Non-associative Quantum Mechanics and Jordan Algebras

by

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Graduate School of Sciences and Engineering

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To my beloved husband,

my parents and my sister

ABSTRACT

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We present a Jordan algebraic formulation of the non-commutative Landau problem coupled to a harmonic potential. To achieve this, we use a new formulation of the Hilbert space version of quantum mechanics. Using this construction, the Hilbert space corresponding to the non-commutative Landau problem is obtained. For that problem, non-commutative parameters are described in terms of an associator in the Jordan algebraic setting. Pure states and density matrices arising from this problem are characterized. This in turn leads us to the Jordan-Schrödinger time-evolution equation for the state vectors for this specific problem.

ÖZETÇE

Birleşmeli Olmayan Kuantum Mekaniği ve Jordan Cebirleri
Ekin Sıla Yörük
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Bu tezde harmonik potansiyel etkisindeki değişmeli olmayan Landau probleminin Jordan cebirsel formülasyonu sunulmuştur. Bunun için kuantum mekaniğinin yeni bir Hilbert uzayı formülasyonu ortaya atılmıştır. Bu yaklaşım kullanılarak değişmeli olmayan Landau problemine karşılık gelen Hilbert uzayı oluşturulmuştur. Daha sonra değişmeli olmayan parametreler birleştirici işlem cinsinden Jordan cebirleri nezninde tanımlanmıştır. Bu problemden ortaya çıkan saf durumlar ve yoğunluk matrisleri de karakterize edilmiştir. Bu kuram bizi eldeki problemin durum vektörleri cinsinden Jordan-Schrödinger zaman evrimi denklemine götürmüştür.

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Chapter 1

INTRODUCTION

In the algebraic formulation of quantum mechanics, observables play a central role. The main motivation for this approach is to formulate quantum mechanics in terms of the essential ingredients only; which are observables, states, expectation values and their time evolution. In this approach which is initiated by Segal [Segal, 1947], observables are taken as the self-adjoint part of a C^*-algebra $\mathcal{A}$, states are positive linear functionals on $\mathcal{A}$, and the real number that a state assings to an observable is the expectation value of that observable in that state (See [Fano, 1957], [Sorkin, 1994], [Kempf A., 1995] for deeper formulations of quantum mechanics). But it turns out that the C^*-algebra we started with contains a lot of elements which are not self-adjoint. As a result of this, one needs a better structure describing the algebra of observables to start with.

Observables form an algebra under the following non-associative Jordan product

$$A \bullet B = \frac{1}{2}(AB + BA)$$

which qualifies as a real Jordan algebra since the Jordan product is commutative and it satisfies the Jordan identity

$$(A \bullet B) \bullet A^2 = A \bullet (B \bullet A^2)$$

which is equivalent to power-associativity of $\mathcal{A}$ when $\mathcal{A}$ is formally real [Jordan P., 1934]. Jordan algebras were originally introduced as a mathematical model of quantum mechanics motivated by the observation below [Jordan P., 1934]. Since the Jordan product can be expressed by squares only

$$A \bullet B = \frac{1}{4}((A + B)^2 - (A - B)^2),$$

the set of observables should be closed under Jordan multiplication. A Jordan algebra is called special if it comes from the Jordan product in an associative algebra, which is the case in our definition of Jordan algebra of observables.

Since the algebras arising in quantum mechanics are typically infinite dimensional, it is of interest to develop a structure theory compatible with the infinite dimensional case. This idea is initiated by [Alfsen E. M., 1978a], where the direct analogues of C^*-algebras that are called as JB-algebras are studied. JB-algebras are Jordan algebras with a Banach space structure. They satisfy the following conditions

$$\|A^2\| = \|A\|^2,$$
$$\|A^2 - B^2\| \leq \max\{\|A^2\|, \|B\|^2\},$$
$$\|A \bullet B\| \leq \|A\|\|B\|.$$

First condition, also known as the C^* condition, is a consequence of second condition (positivity condition) and third condition (an analogue of Banach space axiom). Therefore, the observables, which can be considered as the self adjoint part of a C^*-algebra, form a JC-algebra, in other words, a special JB-algebra. At this point, we can say that if one wants to formulate quantum mechanical problems using the essential ingredients only, it is important to write down a formulation using the Jordan algebraic approach. Also, we can note that all JB-algebras, with the exception of $H_3(\mathbb{O})$, can be seen as the self adjoint part of a C^*-algebra, so they can be represented on a complex Hilbert space [Alfsen E. M., 1978a] (See [Townsend P. K., 2016], [Günaydın, 1980], [Günaydın M., 1984], [Niestegge, 2020] for more detailed work on the exceptional character of $H_3(\mathbb{O})$, and Jordan algebras).

Apart from the Jordan algebras, we should also note other non-associative structures which have applications in physical problems [Günaydın M., Zumino B., 1985], [Jackiw, 1985], [Szabo, 2018], [Szabo R. J., 2019], [Mikelsson J., 2021], [Bojowald M., 2015]. The associative product used in quantum mechanics do not apply to some exotic systems such as magnetic monopoles, which lead to non-associative algebras. It turns out that if we consider dynamics of electrons in uniform distributions of magnetic charge, certain non-associative deformations of quantum mechanics and gravity in three dimensions are needed. The quantitative framework for non-associative quantum mechanics in this setting exhibits new effects compared to ordinary quantum mechanics with sourceless magnetic fields. On the other hand, the non-associativity of translations in a quantum system with magnetic field background has received renewed interest in association with topologically trivial gerbes over $\mathbb{R}^n$. The non-associativity arises when trying to lift the translation group action on the 1-particle system to the second quantized system, and it is described by a 3-cocycle of the group $\mathbb{R}^n$ with values in the unit circle S^1.

In this thesis, our main interest will be the Landau problem. From the physical point of view, Landau problem, an electron interacting a uniform magnetic field, is one of the most important problems in quantum mechanics, and it is inspired by different areas of physics such as quantum optics and condensed matter physics. The minimal coupling of the electron's charge density to the external magnetic field described by the electromagnetic vector potential $\boldsymbol{\mathcal{A}}$ leads to the Hamiltonian

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2 =: \frac{1}{2m}\boldsymbol{P}^2,$$

where $\boldsymbol{P}$ is the kinetical momenta. The Landau problem is related to the motion of a charged particle on the flat plane xy in the presence of a constant magnetic field along the z-axis. In metals, electrons occupy Landau levels given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right),$$

where each level being infinitely degenerate, with Larmour frequency $\omega_L = \frac{eB}{mc}$.

Since Landau problem has numerous applications in different areas of physics, it has been studied before by many different authors in the literature. In particular, Dereli, et al. [Dereli T., 2022] studied the Newton-Hooke duality between the $2D$ hydrogen atom and the Landau problem via the Tits-Kantor-Koecher construction of the conformal symmetries of the Jordan algebra of real symmetric 2×2 matrices.

Non-commuting spatial coordinates and fields can be seen in actual physical situations [Jackiw R., 2001]. Landau models and their quantum Hall limit have become an instrument of intense research activity as a physical realization of the simplest example of non-commutative geometries [Dunne G., 1993], [Dunne G., 1990], [Duval C., 2000], [Duval and Horváthy, 2001], [Horváthy, 2002], [Horváthy, 2006], [Zhang P. H., 2012], [Alvarez P.D., 2008], [Giri P. R., 2008], [Hounkonnou M. N., 2012]. Given an electron moving in a plane in the presence of a constant external magnetic field $\boldsymbol{B}$ perpendicular to that plane, the spectrum of the quantized theory gives infinitely degenerate Landau levels, with separation $\mathcal{O}\left(\frac{eB}{mc}\right)$. The limit $B \to \infty$ or $m \to 0$ projects onto the lowest Landau level. On each of the projected Landau levels, one obtains a non-commuting algebra for the space coordinates giving the center of the cyclotron motion,

$$[X, Y] = -\frac{i\hbar c}{eB} = i\theta \neq 0.$$

Similar structures also arise in the nonperturbative string theory or relativistic quantum theories of gravity. A common expectation for the purpose of unifying gravity with quantum mechanics involves replacing the classical geometry of spacetime by a non-commutative quantum geometry. The main motivation of this thesis is to formulate the non-commutative nature of spatial coordinates in terms of the non-associativity of observables based on the algebraic formulation of quantum mechanics.

In Chapter 2, we briefly review some aspects of operator algebras. In the first section, we will discuss the theory of C^*-algebras and von Neumann algebras, which leads us to Hilbert-Schmidt operator formulation of quantum mechanics. In order to go deeper in the theory of C^*-algebras, see [Bratelli O., 1987]. In the second section, we will present the theory of JB-algebras. This formulation is done by [Alfsen E. M., 2003]. The reader should keep in mind that the self-adjoint part of a C^*-algebra forms a JB-algebra, and the self-adjoint part of a von Neumann algebra forms a JBW-algebra. So, many results in the Jordan context are generalizations of the associative case.

In Chapter 3, we will formulate quantum mechanics using the Jordan algebraic approach. After a review of the usual formalism of quantum mechanics and the postulates given by von Neumann [Neumann, 1955], we give an axiomatic approach by [Emch, 1972] that quantum mechanics operates. In Section 3, Jordan algebraic Hilbert-space formulation, which is an analogue of Gelfand-Naimark-Segal construction, is presented .

In Chapter 4, the non-commutative Landau problem coupled to a harmonic potential will be discussed using the Jordan algebraic approach [Dereli T., Yörük E. S., 2024]. It turns out that, in this problem, one obtains a pair of commuting Hamiltonians, which can be written in terms of two pairs of mutually commuting creation and annihilation operators which are of oscillator type. These operators further generate two mutually commuting von Neumann algebras. Based on the Hilbert-Schmidt operator formulation of the non-commutative Landau problem, one can write down a Jordan algebraic formulation. We will take the associative case as our guideline and mention connections to the associative case whenever possible. At the end, we will derive a Schrödinger-like time evolution equation.

Chapter 2

OPERATOR ALGEBRAS

2.1 Basics on C^*-algebras

In this section, we will denote a Hilbert space over $\mathbb{C}$ by $\mathscr{H}$, and it is assumed to be separable of dimension N, which could be finite or infinite. We denote $\mathcal{B}(\mathscr{H})$ the C^* algebra of all bounded operators on $\mathscr{H}$.

Definition 2.1.1. *Let $\mathcal{A}$ be an algebra. A mapping $* : \mathcal{A} \to \mathcal{A}$, defined by $A \mapsto A^*$ for every $A \in \mathcal{A}$, is called an involution or adjoint operation of $\mathcal{A}$, if it satisfies the properties below:*

- $A^{**} = A$,
- $(AB)^* = B^*A^*$,
- $(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$,

for all $A, B, A^*, B^* \in \mathcal{A}$, and $\alpha, \beta \in \mathbb{C}$.

Definition 2.1.2. *An algebra with an involution is called a *-algebra and a subset $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is called self-adjoint if $A \in \tilde{\mathcal{A}}$ implies $A^* \in \tilde{\mathcal{A}}$.*

Definition 2.1.3. *An algebra $\mathcal{A}$ is called a normed algebra, if for any element $A \in \mathcal{A}$, there is associated a real number $\|A\|$, the norm of A, satisfying the following conditions*

- $\|A\| \geq 0$, *and* $\|A\| = 0$ *if and only if* $A = 0$,
- $\|\alpha A\| = |\alpha|\|A\|$,
- $\|A + B\| \leq \|A\| + \|B\|$,

- $\|AB\| \leq \|A\|\|B\|$,

for all $A, B \in \mathcal{A}$, *and* $\alpha \in \mathbb{C}$.

The norm defines a metric topology on $\mathcal{A}$, which is called as the uniform topology. ϵ-neighborhood of an element $A \in \mathcal{A}$ is defined as

$$N_\epsilon(A) = \{B \in \mathcal{A} : \|B - A\| < \epsilon\}.$$

If $\mathcal{A}$ is complete with respect to the uniform topology, then it is called a Banach space. A complete, normed algebra $\mathcal{A}$ with adjoint operation and having the property

$$\|A\| = \|A^*\|,$$

for all $A \in \mathcal{A}$ is called a Banach *-algebra.

Definition 2.1.4. *A* C^**-algebra* $\mathcal{A}$ *is a Banach *-algebra with the property*

$$\|A^*A\| = \|A\|^2$$

for all $A \in \mathcal{A}$.

In the theory of quantum mechanics, the concepts about representations and states play a central role. Therefore, before going further, we will try to understand those ideas.

Definition 2.1.5. *Let* $\mathcal{A}$ *and* $\tilde{\mathcal{A}}$ *be two *-algebras. The *-morphism between those algebras is the mapping* $\pi : A \in \mathcal{A} \to \pi(A) \in \tilde{\mathcal{A}}$ *satisfying the properties*

- $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,
- $\pi(AB) = \pi(A)\pi(B)$,
- $\pi(A^*) = \pi(A)^*$,

for all $A, B \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$.

Here, it is worth remarking that every *-automorphism is positive. If $A \in \mathcal{A}$ is positive, then $A = B^*B$ for some $B \in \mathcal{A}$. Hence

$$\pi(A) = \pi(B^*B) = \pi(B)^*\pi(B) \geq 0.$$

Definition 2.1.6. *A representation of a C^*-algebra $\mathcal{A}$ is defined to be a pair $(\mathscr{H}, \pi)$, where $\mathscr{H}$ is a complex Hilbert space, and π is a *-morphism of $\mathcal{A}$ into $\mathcal{B}(\mathscr{H})$. A representation is called faithful if and only if it satisfies each of the following equivalent conditions:*

- $ker\pi = \{0\}$,
- π *is a *-isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$,*
- $\|\pi(A)\| = \|A\|$ *for all $A \in \mathcal{A}$,*
- $\pi(A) > 0$ *for all $A > 0$.*

The equivalence of the above conditions can be shown using the proposition below.

Proposition 2.1.1. *Let $\mathcal{A}$ be a C^*-algebra, $\tilde{\mathcal{A}}$ be a Banach *-algebra with identity, and π a *-morphism of $\tilde{\mathcal{A}}$ into $\mathcal{A}$. Then, π is continuous and*

$$\|\pi(A)\| \leq \|A\|$$

for all $A \in \tilde{\mathcal{A}}$. Moreover, if $\tilde{\mathcal{A}}$ is a C^*-algebra, then the range of π is a C^*-subalgebra of $\mathcal{A}$.

Definition 2.1.7. *Let A be an element of a Banach algebra $\mathcal{A}$ with identity, the spectral radius $\rho(A)$ of A is defined as follows*

$$\rho(A) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(A)\},$$

where $\sigma_{\mathcal{A}}(A)$ is the spectrum of A. Spectrum is defined as the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda\mathbb{I})$ does not have an inverse.

Remark. *It turns out that*

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n} \leq \|A\|,$$

and the spectrum of A is a nonempty compact set.

Definition 2.1.8. *A cyclic representation of a C^*-algebra $\mathcal{A}$ is defined to be a triplet $(\mathscr{H}, \pi, \Omega)$, where $(\mathscr{H}, \pi)$ is a representation of $\mathcal{A}$, and Ω is a cyclic vector for π. Since Ω is a cyclic vector for π, $\{\pi(a)\Omega : a \in \mathcal{A}\}$ forms a dense set. If $\mathscr{R}$ is a closed subspace of $\mathscr{H}$, then $\mathscr{R}$ is called a cyclic subspace for $\mathscr{H}$ whenever the set*

$$\Big\{ \sum_i \pi(A_i)\psi_i : A_i \in \mathscr{H}, \psi_i \in \mathscr{R} \Big\}$$

is dense in $\mathscr{H}$.

Definition 2.1.9. *A linear functional ω over the C^*-algebra $\mathcal{A}$ is defined to be positive if*

$$\omega(A^*A) \geq 0$$

for all $A \in \mathcal{A}$. A positive, unit norm, linear functional over a C^*-algebra is called a state.

Here, we should note that every positive element of a C^*-algebra is of the form A^*A, and hence positivity of a state ω is equivalent to ω being positive on positive elements of $\mathcal{A}$.

Definition 2.1.10. *Consider a representation $(\mathscr{H}, \pi)$ of a C^*-algebra $\mathcal{A}$, take a nonzero vector $\Omega \in \mathscr{H}$ and define ω_Ω by*

$$\omega_\Omega(A) = (\Omega, \pi(A)\Omega)$$

for all $A \in \mathcal{A}$. It turns out that ω_Ω is a positive, linear functional over $\mathcal{A}$ since

$$\omega_\Omega(A^*A) = \|\pi(A)\Omega\|^2 \geq 0.$$

It can be shown that $\|\Omega\| = 1$ implies $\|\omega_\Omega\| = 1$, and π is non-degenerate which means $\{\pi(a)h : a \in \mathcal{A}, h \in \mathscr{H}\}$ is dense in $\mathscr{H}$. In this case, ω_Ω is called a vector state for the given representation.

Definition 2.1.11. *The cyclic representation* $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$*, constructed from the state* ω *over the* C^**-algebra* $\mathcal{A}$ *is called the canonical cyclic representation of* $\mathcal{A}$ *associated with* ω*.*

Definition 2.1.12. *A state* ω *over a* C^**-algebra is defined to be pure if for any positive linear functional* $\psi : \mathcal{A} \to \mathbb{C}$ *with* $\psi \leq \omega$*, there exists some* $\lambda \in (0, 1]$ *such that* $\omega = \lambda\psi$*.*

Theorem 2.1.1. *Let* ω *be a state over the* C^**-algebra* $\mathcal{A}$ *and* $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ *be the associated cyclic representation. Then the following conditions are equivalent:*

- $(\mathscr{H}_\omega, \pi_\omega)$ *is irreducible,*
- ω *is pure,*
- ω *is an extreme point of the set of states over* $\mathcal{A}$*.*

2.1.1 von Neumann Algebras

From now on, we will provide some notions related to von Neumann algebras. In order to stress the Hilbert space which the von Neumann algebra $\mathcal{A}$ acts, one can use the notation $\{\mathcal{A}, \mathscr{H}\}$ in the theory of von Neumann algebras. Let $\mathscr{H}$ be a Hilbert space, for each subset $\mathcal{A}$ of $\mathcal{B}(\mathscr{H})$, we denote $\mathcal{A}'$ by the set of all bounded operators on $\mathscr{H}$ commuting with every element of $\mathcal{A}$. Note that $\mathcal{A}'$ is a Banach algebra of operators containing the identity operator $\mathbb{I}$ on $\mathscr{H}$. Furthermore, $\mathcal{A}'$ is called the commutant of $\mathcal{A}$.

Definition 2.1.13. *A von Neumann algebra is a *-subalgebra* $\mathcal{A}$ *of* $\mathcal{B}(\mathscr{H})$ *such that*

$$\mathcal{A} = \mathcal{A}''.$$

$\mathcal{A}$ *is a* C^**-algebra acting on the Hilbert space* $\mathscr{H}$ *that is closed under the weak operator topology, which means*

$$\lim_{n\to\infty} A_n = A$$

if and only if

$$\lim_{n\to\infty} \langle \xi | A_n \eta \rangle = \langle \xi | A\eta \rangle ,$$

for all $\xi, \eta \in \mathscr{H}$*. Last statement can be equivalently formulated under the* σ*-weak topology as follows:*

$$\lim_{n\to\infty} A_n = A$$

if and only if for all sequences (ξ_k), (η_k) *in* $\mathscr{H}$ *such that*

$$\sum_{k=1}^{\infty} \|\xi_k\|^2$$

and

$$\sum_{k=1}^{\infty} \|\eta_k\|^2$$

are finite, we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \langle \xi_k | A_n \eta_k \rangle = \sum_{k=1}^{\infty} \langle \xi_k | A\eta_k \rangle .$$

Remark. *A von Neumann algebra* $\mathcal{A}$ *always contains the identity operator* $\mathbb{I}_{\mathscr{H}}$*, and* $\mathcal{A}$ *is called a factor if*

$$\mathcal{A} \cap \mathcal{A}' = \mathbb{C}\mathbb{I}_{\mathscr{H}}.$$

It is very important to note that $\mathcal{B}(\mathscr{H})$, which is the set of all bounded operators on $\mathscr{H}$, is a von Neumann algebra and a factor. The Hilbert space involution defines an adjoint operation on $\mathcal{B}(\mathscr{H})$, and C^*-norm property follows from

$$\|A\|^2 = \sup_{x\in\mathscr{H}-\{0\}} \frac{\|Ax\|^2_{\mathscr{H}}}{\|x\|^2_{\mathscr{H}}} = \sup_{x\in\mathscr{H}-\{0\}} \frac{\langle Ax|Ax\rangle}{\|x\|^2_{\mathscr{H}}}$$

$$= \sup_{x\in\mathscr{H}-\{0\}} \frac{\langle A^*Ax|x\rangle}{\|x\|^2_{\mathscr{H}}} \leq \sup_{x\in\mathscr{H}-\{0\}} \frac{\|A^*Ax\|_{\mathscr{H}}\|x\|_{\mathscr{H}}}{\|x\|^2_{\mathscr{H}}} = \|A^*\|\|A\|$$

for all $A \in \mathcal{B}(\mathscr{H})$.

Definition 2.1.14. *Let* $\mathcal{A}$ *be a von Neumann algebra, and* $\phi : \mathcal{A} \to \mathbb{C}$ *be a bounded linear functional on* $\mathcal{A}$*, which is denoted by* $\langle \phi; A\rangle$*,* $A \in \mathcal{A}$*,* ϕ *is called a state on this algebra if it satisfies the following conditions:*

- $\langle \phi; A^*A\rangle \geq 0$*, for all* $A \in \mathcal{A}$*,*

- $\langle \phi ; \mathbb{I}_{\mathscr{H}} \rangle = 1$.

The state ϕ is called a vector state if there exists a vector $\Phi \in \mathscr{H}$ with

$$\langle \phi ; A \rangle = \langle \Phi | A \Phi \rangle ,$$

for all $A \in \mathcal{A}$. Such a state is also a normal state.

A state ω on a von Neumann algebra $\mathcal{A}$ is faithful if $\omega(A)$ is positive whenever $A \in \mathcal{A}$ is positive.

Remark. *Let $\mathscr{H}$ be a separable Hilbert space, then every normal state ω over $\mathcal{B}(\mathscr{H})$ is of the form*

$$\omega(A) = tr(\rho A),$$

where ρ is a density matrix which means ρ is positive and it is trace-class. It turns out that ρ is invertible if and only if ω is faithful.

Proposition 2.1.2. *Let tr be the usual trace on $\mathcal{B}(\mathscr{H})$, and $\mathcal{T}(\mathscr{H})$ be the Banach space of trace-class operators on $\mathscr{H}$ with the trace norm. Then it follows that $\mathcal{B}(\mathscr{H})$ is the dual $\mathcal{T}(\mathscr{H})^*$ of $\mathcal{T}(\mathscr{H})$ by the duality*

$$A \times T \in \mathcal{B}(\mathscr{H}) \times \mathcal{T}(\mathscr{H}) \to tr(AT).$$

It would be useful to close this section with the following theorem which will be very important in the theory of quantum mechanics is the following chapters.

Theorem 2.1.2. *Let ω be a state on a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathscr{H}$. The following conditions are equivalent:*

- *ω is normal,*

- *ω is σ-weakly continuous, i.e. there exists sequences $\{\xi_n\}$, $\{\eta_n\}$ of vectors in $\mathscr{H}$ such that*
$$\sum_{n=1}^{\infty} \|\xi_n\|^2$$

and

$$\sum_{n=1}^{\infty} \|\eta_n\|^2$$

are finite, and

$$\omega(A) = \sum_{n=1}^{\infty} \langle \xi_n | A\eta_n \rangle ,$$

- *there exists a density matrix ρ, a positive, trace-class operator on $\mathscr{H}$, with $tr(\rho) = 1$ such that*

$$\omega(A) = tr(\rho A),$$

for any $A \in \mathcal{A}$.

2.2 Jordan-Banach Algebras

In this section, we will give the definition and basic properties of JB-algebras and JBW-algebras. Physically motivating example for a JB-algebra is a JC-algebra, which is the norm closed, real, linear space of self-adjoint operators on a Hilbert space $\mathscr{H}$, it is closed under the symmetric product

$$A \bullet B = \frac{1}{2}(AB + BA). \tag{2.1}$$

This product is called the Jordan product. It is bilinear, commutative, and non-associative.

Definition 2.2.1. *A Jordan algebra over $\mathbb{R}$ is a vector space $\mathcal{A}$ over $\mathbb{R}$ equipped with a commutative, bilinear product $\bullet$ which satisfies the Jordan identity*

$$(A^2 \bullet B) \bullet A = A^2 \bullet (B \bullet A) \tag{2.2}$$

for all $A, B \in \mathcal{A}$.

Here, we should note that if $\mathcal{A}$ is any associative algebra over $\mathbb{R}$ or any subspace of it which is closed under $\bullet$, then $\mathcal{A}$ becomes a Jordan algebra with the symmetrized product (2.1). Furthermore, Jordan algebras which are isomorphic to such algebras are called as the special Jordan algebras. If a Jordan algebra can not be embedded to an associative algebra over $\mathbb{R}$, then it is called exceptional. If $\mathcal{A}$ is a C^*-algebra, then the self-adjoint part of $\mathcal{A}$, which can be denoted as $\mathcal{A}_{sa}$, forms a special Jordan algebra.

In any Jordan algebra, powers are defined recursively

$$A^1 = A, \qquad A^n = A^{n-1} \bullet A.$$

It is important to know whether associative law holds for powers of a single element or not. As an answer to this problem, Jordan, Neumann and Wigner [Jordan P., 1934] proved the following result

Theorem 2.2.1. *Assuming formally reality (positivity) for a Jordan algebra $\mathcal{A}$, which means the following implication*

$$A^2 + B^2 + \cdots = 0 \implies A = B = \cdots = 0,$$

for all $A, B, \cdots \in \mathcal{A}$. Following proporties are equivalent

$$(i)\ A^n A^m = A^{n+m},$$

$$(ii)\ A^2 \bullet (B \bullet A) = (A^2 \bullet B) \bullet A.$$

The classification of formally real, unital, finite dimensional Jordan is also done by Jordan, Neumann and Wigner [Jordan P., 1934].

Theorem 2.2.2. *Every finite dimensional, formally real Jordan algebra is a direct sum of a finite number of simple ideals, and there are five basic types of simple algebras: Four types of hermitian matrix algebras corresponding to the four composition division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, together with the spin factors. Every finite dimensional, formally real, simple Jordan algebra is isomorphic to one of*

$$H_n^1 = \mathcal{H}_n(\mathbb{R}), \quad H_n^2 = \mathcal{H}_n(\mathbb{C}), \quad H_n^4 = \mathcal{H}_n(\mathbb{H}), \quad H_3^8 = \mathcal{H}_3(\mathbb{O}), \quad JSpin_n,$$

where H_n^k is the $n \times n$ Hermitian matrices over the k-dimensional real, composition, division algebra.

$H_3^8 = \mathcal{H}_3(\mathbb{O})$, Hermitian 3×3 matrices with entries in the octonions, is called as the Albert algebra and Albert [Albert, 1934] showed that $H_3^8 = \mathcal{H}_3(\mathbb{O})$ is an exceptional Jordan algebra of dimension 27. Later, Zel'manov [Zel'manov, 1979] gives a full treatment of this exceptional theorem. He showed that finite dimensional Albert algebras are the only simple, exceptional Jordan algebras.

Since the algebras used in quantum mechanics are most of the time infinite dimensional, it is of interest to develop a structure theory for infinite dimensional Jordan algebras which act like Jordan algebras of self-adjoint operators on a Hilbert space. This problem is solved by Alfsen, Schultz and Størmer [Alfsen E. M., 2003] with the introduction of Jordan-Banach algebras.

Definition 2.2.2. *A Jordan-Banach algebra, which is abbreviated as JB-algebra, is a Jordan algebra $\mathcal{A}$ over $\mathbb{R}$ with identity element $\mathbb{I}$ equipped with a complete norm satisfying the following requirements for $A, B \in \mathcal{A}$*

$$\|A \bullet B\| \leq \|A\|\|B\|, \tag{2.3}$$

$$\|A^2\| = \|A\|^2, \tag{2.4}$$

$$\|A^2\| \leq \|A^2 + B^2\|. \tag{2.5}$$

The self-adjoint part of a C^*-algebra is a JB-algebra with the Jordan product defined in (2.1). (2.3), (2.4), and (2.5) follow from the properties of a C^*-algebra. Furthermore, any norm closed Jordan subalgebra of self-adjoint part of $\mathcal{B}(\mathscr{H})$, which is denoted as $\mathcal{B}(\mathscr{H})_{sa}$, is a JB-algebra.

Definition 2.2.3. *A JC-algebra is a JB-algebra $\mathcal{A}$ that is isomorphic to a norm closed Jordan subalgebra of $\mathcal{B}(\mathscr{H})_{sa}$. If $\mathcal{A}$ is a norm closed Jordan subalgebra of $\mathcal{B}(\mathscr{H})_{sa}$, we call it as a JC-algebra acting on $\mathscr{H}$.*

It turns out that a finite dimensional JB-algebra is a JC-algebra if and only if it does not contain Albert algebra as a direct summand. In infinite dimensions, each JB-algebra $\mathcal{A}$ admits an exceptional ideal $\mathcal{J}$ such that $\mathcal{A}/\mathcal{J}$ is a JC- algebra.

Theorem 2.2.3 (Gelfand-Naimark theorem)**.** *Every JB-algebra can be isometrically, isomorphically embedded in some $\mathcal{B}(\mathscr{H})_{sa} \oplus \mathcal{C}(X, \mathcal{H}_3(\mathbb{O}))$, a direct sum of a JC-algebra of all self-adjoint operators on some real Hilbert space, and the purely exceptional algebra of all continuous Albert algebra valued functions on some compact topological space X.*

It turns out that every JB-algebra is not only a Banach space but it is also an ordered linear space, with the positive cone being the set of squares.

Definition 2.2.4. *An order unit space is an ordered normed linear space $\mathcal{A}$ over $\mathbb{R}$ with a closed positive cone and an element e, satisfying*

$$\|a\| = \inf\{\lambda > 0 : -\lambda e \leq a \leq \lambda e\}.$$

The element e is called the distinguished order unit.

With the ordering mentioned above, every Jordan-Banach algebra becomes an order unit space. Jordan algebras initially proposed as a model for quantum mechanics. However, the fundamental axiom (2.2) has no obvious physical interpretation. Therefore, von Neumann [von Neumann, 1936], Iochum and Loupias [Iochum and Loupias, 1985],[Iochum and Loupias, 1991] proposed to replace (2.2) by power associativity. This idea led to the notion of Banach power associative algebra, which is also called as an order unit algebra.

Definition 2.2.5. *A normed algebra $\mathcal{A}$ is an algebra with a norm such that*

$$\|AB\| \leq \|A\|\|B\|$$

for all $A, B \in \mathcal{A}$. Note that it is not necessarily associative. An order unit space $\mathcal{A}$ which is also a power associative, complete, normed algebra and satisfing the following properties: the distinguished order unit 1 is a multiplicative identity, $A^2 \in \mathcal{A}^+$ for all $A \in \mathcal{A}$, is called an ordered unit algebra.

Theorem 2.2.4. *Every JB-algebra $\mathcal{A}$ is a commutative order unit algebra with the given norm and positive cone*

$$\mathcal{A}^2 = \{a^2 : a \in \mathcal{A}\}.$$

In addition to this, every commutative, order unit algebra forms a JB-algebra.

For each element $A \in \mathcal{A}$, where $\mathcal{A}$ is a JB-algebra, we denote the norm closed subalgebra generated by A and 1 by $C(A, 1)$, it turns out that $C(A, 1)$ is associative. The following proposition turns out to be vital in the spectral theory of JB-algebras.

Proposition 2.2.1. *Let $\mathcal{A}$ be a JB-algebra and $\mathcal{B} = C(A, 1)$ for some $A \in \mathcal{A}$, then $\mathcal{B}$ is isometrically isomorphic to $C_{\mathbb{R}}(X)$, norm closed subalgebra generated by X, for some compact Hausdorf space X.*

In any Jordan algebra, we define the triple product as follows

$$\{ABC\} = (A \bullet B) \bullet C + (B \bullet C) \bullet A - (A \bullet C) \bullet B.$$

Observe that triple product is linear in each factor, and $\{ABC\} = \{CBA\}$. In a special Jordan algebra,

$$\{ABC\} = \frac{1}{2}(ABC + CBA),$$

where right hand side is written with associative product. It turns out that the special case $\{ABA\}$ plays an important role in the theory of Jordan algebras. We define $U_A : \mathcal{A} \to \mathcal{A}$ by

$$U_A(B) = \{ABA\} = 2A \bullet (A \bullet B) - A^2 \bullet B.$$

For a special Jordan algebra, we have $U_A(B) = ABA$. The following identities show that triple product $\{ABA\}$ acts like the associative product ABA, and they are valid in any Jordan algebra

$$\{\{ABA\}C\{ABA\}\} = \{A\{B\{ACA\}B\}A\}, \tag{2.6}$$

$$\{BAB\}^2 = \{B\{AB^2A\}B\}. \tag{2.7}$$

Note that (2.7) can be rewritten as

$$U_{\{ABA\}} = U_A U_B U_A.$$

For a JC-algebra (2.6) and (2.7) can be checked easily since $\{ABA\} = ABA$. The famous MacDonald's theorem states that most Jordan identities valid for special Jordan algebras are valid for all Jordan algebras.

Theorem 2.2.5 (MacDonald). *Any polynomial Jordan identity with three or fewer variables, possibly involving the identity* 1*, which is of degree at most* 1 *in one of the variables and which holds for all special Jordan algebras will hold for all Jordan algebras.*

Definition 2.2.6. *An element* A *of a JB-algebra* $\mathcal{A}$ *is invertible if it is invetible in the Banach algebra* $C(A, 1)$*. Its inverse is denoted as* A^{-1}*.*

Proposition 2.2.2. *Let* $\mathcal{A}$ *be an associative algebra, consider associated Jordan algebra under the Jordan product (2.1). Then for* $A, B \in \mathcal{A}$*,* $AB = BA = 1$ *if and only if*

$$A \bullet B = 1 \text{ and } A^2 \bullet B = A. \tag{2.8}$$

If elements A, B in a Jordan algebra satisfy (2.8), we say that A and B are Jordan invertible, and B is the Jordan inverse of A. It turns out that the invertibility in the Jordan algebraic setting is equivalent to the invertibility in Definition (2.2.6)

Proposition 2.2.3. *For elements* $A, B \in \mathcal{A}$*, where* $\mathcal{A}$ *is a Jordan algebra, the following are equivalent:*

- B *is the inverse of* A *in the Banach algebra* $C(A, 1)$
- B *is the Jordan inverse of* A *in* $\mathcal{A}$*.*

Definition 2.2.7. *If* A *is an element of a JB-algebra* $\mathcal{A}$*, the spectrum of* A*, denoted as* $\sigma(A)$*, is the set of all* $\lambda \in \mathbb{R}$ *such that* $\lambda 1 - A$ *is not invertible.*

Definition 2.2.8. *Let* $\mathcal{A}$ *be a JB-algebra. Positive elements* $A, B \in \mathcal{A}$ *are orthogonal if* $\{ABA\} = \{BAB\} = 0$*.*

2.2.1 JBW-algebras

JBW-algebras are the Jordan analogs of von Neumann (W^*) algebras, and they include the self-adjoint part of von Neumann algebras as a special case. In this section, we will develop basic facts about JBW-algebras.

Definition 2.2.9. *An ordered Banach space $\mathcal{A}$ is monotone complete if each increasing net $\{A_\alpha\}$ which is bounded above has a least upper bound $A \in \mathcal{A}$. A bounded linear functional σ on a monotone complete space $\mathcal{A}$ is normal if whenever $\{A_\alpha\}$ is such a net, then $\sigma(A_\alpha) \to \sigma(A)$.*

Definition 2.2.10. *A JBW-algebra is a JB-algebra that is monotone complete and admits a separating set of normal states.*

The definition above is inspired by Kadison's characterization of W^*-algebras as those C^*-algebras whose self-adjoint parts are monotone, complete and admit a separating set of normal states. Self-adjoint part of a W^*-algebra with the Jordan product (2.1) is an example of a JBW-algebra.

Definition 2.2.11. *Let $\mathcal{A}$ be a JBW-algebra, $\mathcal{K}$ be its normal state space, and $\mathcal{V}$ be the linear span of $\mathcal{K}$. The topology on $\mathcal{A}$ defined by the duality of $\mathcal{A}$ and $\mathcal{V}$ is the σ-weak topology. The σ-weak topology is defined by the set of seminorms $A \to |\rho(A)|$ for $\rho \in \mathcal{K}$. The σ-strong topology on $\mathcal{A}$ is defined by the set of seminorms $A \to \rho(A^2)^{1/2}$ for $\rho \in \mathcal{K}$. Therefore, $A_\alpha \to A$ σ-strongly if $\rho((A_\alpha - A)^2) \to 0$ for all $\rho \in \mathcal{K}$.*

Here, we should note that the σ-weak and σ-strong topologies defined above coincide with the topologies defined in the von Neumann algebra setting.

Lemma 2.2.1. *Let $\mathcal{A}$ be a commutative, order unit algebra that is the dual of a base norm space V such that the multiplication in $\mathcal{A}$ is separately w^*- continuous. Then for each $a \in \mathcal{A}$ and each $\epsilon > 0$ there are orthogonal projections $p_1, p_2, \dots, p_n$ in the w^*-closed subalgebra generated by a and 1, and scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that*

$$\|a - \sum_{i=1}^{n} \lambda_i p_i\| < \epsilon.$$

Using Theorem 2.46 of [Alfsen E. M., 2003], and the above lemma, one can show the following theorem.

Theorem 2.2.6. *Let $\mathcal{A}$ be a commutative order unit algebra which is a JB-algebra by Theorem 2.2.4. Then the set of finite linear combinations of orthogonal projections is norm dense in $\mathcal{A}^{**}$, here $\mathcal{A}^{**}$ denotes the bidual of $\mathcal{A}$, which is a JBW-algebra.*

One can work with normal, faithful, semi-finite traces on a JBW algebra $\mathcal{A}$ [Iochum, 2006]. Trace on a Jordan algebra is defined as a weight on $\mathcal{A}$, and it satisfies the cyclicity condition

$$tr(A \bullet (B \bullet C)) = tr((A \bullet B) \bullet C).$$

Also for positive elements $A, B \in \mathcal{A}$, and positive real numbers λ, we have

$$tr(A + B) = trA + trB,$$

$$tr\lambda A = \lambda trA.$$

Chapter 3

JORDAN ALGEBRAIC FORMULATION OF QUANTUM MECHANICS

3.1 Quantum Mechanics

Dirac's [Dirac, 1947] exposition of the fundamentals of quantum mechanics form the traditional framework of quantum mechanics, von Neumann [Neumann, 1955] also formulated these ideas with a different mathematical approach. This theory will be briefly discussed in this section, and in the forthcoming sections, we will give the full details of this approach.

Given a physical system, observables are determined via self-adjoint, linear operators acting on a Hilbert space $\mathscr{H}$, which is a finite or infinite dimensional complex vector space. $\mathscr{H}$ is equipped with a scalar product $\langle \Psi, \Phi \rangle$, scalar product is anti-linear in the first component and linear in the second component. $\mathscr{H}$ is complete with respect to the norm induced by this scalar product, $\|\Psi\| := \langle \Psi, \Psi \rangle^{1/2}$. So, every Cauchy sequence of vectors in $\mathscr{H}$ converges to a vector in $\mathscr{H}$ with respect to the norm defined above.

States of a physical system can also be identified with the positive, linear, self-adjoint operators ρ with unit trace, they are called as the density matrices on $\mathscr{H}$. Using this formulation, one can compute the expectation value of an observable A on a state ρ with the formula

$$\langle A \rangle_{\rho} = tr\rho A$$

It is important to note that for every unit vector Ψ in $\mathscr{H}$, there corresponds a state ψ defined by

$$\langle A \rangle_{\psi} = trP_{\psi}A = \langle \Psi, A\Psi \rangle ,$$

where P_ψ is the projector generated by Ψ is defined by

$$P_\psi = |\Psi\rangle\langle\Psi|$$

in Dirac's notation, i.e. $P_\psi \Phi = \langle \Psi, \Phi \rangle \Psi$ for all Φ in $\mathscr{H}$. The probability that the system will be found in state ψ, when it is known that the system in the state ϕ (transition probability) is given with the formula

$$Pr\{\psi|\phi\} = \langle \Phi, P_\psi \Phi \rangle = |\langle \Phi, \Psi \rangle|^2.$$

Here, we should note that assuming $\mathscr{H}$ is the space of all square-integrable functions on $\mathbb{R}^n$, Ψ is interpreted as the wave function corresponding to the state ψ, and Ψ is determined up to a phase ω i.e. Ψ and $\omega\Psi$ corresponds to the same physical state.

Any density matrix can be thought as a statistical mixture of the states described above:

$$\rho = \sum_{n=0}^{\infty} p_n P_{\psi_n},$$

where p_n ranges between zero and one, and

$$\sum_{n=0}^{\infty} p_n = 1.$$

In Schrödinger picture, the time evolution of Ψ or ρ is described by the action of a continuous, one-parameter group of unitary operators U_t as follows:

$$\Psi_t = U_t \Psi$$

$$\rho_t = U_t \rho U_{-t}.$$

(In Heisenberg picture, observables are time dependent and the time evolution is calculated by $A_t = U_{-t} A U_t$. However, in the rest of the discussion, we will assume the Schrödinger picture.) Time evolution of expectation values can be found as

$$\langle A \rangle_\psi (t) = \langle \Psi_t, A\Psi_t \rangle,$$

and, in general,

$$\langle A \rangle_\rho (t) = \langle A \rangle_{\rho_t} = tr(\rho_t A).$$

The differential form of the evolution laws are

$$\frac{d}{dt}\Psi_t = -iH\Psi_t,$$

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] = -i\mathcal{L}\rho_t,$$

where H is the Hamiltonian of the theory. And, one can define the Liouville operator $\mathcal{L}$ as $\mathcal{L}\rho = [H, \rho]$.

3.2 Jordan Algebra of Observables

The idea of algebraic approach is reflected to the initial matrix formulation of quantum mechanics which the names Dirac, Heisenberg, Born, Jordan... are attached. Von Neumann [Neumann, 1955] made two contributions to this purpose. First he formulated the theory as an eigenvalue problem, second he interpreted the concept of a state via the theory of probability. In this section, we will discuss postulates of quantum mechanics as it was recognized at that time and the structure axioms for the algebra of observables.

Postulate 1. *Given physical system, for each observable, there corresponds a self-adjoint, linear operator A acting on a Hilbert space $\mathscr{H}$.*

If one denotes the set of observable by $\mathcal{A}$, one can observe that
(i) for any sequence $\{A_i\}$ in $\mathcal{A}$, and any sequence $\{\lambda_i\}$ in $\mathbb{R}$,

$$\sum_i \lambda_i A_i$$

also belongs to $\mathcal{A}$;
(ii) for any observable A, and any real function f of one variable, $f(A)$ belongs to $\mathcal{A}$;
(iii) If A, B are two random elements of $\mathcal{A}$, then AB does not necessarily belong to $\mathcal{A}$ but the symmetric product

$$\{A, B\} = \frac{1}{2}(AB + BA)$$

belongs to $\mathcal{A}$ [Emch, 1972]. Note that the symmetric product is commutative and bilinear, and in order to define it we do not require the knowledge of the ordinary product of two noncompatible observables. It can be also formulated as

$$\{A, B\} = \frac{1}{4}((A + B)^2 - (A - B)^2),$$

the right hand side of the latter equation involves operations mentioned in (i) and (ii). So, the symmetric product can be defined without going out of $\mathcal{A}$. Note that the symmetric product is not associative in general. The associator

$$[A, B, C] := \{\{A, B\}, C\} - \{A, \{B, C\}\},$$

can be different than zero for random $A, B, C \in \mathcal{A}$. We note that this lack of associativity could be a candidate for the notion of non-compatibility of two observables. We have

$$[A, B, C] = \frac{1}{4}[B, [A, C]], \tag{3.1}$$

so A and C being compatible implies that the associator in (3.1) vanishes.

Theorem 3.2.1. *For bounded operators in $\mathcal{A}$, the following conditions are equivalent*

- $[A, B, C] = 0$*, for all* $B \in \mathcal{A}$,
- $[A, A, C] = 0$,
- $[A, C, C] = 0$,
- $[A, C] = 0$.

In the case of unbounded operators, one can produce a counter example using canonical commutation relation for the implication "item 1 implies item 4". Using Theorem 3.2.1, it is possible to formulate the compatibility for bounded operators using the associator, which is defined in terms of the symmetrized product. In order to get a completely algebraic formulation of observables, one needs a further concept. Intuitively, the state of a physical system is a way to express the knowledge of the expectation value of all observables on the physical system.

Postulate 2. *Each state ϕ of a physical system corresponds to a density matrix ρ_ϕ acting on the Hilbert space $\mathscr{H}$ of Postulate 1, and the expectation value can be computed as*

$$\langle A\rangle_\phi = tr\rho_\phi A.$$

Here, we should remark that our knowledge has progressed in the meanwhile, and it turns out that normal states can be represented by a density matrix ρ. Our aim is to analyze the structure resulting from postulate 2. In particular, we have
(i) for any sequence $\{A_i\}$ in $\mathcal{A}$, and any sequence $\{\lambda_i\}$ in $\mathbb{R}$,

$$\begin{aligned} tr\left(\rho_\phi \sum_i \lambda_i A_i\right) &= \sum_i \lambda_i tr(\rho_\phi A_i) \\ &= \sum_i \lambda_i \langle A_i\rangle_\phi; \end{aligned}$$

(ii) for any observable A, $tr\rho_\phi A^2 \geq 0$;
(iii) $tr\rho_\phi \mathbb{I} = 1$.

Remark. *For any sequence of states $\{\phi_i\}$, and any sequence of positive real numbers $\{\lambda_i\}$, satisfying $\sum_i \lambda_i = 1$, one can define the state ϕ as*

$$\langle A\rangle_\phi := \sum_i \lambda_i \langle A\rangle_{\phi_i}. \tag{3.2}$$

Note that ϕ satisfies the three properties above. States which can not be properly decomposed as in (3.2) are called pure states.

From now on, we will denote the set of all states satisfying all properties (i),(ii) and (iii), as $\mathcal{S}$.

Now, we are ready to state the axioms of an algebraic approach using the couple $(\mathcal{A}, \mathcal{S})$. It turns out that postulates 1 and 2 are too restrictive, and one has the feeling that the structure of $\mathcal{A}$, and $\mathcal{S}$ can be summarized assuming $\mathcal{A}$ is a real, commutative Jordan algebra.

Axiom 1. *For a given physical system, one can associate the triple $(\mathcal{A}, \mathcal{S}, \langle\rangle)$, where $\mathcal{A}$ is the set of all observables, $\mathcal{S}$ is the set of all states, and the mapping $\langle\rangle : (\mathcal{A}, \mathcal{S}) \to \mathbb{R}$ associates each pair of $(\mathcal{A}, \mathcal{S})$ with a real number, and interpreted as the expectation value of given observable in the given state.*

Hence the states $\phi \in \mathcal{S}$ can be interpreted as mappings from $\mathcal{A}$ to real numbers, and the value $\phi(A) := \langle A \rangle_\phi$ gives the expectation value of A. If $\mathcal{I}$ is a subset of $\mathcal{S}$, $A|_\mathcal{I}$ is the restriction of this mapping to $\mathcal{I}$. We say that $A|_\mathcal{I} \leq B|_\mathcal{I}$ if $\langle A \rangle_\phi \leq \langle B \rangle_\phi$ for all $\phi \in \mathcal{I}$. If this inequality holds for all states in $\mathcal{A}$, we write $A \leq B$. Note that $A \geq 0$ if and only if $\langle A \rangle_\phi \geq 0$ for all $\phi \in \mathcal{S}$. So states are positive functions on $\mathcal{A}$.

Definition 3.2.1. *A subset $\mathcal{I}$ of $\mathcal{A}$ is said to be full with respect to a subset $\mathcal{B} \subseteq \mathcal{A}$ if the inequality $A|_\mathcal{I} \leq B|_\mathcal{I}$ implies $A \leq B$ for all $A, B \in \mathcal{B}$.*

Axiom 2. *The relation $\leq$ is a partial ordering relation on $\mathcal{A}$, in other words $A \leq B$ and $B \leq A$ imply $A = B$.*

Axiom 2 can be viewed as $\langle A \rangle_\phi = \langle B \rangle_\phi$ for all $\phi \in \mathcal{S}$ implies $A = B$. So, we identify two observables whose expectation values coincide on all states of $\mathcal{S}$. Here, one can note that if A and B are two observables, then $\{\langle A \rangle_\phi + \langle B \rangle_\phi : \phi \in \mathcal{S}\}$ defines an observable, which is a measurable quantity.

Axiom 3. *(i) There exists two elements O and $\mathbb{I}$ in $\mathcal{A}$ such that for all $\phi \in \mathcal{S}$, we have $\langle O \rangle_\phi = 0$ and $\langle \mathbb{I} \rangle_\phi = 1$.*
(ii) For each observable $A \in \mathcal{A}$, and $\lambda \in \mathbb{R}$, there exists an element $\lambda A \in \mathcal{A}$ such that $\langle \lambda A \rangle_\phi = \lambda \langle A \rangle_\phi$ for all $\phi \in \mathcal{S}$.
(iii) For each observables $A, B \in \mathcal{A}$, there exists an element $A + B \in \mathcal{A}$ such that $\langle A + B \rangle_\phi = \langle A \rangle_\phi + \langle B \rangle_\phi$ for all $\phi \in \mathcal{S}$.

Note that the latter structure axiom makes $\mathcal{A}$ a real vector space, and $\mathcal{S}$ the set of real, linear functionals on $\mathcal{A}$. After a sequence of independent measurements of an observable A, an observer receives a distribution of real numbers whose proper average is called the expectation value $\langle A \rangle_\phi$ of A in the state ϕ. The state ϕ is called dispersion-free on the observable A, when the distribution of measurements concentrated on $\langle A \rangle_\phi$. We will denote the set of all dispersion-free states on A by $\mathcal{S}_A$, and the set of all the values that A assumes on its dispersion-free states by σ_A. The set σ_A is called as the spectrum of A. Two observables A and B can be simultaneously measure with arbitrary precision whenever the system is in a state

$\phi \in \mathcal{S}_A \cap \mathcal{S}_B$. Note that the set $\mathcal{S}_A \cap \mathcal{S}_B$ can be empty, so that A and B can not be simultaneously measured with arbitrary precision in any state.

Definition 3.2.2. *A subset $\mathcal{I} \subseteq \mathcal{S}$ is called complete if it is full with respect to the subset $\mathcal{A}_{\mathcal{I}} \subseteq \mathcal{A}$, where $\mathcal{A}_{\mathcal{I}} := \{A \in \mathcal{A} : \mathcal{I} \subseteq \mathcal{S}_A\}$. A complete subset is called deterministic for a subset $\mathcal{B} \subseteq \mathcal{A}$ if $\mathcal{B} \subseteq \mathcal{A}_{\mathcal{I}}$. Finally, a subset $\mathcal{B} \subseteq \mathcal{A}$ is called compatible if the set $\mathcal{S}_{\mathcal{B}} := \bigcap_{B \in \mathcal{B}} \mathcal{S}_B$ is complete.*

Axiom 4. *For any observable A, the set $\mathcal{S}_A$ is deterministic for the one dimensional subspace of $\mathcal{A}$ generated by A, for any $A, B \in \mathcal{A}$: $\mathcal{S}_A \cap \mathcal{S}_B \subseteq \mathcal{S}_{A+B}$, and $\mathcal{S}_{\mathbb{I}} = \mathcal{S}$.*

This axiom implies that for any nontrivial theory, σ_A and $\mathcal{S}_A$ are nonempty. Two observables that have the same dispersion-free states and coincide on them are identical. On the other hand, Axiom 4 allows us to formulate the existence of powers of observables.

Axiom 5. *For any $A \in \mathcal{A}$ and any non-negative integer n, there exists at least one element, $A^n \in \mathcal{A}$ such that*
(i) $\mathcal{S}_A \subseteq \mathcal{S}_A n$
(ii) $\langle A^n \rangle_\phi = \langle A \rangle_\phi^n$.

Using these axioms, one can deduce the following
(i) A^n, nth power of A, is unique in $\mathcal{A}$,
(ii) the operation of raising an observable to a power is associative, and distributive laws hold with respect to scalar multiplication,
(iii) $A^2 + B^2 + C^2 + \cdots = 0$ implies $A = B = C = \cdots = 0$,
(iv) Let $\mathscr{B}(A)$ be the subset of $\mathcal{A}$ generated from the powers of A. For any $B \in \mathscr{B}(A)$, we have $\mathcal{S}_A \subseteq \mathcal{S}_B$. Using Axiom 4, one can see that $\mathcal{S}_A$ is deterministic on $\mathscr{B}(A)$, hence $\mathscr{B}(A)$ is a compatible set of observables. It is possible to generalize this statement as follows.

Theorem 3.2.2. *Let $\mathcal{B}$ be a compatible set of observables and $\mathscr{B}(\mathcal{B})$ be the subset of $\mathcal{A}$ generated from the elements of $\mathcal{B}$ by addition, power raising and multiplication by scalars. Then $\mathscr{B}(\mathcal{B})$ is a compatible set of observables.*

In the subset $\mathscr{B}(\mathcal{B})$ the power-raising operation is distributive with respect to the addition. As of now, we can say that $\mathcal{A}$ is equipped with a structure of real vector space, and we impose an associative power structure. Now, our aim is to provide a richer structure for $\mathcal{A}$.

Definition 3.2.3. *With each pair $A, B \in \mathcal{A}$, we associate an element $A \bullet B$ of $\mathcal{A}$ defined with the symmetric product*

$$A \bullet B = \frac{1}{2}((A+B)^2 - A^2 - B^2).$$

It immediately follows that this symmetric product is commutative. Using Axiom 4 and 5,

$$\langle A \bullet B \rangle_\phi = \langle A \rangle_\phi \langle B \rangle_\phi \tag{3.3}$$

for all $\phi \in \mathcal{S}_A \cap \mathcal{S}_B$. We conclude from that $A^n \bullet A^m = A^{n+m}$, since $\mathcal{S}_A$ is deterministic on $\mathscr{B}(A)$. In particular $A \bullet \mathbb{I} = A$. It is very important to determine whether this product is associative and distributive. We define the associator $[A, B, C] = (A \bullet B) \bullet C - A \bullet (B \bullet C)$. Let $\mathcal{B}$ be a subset of $\mathcal{A}$, $\mathcal{S}_\mathcal{B} := \bigcap_{B \in \mathcal{B}} \mathcal{S}_B$, and $\mathcal{A}_{\mathcal{S}_\mathcal{B}}$ be denoted by $\mathcal{A}(\mathcal{B})$. For every triple $A, B, C \in \mathcal{A}(\mathcal{B})$, we conclude by (3.3) that $\langle [A, B, C] \rangle_\phi = 0$ for all $\phi \in \mathcal{S}_\mathcal{B}$. If $\mathcal{B}$ is compatible, $\mathcal{S}_\mathcal{B}$ is complete, which contains in particular $[A, B, C]$. Hence $[A, B, C] = 0$. Similarly, one can see that $A \bullet (\lambda B + \mu C) = \lambda (A \bullet B) + \mu (A \bullet C)$.

Theorem 3.2.3. *The symmetric product is associative and distributive over $\mathcal{A}(\mathcal{B})$ for any compatible set $\mathcal{B}$ of observables. So, $\mathcal{B}$ being compatible implies $\mathscr{B}(\mathcal{B})$ being associative.*

Recall that using Theorem 3.2.1, von Neumann characterized the compatibility of two bounded operators A and C by the relation $[A, B, C] = 0$ for all B, where B is another bounded operator.

Axiom 6. *For any triple $A, B, C \in \mathcal{A}$ in which A and C are compatible the associator $[A, B, C]$ vanishes.*

Later in this section, we will prove that the converse of this axiom is also holds. Here one can also see that for any triple of observables A, B, C, where A and C

are compatible, the associator $[A, B, C]$ is compatible with both A and C. Furthermore, we can note that since all powers of an observable are compatible we have $[A^n, B, A^m] = 0$ for all $A, B \in \mathcal{A}$ and all non-negative integers n, m. In particular,

$$A \bullet (B \bullet A^2) = (A \bullet B) \bullet A^2.$$

Theorem 3.2.4. *The set of all observables $\mathcal{A}$ on a physical system is a commutative, distributive, but not necessarily associative real Jordan algebra.*

At this point, an attractive question reveals itself. Can this structure lead to a more general theory of quantum mechanics? Jordan, von Neumann and Wigner [Jordan P., 1934] attacked this problem in their fundamental paper. For their results, the reader is referred to Chapter 2.2. It turns out that there was weakness in their theory, they assumed that $\mathcal{A}$ has a finite linear basis. This weakness is corrected by the introduction of an topological structure to Jordan algebras. We define

$$\|A\| := \sup_{\phi \in \mathcal{S}} |\langle A \rangle_\phi|.$$

Theorem 3.2.5. $\|.\|$ *is a norm for $\mathcal{A}$ and $|\langle A \rangle_\phi| \leq \|A\|$ for all $(A, \phi) \in (\mathcal{A}, \mathcal{S})$.*

For a proof of this result see page 52 of [Emch, 1972]. Any Cauchy-sequence admits a limit point in $\mathcal{A}$, and $\mathcal{A}$ is now equipped with the structure of a real Banach space relative to the norm above. States are continuous, positive, linear functionals on $\mathcal{A}$ with respect to the topology induced by this norm.

Axiom 7. *The norm of any element A in $\mathcal{A}$ is finite and $\mathcal{A}$ is topologically complete metric space, where the distance between A and B is given by $\|A-B\|$. $\mathcal{S}$ is identified with the set of all continuous, positive, linear functionals ϕ on $\mathcal{A}$ satisfying $\langle \mathbb{I} \rangle_\phi = 1$.*

For all $A, B \in \mathcal{A}$, one can show that

$$\|A^2\| = \|A\|^2,$$

$$\|A^2 - B^2\| \leq \max\{\|A\|^2, \|B\|^2\}.$$

Lemma 3.2.1. *(i) For any state $\phi \in \mathcal{S}$ and any pair $A, B \in \mathcal{A}$, the Schwartz inequality holds*

$$\langle A \bullet B \rangle_\phi^2 \leq \langle A^2 \rangle_\phi \langle B^2 \rangle_\phi.$$

(ii) Let $A \in \mathcal{A}$ and $\phi \in \mathcal{S}_A$, then

$$\langle A \bullet B \rangle_\phi = \langle A \rangle_\phi \langle B \rangle_\phi.$$

for all $B \in \mathcal{A}$.

Using Lemma 3.2.1, one can prove the following theorem.

Theorem 3.2.6. *For any state $\phi \in \mathcal{S}$, and any pair $A, B \in \mathcal{A}$, we have*

$$|\langle A \bullet B \rangle_\phi| \leq \|A\| \|B\|,$$
$$\|A \bullet B\| \leq \|A\| \|B\|,$$
$$\|A^2 - B^2\| \leq \|A - B\| \|A + B\|.$$

In order to see more results on the topology of $\mathcal{A}$, we refer to [Emch, 1972],[Alfsen E. M., 2003]. We will now focus on the concept of compatibility of observables.

Lemma 3.2.2. *An observable P is called a projection if it is an idempotent element of $\mathcal{A}$, i.e. $P^2 = P$, a necessary and sufficient condition for two projections P and Q to be compatible is that there exists three pairwise distinct projections R, P_1, Q_1 such that $P = R + P_1$ and $Q = R + Q_1$.*

Given an arbitrary projection P, and an arbitrary real number λ, one can define

$$N^\lambda(P) = \{A \in \mathcal{A} : (P - \lambda \mathbb{I}) \bullet A = 0\},$$

which is a closed subspace of $\mathcal{A}$. One can polarize the power-associative identity

$$\{A, A, A^2\} = 0$$

by taking $A = P + \mu A$, where $\mu \in \mathbb{R}$. The linear term in μ gives

$$2P \bullet (P \bullet (P \bullet A)) - 3P \bullet (P \bullet A) + P \bullet A = 0.$$

If $A \in N^{\lambda}(P)$, we have

$$(2\lambda^3 - 3\lambda^2 + 1)A = 0.$$

From that, we can conclude that $N^{\lambda}(P) = 0$ unless $\lambda = 0, 1/2, 1$. Now, we form the three elements for each $A \in \mathcal{A}$

$$\begin{aligned}
A^0 &= 2P \bullet (P \bullet A) - 3P \bullet A + A, \\
A^{1/2} &= -4P \bullet (P \bullet A) + 4P \bullet A, \\
A^1 &= 2P \bullet (P \bullet A) - P \bullet A,
\end{aligned}$$

which add up to A. Here, one can realize that for $\lambda = 0, 1/2, 1$, $(P - \lambda\mathbb{I}) \bullet A^{\lambda}$ reproduces the relation

$$2P \bullet (P \bullet (P \bullet A)) - 3P \bullet (P \bullet A) + P \bullet A,$$

so that $(P - \lambda\mathbb{I}) \bullet A^{\lambda} = 0$ for $\lambda = 0, 1/2, 1$. Now, we can conclude that $A^{\lambda} \in N^{\lambda}(P)$. So, we have the following theorem.

Theorem 3.2.7 (Pierce decomposition)**.** *An arbitrary projection $P \in \mathcal{A}$ generates a decomposition of $\mathcal{A}$ into a direct sum $\bigoplus N^{\lambda}(P)$ of the three closed subspaces $N^{\lambda}(P) = \{A \in \mathcal{A} : (P - \lambda\mathbb{I}) \bullet A = 0\}$, with $\lambda = 0, 1/2, 1$.*

As a consequence of this theorem, we have

$$\begin{aligned}
&N^0(P) \bullet N^0(P) \subseteq N^0(P), \\
&N^0(P) \bullet N^1(P) = 0, \\
&N^1(P) \bullet N^1(P) \subseteq N^1(P).
\end{aligned}$$

Also note that for any projection P, P maps an element of $N^{1/2}(P)$ to $N^{1/2}(P)$. Using the Pierce decomposition and Lemma 3.2.2, now we can prove a necessary and sufficient condition for compatibility of projections.

Theorem 3.2.8. *A necessary and sufficient condition for two projections P and Q to be compatible is that $[P, A, Q] = 0$ for all $A \in \mathcal{A}$.*

Proof. By Axiom 6, we know that if P and Q are compatible, then the associator vanishes, which also follows as a consequence of Pierce decomposition. From Theorem 3.2.7, it is sufficient to prove it for $A \in N^{\lambda}(P)$. We first show the particular case in which $P \bullet Q = 0$, so $Q \in N^0(P)$. We get that both sides of the relation $P \bullet (Q \bullet A) = Q \bullet (P \bullet A)$ are respectively $0, 0, \frac{1}{2} Q \bullet A$ for A in $N^0(P), N^1(P), N^{1/2}(P)$, respectively. This proves that the condition of the theorem is satisfied under the condition $P \bullet Q = 0$. Using Lemma 3.2.2, general case can be reduced to this particular case. So, we need to show that if $P \bullet (Q \bullet A) = Q \bullet (P \bullet A)$ for all $A \in \mathcal{A}$, then P and Q are compatible. Using Theorem 3.2.8, $P = P^0 + P^{1/2} + P^1$ with

$$P^{1/2} = -4Q \bullet (Q \bullet P) + 4Q \bullet P = 4[Q, P, P] - 4(Q \bullet Q) \bullet P + 4Q \bullet P.$$

We get $P^{1/2} = 0$, since $Q^2 = Q$ and $[Q, P, P] = 0$. Hence $P = P^0 + P^1$. Since $P^2 = P$, and $N^0(P) \bullet N^1(P) = 0$, we have $(P^0)^2 - P^0 = (P^1)^2 - P^1$. Using $N^0(P)^2 \subseteq N^0(P)$, $N^1(P)^2 \subseteq N^1(P)$, and $N^0(P) \bullet N^1(P) = 0$, we get that P^0 and P^1 are projections. Furthermore, $P^1 = 2Q \bullet (Q \bullet P) - Q \bullet P = 2Q \bullet P - Q \bullet P = Q \bullet P$. Now, we can write $R = Q \bullet P$, $P_1 = P - Q \bullet P$, and $Q_1 = Q - Q \bullet P$. Finally, using Theorem 3.2.2, we can conclude that P and Q are compatible. □

It turns out that $\mathcal{A}$ can be embedded in an associative C^*-algebra.

Axiom 8. *$\mathcal{A}$ can be identified with the set of self-adjoint elements of a complex, involutive, associative algebra $\mathcal{B}$ satisfying*
(i) For each $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B^*B = A^2$,
(ii) $B^*B = 0$ implies $B = 0$.

Next theorem we will present is about Gelfand-Naimark-Segal construction. We refer to Section 3.3 for details of the construction.

Theorem 3.2.9. *Let $\mathcal{A}$ be the algebra of observables on a physical system, and $\mathcal{B}$ be the algebra associated to $\mathcal{A}$ by axiom 8. Then each state $\phi \in \mathcal{S}$ generates a canonical representation π_ϕ of $\mathcal{A}$ by self-adjoint operators acting on a complex Hilbert space.*

Axiom 9. *Let $A, B \in \mathcal{A}$ be a pair of observables. Then there exists an observable $C \in \mathcal{A}$ which provides the actual lower bound to the simultaneous observability of A*

and B

$$(\langle A^2\rangle_\phi - \langle A\rangle_\phi^2)(\langle B^2\rangle_\phi - \langle B\rangle_\phi^2) \geq \left|\frac{1}{2}\langle C\rangle_\phi\right|^2,$$

for all $\phi \in \mathcal{S}$.

Note that this axiom is automatically satisfied if we assume that $\mathcal{A}$ is the set of all self-adjoint elements of a C^*-algebra $\mathcal{B}$. In this case, the uncertainty principle takes its usual form

$$(\langle A^2\rangle_\phi - \langle A\rangle_\phi^2)(\langle B^2\rangle_\phi - \langle B\rangle_\phi^2) \geq \left|\frac{i}{2}\langle [A,B]\rangle_\phi\right|^2.$$

3.3 Jordan Algebraic Hilbert Space Construction

In this section, after reviewing Gelfand-Naimark-Segal (GNS) construction for C^*-algebraic approach, we will present Jordan-algebraic Hilbert space construction. Here we should note that the Hilbert space construction in this section is applicable to all JB-algebras, it is not only valid for JC-algebras.

3.3.1 Hilbert Space Formulation via GNS-construction

Consider a pure state ψ of a C^*-algebra $\mathcal{A}$, and the hermitian form $\langle A|B\rangle = \psi(A^*B)$. It turns out that the set of zero-elements

$$\mathcal{I}_\psi = \{A \in \mathcal{A} : \psi(A^*A) = 0\}$$

forms an ideal in $\mathcal{A}$. Using that, one can obtain a positive-definite, hermitian scalar product on $\mathcal{A}/\mathcal{I}_\psi$ and a norm

$$\|A\| = \sqrt{\langle A|A\rangle}.$$

One can construct a complex, separable Hilbert space $\mathscr{H}$ as follows

$$\mathscr{H} = \overline{\mathcal{A}/\mathcal{I}_\psi}.$$

It is possible to write down a representation between $\mathcal{A}$ and the collection of bounded, linear operators on $\overline{\mathcal{A}/\mathcal{I}_\psi}$ as follows

$$\pi_\psi : \mathcal{A} \to \mathcal{B}(\mathscr{H}), \quad \pi_\psi(A)B = AB.$$

This construction lies at the heart of the celebrated the Gelfand-Naimark-Segal (GNS) theory. In this construction, pure states are represented by single rays in $\mathscr{H}$, and the expectation value of $A \in \mathcal{A}$ is given by

$$\psi_B(A) = \langle B|A|B\rangle = \psi(B^*AB).$$

In the constructed Hilbert space $\mathscr{H}$, the time evolution equation for pure states $\nu \in \mathscr{H}$ is given by the Schrödinger equation

$$i\hbar\dot{v} = Hv,$$

where H is the corresponding Hamiltonian. But this method is not suitable for the Jordan algebraic approach. If we consider the self-adjoint part of $\mathcal{A}$, which forms a JB-algebra, and imitate the same construction with the real, symmetric, bilinear form

$$\langle A|B\rangle = \psi(A \bullet B),$$

it turns out that the set

$$\mathcal{I}_\psi = \{A \in \mathcal{A} : \psi(A^2) = 0\}$$

is not an ideal of the corresponding JB-algebra. In other words, this JB-algebra does not act on the constructed Hilbert space. Therefore, one demands another, more general Hilbert space construction. It turns out that this constructions is based on traces of operators instead of their states.

From now on, let us assume that $\mathcal{A}$ is the algebra $\mathcal{B}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$, which is well-known to be a von Neumann algebra. As a result of this, there exists a semi-finite, faithful, normal trace function on $\mathcal{A}$ [Bratelli O., 1987]. We will then denote positive, trace class elements by $\mathcal{A}_1$, and those $A \in \mathcal{A}$ with $tr(A^*A) < \infty$ by $\mathcal{A}_2$. One can check that $\mathcal{A}_2$ is an ideal in $\mathcal{A}$. In particular, for all $A, B \in \mathcal{A}_2$, we have

$$\langle A|B\rangle = trA^*B, \qquad \|A\|_{tr} = \sqrt{trA^*A}.$$

Inspired by the GNS construction, one obtains the Hilbert space of Hilbert-Schmidt operators in the form

$$\hat{\mathscr{H}} = \overline{\mathcal{A}_2}^{tr}.$$

Since $\mathcal{A}_2$ is an ideal, and $\mathcal{A}$ is associative, $\mathcal{A}$ can be represented over $\mathcal{A}_2$ by the multiplication operator which is defined by

$$\pi(A)B = AB$$

for all $A \in \mathcal{A}$, $B \in \mathcal{A}_2$. Thus, π can be extended to a faithful representation of $\mathcal{A}$ on $\hat{\mathscr{H}}$ [Takesaki, 1979]. If we take a state ψ represented by a density matrix $\rho \in \mathcal{A}_1$, then there is at least one $B \in \mathcal{A}_2$ such that $\rho = BB^*$ holds. At this point, it is worth mentioning that the expectation value of an observable $A \in \mathcal{A}$ is given by

$$\psi(A) = tr\rho A = trBB^*A = trB^*AB = \langle B|A|B\rangle.$$

Next let $U : \mathscr{H} \to \mathscr{H}$ be a bounded, unitary operator. If we take any vector $B \in \mathcal{A}_2$, then BU becomes a vector in $\hat{\mathscr{H}}$ describing the same vector as B does. Therefore, the representation of states in $\hat{\mathscr{H}}$ is not unique, more precisely the states in $\hat{\mathscr{H}}$ are represented by right unitary orbits defined by

$$BUU^*B^* = BB^* = \rho.$$

On the other hand, every Hilbert-Schmidt operator $B \in \mathcal{A}_2$ with nonzero trace leads to a density matrix

$$\rho_B := \frac{BB^*}{tr(BB^*)}.$$

To emphasize the difference between $\mathscr{H}$ and $\hat{\mathscr{H}}$, note that now, every (pure and mixed) state is represented by at least one vector in $\hat{\mathscr{H}}$. It turns out that $\mathscr{H}$ and $\hat{\mathscr{H}}$ are connected in the following algebraic sense [Takesaki, 1979]

$$\hat{\mathscr{H}} = \overline{\mathscr{H} \otimes \mathscr{H}^*}^{tr}.$$

As an example, we will consider the physical system of quantum harmonic oscillator, which turns out to be very crucial in the formulation of the Landau problem. It is then necessary to focus on the application of Hilbert-Schmidt operators, which can be viewed as the collection of bounded operators acting on the configuration space

$$\mathscr{H} := span\left\{|n\rangle : \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\right\}_{n=0}^{\infty}.$$

Furthermore, this space is isomorphic to the boson Fock space $\mathcal{F} = \{|n\rangle\}_{n=0}^{\infty}$, where the annihilation and creation operators obey the Fock algebra $[a, a^{\dagger}] = 1$. Observe that the physical states of the system represented on $\hat{\mathscr{H}}$, the set of Hilbert-Schmidt operators, in other words the set of all bounded operators on $\mathscr{H} = L^2(\mathbb{R})$ equipped with the scalar product

$$\langle X|Y\rangle = tr(X^*Y) = \sum_k \langle \phi_k|X^*Y\phi_k\rangle,$$

where $\{\phi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $\mathscr{H}$. Moreover, the basis vectors of $\hat{\mathscr{H}}$ are given by

$$\phi_{nl} := |\phi_n\rangle\langle\phi_l|, \quad n, l = 0, 1, 2, \dots \ .$$

Here, we should note that $\mathscr{H} \otimes \mathscr{H}^*$ is a pre-Hilbert space containing all finite sums of the form

$$\sum_{j,k=0}^{n} \lambda_{jk}|\phi_j\rangle \otimes |\phi_k\rangle^*,$$

where a basis of $\overline{\mathscr{H} \otimes \mathscr{H}^*}^{tr}$ is

$$\{|\phi_j\rangle \otimes |\phi_k\rangle^*\}_{j,k=0}^{\infty}.$$

Setting $|\phi_j\rangle \otimes |\phi_k\rangle^* = |\phi_j\rangle\langle\phi_k|$, $\hat{\mathscr{H}}$ admits an orthonormal basis $\{\phi_{jk}\}_{j,k=0}^{\infty}$ such that

$$\phi_{jk} := |\phi_j\rangle\langle\phi_k|.$$

Note that $\hat{\mathscr{H}}$ can also be described as the Hilbert space of square integrable functions on $(-\infty, \infty)$ by

$$\hat{\mathscr{H}} = \left\{\psi(x_1, x_2) : \psi(x_1, x_2) \in \mathcal{B}(\mathscr{H}),\ tr(\psi(x_1, x_2)^{\dagger}\psi(x_1, x_2)) < \infty\right\}. \tag{3.4}$$

Recall that $\hat{\mathscr{H}}$ is also the set of bounded operators, described in (3.4), acting on the configuration space $\mathscr{H}$. Clearly, a general element in this space can be written as

$$|\psi\rangle = \sum_{n,m=0}^{\infty} c_{n,m}|n, m\rangle,$$

where $\{|n, m\rangle := |n\rangle\langle m|\}_{n,m=0}^{\infty}$ is a basis of $\hat{\mathscr{H}}$ satisfying the inner product condition

$$\langle n_1, m_1|n_2, m_2\rangle = tr[(|n_1\rangle\langle m_1|)^{\dagger}|n_2\rangle\langle m_2|] = \delta_{n_1,n_2}\delta_{m_1,m_2}.$$

3.3.2 Jordan-GNS Construction

In this section, we will present a Jordan algebraic GNS-type construction. But this time, our development will be based on the JB algebra of observables. The idea that a non-associative algebra of observables equipped with a tri-cyclic trace leading to a Hilbert space construction is very fruitful. For more details on this construction and its applications, we especially recommend the recent work [Schupp P., Szabo R. J., 2023]. It turns out that there is an analogue of von Neumann algebras in the context of JB algebras, which are called as JBW algebras [Hanche-Olsen H., 1984]. One can work with normal, faithful, semi-finite traces on a JBW algebra $\mathcal{A}$ [Iochum, 2006]. Trace on a Jordan algebra is defined as a weight and it satisfies the cyclicity condition

$$tr(A \bullet (B \bullet C)) = tr((A \bullet B) \bullet C).$$

Similarly, we will then denote positive, trace class elements by $\mathcal{A}_1$, and those $A \in \mathcal{A}$ with $tr(A^2) < \infty$ by $\mathcal{A}_2$. One can check that $\mathcal{A}_2$ is a Jordan ideal in $\mathcal{A}$. Trace induces a bilinear, symmetric, real scalar product on $\mathcal{A}_2$, and in particular, we have

$$\langle A|B\rangle = tr(A \bullet B), \qquad \|A\|_{tr} = \sqrt{trA^2}.$$

Similar to the associative case, closure with respect to this norm yields a real Hilbert space

$$\hat{\mathscr{H}}_J = \overline{\mathcal{A}_2}^{tr}.$$

Next we aim to write down a Jordan representation from $\mathcal{A}$ to this Hilbert space. Note that $\mathcal{A}_2$ with the multiplication by elements of $\mathcal{A}$ is a Jordan module since for all $\omega \in \mathcal{A}_2$, and $A, B \in \mathcal{A}$, we have [Jacobson, 1968],[Hanche-Olsen H., 1984]

$$A \bullet \omega = \omega \bullet A,$$

$$A^2 \bullet (A \bullet \omega) = A \bullet (A^2 \bullet \omega),$$

$$2A \bullet (B \bullet (A \bullet \omega)) + (B \bullet A^2) \bullet \omega = 2(A \bullet B) \bullet (A \bullet \omega) + A^2 \bullet (B \bullet \omega).$$

First two relations follow from the definition of a Jordan algebra. Third one can be verified for a special Jordan algebra, and by Theorem 2.2.5, it is valid in every

Jordan algebra. We may then define

$$\pi_J(A) : \mathcal{A}_2 \to \mathcal{A}_2, \quad \pi_J(A)B = A \bullet B$$

for all $A \in \mathcal{A}, B \in \mathcal{A}_2$. It is easy to check that π_J satisfies the following properties

$$\pi_J(A^2) \bullet \pi_J(A) = \pi_J(A) \bullet \pi_J(A^2),$$

$$2\pi_J(A) \bullet \pi_J(B) \bullet \pi_J(A) + \pi_J(B \bullet A^2) = 2\pi_J(A \bullet B) \bullet \pi_J(A) + \pi_J(A^2) \bullet \pi_J(B).$$

Therefore, it becomes a Jordan representation. Moreover, due to the continuity of multiplication, π_J can be extended to whole space $\hat{\mathscr{H}}_J$.

At this point, we should stress the relationship between $\hat{\mathscr{H}}_J$ and $\hat{\mathscr{H}}$. It turns out that $\hat{\mathscr{H}}_J$ is the real subspace of self-adjoint Hilbert-Schmidt operators embedded in $\hat{\mathscr{H}}$ [Bischoff W., 1993]. Turning back to the problem of quantum harmonic oscillator, $\hat{\mathscr{H}}_J$ is a real span of following three family of operators

$$\begin{aligned} &|\phi_j\rangle\langle\phi_j|, \\ &\frac{1}{2}(|\phi_j\rangle\langle\phi_k| + |\phi_k\rangle\langle\phi_j|), \\ &\frac{i}{2}(|\phi_j\rangle\langle\phi_k| - |\phi_k\rangle\langle\phi_j|), \end{aligned} \tag{3.5}$$

where $j > k$, and j, k ranges from 0 to ∞. Note that these elements form a basis for $\hat{\mathscr{H}}_J$ since they are self-adjoint. Here, $\{\phi_k\}_{k=0}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$ which can be written in terms of Hermite polynomials in the position space representation.

In a Jordan algebra, one can define the associator in the form

$$[A, B, C] := (A \bullet B) \bullet C - A \bullet (B \bullet C).$$

Our aim is to find a connection between the commutator defined over the associative algebra $\mathcal{B}$ and the associator over $\mathcal{A}$, which is the self-adjoint part of $\mathcal{B}$. See [Dereli T., Yörük E. S., 2024] for the following proposition.

Proposition 3.3.1. *Every self-adjoint element $H \in \mathcal{A}$ can be written as a finite sum of commutators in the form*

$$H = i\sum_{j=1}^{N}[H_{Lj}, H_{Rj}], \tag{3.6}$$

where H_{Lj}, and H_{Rj} are self-adjoint operators in $\mathcal{A}$.

Proof. Let $Aut^0(\mathcal{A})$ be the connected $1-$component of the Banach Lie group of all automorphisms of $\mathcal{A}$. Let A be a self-adjoint element in $\mathcal{A}$. We define an operator U as $U = \exp(iA)$, which is clearly a unitary operator. By [Upmeier, 1981], $Aut^0(\mathcal{A})$ is known to be algebraically generated by the symmetries in $\mathcal{A}$. Moreover by Theorem 1.6 of [Upmeier, 1981], each unitary operator can be written as a finite product of symmetries in $\mathcal{A}$. Combining these facts, we deduce that $U \in Aut^0(\mathcal{A})$. Armed with this, we can now apply Theorem 2.5 of [Upmeier, 1981] to obtain

$$U = \exp\left(\sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right),$$

where $a_j, b_j \in \mathcal{A}$ for all j, and $L_a b := a \bullet b$ is the left multiplication operator. Consequently, we have

$$\exp\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right) = 1.$$

Our next goal is to verify that

$$\exp\left(t\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right)\right)$$

is a one parameter family for $t \in \mathbb{R}$. First, this operator depends continuously on t. Second, when $t = 0$, the operator reduces to the identity operator. Lastly, for any $t, s \in \mathbb{R}$, it follows from the properties of the exponential of an operator that

$$\begin{aligned}\exp\left((t+s)\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right)\right) &= \exp\left(t\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right) + s\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right)\right) \\ &= \exp\left(t\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right)\right)\exp\left(s\left(iA - \sum_{j=1}^{n}[L_{a_j}, L_{b_j}]\right)\right).\end{aligned}$$

By the uniquness of generators of one-parameter families of operators (see Stone's theorem [Stone, 1932]), we may now conclude that

$$A = i\sum_{j=1}^{n}[L_{a_j}, L_{b_j}].$$

And (3.6) follows. □

Chapter 4

NON-COMMUTATIVE GEOMETRY AND LANDAU PROBLEM

4.1 Landau Problem

In this chapter, we are mainly concerned with the electron in a uniform magnetic field, namely the Landau problem. To simplify the problem, we assume that the electron moves only on xy plane under the magnetic field pointing along the z direction. The Lorentz force causes the electron to make a spiral motion with Larmor frequency

$$\omega_L = \frac{|eB|}{mc}.$$

One common choice for vector potential, which preserves the rotational invariance, is the symmetric gauge

$$\boldsymbol{\mathcal{A}} = (A_x, A_y) = \frac{B}{2}(-y, x).$$

The Hamiltonian can be written as

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2 =: \frac{1}{2m}\boldsymbol{P}^2,$$

where $\boldsymbol{P}$ is the kinetic momenta. One important point to note that is two kinetic momenta do not commute,

$$[P_x, P_y] = [p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y] = i\frac{e\hbar}{c}(\nabla_x A_y - \nabla_y A_x) = i\frac{e\hbar}{c}B_z. \tag{4.1}$$

The commutation relation (4.1) suggests that P_x and P_y play the role of position and momentum operators, respectively. Combining this fact with the Hamiltonian of the problem, the system can be viewed as a harmonic oscillator. Creation and annihilation operators are defined as follows

$$a = \sqrt{\frac{c}{2e\hbar B}}(P_x + iP_y),$$
$$a^\dagger = \sqrt{\frac{c}{2e\hbar B}}(P_x - iP_y).$$

They satisfy

$$[a, a^\dagger] = \frac{c}{2e\hbar B}[P_x + iP_y, P_x - iP_y] = \frac{c}{2e\hbar B} i \frac{e\hbar B}{c}(-i - i) = 1.$$

So, the Hamiltonian becomes

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right),$$

with the classical Larmor frequency. The spectrum is that of a harmonic oscillator

$$E = \hbar\omega \left(N + \frac{1}{2} \right),$$

where N is a non-negative integer. Energy eigenstates are called as Landau levels. One can obtain the ground state wave function by solving $a|0\rangle = 0$, and excited states by acting with $a^\dagger$ like harmonic oscillator problem. It turns out that there are infinitely many states at each landau level N.

The center of the cyclotron motion

$$X = x + \frac{v_y}{\omega} = x + \frac{c}{eB}\left(p_y - \frac{e}{c}A_y \right),$$
$$Y = y - \frac{v_x}{\omega} = y - \frac{c}{eB}\left(p_x - \frac{e}{c}A_x \right),$$

commute with P_x, P_y, a, $a^\dagger$ and the Hamiltonian. But we should note that

$$[X, Y] = [x + \frac{P_y}{m\omega}, y - \frac{P_x}{m\omega}] = -\frac{i\hbar}{m\omega} = -\frac{i\hbar c}{eB} = i\theta \neq 0. \tag{4.2}$$

(4.2) means that it is not possible to determine both x and y coordinates of the center of the cyclotron motion at the same time with arbitrary precision. Furthermore, it is possible to introduce another pair of creation and annihilation operators

$$b = \sqrt{\frac{eB}{2\hbar c}}(X - iY),$$
$$b^\dagger = \sqrt{\frac{eB}{2\hbar c}}(X + iY),$$

satisfying $[b, b^\dagger] = 1$.

4.2 Non-commutativity in the lowest Landau level in classical formulation

The standard Landau model yields all Landau levels. The non-commutative model only yields lowest-Landau level states, as discussed by Peierls [Peierls, 1933]. This fact is used in the description of the ground states of the Fractional Quantum Hall Effect.

We consider a point particle on a plane with an external magnetic field $\boldsymbol{B}$ perpendicular to the plane. The equation for the position vector $\boldsymbol{r} = (x, y)$ is

$$m\ddot{r}^i = \frac{e}{c}\epsilon^{ij}\dot{r}^j B + f^i(\boldsymbol{r}), \tag{4.3}$$

here $\boldsymbol{f}$ gives other forces, and it can be found from a potential V as $\boldsymbol{f} = -\boldsymbol{\nabla} V$. The large B or small m limit projects to the lowest Landau level. And setting m to be zero, equation (4.3) can be written as

$$\dot{r}^j = \frac{c}{eB}\epsilon^{ji} f^i(\boldsymbol{r}). \tag{4.4}$$

Assuming the Hamiltonian is $H_0 = V$, (4.4) can be obtained by taking Poisson brackets of $\boldsymbol{r}$ with H_0, provided

$$\{r^i, r^j\} = \frac{c}{eB}\epsilon^{ij}. \tag{4.5}$$

The non-commutative algebra (4.5) and the dynamics of the motion can be derived as follows. The Lagrangian for the equation of motion is

$$L = \frac{1}{2}mv^2 + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{\mathcal{A}} - V.$$

For simplicity, we choose gauge $\boldsymbol{\mathcal{A}} = (0, Bx)$. In the limit $m \to 0$,

$$L = \frac{eB}{c}x\dot{y} - V.$$

Note that $\left(\frac{eB}{c}x, y\right)$ forms a canonical pair, which implies (4.5) and identifies V as the Hamiltonian.

One can also give canonical derivation of non-commutativity in the limit $m \to \infty$. In this method, non-commuting coordinates appear as Dirac brackets in a system constrained to the lowest Landau level. Let

$$H = \frac{P^2}{2m} + V, \tag{4.6}$$

where $\boldsymbol{P}$ is the kinetical momentum, $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ is the canonical momentum, and $\boldsymbol{P} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}$. This Hamiltonian gives (4.3) by bracketing $\boldsymbol{r}$ and $\boldsymbol{P}$ if the following identities hold

$$\begin{aligned} &\{r^i, r^j\} = 0, \\ &\{r^i, P^j\} = \delta^{ij}, \\ &\{P^i, P^j\} = -\frac{eB}{c}\epsilon^{ij}, \end{aligned} \tag{4.7}$$

Now, we look at the limit $m \to 0$ in (4.6), in order to do that one also needs $P = 0$ as a constraint, which is not possible due to (4.7). For the purpose of resolving this problem, we introduce the following Dirac bracket

$$\{r^i, r^j\}_D = \frac{c}{eB}\epsilon^{ij}.$$

4.3 Non-commutative Landau problem with harmonic potential

Let $\mathcal{B}$ be the C^*-algebra of operators on $\mathscr{H}$ with self-adjoint part $\mathcal{A}$, which is considered as the Jordan algebra of observables. In this section, we will consider the motion of an electron in the xy plane under the influence of a constant magnetic field pointing along the positive z direction in the symmetric gauge $\boldsymbol{\mathcal{A}} = \left(-\frac{B}{2}y, \frac{B}{2}x\right)$. Furthermore, we will assume the presence of a harmonic potential. This problem is described by the following Hamiltonian

$$H_\theta = \frac{1}{2m}\left(p_x + \frac{eB}{2c}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{eB}{2c}x\right)^2 + \frac{m\omega^2}{2}(x^2 + y^2),$$

where θ is the non-commutative parameter, $\omega_L = \frac{eB}{mc}$ is the Larmour frequency. Position and momentum operators satisfy the following nonzero commutation relations of the non-commutative Heisenberg algebra

$$[x, y] = i\theta, \quad [x, p_x] = [y, p_y] = i\hbar. \tag{4.8}$$

Next, we define new operators in $\mathcal{B}$, corresponding to complex variables, related to chiral decomposition of non-commutative Landau problem. It is worth remarking that our approach here is reminiscent of the introduction of complex coordinates in the Landau problem pertaining to the Bloch waves quasi-periodicity on a torus (see [Dereli T., 2021]). Precisely, we have

$$z = \frac{1}{\sqrt{2}}(x + iy), \quad \overline{z} = \frac{1}{\sqrt{2}}(x - iy).$$

Corresponding derivation operators are computed to be

$$\partial_z = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y).$$

These newly defined operators can be shown to satisfy the following commutation relations in $\mathcal{B}$

$$[\partial_z, z] = [\partial_{\overline{z}}, \overline{z}] = 1,$$

$$[\partial_z, \overline{z}] = [\partial_{\overline{z}}, z] = [z, \overline{z}] = [\partial_z, \partial_{\overline{z}}] = 0.$$

We set two pairs of creation and annihilation operators in $\mathcal{B}$

$$A_+ = \xi\frac{\overline{z}}{2} + \frac{i}{\xi\hbar}p_z, \quad A_+^\dagger = \xi\frac{z}{2} - \frac{i}{\xi\hbar}p_{\overline{z}},$$

$$A_- = \xi\frac{z}{2} + \frac{i}{\xi\hbar}p_{\overline{z}}, \quad A_-^\dagger = \xi\frac{\overline{z}}{2} - \frac{i}{\xi\hbar}p_z,$$

where

$$\xi = \sqrt[4]{\frac{m^2\omega^2/\hbar^2 + m^2{\omega_L}^2/4\hbar^2}{1 - (m\omega_L\theta/2) + (m^2\omega^2\theta^2/16) + (m^2{\omega_L}^2\theta^2/64)}}.$$

The nonzero commutation relations between these operators are

$$[A_+, A_+^\dagger] = 1, \ [A_-, A_-^\dagger] = 1.$$

Finally, defining $\Omega_\pm := \Omega \pm \frac{\tilde{\omega}_L}{2}$, where

$$\Omega = \sqrt{\omega^2 + \frac{\omega_L^2}{4} - \frac{m\omega_L\omega^2\theta}{2} - \frac{m\omega_L^3\theta}{8} + \left(\omega^2 + \frac{\omega_L^2}{4}\right)\left(\left(\frac{m\omega\theta}{4}\right)^2 + \left(\frac{m\omega_L\theta}{8}\right)^2\right)},$$

$$\tilde{\omega}_L = \omega_L\left(1 - \left(\frac{\omega_L}{4} + \frac{\omega^2}{\omega_L}\right)m\theta\right),$$

one can write H_θ as follows [Aremua I., Baloitcha E., Hounkonnou M. N., Sodoga K., 2017], [Hounkonnou M. N., 2012]

$$H_\theta = \hbar\Omega_+ \left(N_+ + \frac{1}{2}\right) + \hbar\Omega_- \left(N_- + \frac{1}{2}\right), \quad N_\pm = A_\pm^\dagger A_\pm. \tag{4.9}$$

Here note that $\Omega_\pm$ is real since θ is negative. Let

$$H_\pm = \hbar\Omega_\pm \left(N_\pm + \frac{1}{2}\right).$$

We define the operator $H_+ \otimes \mathbb{I} + \mathbb{I} \otimes H_-$ to be the infinitesimal generator of the strongly continuous one-parameter unitary group $e^{itH_+} \otimes e^{itH_-}$. Thus by Stone's theorem, $H_+ \otimes \mathbb{I} + \mathbb{I} \otimes H_-$ is self-adjoint. Using that, (4.9) can be written as

$$H_\theta = H_+ \otimes \mathbb{I}_{\hat{\mathscr{H}}_{J,-}} + \mathbb{I}_{\hat{\mathscr{H}}_{J,+}} \otimes H_-.$$

Here $\hat{\mathscr{H}}_{J,+} \otimes \hat{\mathscr{H}}_{J,-}$ is the real Jordan-Hilbert space for this problem described in (3.5), and $\mathbb{I}_{\hat{\mathscr{H}}_{J,\pm}}$ are the corresponding identity operators. For all the forthcoming results after this point, see [Dereli T., Yörük E. S., 2024].

Theorem 4.3.1. *Nonzero commutation relations in $\mathcal{B}$, given in* (4.8)*, can be written as associator relations in $\mathcal{A}$ as follows*

$$[y_L, x, y_R] = \theta, \quad [p_{xL}, x, p_{xR}] = \hbar, \quad [p_{yL}, y, p_{yR}] = \hbar, \tag{4.10}$$

where

$$\begin{aligned} y_L = p_{xL} &= p_{yL} = 4H_\theta, \\ y_R &= \left(\frac{1}{2\hbar\Omega_-} - \frac{1}{2\hbar\Omega_+}\right) x - \left(\frac{1}{\xi^2\hbar^2\Omega_+} + \frac{1}{\xi^2\hbar^2\Omega_-}\right) p_y, \\ p_{xR} &= \left(\frac{\xi^2}{4\Omega_+} + \frac{\xi^2}{4\Omega_-}\right) x + \left(\frac{1}{2\hbar\Omega_+} - \frac{1}{2\hbar^2\Omega_-}\right) p_y, \\ p_{yR} &= \left(\frac{\xi^2\hbar}{4\Omega_+} + \frac{\xi^2\hbar}{4\Omega_-}\right) y + \left(\frac{1}{2\Omega_-} - \frac{1}{2\Omega_+}\right) p_x. \end{aligned}$$

Proof. In the case of a special Jordan algebra, an associator can be expressed as a double commutator

$$[A, B, C] = \frac{1}{4}[B, [A, C]]. \tag{4.11}$$

In order to find the operators explicitly, one needs to write x, y, p_x, and p_y in terms of newly defined creation and annihilation operators. It turns out that

$$x = \frac{1}{\sqrt{2}\xi}(A_- + A_+ + A_+^\dagger + A_-^\dagger),$$

$$y = \frac{i}{\sqrt{2}\xi}(-A_- + A_+ - A_+^\dagger + A_-^\dagger),$$

$$p_x = \frac{i\xi\hbar}{2\sqrt{2}}(-A_- - A_+ + A_+^\dagger + A_-^\dagger),$$

$$p_y = \frac{\xi\hbar}{2\sqrt{2}}(-A_- + A_+ + A_+^\dagger - A_-^\dagger).$$

The following commutation relation can be written for y

$$\left[\frac{H_\theta}{\Omega_+}, A_+ + A_+^\dagger\right] + \left[\frac{H_\theta}{\Omega_-}, -A_- - A_-^\dagger\right]$$

$$= \hbar([A_+^\dagger A_+, A_+] + [A_+^\dagger A_+, A_+^\dagger] + [A_-^\dagger A_-, -A_-] + [A_-^\dagger A_-, -A_-^\dagger])$$

$$= \hbar(-A_+ + A_+^\dagger + A_- - A_-^\dagger) = i\sqrt{2}\xi\hbar y.$$

So we have,

$$i\left[H_\theta, \left(\frac{1}{2\hbar\Omega_-} - \frac{1}{2\hbar\Omega_+}\right)x - \left(\frac{1}{\xi^2\hbar^2\Omega_+} + \frac{1}{\xi^2\hbar^2\Omega_-}\right)p_y\right] = y.$$

Similarly, one can check that

$$i\left[H_\theta, \left(\frac{\xi^2}{4\Omega_+} + \frac{\xi^2}{4\Omega_-}\right)x + \left(\frac{1}{2\hbar\Omega_+} - \frac{1}{2\hbar^2\Omega_-}\right)p_y\right] = p_x,$$

$$i\left[H_\theta, \left(\frac{\xi^2\hbar}{4\Omega_+} + \frac{\xi^2\hbar}{4\Omega_-}\right)y + \left(\frac{1}{2\Omega_-} - \frac{1}{2\Omega_+}\right)p_x\right] = p_y.$$

Using (4.11) and the commutation relation provided for y, p_x and p_y, associator relations (4.10) follow. □

4.3.1 Density Matrices in the JBW-algebra Setting

It turns out that one needs a notion of density matrix in the JBW-algebra $\mathcal{A}$ in order to describe the state of a quantum system. For the purpose of applications to JBW-algebras, we will use the results from [Alfsen E. M., 1978b], [Alfsen E. M., 2003] given in Lemma 2.2.1 and Theorem 2.2.6. We will assume in Theorem 4.3.3 that any element can be written as a finite linear combination of orthonormal projection operators. In particular, for any $B \in \hat{\mathscr{H}}_J$ we have

$$B = \sum_{i=1}^{n} \lambda_i P_i, \tag{4.12}$$

where λ_i's are real scalars, and P_i's denoting the orthonormal operators. And for a general element, we use the fact that it can be approximated in norm by linear combinations of orthonormal projections. So, the general case follows from the norm continuity of multiplication.

Definition 4.3.1. *A self-adjoint operator $\rho \in \mathcal{B}(\mathscr{H})_{sa}$ is a density matrix if ρ is non-negative and $tr(\rho) = 1$.*

Typically, quantum observables we consider are unbounded self-adjoint operators. But one can also take expectation values of bounded self-adjoint operators. It turns out that expectations for bounded operators determine those for unbounded operators. As an example, assume A is an unbounded self-adjoint operator and we know the expectation value for $1_E(A)$ for every Borel set $E \subset \mathbb{R}$, where 1_E is the indicator function of E and $1_E(A)$ is defined as follows:

$$1_E(A) = \int_{\sigma(A)} 1_E(\lambda) d\mu^A(\lambda),$$

where $\sigma(A)$ is the spectrum of A, and μ^A is the projection valued measure. The expectation value for $1_E(A)$ is the probability of obtaining a value in E for a measurement of A. If one knows the probability for each E, then the full probability distribution can be found. Therefore, the general notion of the state of a quantum system can be viewed as a list of expectation values.

Theorem 4.3.2. *Assume that ρ is a density matrix on $\hat{\mathscr{H}}_J$, then the map $\Phi_\rho : \mathcal{A} \to \mathbb{R}$ given by*

$$\Phi_\rho(A) = tr(\rho \bullet A)$$

is a family of expectation values.

Proof. In order to show this, we need to verify the following three properties of Φ.
(i) $\Phi_\rho(\mathbb{I}) = 1$,

(ii) $\Phi_\rho(A)$ is real for all $A \in \mathcal{A}$, and $\Phi_\rho(A)$ is non-negative whenever A is non-negative.
(iii) For any sequence $A_n \in \mathcal{A}$, if

$$\lim_{n\to\infty} \|A_n \bullet \psi - A \bullet \psi\| = 0$$

holds for all $\psi \in \hat{\mathscr{H}}_J$, then $\Phi_\rho(A_n)$ converges to $\Phi_\rho(A)$.
Note that (i) is immediate from the definition of a density matrix. Since $A \in \mathcal{A}$ is self-adjoint, $tr(\rho \bullet A)$ is real by definition. Let $B \in \hat{\mathscr{H}}_J$ be the non-negative, self-adjoint square root of ρ. It follows that $tr(A \bullet (B \bullet B)) = tr(B \bullet (A \bullet B)) \geq 0$, since $B \bullet (A \bullet B) \in \hat{\mathscr{H}}_J$ is self-adjoint and non-negative. This verifies condition (ii). Finally, assume $A_n \bullet \psi$ converges in norm to $A \bullet \psi$ for all $\psi \in \hat{\mathscr{H}}_J$. Then, $\|A_n \bullet \psi\|$ is bounded for each fixed ψ. By the principle of uniform boundedness, there is a constant C such that $\|A_n\| \leq C$. Let P_i's denote the orthonormal projection operators introduced in (4.12), we have

$$|\langle P_i, (B \bullet (A_n \bullet B)) \bullet P_i\rangle| = |\langle P_i, (A_n \bullet (B \bullet P_i)) \bullet B\rangle|$$

$$= |\langle (B \bullet P_i), A_n \bullet (B \bullet P_i))\rangle| \leq C\|B \bullet P_i\|^2,$$

$$\lim_{m\to\infty} \sum_{i=1}^{m} \|B \bullet P_i\|^2 = \lim_{m\to\infty} \sum_{i=1}^{m} \langle B \bullet P_i, B \bullet P_i\rangle = \lim_{m\to\infty} \sum_{i=1}^{m} \langle P_i, \rho \bullet P_i\rangle = tr(\rho) < \infty.$$

Since $A_n \bullet (B \bullet P_i)$ converges to $A \bullet (B \bullet P_i)$ for each i, by dominated convergence

$$tr(B^2 \bullet A) = tr(B \bullet (A \bullet B)) = \lim_{m\to\infty} \sum_{i=1}^{m} \langle P_i, (B \bullet (A \bullet B)) \bullet P_i\rangle$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \langle P_i, (B \bullet (A_n \bullet B)) \bullet P_i\rangle$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} \langle P_i, (B \bullet (A \bullet B)) \bullet P_i\rangle = \lim_{n\to\infty} tr(B \bullet (A_n \bullet B)) = \lim_{n\to\infty} tr(B^2 \bullet A_n).$$

□

For a normal state $B \in \hat{\mathscr{H}}_J$, the expectation value of an observable $A \in \mathcal{A}$ is given by

$$\Phi_B(A) = \langle B, A \bullet B\rangle = tr(B \bullet (A \bullet B)) = tr(B^2 \bullet A) = tr(\rho \bullet A),$$

where $\rho = B^2$ is the corresponding density matrix.

Definition 4.3.2. *A density matrix is called a pure state if it can be written as a projection operator in $\mathcal{A}$, otherwise it is called a mixed state.*

Now, we can revisit the non-commutative Landau problem, where the Hilbert space for the composite system is the Hilbert tensor product $\hat{\mathscr{H}}_{J,+} \otimes \hat{\mathscr{H}}_{J,-}$ of the Hilbert spaces $\hat{\mathscr{H}}_{J,+}$ and $\hat{\mathscr{H}}_{J,-}$ describing the subsystems. Assuming ρ is a density matrix on $\hat{\mathscr{H}}_{J,+} \otimes \hat{\mathscr{H}}_{J,-}$, there exists a unique density matrix ρ_+ on $\hat{\mathscr{H}}_{J,+}$ with the property that $tr(\rho_+ \bullet A) = tr(\rho \bullet (A \otimes \mathbb{I}_{\hat{\mathscr{H}}_{J,-}}))$ for all $A \in \mathcal{B}(\hat{\mathscr{H}}_{J,+})$. Similarly, there exists a unique density matrix ρ_- on $\hat{\mathscr{H}}_{J,-}$ with the property that $tr(\rho_- \bullet B) = tr(\rho \bullet (\mathbb{I}_{\hat{\mathscr{H}}_{J,+}} \otimes B))$ for all $B \in \mathcal{B}(\hat{\mathscr{H}}_{J,-})$. Since, the state of the first system is independent of the state of the second system, the density matrix ρ is of the form $\rho = \rho_+ \otimes \rho_-$ for this problem.

Theorem 4.3.3. *A pure state is being represented in $\hat{\mathscr{H}}_{J,+} \otimes \hat{\mathscr{H}}_{J,-}$ by $\pm P_{0,+} \otimes \pm P_{0,-}$, where $P_{0,+}$ and $P_{0,-}$ are idempotent elements on $\hat{\mathscr{H}}_{J,+}$ and $\hat{\mathscr{H}}_{J,-}$, respectively.*

Proof. To prove our claim, we consider a pure state $\rho = \rho_+ \otimes \rho_-$, given by an idempotent $P_{0,+} \otimes P_{0,-}$. Without loss of generality, we can take $P_{0,+}$ as the first basis vector of $\hat{\mathscr{H}}_{J,+}$, and $P_{0,-}$ as the first basis vector of $\hat{\mathscr{H}}_{J,-}$. We denote the other basis vectors as $P_{i,\pm}$. Let B be the square root of ρ. Then we have

$$B = \left(\sum_{i=0}^{n} \lambda_i P_{i,+} \right) \otimes \left(\sum_{i=0}^{n} \mu_i P_{i,-} \right).$$

Using this, ρ can be written as

$$\rho = B^2 = \left(\sum_{i=0}^{n} \lambda_i^2 P_{i,+} \right) \otimes \left(\sum_{i=0}^{n} \mu_i^2 P_{i,-} \right),$$

Taking traces of both sides, we obtain

$$\left(\sum \lambda_i^2 \right) \left(\sum \mu_i^2 \right) = 1.$$

Finally, we have

$$P_{0,+} \otimes P_{0,-} = \left(\sum_{i=0}^{n} \lambda_i^2 P_{i,+} \right) \otimes \left(\sum_{i=0}^{n} \mu_i^2 P_{i,-} \right).$$

The above formula clearly implies that $\lambda_0 = \pm 1$, $\mu_0 = \pm 1$, and $P_{i,\pm} = 0$ for $i > 0$. □

Now, we will define new observables before stating our next theorem giving the time evolution equation for the state vectors of non-commutative Landau problem. Let

$$X_+ = C_{1_+}(A_+^\dagger + A_+), \qquad X_- = C_{1_-}(A_-^\dagger + A_-),$$

be the position operators corresponding to subsystems. And

$$P_+ = iC_{2_+}(A_+^\dagger - A_+), \qquad P_- = iC_{2_-}(A_-^\dagger - A_-),$$

be the corresponding momentum operators, where

$$C_{1_+} = \sqrt{\frac{\hbar}{2m\Omega_+}}, \quad C_{2_+} = \sqrt{\frac{\hbar m\Omega_+}{2}}, \quad C_{1_-} = \sqrt{\frac{\hbar}{2m\Omega_-}}, \quad C_{2_-} = \sqrt{\frac{\hbar m\Omega_-}{2}}.$$

Theorem 4.3.4. *Given a state vector $v_+ \otimes v_- \in \hat{\mathscr{H}}_{J,+} \otimes \hat{\mathscr{H}}_{J,-}$, the time evolution equation can be written as follows*

$$\frac{d}{dt}(v_+\otimes v_-) = \frac{-4}{\hbar}((([L_{S_{1_+}}, L_{R_{1_+}}]+[L_{S_{2_+}}, L_{R_{2_+}}])v_+\otimes v_-)+(v_+\otimes([L_{S_{1_-}}, L_{R_{1_-}}]+[L_{S_{2_-}}, L_{R_{2_-}}])v_-)),$$

where $H_+ = i[R_{1_+}, S_{1_+}] + i[R_{2_+}, S_{2_+}]$, $H_- = i[R_{1_-}, S_{2_-}] + i[R_{2_-}, S_{2_-}]$,

$$R_{1_+} = -\frac{\Omega_+}{4}(X_+A_+ + A_+^\dagger X_+),$$

$$R_{2_+} = -\frac{\Omega_+}{4}(2C_{1_+}P_+A_+ + 2C_{1_+}A_+^\dagger P_+),$$

$$S_{1_+} = P_+A_+ + A_+^\dagger P_+,$$

$$S_{2_+} = 2C_{1_+}A_+^\dagger A_+,$$

$$R_{1_-} = -\frac{\Omega_-}{4}(X_-A_- + A_-^\dagger X_-),$$

$$R_{2_-} = -\frac{\Omega_-}{4}(2C_{1_-}P_-A_- + 2C_{1_-}A_-^\dagger P_-),$$

$$S_{1_-} = P_-A_- + A_-^\dagger P_-,$$

$$S_{2_-} = 2C_{1_-}A_-^\dagger A_-,$$

Proof. Let us first see that $H_+ = i[R_{1_+}, S_{1_+}] + i[R_{2_+}, S_{2_+}]$, $H_- = i[R_{1_-}, S_{2_-}] + i[R_{2_-}, S_{2_-}]$. We will only show the first equality, since two subsystems are independent identical copies. Note that

$$\begin{aligned}
&\frac{-i}{C_{1_+}C_{2_+}}[X_+A_+ + A_+^\dagger X_+, P_+A_+ + A_+^\dagger P_+] \\
&= [2A_+^\dagger A_+ + A_+^2 + A_+^{\dagger\,2}, A_+^{\dagger\,2} - A_+^2] \\
&= [2A_+^\dagger A_+, A_+^{\dagger\,2}] - [2A_+^\dagger A_+, A_+^2] + 2[A_+^2, A_+^{\dagger\,2}] \\
&= 2(A_+^\dagger[A_+, A_+^{\dagger\,2}] - [A_+^\dagger, A_+^2]A_+ + [A_+, A_+^{\dagger\,2}]A_+ + A_+[A_+, A_+^{\dagger\,2}]) \\
&= 4A_+^{\dagger\,2} + 4A_+^2 + 8A_+^\dagger A + 4\mathbb{I}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{-i}{C_{2_+}}[P_+A_+ + A_+^\dagger P_+, A_+^\dagger A] \\
&= [A_+^{\dagger\,2}, A_+^\dagger A_+] - [A_+^2, A_+^\dagger A_+] \\
&= -[A_+^\dagger A_+, A_+^\dagger]A_+^\dagger - A_+^\dagger[A_+^\dagger A_+, A_+^\dagger] + [A_+^\dagger A_+, A_+]A_+ + A_+[A_+^\dagger A_+, A_+] \\
&= -2A_+^{\dagger\,2} - 2A_+^2.
\end{aligned}$$

Combining these, we get the required result. The time evolution equation for density matrices in a C^*-algebra is given by

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho], \tag{4.13}$$

where H denotes the Hamiltonian. In the case of Jordan algebras, inner derivations are given in terms of the associator, which acts as a derivation on the second argument [Hanche-Olsen H., 1984]. Assuming $H = i[R, S]$, where R and S are self-adjoint, (4.13) transforms into the following equation

$$\dot{\rho} = \frac{-4}{\hbar}[R, \rho, S] = \frac{-4}{\hbar}[L_S, L_R]\rho,$$

where L_S and L_R are left multiplication operators with S and R, respectively. This amounts to the Jordan-algebraic evolution equation for the density matrices. All density matrices can be written in the form $\rho = B^2$, for some $B \in \mathcal{A}_2$. Taking the derivative of both sides

$$\dot{\rho} = 2B \bullet \dot{B}.$$

On the other hand, we also have

$$\dot{\rho} = \frac{-4}{\hbar}[R, \rho, S] = \frac{-4}{\hbar}[R, B^2, S] = \frac{-4}{\hbar} 2B \bullet [R, B, S].$$

Comparing them, one obtains

$$\dot{B} = \frac{-4}{\hbar}[R, B, S] = \frac{-4}{\hbar}[L_S, L_R]B.$$

Let us remark that this equation can be extended to whole space $\hat{\mathscr{H}}_J$ by

$$\dot{v} = \frac{-4}{\hbar}[R, v, S] = \frac{-4}{\hbar}[L_S, L_R]v.$$

Next, let $v_+ \otimes v_-$ be an element of $\hat{\mathscr{H}}_{J,+} \otimes \hat{\mathscr{H}}_{J,-}$ corresponding to the density matrix $\rho_+ \otimes \rho_-$, and let $H_\theta = H_+ \otimes \mathbb{I}_{\hat{\mathscr{H}}_{J,-}} + \mathbb{I}_{\hat{\mathscr{H}}_{J,+}} \otimes H_-$ be the Hamiltonian, where $H_+ = i[R_{1_+}, S_{1_+}] + i[R_{2_+}, S_{2_+}]$, $H_- = i[R_{1_-}, S_{2_-}] + i[R_{2_-}, S_{2_-}]$. Based on these, we may deduce that

$$\begin{aligned}
\frac{d}{dt}(\rho_+ \otimes \rho_-) &= -\frac{i}{\hbar}[H_+ \otimes \mathbb{I}_{\hat{\mathscr{H}}_{J,-}} + \mathbb{I}_{\hat{\mathscr{H}}_{J,+}} \otimes H_-, \rho_+ \otimes \rho_-] \\
&= -\frac{i}{\hbar}([H_+ \otimes \mathbb{I}_{\hat{\mathscr{H}}_{J,-}}, \rho_+ \otimes \rho_-] + [\mathbb{I}_{\hat{\mathscr{H}}_{J,+}} \otimes H_-, \rho_+ \otimes \rho_-]) \\
&= -\frac{i}{\hbar}((([H_+, \rho_+] \otimes \rho_-) + (\rho_+ \otimes [H_-, \rho_-])).
\end{aligned}$$

Using the discussion above, we may replace $-i[H, \rho]$ with $-4[L_{S_1}, L_{R_1}]v - 4[L_{S_2}, L_{R_2}]v$ to arrive at the formula

$$\frac{d}{dt}(v_+ \otimes v_-) = \frac{-4}{\hbar}((([L_{S_{1_+}}, L_{R_{1_+}}] + [L_{S_{2_+}}, L_{R_{2_+}}])v_+ \otimes v_-) + (v_+ \otimes ([L_{S_{1_-}}, L_{R_{1_-}}] + [L_{S_{2_-}}, L_{R_{2_-}}])v_-)).$$

Finally, the solution to this Jordan-Schrödinger equation can be written as

$$v_+(t) \otimes v_-(t) = e^{\frac{-4t}{\hbar}([L_{S_{1_+}}, L_{R_{1_+}}] + [L_{S_{2_+}}, L_{R_{2_+}}])} v_+(0) \otimes e^{\frac{-4t}{\hbar}([L_{S_{1_-}}, L_{R_{1_-}}] + [L_{S_{2_-}}, L_{R_{2_-}}])} v_-(0).$$

□

Chapter 5

CONCLUSION

In this thesis, a review for the algebraic approach to quantum mechanics is given. As an example, the non-commutative Landau problem is studied. For a two dimensional non-commutative space, a Jordan algebraic formulation of the Hamiltonian associated to the motion of an electron under a uniform magnetic field with a confining harmonic potential has been provided. The main result is given in Theorem 4.3.1, and the non-commutative parameter θ is written in terms of an associator in the Jordan algebra of observables. Secondly, a general understanding for the expectation value of an observable is provided in terms of the Jordan algebraic setting. In addition, the pure states are characterized for this specific problem. Lastly, a Jordan algebraic time evolution equation for the state vectors in the constructed Hilbert space and its solution are presented.

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