensor in the geometry of almost complex and almost Hermitian manifolds

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