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TOKAT GAZİOSMANPAŞA UNIVERSITY

INSTITUTE OF GRADUATE EDUCATION

DEPARTMENT OF PHYSICS

MASTER OF SCIENCE

ANALYTICAL SOLUTION OF BOHR HAMILTONIAN WITH DAVIDSON POTENTIAL FOR E(5) MODEL USING NIKIFOROV-UVAROV METHOD

MASTER'S THESIS

Lava MOHAMMED

Advisor: Prof.Dr. İbrahim YİĞİTOĞLU

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Scientific Ethics Page

According to the thesis writing guide Tokat Gaziosmanpaşa University Institute of Graduate Education, My Master's thesis named **"Analytical Solutions of Bohr Hamiltonian with Davidson Potential for E(5) Model using Nikiforov- Uvarov Method"** that l have prepared under the consultancy of **" Prof. Dr. İbrahim YİĞİTOĞLU"** is based on scientific ethical values and rules. I hereby declare that it is an appropriate original work and l will accept any legal sanctions if it is determined otherwise.

February 1st 2024

Lava Mohammed

JURİ KABUL VE ONAY

Lava MOHAMMED tarafından hazırlanan **“Analytical Solution of Bohr Hamiltonian With Davidson Potential For E(5) Model Using Nikiforov-Uvarov Method”** adlı tez çalışmasının savunma sınavı 01.02.2024 tarihinde yapılmış olup aşağıda verilen Jüri tarafından Oy Birliği ile Tokat Gaziosmanpaşa Üniversitesi Lisansüstü Eğitim Enstitüsü Fizik Anabilimdalı’nda Yüksek Lisans Tezi olarak kabul edilmiştir.

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ÖZET

YÜKSEK LİSANS TEZİ

NIKIFOROV-UVAROV YÖNTEMİ KULLANILARAK E(5) MODELİ İÇİN BOHR HAMİLTONYENİNİN DAVIDSON POTANSİYELİ İLE ANALİTİK ÇÖZÜMÜ

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Bu araştırmanın amacı, atom çekirdeğinin faz geçiş bölgesindeki nükleer yapılarını açıkça gösterecek benzersiz Bohr Hamiltonian çözümlerini bulmaktır. Bu bağlamda E(5) kritik nokta simetrisi için yeni bir model geliştirmek amacıyla Bohr Hamiltoniyenindeki Davidson potansiyeli kullanıldı. E(5)-D, Davidson potansiyeli kullanılarak türetilen $\gamma = 0°$ için γ-katı Bohr Hamiltonian çözümünün adıdır. PYTHON 3 Jupyter (data science) programını kullanarak enerji özdeğerlerini belirlemek için Nikiforov ve Uvarov tarafından oluşturulan analitik bir teknik kullanıldı. E(5)-D modelinin temel düzey bantları; ^{110}Xe ve ^{134}Ba çekirdekleri için mevcut deneysel verilerle uyumludur. Bu uyumsuzluğun nedeni çözümün γ kat çözüm olmasıdır. En yüksek spin seviyelerine kadar E(5)-D modelinin tahminleri, ^{156}Dy, ^{154}Gd, ^{154}Sm, ^{102}Pd, ^{150}Nd ve ^{74}Ge çekirdeklerinin deneysel sonuçlarıyla örtüşmektedir.

Anahtar Kelimeler: Kritik Nokta Simetrileri, E(5) model, Davidson Potansiyeli, Nikiforov-Uvarov Metodu

ABSTRACT

MASTER'S THESIS

ANALYTICAL SOLUTION OF BOHR HAMILTONIAN WITH DAVIDSO POTENTIAL FOR E(5) MODEL USING NIKIFOROV-UVAROV METHOD

LAVA MOHAMMED

Lava , Mohammed
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Advisor: .Prof.Dr. İbrahim YİĞİTOĞLU
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The goal of this research is to find unique Bohr Hamiltonian solutions that will clearly show the atomic nuclei's nuclear structures in the phase transition area. In this context, the Davidson potential in the Bohr Hamiltonian was used to develop a new model for the E(5) critical point symmetry. E(5)-D is the name of the γ-solid Bohr Hamiltonian solution for $\gamma = 0°$ that was derived using the Davidson potential. An analytical technique created by Nikiforov and Uvarov was used to determine the energy eigenvalues by using the PYTHON 3 Jupyter (data science) program. Fundamental level bands of the E(5)-D model; it aligns with the available experimental data for the nuclei ^{110}Xe and ^{134}Ba. The reason for this incompatibility is that this solution is a γ-fold solution. Up to the highest spin levels, the predictions of the E(5)-D model overlap with the experimental results of the nuclei ^{156}Dy, ^{154}Gd, ^{154}Sm, ^{102}Pd, ^{150}Nd, and ^{74}Ge.

Key Words: Critical point symmetries, E(5) MOdel, Davidson Potential, Nikiforov- Uvarov Method.

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Lava Mohammed

February, 2024

CONTENTS

SYMBOLS AND ABBREVIATIONS

Symbols	Explanation
$\psi(x)$	The wave function
E	The energy eigenvalue
$\boldsymbol{\beta}$	Measure of the total deformation of the nucleus
γ	Measure of the deviation from rotational symmetry
R_L	Ratio of energies at different momentum
L	Angular momentum
ħ	The Planck's constant
B	Induce mass
DM	Diatomic molecule
IBM	Interacting Boson Model
NU	Nikiforov-Uvarov
CPS	Critical Point Symmetries

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1. INTRODUCTION

The study delves into the intricacies of atomic nuclei, particularly their equilibrium shapes, which undergo phase transitions between stable and unstable states. The challenge lies in defining the nuclei's structures during these transitions, a task addressed within the framework of the Interacting Boson Model (IBM). Dynamic symmetries, described by the Hamiltonian operator using Casimir operators, play a pivotal role.

The IBM views nuclei as systems composed of bosons with angular momentum L=0 (s-bosons) or L=2 (d-bosons). The boson number $N = \frac{n}{2}$, where n is half the number of valence fermions, is conserved and rotation invariant. The model, rooted in the U(6) group structure, embodies dynamic symmetries such as U(5), SU(3), and O(6), corresponding to spherical vibration, axisymmetric rotation, and γ-unstable rotation, respectively.

Phase transitions in atomic nuclei, influenced by proton and neutron numbers, manifest as quantum transitions between ground and low-energy structures. The study introduces the concept of 'critical point symmetries' to describe nuclei's structures at these transition points. Iachello's work led to the identification of dynamic symmetries like E(5), X(5), and Y(5), characterizing critical points between different nuclear shapes. The Z(4) and Z(5) models represent axially deformed structures.

The Bohr Hamiltonian, which governs the system, yields both exact and approximate solutions. The E(5) model, an exact solution, corresponds to the transition from a spherical vibrator to γ-unstable shapes (U(5)↔O(6)). The X(5) model, an approximate solution for $\gamma \approx 0°$, and the Z(5) model for $\gamma \approx 30°$ are derived. These models consider five degrees of freedom, including collective variables β, γ, and three Euler angles.

Critical point symmetries, particularly E(5), provide an exact solution of the Bohr Hamiltonian with an independent potential for γ. The X(5) model serves as an approximate solution for $\gamma \approx 0°$, while the Z(5) model addresses $\gamma \approx 30°$. The Z(4) model offers an exact solution for $\gamma = 30°$, focusing solely on four degrees of freedom. These analytical solutions significantly contribute to understanding nuclear structures during phase transitions, providing a basis for comparison with experimental data.

Iachello, in the early 2000s, harnessed the Bohr Hamiltonian to derive two distinct analytical models elucidating nuclear structures at critical transition points. The proposed solution incorporates diverse potentials, such as Morse, Kratzer, Coulomb, Davidson, Eckart, Manning-Rosen, and Killingbeck, each contributing to the eigenvalue-eigenfunction problem. This analytical exploration is grounded in the Nikiforov-Uvarov method, focusing on the collective dynamics of atomic nuclei.

Dynamic symmetry, a cornerstone in understanding nuclear structural properties, manifests through U(5) and O(6) symmetries, characterizing the transition from vibrational to γ-labile nuclei (E(5)), and U(5) vibrational to SU(3) axial deformed nuclei (X(5)). The Bohr Hamiltonian offers exact solutions for γ-soft (E(5)) and approximate solutions for $\gamma \approx 0°$ (X(5)).

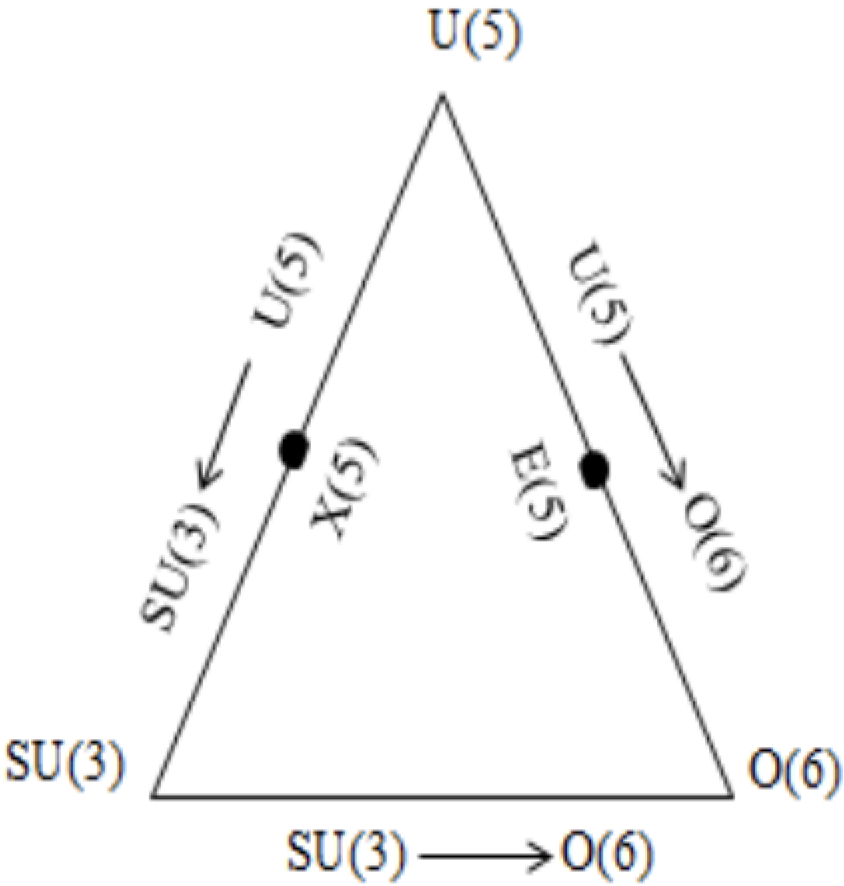

Figure 1.1: Casten Triangle

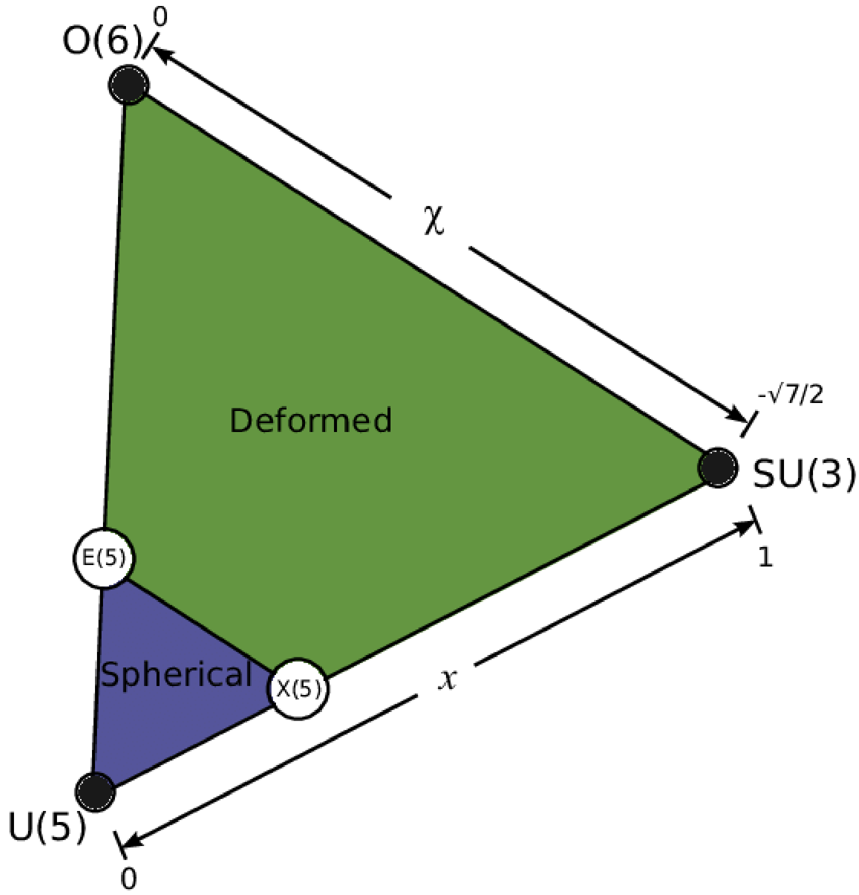

Figure 1.2: Casten Triangle, dynamic symmetry.

The Casten Triangle, represented in Figure 1.2, illustrates dynamic symmetries denoted by U(5), SU(3), and O(6) within the Interacting Boson Model (IBM). Each symmetry corresponds to distinct nuclear structures, providing insights into their deformations. The Ratios $R_{4/2}$, exemplified in Figure 1.4, serve as indicators of deformation magnitude, crucial in classifying nuclei.

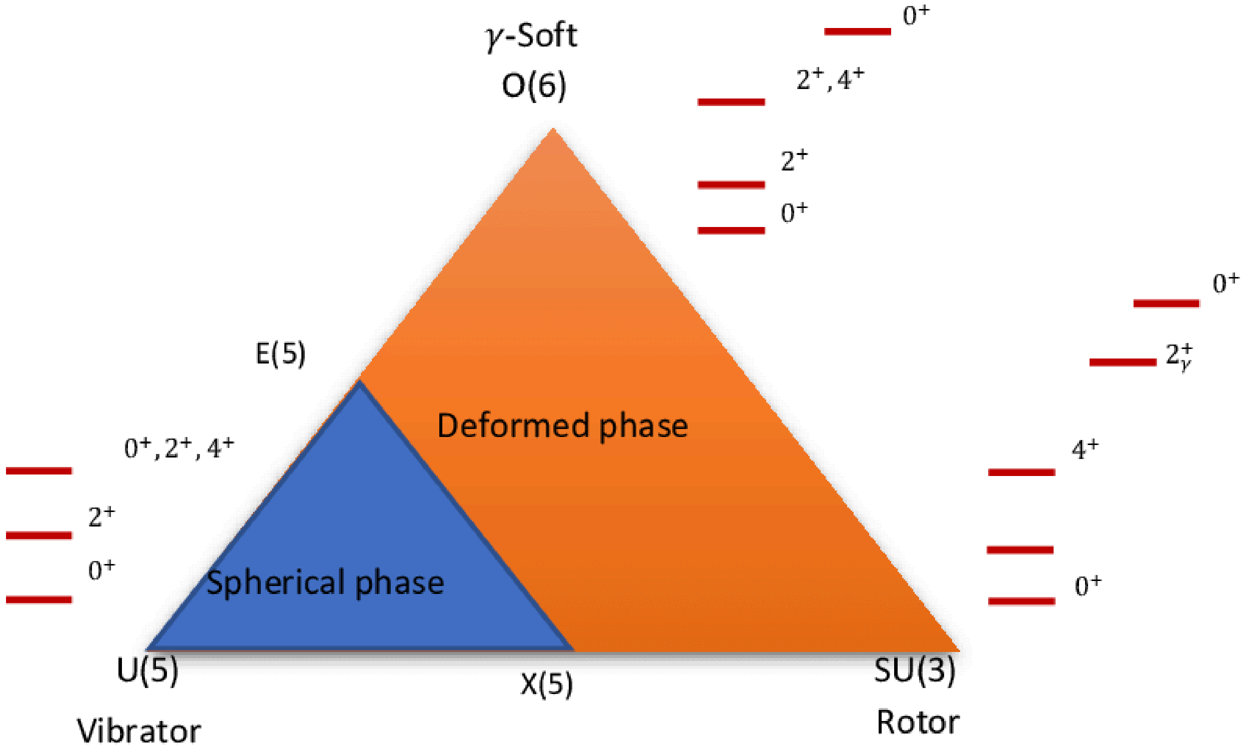

Figure 1.3: The IBM phase diagram represented as a Casten triangle.

The Extended Casten Triangle (Figure 1.4) introduces critical point symmetries, marking first and second-order phase transitions a nuclear triple point signifying a second-order transition, emphasizing the coexistence of deformed structures.

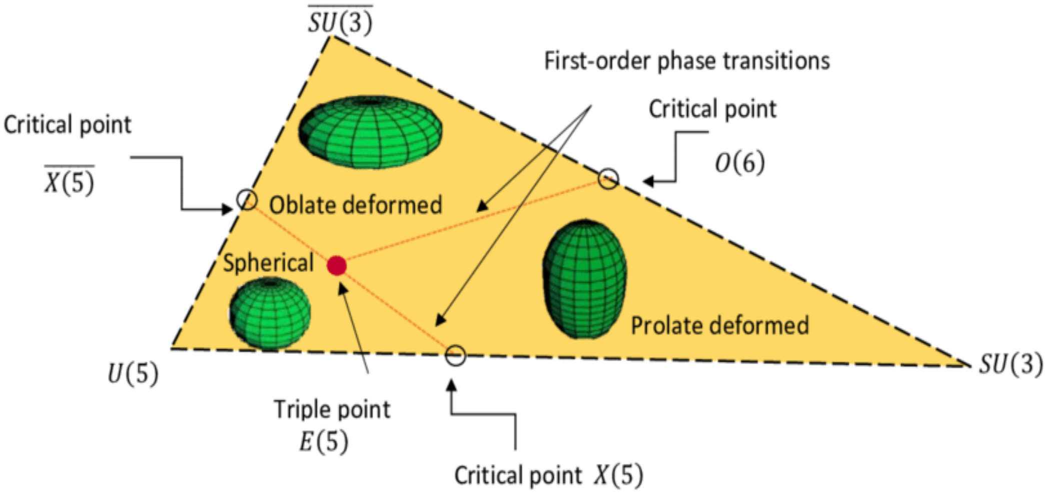

Figure 1.4: Extended Casten Triangle.

further elucidate second-order phase transitions, showcasing the sensitivity to deformation and the emergence of deformed equilibrium shapes during first-order transitions.

The Bohr Hamiltonian for ellipsoid nuclei involves structural variables $\beta - \gamma$ and Euler angles, reflecting departure from spherical shape and axial symmetry. The study explores Schrödinger equations for X(3) and X(5) models, deriving analytical solutions for the partial Kratzer potential, named X(3)-K and X(5)-K.

In the study's first part, fundamental level bands, β_1-bands, are analyzed for X(3)-K at various β_0 values, compared with X(3) and X(5) predictions. The second part investigates base level bands, $\beta 1$-bands, and γ-bands, exploring X(5)-K for different β_0 values, juxtaposed with X(3) and X(5) model predictions. This comprehensive analysis contributes to the understanding of nuclear structures during phase transitions, showcasing the applicability of the Nikiforov-Uvarov method.

The direction of the ellipsoid in space (as shown in Figure 1-5) is expressed by the Euler angles (θ_1, θ_2, θ_3). Different major axes of symmetry are associated with distinct colors: green for the z-axis, red for the y-axis, and blue for the x-axis. In Figure 1.5, the Hill-Wheeler coordinates $\{\beta, \gamma\}$ correspond to $\beta = 0.4$ and $\gamma = n\pi/3$. These coordinates represent deformed quadrilateral shapes for various values of n, ranging from 0 to 5 (figure 1.5).

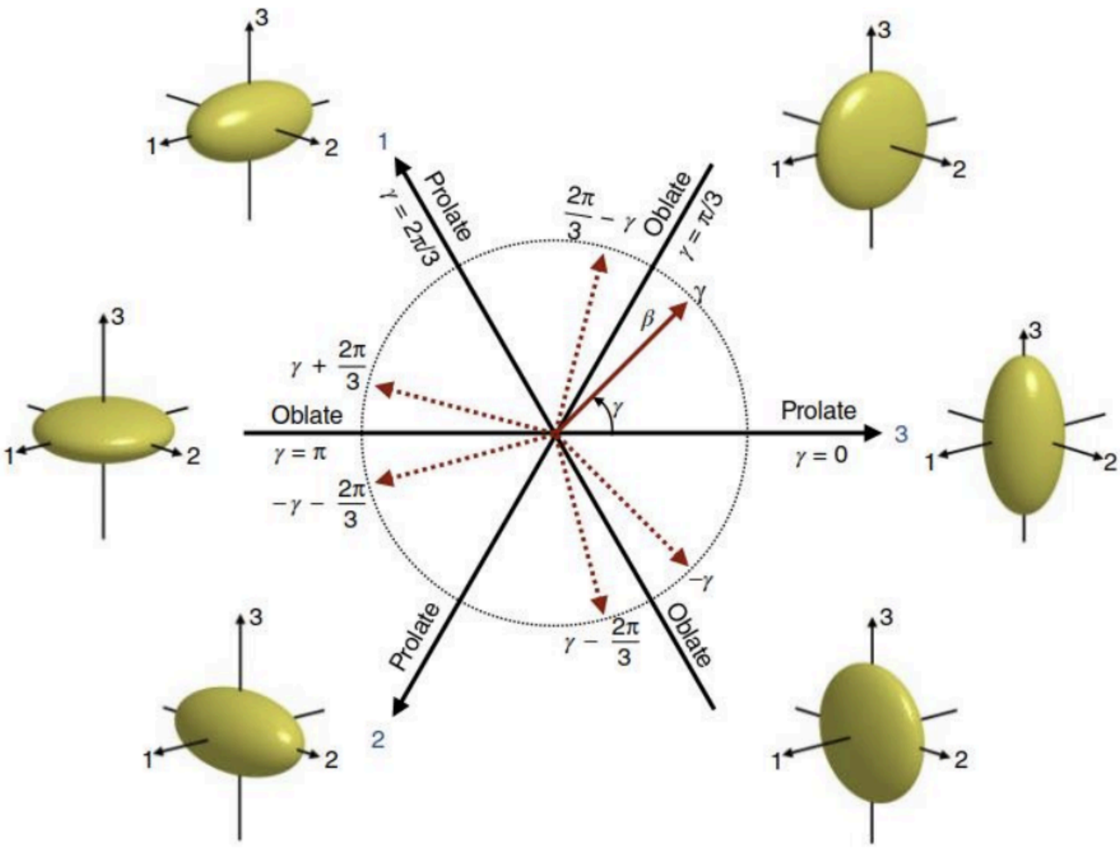

Figure 1.5: A polar diagram in the deformation plane (β, γ). Any quadrupole shape can be identified within a region $\beta \geq 0$ and $0 \leq \gamma \leq \pi/3$

Nuclei exhibit either quadrupole or ellipsoidal shapes, which are characterized by the parameters β and γ. When considering nuclei, the ellipsoidal deformation parameter β assumes a value of 0 for spherical nuclei and about 0.3 for typical prolate (football-shaped) nuclei. The angle γ reflects the symmetry of the ellipsoidal shape around the axis of symmetry, with 0° indicating an axisymmetric shape, maximal asymmetry (squashed football-shaped) at 30°, and an oblate (disc-shaped) shape at 60°.

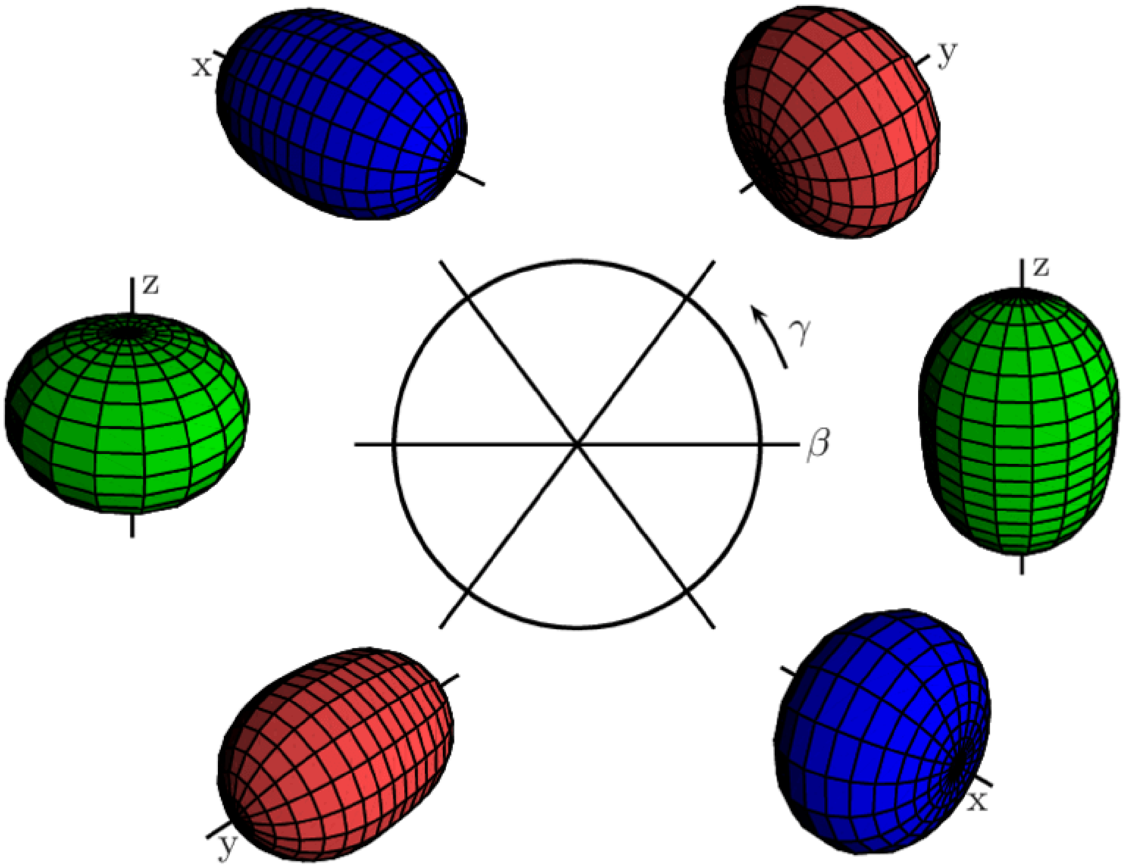

Figure 1.6: Hill-Wheeler coordinates { β, γ }. Corresponding to β = 0.4 and $\gamma = n\pi/3$

For more than 50 years, the collective model description of nuclei has been based on the Bohr Hamiltonian. Since its derivation, many solutions have been provided by selecting various forms of the potential V (β,γ) and solving the related eigenvalue equation analytically or roughly. This strategy has recently attracted increased attention, partly as a result of the growth of the critical point symmetries (CPS) notion. The two models, E(5) and X(5), are Bohr Hamiltonian special solutions intended to characterize nuclei at the crucial point of the shape/phase transition between axially symmetric deformed structures and vibrational and γ-soft structures, respectively.

In this model (E(5)) , a γ-independent potential of the form u(β) is used, leading to exact separation of β from γ and the Euler angles.

Since there is increasing evidence from microscopic studies that the potential at the transition point between distinct forms should be flat, an infinite square well potential has been used as u(β) in both E(5) and X(5) so there is no exception in the case of critical point symmetries. Square wells in β and flat or harmonic oscillator potentials in the γ degree of freedom, regardless of their simplicity.

There are two classes that each focus on a certain kind of potential that is separate from the beginning. These potentials take the form of

$$u(\beta,\gamma) = u(\beta) + \frac{u(\gamma)}{\beta^2} \quad \text{.......}(1.1)$$

when u(β,γ) can be separated. In the first of these, the square well in β and the γ dependent potential provided by a harmonic oscillator in γ are the same as in X(5) itself. This is the solution known as X(5). The Davidson potential in β is used by the second group of the exactly separable class of potentials, specifically:

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2} \quad \text{.......}(1.2)$$

where β_0 is the free parameter and gives the position of the minimum of the potential in β.

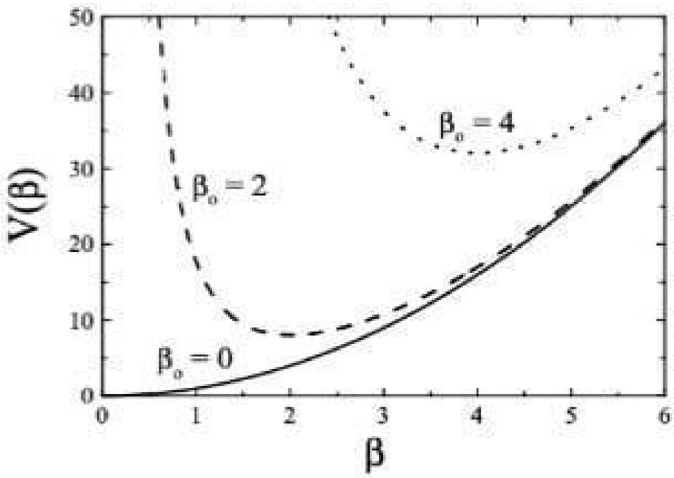

Figure 1.7: The Davidson potential

Figure 1.7 shows Davidson potential in the β degree of freedom for a few values of the parameter β_0.

2. Literature Review

2.1. Overview

Nuclear models are any of several hypotheses that describe how atomic nuclei are made and function. All models rely on a workable analogy that connects a significant quantity of data and makes predictions about the properties of nuclei possible.

Nuclear models fall into two primary categories:

- Models known as independent particles, like the Shel model
- Models known as intense interaction, like the liquid drop concept.

Some nuclear models, like the collective model, combine elements from both categories.

The nuclear shell model is a model of the atomic nucleus that describes the nuclear structure as a set of concentric shells. Each shell can contain a certain number of nucleons (protons and neutrons) before the next shell begins to fill. The nucleons in each shell are bound together by the strong nuclear force. The shell model is based on the observation that certain nuclei are more stable than others. For example, the nucleus of helium-4, which has two protons and two neutrons, is very stable. This is because the two protons and two neutrons can be arranged in a single, full shell. On the other hand, the nucleus of lithium-6, which has three protons and three neutrons, is less stable. This is because the three protons and three neutrons cannot be arranged in a single, full shell.

Nuclear energy levels are computed in the shell model using the assumption that a single proton or neutron is moving in a potential field created by all the other nucleons.

The nuclear structure and behavior in the liquid-drop model are explained by the statistical contributions of each nucleon (much as the molecules in a spherical drop of water contribute to the surface tension and overall energy).

A description of atomic nuclei known as the Collective model combines elements of the shell nuclear model with the liquid-drop model to account for some magnetic and electric phenomena that neither model alone can account for.

Analytical solutions for the atomic nucleus using the Nikiforov-Ovarov method have been developed for a variety of nuclear potentials, including the Woods-Saxon potential, the Hulthén potential, and the Kratzer potential. These solutions provide a way to calculate the energy levels and wave functions of the nucleons in the nucleus.
The Nikiforov-Ovarov method is a powerful tool for solving the Schrödinger equation, which is the equation that describes the quantum behavior of the nucleons in the nucleus. The method is based on a transformation that converts the Schrödinger equation into a hypergeometric equation. The hypergeometric equation is a well-known equation with known solutions.

Analytical solutions for the atomic nucleus using the Nikiforov-Ovarov method have been used to study a variety of nuclear phenomena, including nuclear stability, nuclear excitations, and nuclear reactions. The solutions have also been used to develop new nuclear models.

The following are some examples of how analytical solutions for the atomic nucleus using the Nikiforov-Uvarov method have been used to study nuclear phenomena:

- The solutions have been used to calculate the energy levels of the nucleons in the nucleus. This information can be used to predict nuclear stability and radioactivity.
- The solutions have been used to study nuclear excitations, such as the giant dipole resonance. This resonance is important for understanding the interaction of gamma rays with nuclei.
- The solutions have been used to develop new nuclear models, such as the cluster model. The cluster model describes the nucleus as a collection of smaller clusters of nucleons.

The atomic nucleus is a complex system, and its analytical solution is a challenging task. However, there have been some promising advances in this area, particularly using the Nikiforov-Uvarov (NU) method. The NU method is a powerful mathematical technique that can be used to solve a wide range of quantum mechanical problems, including the Schrödinger equation for the atomic nucleus. It is based on a transformation that converts the Schrödinger equation into a more tractable form.

To determine energies using the NU method, it first needs to choose a suitable potential function for the atomic nucleus. A common choice is the Woods-Saxon potential, which is a realistic potential that takes into account the various forces that act between nucleons. Once we have chosen a potential function, we can apply the NU method to solve the Schrödinger equation. This will give us a set of energy eigenvalues and Eigenfunctions. The energy eigenvalues represent the possible energies of the nucleus, while the Eigen functions represent the corresponding wave functions. The NU method has been successfully used to solve the Schrödinger equation for a variety of atomic nuclei.

For example, it has been used to calculate the binding energies of magic nuclei, such as ^{16}O and ^{40}Ca. It has also been used to study the excitation states of nuclei and to calculate their radiative transition probabilities.

The NU method is a powerful tool for studying the atomic nucleus. It is still under development, but it has already made significant contributions to our understanding of this complex system. Current state of knowledge about analytical solutions of the atomic nucleus The analytical solution of the atomic nucleus is a challenging task, but there have been some promising advances in this area, particularly using the Nikiforov-Uvarov (NU) method. The NU method has been successfully used to solve the Schrödinger equation for a variety of atomic nuclei, including magic nuclei, such as ^{16}O and ^{40}Ca. It has also been used to study the excitation states of nuclei and to calculate their radiative transition probabilities.

$$[\frac{d2}{dy2} + (\lambda + \sigma y + \frac{\tau}{y^2}]\psi(y) = -EY\psi(y) \quad \ldots\ldots\ldots\ldots\ldots(2.1)$$

This is the transformed Schrödinger equation that is obtained using the NU method. The parameters λ, σ, and τ are determined by the potential function.

y = tan(x) ……………………………………………………………………….(2.2)

This is the transformation that is used to convert the Schrödinger equation into the form shown in Equation 1. The nuclear shell model is a model of the atomic nucleus that describes the behavior of nucleons in terms of single-particle energy levels. The model was developed independently in the late 1940s by Maria Goeppert Mayer and J. Hans D. Jensen, who shared the Nobel Prize in Physics in 1963 for their work.

The shell model is based on the analogy with the Bohr model of the atom, in which electrons occupy energy levels around the nucleus. In the shell model, nucleons occupy energy levels around the center of the nucleus. The energy levels are determined by the nuclear potential, which represents the attractive and repulsive forces between the nucleons. The shell model is able to explain many of the observed properties of atomic nuclei, such as the existence of magic numbers, which are numbers of protons or neutrons that correspond to particularly stable nuclei. The model is also able to predict the energies of excited states of nuclei.

The energy levels of a nucleon in the shell model ($E = \ -\varepsilon + (n + \frac{1}{2})\omega$) where E is the energy of the nucleon, ε is the depth of the nuclear potential, n is the principal quantum number of the nucleon, ω is the spin-orbit splitting energy. The spin-orbit splitting energy is a term that arises from the interaction between the spin of the nucleon and the spin of the nucleus. It causes the energy levels of nucleons with different spins to be split apart. The shell model has been very successful in explaining the properties of atomic nuclei. However, it is an approximation, and it does not take into account all of the interactions between the nucleons. For example, the shell model does not take into account the collective motion of the nucleus.

Analytical solutions for the atomic nucleus using the Nikiforov-Ovarov method are a powerful tool for understanding the structure and behavior of the atomic nucleus. The solutions have been used to study a variety of nuclear phenomena and to develop new nuclear models.

2.2 Rated Literature

In other studies, it aims to derive specialized Bohr-Hamiltonian solutions to accurately represent nuclear structures within the phase transition zone. Two novel models, based on the Kratzer potential, were developed within the Bohr Hamiltonian framework, focusing on the symmetries of critical points X(3) and X(5). The study comprises two distinct parts. In the first part, a γ-rigid Bohr Hamiltonian was solved for $\gamma = 0°$, yielding the X(3)-Kratzer solution.

Energy eigenvalues and wave functions were computed using Nikiforov and Uvarov analytical methods. X(3)-Kratzer's ground state properties align with experimental data for ^{112}Cd, ^{142}Sm, and ^{156}Gd nuclei, although discrepancies exist in the $\beta 1$ range due to its γ-rigid nature. In the second part, an approximate solution for $\gamma \approx 0°$ was obtained from the Kratzer potential, termed X(5)-Kratzer. Eigenvalues of energy and wave functions were obtained, and X(5)-Kratzer's predictions concurred with experimental data for ^{150}Nd, ^{152}Sm, ^{154}Gd, ^{156}Dy, and $^{176\text{-}178}$Os nuclei, especially in their upper spin levels.

Analytical solutions for the radial D-dimensional Schrödinger equation were derived, considering the Yukawa potential along with spin-orbit and Coulomb interaction terms. The Nikiforov-Uvarov method, coupled with the Greene-Aldrich approximation for the centrifugal barrier, was employed. The resulting energy eigenvalues were used to compute the single-energy spectrum of ^{56}Ni and ^{116}Sn across different quantum states. Furthermore, normalized wave functions for these "magic nuclei" were expressed in terms of Jacobi polynomials. Intriguingly, the energy spectrum in the absence of spin-orbit and Coulomb interactions perfectly matches the quantum mechanical system of the Yukawa potential field at any state.

The Nikiforov-Uvarov method is a mathematical technique used to determine energy eigenvalues for quantum systems. It's often used in the context of solving the Schrödinger equation for various quantum systems, and it provides a systematic approach to finding energy levels associated with a given Hamiltonian.

The expression of Nikiforov- Uvarov method is:

$$\psi''(s) + \frac{\hat{\tau}(s)}{\sigma(s)}\psi''(s) + \frac{\hat{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

where $\sigma(s)$ and $\hat{\sigma}(s)$ are the polynomials of at most second-degree; $\hat{\tau}(s)$ is of the first-degree; thus $\psi(s)$ is the function of the hypergeometric type.

The Nikiforov-Uvarov method is a powerful tool for solving quantum mechanical problems. It is based on a coordinate transformation that converts the Schrödinger equation into a simpler form. This makes it possible to find exact solutions to a wide range of problems that would be difficult or impossible to solve using other methods. The advantages of this method

1. Developing new and more efficient ways to solve quantum mechanical problems using the Nikiforov-Uvarov method.
2. Extending the applicability of the Nikiforov-Uvarov method to a wider range of problems, including those involving nonlinear potentials.
1. Providing new insights into the physical nature of the Nikiforov-Uvarov method and its connection to other areas of physics.
2. Demonstrating the power of the Nikiforov-Uvarov method as a tool for solving real-world problems.

Here are some specific examples of how this research could contribute to the field of physics:

1. It could develop new methods for calculating the energies of molecules and other complex physical systems. This could lead to the development of new materials with desirable properties, such as high-temperature superconductors or efficient solar cells.

2. It could also develop new methods for calculating the energies of quantum states in materials. This could lead to the development of new semiconductor devices and other electronic devices.

3. It could also develop new methods for calculating the energies of particles in accelerators. This could lead to the development of new particle accelerators and new discoveries in physics.

Overall, this research on determining energies using the Nikiforov-Uvarov method will make a significant contribution to the field of physics. This research has the potential to impact a wide range of areas of physics, including materials science, semiconductor physics, and particle physics.

3. Methodology and Results

3.1. Describing the mathematical framework of the Nikiforov-Uvarov method:

The Nikiforov-Uvarov method, also known as the Nikiforov-Uvarov (NU) method or the quantum mechanical mathematical framework, is a mathematical technique used to solve certain types of differential equations that arise in quantum mechanics, particularly in the context of one-dimensional Schrödinger equations. This method is named after the Russian mathematicians A. F. Nikiforov and V. B. Uvarov, who developed it.

The NU method is primarily used to find exact solutions to second-order linear differential equations with special potential functions that can be written as a product of two functions. The general form of such an equation is:

$$-\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \quad \text{………………………………………………………(3.1)}$$

Here, $\psi(x)$ represents the wave function, V(x) is the potential energy, E is the energy eigenvalue, and x is the spatial coordinate. The potential function V(x) is expressed as:

$$V(x) = R(x) + S(x) \quad \text{……………………………………………………………...…(3.2)}$$

The key idea of the NU method is to transform the Schrödinger equation into a standard form by making a change of variable and then using a set of functions, called the Nikiforov-Uvarov functions (or sometimes known as auxiliary functions), to simplify the equation. The transformation is performed using the following substitutions:

Let:

$$\psi(x) = e^{f(x)}\chi(x) \quad \text{…………………………..……………………………………...(3.3)}$$

2. Substitute this into the Schrödinger equation and perform algebraic manipulations to obtain a differential equation for the function f(x). The equation for f(x) depends on the potential terms R(x) and S(x).

3. Solve the resulting equation for f(x).

4. After obtaining the solution for f(x), you can find the function χ(x) by solving a second differential equation, which depends on the same potential terms R(x) and S(x). The solution for χ(x) is usually expressed in terms of special functions like hypergeometric functions.

5. Finally, using the solutions for f(x) and χ(x), you can construct the overall solution for the original Schrödinger equation, ψ(x).

The success of the NU method depends on the specific form of the potential V(x) and the ability to find suitable solutions for the auxiliary functions f(x) and χ(x). This method is particularly useful for finding exact solutions to certain quantum mechanical problems, including the one-dimensional harmonic oscillator and the hydrogen atom, as well as other problems with separable potentials. However, it may not be applicable to all potential functions, and in some cases, numerical methods or other approximation techniques are needed.

3.2. How can the Nikiforov-Uvarov method be used to determine the energies of the atomic nucleus?

The Nikiforov-Uvarov method is a mathematical technique used to determine energy eigenvalues for quantum systems. It's often used in the context of solving the Schrödinger equation for various quantum systems, and it provides a systematic approach to finding energy levels associated with a given Hamiltonian. In this case, it provided the Bohr Hamiltonian with a specific form, and it understood how to use the Nikiforov-Uvarov method to determine the energies associated with it.

The Bohr Hamiltonian is given as:

$$H =- \frac{\hbar^2}{2B} [\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\Sigma_K\frac{Q_{1L}^2}{\sin^2\left(\gamma-\frac{2\pi k}{J}\right)}] + V(\beta,\gamma) \quad ...(3.4)$$

$$V(\beta,\gamma) = u(\beta) + v(\gamma) \quad ....(3.5)$$

Here, β and γ are the variables, ħ is the planck constant, and there's a sum over K and other terms in the Hamiltonian.

To use the Nikiforov-Uvarov method to determine the energies, you would follow these general steps:

1. Prepare the Schrödinger Equation: You would start with the Schrödinger equation for your system, using the given Hamiltonian. The Schrödinger equation is typically of the form HΨ = EΨ, where H is the Hamiltonian operator, Ψ is the wave function, and E is the energy eigenvalue.
2. Separate Variables: If the Hamiltonian allows, try to separate the variables in your Schrödinger equation. In your case, β and γ are the variables, so you want to find a way to separate them.
3. Apply the Nikiforov-Uvarov Method: The Nikiforov-Uvarov method involves transforming the Schrödinger equation into a standard differential equation with known solutions. You would apply this method to the separated variables to make the equation solvable.
4. Solve the Standard Differential Equation: The transformed equation should correspond to a known standard differential equation (like Hermite or Laguerre equations), and you would use known methods to solve this standard equation. The solutions of this standard equation will be functions of the energy eigenvalues.
5. Determine the Energies: Once you've found the solutions to the standard differential equation, these solutions will provide you with the energy eigenvalues for the system. The specific form and nature of these solutions will depend on the details of your Hamiltonian.

$E(5)$, Kinetic Energy (K) Term is:

$$-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\ \frac{1}{4\beta^2}\Sigma_K\frac{Q_{1L}^2}{\sin^2\left(\gamma\frac{2\pi k}{J}\right)}\right]\text{………...………………….....(3.6)}$$

If we assign the wave function as: $\Psi(\beta,\ \gamma,\theta_i)=f(\beta)\phi\left(\gamma,\theta_i\right)$

HΨ = EΨ

$$\mathrm{K}\,f(\beta)\phi\left(\gamma,\theta_i\right)+\ u(\beta)f(\beta)\phi\left(\gamma,\theta_i\right)=\ Ef(\beta)\phi\left(r,\theta_i\right)\text{....………………...……………..(3.7)}$$

$$-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\frac{1}{4\beta^2}\Sigma_K\frac{Q_{1L}^2}{\sin^2\left(\gamma-\frac{2\pi k}{J}\right)}\right]f(\beta)\phi\left(\gamma,\theta_i\right)$$
$$+\ u(\beta)f(\beta)\phi\left(\gamma,\theta_i\right)=Ef(\beta)\phi\left(r,\theta_i\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(3.8)}$$

$$-\frac{\phi\left(r,\theta_i\right)}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial f}{\partial f}-\frac{f(\beta)}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial\phi}{\partial\gamma}+\frac{1}{u\beta^2}\Sigma_K\frac{Q_k^2}{\sin^2\left(r-\frac{2\pi_k}{\beta}\right)}f(\beta)\phi\left(\gamma,\theta_i\right)$$
$$+\frac{2\beta}{\hbar^2}u(\beta)f(\beta)\phi\left(\gamma,\theta_i\right)-\frac{2\beta}{\hbar^2}Ef(\beta)\phi\left(\gamma,\theta_i\right)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(3.9)}$$

$$-\frac{1}{f(\beta)}\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^4\frac{\partial f}{\partial\beta}-\frac{1}{\phi\left(\gamma,\theta_i\right)}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}+\frac{1}{4}\Sigma_K\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2\pi_k}{\beta}\right)}$$
$$+\frac{2B}{\hbar^2}u(\beta)\beta^2-\frac{2B}{\hbar^2}E\beta^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(3.10)}$$

The goal of solving this equation is to determine the allowed energy eigenvalues E and the corresponding wave function $\psi(\beta,\gamma)$ that describe the quantum state of the system. This involves solving for the wave function under the influence of the kinetic and potential energy operators.

The specific form and solutions will depend on the details of the potential energy function $u(\beta)$, the functions) $f(\beta)$ and $\phi(\gamma,\theta i)$, and the constants involved.

When $\frac{2\beta}{\hbar^2}u(\beta)=u$, And $\frac{2B}{\hbar^2}E=\varepsilon$

$$-\frac{1}{f(\beta)}\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^4\frac{\partial f}{\partial\beta}+u(\beta)\beta^2-\Sigma\beta^2=-\Lambda.$$

$$-\frac{1}{\phi\left(\gamma,\theta_i\right)}\frac{1}{\sin 3\gamma}\sin 3\gamma\frac{\partial\phi}{\partial\gamma}+\frac{1}{4}\Sigma_K\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2\pi_k}{\beta}\right)}=-\Lambda.$$

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{\Lambda}{\beta^2}+u(\beta)\right]f(\beta)=\varepsilon f(\beta)$$

$$\left[-\frac{1}{\phi(\gamma,\theta_i)}\frac{1}{\sin 3\gamma}\sin 3\gamma\,\frac{\partial\phi}{\partial\gamma}+\frac{1}{4}\Sigma_K\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2\pi_k}{\beta}\right)}\right]\phi(\gamma,\theta_i)=\Lambda\phi(\gamma,\theta_i)\;\ldots\ldots\ldots\ldots\ldots\ldots..(3.11)$$

when $\Lambda = \tau(\tau+3)$

$$[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{\tau(\tau+3)}{\beta^2}+u(\beta)]\,\varepsilon f(\beta)=\varepsilon f(\beta)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.12)$$

$$u(\beta)=\beta^2+\frac{\beta_0^4}{\beta^2}\quad(aliner)$$

$$\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial f(\beta)}{\partial\beta}-\frac{\tau(\tau+3)}{\beta^2}f(\beta)-\beta^2 f(\beta)-\frac{\beta_0^4}{\beta^2}f(\beta)+\varepsilon f(\beta)=0\;\ldots\ldots\ldots\ldots\ldots\ldots(3.13)$$

$$\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial(\beta)}{\partial\beta}=\frac{1}{\beta^4}[4\beta^3\frac{\partial}{\partial\beta}+\beta^4\frac{\partial^2}{\partial\beta^2}=\frac{4}{\beta}\frac{\partial}{\partial\beta}+\frac{\partial^2}{\partial\beta^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.14)$$

$$\frac{d^2f(\beta)}{d\beta^2}+\frac{4}{\beta}\frac{df(\beta)}{d\beta}+[\varepsilon-\frac{\tau(\tau+3)}{\beta^2}-\beta^2-\frac{\beta_0^4}{\beta^2}]\,f(\beta)=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.15)$$

Where , $\beta^2 = s$, $2\beta d\beta = ds$ and $\beta=\sqrt{s}$

$$\frac{d}{d\beta}=\frac{d}{ds}.\frac{ds}{d\beta}=2\beta\frac{d}{ds}=2\sqrt{s}\,\frac{d}{ds}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.16)$$

$$\frac{d^2}{d\beta^2}=2\sqrt{s}\,\frac{d}{ds}\left(2\sqrt{s}\,\frac{d}{ds}\right)=4s\frac{d^2}{ds^2}+2\frac{d}{ds}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.17)$$

$$4s\,\frac{d^2f(s)}{ds^2}+2\frac{df(s)}{ds}+\frac{4}{\sqrt{s}}\,2\sqrt{s}\frac{df(s)}{ds}+\left[\varepsilon-\frac{\tau(\tau+3)}{s^2}-s-\frac{\beta_0^4}{s}\right]f(s)=0\ldots\ldots\ldots\ldots(3.18)$$

$$4s\frac{d^2f(s)}{ds^2} + 10\frac{df(s)}{ds} + \left[\varepsilon - \frac{\tau(\tau+3)}{s^2} - s - \frac{\beta_0^4}{s}\right]f(s) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.19)$$

In the above equation, additional terms and derivatives have been introduced in the context of solving the differential equation. Here are the definitions of coefficients and parameters:

ϵ is a parameter.

τ is a parameter.

β is a variable.

By dividing eq. (3.19) by 4s:

$$\frac{d^2f(s)}{ds^2} + \frac{5}{2s}\frac{df(s)}{ds} + \left[\frac{\varepsilon}{4s} - \frac{\tau(\tau+3)}{4s^2} - \frac{1}{4} - \frac{\beta_0^4}{4s^2}\right]f(s) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.20)$$

$$\frac{d^2f(s)}{ds^2} + \frac{5}{2s}\frac{df(s)}{ds} + \frac{1}{4s^2}[- s^2 + \varepsilon s - [\beta_0^4 + \tau(\tau + 3)]\, f(s) = 0 \;\ldots\ldots\ldots\ldots(3.21)$$

$$\text{At } \delta = \beta_0^4 + \tau(\tau + \beta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.22)$$

$$\frac{d^2f(s)}{ds^2} + \frac{5}{2s}\frac{df(s)}{ds} + \frac{1}{4s^2}[- s^2 + \varepsilon s - \delta]\, f(s) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.23)$$

The Nikiforov-Uvarov method in (3.1)

$$\psi''(s) + \frac{\hat{\tau}(s)}{\sigma(s)}\psi''(s) + \frac{\hat{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.24)$$

This equation represents the final form of the differential equation, which it can solve to find solutions for the function f(s).

Equitable (3.12) in (3.24) at $f(s) \to \psi(s)$

$\hat{\tau}(s) = 5 \quad , \quad \sigma(s) = 2s \quad , \quad \hat{\sigma} = -s^2 + \varepsilon s - \delta$

$$\pi(s) = \frac{\sigma'(s) - \hat{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \hat{\tau}(s)}{2}\right)^2 - \hat{\sigma}(s) + k\sigma(s)} \quad \text{……………………………………………}(3.25)$$

$$\pi(s) = \frac{2-5}{2} \pm \sqrt{\frac{9}{4} + s^2 - \varepsilon s + \delta + 2ks} \quad \text{……………………………………………}(3.26)$$

$$\pi(s) = -\frac{3}{2} \pm \sqrt{s^2 + s(2k - \varepsilon) + \delta + \frac{9}{4}} \quad \text{……………………………………}(3.27)$$

where $\quad , \quad \Delta = 0 \quad , \quad \Delta = (b^2 - 4ac)$

$$\Delta = (2k - \varepsilon)^2 - 4\left(\delta + \frac{9}{4}\right) = 0 \quad \text{……………………………………………}(3.28)$$

$$\Delta = 4k^2 - 4k\varepsilon + \varepsilon^2 - 4\left(\delta + \frac{9}{4}\right) = 0 \quad \text{……………………………………………}(3.29)$$

$$k^2 - k\varepsilon + \left[\frac{\varepsilon^2}{4} - \delta - \frac{9}{4}\right] = 0 \quad \text{……………………………………………}(3.30)$$

Where:

$$k_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2} \quad \text{……………………………………………………………}(3.31)$$

$$k_{\pm} = \frac{\varepsilon}{2} \pm \frac{1}{2}\sqrt{\varepsilon^2 - 4\left(\frac{\varepsilon^2}{4} - \delta - \frac{9}{4}\right)} \quad \text{……………………………………}(3.32)$$

$$k_{\pm} = \frac{\varepsilon}{2} \pm \sqrt{4\varepsilon + 9} \quad \text{……………………………………………………}(3.33)$$

$$k_{+} = \frac{\varepsilon}{2} + \sqrt{\left(\delta + \frac{9}{4}\right)} \quad , \quad k_{-} = \frac{\varepsilon}{2} - \sqrt{\left(\delta + \frac{9}{4}\right)}$$

Put $k_{\pm}$ in $\pi(s)$:

For k_+:

$$\pi(s) = -\frac{3}{2} \pm \sqrt{s^2 + (\varepsilon + 2\sqrt{\delta + \frac{9}{4}} - \varepsilon) + \delta + \frac{9}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.34)$$

$$\pi(s) = -\frac{3}{2} \pm [\sqrt{s^2 + 2s\sqrt{\delta + \frac{9}{4}} + \delta + \frac{9}{4}}\ ] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.35)$$

This equation calculates the values of π(s) based on the values of ϵ, δ, and other parameters. It appears to be solving a quadratic equation for π(s).

$$\sqrt{s^2 + 2s\sqrt{\delta + \frac{9}{4}} + \delta + \frac{9}{4}} = s + \sqrt{\delta + \frac{9}{4}}$$

$$\pi(s) = -\frac{3}{2} \pm [s + \sqrt{\delta + \frac{9}{4}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.36)$$

While $\delta = \beta_0^4 + \tau(\tau + 3)$.

For k_-:

$$\pi(s) = -\frac{3}{2} \pm [s - \sqrt{\delta + \frac{9}{4}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.37)$$

The negative root is taken:

$$\pi(s) = -\frac{3}{2} - [s - \sqrt{\delta + \frac{9}{4}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.38)$$

$$\lambda = k + \pi'(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.39)$$

$$\lambda = [\frac{\varepsilon}{2} - \sqrt{\delta + \frac{9}{4}}] - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.40)$$

$$\tau(s) = \hat{\tau}(s) + 2\pi(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.41)$$

$$\tau(s) = 2 - 2[s - \sqrt{\delta + \frac{9}{4}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.42)$$

$\lambda_n = -n\,\tau'(s) - \frac{n(n-1)}{2}\sigma''(s)$ at $\sigma(s) = 2s$

$\lambda_n = 2n$ …………………………………………………………………………………(3.43)

$\frac{\varepsilon}{2} - \sqrt{\delta + \frac{9}{4}} - 1 = 2n \Rightarrow \frac{\varepsilon}{2} = 2n + 1 + \sqrt{\delta + \frac{9}{4}}$

$\frac{2\beta}{\hbar^2}E = \varepsilon \Rightarrow E = \frac{\hbar^2}{B}\frac{\varepsilon}{2}$

$\delta = \beta_0^4 + \tau(\tau + 3)$

$E_{n,L} = \frac{\hbar^2}{B}(2n + 1) + \sqrt{\beta_0^4 + \tau(\tau + 3) + \frac{9}{4}}$ ………………………………………(3.44)

Equation (E5): This equation represents a relationship between two functions, f(s) and $\phi(s)y_n(s)$.

- f(s) is a function of s.
- $\phi(s)$ and) $y_n(s)$ are also functions of s.

The equation suggests that f(s) can be expressed as the product of $\phi(s)$ and $y_n(s)$.

$f(s) = \phi(s)y_n(s)$ ………………………………………………………………(3.45)

$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}$ ……………………………………………………………………(3.46)

$\frac{\phi'(s)}{\phi(s)} = \frac{-\frac{3}{2} - s + \sqrt{\delta + \frac{9}{4}}}{2s}$ ……………………………………………………………(3.47)

$\frac{\phi'(s)}{\phi(s)} = [-\frac{3}{4S} - \frac{1}{2} + \frac{\sqrt{\delta + \frac{9}{4}}}{2s}]$ ds ……………………………………………………(3.48)

$ln\,\phi(s) = -\frac{3}{4}\,ln\,s - \frac{s}{2} + \frac{\sqrt{\delta + \frac{9}{4}}}{2}\,ln\,s$ ……………………………………………(3.49)

$\phi(s) = s^{-3/4}\,e^{-s/2}\,s^{\frac{\sqrt{\delta + \frac{9}{4}}}{2}}$ ………………………………………………………(3.50)

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n}[\sigma^n(s)\,\rho(s)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.51)$$

$$\frac{\rho'(s)}{\rho(s)} = [-\ 1\ + \frac{\sqrt{\delta+\frac{9}{4}}}{s}]\mathrm{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.52)$$

$$ln\,\rho(s) \ = \ -\ s\ + \frac{\sqrt{\delta+\frac{9}{4}}}{s}\ ln\,s \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.53)$$

$$\rho(s) \ = e^{-s}s^{\sqrt{\delta+\frac{9}{4}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.54)$$

$$y_n(s){=}B_n e^{s} s^{-\sqrt{\delta+\frac{9}{4}}} \frac{d^n}{ds^n}\,[2s^n e^{-s} s^{\sqrt{\delta+\frac{9}{4}}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.55)$$

$$y_n(s){=}B_n\, 2^n e^{s} s^{-\sqrt{\delta+\frac{9}{4}}} \frac{d^n}{ds^n}\,[\,e^{-s} s^{n+\sqrt{\delta+\frac{9}{4}}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.56)$$

$$L_n^k(x) \ = \ (x)\frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n}\,[e^{-x} x^{n+k}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.57)$$

$$\frac{1}{n!} = B_n\, 2^n$$

$$y_n(s) = L_n^{\sqrt{\delta+\frac{9}{4}}}\ (s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.58)$$

$$f(s) = s^{-3/4}\ e^{-s/2}\ s^{\frac{\sqrt{\delta+\frac{9}{4}}}{2}}\ \ Ln^{\sqrt{\delta+\frac{9}{4}}}(s) \quad , \quad \beta^2 \ = \ s$$

$$f(\beta) \ {=}C_N\, \beta^{-3/4} e^{-\beta/2} s^{\sqrt{\delta+\frac{9}{4}}} L_n^{\sqrt{\delta+\frac{9}{4}}}\ (\beta^2) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.59)$$

The equations describe the relationships and definitions of various functions, including) $\phi(s)$, $y_n(s)$, and(β), in terms of different variables such as s and β.

The value of C_N is expressed in terms of parameters such as α, n, and δ in a specific conditional form. The equations involve integration and manipulation of functions over specified intervals to arrive at the final result for C_N.

We are also given that the integral of $\beta^4 f^2(\beta)$ from 0 to ∞ is equal to 1:

$$\int_0^\infty \beta^4 f^2(\beta) d\beta = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.60)$$

Let's break down the steps for using the Nikiforov-Uvarov method to determine the energies (eigenvalues) associated with this function:

1. First, it needs to normalize the given function f(β) by determining the constant C_N. This constant ensures that the integral of $\beta^4 f^2(\beta)$ from 0 to ∞ is equal to 1 The normalization condition is used to make the wavefunction physically meaningful.
2. Normalize the function:

$$\left|C_N\right|^2 \int_0^\infty \beta^4 \beta^{-3} e^{-\beta^2} \beta^{2\sqrt{\delta+\frac{9}{4}}} L_n^{\sqrt{\delta+\frac{9}{4}}}(\beta^2)\, L_n^{\sqrt{\delta+\frac{9}{4}}}(\beta^2)\, d\beta = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.61)$$

$$\left|C_N\right|^2 \int_0^\infty \beta\, e^{-\beta^2} \beta^{2\sqrt{\delta+\frac{9}{4}}} L_n^{\sqrt{\delta+\frac{9}{4}}}(\beta^2)\, L_n^{\sqrt{\delta+\frac{9}{4}}}(\beta^2)\, d\beta = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.62)$$

Where: $\beta^2 = s \quad , \quad 2\beta\, d\beta = ds$

$$\sqrt{\delta + \frac{9}{4}} = a \quad , \quad \delta = \beta_0^4 + \tau(\tau + 3)$$

$$\left|C_N\right|^2 \int_0^\infty \sqrt{s}\, e^{-s} s^a L_n^a(s)\, L_n^a(s) \frac{ds}{2\sqrt{s}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.63)$$

$$\frac{|C_N|^2}{2} = \int_0^\infty e^{-s} s^a L_n^a(s) L_n^a(s) ds = 1 \quad \ldots\ldots\ldots(3.64)$$

$$\frac{|C_N|^2}{2} = \int_0^\infty e^{-x} x^\alpha \; L_n^\alpha\;(x)\;\; L_n^\alpha\;(x)\; dx \;=\; \{0\,,\, M \neq n \; or \left[\Gamma\;(1\;+\;\alpha)\binom{n+\alpha\, n}{n},\; M\;=\;n\right]\} \ldots(3.65)$$

$$\frac{|C_N|^2}{2} \Gamma\;(1+a)\;\binom{n+a\,n}{n} = 1 \quad \ldots\ldots\ldots(3.66)$$

$$C_N = \left(\frac{2}{\Gamma\;(1+a)\binom{n+an}{n}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots(3.67)$$

at a= $\sqrt{\delta\;+\;\frac{9}{4}}$

$\delta\;=\;\beta_0^4 + \tau(\tau\;+\;3)$

in this equation:

x is introduced as a new variable.

α is used to represent $\sqrt{\delta\;+\;\frac{9}{4}}$

The integral now represents the integral of the given function over the interval [0, ∞)[0,∞).

The equation introduces a conditional statement, stating that the result is zero when M is not equal to n and a specific expression when M is equal to n.

Equation 3.22 is part of the wave function solution within the Nikiforov-Uvarov method, and it contributes to the determination of energy levels for atomic systems based on the specific quantum state being considered. The method provides a framework for systematically finding the energies associated with various quantum states of the system.

$$f(\beta) = \left(\frac{2}{\Gamma\;(1+a)\binom{n+an}{n}}\right)^{\frac{1}{2}} \beta^{-3/2} e^{-\beta^2/2} \beta^a L_n^a\left(\beta^2\right) \quad \ldots\ldots\ldots(3.68)$$

The equation you've given is part of the solution for the wave function for a specific quantum state of the atomic system. It describes the radial part of the wave function (f(β)) as a function of the dimensionless variable β. The equation involves several parameters, including 'a,' 'n,' and 'Ln,' which have specific meanings in this context. Here's a breakdown of the key components:

1. f(β): This represents the radial part of the wave function, which is part of the solution to the Schrödinger equation. The form of f(β) is determined by the specific potential energy in the Schrödinger equation, and it plays a crucial role in defining the energy levels of the quantum system.

2. $\left(\frac{2}{\ulcorner\,(1+a)\binom{n+an}{n}}\right)^{\frac{1}{2}}$: This factor is related to the normalization of the wave function.

3. It ensures that the wave function is properly scaled so that the integral of the absolute square of the wave function is equal to 1. In quantum mechanics, wave functions must be normalized to represent probabilities correctly.

4. β: The variable β is dimensionless and is often introduced as part of a change of variables to simplify the Schrödinger equation. It is specific to the potential energy being considered.

5. 'a' and 'n': These parameters are determined by the specific quantum state you are interested in. They depend on the quantum numbers of the system, and they define the behavior of the wave function for that particular state.

The Nikiforov-Uvarov method involves transforming the Schrödinger equation into a standard form and solving it by introducing auxiliary functions and variables. The solutions for these auxiliary functions, such as f(β), are key to determining the energy levels (E) of the atomic system.

The method is applied to different quantum states, each corresponding to a different set of quantum numbers 'a' and 'n.' By finding the solutions for the wave function for each quantum state and calculating the associated energies, you can determine the energy spectrum of the atomic system, which represents the allowed energy levels for that system.

3.3. Discussion of the numerical methods that will be used to solve the equations of motion:

The provided equations appear to be part of a mathematical framework or methodology used to determine the energies of an atomic nucleus, specifically in the context of the Nikiforov-Uvarov method applied to the Bohr Hamiltonian. This method aims to find exact solutions to the Schrödinger equation for the atomic nucleus. Here's a summary of the key steps and equations involved:

1. Schrödinger Equation: The methodology begins with a Schrödinger equation (3. Methodology, equation 3.1) representing the atomic nucleus's quantum behavior. This equation is specific to the Bohr Hamiltonian.

2. Transformation: The methodology involves transforming the Schrödinger equation using a change of variables and auxiliary functions to simplify it (equation 3.4).

3. Differential Equations: After transformation, the Schrödinger equation is rewritten as a set of differential equations for different functions, such as $f(\beta)$, $\phi\left(\gamma, \theta_i\right)$, and others.

4. Solving Differential Equations: The methodology then solves these differential equations to find the functions $f(\beta)$, $\phi\left(\gamma, \theta_i\right)$, and others. This process involves various substitutions, manipulations, and transformations (equations 3.7, 3.8, 3.9, 3.10, 3.11, and others).

5. Energy Calculation: The energies of the atomic nucleus are determined by obtaining specific values or expressions for ε in equation 3.44, which represents the energy levels.

6. Normalization: The methodology includes normalization of the functions, which is typically done to ensure that the wave functions are properly scaled (equation 3.68).

It's important to note that this methodology involves a series of mathematical manipulations and solutions to obtain the energy levels of the atomic nucleus. The specific details of the method depend on the given potential functions, and the method appears to be particularly suitable for cases where the potential can be expressed as a product of two functions. The method aims to find exact solutions, which can be challenging for general potential functions but is achievable for specific cases using the Nikiforov-Uvarov method.

4. Results and Discussion

Some of the atomic nuclei properties such as level energies can be achieved by using the Nikiforov-Uvarov Method.

The NU method offers a systematic way to solve certain types of differential equations in quantum mechanics. It involves transforming the Schrödinger equation into a standard form by employing specific functions called Nikiforov-Uvarov functions. These functions simplify the equation, allowing for exact solutions for cases with separable potentials.

Application of the NU Method in order to determine the level energies of the atomic nuclei can be achieved by transforming Bohr Hamiltonian. The provided Bohr Hamiltonian is manipulated using the NU method to extract the energies associated with the system. Determination of the energies involves separating variables, transforming the Schrödinger equation, and solving resulting differential equations. The solutions provide energy eigenvalues specific to the system. Specific equations and solutions can be obtained using **Final Differential Equation (3.24)** that represents the transformed equation obtained through the NU method, where solving it yields the function related to the quantum state. **Function Relationships (3.59)** describes the wave function's radial part in terms of specific parameters like 'a' and 'n.' **Normalization (3.68)** Shows the normalized wave function, crucial for ensuring the proper scaling and physical interpretation of the wave function.

For getting more specific results, Python 3 with Jupyter (data science) have been used for calculating the energy eigenvalues, determining the energy ratios (R_L), with making graphs between (R_L) and angular momentum (L) then making comparison with experimental data of different nuclei as follow:

In [2]:
```
#Table 4.1: The energy ratio RL of E(5)-D model at β0 = 1
import pandas as pd
df=pd.read_excel("bo=1.xlsx")
df
```

Out[2]:

	n	τ	βo	L	E-110Xe	E-134Ba	E-156Dy	E-154Gd	E-154Sm	E-102Pd	E-150Nd	E-74Ge
0	0	0	1	0	0.0	0.0000	0.00	0.0000	0.000	0.00	0.00	0.000
1	0	1	1	2	469.7	604.7223	137.77	123.0709	81.981	556.44	130.21	595.850
2	0	2	1	4	1113.1	1400.5900	404.19	370.9998	266.817	1275.91	381.10	1463.759
3	0	3	1	6	1889.7	2211.3000	770.44	717.6620	544.100	2111.41	720.16	2569.329
4	0	4	1	8	NaN	2835.9000	1215.61	1144.4400	902.750	3013.13	1129.60	3681.000
5	0	5	1	10	NaN	NaN	1725.02	1637.0500	1333.000	3992.78	1598.50	NaN
6	0	6	1	12	NaN	NaN	2285.88	2184.6900	1825.900	5055.17	2118.70	NaN
7	0	7	1	14	NaN	NaN	2887.82	2777.3200	2373.000	6179.90	2681.60	NaN
8	0	8	1	16	NaN	NaN	3523.30	3404.4500	2968.200	7428.90	3279.60	NaN
9	0	9	1	18	NaN	NaN	4178.10	4087.1500	3609.300	8707.10	NaN	NaN
10	0	10	1	20	NaN	NaN	4859.00	4782.3000	4295.700	10223.10	NaN	NaN

Table 4.1: Energy levels of the ground state band (gsb) of different nuclei with E(5)-D model at $\beta_0 = 1$.

In [16]:
```
%matplotlib inline
import matplotlib.pyplot as plt


import numpy as np
t=np.array(df['□'])
b=np.array(df['βo'])
n=np.array(df['n'])
E110Xe=np.array(df['E-110Xe'])
E134Ba=np.array(df['E-134Ba'])
E156Dy=np.array(df['E-156Dy'])
E154Gd=np.array(df['E-154Gd'])
E154Sm=np.array(df['E-154Sm'])
E102Pd=np.array(df['E-102Pd'])
E150Nd=np.array(df['E-150Nd'])
E74Ge=np.array(df['E-74Ge'])
```

```
E5D=np.array((2*n+1)+np.sqrt((b**4)+(t*t)+(t*3)+(9/4)))
print(E5D)

R5D=np.array((E5D-E5D[0])/(E5D[1]-E5D[0]))
print(R5D)

R110Xe=np.array((E110Xe-E110Xe[0])/(E110Xe[1]-E110Xe[0]))
print(R110Xe)

R134Ba=np.array((E134Ba-E134Ba[0])/(E134Ba[1]-E134Ba[0]))
print(R134Ba)

R156Dy=np.array((E156Dy-E156Dy[0])/(E156Dy[1]-E156Dy[0]))
print(R156Dy)

R154Gd=np.array((E154Gd-E154Gd[0])/(E154Gd[1]-E154Gd[0]))
print(R154Gd)

R154Sm=np.array((E154Sm-E154Sm[0])/(E154Sm[1]-E154Sm[0]))
print(R154Sm)

R102Pd=np.array((E102Pd-E102Pd[0])/(E102Pd[1]-E102Pd[0]))
print(R102Pd)

R150Nd=np.array((E150Nd-E150Nd[0])/(E150Nd[1]-E150Nd[0]))
print(R150Nd)

R74Ge=np.array((E74Ge-E74Ge[0])/(E74Ge[1]-E74Ge[0]))
print(R74Ge)

plt.plot(R156Dy,color="purple")

plt.plot(R154Gd,color="green")

plt.plot(R154Sm,color="yellow")

plt.plot(R102Pd,color="red")

plt.plot(R150Nd,color="pink")

plt.plot(R74Ge,color="black")

plt.plot(R134Ba,color="brown")

plt.plot(R110Xe,color="orange")

plt.plot(R5D,color="blue")

Plt.show
```

```
[ 2.80277564  3.6925824   4.64005494  5.60977223  6.59016994  7.57647322
  8.56637298  9.55862138 10.55248659 11.54751155 12.54339638]
[ 0.          1.          2.06480708  3.15461367  4.25642336  5.36486995
  6.47735841  7.59248638  8.70943136  9.82767973 10.94689445]
[0.         1.         2.36981052 4.0232063         nan        nan
        nan        nan        nan        nan        nan]
[0.         1.         2.3160879  3.65671979 4.68959058        nan
        nan        nan        nan        nan        nan]
[ 0.          1.          2.93380271  5.59221892  8.82347391 12.52101328
 16.59200116 20.96116716 25.57378239 30.32663134 35.26892647]
[ 0.          1.          3.0145209   5.83128912  9.29903007 13.3016822
 17.75147496 22.56682936 27.66250998 33.20971895 38.85808912]
[ 0.          1.          3.25461997  6.63690367 11.01169783 16.25986509
 22.27223381 28.94573133 36.20595016 44.02605482 52.39872653]
[ 0.          1.          2.29298756  3.79449716  5.4150133   7.17558048
  9.08484293 11.10613903 13.35076558 15.64786859 18.37233125]
[ 0.          1.          2.92681054  5.53075801  8.67521696 12.27632286
 16.27140773 20.59442439 25.18700561         nan         nan]
[0.         1.         2.45658975 4.31203994 6.17772929        nan
        nan        nan        nan        nan        nan]
```

Out[3]: <function matplotlib.pyplot.show(close=None, block=None)>

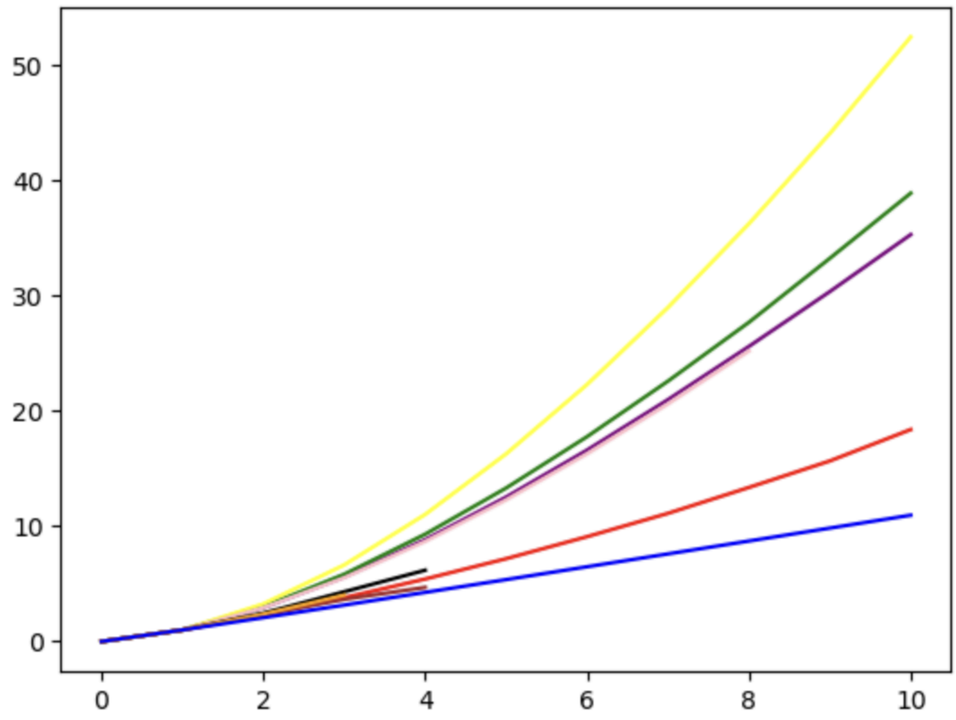

Figure 4.1: Comparison of the energy levels of the ground state band (gsb) of different nuclei with E(5)-D model at $\beta_0 = 1$.

```
In [6]: #Table 4.2: The energy ratio RL of E(5)-D model at β0 = 1.5
        import pandas as pd
        df=pd.read_excel("bo=1.5.xlsx")
        df
```

Out[6]:

	n	τ	βo	L	E-110Xe	E-134Ba	E-156Dy	E-154Gd	E-154Sm	E-102Pd	E-150Nd	E-74Ge
0	0	0	1.5	0	0.0	0.0000	0.00	0.0000	0.000	0.00	0.00	0.000
1	0	1	1.5	2	469.7	604.7223	137.77	123.0709	81.981	556.44	130.21	595.850
2	0	2	1.5	4	1113.1	1400.5900	404.19	370.9998	266.817	1275.91	381.10	1463.759
3	0	3	1.5	6	1889.7	2211.3000	770.44	717.6620	544.100	2111.41	720.16	2569.329
4	0	4	1.5	8	NaN	2835.9000	1215.61	1144.4400	902.750	3013.13	1129.60	3681.000
5	0	5	1.5	10	NaN	NaN	1725.02	1637.0500	1333.000	3992.78	1598.50	NaN
6	0	6	1.5	12	NaN	NaN	2285.88	2184.6900	1825.900	5055.17	2118.70	NaN
7	0	7	1.5	14	NaN	NaN	2887.82	2777.3200	2373.000	6179.90	2681.60	NaN
8	0	8	1.5	16	NaN	NaN	3523.30	3404.4500	2968.200	7428.90	3279.60	NaN
9	0	9	1.5	18	NaN	NaN	4178.10	4087.1500	3609.300	8707.10	NaN	NaN
10	0	10	1.5	20	NaN	NaN	4859.00	4782.3000	4295.700	10223.10	NaN	NaN

Table 4.2: Energy levels of the ground state band (gsb) of different nuclei with E(5)-D model at $\beta_0 = 1.5$.

```
In [2]:
%matplotlib inline
import matplotlib.pyplot as plt


import numpy as np
t=np.array(df['▯'])
b=np.array(df['βo'])
n=np.array(df['n'])
E110Xe=np.array(df['E-110Xe'])
E134Ba=np.array(df['E-134Ba'])
E156Dy=np.array(df['E-156Dy'])
E154Gd=np.array(df['E-154Gd'])
E154Sm=np.array(df['E-154Sm'])
E102Pd=np.array(df['E-102Pd'])
E150Nd=np.array(df['E-150Nd'])
E74Ge=np.array(df['E-74Ge'])
```

```
E5D=np.array((2*n+1)+np.sqrt((b**4)+(t*t)+(t*3)+(9/4)))
print(E5D)

R5D=np.array((E5D-E5D[0])/(E5D[1]-E5D[0]))
print(R5D)

R110Xe=np.array((E110Xe-E110Xe[0])/(E110Xe[1]-E110Xe[0]))
print(R110Xe)

R134Ba=np.array((E134Ba-E134Ba[0])/(E134Ba[1]-E134Ba[0]))
print(R134Ba)

R156Dy=np.array((E156Dy-E156Dy[0])/(E156Dy[1]-E156Dy[0]))
print(R156Dy)

R154Gd=np.array((E154Gd-E154Gd[0])/(E154Gd[1]-E154Gd[0]))
print(R154Gd)

R154Sm=np.array((E154Sm-E154Sm[0])/(E154Sm[1]-E154Sm[0]))
print(R154Sm)

R102Pd=np.array((E102Pd-E102Pd[0])/(E102Pd[1]-E102Pd[0]))
print(R102Pd)

R150Nd=np.array((E150Nd-E150Nd[0])/(E150Nd[1]-E150Nd[0]))
print(R150Nd)

R74Ge=np.array((E74Ge-E74Ge[0])/(E74Ge[1]-E74Ge[0]))
print(R74Ge)

plt.plot(R156Dy,color="purple")
plt.plot(R154Gd,color="green")
plt.plot(R154Sm,color="yellow")
```

```
plt.plot(R102Pd,color="red")
plt.plot(R150Nd,color="pink")
plt.plot(R74Ge,color="black")
plt.plot(R134Ba,color="brown")
plt.plot(R110Xe,color="orange")
plt.plot(R5D,color="blue")
plt.show
```

```
[ 3.70416346  4.36340601  5.16082924  6.03115295  6.94243216  7.87840825
  8.83022988  9.7927527  10.76281209 11.7383658  12.71804165]
[ 0.          1.          2.20960521  3.5297926   4.91210508  6.33188006
  7.77569103  9.23573455 10.70721024 12.18702021 13.67308302]
[0.         1.         2.36981052 4.0232063         nan        nan
        nan        nan        nan        nan        nan]
[0.         1.         2.3160879  3.65671979 4.68959058        nan
        nan        nan        nan        nan        nan]
[ 0.          1.          2.93380271  5.59221892  8.82347391 12.52101328
 16.59200116 20.96116716 25.57378239 30.32663134 35.26892647]
[ 0.          1.          3.0145209   5.83128912  9.29903007 13.3016822
 17.75147496 22.56682936 27.66250998 33.20971895 38.85808912]
[ 0.          1.          3.25461997  6.63690367 11.01169783 16.25986509
 22.27223381 28.94573133 36.20595016 44.02605482 52.39872653]
[ 0.          1.          2.29298756  3.79449716  5.4150133   7.17558048
  9.08484293 11.10613903 13.35076558 15.64786859 18.37233125]
[ 0.          1.          2.92681054  5.53075801  8.67521696 12.27632286
 16.27140773 20.59442439 25.18700561         nan         nan]
[0.         1.         2.45658975 4.31203994 6.17772929        nan
        nan        nan        nan        nan        nan]
```

```
Out[7]: <function matplotlib.pyplot.show(close=None, block=None)>
```

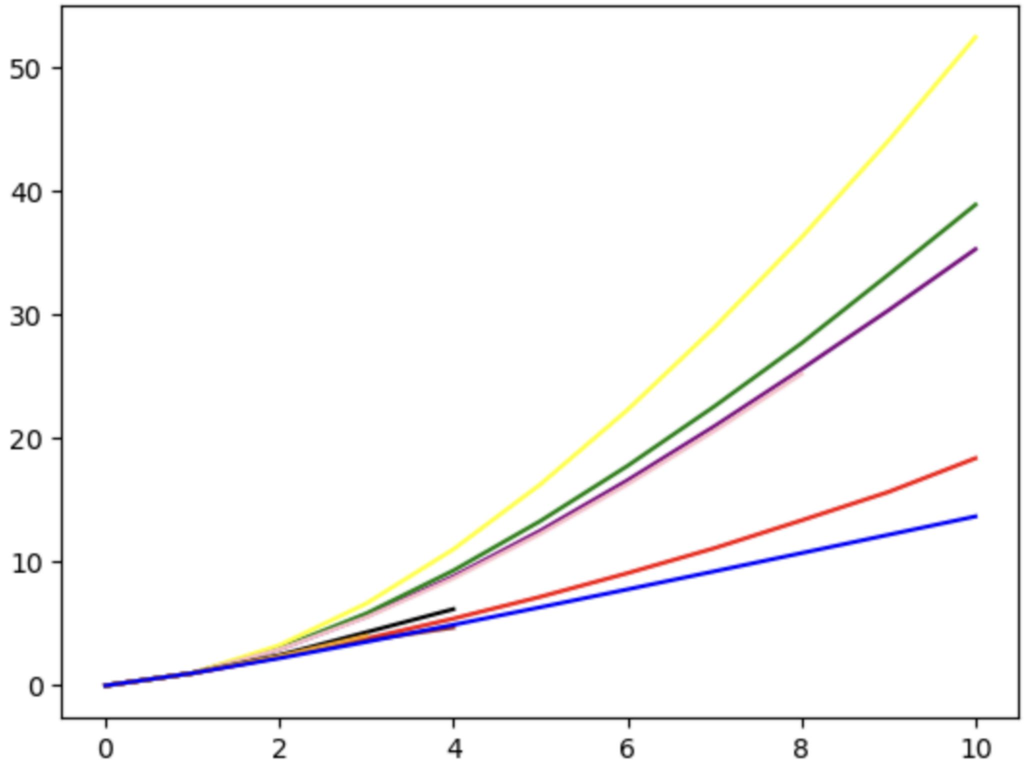

Figure 4.2: Comparison of the energy levels of the ground state band (gsb) of different nuclei with E(5)-D model at $\beta_0 = 1.5$.

In [1]:
```
#Table 4.3: The energy ratio RL of E(5)-D model at β0 = 2
import pandas as pd
df=pd.read_excel("bo=2.xlsx")
df
```

Out[1]:

	n	τ	βo	L	E-110Xe	E-134Ba	E-156Dy	E-154Gd	E-154Sm	E-102Pd	E-150Nd	E-74Ge
0	0	0	2	0	0.0	0.0000	0.00	0.0000	0.000	0.00	0.00	0.000
1	0	1	2	2	469.7	604.7223	137.77	123.0709	81.981	556.44	130.21	595.850
2	0	2	2	4	1113.1	1400.5900	404.19	370.9998	266.817	1275.91	381.10	1463.759
3	0	3	2	6	1889.7	2211.3000	770.44	717.6620	544.100	2111.41	720.16	2569.329
4	0	4	2	8	NaN	2835.9000	1215.61	1144.4400	902.750	3013.13	1129.60	3681.000
5	0	5	2	10	NaN	NaN	1725.02	1637.0500	1333.000	3992.78	1598.50	NaN
6	0	6	2	12	NaN	NaN	2285.88	2184.6900	1825.900	5055.17	2118.70	NaN
7	0	7	2	14	NaN	NaN	2887.82	2777.3200	2373.000	6179.90	2681.60	NaN
8	0	8	2	16	NaN	NaN	3523.30	3404.4500	2968.200	7428.90	3279.60	NaN
9	0	9	2	18	NaN	NaN	4178.10	4087.1500	3609.300	8707.10	NaN	NaN
10	0	10	2	20	NaN	NaN	4859.00	4782.3000	4295.700	10223.10	NaN	NaN

Table 4.3: Energy levels of the ground state band (gsb) of different nuclei with E(5)-D model at $\beta_0 = 2$.

In [4]:
```
%matplotlib inline
import matplotlib.pyplot as plt

import numpy as np
t=np.array(df['?'])
b=np.array(df['βo'])
n=np.array(df['n'])
E110Xe=np.array(df['E-110Xe'])
E134Ba=np.array(df['E-134Ba'])
E156Dy=np.array(df['E-156Dy'])
E154Gd=np.array(df['E-154Gd'])
E154Sm=np.array(df['E-154Sm'])
E102Pd=np.array(df['E-102Pd'])
E150Nd=np.array(df['E-150Nd'])
E74Ge=np.array(df['E-74Ge'])


E5D=np.array((2*n+1)+np.sqrt((b**4)+(t*t)+(t*3)+(9/4)))
print(E5D)

R5D=np.array((E5D-E5D[0])/(E5D[1]-E5D[0])) print(R5D)


R110Xe=np.array((E110Xe-E110Xe[0])/(E110Xe[1]-E110Xe[0]))
print(R110Xe)
```

```
R134Ba=np.array((E134Ba-E134Ba[0])/(E134Ba[1]-E134Ba[0]))print
(R134Ba)


R156Dy=np.array((E156Dy-E156Dy[0])/(E156Dy[1]-E156Dy[0]))print
(R156Dy)


R154Gd=np.array((E154Gd-E154Gd[0])/(E154Gd[1]-E154Gd[0]))print
(R154Gd)


R154Sm=np.array((E154Sm-E154Sm[0])/(E154Sm[1]-E154Sm[0]))print
(R154Sm)


R102Pd=np.array((E102Pd-E102Pd[0])/(E102Pd[1]-E102Pd[0]))print
(R102Pd)


R150Nd=np.array((E150Nd-E150Nd[0])/(E150Nd[1]-E150Nd[0]))print
(R150Nd)


R74Ge=np.array((E74Ge-E74Ge[0])/(E74Ge[1]-E74Ge[0]))
print(R74Ge)


plt.plot(R156Dy,color="purple")

plt.plot(R154Gd,color="green")

plt.plot(R154Sm,color="yellow")

plt.plot(R102Pd,color="red")

plt.plot(R150Nd,color="pink")

plt.plot(R74Ge,color="black")

plt.plot(R134Ba,color="brown")

plt.plot(R110Xe,color="orange")

plt.plot(R5D,color="blue")

plt.show
```

```
[ 5.27200187  5.71699057  6.31507291  7.02079729  7.80073525  8.63216876
  9.5        10.39414711 11.30776406 12.23610253 13.17579566]
[ 0.          1.          2.34403941  3.92997719  5.68269131  7.55112869
  9.5013608  11.51073121 13.56385518 15.65006203 17.76178565]
```

```
[0.         1.         2.36981052 4.0232063         nan        nan
        nan        nan        nan        nan        nan]
[0.         1.         2.3160879  3.65671979 4.68959058        nan
        nan        nan        nan        nan        nan]
[ 0.          1.          2.93380271  5.59221892  8.82347391 12.52101328
 16.59200116 20.96116716 25.57378239 30.32663134 35.26892647]
[ 0.          1.          3.0145209   5.83128912  9.29903007 13.3016822
 17.75147496 22.56682936 27.66250998 33.20971895 38.85808912]
[ 0.          1.          3.25461997  6.63690367 11.01169783 16.25986509
 22.27223381 28.94573133 36.20595016 44.02605482 52.39872653]
[ 0.          1.          2.29298756  3.79449716  5.4150133   7.17558048
  9.08484293 11.10613903 13.35076558 15.64786859 18.37233125]
[ 0.          1.          2.92681054  5.53075801  8.67521696 12.27632286
 16.27140773 20.59442439 25.18700561         nan        nan]
[0.         1.         2.45658975 4.31203994 6.17772929        nan
        nan        nan        nan        nan        nan]
```

Out[11]: `<function matplotlib.pyplot.show(close=None, block=None)>`

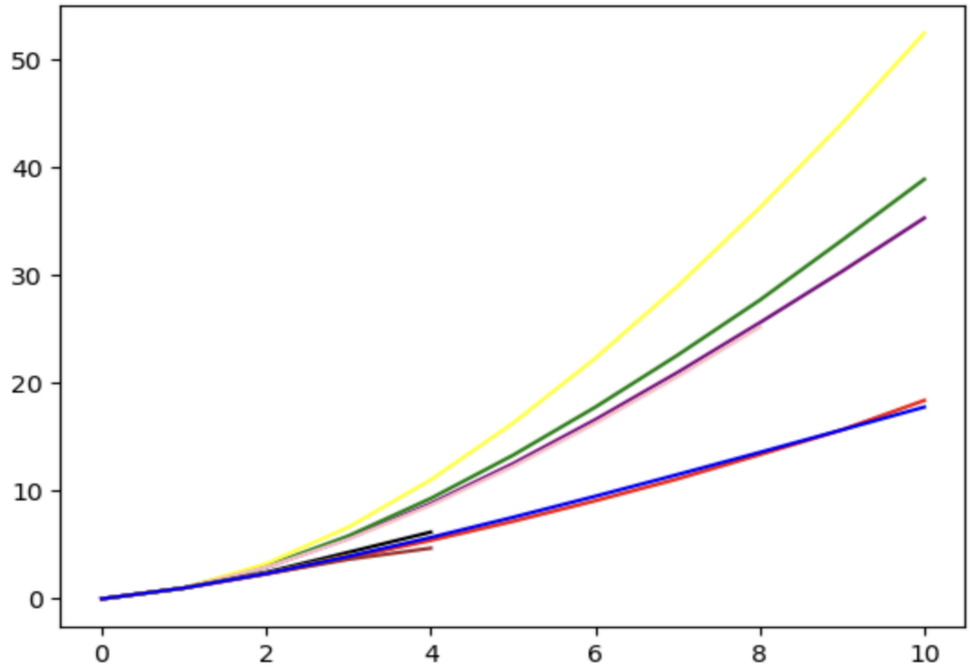

Figure 4.3: Comparison of the energy levels of the ground state band (gsb) of different nuclei with E(5)-D model at $\beta_0 = 2$.

Determining $E_{n,L}$ values which came from eq. (3.39), □ is angular momentum, L=2□, R_L is the energy ratio, the values came from this equation:

$R_{n,L} = \frac{En,L - E0,0}{E0,2 - E0,0}$.

Table (4.1) shows the results of gsb energies of E(5)-D model obtained from equation(3.39) at β_0=1

n	□	$\boldsymbol{\beta}_o$	$E_{n,L}$	L	R_L
0	0	1	2.802775638	0	0.00
0	1	1	3.692582404	2	1.00
0	2	1	4.640054945	4	2.06
0	3	1	5.609772229	6	3.15
0	4	1	6.590169944	8	4.26
0	5	1	7.576473219	10	5.36
0	6	1	8.566372975	12	6.48
0	7	1	9.558621384	14	7.59
0	8	1	10.55248659	16	8.71
0	9	1	11.54751155	18	9.83
0	10	1	12.54339638	20	10.95

Table 4.4: The energy ratio R_L of E(5)-D model at β_0 = 1.

Then this result is compared with gsb energies of different nuclei such as ^{110}Xe, ^{134}Ba, ^{156}Dy, ^{154}Gd, ^{154}Sm, ^{102}Pd, ^{150}Nd and ^{74}Ge.

The International Nuclear Forces website is a valuable resource for nuclear physics data. It includes a variety of data, including nuclear masses, binding energies, and transition rates. The website is also regularly updated with new data. The result of E(5)-D model at different β_0 such as β_0 =1,1.5,2. is compared with gsb energies from experimental data from Nuclear Forces website as follow:

For ^{110}Xe:

^{110}Xe	E	L	R_L
	0	0	0.00
	469.7	2	1.00
	1113.1	4	2.37
	1889.7	6	4.02

Table 4.5: The energy ratio R_L of ^{110}Xe from experimental data.

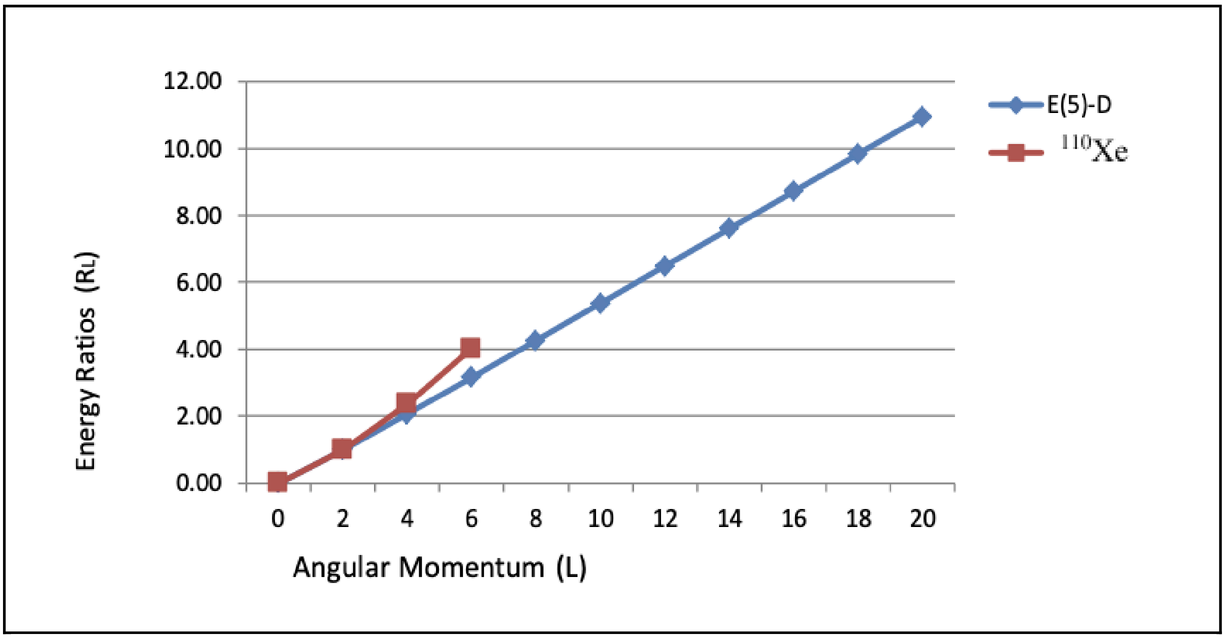

Figure 4.4: Comparison of the energy levels of the ground state band (gsb) of the ^{110}Xe nuclei with E(5)-D model at $\beta_0 = 1$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{110}Xe nucleus up to L = 6 angular momentum they match with the this model. Because; The ^{120}Xe isotope can be called the E(5)-D model nucleus.

For ^{134}Ba:

^{134}Ba	E	L	R
	0	0	0.00
	604.7223	2	1.00
	1400.59	4	2.32
	2211.3	8	3.66
	2835.9	10	4.69

Table 4.6: The energy ratio R_L of ^{134}Ba from experimental data.

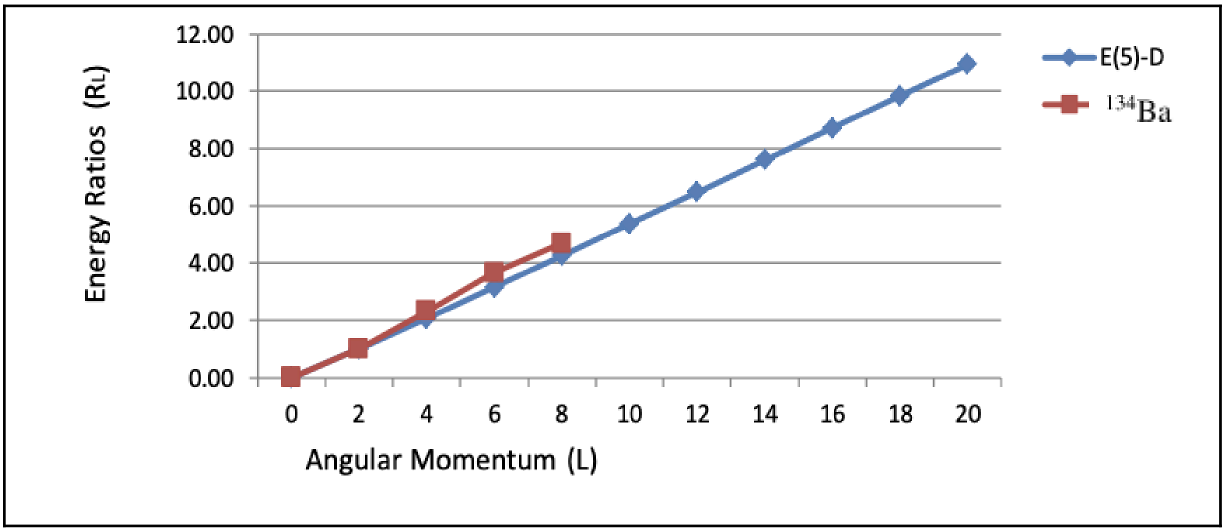

Figure 4.5: Comparison of the energy levels of the ground state band (gsb) of the ^{134}Ba nuclei with E(5)-D model at $\beta_0 = 1$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{134}Ba nucleus up to L = 8 angular momentum they match with the this model. Because; The ^{134}Ba isotope can be called the E(5)-D model nucleus.

For ^{74}Ge:

^{74}Ge	E	L	R
	0	0	0.00
	595.85	2	1.00
	1463.759	4	2.46
	2569.329	6	4.31
	3681	8	6.18

Table 4.7: The energy ratio R_L of ^{74}Ge from experimental data.

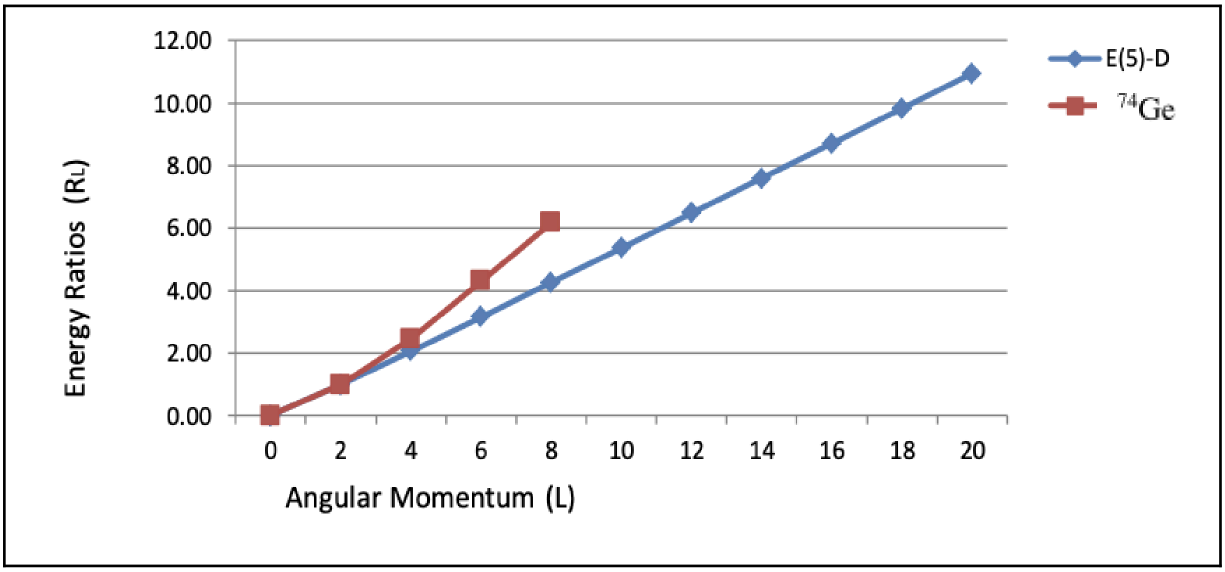

Figure 4.6: Comparison of the energy levels of the ground state band (gsb) of the ^{74}Ge nuclei with E(5)-D model at $\beta_0 = 1$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{74}Ge nucleus up to L = 4 angular momentum they match with the this model after that they overlap.

For ^{102}Pd:

^{102}Pd	E	L	R
	0	0	0.00
	556.44	2	1.00
	1275.91	4	2.29
	2111.41	6	3.79
	3013.13	8	5.42
	3992.78	10	7.18
	5055.17	12	9.08
	6179.9	14	11.11
	7428.9	16	13.35
	8707.1	18	15.65
	10223.1	20	18.37

Table 4.8: The energy ratio R_L of ^{102}Pd from experimental data.

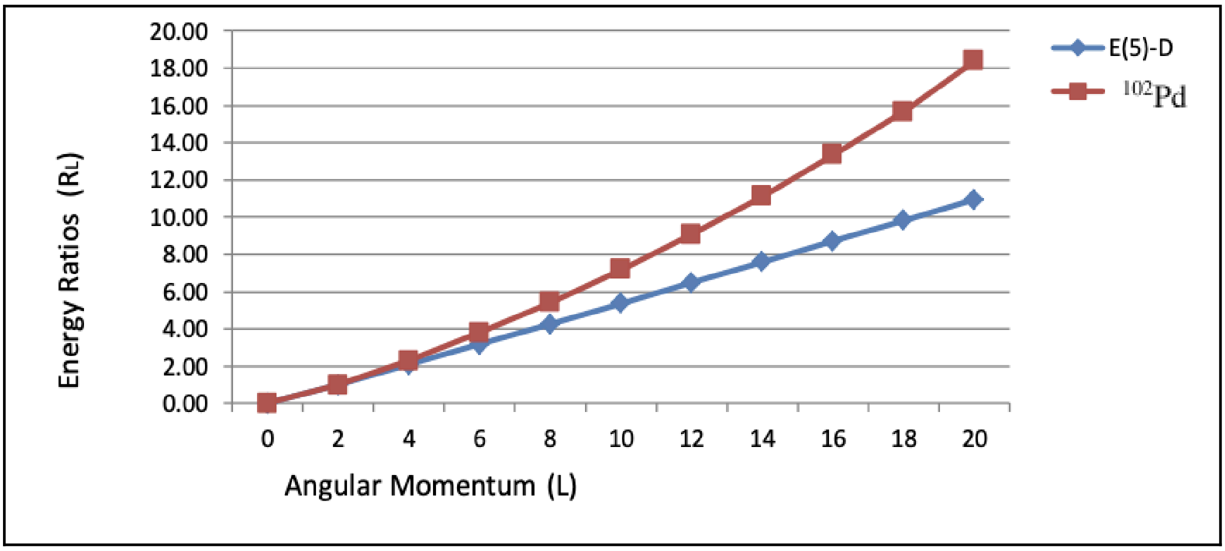

Figure 4.7: Comparison of the energy levels of the ground state band (gsb) of the ^{102}Pd nuclei with E(5)-D model at β_0 = 1.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{102}Pd nucleus up to L = 8 angular momentum they match with the this model after that they overlap.

For ^{156}Dy:

^{156}Dy	E	L	R
	0	0	0.00
	137.77	2	1.00
	404.19	4	2.93
	770.44	6	5.59
	1215.61	8	8.82
	1725.02	10	12.52
	2285.88	12	16.59
	2887.82	14	20.96
	3523.3	16	25.57
	4178.1	18	30.33
	4859	20	35.27

Table 4.9: The energy ratio R_L of ^{156}Dy from experimental data.

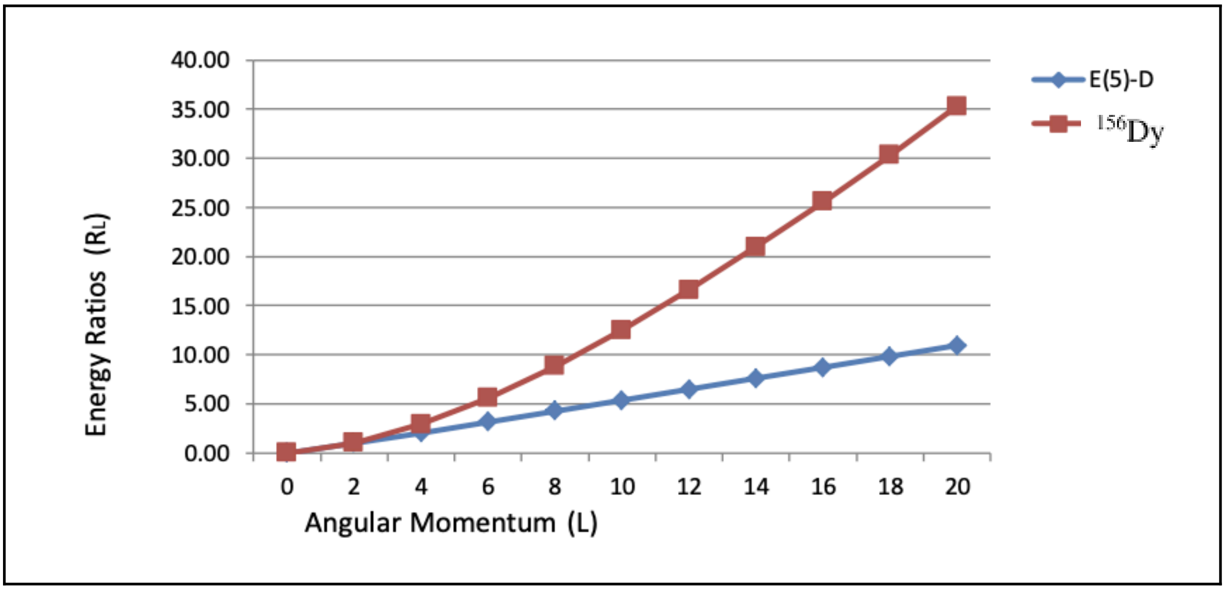

Figure 4.8: Comparison of the energy levels of the ground state band (gsb) of the ^{156}Dy nuclei with E(5)-D model at $\beta_0 = 1$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{156}Dy nucleus, they overlap.

For ^{154}Sm:

^{154}Sm	E	L	R
	0	0	0.00
	81.981	2	1.00
	266.817	4	3.25
	544.1	6	6.64
	902.75	8	11.01
	1333	10	16.26
	1825.9	12	22.27
	2373	14	28.95
	2968.2	16	36.21

3609.3	18	44.03
4295.7	20	52.40

Table 4.10: The energy ratio R_L of ^{154}Sm from experimental data.

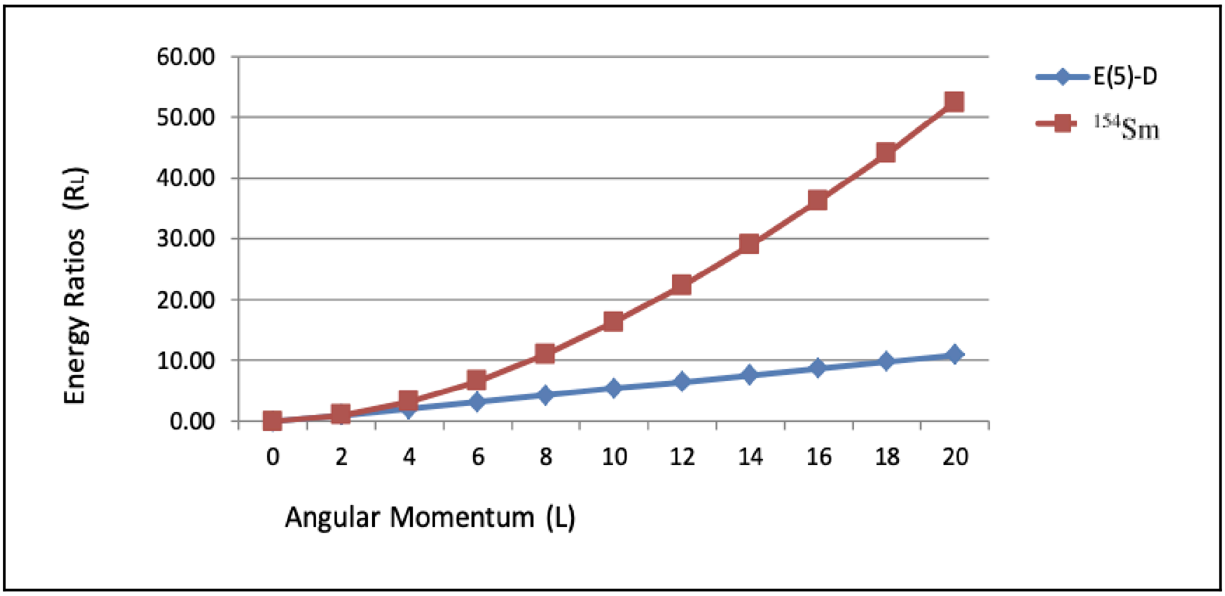

Figure 4.9: Comparison of the energy levels of the ground state band (gsb) of the ^{154}Sm nuclei with E(5)-D model at $\beta_0 = 1$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{154}Sm nucleus up to $L = 4$ angular momentum they match with the this model after that they overlap.

For ^{154}Gd:

^{154}Gd	E	L	R
	0	0	0.00
	123.0709	2	1.00
	370.9998	4	3.01
	717.662	6	5.83
	1144.44	8	9.30
	1637.05	10	13.30
	2184.69	12	17.75
	2777.32	14	22.57
	3404.45	16	27.66
	4087.15	18	33.21
	4782.3	20	38.86

Table 4.11: The energy ratio R_L of ^{154}Gd from experimental data.

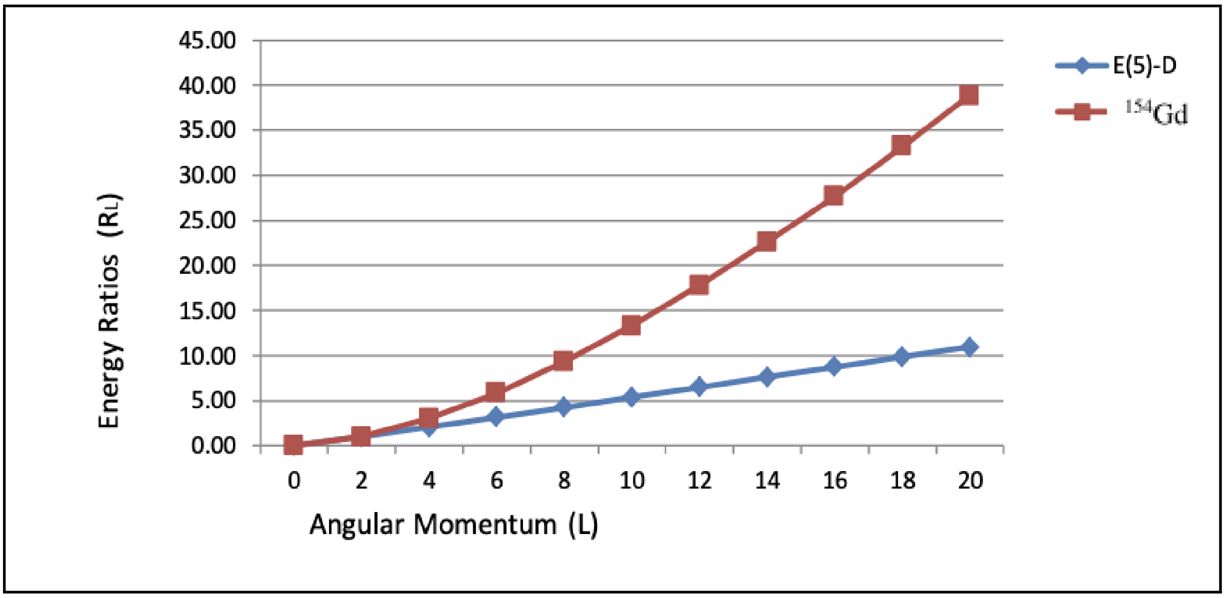

Figure 4.10: Comparison of the energy levels of the ground state band (gsb) of the ^{154}Gd nuclei with E(5)-D model at $\beta_0 = 1$.

When the figure above is examined; It has been observed that for the value β_0 = 1, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{154}Gd nucleus, they overlap.

For ^{150}Nd:

^{150}Nd	E	L	R
	0	0	0.00
	130.21	2	1.00
	381.1	4	2.93
	720.16	6	5.53
	1129.6	8	8.68
	1598.5	10	12.28
	2118.7	12	16.27
	2681.6	14	20.59
	3279.6	16	25.19

Table 4.12: The energy ratio R_L of ^{150}Nd from experimental data.

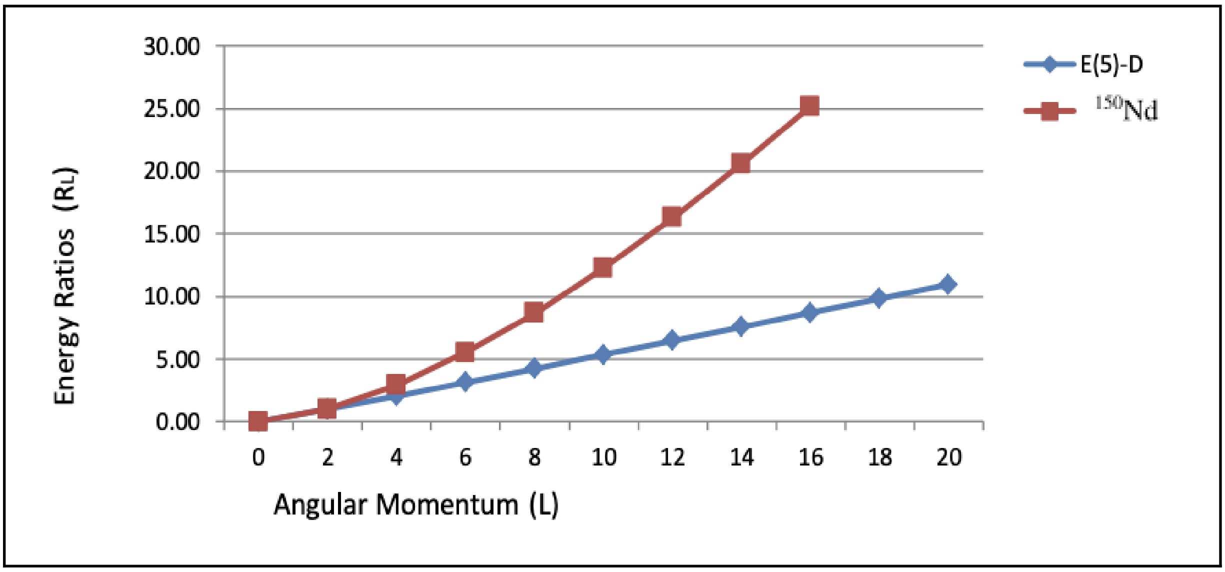

Figure 4.11: Comparison of the energy levels of the ground state band (gsb) of the ^{150}Nd nuclei with E(5)-D model at β_0 = 1.

When the figure above is examined; It has been observed that for the value β_0 = 1, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{102}Pd nucleus, they overlap.

Now table (4.10) shows the results of gsb energies of E(5)-D model obtained from equation(3.39) at β_0=1.5.

n	□	β_o	$E_{n,L}$	L	R_L
0	0	1.5	3.704163457	0	0.00
0	1	1.5	4.363406012	2	1.00
0	2	1.5	5.160829244	4	2.21
0	3	1.5	6.031152949	6	3.53
0	4	1.5	6.942432162	8	4.91
0	5	1.5	7.878408246	10	6.33
0	6	1.5	8.830229882	12	7.78
0	7	1.5	9.792752698	14	9.24
0	8	1.5	10.76281209	16	10.71
0	9	1.5	11.7383658	18	12.19
0	10	1.5	12.71804165	20	13.67

Table 4.13: The energy ratio R_L of E(5)-D model at β_0 = 1.5

Then this result compared with experimental data for different nuclei as follow:

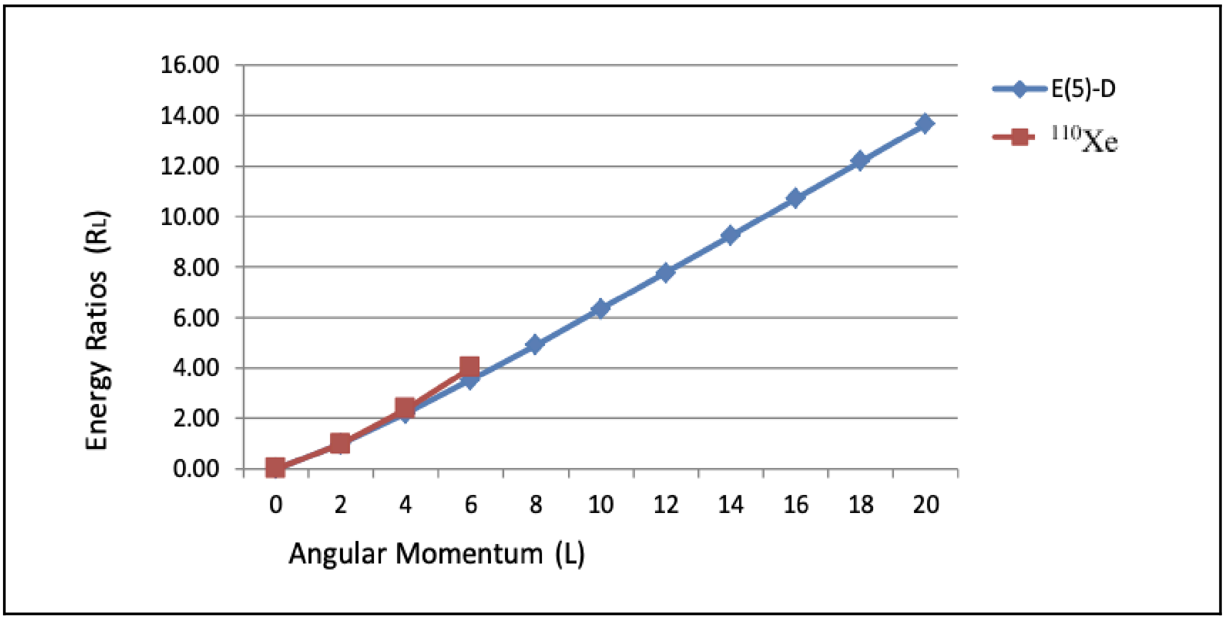

Figure 4.12: Comparison of the energy levels of the ground state band (gsb) of the ^{110}Xe nuclei with E(5)-D model at β_0 = 1.5.

When the figure above is examined; It has been observed that for the value β_0 = 1.5, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{110}Xe nucleus up to L = 6 angular momentum they match with the this model. Because; The ^{120}Xe isotope can be called the E(5)-D model nucleus.

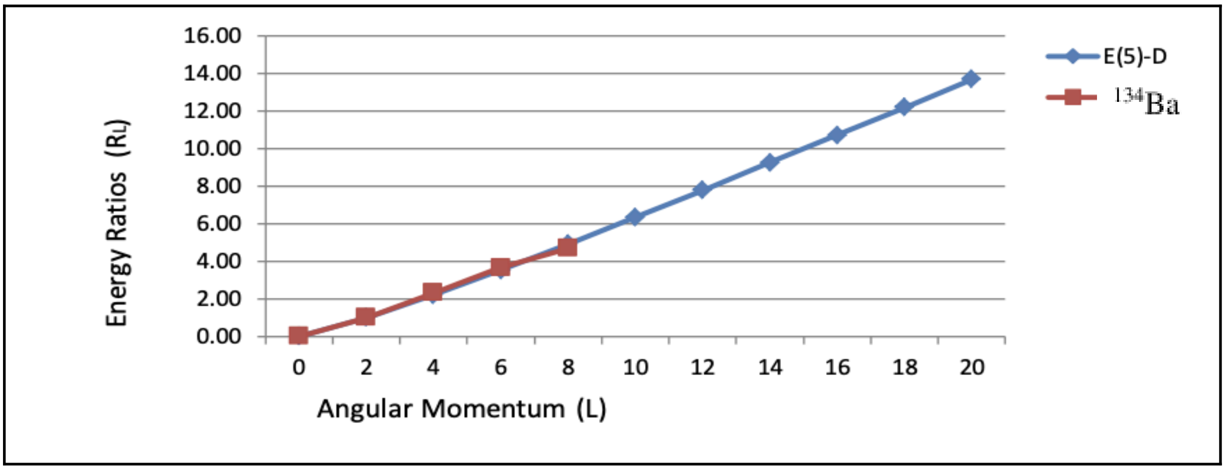

Figure 4.13: Comparison of the energy levels of the ground state band (gsb) of the ^{134}Ba nuclei with E(5)-D model at β_0 = 1.5.

When the figure above is examined; It has been observed that for the value β_0 = 1.5 , the energy levels of the ground state band of the E(5)-D model are compatible with the experimental data of the ^{134}Ba nucleus up to L = 8 angular momentum they match with the this model. Because; The ^{134}Ba isotope can be called the E(5)-D model nucleus.

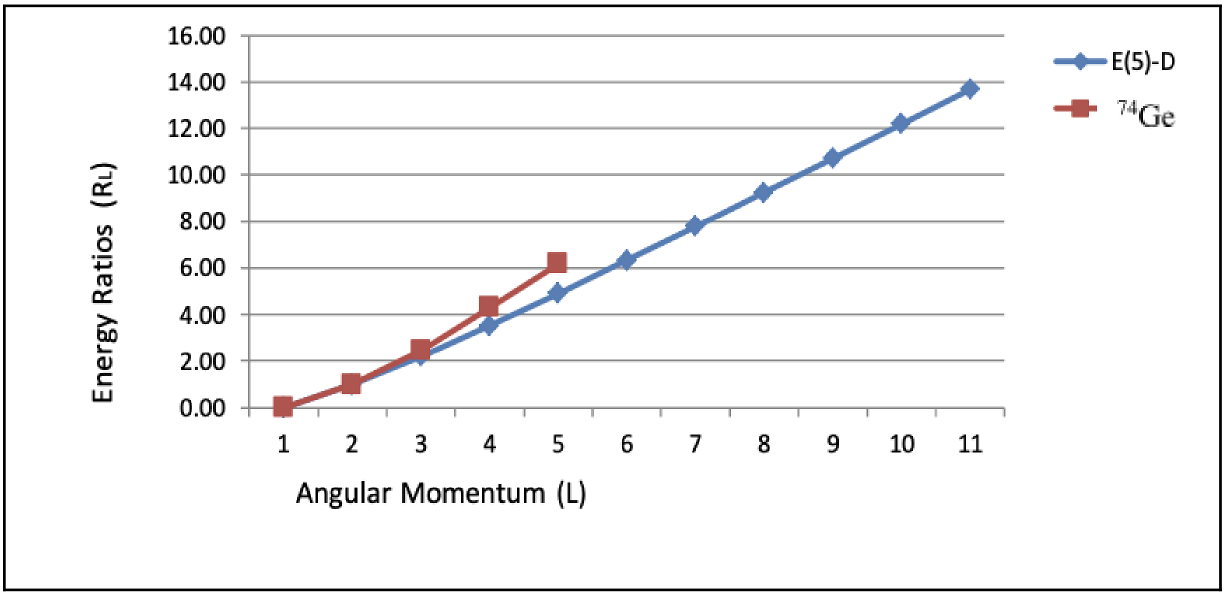

Figure 4.14: Comparison of the energy levels of the ground state band (gsb) of the ^{74}Ge nuclei with E(5)-D model at $\beta_0 = 1.5$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1.5$, the energy levels of the ground state band of the E(5)-D model are compatible with the experimental data of the ^{74}Ge nucleus up to L = 4 angular momentum they match with the this model after that they overlap. They are much closer than $\beta_0 = 1$.

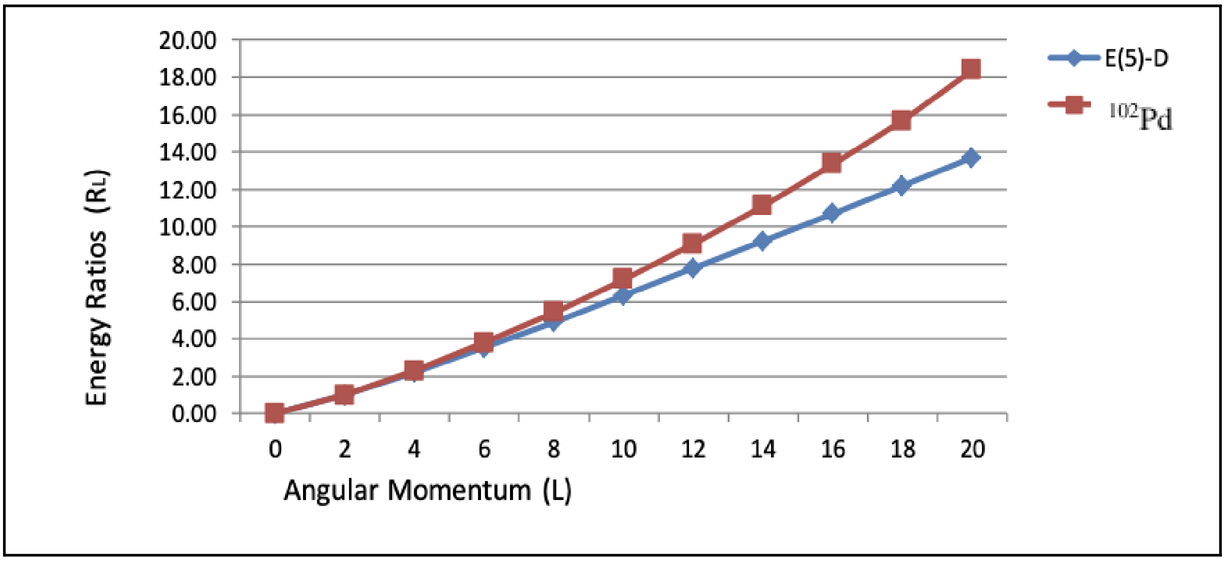

Figure 4.15: Comparison of the energy levels of the ground state band (gsb) of the ^{102}Pd nuclei with E(5)-D model at $\beta_0 = 1.5$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 1.5$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{102}Pd nucleus up to L = 10 angular momentum they match and are so close with this model after that they overlap.

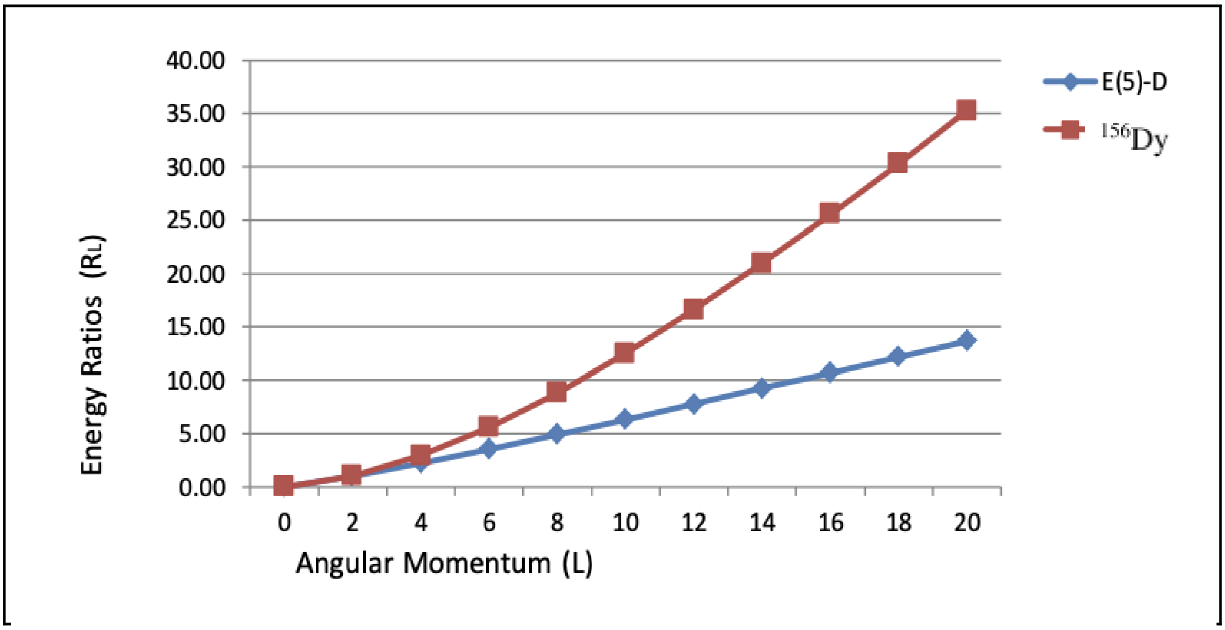

Figure 4.16: Comparison of the energy levels of the ground state band (gsb) of the ^{156}Dy nuclei with E(5)-D model at $\beta_0 = 1.5$.

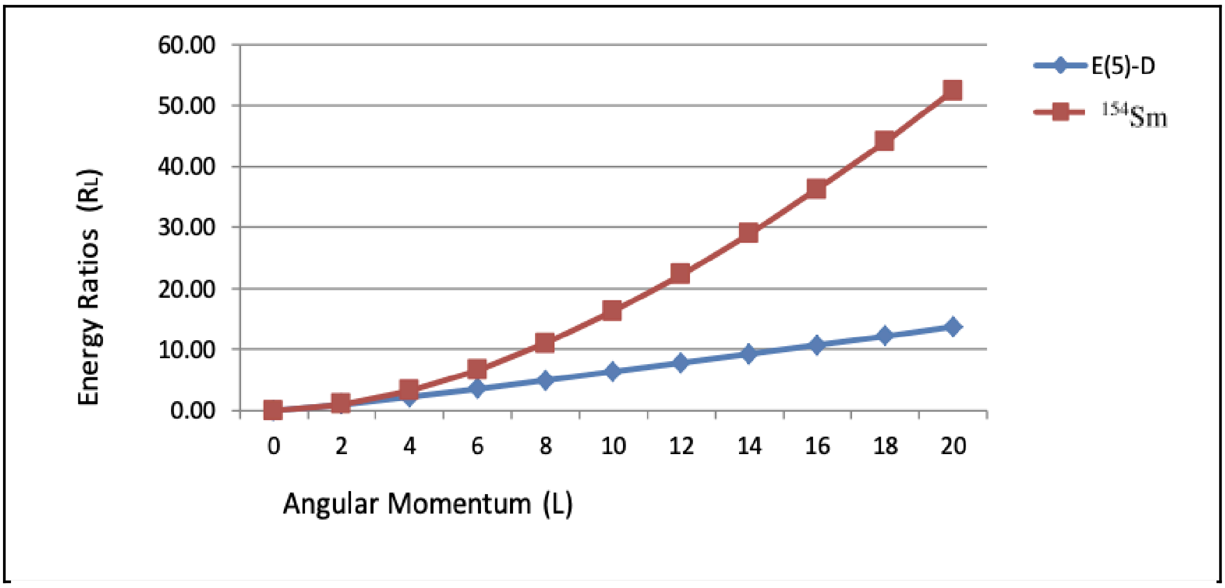

Figure 4.17: Comparison of the energy levels of the ground state band (gsb) of the ^{154}Sm nuclei with E(5)-D model at $\beta_0 = 1.5$.

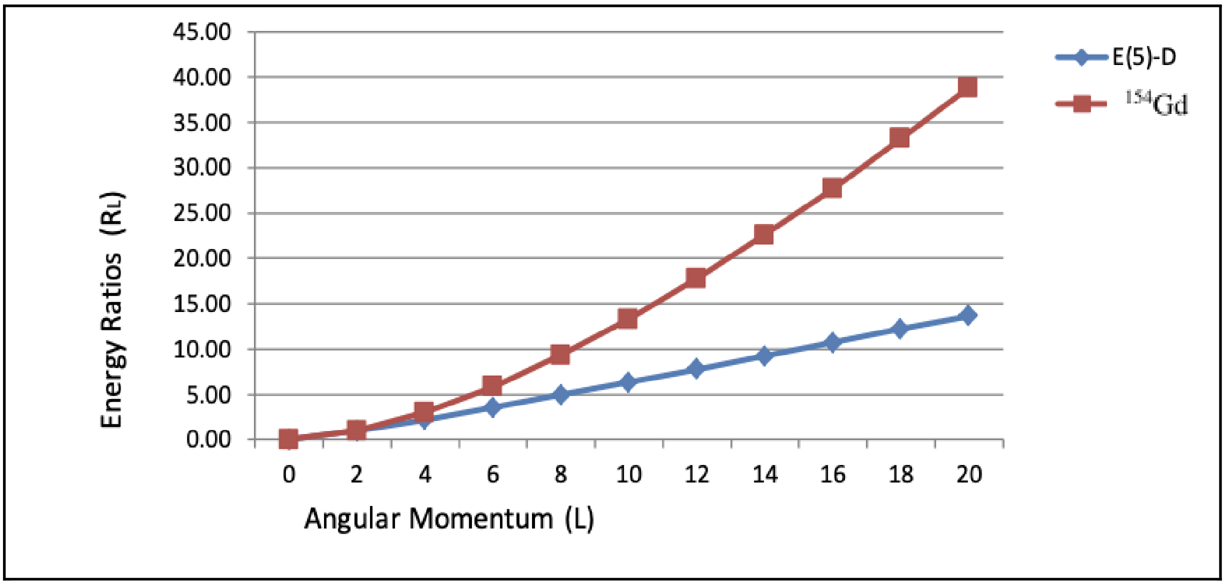

Figure 4.18: Comparison of the energy levels of the ground state band (gsb) of the ^{154}Gd nuclei with E(5)-D model at β_0 = 1.5.

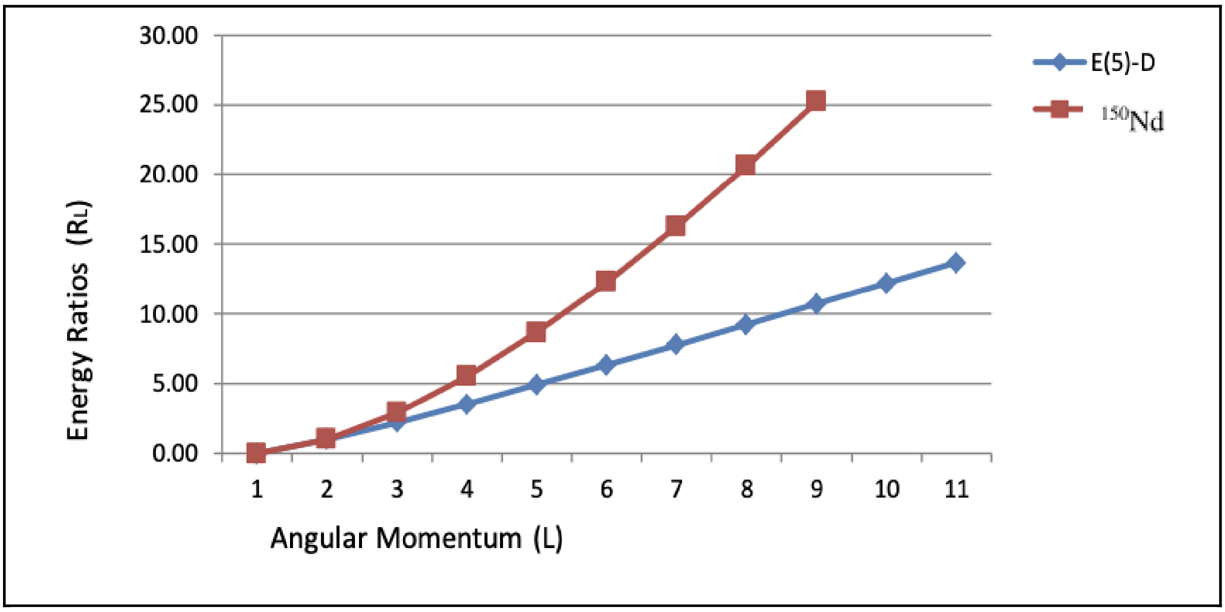

Figure 4.19: Comparison of the energy levels of the ground state band (gsb) of the ^{150}Nd nuclei with E(5)-D model at β_0 = 1.5.

For each of figures 4.16,17,18, and 19 for ^{156}Dy, ^{154}Sm, ^{154}Gd, and ^{150}Nd respectively are examined; It has been observed that for the value β_0 = 1.5, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of these nuclei, they overlap.

Now table (4.11) shows the results of gsb energies of E(5)-D model obtained from equation(3.39) at β_0=2.

n	?	β_o	$E_{n,L}$	L	R_L
0	0	2	5.272001873	0	0.00
0	1	2	5.716990566	2	1.00
0	2	2	6.315072906	4	2.34
0	3	2	7.020797289	6	3.93
0	4	2	7.800735254	8	5.68
0	5	2	8.632168761	10	7.55
0	6	2	9.5	12	9.50
0	7	2	10.39414711	14	11.51
0	8	2	11.30776406	16	13.56
0	9	2	12.23610253	18	15.65
0	10	2	13.17579566	20	17.76

Table 4.14: The energy ratio R_L of E(5)-D model at β_0 = 2

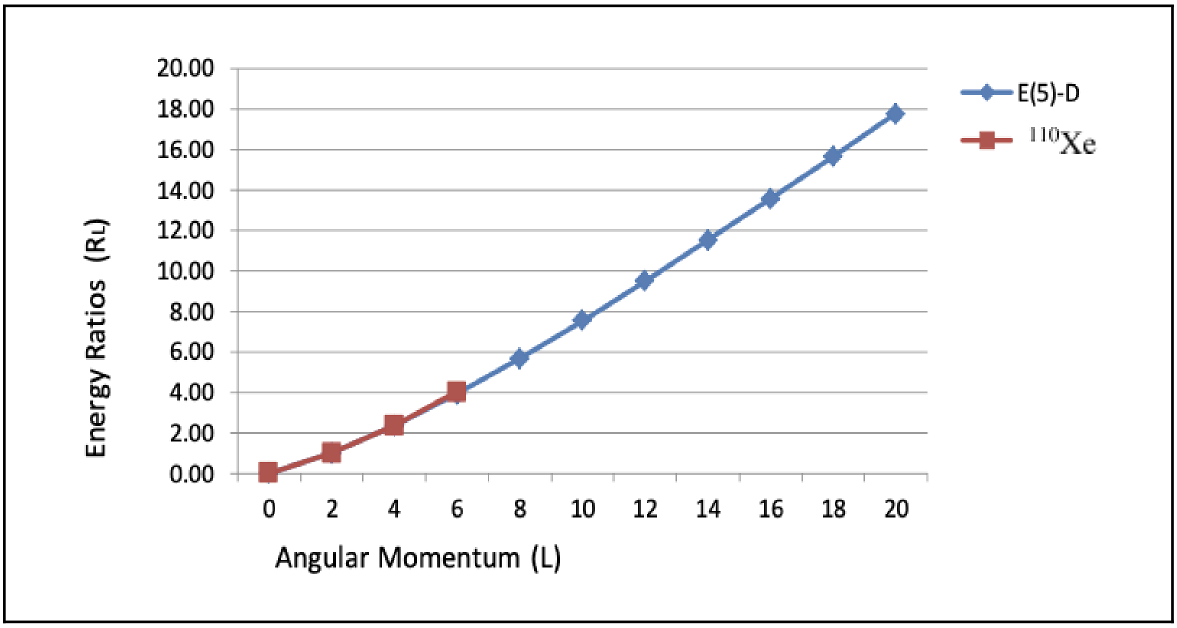

Figure 4.20: Comparison of the energy levels of the ground state band (gsb) of the ^{110}Xe nuclei with E(5)-D model at $\beta_0 = 2$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 2$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{110}Xe nucleus up to L = 6 angular momentum they match more compared to β_0=1and1.5 with the this model. Because; The ^{120}Xe isotope can be called the E(5)-D model nucleus.

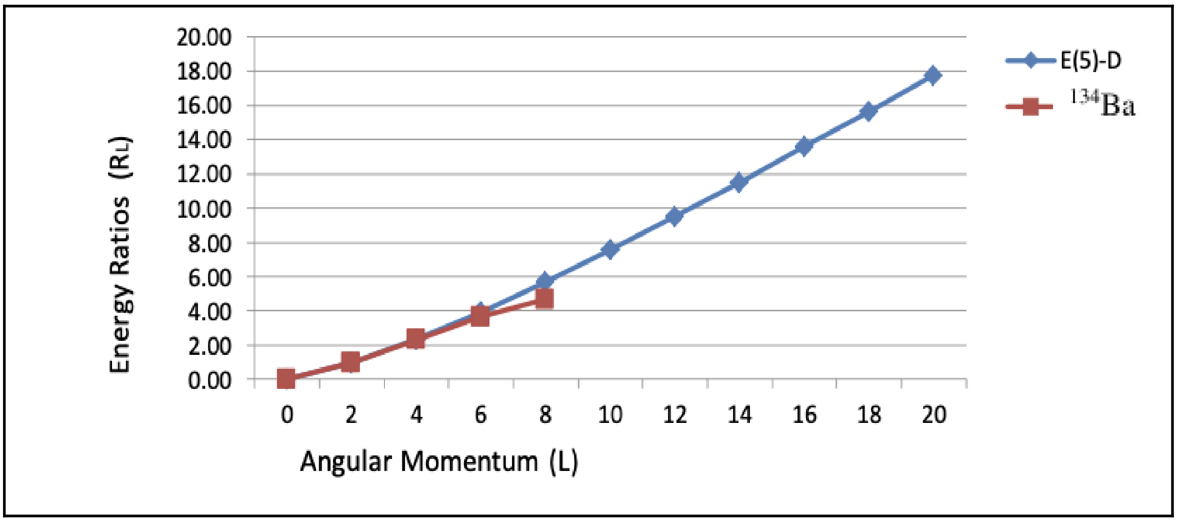

Figure 4.21: Comparison of the energy levels of the ground state band (gsb) of the ^{134}Ba nuclei with E(5)-D model at $\beta_0 = 2$.

When the figure above is examined; It has been observed that for the value $\beta_0 = 2$, the energy levels of the ground state band of the E(5)-D model are compatible with the experimental data of the ^{134}Ba nucleus up to L = 8 angular momentum they match exactly compared to β_0=1and1.5 with the this model. Because; The ^{134}Ba isotope can be called the E(5)-D model nucleus.

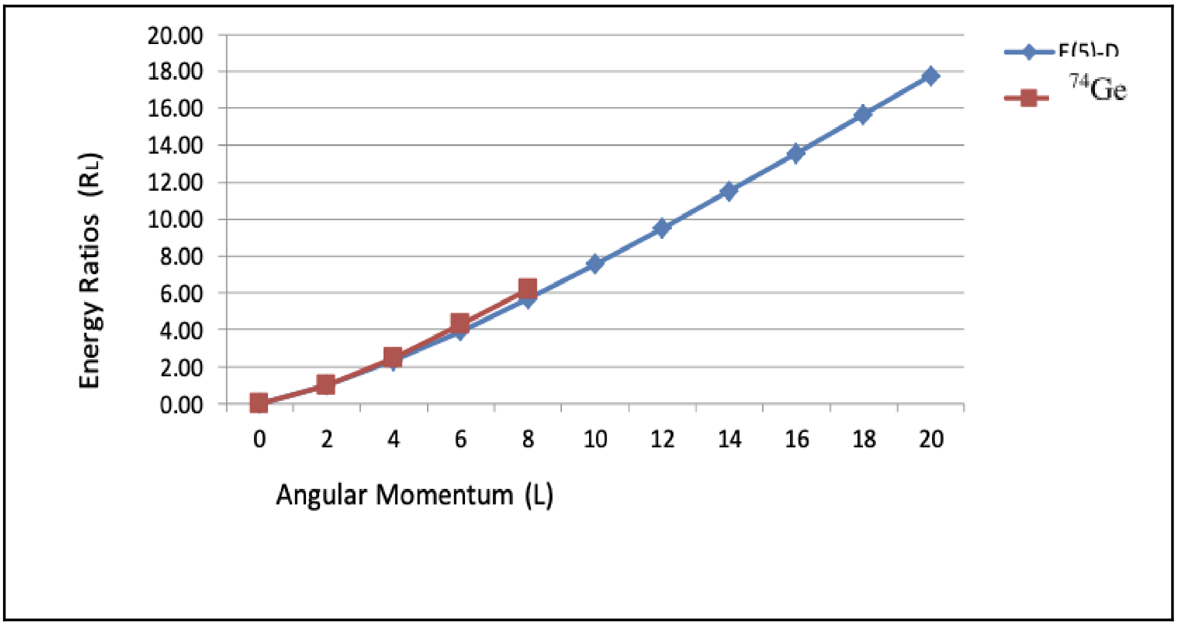

Figure 4.22: Comparison of the energy levels of the ground state band (gsb) of the ^{74}Ge nuclei with E(5)-D model at β_0 =2.

When the figure above is examined; It has been observed that for the value $\beta_0 = 2$, the energy levels of the ground state band of the E(5)-D model are compatible with the experimental data of the ^{74}Ge nucleus up to L = 6 angular momentum they match with the this model after that they overlap slightly. They are much closer than β_0 = 1 and 1.5.

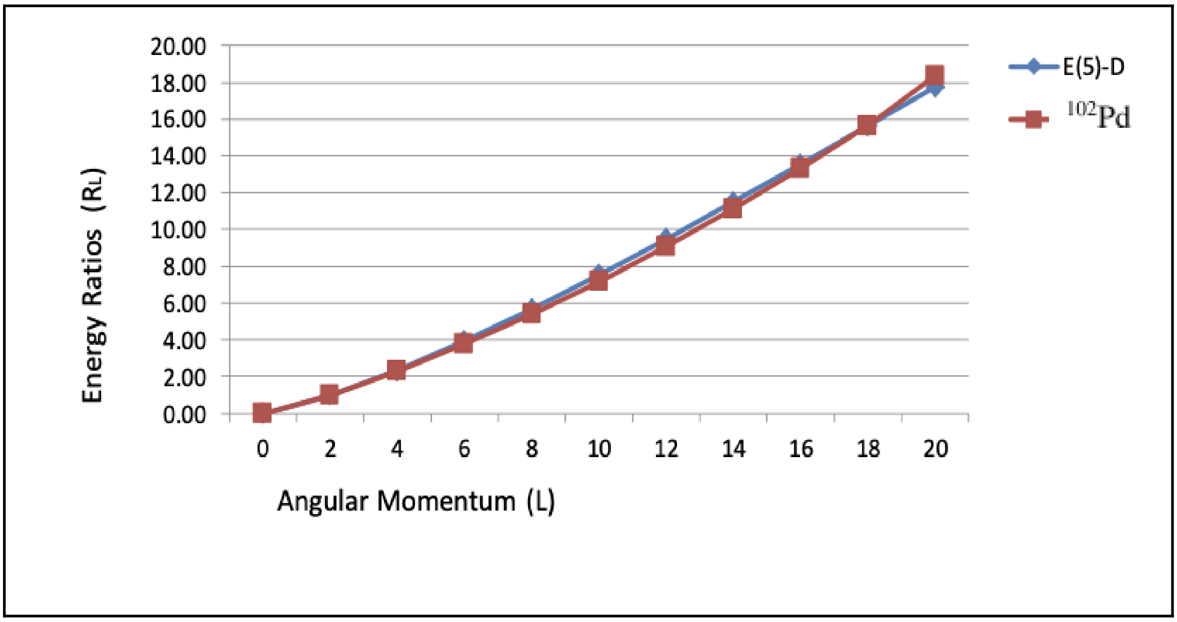

Figure 4.23: Comparison of the energy levels of the ground state band (gsb) of the ^{102}Pd nuclei with E(5)-D model at β_0 =2.

When the figure above is examined; It has been observed that for the value $\beta_0 = 2$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of the ^{102}Pd nucleus up to L = 12 angular momentum they match and are so close with this model compared to β_0=1and1.5, after that they overlap slightly.

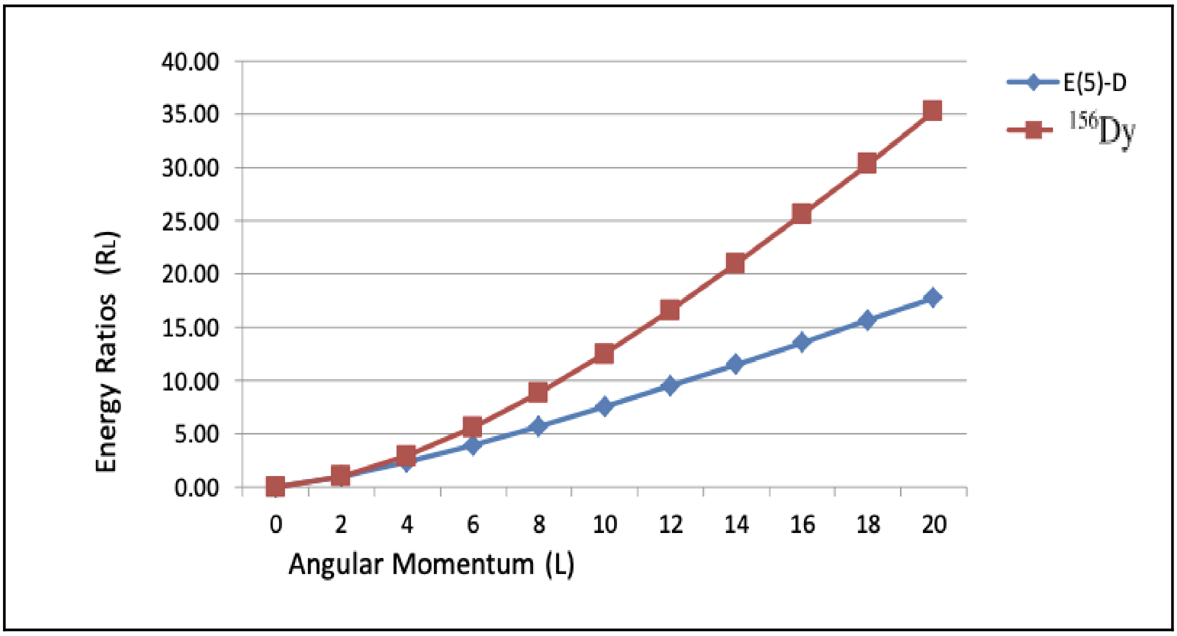

Figure 4.24: Comparison of the energy levels of the ground state band (gsb) of the ^{156}Dy nuclei with E(5)-D model at β_0 =2.

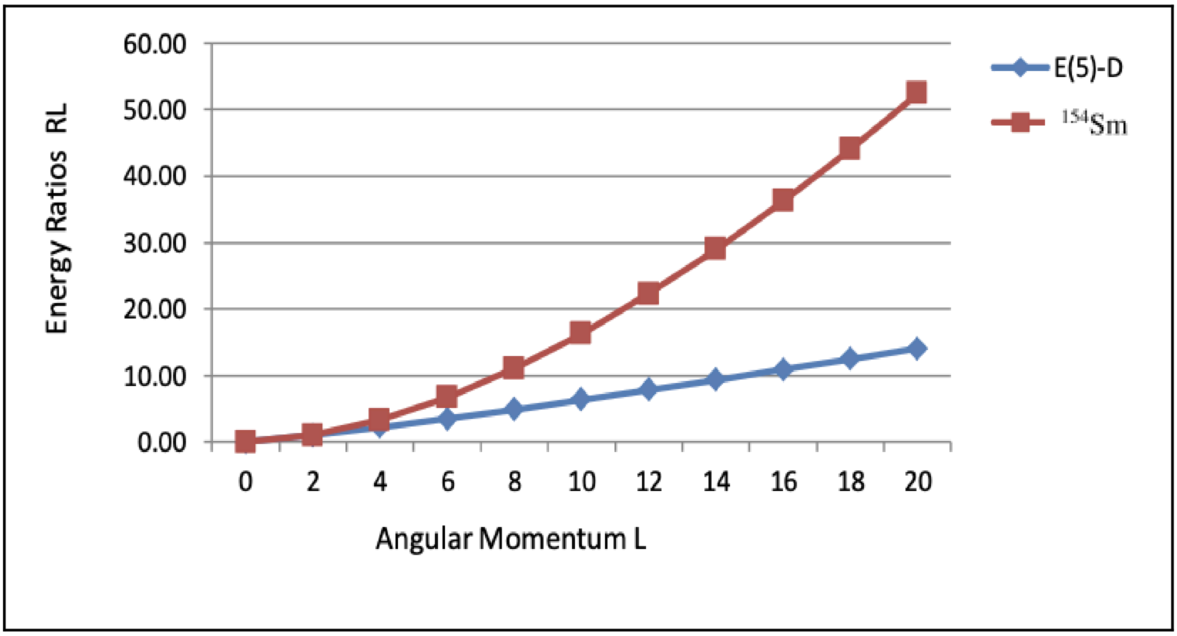

Figure 4.25: Comparison of the energy levels of the ground state band (gsb) of the ^{154}Sm nuclei with E(5)-D model at β_0 =2.

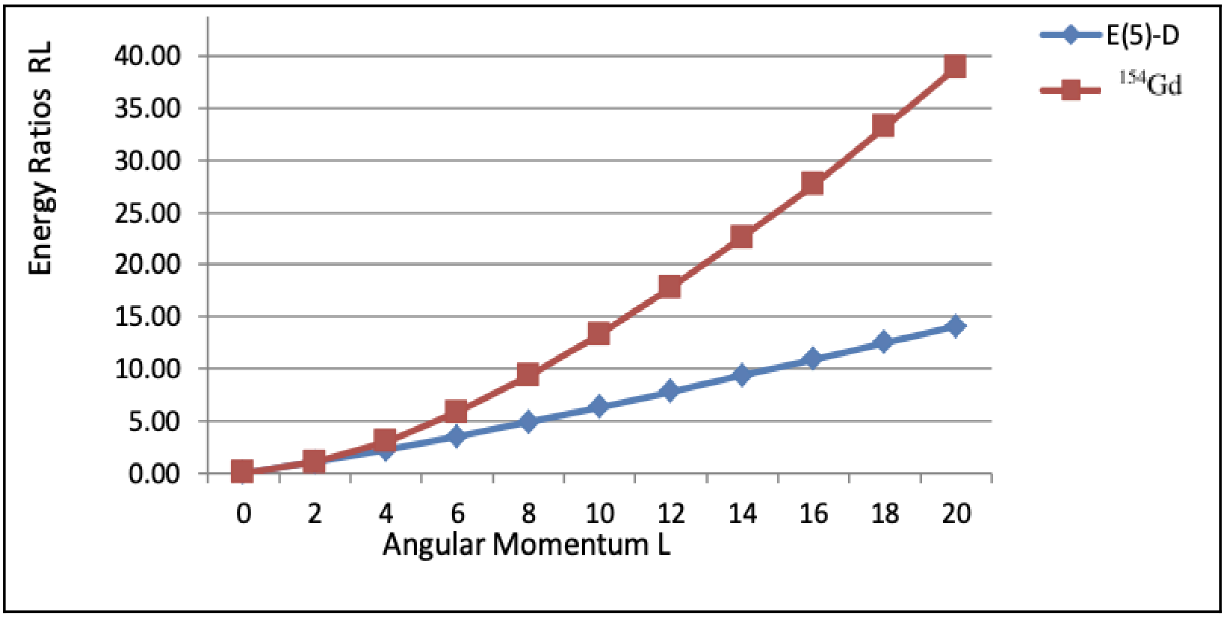

Figure 4.26: Comparison of the energy levels of the ground state band (gsb) of the ^{154}Gd nuclei with E(5)-D model at β_0 =2.

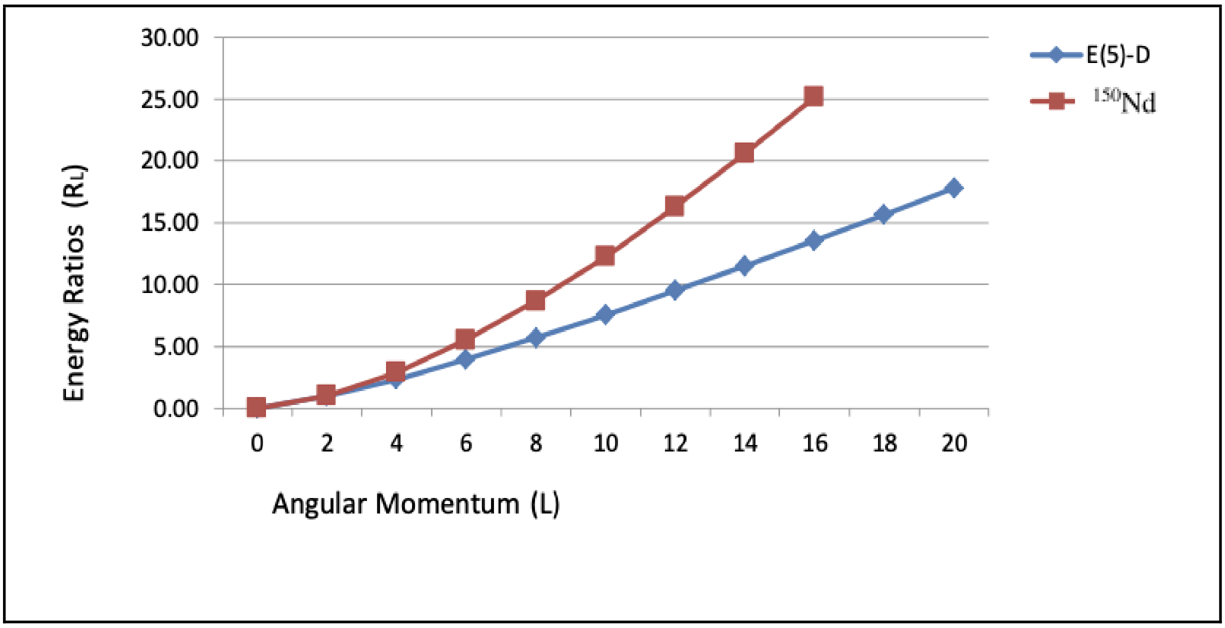

Figure 4.27: Comparison of the energy levels of the ground state band (gsb) of the ^{150}Nd nuclei with E(5)-D model at β_0 = 2.

For each of figures 4.24, 25, 26, and 27 for ^{156}Dy, ^{154}Sm, ^{154}Gd, and ^{150}Nd respectively are examined; It has been observed that for the value $\beta_0 = 2$, the energy levels of the ground level band of the E(5)-D model are compatible with the experimental data of these nuclei, they overlap.

5. CONCLUSION

In this study, the part of the Schrödinger equation depending on the β variable was solved for a Davidson potential by taking the Bohr Hamiltonian as γ = 0° and $\gamma \simeq$ 0°, and an analytical solution was obtained. The solution was called the E(5)-D model. Energy eigenvalues and wave functions were obtained using an analytical method developed by Nikiforov and Uvarov.

It is known that the E(5) model depends on only three variables and can be completely separated into its variables, while the solution E(5)-D of the Bohr-Hamiltonian equation may be shown, with particular emphasis on solving the part β with an inverse square probability. Using the PYTHON 3 Jupyter (data science) program for calculating Energy eigenvalues with angular momentum for ground state band of E(5)-D model , it obtained this solution works for some nuclei such as ^{110}Xe, ^{134}Ba and it won't for others such as ^{56}Dy, ^{154}Gd, ^{154}Sm, ^{102}Pd, ^{150}Nd and ^{74}Ge.

At β_0 = 1.5 the experimental data for ^{110}Xe and^{134}Ba show good agreement with E(5)-D model as it is shown in the figures.

At β_0 = 2 E(5)-D model predictions are in agreement to experimental values of the data are ^{110}Xe, ^{134}Ba, ^{74}Ge, and ^{102}Pd nuclei.

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