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A NOVEL APPROACH FOR HEAVY QUARKONIUM SPECTRA

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BY

YÜCEL CANÇELİK

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A Novel Approach
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Heavy Quarkonium Spectra

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Supervisor

Prof. Dr. Bülent GÖNÜL

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Yücel CANÇELİK

ABSTRACT

A NOVEL APPROACH FOR HEAVY QUARKONIUM SPECTRA

CANÇELİK, Yücel

Ph.D. in Engineering Physics

Supervisor: Prof. Dr. Bülent Gönül

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In this work, we attempt to investigate heavy quark systems ($c\bar{c}$ and $b\bar{b}$) in the non-relativistic framework using phenomenological inter-quark potentials. For this reason, we have developed a powerful and flexible theoretical formalism, within the frame of the Schrödinger equation, to use for the analysis of heavy quarkonium systems involving exactly solvable energy-dependent potentials, together with quasi-/conditionally-solvable and non-solvable interaction potentials.

Our first application to exactly solvable systems has clarified that energy dependent term in the potential leads to saturation of the spectra and degree of saturation is governed by the magnitude of perturbation. Second application to quasi-exactly solvable potentials, which is the Coulomb plus linear potential with a quadratic term, has justified the success of the present formalism due to its wide range of applicability, considering the comparison of the findings obtained by the present approach with those of experimental results and the other theoretical predictions in the related literature. Finally, the application of this new technique to also non-solvable systems involving Cornell potential has shed some light on the questions concerning with the limitations of the traditional perturbation techniques.

Key Words: Heavy Quarkonium, Phenomenological Inter-Quark Potentials, Energy Dependent-Exactly Solvable Potentials, Quasi-Exactly Solvable Potentials, Non-Solvable Potentials, Perturbation Theories

ÖZET

AĞIR KUARK-ANTIKUARK BAĞLI DURUM ENERJİ ANALİZİ İÇİN YENİ BİR YAKLAŞIM

CANÇELİK, Yücel

Doktora Tezi, Fizik Mühendisliği Bölümü

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Haziran 2016, 91 sayfa

Bu çalışmada, kuarklar arası etkileşim potasiyelleri kullanılarak relativistik olmayan çerçevede ağır ve bağlı kuark-antikuark sistemlerine ($c\bar{c}$ and $b\bar{b}$) ait enerji durumları farklı bir kuramsal çerçevede analiz edildi. Bu amaçla ilkin, ağır kuark-antikuark bağlı sistemlerinin analizinde kullanılmak üzere Schrödinger denklemi yardımı ile literatürde sıklıkla karşılaşılan tam çözülebilen enerji-bağımlı potansiyeller ile kısmen/şartlı çözülebilir ve tam çözülemeyen etkileşim potansiyellerinin kuramsal analizini verebilen güçlü ve esnek bir formalizm geliştirildi.

Tam çözülebilen sistemler üzerine yapılan ilk uygulama, potansiyeldeki enerjiye bağımlı terimin enerji değerlerini doyum noktasına ulaştırdığına ve bu doyma derecesinin pertürbasyonun büyüklüğü ile belirlendiğine açıklık getirdi. Tez kapsamında geliştirilen modelimizin ikinci uygulaması ise, lineer potansiyele ek olarak kuadratik ifade içeren üç terimli Coulomb potansiyeline ait elde edilen teorik bulgularla ilgili deneysel sonuçların ve literatürdeki diğer kuramsal tahminlerin karşılaştırılması dikkate alındığında, bahsedilen formalizmin farklı alanlarda başarıyla uygulanabilirliğini kanıtlamıştır. Son olarak, bu yeni kuramsal tekniğin tam çözülemeyen ve ilgili literatürde çok iyi bilinen Cornell potansiyeli için uygulama sonuçları ise geleneksel pertürbasyon tekniklerinin sınırlamaları ile ilgili literatürde sıkça tartışılan bazı hususlar üzerine ışık tutarak okuyucuya anlam derinliği kazandırmıştır.

Anahtar Kelimeler: Ağır Kuark-Antikuark Çifti, Kuarklararası Kuramsal Potansiyeller, Enerjiye Bağlı Tam Çözülebilir Potansiyeller, Kısmen-Kesin Çözülebilir Potansiyeller, Tam Çözülemeyen Potansiyeller, Yaklaşım Teorileri

I dedicated this work to my grandfather

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CHAPTER 1

INTRODUCTION

When I was in high school, during a lecture about the structure of atom, I asked to my teacher about the possibility to break up the neutron, proton and electron. I got no answer but a tight scold from my teacher. I understand now what made him so angry. Even if my teacher came up with a positive answer with some new particle as a building block of those, I would have asked the same question again. The tight scold was not as painful as the thought of making something infinitely divisible which made my teacher restless.

At the beginning of the twentieth century, scientists believed that they understood the most of the fundamental principles of nature. Atoms were solid building blocks of nature; people trusted Newtonian laws of motion; most of the problems of physics seemed to be solved. However, starting with Einstein's theory of relativity which replaced Newtonian mechanics, scientists gradually realized that their knowledge was far from complete. Of particular interest was the growing field of quantum mechanics, which completely altered the fundamental laws of physics.

By the mid-1960's, physicists realized that their earlier understanding, where all matter is composed of the nucleons and electron, was insufficient to explain the new particles being discovered. Gell-Mann's and Zweig's quark theory solved these problems. Over the last years, the theory that is now called the Standard Model of particles and interactions has gradually grown and gained increasing acceptance with new evidence from new particle accelerators.

Therefore, one of the main objectives of the study of nuclear physics is understanding the structure of nuclei. Since the exact character of nuclear forces is not known, nuclear models are used in investigations of regularities in the systematics of nucleiEven if the exact nature of nuclear forces were known, it would still not be possible to accurately solve the resulting many-body problem. The main object would still be to make approximations which reduce the exact, but unsolvable, problem to a solvable one, allowing as little error as possible. Nuclear models, therefore, approximate the actual nuclear system as closely as possible while still being easily controlled mathematically.

In connection with this, today quarkonium is a key issue in most of the accelerator experiments where millions and in some cases billions of heavy quarkonium states are being produced. Historically quarkonium has been one of the most important playgrounds for our understanding of the strong interactions.

When the quark model was first formulated very few physicists considered quarks as particles and tried to calculate the hadron spectrum from them. Among those who did we could mention Dalitz [1], almost 'preaching in the desert' at the Oxford conference in 1965, and Gerasimov [2]. The situation changed drastically after the '1974 October Revolution'. As soon as the J/ψ ($c\bar{c}$) [3, 4] and the Υ ($b\bar{b}$)[5] had been discovered, the interpretation of these states as charm-anticharm bound states was universally accepted and potential models using the Schrodinger equation were proposed [6, 7].

After November 1974, when it was understood that the J/ψ was made of heavy quark-antiquark pairs, there was a renewed interest in potential models of hadrons, which continued with the discovery of the *b* quark in 1977. In parallel with positronium was obvious; this is the origin of the neologism "quarkonium". However, in contrast to positronium, which is dominated by the Coulomb potential, the potential between quarks was not known and outside explicit numerical calculations with specific models, there was a definite need for new theoretical tools to study the energy levels, partial widths, radiative transitions, etc. for large classes of potentials. This led to the discovery of a large number of completely new rigorous results on the Schrodinger equation which are interesting not only for the qualitative understanding of quarkonium and more

generally hadrons but also in themselves and which can be in turn applied to other fields such as atomic physics.

name(symbol)	**electric charge (*e*)**	**mass (mc^2)**
up (u)	2⁄3	$2.49^{+0.81}_{-0.79}$ MeV
down (d)	−1⁄3	$5.05^{+0.75}_{-0.95}$ MeV
charm (c)	2⁄3	$1.27^{+0.07}_{-0.11}$ GeV
strange (s)	−1⁄3	104^{+26}_{-34} MeV
top (t)	2⁄3	$171.2^{+2.1}_{-2.1}$ GeV
bottom (b)	−1⁄3	$4.20^{+0.17}_{-0.07}$ GeV

Table 1.1: A summary of the quark properties in the SM. [8].

Because strong interactions are flavour independent, it would appear that they can be equally well studied in the spectroscopy of light or heavy quarks. However, the highly relativistic nature of quarks in light-quark (*u, d, s*) hadrons, the large value of the strong coupling constant $\alpha_s(\geq 0.6)$, and the near equality of the masses of the *u, d, s* quarks, which are shown in Table 1.1, makes the light quark states strongly overlapping, with small spacing ≈ 14 MeV, large widths, $\Gamma \approx 150$ MeV, and mostly mixtures of all three flavours. In contrast, $b\bar{b}$ and $c\bar{c}$ hadrons are relatively free from these problems, with $\langle v^2/c^2 \rangle \approx 0.1-0.2$, $\alpha_s = 0.2-0.3$, and have well resolved states (spacing ≈ 15−40 MeV, $\Gamma \approx 0.05-5$ MeV). The spectra of $|c\bar{c}\rangle$ charmonium (*J/ψ* etc.) and $|b\bar{b}\rangle$ bottomium (ϒ *etc.*) are illustrated in Fig. 1.1, which are the subject of this section.

Within this context, energy spectrums of heavy quarkonium are a rich source of information on the nature of the inter-quark force, which leads to understanding the interaction mechanism behind nuclear forces. Many features such as mass spectra and decay properties of heavy quarkonium could be described by applying the ordinary non-relativistic Schrödinger wave equation to the two body quark- antiquark system.

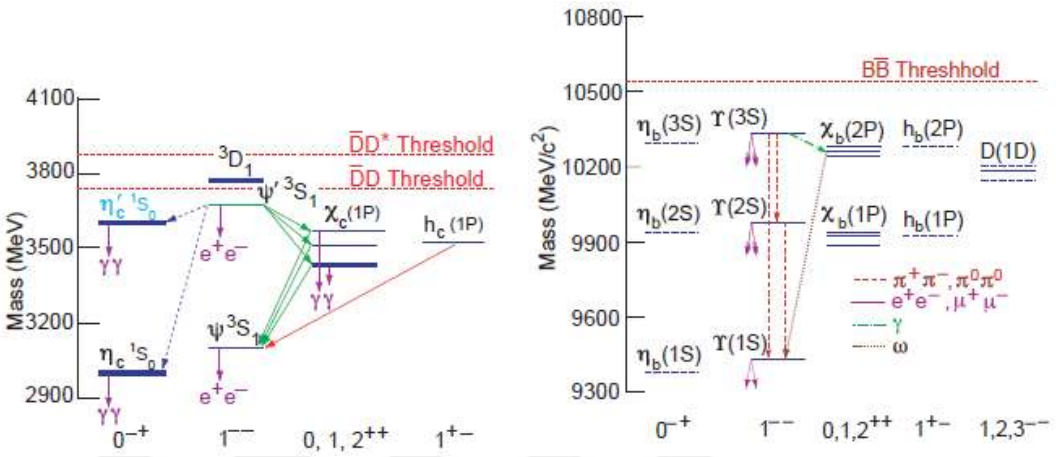

Figure 1.1: The spectrum of quarkonium states, (left for charmonium, right bottomium)

It is well known that the non-relativistic approach is justified when large quark masses are involved and level spacing between the energy levels is less than the constituent masses.

A variety of forms for the interquark potential have been used in heavy quarkonium mass spectroscopy. These can be broadly classified as

1) Quantum Chromodynamics (QCD) motivated potential [9-13] and

2) Purely phenomenological potential [14-19].

A comprehensive list of potential models is described in the work of Lichtenberg [20]. All the potentials have almost similar behavior in the range of 0.1 fm $\leq r \leq$ 1 fm, the characteristic interval of $c\bar{c}$ and $b\bar{b}$ systems but differ from each other outside this range. At present it is not possible to obtain exact inter-quark potential in the whole range of distances from the first principal of QCD. Moreover major shortcoming of the existing potentials is that they have not been able to account for the observed saturation pattern in the experimental spectra.

A different category of potentials, which are energy dependent, have been known in physics for a long time. They encountered in relativistic quantum mechanics

at various places like with Pauli Schrodinger equation [21], in the Hamiltonian formulation of relativistic many body problem with constraints in nonlinear Hamiltonian evolution equation, and also in solution propagation. In the non-relativistic physics energy dependent potentials offer the possibility of studying non-linear effects in the framework of Schrodinger equation. Lombard and his co-workers [9] have for the first time used an energy dependent potential to study the bound state properties of $c\bar{c}$ and $b\bar{b}$ systems. Initially they had used one dimensional harmonic oscillator potential with a linear energy dependent term as perturbation to study the effect of energy dependence on the energy eigenvalues. Later they solved the problem for three dimensions with different power law potentials (Harmonic, Linear and Coulomb) with energy dependence. Their main conclusion has been that the energy dependence saturates the mass spectra. It is well known that any realistic inter-quark potential has two components: asymptotic and confinement. However Lombard's potential has only one component either confinement (Harmonic and Linear) or asymptotic (Coulomb). In view of this point, Chapter 1 of the present thesis work discusses in detail such interactions with small linear energy dependence on the confining harmonic oscillator and Coulomb potentials.

The choice of linear dependence is motivated by the fact that it leads to a coherent theory [10]. The combination of harmonic oscillator and inverse square potential was first of all adopted by Joshi and Mitra [11] in a theory based on the Schrödinger equation for studying the heavy meson spectroscopy. Later it has been used by Iyer *et al.* [12] and Reyes *et al.* [13] in the study of hadron spectroscopy. In all these studies a better fit was obtained for the potential with a term proportional to $1/r^2$ because it has singularity for $r \to 0$ that improves the behavior in this region. Also one great advantage is that such potential admits exact analytical solution for the radial part of the Schrodinger equation. This is a great advantage in view of the high nonlinearity of the differential equation to be solved. Using this potential researchers have calculated the mass spectrum, the root mean square radii, average radii, leptonic decay width (Γ) and $\langle 1/r \rangle$ for $c\bar{c}$ and $b\bar{b}$ systems to see the reliability of such potentials. The latter two properties are sensitive to the asymptotic part of the potential. The another significant aim in such studies is also to search the effect of energy dependence on the low as well as high excitation states of the system by accounting for the properties of charmonium

and bottomium, which is the main motivation behind the work in Chapter 1 in this thesis work.

Since their discoveries, investigation of heavy quarkonium systems ($c\bar{c}$ and $b\bar{b}$) provides us a crucial role for quantitative tests of QCD and the standard model. For a detailed review of heavy quarkonium physics recent progresses, see e.g. [22, 23, 24] and the related references therein. Because of the heavy masses of the constituent quarks which are illustrated in Table 1.1 ($m_q \gg \Lambda_{QCD}$, i.e. than about 200 MeV, where Λ_{QCD} is the hadronization scale), many features of these systems can be investigated within the framework of nonrelativistic Schrodinger equation, where one assumes that the quark-antiquark strong interaction is described by a phenomenological potential. There are many potential models that are commonly used to study heavy quarkonium spectra; for instance, Martin, logarithmic, and Cornell potentials [19, 24, 25, 26]. Any of these potential should take into account the two distinctive features of the strong interaction, namely, asymptotic freedom and confinement. A successful potential model for such systems is the one that produces its mass spectra in agreeing with the experimental data within about 20 MeV and leptonic decay widths within a factor of two [27]. Additionally, the main obstacle in such studies arises due to the absence of the exact solutions of Schrödinger equation for such systems, particularly when the centrifugal potential $\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$ is included. For $\ell \neq 0$, some approximation techniques, analytical and numerical, have been developed, such as Supersymmetry [28], 1/N expansion [29], variational methods [30, 31], and asymptotic iteration methods [32].

In fact, a simple method of investigating the solution of the Schrödinger equation [33] has been known for a long time. These authors applied their method to the hypergeometric, confluent hypergeometric and Bessel equations. Later it turned out that it can be related to algebraic techniques of solving differential equations [34]. Another systematic application of this method (to the hypergeometric functions) has been carried out by Natanzon [35] independently. In the following years, there has been also renewed interest in simple quantum mechanical systems as a result of the introduction of two important concepts: Supersymmetric Quantum Mechanics (SUSYQM) and shape invariance. For a comprehensive review on this topic, the reader is referred to [36] and the related references therein. In the light of this progress and the previous works mentioned, a significant question has then arised regarding if there are any other special

functions which are solutions of the Schrödinger equation with shape invariant potentials. This question has been answered in detail by Lévai [37] through the consideration of the link between the works in [33, 34, 35] and the formalism of SUSYQM, deducing a condition which has to be satisfied by any special function leading to the orthogonal polynomials and exactly solvable shape invariant potentials. Besides the results obtained, the combination of SUSYQM with traditional approaches to solvable potentials proved to be fruitful. For instance, [38, 39, 40, 41, 42, 43, 44, 45, 46, 47] involve some applications of the original idea discussed in [37].

However, to our knowledge, this formalism up to now has been used only to study exactly solvable systems. Therefore, it needs a meticulous modification to also solve more realistic other systems, such as the ones of interest in this thesis work. Within this context the main motivation behind the present investigation, bearing in mind that realistic physical problems can practically never be solved exactly, is to suggest a more comprehensive and generalized model using the spirit of the investigation in [37].

For this reason, in the present thesis work, considering all the previous investigations mentioned above, we have developed a more powerful and flexible theoretical formalism within the non-relativistic frame for the treatments of heavy quark systems ($c\bar{c}$ and $b\bar{b}$).

In Chapter 2, we present the applications on the energy dependent-exactly solvable potentials. Theoretical description of energy-dependent interactions have been subjected to intensive investigations during the last decades and the use of such phenomenological potentials in wave equations are proven to be useful in dealing with problems in atomic, molecular as well as nuclear and particle physics [48-53]. In particular, the presence of energy-dependent contribution in the potential has several implications modifying the usual rules of quantum mechanics. To get an insight into the clean route visualizing such modifications within the frame of the new scheme, linear energy dependency are considered which has not been previously studied under the traditional models discussed above. This work provides a benchmark test for the present model calculations; too, as the full result obtained within the new formalism should reduce to the familiar solutions concerning with the exactly solvable energy-independent potentials in case the potential parameter related to the energy vanishes.

This chapter has clarified that energy dependent term in the potential leads to saturation of the spectra and degree of saturation is governed by the magnitude of perturbation.

In Chapter 3, we apply our new model to also the quasi-exactly solvable three-term perturbed Coulomb potential. The corresponding energy eigenvalues and eigenfunctions are obtained in compact forms for any l-value, but for low-lying states. The obtained results are used to produce potential parameters (a, b and c) for the charmonium and bottomonium systems, from which then their full mass spectra are determined. The findings are then compared to the available experimental results and also to the predictions of the other models. Remarkable agreement in these comparisons has been observed. This successful application has justified the flexibility of the novel model.

For completeness, in Chapter 4, we finally attempt to apply the model to non-solvable Cornell potential, too. The Schrödinger equation with the Cornell potential is an important non-relativistic model to study quark-antiquark systems. This Coulomb-plus-linear pair potential was originally proposed for describing quarkonia with heavy quarks. It takes into account general properties expected from the interquark interaction, namely Coulombic behavior at short distances and a linear confining term at long distances [54]. The present model application to this potential has been well discussed in this chapter involving a deficiency arised naturally during the application. An elegant use of this defect is proposed through this chapter to shed some light on the questions concerning with the limitations of traditional perturbation theories.

Finally, concluding remarks are presented in Chapter 5.

CHAPTER 2

APPLICATION TO EXACTLY SOLVABLE POTENTIALS

Here, we investigate heavy quark systems ($c\bar{c}$ and $b\bar{b}$) in the non-relativistic framework using energy dependent inter-quark potentials. For this reason, we use a proper algebraic technique developed recently [53], within the frame of the Schrödinger equation, considering physically meaningful interaction potentials to provide analytical solutions in an explicit manner for the system of interest. The present work has clarified that energy dependent term in the potential leads to saturation of the spectra and degree of saturation is governed by the magnitude of perturbation.

2.1. Formalism

It is well known that the general framework of non-relativistic quantum mechanics is by now well understood and its predictions have been carefully proved against observations. Physics is continuously developing in a tight interplay with mathematics. It is of importance to know therefore whether some familiar problems are of particular case of a more general scheme or to search if a map between the radial equations of two different scenarios exists. It is hence worthwhile to devote ourselves to the clarification of this point through the rest of this thesis.

Considering the Schrödinger equation $(\hbar=2m=1)$

$$\frac{d^2\Psi}{dr^2}+\big(E\text{-}V(r)\big)\Psi=0 \tag{2.1}$$

we suggest, for a generalized formalism, that

$$\Psi(r) = f(r)F\big(g(r)\big)h(r) \tag{2.2}$$

where $f(r)F\big(g(r)\big)$ yields an algebraic closed solution for exactly solvable potentials with $F(g)$ being a special function which satisfies a second-order differential equation

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0 \tag{2.3}$$

while $h(r)$ is the moderating function in connection with a perturbing piece of the full potential corresponding to (2.2). The form of $Q(g)$ and $R(g)$ is already well defined for any special function $F(g)$ when dealing with analytically solvable potentials. However, in case of the consideration of a realistic non-exactly solvable problem one should derive reliable expressions, in an explicit form, for plausible definitions of the related $Q(g)$ and $R(g)$. This is the significant point in the framework of the new formalism to reach physically meaningful solutions.

Substituting Eq. (2.2) into (2.1) leads to

$$\frac{d^2F}{dg^2} + \frac{dF}{dg}\left(\frac{g''}{(g')^2} + \frac{2f'}{f\,g'} + \frac{2h'}{hg'}\right) + F(g)\left(\frac{f''}{f(g')^2} + \frac{2f'h'}{f\,h(g')^2} + \frac{h''}{h(g')^2} + \frac{E-V(r)}{(g')^2}\right) = 0 \tag{2.4}$$

From the comparison of Eqs. (2.3) and (2.4) it follows that

$$Q\big(g(r)\big) = \frac{g''}{(g')^2} + \frac{2f'}{fg'} + \frac{2h'}{hg'} \tag{2.5}$$

and

$$R(g(r)) = \frac{f''}{f(g')^2} + \frac{2f'h'}{f\,h(g')^2} + \frac{h''}{h(g')^2} + \frac{E-V(r)}{(g')^2} \tag{2.6}$$

Obviously, Eqs. (2.4-2.6) reduce to Eqs. (3.4-3.6) in Ref. [37] for the consideration of exact solvability, in which case $h(r)$ in the equations above goes to a constant value. Gaining confidence from this observation we proceed with

$$\begin{aligned} V(r) &= V_{ES}(r) + \Delta V\left(r\right) \\ E &= E_{ES} + \Delta E \end{aligned} \tag{2.7}$$

in accordance with our choice in (2.2), which means that potentials considered in this article are admitted as the sum of an exactly solvable potential with a perturbation or a moderating piece. Hence, the aim in this perspective is to reveal the corrections to energy (ΔE) and wavefunction $h(r)$ for a given $\Delta V\left(r\right)$, as the main piece of the solutions leading to exact solvability can easily be found from the literature.

The use of (2.7) within Eq. (2.6) produces coupled equations in the form of

$$E_{ES} - V_{ES}(r) = R_{ES}\left(g\left(r\right)\right)\left(g'\right)^2 - f''/f \tag{2.8}$$

and

$$\Delta E - \Delta V\left(r\right) = \Delta R\left(g\left(r\right)\right)\left(g'\right)^2 - 2f'h'/(fh) - h''/h \tag{2.9}$$

where $R_{ES}\left(g\left(r\right)\right) + \Delta R\left(g\left(r\right)\right)$ should certainly reproduce Eq. (2.6). Similarly, Eq. (2.5) can be decomposed as

$$Q_{ES}\left(g\left(r\right)\right) = g''/\left(g'\right)^2 + 2f'/fg' \ , \ \Delta Q(g(r)) = 2h'/hg' \quad \Rightarrow \quad Q = Q_{ES} + \Delta Q \tag{2.10}$$

to be more practical it is reminded that $f''/f = (f'/f)^2 + (f'/f)'$ and the same is valid for h''/h in the equations above, which transform Eqs. (2.8) and (2.9) into more applicable forms

$$E_{ES} - V_{ES}\left(r\right) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \left(g'\right)^2\left[R_{ES}\left(g\left(r\right)\right) - \frac{1}{2}\frac{dQ_{ES}}{dg} - \frac{1}{4}Q_{ES}\left(g\left(r\right)\right)^2\right] \tag{2.11}$$

and

$$\Delta E-\Delta V\left(r\right)=-\left(\frac{g''}{2}+\frac{f'g'}{f}\right)\Delta Q\left(g\left(r\right)\right)+\left(g'\right)^{2}\left[\Delta R\left(g\left(r\right)\right)-\frac{1}{2}\frac{d\left(\Delta Q\right)}{dg}-\frac{1}{4}\Delta Q\left(g\left(r\right)\right)^{2}\right] \tag{2.12}$$

The result of this brief investigation opens a gate to the reader for the visualization of the explicit form of the correction (ΔE) to the energy. Unfortunately, there seems a problem naturally arised in calculating the correction term owing to the presence of two unknown: $\Delta Q\left(g\left(r\right)\right)$ and $\Delta R\left(g\left(r\right)\right)$ on the right hand side of Eq. (2.12). To circumvent the resulting drawback and proceed safely we need to go back Eq. (2.4) and substitute the definitions given by (2.7) in it, which leads us to handle

$$\Delta E-\Delta V(r)=-\frac{2h'}{h}\left(\frac{f'}{f}+\frac{F'\left(g\left(r\right)\right)\,g'}{F\left(g\left(r\right)\right)}\right)-\frac{h''}{h} \tag{2.13}$$

that is another form of (2.9). Thus, equating (2.9) and (2.13) and remembering the form of ΔQ in Eq. (2.10) we arrive at

$$\Delta R\left(g\right)=-\Delta Q\left(g\right)\frac{F'\left(g\right)}{F\left(g\right)} \tag{2.14}$$

which is vital to overcome the problem encountered in (2.12). As $F\left(g\right)$ is well defined for a given exactly solvable potential, evidently one need here to find only an appropriate expression for $\Delta Q\left(g\right)$ to be employed in (2.12) that reveals clearly the full solution.

Before closing this section, we should remark that once choosing carefully $Q_{ES}(g)$ and $R_{ES}\left(g\right)$ for the analytically solvable part ($V_{ES}\left(r\right)$) of the full potential under investigation we can easily set a proper internal function $g_{(r)}$ and considering Eq. (2.5)

$$f\left(r\right)=\left(g'\right)^{-\frac{1}{2}}\exp\left[\frac{1}{2}\int^{g(r)}Q_{ES}\left(g\right)dg\right] \tag{2.15}$$

as discussed in Ref. [37], which are used in Eqn.(2.12) to find corrections to the solutions of the exactly solvable piece.

The application of the model to specifically chosen different potentials is discussed in the following section.

2.2. Applications

Special care has to be taken in the application of the model as the results obtained are crucial in the interpretation of the system behaviour in terms of the Hamiltonian described in this work. To reveal especially the flexibility of the scheme used particular cases are discussed İn the next section.

2.2.1. Energy-dependent Potentials

Considering an ongoing belief that standard techniques for approximating a given potential with a separable potential are only applicable to energy independent potentials, a significant extension of the model applications is achieved in this section by incorporating such conventional considerations to those accommodating explicit energy dependence in potentials (EDP) with emphasis on power-law potentials as examples admitting analytical solutions. Heavy quark systems in particular constitute a natural domain for the application of such interactions. Comparing the results of EDP with those of conventional potentials the new features appeared [48, 49, 50, 51, 52] in a deep understanding of the systems in high energy physics. For instance, it is now clear that the energy dependent component in the potential has a significant influence on the calculated observables of charmonium and bottomium, unlike the conventional ones. Also the existence of analytical solutions presents a good opportunity in tackling such problems [52].

However, the physical discussion behind EDP applications is not our goal at the present stage. The real question is to know here if there is a failure in the application of our extended formalism to the systems involving EDP, which is the subject of the next

section. For the sake of simplicity, we assume a spherical symmetry and a linear energy dependency in the three illustrative examples discussed in the next sections.

2.2.1.1. Harmonic Oscillator

Solutions of such equations with EDP exhibit properties quite unusual with respect to the known solutions of the ordinary Schrödinger equation for the same potential shape. This is particularly spectacular in the case of harmonic oscillator with a linear energy dependence, which is well discussed in [48,49].

The related reduced radial wave equation reads

$$\frac{\Psi''_{n\ell}(r)}{\Psi_{n\ell}(r)} = \left[V\left(r, E_{n\ell}\right) + \frac{\ell(\ell+1)}{r^2} \right] - E_{n\ell} \tag{2.16}$$

where

$$V\left(r, E_{n\ell}\right) = V_{ES}\left(r\right) + \Delta V\left(r, E_{n\ell}\right) = \left(\frac{\omega^2 r^2}{4} + \frac{\ell(\ell+1)}{r^2} \right) + \frac{\omega^2 r^2}{4}\left(\gamma E_{n\ell}\right) \tag{2.17}$$

The familiar solutions of the energy independent piece of the spherical harmonic oscillator potential $V_{ES}\left(r\right)$ in three dimensions are [55]

$$\begin{aligned} E_{ES} &= \left(2n + \ell + \frac{3}{2}\right)\omega \\ \Psi_{ES}\left(r\right) &= f\left(r\right)F\left(g\left(r\right)\right) \sim g^{\frac{\ell+1}{2}} \exp\left(-\frac{g}{2}\right) L_n^{\left(\ell+\frac{1}{2}\right)}\left(g\right) \end{aligned} \tag{2.18}$$

due to the choice of the generalized Laguerre polynomials $F(g) = L_n^{\left(\ell+\frac{1}{2}\right)}(g)$ which leads us to consider $Q_{ES} = \left(\ell - g + \frac{3}{2}\right)/g$ and $R_{ES} = n/g$ in dealing with the algebraic solutions of the corresponding differential equations. This ends up with the strict definitions of the interval functions such as $g(r) = \frac{\omega}{2} r^2$ and $f = \left(2\omega\right)^{-1/4} g^{(\ell+1)/2} \exp\left(-g/2\right)$. Obviously, the substitution of these findings in (2.8) or

(2.11), as previously discussed in section 2.1, reproduces Eq. (2.18). Though this reveal anything new, it would be helpful in arriving at the modification terms in their explicit form for understanding the influence of energy dependence of the interaction potential.

From the expertise gained in the analyses of the quartic anharmonic oscillator problem [53], we can safely set

$$\Delta Q = 1 - \sqrt{1+\gamma E_{n\ell}} \tag{2.19}$$

to obtain certain expressions for the corrections brought by the energy component of the potential. It is noted that (2.19) dies away in the case of $\gamma = 0$, from which Eqs. (2.14) and subsequently (2.12) vanish. This confirms the reliability of the choice in (2.19) which is the key point for benchmark tests when compared to the solutions in connection with the conventional energy independent potentials. The use of (2.19) in Eqs. (2.14) and then (2.12) reveals the modifying term as

$$\Delta E = \left(\sqrt{1+\gamma E_{n\ell}} - 1\right)\left(2n + \ell + \frac{3}{2}\right)\omega \tag{2.20}$$

for the energy, and similarly one can combine the form of ΔQ in Eq. (2.10) with (2.19) to get

$$h\left(r, E_{n\ell}\right) = \exp\left[\frac{\omega r^2}{4}\left(1 - \sqrt{1+\gamma E_{n\ell}}\right)\right] \tag{2.21}$$

for the correction to the wave function of the entire system that takes, from Eq. (2.2), its final form as

$$\begin{aligned}\Psi\left(r, E_{n\ell}\right) &= \Psi_{ES}\left(r\right) h\left(r, E_{n\ell}\right) \\ &= C_{n\ell}\, r^{\ell+1} \exp\left(-\frac{\omega r^2}{4}\right) \exp\left[\left(-\frac{\omega r^2}{4}\right)\left(1 - \sqrt{1+\gamma E_{n\ell}}\right)\right]\end{aligned} \tag{2.22}$$

where $C_{n\ell}$ is the normalization constant.

Proceeding with (2.20) to observe the structure of the full energy spectra, we have

$$E_{n\ell} = E_{ES} + \Delta E = \left(2n + \ell + \frac{3}{2}\right)\omega + \left(\sqrt{1+\gamma E_{n\ell}} - 1\right)\left(2n + \ell + \frac{3}{2}\right)\omega \tag{2.23}$$

The nonlinear character of the wave equation in (2.16) is seen explicitly in the above equation. It thus results in a quadratic equation for the eigenvalues which are then given by,

$$E_{n\ell}^{\pm}=\frac{\omega^2\gamma}{2}\left(2n+\ell+\frac{3}{2}\right)^2\pm\frac{\omega}{2}\left(2n+\ell+\frac{3}{2}\right)\sqrt{\omega^2\gamma^2\left(2n+\ell+\frac{3}{2}\right)^2+4} \tag{2.24}$$

The algebraic details for the derivation of (2.24) are given in Appendix A.

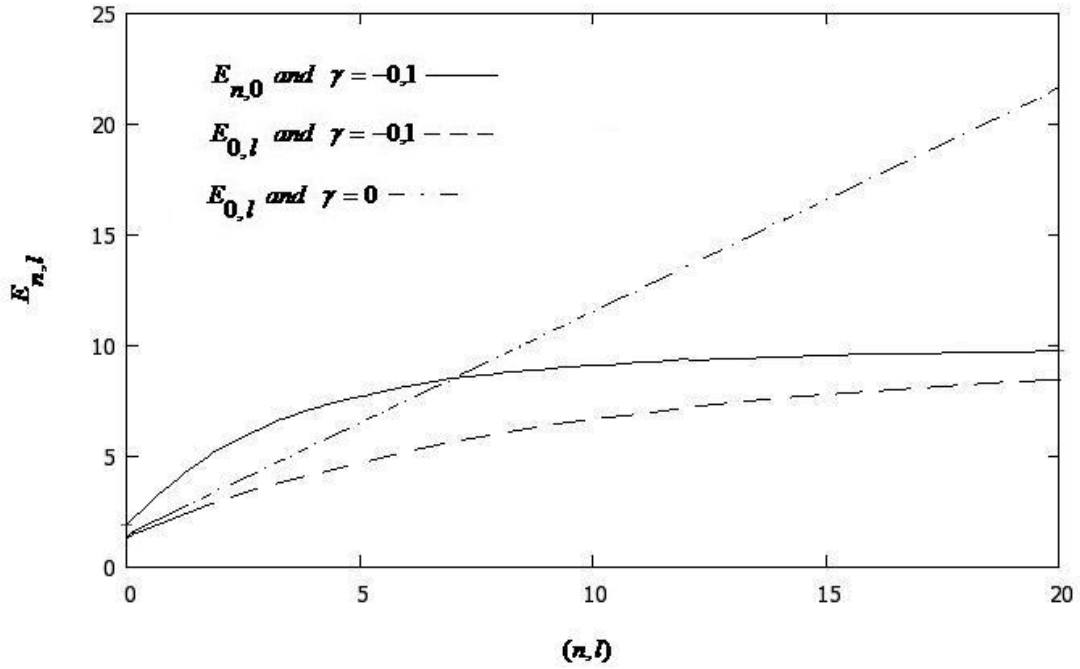

Figure 2.1: Behaviors of the spectrum of the harmonic oscillator potential $V_{(r,E_{n\ell})}$ and the eigenvalues $E_{n,0}$ compared to $E_{n,\ell}$ and are plotted for different quantum numbers for $\gamma=-0.1$ compared to $\gamma=0$, and $\omega=\sqrt{2}$

The requirement of normalizable wavefunction imposes discarding the negative roots. Further, as discussed earlier [48, 49, 50, 51, 52], a coherent model is met only for $\gamma<0$. The results in (2.22) and (2.24) are in agreement with those in Refs. [49, 51, 52], for which it is reminded that the principal quantum number n_p is related to the radial quantum number $(n=0,1,2,\ldots.)$ used here as $n_p=n+\ell+1$. Finally, we note that the solutions in (2.22) and (2.24) reduce explicitly to those concerning with the conventional harmonic oscillator potential in case $\gamma=0$ which serves as a testing ground.

The interesting feature of the spectrum in the figure given is the saturation effect. As the quantum number increase, the eigenvalues reach an upper limit. This beheviour is certainly due to the dependence of interaction potential for the energy values.

2.2.1.2. Coulomb Potential

This potential is given by

$$V\left(r,E_{n\ell}\right)=V_{ES}\left(r\right)+\Delta V\left(r,E_{n\ell}\right)=\left(\frac{\lambda}{r}+\frac{\ell\left(\ell+1\right)}{r^{2}}\right)+\frac{\lambda}{r}\left(\gamma E_{n\ell}\right)\qquad\lambda<0 \tag{2.25}$$

as having a negative strength, it requires $\gamma>0$. The analytically solvable energy independent piece of the Coulomb potential has the known solutions [55]

$$E_{ES}=\frac{\lambda}{4\left(n+\ell+1\right)^{2}}\quad,$$

$$\Psi_{ES}\left(r\right)=f(r)F\left(g\left(r\right)\right)\sim r^{(\ell+1)}\exp\left(\frac{\lambda}{2\left(n+\ell+1\right)}r\right)L_{n}^{(2\ell+1)}\left(g\right) \tag{2.26}$$

for which we choose

$$Q\left(g\right)=0\quad,\quad R\left(g\right)=\frac{n+\ell+1}{g}+\frac{\ell\left(\ell+1\right)}{g}-\frac{1}{4} \tag{2.27}$$

that leads to $g\left(r\right)=-\frac{\lambda r}{n+\ell+1}$

For the calculations of the modifying terms, which are finally added to the energy and the reduced wavefunction given in (2.26) due to the energy dependent part of the potential (ΔV), we set $\Delta Q=-\gamma E_{n\ell}$ and the use of which into Eq. (2.12), together with the consideration of (2.14), reproduces

$$\Delta E=-\frac{\lambda^{2}\gamma E_{n\ell}}{2\left(n+\ell+1\right)^{2}}\left(1+\frac{\gamma E_{n\ell}}{2}\right) \tag{2.28}$$

and remembering that $\Delta Q\left(g\left(r\right)\right)=\dfrac{2h'}{hg'}$ from Eq. (2.10), we arrive at

$$h\left(r,E_{n\ell}\right)=\exp\left(\frac{\gamma E_{n\ell}\lambda}{2\left(n+\ell+1\right)}r\right) \tag{2.29}$$

Consequently, the whole of the actual solutions are

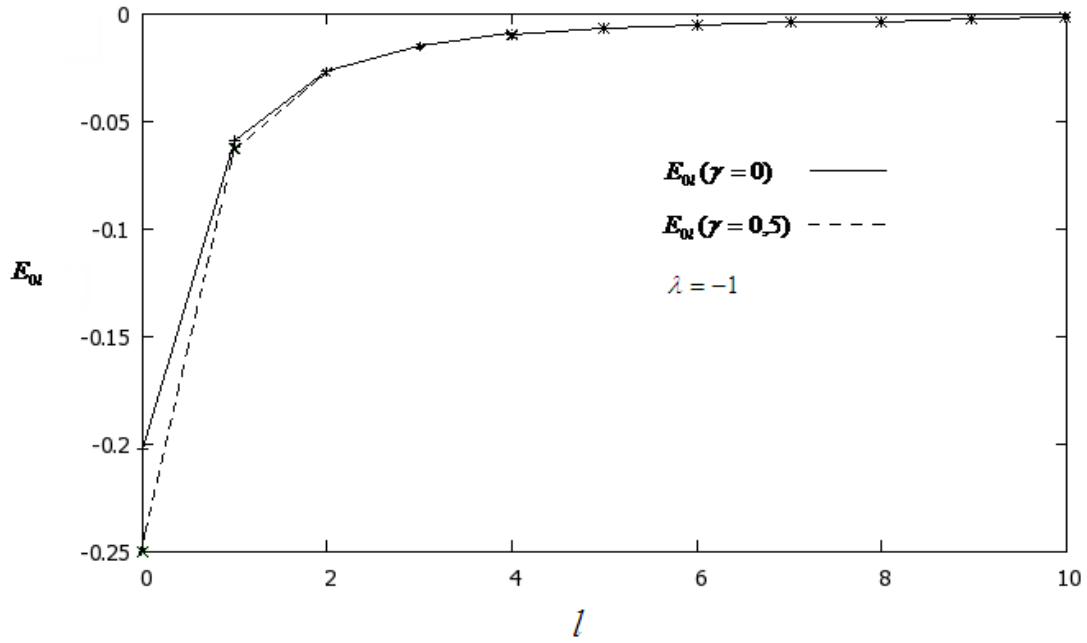

Figure 2.2: Shows the spectrum of the Coulomb potential $V_{(r,E_{n\ell})}$ for $\gamma=0,5$ compared to $\gamma=0$ for $\lambda=-1$.

$$\Psi\left(r,E_{n\ell}\right)=f\left(r\right)F\left(g\right)h\left(r,E_{n\ell}\right)\sim r^{\ell+1}\exp\left[\frac{\lambda r}{2\left(n+\ell+1\right)}\left(1+\gamma E_{n\ell}\right)\right]L_n^{(2\ell+1)}\left(g\right) \tag{2.30}$$

And the sum of the two different energy contributions, $E_{n\ell}=E_{ES}+\Delta E$, gives the energy eigenvalues as the solutions of a second order equation with two roots

$$E_{n\ell}^{\pm}=\frac{1}{\gamma^2\lambda^2}\left[-2\left(n+\ell+1\right)^2-\gamma\lambda^2\pm2\left(n+\ell+1\right)\sqrt{\left(n+\ell+1\right)^2+\gamma\lambda^2}\right] \tag{2.31}$$

where $E_{n\ell}^{+}$ is the physically acceptable one. The results in Eqs. (2.30) and (2.31) agree with those of Ref. [52].

From Figure 2.2, it is clear that the energy dependence affect only the lowest levels.

As a conclusion, the works in this section with two specific examples justify the power and flexibility of the new model introduced. The next section discusses the improvement of this algebraic model in the light of experimental observations.

2.2.1.3. An Alternative Application to Heavy Quark Systems

The qualitative features of the quark-antiquark excitation spectrum can be inferred by visual interpolation, as shown in figure. Imagine a continuous deformation of an oscillator potential into a Coulomb field, and the associated change of the bound states. The potential of interest in mesonic spectroscopy is between the two extremes, whence the spectrum labeled "hadron". The parameters of the potentials are chosen so that the 2S-1S difference is fixed. In the case of the "hadron", not all levels are shown for the sake of clarity. Note that we label states by (n-1), where n is the number of the radial nodes.

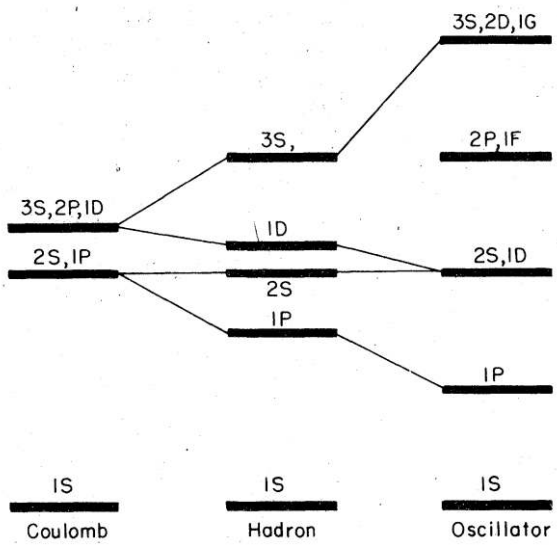

Figure 2.3: The qualitative features of the quark-antiquark excitation spectrum.

Along this line, according to Figure 2.3. the harmonic oscillator is in fact not a good starting point to describe $q\bar{q}$ hadrons. Experimentally, the 1d state is above the 2s

state, and the energy dependence does not remove their degeneracy (see Figure 2.1). However due to experimental observation shown in Figure 2.3., harmonic oscillator seems still the best candidate for such considerations. Since it is often useful to have a simple model at our disposal, we keep a linear energy dependence and start with the following another consideration: In this case the corresponding potential,

$$V\left(r,E_{n,\ell}\right)=\frac{1}{2}\mu\omega^2 r^2\left(1+\gamma E_{n,\ell}\right)+\frac{b}{r^2} \tag{2.32}$$

where ω,b,γ are constants, μ is the reduced mass and m_q is quark mass, $m_{\bar{q}}$ is antiquark mass. As

$$\mu=\frac{m_q m_{\bar{q}}}{m_q+m_{\bar{q}}} \tag{2.33}$$

the three dimensional Schrödinger equation in the center of mass system is,

$$\left[-\frac{1}{2\mu}\nabla^2+V\left(r,E_{n,\ell}\right)\right]\Psi_{n,\ell,m}\left(r,\theta,\phi\right)=E_{n,\ell}\,\Psi_{n,\ell,m}\left(r,\theta,\phi\right) \tag{2.34}$$

where $\Psi_{n,\ell,m}\left(r,\theta,\phi\right)$ is the Schrödinger wave function. For a central potential it is convenient to write

$$\Psi_{n,\ell,m}\left(r,\theta,\phi\right)=R_{n,\ell}\left(r\right)Y_{\ell,m}\left(\theta,\phi\right) \tag{2.35}$$

where $R_{n,\ell}\left(r\right)$ is the radial wave function and $Y_{\ell,m}\left(\theta,\phi\right)$ is a spherical harmonic. Here n is the radial quantum number, ℓ and m are the orbital angular momentum and its projection respectively.

Then the radial wave function clearly satisfies

$$-\frac{1}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)R_{n,\ell}\left(r\right)-\left[E_{n,\ell}-V\left(r,E_{n,\ell}\right)-\frac{\ell\left(\ell+1\right)}{2\mu r^2}\right]R_{n,\ell}\left(r\right)=0 \tag{2.36}$$

The radial equation can be put in formal correspondence with one-dimensional Schrödinger equation by means of the substitution of

$U_{n,\ell}\left(r\right)=rR_{n,\ell}\left(r\right)$ which defines the reduced radial equation,

$$U''_{n,\ell}(r)+2\mu\left[E_{n,\ell}-V\left(r,E_{n,\ell}\right)-\frac{\ell(\ell+1)}{2\mu r^2}\right]U_{n,\ell}(r)=0 \tag{2.37}$$

in which we rearrange as

$$V_{eff}\left(r,E_{n,\ell}\right)=V\left(r,E_{n,\ell}\right)+\frac{\ell(\ell+1)}{2\mu r^2} \tag{2.38}$$

$$V_{eff}\left(r,E_{n,\ell}\right)=\frac{1}{2}\mu\omega^2 r^2\left(1+\gamma E_{n,\ell}\right)+\frac{b}{r^2}+\frac{\ell(\ell+1)}{2\mu r^2} \tag{2.39}$$

Then the radial equation becomes

$$U''_{n,\ell}(r)+2\mu\left[E_{n,\ell}-\frac{1}{2}\mu\omega^2 r^2\left(1+\gamma E_{n,\ell}\right)-\frac{b}{r^2}-\frac{\ell(\ell+1)}{2\mu r^2}\right]U_{n,\ell}(r)=0 \tag{2.40}$$

where

$$2\mu E_{n,\ell}=\tilde{E}_{n,\ell} \tag{2.41}$$

$$2\mu V_{eff}\left(r,E_{n,\ell}\right)=\tilde{V}_{eff}\left(r,E_{n,\ell}\right) \tag{2.42}$$

due to the choice of the generalized Laguerre polynomials $F(g)=L_n^{(\ell+1/2)}$ which leads us to consider $Q_{ES}=(\ell-g+3/2)/g$ and $R_{ES}=n/g$ in dealing with the algebraic solutions of the corresponding differential equations. This ends up with the strict definitions of the interval functions such as $g=\mu\omega r^2$ and $f=(2\omega)^{-1/4}g^{(\ell+1)/2}\exp(-g/2)$. Obviously, the substitution of these findings in (2.8) or (2.11), as previously discussed in section 2.1., reproduces energy independent part as

$$E_{n,\ell}=\omega\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right] \tag{2.43}$$

$$V_{eff}\left(r,E_{n,\ell}\right)=\frac{1}{2}\mu\omega^2 r^2+\frac{b}{r^2}+\frac{\ell(\ell+1)}{2\mu r^2} \tag{2.44}$$

Afterwards, we can obtain readily certain expressions for the corrections brought by the energy dependent component of the potential. Once again we remind that Eq. (2.19) dies away in the case of $\gamma=0$, from which Eqs. (2.14) and subsequently (2.12)

vanish. This confirms the reliability of the choice in (2.19) which is the key point for benchmark tests when compared to the solutions in connection with the conventional energy independent potentials. The use of (2.19) in Eqs. (2.14) and then (2.12) reveals the modifying terms, for the present consideration, as

$$\Delta E_{n,\ell} = \omega\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right]\left(\sqrt{1+\gamma E_{n,\ell}}-1\right) \tag{2.45}$$

$$\Delta V\left(r,E_{n,\ell}\right) = \frac{1}{2}\mu\omega^2 r^2\left(\gamma E_{n,\ell}\right)+2\omega\left(n+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right)\frac{L_{n-1}}{L_n}\left(\sqrt{1+\gamma E_{n,\ell}}-1\right) \tag{2.46}$$

The algebraic details of the above equations are given in Appendix B. Hence the eigenvalues of the whole system now becomes,

$$E_{n,\ell} = \omega\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right]\left(\sqrt{1+\gamma E_{n,\ell}}\right) \tag{2.47}$$

To proceed further we need to take the square of each side of the equation above

$${E_{n,\ell}}^2 = \omega^2\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right]^2\left(1+\gamma E_{n,\ell}\right) \tag{2.48}$$

from which one easily obtains that

$$\begin{aligned} E_{n,\ell} = {} & \frac{\omega^2\gamma}{2}\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right]^2 \\ & \mp\frac{\omega}{2}\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right]\sqrt{\omega^2\gamma^2\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right]^2+4} \end{aligned} \tag{2.49}$$

which is in agreement with the related literature. In addition, the above equation reduces smoothly to Eq.(2.24) in case b=0, which justifies once more the reliability of Eq.(2.49).

2.3. Summary

An attempt first has been made to generalize the work in [37] and shown that the mathematically rigorous new scheme unifies different theories for the solution of Schrödinger equation with analytically solvable conventional and energy-dependent potentials. Although the literature covered similar problems, to our knowledge an investigation such as the one presented here was missing.

To go beyond qualitative aspects, the applications are devoted to the study of the wave equation with potentials depending on the energy which is essential in understanding the interaction mechanism in heavy quark systems. It has been clarified that such investigations can be performed safely through our schematical model without causing any physical problem. Although we have limited ourselves to three illustrative examples, the range of application of the method is rather large and appears to be straightforward**.**

Beyond its intrinsic importance as a new solution for a fundamental equation in physics, we expect that the present simple method would also find a widespread application in the study of different quantum mechanical systems with constant and position-dependent masses. In particular, the present discussion would be useful in perturbational treatments of the exact spectra of a few particle systems, and thus provide a further insight on discussion of the fractional nature of such systems.

CHAPTER 3

APPLICATION TO QUASI-EXACTLY SOLVABLE POTENTIALS

Gauge theories of the strong interactions suggest that the coupling between quarks is weak at short distance but becomes very strong at large distances. This explains the paradox where quarks appear to be have as quasi-free particles within hadrons but cannot be liberated from the hadrons. On the basis of these arguments it was conjectured that heavy quarks would move non-relativistically within hadrons. Thus bound states of a heavy quark and anti-quark should be a hadronic analogue of the positronium system of a bound electron and positron. This so-called quarkonium system might then be interpreted according to the familiar rules of non-relativistic quantum mechanics using a potential to describe the interquark force. At very short distances, a non-relativistic Coulomb potential with a strength proportional to the strong coupling constant is derived from the one-gluon-exchange interaction in QCD. At long distances, nonperturbative effects such as confinement has to be considered. For instance, lattice calculations [56] in the so-called quenched approximation suggest a linearly increasing potential. However, a specific form of the QCD potential in the whole range of distances is not known. Taking into consideration the this factor, in this work, we have used a non-relativistic potential model which consists of Coulomb plus linear potential, together with a quadratic term, to take well care of the confinement, for describing the quarkonia spectra and its decays, which provides some advantages for quasi-exactly solvable analytical treatments of the quarkonium problems in terms of special polynomials of mathematical physics. Additionaly, such class of potentials is a possible candidate for the quarkonium potential as has been indicated by the quarkonium spectroscopy [57].

Here an application of the present method, discussed in detail in the previous chapter, to the bound-state eigenvalue problem in the elementary quarkonium potential $V(r) = -a/r + br + cr^2$ is described, proved and illustrated for $c\bar{c}$ and $b\bar{b}$ systems. The quasi-exactly solvable spin-averaged mass spectra of heavy quarkonia are obtained in compact forms. The comparison of the present predictions with those of other theories in the related literature, together with the available data, has shown the success of the model used in this work and also revealed that the use of different confinings in the perturbed Coulomb potential descriptions has no considerable effect on the mass spectra of such systems.

3.1. Formalism

Considering the improved description [53] for the treatment of non-relativistic Schrödinger equation $(\hbar = 1)$

$$\frac{d^2\Psi}{dr^2} + 2m\big(E - V(r)\big)\Psi(r) = 0 \qquad (3.1)$$

where, for a generalized formalism, in this new case,

$$\Psi(r) = f(r)F\big(g(r)\big)h(r)$$

$$\Psi(r) = \big[f(r)F\big(g(r)\big)\big]h(r) \qquad (3.2)$$

in which $f(r)F(g)$ yields an algebraic closed solution for exactly solvable potentials [53], and the related references therein] with $F\big[g(r)\big]$ being a special function which satisfies a second-order differential equation

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0 \qquad (3.3)$$

while $h(r)$ is the moderating function in connection with a perturbing piece of the full potential in Eq. (3.1). The form of $Q(g)$ and $R(g)$ is already well defined for any

special function $F(g)$ when dealing with analytically solvable potentials. Substituting Eq. (3.2) into (3.1) leads to

$$\frac{d^2F}{dg^2}+\frac{dF}{dg}\left(\frac{g''}{(g')^2}+\frac{2f'}{fg'}+\frac{2h'}{hg'}\right) + F(g)\left(\frac{f''}{f(g')^2}+\frac{2f'h'}{fh(g')^2}+\frac{h''}{h(g')^2}+\frac{2m\left[E-V(r)\right]}{(g')^2}\right)=0 \tag{3.4}$$

From the comparison of Eqs. (3.3) and (3.4) it follows that

$$Q(g(r))=\frac{g''}{(g')^2}+\frac{2f'}{fg'}+\frac{2h'}{hg'} \tag{3.5}$$

and

$$R(g(r))=\frac{f''}{f(g')^2}+\frac{2f'h'}{fh(g')^2}+\frac{h''}{h(g')^2}+\frac{2m\left[E-V(r)\right]}{(g')^2} \tag{3.6}$$

Obviously, Eqs. (3.4-3.6) reduce to Eqs.(3.4-3.6) of Ref. [37] for the case of exact solvability, in which case $h(r)$ in the equations above goes to a constant value. Gaining confidence from this observation we proceed with

$$\begin{aligned} V(r)&=V_{ES}(r)+\Delta V(r)\\ E&=E_{ES}+\Delta E \end{aligned} \tag{3.7}$$

in accordance with our choice in (3.2), which means that the interaction potentials considered in this article to describe the spectra of charmonium and bottomonium systems are admitted as the sum of an exactly solvable Coulomb like $\left(V_{ES}=\frac{-a}{r}+\frac{\ell(\ell+1)}{2mr^2}\right)$ potential with a perturbation or a moderating piece related to the confining part $\Delta V\left(=br+cr^2\right)$ of the potential function. Hence the aim in this perspective is to reveal the corrections to energy (ΔE) and wavefunction $\left(h(r)\right)$ for a given ΔV, as the main piece of the solutions leading to exact solvability can easily be found from the literature.

The use of (3.7) within Eq. (3.6) produces coupled equations in the form of

$$2m[E_{ES}-V_{ES}(r)]=R_{ES}\left(g(r)\right)\left(g'\right)^2-f''/f \tag{3.8}$$

and

$$2m\left[\Delta E-\Delta V(r)\right]=\Delta R\left(g(r)\right)\left(g'\right)^2-2f'h'/\left(fh\right)-h''/h \tag{3.9}$$

where $R_{ES}+\Delta R$ $\mathrm{R_{ES}(g(r))}+\Delta\mathrm{R(g(r))}$ should certainly reproduce Eq. (3.6). Similarly, Eq. (3.5) can be decomposed as

$$Q_{ES}\left(g(r)\right)=g''/\left(g'\right)^2+2f'/fg'\ ,\ \Delta Q\left(g(r)\right)=2h'/hg'\quad\Rightarrow\quad Q=Q_{ES}+\Delta Q \tag{3.10}$$

After some algebra, Eqs. (3.8) and (3.9) are transformed into more applicable forms

$$2m[E_{ES}-V_{ES}(r)]=\frac{g'''}{2g'}-\frac{3}{4}\left(\frac{g''}{g'}\right)^2+\left(g'\right)^2\left[R_{ES}\left(g(r)\right)-\frac{1}{2}\frac{dQ_{ES}}{dg}-\frac{1}{4}Q_{ES}\left(g(r)\right)^2\right] \tag{3.11}$$

and

$$\begin{aligned}2m\left[\Delta E-\Delta V(r)\right]=&-\left(\frac{g''}{2}+\frac{f'g'}{f}\right)\Delta Q\left(g(r)\right)\\&+\left(g'\right)^2\left[\Delta R\left(g(r)\right)-\frac{1}{2}\frac{d(\Delta Q)}{dg}-\frac{1}{4}\Delta Q\left(g(r)\right)^2\right]\end{aligned} \tag{3.12}$$

The result of this brief investigation opens a gate to the reader for the visualization of the explicit form of the correction (ΔE) to the energy. Unfortunately, there seems a problem naturally arose in calculating the correction term owing to the presence of two unknown: ΔQ and ΔR on the right hand side of Eq. (3.12). To circumvent the resulting drawback and proceed safely we need to go back Eq. (3.4) and substitute the definitions given by (3.7) in it, which leads us to a more practical expression

$$2m\left[\Delta E-\Delta V(r)\right]=-\frac{2h'}{h}\left(\frac{f'}{f}+\frac{F'(g)\,g'}{F(g)}\right)-\frac{h''}{h} \tag{3.13}$$

that is in fact another form of (3.9). Thus, equating (3.9) to (3.13) and remembering the form of ΔQ in Eq. (3.10) we arrive at

$$\Delta R(g) = -\Delta Q(g)\frac{F'(g)}{F(g)} \tag{3.14}$$

which is vital to overcome the problem encountered in (3.12). As $F(g)$ is well defined for a given exactly solvable potential, evidently one needs here to find only an appropriate expression for $\Delta Q(g)$ to be employed in (3.12) that reveals clearly the full solution.

As discussed earlier, before closing this section we should again remark that once choosing carefully $Q_{ES}(g)$ and $R_{ES}(g)$ for the analytically solvable part (V_{ES}) of the full potential under investigation, we can subsequently easily set a proper internal function $g(r)$, and considering Eq. (3.5)

$$f(r) = (g')^{-\frac{1}{2}} \exp\left[\frac{1}{2}\int^{g(r)} Q_{ES}(g)\,dg\right] \tag{3.15}$$

as well discussed in Ref. [37], which are then used in (3.13) to find corrections to the solutions of the exactly solvable piece.

The application of the model to a specifically chosen potential is discussed in the following section, which would shed a light to ongoing discussions through the literature of heavy quarkonium physics.

3.2. Application

As a phenomenological model for a heavy quarkonium system $(q\bar{q})$, we consider here the following spherically symmetric spin-averaged potential $(\hbar = 1)$

$$V(r) = -\frac{a}{r} + br + cr^2 + \frac{\ell(\ell+1)}{2mr^2} \tag{3.16}$$

where a, b and c are non-negative constants and r is the interquark distance. This potential has two parts: The first is $-\frac{a}{r}$ which corresponds to the potential induced by

one-gluon exchange between the quark and its anti-quark that dominated at short distances, while the second part $br+cr^2$ accounts for quark confinement at large distances. Within this context, starting with Eq. (3.1) in which

$$V(r)=V_{ES}(r)+\Delta V(r)=\left(-\frac{a}{r}+\frac{l(l+1)}{2mr^2}\right)+\left(br+cr^2\right) \tag{3.17}$$

with $m\left(=\frac{m_q m_{\bar{q}}}{(m_q+m_{\bar{q}})}=\frac{m_q}{2}\right)$ being the reduced mass of $q\bar{q}$ system, the well known closed solutions of the first part of the full potential (V_{ES}) can be given in an explicit form [37, 53], within the framework of Eqs. (3.11) and (3.15),

$$E_{ES}=-\frac{ma^2}{2(n+l+1)^2} \tag{3.18}$$

$$\Psi_{ES}(r)\propto f(r)\left[F(g(r))\right]=\left(\frac{n+l+1}{2ma}\right)^{1/2}\left[e^{-g/2}\,g^{-(\alpha+1)/2}\,L_n^{\alpha}(g)\right] \tag{3.19}$$

for which we have chosen

$$Q(g)=0 \quad , \quad R(g)=\frac{n+l+1}{g}-\frac{l(l+1)}{g^2}-\frac{1}{4} \tag{3.20}$$

in connection with the generalized Laguerre polynomial $\left(L_n^{\alpha}(g)\right)$ appeared in (3.19), that leads to the certain definitions of $g(r)=\frac{(2ma)r}{(n+l+1)}$ and $\alpha=2l+1$.

For completeness, one needs also to express the corresponding modifying terms due to the confining potential $\left(\Delta V=br+cr^2\right)$, for which we return back to Eq. (3.13) that in this case turns out to be

$$2m\left(br+cr^2-\Delta E\right)=\frac{h''}{h}+\frac{2h'}{h}\left[\frac{ma}{(n+l+1)}\left(2\frac{(L_n^{2\ell+1})'}{L_n^{2\ell+1}}-1\right)+\left(\frac{l+1}{r}\right)\right] \tag{3.21}$$

with the use of exact definitions for $f(r)$, $F(g)$ and $g(r)$ just discussed above in defining the contributions of exactly soluable piece of the potential. We also remind that

$\left(L_n^{\alpha}\right)'$ in (3.21) denotes the derivative of the Laguerre polynomial with respect to g, $dL_n^{\alpha}(g)/g$, unlike the other primes in there which represent the usual derivatives depending on r.

At this stage, for the solution of (3.21), which is a kind of Riccati equation, we suggest that

$$h(r)=\exp\left(-\gamma r^2\right) \tag{3.22}$$

leading to

$$\Delta E_{n=0}=\sqrt{\frac{c}{2m}}\left(2\ell+3\right) \tag{3.23}$$

for the ground state with the constraints in the parameters

$$b=\frac{2a\gamma}{\ell+1} \quad , \quad \gamma=\sqrt{\frac{mc}{2}} \tag{3.24}$$

This, and other constraints for excited states ($n \succ 0$) discussed below, clarify that the linear and quadratic terms in confining part of the potential in (3.17) destroy naturally the general symmetry, reducing the problem to the non-exactly solvable case as generally expected, nevertheless the restrictions in the confining potential parameters, Eq. (3.24), can restore the symmetries again in an elegant manner for its particularly chosen values by yielding the corrections through the explicit expressions, Eqs. (3.22) and (3.23).

Consequently, the full solutions for the ground state of $q\bar{q}$ potential in (3.17) are

$$\Psi_{n=0}(r)=\left[f(r)F(g)\right]h(r)\propto r^{l+1}\exp\left[-\sqrt{\frac{mc}{2}}r^2-\frac{ma}{(l+1)}r\right] \tag{3.25}$$

and

$$E_{n=0}=\left(E_{ES}+\Delta E\right)_{n=0}=-\frac{ma^2}{2(l+1)^2}+\sqrt{\frac{c}{2m}}\left(2\ell+3\right)=-\frac{b^2}{4c}+\sqrt{\frac{c}{2m}}\left(2\ell+3\right) \tag{3.26}$$

where $m = m_q/2$ as discussed above.

Obviously, the most significant piece in this model is Eq. (3.13), and its application to the present case Eq. (3.21), which are responsible for the calculation of corrections related to the interaction at large distances, Eqs. (3.22) and (3.23), which modify the solutions given by (3.18) and (3.19) describing the short range heavy quark interactions. When, however, the modifying function $h(r)$ becomes constant, it can easily be seen that Eq. (3.13) dies away and the whole calculations reduce to the exactly solvable case corresponding to the Coulomb like interaction of $q\bar{q}$ interaction, which provides us a testing ground for our calculation results. More specifically, a constant $h(r)$ requires $\gamma = 0$ in (3.22) and, subsequently, the confining potential parameters (b, c) vanish naturally. This serves us a benchmark testing of Eqs. (3.25) and (3.26), which in this case reduce smoothly to the solutions of a Coulomb like potential [37, 53] justifying our calculation results.

Furthermore, through the discussion in this section, we have started with a natural consideration that a Coulomb like potential, together with a barrier term, in Eq. (3.17) represents an exactly solvable piece and appropriate prescriptions for the modifications are given for the rest of the potential in (3.17). Instead, on the contrary, one can also start with an alternative approach to the same problem, namely the harmonic like potential (cr^2) with the barrier term may be mathematically, not physically, considered as the exactly solvable part in (3.17) and all the other procedure is then repeated for the new perturbing potential $\left(\Delta V = -\frac{a}{r} + br\right)$. This significant alternative consideration, which supports once more the flexibility of the present formalism, can be safely used to check again the reliability of the calculation results obtained, which reproduces indeed exactly the same algebraic expressions being identical to (3.25) and (3.26).

As a final remark, Eqs. (3.22) through (3.26) developed by the present formalism are in a remarkable agreement with the works in Refs. [58-62] carried out only for the ground state calculations $(n = 0)$. The comparison is presented in Table 3.1.

a (Eq. 3.24)	b	c	ℓ	Eq. (3.26)	Ref. [61]
4	1	1/32	0	–7.625	–7.625
8	1	1/32	1	–7.375	–7.375
12	1	1/32	2	–7.125	–7.125

Table 3.1: Lowest energy eigenvalues (in GeV) in three-dimension for the potential in Eq. (3.17) with $m=1$.

The wavefunctions introduced by [61] concerning the quantum states in Table 3.1 overlap with the definition given by (3.25) in the present work. We stress also that the model used here describes explicitly the asymptotic and short distance contributions of the potential indivudually, unlike the previous models reported in the literature, without using an ansatz for the wavefunction to be able to initiate the whole calculation procedure as is the case in general encountered in the related algebraic models. In connection with this, the natural appearance of the Laguerre polynomials in (3.19), and consequently in (3.25), justifies and clarifies the comment given by [63] on the recent work [64] in which the authors used a physically inappropriate power series expansion as an ansatz for the wavefunction description corresponding to the same interaction potential discussed in this article.

However, the situation for the excited states are slightly different. Because, the generalized Laquerre polynomial, appeared through Eq. (3.19) in describing the behavior of exactly solvable Coulomb case, now is not constant and disturbed by the confining potentials. For instance, consideration of the first excited state ($n=1$) for the purely Coulumb case requires naturally the usually known corresponding function $F\left(g\right)=L_n^{2\ell+1}(g)=2(\ell+1)-\dfrac{2ma}{\left(l+2\right)}\,r$. However, for the present consideration, Eq. (3.17), the change in the Coulomb type potential parameter through the restrictions mentioned above forces us to reconsider the structure of $g(r)=\dfrac{2ma}{\left(\ell+2\right)}r$ which now should be $\tilde{g}\left(r\right)=\lambda\left(a,c,\ell\right)r$ due to the presence of the linear and quadratic terms in the potential. This behaviour thus implies the replacement of $L_{n=1}^{2\ell+1}(g)=2(\ell+1)-\dfrac{2ma}{\left(l+2\right)}\,r$

with an appropriate another orthogonal Laguerre-like polynomial $\tilde{L}_{n=1}^{2\ell+1}(\tilde{g})=2(\ell+1)-\lambda r$, involving deformed Laguarre functions. With this new consideration, the full wavefunction for the first excited state becomes

$$\Psi_{n=1}(r)\propto r^{l+1}\exp\left[-\sqrt{\frac{mc}{2}}r^2-\frac{ma}{(l+1)}r\right]\left(1-\frac{\lambda}{2(\ell+1)}r\right) \tag{3.27}$$

For the related energy expression, one needs to go back to Eq. (3.21) and replace the standard Laguerre polynomial with the deformed one, $L_{n=1}^{2\ell+1}\to\tilde{L}_{n=1}^{2\ell+1}=2(\ell+1)-\lambda r\propto\left(1-\frac{\lambda}{2(\ell+1)}r\right)$, which guides us to use the exact treatment $\frac{\Psi''}{\Psi}=2m(V-E)$ for defining λ precisely to initiate the calculations in (3.21) for $n=1$ case. The use of (3.27), for this purpose, in the exact treatment yields

$$\lambda=\frac{ma+\sqrt{(ma)^2+4(\ell+1)(\ell+2)^2\sqrt{2cm}}}{(\ell+2)} \tag{3.28}$$

as one of the physically meaningful root of the quadratic equation $\lambda\left[\lambda(\ell+1)(\ell+2)-ma\right]-(\ell+2)\sqrt{2mc}=0$. Obviously, Eq. (3.28) reduces the purely Coulomb case, $\tilde{g}(r)=\lambda r\to g(r)=2mar/(\ell+2)$, as if the system is not perturbed $(c=0)$, justifying the reliability of the expression. The proper consideration of this new function in (3.21), together with the correct derivatives in there, such as

$$2m\left(br+cr^2-\Delta E_{n=1}\right)=\frac{h''}{h}+\frac{2h'}{h}\left[\left(-\frac{1}{2}+\frac{l+1}{\lambda r}-\frac{1}{(2\ell+1)-\lambda r}\right)\lambda\right] \tag{3.29}$$

produces the first excited state energy as

$$\Delta E_{n=1}=\sqrt{\frac{c}{2m}}(2\ell+5)\quad\to \tag{3.30}$$

$$E_{n=1}=-\frac{ma^2}{2(l+2)^2}+\sqrt{\frac{c}{2m}}(2\ell+5)=-\frac{b^2}{4c}+\sqrt{\frac{c}{2m}}(2\ell+5)$$

but with a new constraint $b=\frac{a\sqrt{2mc}}{\ell+2}$ regarding the coupling potential parameter. We note here that the confining potential parameters b and c are always constant for a given quantum state while the strength of the Coulomb like interaction (a) depends on n and ℓ and changes from state to state as shown in Table 3.1. Overall, the use of (3.27) in the analytical treatment $\frac{\Psi''_{n=1}}{\Psi_{n=1}}=2m\left(V-E_{n=1}\right)$ confirms explicitly the obtained result.

Larger values of n can also be considered, for which the problem is reduced to rather complicated polynomials leading to messy equations, but $n=0,1$ are sufficient for our purposes clarifying the flexibility of the model used in this paper and revealing the possibility of additional restrictions on the potential parameters in each quantum state as a redemption to be able to have closed algebraic solutions that is in agreement with [59]. A complete study including more solutions in terms of explicit closed forms and applications to few-body systems is under way.

3.2.1. Mass Spectra of Heavy Quarkonium

Within the framework of the present formalism, we are now ready to examine the possibility of producing the mass spectra of heavy quarkonium systems such as charmonium and bottomonium having quark and antiquark of same flavor. For determining the mass spectra in three dimensions, we use the following familiar relation

$$M\left(q\bar{q}\right)=2m_q+E \tag{3.31}$$

where E is given by Eqs. (3.26) and (3.30). Using the masses of charm and bottom quarks, $m_c=1.488$ GeV and $m_b=4.686$ GeV, and the most reliable ground state experimental mass values ($1S$ and $1D$ states), one can determine the potential parameters in safe. Then the rest of the mass spectra are described through Eqs. (3.26), (3.30) and (3.31) as discussed in the Appendix D. The results are shown in Tables 3.2 and 3.3.

State	Ref. [65]	Ref. [66]	Present	Experiment
1S	9.434	9.460	9.460	9.460
1P	9.905	9.840	9.811	9.900
2S	9.994	10.023	10.023	10.023
1D	10.142	10.140	10.161	10.161
2P	10.242	10.160	10.373	10.260

Table 3. 2: Bound-state spectrum of the $c\bar{c}$ system (in GeV). Experimental data are taken from Ref. [8]. The way of fixing the parameters are explained in the Appendix D.

State	Ref. [65]	Ref. [66]	Present	Experiment
1S	3.097	3.096	3.097	3.097
1P	3.592	3.433	3.434	3.525
2S	3.686	3.686	3.686	3.686
1D	3.841	3.767	3.770	3.770
2P	3.946	3.910	4.022	_

Table 3.3: Bound-state spectrum of the $b\bar{b}$ system (in GeV). Experimental data are taken from Ref. [8]. The way of fixing the parameters are explained in the Appendix D.

Note that the work in [65] was fitted to the data of Barik&Barik in [8] while the recent work in [66] has considered the new experimental results of Nakamura *et al*. in [8] as in the present case. Overall, as a non-relativistic model, the agreement with the data, together with the other predictions, is fairly good and also the comparison with the other findings reveals that the type of confining in such systems is not so significant due to the consideration of $-a/r+br$ in [65] while $-a/r+br^2$ in [66]. In addition, the present comparison also tells us that the mass spectra of heavy quarkonium is model independent as theoretical predictions in Tables 3.2 and 3.3 were carried out by different approaches within the frame of non-relativistic quantum mechanical treatment.

The prediction of the mass spectrum in accordance with the experimental results does not however guarantee the validity of a model for describing hadronic interactions. Different potentials, as illustrated in Tables 3.2 and 3.3, can reproduce the same spectra. Hence, in a given model, one must be able to calculate also other observables accurately like the decay constants, leptonic decay widths, radiative decay widths, etc., to make sure from the accuracy of the wavefunction introduced by the model and the sensitivity of the asymptotic part of the potential. One should, therefore, not forget in such calculations that heavy quarkonium decays provide a deeper insight into the exact nature of the inter quark forces and decay mechanisms. For example, the leptonic decay widths are a probe of the quarkonium system and provide important information complementary to level spacings [67]. Within this context, a theoretical model needs also to be extended to incorporate such calculations, which is one of our future works.

3.3. Summary

We have shown that quasi-exact solutions to the Schrödinger equation for the three-term perturbing Coulomb potential for inter quark interactions can be found subject to constraints on the potential parameters. This means that the symmetries of the Hamiltonian for such systems can be recovered for particularly chosen values of the strengths of the confining part of the interactions in the asymptotic region. Taking this point of view, it would be interesting to extend the present scenario, which has proven its success for the present consideration, to other more complex systems. In particular, the present prescription would be useful in perturbational treatments of the exact or quasi-exact spectra of a few particle systems, and thus provide a further insight on discussion of the fractional nature of such systems. Along this line, the works are in progress.

Furthermore, the major shortcoming of the potential used in the present work is that it has not been able to account for the observed saturation pattern in the experimental spectra. Therefore, considering our expertise in [53], the calculations discussed in this article needs to be repeated by involving the energy dependency of the interaction potential in order to remove the drawback mentioned above, which is to be adressed in a further work.

CHAPTER 4

APPLICATION TO NON-SOLVABLE POTENTIALS

The Schrödinger equation with the Cornell potential, $V(r) = -\frac{a}{r} + br$, also known as the Coulomb plus linear potential, has received a great deal of attention [15, 54, 68, 69] as an important non-relativistic model in both particle physics, more precisely in the context of meson spectroscopy where it is used to describe systems of quark and antiquark bound states, and in atomic and molecular physics where it represents a radial Stark effect in hydrogen. Aside of the physical relevance, the solutions of the Schrödinger equation for the Coulomb plus linear potential have been rigorously investigated with a large number of techniques due to its also nontrivial mathematical properties. In addition, this potential has an advantage that leads naturally to two choices of parent Hamiltonian in perturbative treatments, one based on the Coulomb part and the other on the linear term, which can be usefully compared. Although such models have been studied for many years, exact solutions of Schrödinger equation with this potential are still unknown to a great extent. Most of the earlier work either relies on direct numerical integration of the Schrödinger equation or various techniques for approximating the eigenenergies.

Within this context, and also gaining confidience from the succesful applications [53, 70] of the recently developed formalism appeared in a series of papers for the analytical treatment of exactly- and quasi-/conditionaly-exactly solvable potentials used in analysing heavy quarkonium observations, we aim in this chapter at indicating how the previous recipe in [53, 70] can be extended to give a further prescription for also non-exactly solvable more realistic potentials employed in different disciplines of physics and engineering as well.

This work therefore is the completeness of the preceding reports in [53, 70] and the justification of the flexibility of the model developed and used for various treatments.

The method we propose in this article do not require any particular constraint on the potential parameters, suggesting a simple expression for an explicit analysis of the results obtained exactly but without providing a deep physical interpretation of the findings such as precise numerical treatments or reliable experimental observations for any physical problem that may be modelled by this class of potential. More interestingly, the present communication emphasizes a relationship between the critical $r-$values, leading to exact energies, in the problem considered and the limitations of the known perturbation theories which depend on finding an appropriate parent Hamiltonian for which no general procedure is available even though the choice may be crucial to the success of such approximate schemes. In this respect, the comments on our tabulated results provide intuition on this crucial choice, which seems to have been insufficiently appreciated hitherto.

The recently introduced scheme [53, 70] is extended to propose an algebraic non-perturbative approach for the analytical treatment of Schrödinger equations with non-solvable potentials involving an exactly solvable potential form together with an additional piece. As an illustration the procedure is successfully applied to the Cornell potential by means of very simple algebraic manipulations. The model provides a clean route to interpret related experimental or precise numerical results. We hope this new technique will shed some light on the questions concerning with the limitations of the traditional perturbation techniques.

In the next section, we extend the scenario in [53, 70] to *N*-dimensional space for the treatment of non-solvable potentials. The main features of the formalism used and its application to the Cornell type potentials are given in following section and finally, we draw our conclusions.

4.1. Formalism

Here, for the stringent test of the present model with the others available in the literature, the usual one-dimensional solution of non-relativistic Schrödinger equation is extended to arbitrary dimensions considering the frame in [53, 70].

The radial Schrödinger equation for a spherically symmetric potential in *N*-dimensional space $(\hbar = 1)$ reads [71]

$$\frac{d^2R}{dr^2} + \frac{N-1}{r}\frac{dR}{dr} = 2m\left[\left(V(r) + \frac{\ell(\ell+1)}{2mr^2}\right) - E\right]R \tag{4.1}$$

which is transform to

$$\frac{d^2\Psi}{dr^2} = 2m\left[\left(V(r) + \frac{(M-1)(M-3)}{8mr^2}\right) - E\right]\Psi \ , \tag{4.2}$$

where $\Psi(r) = r^{(N-1)/2}R(r)$, being the reduced radial wavefunction, and $M = N + 2\ell$. Eq. (4.2) can also be expressed as

$$\frac{d^2\Psi}{dr^2} = 2m\left[\left(V(r) + \frac{(\Lambda)(\Lambda+1)}{2mr^2}\right) - E\right]\Psi \ , \tag{4.3}$$

where $\Lambda = (M-3)/2$. We see that the radial Schrödinger equation in *N*-dimensions has the same form as the three-dimensional one. Consequently, given that the potential has the same form in any dimension, the solution in three dimensions can be employed to obtain the solution in any dimension simply by using the substitution $\ell \to \Lambda$.

At this stage, we extend our previous formalism assuming, in Eq. (4.3), that $\Psi(r) = F[g(r)]f(r)$ where $F(g)$ yields an algebraic closed solution for an exactly solvable potential with $F(g)$ being a special function satisfying a second-order differential equation

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0 \ , \tag{4.4}$$

while $f(r)$ is the moderating function in connection with a perturbing/modifying piece of the entire potential given. The form of $Q(g)$ and $R(g)$ is already well defined [55] for any special function $F(g)$ when dealing with analytically solvable potentials. However, in case of a realistic non-solvable problem consideration one should derive a reliable scheme to obtain the corrections more accurately due to the moderating piece of the potential in the light of exact part of the calculations. This is the significant point in the framework of the new formalism to reach physically meaningful solutions.

After some algebra, Eq. 4.3 turns out to be

$$\frac{F''}{F}(g')^2 + 2\frac{F'}{F}\frac{f'}{f}g' + \frac{F'}{F}g'' + \frac{f''}{f} = 2m\left(V_{eff} - E\right) \quad , \tag{4.5}$$

where $V_{eff}(r) = V(r) + \dfrac{\Lambda(\Lambda+1)}{2mr^2}$ and the primes denote derivatives with respect to r. To make a relation between Eqs. (4.4) and (4.5), we transform the above equation to the form of

$$F'' + F'\left(\frac{2}{g'}\frac{f'}{f} + \frac{g''}{(g')^2}\right) + F\left[\frac{f''}{f(g')^2} + \frac{2m}{(g')^2}\left(E - V_{eff}\right)\right] = 0 \ . \tag{4.6}$$

Bearing in mind that the full spectra of the system underlined is given by $E = E_{ES} + \Delta E$, Eq. (4.6) can be transformed easily to a coupled equation leading to the explicit solutions for exactly solvable part of the potential

$$E_{ES} - \left[V_{ES}(r) + \frac{\Lambda(\Lambda+1)}{2mr^2}\right] = \frac{(g')^2}{2m}R(g) \tag{4.7}$$

and

$$2\left(\frac{F'g'}{F}\right)\frac{f'}{f} + \frac{f''}{f} = 2m\left[\Delta V(r) - \Delta E\right] \tag{4.8}$$

which is responsible for the computation of the rectifications. More clearly, the solution of (4.8) in either an explicit or approximate form depending on the structure of ΔV,

provides us to see the modifications $(\Delta E, f)$ to the exact solutions $\left(E_{ES}, F(g)\right)$ due to the consideration of a modifying/perturbing interaction (ΔV) in a realistic application.

4.2. Application

As the main aim of this chapter is in general to obtain a physically reliable scheme for more accurate solutions of non-solvable potentials, and also to illustrate the flexible applicability of the model, we focus here a specifally chosen example that is the Cornell potential in arbitrary dimensions,

$$V_{eff}(r) = V_{ES}(r) + \Delta V(r) = \left(-\frac{a}{r} + \frac{\Lambda(\Lambda+1)}{2mr^2} \right) + br \quad , \tag{4.9}$$

in which the first term is an exactly solvable Colulomb-like potential that has the well-known solutions [70] along the frame of (4.7) which provides a safe basis for the computation of the connected solutions

$$E_n^{ES} = -\frac{m a^2}{2(n+\Lambda+1)^2} \quad , \quad \Psi_n^{ES}(r) \sim F_n(g) = \left[e^{-g/2}\, g^{(\alpha+1)/2}\, L_n^{\alpha}(g) \right] \tag{4.10}$$

for which we have an appropriate choice [55] such as

$$Q(g) = 0 \quad , \quad R(g) = \frac{n+\Lambda+1}{g} - \frac{\Lambda(\Lambda+1)}{g^2} - \frac{1}{4} \quad , \quad n = 0,1,2,\ldots, \tag{4.11}$$

where $g(r) = \dfrac{2mar}{n+\Lambda+1}$, $\alpha = 2\Lambda+1$, and finally $L_n^{\alpha}(g)$ is the generalized Laguerre polynomial.

Obviously, the task here is to solve Eq. (4.8). Tracking down approximate forms of its solutions has always aroused interest [15, 54, 68, 69, 72, 73, 74, 75, 76, 77] [78, 79, 80, 81, 82, 83, 84, 85, 86]. Apart from being useful in understanding of many physical phenomena, the importance of searching for them also stems from the fact that they very often provide a good starting point for undertaking perturbative calculations of more complex systems. Considering all these earlier works and the experience gained from such calculations, we propose below in section 4.2.2. an alternative simple recipe,

having a physically motivated visualizable framework, which enables one to give secure physical explanations behind the precise numerical findings or experimental measurements concerning with the interaction of interest.

4.2.1. Attempts for the solution of Eq. (4.8)

In the last three decades with the rapid development of nonlinear science, there has appeared ever increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. In particular, most of the one-particle potentials that are encountered in such quantum mechanical applications do not allow closed exact solutions either for the energy eigenvalues or for the wavefunctions as in the present case. In these circumstances, one resorts to approximate methods which may be suitable to the particular situation to obtain approximate solutions. Perhaps the most useful solutions in this context are the perturbative solutions. This is due to the observation that in most of these problems, the Hamiltonian can be written as a sum of a major term which allows exact solutions and a small perturbing term. One may then obtain solutions in the form of perturbation series which agree with the exact solutions to an accuracy of increasing powers of the perturbation parameters. However, in many practical cases we encounter divergent or asymptotically convergent series, depending on the nature and strength of the perturbation [87], and the references therein]. Improving the convergence of the standard Rayleigh–Schrödinger perturbative expansion (RSPE) has been the subject of many studies in the past, see e.g. Refs. [88, 89, 90, 91] and many variants of optimized expansions were proposed. These efforts have been aimed to calculate either energy eigenvalues or wavefunctions and are limited in practice by the rapidly growing complexity of generating corrections beyond the first few orders in the perturbative expansion. With the consideration of all these works we have attempted to solve Eq. (4.8), first, within a perturbative scheme which is discussed through in Appendix E. However our exhaustive application results, which are not given here for the sake of clarity, have revealed once more that such approaches inherently work well for only quite small parameters of the linear potential in (4.9).

We have then drawn our attention to the logarithmic perturbation theory (LPT) [92, 93, 94], a lesser-known alternative to RSPE in quantum mechanics. The virtue of LPT is its avoidance of the cumbersome summation over states for second- and higher-

order energy corrections in RSPE. Unluckily, it has problems of its own in calculating corrections to excited states, owing to the presence of nodes in the wavefunctions. Various schemes have been proposed to circumvent the resulting singularities [95, 96, 97, 98, 99, 100]. Meanwhile, several approximation methods using supersymmetric quantum theory [36] have been developed [101, 102, 103]. Despite their developments in the literature have been completely independent, the work in [102] has explicitly clarified that supersymmetric perturbation theories (SSPT) [101, 102] and LPT are entirely equivalent, in which fortuitously each turns out to resolve adversities encountered in the other. Considering these earlier calculations, in addition to our first attempt mentioned above, the whole formalism discussed in this article has then been expressed within the framework of supersymmetric quantum theory along the work in [102] defining the forms of the superpotentials in there as $W_n^{ES} = -\frac{1}{\sqrt{2m}} \frac{F_n' g'}{F_n(g)}$, corresponding to the exact (Coulomb like) potential, and $\Delta W_n = -\frac{1}{\sqrt{2m}} \frac{f_n'}{f_n}$ concerning the modifying (linear potential) solutions, for especially an alternative treatment of Eq.(4.8). Later, as the next step, the familiar expression of the supersymmetric quantum mechanics

$\left(W_n^{ES} + \Delta W_n\right)^2 - \left(W_n^{ES} + \Delta W_n\right)' \Big/ \sqrt{2m} = V - E_n$ has been splitted in two parts as $\left(W_n^{ES}\right)^2 - \left(W_n^{ES}\right)' \Big/ \sqrt{2m} = V_{ES} - E_n^{ES}$ and $2W_n^{ES}\Delta W_n + \Delta W_n^2 - \left(\Delta W_n\right)' \Big/ \sqrt{2m} = \Delta V - \Delta E_n$ generating strictly Eqs. (4.7) and (4.8), respectively, for individual quantum states. Nevertheless, the expansion of ΔW as described in [102], together with the proper expansions of ΔV and ΔE at each successive perturbation order for the approximate solution of the refinements makes the calculation scheme considerably cumbersome due to a tedious iterative procedure used for the description of the partner potentials in the model. Beside of this inconvenience, which is not illustrated here as it is out of the scope of the present work, we have seen again that this type of calculations work only rather small confining potential parameters $b \ll 0.1$. These two points make the calculation scheme naturally useless and impracticable as in our previous attempt presented in Appendix E.

To sum up, with the experience gained from these our earlier calculations and also taking full advantages of the approaches in Refs. [95, 103] we present here a more

economical but instructive prescription for the interpretation of accurate solutions of Eq. (4.8) in connection with the analysis of Cornell like funnel potentials.

4.2.2. Alternative approach to Eq. (4.8)

Clearly, the total wavefunction of the system under consideration assumes its asymptotic behaviour at very large distance that is completely determined by the modifying linear piece (br) in the full potential appeared in (4.9), which is the same for all (ground and excited) states, while at intermediate distances the wavefunction still decays exponentially but now governed by the Coulomb term. Hence, the lack of a unique frame for the equal treatments of the main and modifying parts of the interaction potential considered in perturbation theories, unlike the model presented here, does not decisevly affect the description of especially the ground-state with small parameters, but destroys the ability of the potential to describe such interactions beyond small perturbation domains. In this respect, the present consideration would offer the opportunity to improve such calculations due to the clear visualization of the refinements, in an explicit manner, in connection with the confinement potential term.

As the full solution for the Coulomb like potentials at intermediate distances of the present interaction is known, Eq. (4.10), and the both sides of Eq. (4.7) are equal to each other in the asymptotic region due to the structure of $R(g)$ in (4.11) when $r \to \infty$, clarifying the non-contribution of the Coulomb potential at large distances, one needs to focus at just Eq. (4.8) to obtain the corrections appeared in the asymptotic domain. In this case, evidently the unnormalized solution in this region is given by the Airy function as in Ref. [54], $f(r) = Ai\left((2mb)^{1/3} r\right)$, if

$$\Delta E_n \simeq -\frac{1}{m}\left(F_n' g' / F_n\right) f'/f \quad \Rightarrow \quad f''(r) - \left(2mbr\right) f'(r) = 0 \tag{4.12}$$

Thus, this assumption guides us to define the entire solution as $\Psi_n(r) = F_n(g) f(r) = \Psi_n^{ES}(r) Ai\left((2mb)^{1/3} r\right)$ where Ψ_n^{ES} is given by (4.10).

To investigate the validity of the above assumption, that is the most significant piece in the formalism, we start first with the ground-state solution in arbitrary

dimensions. In this circumstance, the correction term for the energy contribution takes the following form

$$\Delta E_{n=0}=\left(\frac{a}{\Lambda+1}-\frac{\Lambda+1}{m\,r}\right)\frac{f'(r)}{f(r)} \tag{4.13}$$

which depends on r. This peculiar behaviour is of course due to the present consideration of the non-exactly solvable nature of the Cornell potential. Before proceeding, however, it is worthwhile to pay some attention on (4.13). Recalling our previous second attempt to propose an approximate scheme for the solution of (4.8) within the frame of the supersymmetric perturbation theory, that has been discussed in the previous section, it is clear that

$$2W_n^{ES}\Delta W+\left(\Delta W_n^2-\frac{(\Delta W)'}{\sqrt{2m}}\right)=br-\Delta E_n \tag{4.14}$$

where $W_n^{ES}=-F_n'g'/\sqrt{2m}F_n(g)$ and $\Delta W=-f'/\sqrt{2m}f$ for the present treatment case. The substitution of these superpotentials into (14) yields explicitly Eq.(4.8) and subsequently Eqs. (4.12) and (4.13) for $n=0$ consideration, as $\Delta V=\Delta W_{n=0}^2-(\Delta W)'/\sqrt{2m}=f''/2mf=br$ and

$$\Delta E_{n=0}=2W_{n=0}^{ES}\Delta W=\left(\frac{a}{\Lambda+1}-\frac{\ell+1}{m\,r}\right)\frac{f'(r)}{f(r)} \tag{4.15}$$

From (4.15), one sees that a proper choice of ΔW in connection with the pertinent wavefunction (f) yields a constant value for ΔE reducing the problem to an analytically solvable case. Otherwise, as in the present non-solvable potential consideration, ΔW and ΔE should be expanded, similar to (Appendix E.2), as in either the usual perturbation or supersymmetric perturbation theories to solve the problem in an approximate form. Unfortunately, such treatments have some drawbacks mentioned in 4.2.1.

Therefore, instead of removing this apparent disadvantage in (4.15) using an iterative or perturbative scheme, we adopt here a different strategy concerning with the clarification of the physics behind nearly precise numerical calculations in the light of Eq. (4.13). This will reveal an interesting relation between the critical $r-$values in the

wavefunction, yielding exact energies, and the choice of an appropriate parent potential in perturbation like theories. A quick analysis of (4.13), at this stage, tells us that if $r = r_0 = \dfrac{(\Lambda+1)^2}{m\,a}$ then $\Delta E = 0$, in which condition all the refinements disappear, reducing the entire system to the purely exactly solvable Coulombic case where $E \to E^{ES}_{n=0} = -a/2r_0 = -ma^2/2(\Lambda+1)^2$. This location in the corresponding energy expression is in fact the maximum of the ground-state Coulomb wavefunction, $\Psi^{ES}_{n=0} = r^{\Lambda+1}\exp\left(-\dfrac{ma}{\Lambda+1}r\right)$, that can be easily checked through $\left(\partial\Psi^{ES}_{n=0}/\partial r\right) = 0$.

From this short instructive discussion, a good choice is to proceed with the another maximum (r_0) in connection with the the ground-state of the full wavefunction in the asymptotic domain, $\Psi \sim r^{\Lambda+1} f(r)$, for the precise calculations of ΔE. In fact, for the Cornell potential, because of the attractive Coulomb term, the potential function is in general not bounded below and therefore the choice of r_0 to be the location of the maximum of the ground-state wavefunction in the asymptotic region, as a initial guess, seems reasonable. Hence, one should expect the physically reasonable $r-$values in (4.13), responsible for the exact energies, in the vicinity of this maximum point. To define exactly the positions of these shifted $r = r_{\Delta E}$ values, we need to consider the exact energies calculated through precise numerical techniques such as the ones in Refs. [81, 54] and the related references therein. As $\Delta E = E_{exact} - E_{ES}$, Eq. (4.13) can be expressed as

$$E^{n=0}_{exact} = -\frac{ma}{2(\Lambda+1)^2} + \left(\frac{a}{\Lambda+1} - \frac{\Lambda+1}{m r_{\Delta E}}\right)\frac{f'(r)}{f(r)} \tag{4.16}$$

which serves us to find exact locations for $r-$values related to the exact energy eigenvalues of the individual $\ell-$states. The systematic calculation of $r_{\Delta E}$ value via (4.16) offers no difficulty if we resort a computer algebra system like Mathematica, Mapple or Reduce.

The results obtained are tabulated through the three tables. For the purpose of consistency, we have calculated each $r_{\Delta E}$ value to 16 significant figures, unlike the other

two values rounded in there for clarity. In Tables 4.1 and 4.2 we report the eigenvalues for the Schrödinger equation in distinct dimensions with the potential $V=-1/r+br$ considering the different sizes of the linear potential strength. Table 4.3 illustrates $S-$ wave ground-state eigenenergies for the potential $V=-a/r+r$ with dissimilar values of the parameter a. From these results, the following comments apply:

1- All $r_{\Delta E}-$values obtained satisfy $r_{\Delta E} \prec \dfrac{(\Lambda+1)^2}{m\,a}$ condition, as ΔE is always positive whereas f'/f has a negative structure in Eq. (4.13), except $a=0$ case.

2- The first $r_{\Delta E}-$values in Tables 4.1 and 4.2 for $b=0.01$ reveal why one naturally expects a bound only for this situations, which are very close to the critical $r-$value, $r=r_0=(\Lambda+1)^2/ma$, where the Coulomb like potential is dominant and the linear part of the Cornell potential may be treated as a perturbation in this case.

3- From Table 4.1, as r_0 and $r_{\Delta E}$ depend on $b-$ and $\ell-$values due to the form of the related function, $r^{\ell+1}f(b,r)$, in the asymptotic region, the decrease in these $r-$values can be readily observed through the increase in the parameter b. However, a considerable increment in r_0 and $r_{\Delta E}$ are clearly visible when ℓ increases for each individual $b-$value. Since the asymptotic solution is independent of the Coulombic parameter a, the $r-$values of interest are not affected by the changes in a as in Table 4.3.

4- When b increases ΔE gets larger values. In particular in case of $b=100$ in Table 4.1, where the linear confinement is more active, the energy corrections are striking with the rise of $\ell-$values. In this circumstance, the Colombic piece may be regarded as a perturbative term. In contrast to this observation, Table 4.3 shows that ΔE decreases with increasing $a-$values which makes the Coulomb potential more stronger than the confining part. Consequently, the refinements in eigenvalues due to the linear potential diminish as expected. These results confirm the comments given above.

b	ℓ	r_0	$r_{\Delta E}$ (Eq.4.13)	$\Delta E_{n=0}$
0.01	0	4.103	1.7436481087936350	0.029
	1	6.873	4.9460678940048420	0.080
	2	9.195	7.9252834202633755	0.130
	3	11.265	10.555231364572375	0.175
	4	13.163	12.921702562396474	0.216
	5	14.935	15.094068452133266	0.254
1	0	0.884	0.7994448794104599	1.648
	1	1.481	1.5319979780777233	2.888
	2	1.981	2.1271924940550924	3.878
	3	2.427	2.6476825358214430	4.742
	4	2.836	3.120002415656866	5.527
	5	3.218	3.5579114353323114	6.255
100	0	0.190	0.20718831032409812	46.652
	1	0.319	0.3568814759026067	70.079
	2	0.427	0.48025853095063736	89.743
	3	0.523	0.5892382922692437	107.350
	4	0.611	0.6887636263960271	123.572
	5	0.693	0.7814337345649496	138.768

Table 4. 1: Ground-state $r_{\Delta E}$ − values for $V(r)=\frac{-1}{r}+br$ in $N=3$ dimensional space.

b	ℓ	r_0	$r_{\Delta E}$	$\Delta E_{n=0}$
0.01	0	5.566	3.3312700304565390	0.534
	1	8.074	6.4834209674432110	0.106
	2	10.255	9.278604516903260	0.153
	3	12.232	11.766427983937469	0.196
	4	14.063	14.028816948623492	0.235
	5	15.783	14.585313802191063	0.293
1	0	1.199	1.1896870585180375	2.314
	1	1.739	1.8415039963613402	3.404
	2	2.209	2.394675709114608	4.322
	3	2.635	2.888823024670750	5.143
	4	3.030	2.4699633437009440	7.094
	5	3.400	2.3228580785446677	8.805

Table 4. 2: Ground-state $r_{\Delta E}-$values for $V(r)=\frac{-1}{r}+br$ in $N=4$ dimensional space.

5- Additionally, there is an impressive correlation between the $r-$values underlined. In Table 1, $r_0 \succ r_{\Delta E}$ is valid for $b=0.01$ through $\ell=0$ to $\ell=4$. The rapid rise in $r_{\Delta E}$ due to increase in $\ell-$values makes them comparable, even larger than r_0 as in the case of $\ell=5$. This can be understandable as the angular momentum barrier with large ℓ provides more positive contribution to the whole potential, which decreases effects of the Coulombic part while the modifications in ΔE gets larger. On the other hand, for larger values of b, $r_{\Delta E} \succ r_0$ through the all $\ell-$values except only $b=1$ with $\ell=0$ case where both potentials are comparable. From this short discussion, we conclude that $r_0 \succ r_{\Delta E}$ signalizes the domains where the Coulombic part is dominant while the linear piece behaves as a perturbating potential whereas $r_{\Delta E} \succ r_0$ indicates the regions where

the confining potential is more active and the Coulomb like potential can be treated as a modifying term. Roughly speaking, $r_{\Delta E}$ measures the distance at which the potential changes from being dominantly Coulombic $\left(r_{\Delta E} \prec r_0\right)$ to dominantly linear $\left(r_{\Delta E} \succ r_0\right)$. In sum, as in a perturbation procedure the value of the model parameters play a crucial role in choosing the parent and perturbative terms, reliable estimations of $r_{\Delta E}$ would be helpful in providing perception on this choice and on the energy corrections as well through Eq. (4.13). Some such physical insight is usually necessary in any approach which is based on a judicious choice of the either related potential functions or concerning wavefunctions. Within this context, the present comments on our findings would be helpful on clarifying the questions concerning with the limitations of the traditional perturbation techniques.

a	$r_{\Delta E}$	$\Delta E_{n=0}$	a	$r_{\Delta E}$	$\Delta E_{n=0}$
0.0	1.0092498710582083	2.338	1.0	0.7994448794104599	1.648
0.1	0.9871174720215355	2.254	1.1	0.7806100091335577	1.593
0.2	0.9650736643159619	2.177	1.2	0.7622476487440810	1.541
0.3	0.9432041027784062	2.101	1.3	0.7443630403947710	1.491
0.4	0.9215784931711197	2.028	1.4	0.7269585604248531	1.443
0.5	0.9002533954125955	1.958	1.5	0.7100341340504672	1.397
0.6	0.8792744720957341	1.891	1.6	0.6935875917754227	1.353
0.7	0.8586783015732216	1.826	1.7	0.6776149777440374	1.311
0.8	0.8384938486996074	1.764	1.8	0.6621108181210410	1.270
0.9	0.8187436656830261	1.705	1.9	0.6470684263448296	1.232

Table 4. 3: The lowest level $r_{\Delta E}-$values for $V\left(r\right)=\dfrac{-a}{r}+r$ in $N=3$ dimension with $r_0=0.884$

6- The situation is nonetheless not similar as the dimension increases. In particular, considering $N=4$ case for $b=1$ with $\ell=4$ and $\ell=5$ in Table 4.2, the trend in $r-$ values is opposite to those of $N=3$. The change in the definition of the angular momentum concept in this dimension where $\ell \to \ell+1/2=\Lambda$, which has been debated in section 4.1, causes this conflicting behaviour due to the new shifted larger shapes of the Coulomb wavefunction, $r^{\ell+3/2}e^{-mar/(\ell+3/2)}$, and of the concerning asymptotic wavefunction , $r^{\ell+3/2}f(r)$, for the ground-state solutions in this larger dimension. In addition, unlike the case in Table 4.1, Columbic interaction seems more powerful than the linear end of the potential for even $\ell=5$ in case of $b=1$ shown in Table 4.2, as still $r_0 \succ r_{\Delta E}$ because of the same reason regarding the another forms of the wavefunctions.

Aitchison and Dudek in Ref. [104] put an argument about the significance of some critical $r-$values in the analysis of heavy quarkonium spectra and showed that with the Coulombic part, as parent in the Cornell potential, a bottomonium spectra are well explained than charmonium whereas charmonium states are well elucidated with linear part as the main piece of the same potential, which makes the the above analysis noteworthy in this context.

However, the situation for the excited states is different. Because the generalized Laquerre polynomial in (4.10), defining the behaviour of the Coulomb like wavefunction, now is not constant and might be deformed by the linear potential unlike the case in (4.10) such as, for instance, $L_{n=1}^{2\Lambda+1}=2(\Lambda+1)-\beta(a,b,\Lambda)r$ for the first excited state. Using the exact treatment $H\Psi_{n=1}/\Psi_{n=1}$ where, from the discussion through this subsection,

$$\Psi_{n=1} \sim F_{n=1}(\beta)f=\left\{r^{\Lambda+1}\,\mathrm{e}^{-mar/(\Lambda+2)}\,\left[1-\left(\beta r/2(\Lambda+1)\right)\right]\right\}Ai\left((2mb)^{1/3}\,r\right),$$

one first should extract the structure of $\beta(a,b,\Lambda)$. The procedure for this computation has been well decribed in our previous works through [53, 70]. Our application results have however clarified that there has been certainly no deformation, in the present case, on the Laguere polynomial, $\beta=ma/(\ell+1)(\ell+2)$, as no dependency appears on *b-value*. The reason behind this observation is in fact hidden in the nature of the current potential parameters which do not involve any constraint. Thus, after the construction of

the correct form of the Laquerre polynomial relating to $n=1$ state, subsequently the full wavefunction $F_{n=1}(\beta)=F_{n=1}(g)$, the calculation of $\Delta E_{n=1}$ is straightforward

$$\Delta E_{n=1}=\left[\frac{a}{\Lambda+2}-\frac{a}{(\Lambda+1)(\beta r-1)}+\frac{\Lambda+1}{mr(\beta r-1)}\right]\frac{f'(r)}{f(r)} \tag{4.17}$$

if we recall Eq. (4.8) and substitute $F_{n=1}(\beta)$ in it. This process can be applied to higher excited states in the same manner, but involving lengthy equations that make the calculation procedure tedious and cumbersome.

4.3. Summary

We have presented an alternative technique for the solution of the Schrödinger equation with non-solvable potentials having an analytically solvable part, together with a modifying piece. Although we have limited ourselves to one illustrative example, the range of application of the method is rather large and appears to be straightforward. The formalism has a structure for the analysis of related experimental results and of precise numerical solutions, which can be accomplished with ease. Through the formalism used we have shown how estimates of $r_{\Delta E}$ could help to provide intuition on whether a given state is likely to be modelled better by the Coulomb or the linear part of the Cornell potential, together with reasonable predictions for the energy refinements with the help of Eq. (4.13), which are significant in the analysis of heavy quarkonium spectra. In view of the importance in analysing such corrections in a simple manner, we believe that the present model would serve as a useful toolbox to interpret even properly chosen more realistic situations, which now occur in experimental observations with the advent of the quantum technology. Additionally, when the structure of a critically stable quantum system is analyzed, understanding the analytic behavior of the solution as a function of different physical parameters is often of decisive importance. This fact justifies the significance of the analytical treatments, such as the one discussed here, in different disciplines of the science.

Aside of the problem posed and discussed through the present article having its own inner mathematical beauty and ability for providing a good starting point for doing calculations perturbatively/non-perturbatively for more complex systems, the present

work once more reveals that a proper perturbation method, requiring no small parameters in the equations, which can readily eliminate the limitations of the traditional perturbation techniques which often fails miserably for systems on the border of stability, is urgently required as most nonlinear equations have no small parameter at all. Within this context, nonlinear analysis methods involving homotopy perturbation techniques [105, 106, 107] and the quasilinearization method [108, 109, 110, 111, 112, 113] which approximates solution of nonlinear differential equations by treating the nonlinear terms as a perturbation about the linear ones, are promising. These models are iterative but not perturbative and give generally stable solutions to nonlinear problems without depending on the existence of a smallness parameter. The use of such powerful treatments for the solution of Eq. (4.8) within the present framework and its extension to other similar potentials is underway.

CHAPTER 5

CONCLUSION

In the present thesis work, we have investigated the heavy quark systems ($c\bar{c}$ and $b\bar{b}$) in the non-relativistic framework within a novel formalism using phenomenological inter-quark potentials involving exactly solvable energy-dependent potentials, together with quasi-/conditionally-solvable and non-solvable interaction potentials.

In Chapter 2, our first application to exactly solvable systems has been made to generalize the work in [37] and shown that the mathematically rigorous new scheme unifies different theories for the solution of Schrödinger equation with analytically solvable conventional and energy-dependent potentials. Although the literature covered similar problems, to our knowledge an investigation such as the one presented here was missing. To go beyond qualitative aspects, the applications are devoted to the study of the wave equation with potentials depending, especially, on the energy which is essential in understanding the interaction mechanism in heavy quark systems. It has been clarified that such investigations can be performed safely through our schematical model without causing any physical problem [53]. Although we have limited ourselves to a few illustrative examples, the range of application of the method is rather large and appears to be straightforward**.**

Chapter 3, has discussed the spin-averaged mass spectra of heavy quarkonia, charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), with a conditionally solvable Coulomb plus linear and quadratic potential within the framework of nonrelativistic Schrodinger equation. The exact energy eigenvalues and eigenfunctions have been obtained analytically for $n = 0$ employing the present novel method. Using the available experimental data, the obtained energy formula has been used to fit charmonium and bottomonium mass spectra from which then the potential parameters (a, b, and c) were easily determined.

These parameters have then been employed to reproduce the mass spectra.

The well agreement has been observed when the predictions of our model have been compared with the related data. This justifies the flexibility of the model used when one considers the wide applicability of the formalism developed.

In Chapter 4, we have extended the present novel model for the solution of the non-solvable problems, which in this case has a structure for the analysis of related experimental results and of precise numerical solutions. Through the formalism used we have shown how estimates of $r_{\Delta E}$ could help to provide intuition on whether a given state is likely to be modelled better by the Coulomb or the linear part of the Cornell potential, together with reasonable predictions for the energy refinements with the help of Eq. (4.13), which are significant in the analysis of heavy quarkonium spectra

In conclusion, when the structure of a critically stable quantum system is analyzed, understanding the analytic behavior of the solution as a function of different physical parameters is often of decisive importance. This fact justifies the significance of the analytical treatments, such as the ones discussed in this thesis, in different disciplines of the science. Nevertheless, the present work in particular through Chapter 4 once more reveals that a proper perturbation method, requiring no small parameters in the equations, which can readily eliminate the limitations of the traditional perturbation techniques, is urgently required as most nonlinear equations have no small parameter at all. The use of such powerful treatments for the solution of Eq. [4.8] within the present framework and its extension to other similar potentials is underway.

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APPENDIX A

The full derivation of Eq. (2.24) in section 2.2.1.1 is discussed here. As we know the solution of the energy independent piece of the spherical harmonic oscillator potential from Eq. (2.18), we can solve the energy dependent piece and full solution readily,

$$\Delta E - \Delta V(r) = -\left(\frac{g''}{2} + \frac{f'g'}{f}\right)\Delta Q\left(g(r)\right) + \left(g'\right)^2\left[\Delta R\left(g(r)\right) - \frac{1}{2}\frac{d(\Delta Q)}{dg} - \frac{1}{4}\Delta Q\left(g(r)\right)^2\right] \tag{A.1}$$

For this purpose we start with the Eq. 2.12,

where

$$g = \frac{\omega}{2}r^2 \quad \Rightarrow \quad g' = \omega r \quad \Rightarrow \quad g'' = \omega \tag{A.2}$$

$$\Delta Q = 1 - \sqrt{1 + \gamma E_{n\ell}} \tag{A.3}$$

$$\frac{f'}{f} = \frac{\ell + 1}{r} - \frac{\omega r}{2} \tag{A.4}$$

$$\Delta R(g) = -\Delta Q(g)\frac{F'(g)}{F(g)} \tag{A.5}$$

and $F(g) = L_n^{\alpha}(g)$

leading to

$$\frac{F'(g)}{F(g)} = -\frac{L_{n-1}^{\alpha+1}(g)}{L_n^{\alpha}(g)} \tag{A.6}$$

From the following recurrence relation

$$n L_n^{\alpha}(g) = (n+\alpha) L_{n-1}^{\alpha}(g) - g\, L_{n-1}^{\alpha+1}(g)$$

which gives;

$$-L_{n-1}^{\alpha+1}(g) = \frac{n L_n^{\alpha}(g) - (n+\alpha) L_{n-1}^{\alpha}(g)}{g} \tag{A.7}$$

By the substitution of Eq. (A.7) into Eq. (A.6), we have

$$\frac{F'(g)}{F(g)} = \frac{n L_n^{\alpha}(g) - (n+\alpha) L_{n-1}^{\alpha}(g)}{g L_n^{\alpha}(g)}$$

And finally one readily sees that

$$\frac{F'(g)}{F(g)} = \frac{n}{g} - \frac{(n+\alpha) L_{n-1}^{\alpha}(g)}{g L_n^{\alpha}(g)} \quad , \quad \Rightarrow \quad \frac{F'_{n\geq 1}(g)}{F_{n\geq 1}(g)} \approx \frac{n}{g}$$

(A.8)

Thus,

$$\Delta R(g) = -\Delta Q(g) \frac{n}{g}$$

$$\Delta R(g) = -\left(1 - \sqrt{1+\gamma E_{n\ell}}\right) \frac{n}{g}$$

$$\Delta R(g) = \left(\sqrt{1+\gamma E_{n\ell}} - 1\right) \frac{n}{g} \tag{A.9}$$

Furthermore, the substitution of Eqs. (A2, A.3, A.4 and A.9) into Eq. (A.1), yields

$$\Delta E - \Delta V(r) = \omega \left(2n + \ell + \frac{3}{2}\right)\left(\sqrt{1+\gamma E_{n\ell}} - 1\right) - \omega^2 r^2 \frac{\gamma E_{n\ell}}{4} \tag{A.10}$$

which means that

$$\Delta E = \omega \left(2n + \ell + \frac{3}{2}\right)\left(\sqrt{1+\gamma E_{n\ell}} - 1\right) \tag{A.11}$$

$$\Delta V\left(r\right)=\omega^2 r^2\,\frac{\gamma\,E_{n\ell}}{4} \tag{A.12}$$

Remembering,

$$E_{n\ell}=E_{ES}+\Delta E$$

$$E_{n\ell}=\omega\left(2n+\ell+\frac{3}{2}\right)\left(\sqrt{1+\gamma E_{n\ell}}-1\right)+\omega\left(2n+\ell+\frac{3}{2}\right)$$

$$E_{n\ell}=\omega\left(2n+\ell+\frac{3}{2}\right)\left(\sqrt{1+\gamma E_{n\ell}}\right) \tag{A.13}$$

To finalize, let us take the square of each side

$$\left(E_{n\ell}\right)^2=\left[\omega\left(2n+\ell+\frac{3}{2}\right)\left(\sqrt{1+\gamma E_{n\ell}}\right)\right]^2 \tag{A.14}$$

that can be rearranged as

$$\left(E_{n\ell}\right)^2-E_{n\ell}\left[\omega\left(2n+\ell+\frac{3}{2}\right)\right]^2\gamma+\left[\omega\left(2n+\ell+\frac{3}{2}\right)\right]^2=0 \tag{A.15}$$

Then the solution of this quadratic equation is

$$E_{n\ell}^{\pm}=\frac{\omega^2\gamma}{2}\left(2n+\ell+\frac{3}{2}\right)^2\pm\frac{\omega}{2}\left(2n+\ell+\frac{3}{2}\right)\sqrt{\omega^2\gamma^2\left(2n+\ell+\frac{3}{2}\right)^2+4} \tag{A.16}$$

APPENDIX B

We discuss here the way of clear derivation of Eq. (2.43) appeared in section 2.2.1.3. Starting with Eq. (2.11),

$$E_{ES} - V_{ES}(r) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + (g')^2\left[R_{ES}(g(r)) - \frac{1}{2}\frac{dQ_{ES}}{dg} - \frac{1}{4}Q_{ES}(g(r))^2\right] \tag{B.1}$$

where

$$Q(g) = \frac{(\alpha + 1 - g)}{g} \tag{B.2}$$

$$R(g) = \frac{n}{g} \tag{B.3}$$

$$F(g) = L_n^{\alpha}(g) \tag{B.4}$$

$$\frac{(g')^2}{g} = a^2 \qquad \text{and} \qquad a^2 = 4\mu\omega \tag{B.5}$$

leading to

$$\alpha = \left(\frac{1}{2}\sqrt{(2\ell+1)^2 + 8\mu b}\right) \tag{B.6}$$

and with the substitution of Eqs. (B.2- B.5) into Eq. (B.1), one obtains

$$\tilde{E} - \tilde{V}(r) = 2\mu\omega\left[2n + 1 + \frac{1}{2}\sqrt{(2\ell+1)^2 + 8\mu b}\right] - \mu\omega^2 r^2 - \frac{1}{r^2}\left[\ell(\ell+1) + 2\mu b\right] \tag{B.7}$$

in which

$$\tilde{E} = 2\mu E = 2\mu\omega\left[2n + 1 + \frac{1}{2}\sqrt{(2\ell+1)^2 + 8\mu b}\right]$$

Therefore,

$$E = \omega\left[2n+1+\frac{1}{2}\sqrt{(2\ell+1)^2+8\mu b}\right] \tag{B.8}$$

Similarly,

$$\tilde{V}(r) = 2\mu V(r) = \mu\omega^2 r^2 + \frac{1}{r^2}\left[\ell(\ell+1)+2\mu b\right]$$

which gives the correct result as in (2.44)

$$V(r) = \frac{1}{2}\omega^2 r^2 + \frac{\ell(\ell+1)}{2\mu r^2} + \frac{b}{r^2} \tag{B.9}$$

This observation justifies the reliability of the model calculations.

APPENDIX C

Here, the full derivation of Eqs. (2.45) and (2.46) appeared in section 2.2.1.3 are presented. In analogy to the previous sections, we start here with the Eq. (2.12),

$$\Delta E' - \Delta V'(r) = -\left(\frac{g''}{2} + \frac{f'g'}{f}\right)\Delta Q\left(g(r)\right) + \left(g'\right)^2\left[\Delta R\left(g(r)\right) - \frac{1}{2}\frac{d(\Delta Q)}{dg} - \frac{1}{4}\Delta Q\left(g(r)\right)^2\right] \tag{C.1}$$

in which

$$g = \frac{a^2 r^2}{4} \quad \Rightarrow \quad g' = \frac{a^2}{2}r \quad \Rightarrow \quad g'' = \frac{a^2}{2} \quad \Rightarrow \quad g''' = 0 \tag{C.2}$$

$$a^2 = 4\mu\omega \tag{C.3}$$

$$\Delta Q = 1 - \sqrt{1 + \gamma E_{n\ell}} \tag{C.4}$$

$$\frac{f'}{f} = \frac{2\alpha + 1}{2r} - \frac{1}{4}a^2 r \tag{C.5}$$

$$\alpha = \left(\frac{1}{2}\sqrt{(2\ell+1)^2 + 8\mu b}\right) \tag{C.6}$$

and therefore

$$\Delta R(g) = -\Delta Q(g)\frac{F'(g)}{F(g)} \tag{C.7}$$

where $F(g) = L_n^{\alpha}(g)$ that leads to

$$\frac{F'(g)}{F(g)} = -\frac{L_{n-1}^{\alpha+1}(g)}{L_n^{\alpha}(g)} \tag{C.8}$$

From the following recurrence relation

$$n L_n^{\alpha}(g) = (n+\alpha) L_{n-1}^{\alpha}(g) - g\, L_{n-1}^{\alpha+1}(g)$$

Namely,

$$-L_{n-1}^{\alpha+1}(g) = \frac{n L_n^{\alpha}(g) - (n+\alpha) L_{n-1}^{\alpha}(g)}{g} \tag{C.9}$$

The substitution of the Eq. (C.9) into Eq. (C.8) gives,

$$\frac{F'(g)}{F(g)} = \frac{n L_n^{\alpha}(g) - (n+\alpha) L_{n-1}^{\alpha}(g)}{g L_n^{\alpha}(g)}$$

And finally,

$$\frac{F'(g)}{F(g)} = \frac{n}{g} - \frac{(n+\alpha) L_{n-1}^{\alpha}(g)}{g L_n^{\alpha}(g)} \qquad \Rightarrow \qquad \frac{F'_{n\geq 1}(g)}{F_{n\geq 1}(g)} \approx \frac{n}{g} \tag{C.10}$$

that shapes the related expressions as

$$\Delta R(g) = -\Delta Q(g) \frac{n}{g}$$

or,

$$\Delta R(g) = \left(\sqrt{1+\gamma E_{n\ell}} - 1\right) \frac{n}{g} \tag{C.11}$$

Moreover, the substitution of Eqs. (C.2-C.5), (C.7) and (C.11) into (C.1) yields

$$\tilde{E} - \tilde{V}(r) = 2\mu\omega(2n+\ell+\alpha)\left(\sqrt{1+\gamma E_{n\ell}} - 1\right) - \mu^2\omega^2 r^2 (\gamma E_{n\ell}) \tag{C.12}$$

Remembering,

$$\Delta\tilde{E} = 2\mu\Delta E = 2\mu\omega(2n+\ell+\alpha)\left(\sqrt{1+\gamma E_{n\ell}} - 1\right)$$

one can express the correction term to the energy as

$$\Delta E = \omega(2n+\ell+\alpha)\left(\sqrt{1+\gamma E_{n\ell}} - 1\right) \tag{C.13}$$

In a similar manner, it is not hard to see that

$$\Delta\tilde{V}(r) = 2\mu\Delta V(r) = \mu^2\omega^2 r^2(\gamma E_{n\ell}) \tag{C.14}$$

$$\Delta V(r) = \frac{1}{2}\mu\omega^2 r^2(\gamma E_{n\ell})$$

in which it is clear that

$$E_{n\ell} = E_{ES} + \Delta E$$

or

$$E_{n\ell} = \omega(2n+\ell+\alpha)\left(\sqrt{1+\gamma E_{n\ell}} - 1\right) + \omega(2n+\ell+\alpha)$$

or more clearly,

$$E_{n\ell} = \omega(2n+\ell+\alpha)\left(\sqrt{1+\gamma E_{n\ell}}\right) \tag{C.15}$$

In order to finalize the problem, we need to take the square of each side

$$(E_{n\ell})^2 = \left[\omega(2n+\ell+\alpha)\left(\sqrt{1+\gamma E_{n\ell}}\right)\right]^2$$

that can be re-expressed as

$$(E_{n\ell})^2 - \left[\omega(2n+\ell+\alpha)\right]^2\gamma E + \left[\omega(2n+\ell+\alpha)\right]^2 = 0 \tag{C.16}$$

Obviously, the solution of this quadratic equation can be given as

$$E_{n\ell}^{\pm} = \frac{\omega^2\gamma}{2}(2n+\ell+\alpha)^2 \pm \frac{\omega}{2}(2n+\ell+\alpha)\sqrt{\omega^2\gamma^2(2n+\ell+\alpha)^2 + 4} \tag{C.17}$$

APPENDIX D

The free parameters in the model are to be adjusted so as to give an agreement between the more accurate and reliable experimentally observed values of the lowest states, $1S$ and $1D$, and the corresponding theoretical values determined by (3.31) in the light of (3.26) and (3.30). The difference between these states is

$$E_{1D}-E_{1S}=\left(-\frac{b^2}{4c}+7\sqrt{\frac{c}{2m}}\right)-\left(-\frac{b^2}{4c}+3\sqrt{\frac{c}{2m}}\right)=4\sqrt{\frac{c}{2m}} \tag{D.1}$$

which enables us to compute $c-$value in connection with the related data. From Eq. (3.31), the above expression can be transformed to a more practical form leading to

$$M_{1D}^{\exp}-M_{1S}^{\exp}=4\sqrt{\frac{c}{2m}}=4\sqrt{\frac{c}{m_q}} \quad \Rightarrow \quad c=\frac{\left(M_{1D}^{\exp}-M_{1S}^{\exp}\right)^2}{16}m_q \tag{D.2}$$

As the experimental values in (D.2) for the $c\bar{c}$ and $b\bar{b}$ systems are known, the coupling constant can be readily calculated. Then the use of $c-$value in (3.26) for the ground state calculations, either for $1S$ or $1D$ state, identically reproduces the other coupling parameter, from which the strength of the unperturbed potential (a) can be computed through $b=\dfrac{a_{n,\ell}\sqrt{2mc}}{n+\ell+1}$. Obviously, the two expressions in Eq. (3.26) yields the same results, which can be used as a testing ground.

For the first excited state calculations, bearing the insufficiency and inaccuracy of the available data in mind, we start with the $c-$value obtained for $n=0$ as an input to be able to initiate the related calculations. Following the numerical definition of a_{10} appeared in E_{10} given by (3.30), the strength of the linear potential (b) is calculated for $n=1$ state which enables us to compute a_{11} leading to the calculation of E_{11} which can also be evaluated in terms of only b and c values defined for $n=1$.

Both definitions produce the same E_{11} as in (3.30).

Finally, the bound-state heavy quarkonium mass spectra are described smoothly by (3.31). As a closing remark here, the experimental data for the $2P-b\bar{b}$ bound-system seems inaccurate, because the definition of the coupling constant

$$c=\frac{\left(M_{2P}^{\exp}-M_{2S}^{\exp}\right)^2}{4}m_q \tag{D.3}$$

similar to Eqs. (D.1) and (D.2), yields an imaginary $a_{1\ell}-$value within the frame of (3.30). However, the use of a slightly larger value for the mass spectrum of the bottomonium system for $2P$ level, as the present prediction in Table 3.2, reproduces reasonable results.

APPENDIX E

Here, we show that Eq. (4.8) may also be transformed to another formalism based on an alternative perturbation procedure. Reminding the full wavefunction description $\Psi = F(g) f$ used through the article, we first rearrange Eq. (4.8) as

$$F_n(g) f_n'' + 2(F_n' g') f_n' = 2m\left[\Delta V(r) - \Delta E\right] F_n(g) f_n \quad , \tag{E.1}$$

where $F_n(g)$ is explicitly defined by (4.10), in order to remove the logarithm of exactly solvable Coulomb like wavefunction in (4.8) for not dealing with tedious factoring out of the zeros of $F_n(g)$.

Using the spirit of perturbation theories, one can now expand $f(r)$ and ΔE in terms of the perturbation parameter λ,

$$f_n(r,\lambda) = 1 + \sum_{j=1}^{\infty} \lambda^j f_{nj}(r), \quad \Delta E_n(\lambda) = \sum_{j=1}^{\infty} \lambda^j \Delta E_{nj}, \quad \Delta V(r) = \lambda^1 (br) \quad , \tag{E.2}$$

where j denotes the perturbation order. Substitution of the above expansions into (E.1) by equating terms with the same power of λ on both sides yields

$$F_n(g) f_{n1}'' + 2(F_n' g') f_{n1}' = 2m\left[\Delta V(r) - \Delta E_{n1}\right] F_n(g) \ , \tag{E.3}$$

$$F_n(g) f_{n2}'' + 2(F_n' g') f_{n2}' = 2m\left\{\left[\Delta V(r) - \Delta E_{n1}\right] f_{n1} - \Delta E_{n2}\right\} F_n(g) \ , \tag{E.4}$$

$$F_n(g) f_{n3}'' + 2(F_n' g') f_{n3}' = 2m\left\{\left[\Delta V(r) - \Delta E_{n1}\right] f_{n2} - \Delta E_{n2} f_{n1} - \Delta E_{n3}\right\} F_n(g) \ , \tag{E.5}$$

$$F_n(g) f_{nk}'' + 2(F_n' g') f_{nk}' = 2m\left\{\left[\Delta V(r) - \Delta E_{n1}\right] f_{n(k-1)} - \sum_{i=2}^{k} \Delta E_{ni} f_{n(k-i)}\right\} F_n(g) \ , \tag{E.6}$$

where $f_{n0}(r) = 1$. The results obtained here, interestingly, overlap with those of [95], which reveals that the present approach and the one in [95] are identical though their developments are quite different, such us the relation between LPT and SSPT mentioned above.

Considering [95], and also [102], one can readily see that each of the above linear differential equations is trivially solved by using an integrating factor $F_n(g)$. For instance, multiplying each side of Eq. (E.3) by $F_n(g)$ gives $\left(F_n^2 f_{n1}'\right)' = 2m\left(\Delta V - \Delta E_{n1}\right)F_n^2$. Since $F_n(g) \to 0$ when $r \to \infty$, one obtains first-order corrections as

$$\Delta E_{n1} = \int_0^\infty \Delta V(r) F_n^2\left(g(r)\right) dr \ , \tag{E.7}$$

and

$$f_{n1}(r) = 2m \int^r \frac{dr'}{F_n^2\left(g(r')\right)} \int^{r'} \left[\Delta V(r'') - \Delta E_{n1}\right] F_n^2\left(g(r'')\right) dr'' \ . \tag{E.8}$$

Note that the lower limit of the integration for energy calculations is taken as 0, instead of $-\infty$ that is valid only for one-dimensional cases, in order to accommodate the fact that r is always positive. Repeating the same procedure as in [95, 102], one arrives at the following higher-order results

$$\Delta E_{n2} = \int_0^\infty \left[\Delta V(r) - \Delta E_{n1}\right] f_{n1}(r) F_n^2\left(g(r)\right) dr \ , \tag{E.9}$$

$$f_{n2}(r) = 2m \int^r \frac{dr'}{F_n^2\left(g(r')\right)} \int^{r'} \left\{ \left[\Delta V(r'') - \Delta E_{n1}\right] f_{n1}(r'') - \Delta E_{n2} \right\} F_n^2\left(g(r'')\right) dr'' , \tag{E.10}$$

and

$$\Delta E_{nk} = \int_0^\infty \left\{ \left[\Delta V(r) - \Delta E_{n1}\right] f_{n(k-1)}(r) - \sum_{i=2}^{k-1} \Delta E_{ni} f_{n(k-i)}(r) \right\} F_n^2\left(g(r)\right) dr \ , \tag{E.11}$$

$$f_{nk}(r) = 2m \int^r \frac{dr'}{F_n^2\left(g(r')\right)} \int^{r'} \left\{ \left[\Delta V(r'') - \Delta E_{n1}\right] f_{n(k-1)}(r'') - \sum_{i=2}^{k} \Delta E_{ni}\, f_{n(k-i)}(r'') \right\} F_n^2\left(g(r'')\right) dr'' \tag{E.12}$$

As reported by [95], and from our calculations, the modifying function $f_{nk}(r)$ that is calculated individually at each perturbation order is in general singular at zeros of $F_n\left[g(r)\right]$ while the entire function $\Psi = Ff$ remains finite at these points by shifting the position of zeros of $F_n(g)$ under the influence of a perturbing potential. Also, the

indefiniteness of integrals in the expression for f_{nk} is justified when one requires $F_n(g)$ is orthogonal to $F_n(g) f_{nk}$.

However, in spite of the all success behind this analysis, the application of the model to the potential under consideration has shown that the present formalism, as discussed above, works well for only small perturbation parameters $(b \leq 0.01)$ as is the case for the other perturbation theories, drawing our attention to more applicable alternative procedures.

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1. B. Gönül, O. Özer, Y. Cançelik and M. Koçak (2000). Hamiltonian hierarchy and the Hulthen potential. *Phys. Lett. A*. **275**, 238.
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