BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OPIAL TYPE INTEGRAL INEQUALITIES

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

ELİF ARSLAN

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ADVISOR: Assoc. Prof. Yüksel SOYKAN

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

Elif ARSLAN

ABSTRACT

Master of Science Thesis

OPIAL TYPE INTEGRAL INEQUALITIES

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Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Yüksel SOYKAN

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Integral inequalities have been frequently employed in the theory of functional analysis, differential equations and applied sciences such as probability and statistics. There are a lot of types of inequalities such as Hermite-Hadamard type inequalities, Lyapunov type inequalities, Wirtinger type inequalities and Qi type inequalities. They have been the center of attention in many papers. Especially, Opial inequalities have been studied by many authors.

In this thesis, we are concerned with Opial type integral inequalities. We present some theorems on generalizations of Opial inequality and Opial type inequalities on time scale. We survey the literature on those type inequalities.

The organization of this thesis is as follows:

In Chapter 1, we give a brief overview of what this thesis is about and also introduce the necessary inequalities and some definition.

ABSTRACT (continued)

In Chapter 2, we investigate results on generalizations of Opial inequality.

In Chapter 3, we consider discrete Opial inequalities.

In Chapter 4, we present Opial dynamic inequalities on time scales.

In Chapter 5, we give some general weighted Opial type inequalities on time scales.

In Chapter 6, we establish some Opial type inequalities on time scales via the notion of the diamond-alpha derivative.

In Chapter 7, we consider dynamic Opial type diamond-alpha inequalities in time scales.

Keywords: Opial type inequalities, inequalities, time scales.

Science Code: 403.03.01

ÖZET

Yüksek Lisans Tezi

OPIAL TİPİ İNTEGRAL EŞİTSİZLİKLERİ

Elif ARSLAN

Bülent Ecevit Üniversitesi
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İntegral eşitsizlikleri fonksiyonel analiz teorisinde, diferansiyel denklemlerde ve olasılık istatistik gibi uygulamalı bilimlerde sıkça kullanılır. Hermite-Hadamard tipi eşitsizlikler, Lyapunov tipi eşitsizlikler, Wirtinger tipi eşitsizlikler ve Qi tipi eşitsizlikler gibi bir çok eşitsizlik tipleri vardır. Eşitsizlikler bir çok makalenin merkezinde yer almaktadır. Özellikle Opial eşitsizlikleri birçok yazar tarafından çalışılmıştır.

Bu tezde, Opial tipi integral eşitsizlikleri ile ilgileniyoruz. Zaman skalasında Opial tipi eşitsizlikler ve Opial eşitsizliğinin genellemeleri üzerine bazı teoremler sunuyoruz. Bu tip eşitsizliklerle ilgili literatürü inceliyoruz.

Bu tezin organizasyonu aşağıdaki gibidir:

Birinci bölümde, tezin ne ile alakalı olduğu özet olarak verilmiştir ve ayrıca gerekli eşitsizlikler ve bazı tanımlar sunulmuştur.

ÖZET (devam ediyor)

İkinci bölümde, Opial eşitsizliğinin genellemeleri üzerine sonuçları inceliyoruz.

Üçüncü bölümde, ayrık Opial eşitsizliklerini ele alıyoruz.

Dördüncü bölümde, zaman skalasında Opial dinamik eşitsizlikleri sunuyoruz.

Beşinci bölümde, zaman skalasında bazı genel ağırlıklı Opial tipi eşitsizlikler veriyoruz.

Altıncı bölümde, elmas-alfa türevi kavramı aracılığıyla zaman skalasında bazı Opial tipi eşitsizlikleri gözden geçiriyoruz.

Yedinci bölümde, zaman skalasında dinamik Opial tipi elmas-alfa eşitsizliklerini ele alıyoruz.

Anahtar Kelimeler: Opial tipi eşitsizlikler, eşitsizlikler, zaman skalası.

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CHAPTER 1

PRELIMINARIES AND INTRODUCTION

1.1 THE TIME SCALES CALCULUS

The theory of time scales was first introduced by S. Hilger [1] in 1988 in order to unify continuous and discrete analysis and to extend those theories to cases "in between". Here in this section, we recall some basic concepts from the theory of time scales.

Definition 1.1 *A time scale is any nonempty closed subset of the reals, $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$.*

The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.
We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$\sigma(t) = \inf\{s \in \mathbb{T} |\ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} |\ s < t\}$

respectively, each of which is being supplemented by

$\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

A point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, left-dense, if

$\sigma(t) > t, \sigma(t) = t, \rho(t) < t, \rho(t) = t$

is valid, respectively. In other words, If $\sigma(t) > t$, then we say that t is right-scattered (otherwise, except when $t = \sup \mathbb{T}$: right-dense), while if $\rho(t) < t$, then t is called left-scattered (otherwise, except when $t = \inf \mathbb{T}$: left-dense).
The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum. In other words:

Definition 1.2 *Suppose that the time scale $\mathbb{T}$ have a right-scattered minimum m. Then we define the set $\mathbb{T}^\kappa$ by*

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}\backslash\{m\} & , \quad m \textit{ exists} \\ \mathbb{T} & , \quad m \textit{ does not exist.} \end{cases}$$

On the other hand, if the time scale $\mathbb{T}$ have a left-scattered maksimum M then we define the set $\mathbb{T}^{\kappa}$ by

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T}\backslash\{M\} & , \quad M \text{ exists} \\ \mathbb{T} & , \quad M \text{ does not exist.} \end{cases}$$

Moreover, the (forward) graininess $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t$$

and the backward graininess $\upsilon : \mathbb{T} \to [0, \infty)$ is defined by

$$\upsilon(t) := t - \rho(t).$$

Example 1.1 *If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ then*

$$\mu(t) = \begin{cases} 0 & , \quad \mathbb{T} = \mathbb{R} \\ 1 & , \quad \mathbb{T} = \mathbb{Z} \end{cases}$$

and

$$\upsilon(t) = \begin{cases} 0 & , \quad \mathbb{T} = \mathbb{R} \\ 1 & , \quad \mathbb{T} = \mathbb{Z} \end{cases}$$

From above example we see that the (forward) graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$ while it is constant 1 for $\mathbb{T} = \mathbb{Z}$. But, a time scale $\mathbb{T}$ could have nonconstant forward graininess.

Definition 1.3 *A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be regressive if*

$$1 + \mu(t)f(t) \neq 0 \quad (t \in \mathbb{T}).$$

Furthermore if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the functions $f^{\sigma} = f \circ \sigma : \mathbb{T} \to \mathbb{R}$ and $f^{\rho} = f \circ \rho : \mathbb{T} \to \mathbb{R}$ by

$$f^{\sigma}(t) = f(\sigma(t)), \quad t \in \mathbb{T}$$

and

$$f^{\rho}(t) = f(\rho(t)), \quad t \in \mathbb{T}$$

respectively.

With this terminology, the delta (Δ) and nabla (∇) derivatives are defined as follows.

Definition 1.4 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then f is called Δ-differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists a number a such that for every $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$|f^\sigma(t) - f(s) - a(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U$$

and then we write $f^\Delta(t) = a$.

In this case we say that $f^\Delta(t)$ is the delta derivative of $f(t)$ at the point $t \in \mathbb{T}^\kappa$. If f is delta differentiable for every $t \in \mathbb{T}^\kappa$, then f is said to be delta differentiable on $\mathbb{T}$ and f^Δ is a new function defined on $\mathbb{T}^\kappa$.
If f is delta differentiable at $t \in \mathbb{T}^\kappa$, then it is easy to see that

$$f^\Delta(t) = \begin{cases} \lim_{s\to t, s\in\mathbb{T}} \frac{f(t)-f(s)}{t-s} & \text{if } \mu(t) = 0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t) > 0. \end{cases}$$

Similarly we can define ∇-differentiability.

Definition 1.5 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then f is called ∇-differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists a number b such that for every $\varepsilon > 0$, there exists a neighborhood V of t such that*

$$|f^\rho(t) - f(s) - b(\rho(t) - s)| \leq \varepsilon |\rho(t) - s| \text{ for all } s \in V,$$

and then we write $f^\nabla(t) = b$.

We refer the reader to [2] for a comprehensive development of the calculus of the Δ-derivative and to [3] for an account of the calculus corresponding to the ∇-derivative.
Other useful delta derivative formulas are given as follows:

Lemma 1.1 *Delta derivatives satisfy the following properties:*

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t) f^\Delta(t) && (1.1) \\ (fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) && (1.2) \\ \left(\frac{f}{g}\right)^\Delta(t) &= \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. && (1.3) \end{aligned}$$

A function is called rd-continuous if it is continuous at right-dense points and has finite left-sided limits at left-dense points. In other words:

Definition 1.6 *A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it satisfies each of the following conditions:*

(i) *f is continuous at every right-dense point or maximal point of $\mathbb{T}$;*

(ii) *The left-sided limit:*

$$\lim_{\tau \to t-} f(\tau) = f(t-)$$

exists at every left-dense point of $\mathbb{T}$.

The space of all rd-continuous functions $\mathbb{T} \to \mathbb{R}$ is denoted as follows:

$C_{rd}(\mathbb{T}, \mathbb{R}) := \{f |\ f : \mathbb{T} \to \mathbb{R}$ and $f(t)$ is an rd-continuous function$\}$.

A function is called ld-continuous if it is continuous at left-dense points and has finite right-sided limits at right-dense points.

Lemma 1.2 *If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous at $t \in \mathbb{T}$ and t is right scattered then*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

We now recall the concepts of delta and nabla integrations. The significance of rd-continuous functions is revealed by the following existence result by Hilger [1]: Every rd-continuous function possesses an antiderivative. Here, F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^{\Delta} = f$ holds on $\mathbb{T}^{\kappa}$. In fact,

Definition 1.7 *A function $F : \mathbb{T} \to \mathbb{R}$ is said to be Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if*

$$F^{\Delta}(t) = f(t), \quad t \in \mathbb{T}^{\kappa},$$

i.e., $F^{\Delta} = f$.

In this case we define the Δ-integral of f by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \quad (s, t \in \mathbb{T})$$

and we say f is integrable on $\mathbb{T}$.

Lemma 1.3 *The following integral formulas holds true:*

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\Delta} = f(t) \tag{1.4}$$

and

$$\int_a^t f(\tau)g^{\Delta}(\tau)\Delta\tau = f(\tau)g(\tau)|_{\tau=a}^{t} - \int_a^t f^{\Delta}(\tau)g\left(\sigma(\tau)\right)\Delta\tau \tag{1.5}$$

for any constant $a \in \mathbb{T}$.

Definition 1.8 *A function $G : \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $G^{\nabla} = f$. In this case, we define ∇-integral of f by*

$$\int_s^t f(\tau)\nabla\tau = G(t) - G(s), \ (s, t \in \mathbb{T})$$

and we say f is integrable on $\mathbb{T}$.

It is known that rd-continuous functions have Δ-antiderivatives and ld-continuous functions have ∇-antiderivatives.

Combining the above derivatives and integrations, Sheng et al. [4] (see, also, [5- 7]) have introduced the concept of the diamond-alpha ($\Diamond_\alpha$) derivative on time scales. Suppose $\alpha \in [0, 1]$ and $f : \mathbb{T} \to \mathbb{R}$ be a function. Then the $\Diamond_\alpha$-differentiation (diamond-alpha differentiation) of f at a point $t \in \mathbb{T}$ is defined by

$$f^{\Diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t).$$

It is obvious that $f^{\Diamond_\alpha}$ exists for $\alpha \in (0, 1)$ if and only if f^{Δ} and f^{∇} exist. Observe also $f^{\Diamond_1} = f^{\Delta}$ and $f^{\Diamond_0} = f^{\nabla}$ if these derivatives exist. Moreover, for the other values of α, one has other dynamic derivatives (e.g., a centralized derivative with $\alpha = 1/2$). The $\Diamond_\alpha$-integral of f is defined by

$$\int_a^t f(s)\Diamond_\alpha s = \alpha \int_a^t f(s)\Delta s + (1-\alpha) \int_a^t f(s)\nabla s.$$

Note that the cases of $\alpha = 1$ and $\alpha = 0$ reduce to the Δ-integral and ∇-integral of the function f, respectively.

We say that a function $f : [0, h] \to \mathbb{R}$ is in the class $C^1_{\Diamond_\alpha}$ if f is $\Diamond_\alpha$-differentiable such that αf^{Δ} is rd-continuous, $(1-\alpha)f^{\nabla}$ is ld-continuous, and $\alpha(1-\alpha)f^{\Diamond_\alpha}$ is continuous. This guarantees the existence of all occurring integrals. We note that $C^1_{\Diamond_\alpha}$ is, for $\alpha \in (0, 1)$,

equal to the class of functions that are Δ-differentiable and ∇-differentiable such that f^{Δ} is rd-continuous, f^{∇} is ld-continuous, and $f^{\Diamond_\alpha}$ is continuous. Furthermore, $C^1_{\Diamond_0}$ is equal to the class of functions that are ∇-differentiable such that f^{∇} is ld-continuous, and $C^1_{\Diamond_1}$ is equal to the class of functions that are Δ-differentiable such that f^{Δ} is rd-continuous. Results about $\Diamond_\alpha$-derivatives and $\Diamond_\alpha$-integrals may be found in the papers [4, 6, 7].
For further basic results about the time scales calculus we refer the reader to [1, 8- 10].

1.2 OPIAL INEQUALITIES

In the year 1960, the Polish mathematician Zdzidlaw Opial [11] published the following fascinating integral inequality which is an inequality involving integrals of a function and its derivative:

Theorem 1.4 *[12, Theorem 1.1.1.](Opial Inequality) Suppose $f(t) \in C^{(1)}[0,h]$ is such that $f(0) = f(h) = 0$, and $f(t) > 0$ in $(0,h)$. Therefore, the following inequality is valid*

$$\int_0^h |f(t)f'(t)|\,dt \leq \frac{h}{4}\int_0^h (f'(t))^2 dt. \tag{1.6}$$

Moreover, in (1.6), the constant $h/4$ is the best possible.

A weaker form of (1.6) can be obtained rather easily by combining the Cauchy-Schwarz inequality and Wirtinger's inequality which says that: for $f(t) \in C^{(1)}[0,h]$ such that $f(0) = f(h) = 0$,

$$\int_0^h f^2(t)dt \leq \frac{h^2}{\pi^2}\int_0^h (f'(t))^2 dt. \tag{1.7}$$

Actually, we get

$$\begin{aligned}\int_0^h |f(t)f'(t)|\,dt &\leq \left(\int_0^h f^2(t)dt\right)^{\frac{1}{2}}\left(\int_0^h (f'(t))^2\,dt\right)^{\frac{1}{2}}\\ &\leq \frac{h}{\pi}\int_0^h (f'(t))^2 dt.\end{aligned}$$

Inequalities which involve integrals of functions and their derivatives are of great importance in mathematics with applications in the theory of differential equations, approximations and probability. It has been shown that inequalities of the form (1.6) can be deduced from those of Wirtinger and Hardy type, but the importance of Opial's result is in the establishment of the best possible constant. The monograph [12] is the first book dedicated to the theory of Opial type inequalities.

A function $f : [a, b] \to \mathbb{R}$ is said to be absolutely continuous on $[a, b]$ if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon,$$

whenever $\{[x_i, y_i] : i = 1, ..., n\}$ is a finite collection of mutually disjoint subintervals of $[a, b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$. Obviously, an absolutely continuous function on $[a, b]$ is uniformly continuous. Moreover, a Lipschitz continuous function on $[a, b]$ is absolutely continuous. Suppose f and g are two absolutely continuous functions on $[a, b]$. Therefore $f + g$, $f - g$, and fg are absolutely continuous on $[a, b]$. If, moreover, there exists a constant $C > 0$ such that $|g(x)| \geq C$ for all $x \in [a, b]$, then f/g is absolutely continuous on $[a, b]$. If f is integrable on $[a, b]$, therefore the function F defined by

$$F(x) := \int_a^x f(t)dt, \; a \leq x \leq b,$$

is absolutely continuous on $[a, b]$.

The assumption in Theorem 1.4 can be weakened. The positivity requirement of $f(t)$ in the original proof of Opial was shown to be unnecessary later by Olech [13] where he proved that the inequality (1.6) holds even for functions $f(t)$ which are only absolutely continuous in $[0, h]$. In other words, C. Olech [13] showed that (1.6) is valid for any function $f(t)$ which is absolutely continuous on $[0, h]$, and satisfies the boundary conditions $f(0) = f(h) = 0$. Moreover, Olech's proof of (1.6) was much simpler than that of Opial.

Theorem 1.5 *[13] Suppose that $f(t)$ is absolutely continuous in $[0, h]$ and $f(0) = f(h) = 0$. Then the following inequality is valid*

$$\int_0^h |f(t)f'(t)| \, dt \leq \frac{h}{4} \int_0^h (f'(t))^2 \, dt. \tag{1.8}$$

In his paper, P. R. Beesack [14] give an even simpler proof of (1.6), and then show how this method of proof can be made to yield more general inequalities of the same type.

Theorem 1.6 *[13], [12, Theorem 1.3.1.] Suppose that $f(t)$ is absolutely continuous in $[0, h]$, and $f(0) = f(h) = 0$. Then, the inequality (1.6) is valid. Furthermore, equality is valid if and only if $f(t) = f_0(t)$, where $f_0(t)$ is defined in*

$$f_0(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2} \\ c(h - t), & \frac{h}{2} \leq t \leq h \end{cases} \tag{1.9}$$

and c is an arbitrary constant.

In 1960, Olech [13] extended (1.6) and present the following result. The inequality in the following theorem is also known in the literature as Opial's inequality.

Theorem 1.7 *[14], [15, Theorem 1.1.](Opial Inequality) If $f \in C^1([0,h],\mathbb{R})$ with $h > 0$ fulfills $f(0) = 0$, then*

$$\int_0^h |f(t)f'(t)|\,dt \le \frac{h}{2}\int_0^h (f'(t))^2 dt.$$

The above inequality has generated a lot of research, both in the continuous and the discrete cases (see the monograph [12] and the references therein).

Many authors applied this above inequality to many branches of mathematics such as differential and difference equations [2, 16-22], fractional calculus [23- 27], multivariate analysis [28], semigroups [29, 30], and dynamic equations on time scales [2, 31- 35].

Remark 1.1 *Note that the expression of Theorem 1.7 can be written as follows: Suppose $h > 0$. For any differentiable function $f : [0,h] \to \mathbb{R}$ such that f' is continuous and $f(0) = 0$, we get*

$$\int_0^h |f(t)f'(t)|\,dt \le \frac{h}{2}\int_0^h \left(f'(t)\right)^2 dt.$$

In 1962, Beesack [14] refined Theorem 1.7 as follows (see also [36, Theorem 3]).

Theorem 1.8 *[15, Theorem 1.2.](Beesack Inequality) If $f \in C^1([0,h],\mathbb{R})$ with $h > 0$ fulfills $f(0) = 0$, then*

$$\int_0^h |f(t)f'(t)|\,dt + \frac{h}{2}\int_0^h \frac{g(t)}{t^2}dt \le \frac{h}{2}\int_0^h (f'(t))^2 dt, \tag{1.10}$$

where

$$g(t) := 2\int_0^t |f(\tau)f'(\tau)|\,d\tau - (f(t))^2 \ge 0.$$

Note that Theorem 1.8 includes Theorem 1.7. The next one is Olech's result under weaker conditions and gives an estimate with a bound which is less sharp.

Theorem 1.9 *[12, Theorem 1.4.1.](Continuous Opial inequality) Suppose $f(t)$ is absolutely continuous in $[0,a]$, and $f(0) = 0$. Then, the following inequality is valid*

$$\int_0^a |f(t)f'(t)|\,dt \le \frac{a}{2}\int_0^a (f'(t))^2 dt. \tag{1.11}$$

Furthermore, in (1.11), equality is valid if and only if $f(t) = ct$.

1.3 OPIAL'S INEQUALITY ON TIME SCALES

Opial inequalities and many of their generalizations have various applications in the theories of differential and difference equations. This is very nicely illustrated in the book [12] "Opial Inequalities with Applications in Differential and Difference Equations" by Agarwal and Pang, which is the only book devoted solely to continuous and discrete versions of Opial inequalities. In this section, following [37], we give present several Opial inequalities that are valid on time scales. Throughout we suppose $0 \in \mathbb{T}$ and let $h \in \mathbb{T}$ with $h > 0$.

We will need two simple consequences of the product rule: First,

$$\left(f^2\right)^{\Delta} = (f.f)^{\Delta} = f^{\Delta} f + f^{\sigma} f^{\Delta} = (f + f^{\sigma}) f^{\Delta}, \tag{1.12}$$

and in general, one can use mathematical induction to prove the formula

$$\left(f^{l+1}\right)^{\Delta} = \left\{ \sum_{k=0}^{l} f^k (f^{\sigma})^{l-k} \right\} f^{\Delta}, \; l \in \mathbb{N}. \tag{1.13}$$

Theorem 1.10 *[33], [38, Theorem 6.1.](Delta Dynamic Opial Inequality) For a delta differentiable $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = 0$ we get*

$$\int_0^h \left|(f + f^{\sigma}) f^{\Delta}\right|(t)\Delta t \le h \int_0^h \left|f^{\Delta}\right|^2(t)\Delta t,$$

with equality when $f(t) = ct$, provided all Δ-anti derivatives exist.

We refer to [33] for the proof of Theorem 1.10.

We next state a generalization of Theorem 1.10 where $f(0)$ need not be equal to 0.

Theorem 1.11 *[38, Theorem 6.2.] Suppose $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable and rd-continuous function. Then*

$$\int_0^h \left|(f + f^{\sigma}) f^{\Delta}\right|(t)\Delta t \le \alpha \int_0^h \left|f^{\Delta}(t)\right|^2 \Delta t + 2\beta \int_0^h \left|f^{\Delta}(t)\right| \Delta t,$$

where

$$\alpha \in \mathbb{T} \textit{ with } \operatorname{dist}(h/2, \alpha) = \operatorname{dist}(h/2, \mathbb{T}) \tag{1.14}$$

and $\beta = \max\{|f(0)|, |f(h)|\}$.

A consequence of Theorem 1.11 is the following result.

Theorem 1.12 *[38, Theorem 6.3.] Assume* $f : [0,h] \cap \mathbb{T} \to \mathbb{R}$ *is differentiable and rd-continuous with* $f(0) = f(h) = 0$*. Then*

$$\int_0^h \left|(f+f^\sigma)f^\Delta\right|(t)\Delta t \le \alpha \int_0^h \left|f^\Delta(t)\right|^2 \Delta t,$$

where α *is given in (1.14).*

The following are two time scales versions of Theorem 1.7. We suppose that 0 is contained in the time scale in these two theorems.

Theorem 1.13 *([2, 33]) Assume that* $h \in \mathbb{T}$ *with* $h > 0$*. For any* Δ*-differentiable function* $f : [0,h] \to \mathbb{R}$ *such that* f^Δ *is rd-continuous and* $f(0) = 0$*, we get*

$$\int_0^h \left|(f+f^\sigma)\, f^\Delta\right|(t)\Delta t \le h \int_0^h \left(f^\Delta\right)^2(t)\Delta t.$$

Theorem 1.14 *([3, 32]) Suppose* $h \in \mathbb{T}$ *with* $h > 0$*. For any* ∇*-differentiable function* $f : [0,h] \to \mathbb{R}$ *such that* f^∇ *is ld-continuous and* $f(0) = 0$*, we get*

$$\int_0^h \left|(f+f^\rho)\, f^\nabla\right|(t)\nabla t \le h \int_0^h \left(f^\nabla\right)^2(t)\nabla t.$$

Now we offer some of the possible generalizations of the inequalities presented above. The continuous or/and discrete versions of these results may be found in [12]. We have not included all of such results, but most of them may be proved by using similar techniques as the ones presented in this section.

Theorem 1.15 *[12, Theorem 2.5.1.] Suppose* p*,* q *are positive and continuous on* $[0,h]$*,* $\int_0^h \Delta t p(t) < \infty$*, and* q *non-increasing. For a differentiable* $f : [0,h]\cap\mathbb{T} \to \mathbb{R}$ *with* $f(0) = 0$ *we get*

$$\int_0^h \left[q^\sigma \left|(f+f^\sigma)f^\Delta\right|\right](t)\Delta t \le \left\{\int_0^h \frac{\Delta t}{p(t)}\right\}\left\{\int_0^h p(t)q(t)\left|f^\Delta(t)\right|^2 \Delta t\right\}.$$

Theorem 1.16 *[12, Chapter 3] Assume* $l, n \in \mathbb{N}$*. For a* n*-times differentiable* $f : [0,h] \cap \mathbb{T} \to \mathbb{R}$ *with* $f(0) = f^\Delta(0) = ... = f^{\Delta^{n-1}}(0) = 0$ *we get*

$$\int_0^h \left|\left\{\sum_{k=0}^{l} f^k\left(f^\sigma\right)^{l-k}\right\} f^{\Delta^n}\right|(t)\Delta t \le h^{ln} \int_0^h \left|f^{\Delta^n}(t)\right|^{l+1}\Delta t.$$

Theorem 1.17 *[12, Theorem 3.2.1.] Assume that $n \in \mathbb{N}$. For a n-times differentiable $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = f^{\Delta}(0) = ... = f^{\Delta^{n-1}}(0) = 0$ we get*

$$\int_0^h \left|(f + f^{\sigma})f^{\Delta^n}\right|(t)\Delta t \leq h^n \int_0^h \left|f^{\Delta^n}(t)\right|^2 \Delta t.$$

Theorem 1.18 *[12, Theorem 2.3.1.] Assume $l \in \mathbb{N}$. For a differentiable $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = 0$ we obtain*

$$\int_0^h \left|\left\{\sum_{k=0}^{l} f^k (f^{\sigma})^{l-k}\right\} f^{\Delta}\right|(t)\Delta t \leq h^l \int_0^h \left|f^{\Delta}(t)\right|^{l+1} \Delta t.$$

We conclude this section with a Wirtinger type inequality from [39].

Theorem 1.19 *[38, Theorem 6.8.](Wirtinger's Inequality) Suppose M is positive and strictly monotone such that M^{Δ} exists and is rd-continuous. Therefore we get*

$$\int_a^b \left|M^{\Delta}(t)\right| (y^{\sigma}(t))^2 \Delta t \leq \Psi \int_a^b \frac{M(t)M^{\sigma}(t)}{|M^{\Delta}(t)|} \left(y^{\Delta}(t)\right)^2 \Delta t$$

for any y with $y(a) = y(b) = 0$ and such that y^{Δ} exists and is rd-continuous, where

$$\Psi = \left\{\left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{M(t)}{M^{\sigma}(t)}\right)^{\frac{1}{2}} + \left[\left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\mu(t)\left|M^{\Delta}(t)\right|}{M^{\sigma}(t)}\right) + \left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{M(t)}{M^{\sigma}(t)}\right)\right]^{\frac{1}{2}}\right\}^2.$$

Example 1.2 *[38, Example 6.1.] Suppose $a > 0$ and*

$$\Psi = \left\{\left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\sigma(t)}{t}\right)^{\frac{1}{2}} + \left[\left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\mu(t)}{t}\right) + \left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\sigma(t)}{t}\right)\right]^{\frac{1}{2}}\right\}^2.$$

Therefore

$$\int_a^b \left(y^{\Delta}(t)\right)^2 \Delta t \geq \frac{1}{\Psi} \int_a^b \frac{(y^{\sigma}(t))^2}{t\sigma(t)}\Delta t. \tag{1.15}$$

To establish this we remark that $M(t) = \frac{1}{t}$ satisfies the assumptions of Theorem 1.19, and

$$\frac{M(\sigma(t)) - M(s)}{\sigma(t) - s} = \frac{1/\sigma(t) - 1/s}{\sigma(t) - s} = \frac{(s - \sigma(t))\,/\,(s\sigma(t))}{\sigma(t) - s} = -\frac{1}{s\sigma(t)}$$

implies

$$M^{\Delta}(t) = -\frac{1}{t\sigma(t)}.$$

Hence

$$\frac{M(t)}{M^{\sigma}(t)} = \frac{\sigma(t)}{t}, \quad \frac{\mu(t)\left|M^{\Delta}(t)\right|}{M^{\sigma}(t)} = \frac{\mu(t)}{t}, \quad \text{and} \quad \frac{M(t)M^{\sigma}(t)}{|M^{\Delta}(t)|} = 1,$$

and (1.15) follows from Theorem 1.19.

As an application of Theorem 1.19 and Example 1.2 we now state a sufficient criterion for nonoscillation of a certain second order dynamic equation, see [39, Theorem 3].

Theorem 1.20 *[38, Theorem 6.9.] Suppose $N \in \mathbb{T}$ and define*

$$\Psi_N = \left\{ \left(\sup_{t \geq N, t \in \mathbb{T}} \frac{\sigma(t)}{t} \right)^{\frac{1}{2}} + \left[\left(\sup_{t \geq N, t \in \mathbb{T}} \frac{\mu(t)}{t} \right) + \left(\sup_{t \geq N, t \in \mathbb{T}} \frac{\sigma(t)}{t} \right) \right]^{\frac{1}{2}} \right\}^2 .$$

If

$$0 < \limsup_{N \to \infty} \Psi_N = \Psi < \infty,$$

so the equation

$$y^{\Delta\Delta} + \frac{1}{\Psi t \sigma(t)} y^{\sigma} = 0$$

is nonoscillatory.

1.4 SOME SPECIAL INEQUALITIES ON TIME SCALES: A SURVEY

1.4.1 Hölder's Inequality

The following version of Hölder's inequality on time scales appears in [40, Lemma 2.2 (iv)], and it's proof is similar to that of the classical inequality as given e.g. in [41, Theorem 188].

Theorem 1.21 *[38, Theorem 3.1.](Hölder's Inequality) Suppose $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we get*

$$\int_a^b |f(t)g(t)| \, \Delta t \leq \left\{ \int_a^b |f(t)|^p \, \Delta t \right\}^{\frac{1}{p}} \left\{ \int_a^b |g(t)|^q \, \Delta t \right\}^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p-1)$.

The special case $p = q = 2$ reduces to the Cauchy-Schwarz inequality.

Theorem 1.22 *[38, Theorem 3.2.](Cauchy-Schwarz Inequality) Assume $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we get*

$$\int_a^b |f(t)g(t)| \, \Delta t \leq \sqrt{\left\{ \int_a^b |f(t)|^2 \, \Delta t \right\} \left\{ \int_a^b |g(t)|^2 \, \Delta t \right\}}.$$

Next, we can use Hölder's inequality to deduce Minkowski's inequality.

Theorem 1.23 *[38, Theorem 3.3.](Minkowski's Inequality) Assume $a, b \in \mathbb{T}$ and $p > 1$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we get*

$$\left\{\int_a^b |(f+g)(t)|^p \, \Delta t\right\}^{\frac{1}{p}} \leq \left\{\int_a^b |f(t)|^p \, \Delta t\right\}^{\frac{1}{p}} + \left\{\int_a^b |g(t)|^p \, \Delta t\right\}^{\frac{1}{p}}.$$

Proof. We apply Hölder's inequality, Theorem 1.21, with $q = p/(p-1)$ to obtain

$$\begin{aligned}
\int_a^b |(f+g)(t)|^p \, \Delta t &= \int_a^b |(f+g)|^{p-1} \, |(f+g)(t)| \, \Delta t \\
&\leq \int_a^b |f(t)| \, |(f+g)(t)|^{p-1} \, \Delta t + \int_a^b |g(t)| \, |f+g|^{p-1} \, (t) \Delta t \\
&\leq \left\{\int_a^b |f(t)|^p \, \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |(f+g)(t)|^{(p-1)q} \, \Delta t\right\}^{\frac{1}{q}} \\
&\quad + \left\{\int_a^b |g(t)|^p \, \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |(f+g)(t)|^{(p-1)q} \, \Delta t\right\}^{\frac{1}{q}} \\
&= \left[\left\{\int_a^b |f(t)|^p \, \Delta t\right\}^{\frac{1}{p}} + \left\{\int_a^b |g(t)|^p \, \Delta t\right\}^{\frac{1}{p}}\right] \left[\int_a^b |(f+g)(t)|^p \, \Delta t\right]^{\frac{1}{q}}.
\end{aligned}$$

We divide both sides of the obtained inequality by $\left[\int_a^b |(f+g)(t)|^p \, \Delta t\right]^{\frac{1}{q}}$ to arrive at Minkowski's inequality. ■

1.4.2 Jensen's Inequality

The proof of Jensen's inequality on time scales follows closely to the proof of the classical Jensen's inequality (see for example [42, Problem 3.42]). If $\mathbb{T} = \mathbb{R}$, then our version is the same as the classical Jensen inequality. But, if $\mathbb{T} = \mathbb{Z}$, then it reduces to the well-known arithmetic-mean geometric-mean inequality.

Theorem 1.24 *[38, Theorem 4.1.](Jensen's Inequality) Assume $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Assume $g : [a, b] \to (c, d)$ is rd-continuous and $F : (c, d) \to \mathbb{R}$ is convex. Therefore*

$$F\left(\frac{\int_a^b g(t)\Delta t}{b-a}\right) \leq \frac{\int_a^b F(g(t))\Delta t}{b-a}.$$

Proof. Let $x_0 \in (c, d)$. Therefore (e.g., by [42, p. 109]) there exists $\beta \in \mathbb{R}$ such that

$$F(x) - F(x_0) \geq \beta(x - x_0) \text{ for all } x \in (c, d). \tag{1.16}$$

Note that,

$$x_0 = \frac{\int_a^b g(\tau)\Delta\tau}{b-a}$$

is well-defined because g is rd-continuous. $F \circ g$ is also rd-continuous, and thus we may apply (1.16) with $x = g(t)$ and integrate from a to b to get

$$\begin{aligned}
\int_a^b F(g(t))\Delta t - (b-a)F\left(\frac{\int_a^b g(\tau)\Delta\tau}{b-a}\right) &= \int_a^b F(g(t))\Delta t - (b-a)F(x_0) \\
&= \int_a^b [F(g(t)) - F(x_0)] \\
&\geq \beta \int_a^b [g(t) - x_0]\,\Delta t \\
&= \beta \left[\int_a^b g(t)\Delta t - x_0(b-a)\right] \\
&= 0.
\end{aligned}$$

This directly gives Jensen's inequality. ∎

Example 1.3 *[38, Example 4.1.] Let* $\mathbb{T} = \mathbb{R}$*. Clearly,* $F = -\log$ *is convex and continuous on* $(0, \infty)$*, thus we may apply Theorem 1.24 with* $a = 0$ *and* $b = 1$ *to get*

$$\log \int_0^1 g(t)dt \geq \int_0^1 \log g(t)dt$$

and then

$$\int_0^1 g(t)dt \geq \exp\left[\int_0^1 \log g(t)dt\right]$$

whenever $g : [0,1] \to (0,\infty)$ *is continuous.*

Example 1.4 *[38, Example 4.2] Suppose* $\mathbb{T} = \mathbb{Z}$ *and* $N \in \mathbb{N}$*. Again we apply Jensen's inequality, Theorem 1.24, with* $a = 1$*,* $b = N+1$*,* $F = \log$*, and* $g : \{1, 2, ..., N+1\} \to (0, \infty)$ *to find*

$$\begin{aligned}
\log\left\{\frac{1}{N}\sum_{t=1}^{N} g(t)\right\} &= \log\left\{\frac{1}{n}\int_1^{N+1} g(t)\Delta t\right\} \\
&\geq \frac{1}{N}\int_1^{N+1} \log g(t)\Delta t \\
&= \frac{1}{N}\sum_{t=1}^{N} \log g(t) \\
&= \log\left\{\prod_{t=1}^{N} g(t)\right\}^{\frac{1}{N}}.
\end{aligned}$$

and then

$$\frac{1}{N}\sum_{t=1}^{N} g(t) \geq \left\{\prod_{t=1}^{N} g(t)\right\}^{\frac{1}{N}}.$$

This is the well-known arithmetic-mean geometric-mean inequality.

Example 1.5 *[38, Example 4.3.] Assume that $\mathbb{T} = 2^{\mathbb{N}_0}$ and $N \in \mathbb{N}$. We apply Theorem 1.24 with $a = 1$, $b = 2^N$, $F = \log$, and $g : \{2^k | 0 \leq k \leq N\} \to (0, \infty)$ to find*

$$\begin{aligned}
\log\left\{\frac{\sum_{k=0}^{N-1} 2^k g(2^k)}{2^N - 1}\right\} &= \log\left\{\frac{\int_1^{2^N} g(t)\Delta t}{2^N - 1}\right\} \\
&\geq \frac{\int_1^{2^N} \log g(t)\Delta t}{2^N - 1} \\
&= \frac{\sum_{k=0}^{N-1} 2^k \log g(2^k)}{2^N - 1} \\
&= \frac{\sum_{k=0}^{N-1} \log \left(g(2^k)\right)^{2^k}}{2^N - 1} \\
&= \frac{\log\left\{\prod_{k=0}^{N-1} \left(g(2^k)\right)^{2^k}\right\}}{2^N - 1} \\
&= \log\left\{\prod_{k=0}^{N-1} \left(g(2^k)\right)^{2^k}\right\}^{1/(2^N-1)}
\end{aligned}$$

and thus

$$\frac{\sum_{k=0}^{N-1} 2^k g(2^k)}{2^N - 1} \geq \left\{\prod_{k=0}^{N-1} \left(g(2^k)\right)^{2^k}\right\}^{1/(2^N-1)}.$$

1.4.3 Gronwall's Inequality

Definition 1.9 *[38, Definition 5.1] An rd-continuous function f is called regressive provided*

$1 + \mu(t)f(t) \neq 0$ *for all* $t \in \mathbb{T}$.

The set of all rd-continuous functions f that satisfy $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$ will be denoted by $\mathcal{R}^+$.

The following existence theorem has been proved by Hilger, see [1].

Theorem 1.25 *[38, Theorem 5.1.] Suppose $t_0 \in \mathbb{T}$. If p is rd-continuous and regressive, then*

$$y^{\Delta} = p(t)y, \; y(t_0) = 1$$

has a unique solution.

We call the unique solution from Theorem 1.25 the exponential function and denote it by $e_p(\cdot, t_0)$. In fact, there is an explicit formula for $e_p(t,s)$, using the so-called cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{z} & if \quad h \neq 0 \\ z & if \quad h = 0. \end{cases}$$

The formula, see [43], reads

$$e_p(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}\left(p(\tau)\right) \Delta\tau \right\}.$$

We now proceed to give some fundamental properties of the exponenetial function. To do so it is necessary to introduce the following notation: For regressive $p, q : \mathbb{T} \to \mathbb{R}$ we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \; p \ominus q := p \oplus (\ominus p).$$

We note that the set of all regressive and rd-continuous functions together with the addition $\oplus$ is an Abelian group.

Theorem 1.26 *[38, Theorem 5.2.] Suppose $p, q : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous, then the followings are valid:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))\, e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

Next we note the following result from [44].

Theorem 1.27 *[38, Theorem 5.3.] If p and f are rd-continuous and p is regressive, hence the unique solution of the initial value problem*

$$y^{\Delta} = p(t)y + f(t),\ \ y(t_0) = y_0$$

is given by

$$y(t) = y_0 e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

A comparison theorem follows.

Theorem 1.28 *[38, Theorem 5.4.] Suppose y and f are rd-continuous and $p \in \mathcal{R}^+$. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq y(a)e_p(t,a) + \int_{a}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

The above comparison Theorem 1.28 gives the following interesting results.

Theorem 1.29 *[38, Theorem 5.5.](Bernoulli's Inequality) Suppose $\alpha \in \mathbb{R}$ with $\alpha \in \mathcal{R}^+$. Then*

$$e_\alpha(t,s) \geq 1 + \alpha(t-s) \text{ for all } t,s \in \mathbb{T}.$$

Theorem 1.30 *[38, Theorem 5.6.](Gronwall's Inequality) Suppose y and f are rd-continuous and $p \in \mathcal{R}^+$, $p \geq 0$. Then*

$$y(t) \leq f(t) + \int_{a}^{t} y(\tau)p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq f(t) + \int_{a}^{t} e_p(t,\sigma(\tau))f(\tau)p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Example 1.6 *[38, Example 5.1.] If $\mathbb{T} = h\mathbb{Z}$ and y is a nonnegative function on $[0, \infty)$ and $b > 0$ is a constant such that*

$$y(t) \leq c(t) + b \sum_{k=0}^{\frac{t}{h}-1} y(kh)$$

for $t \in [0, \infty)$, hence

$$y(t) \leq c(t) + b \sum_{k=0}^{\frac{t}{h}-1} c(kh)(1 + bh)^{\frac{t-h(k+1)}{h}}.$$

If we let $h = 1$ in the above example we have the following.

Example 1.7 *[38, Example 5.2.] Suppose $\{y_n\}_{n=0}^{\infty}$ is a sequence of nonnegative real numbers and $b > 0$ is a constant such that*

$$y_n \leq c_n + b \sum_{k=0}^{n-1} y(k)$$

for $n \in \mathbb{N}_0$, so it follows that

$$y_n \leq c_n + b \sum_{k=0}^{n-1} c_k(1 + b)^{n-k-1}.$$

Corollary 1.1 *[38, Corollary 5.1.] Assume y is rd-continuous and $p \in \mathcal{R}^+$ with $p \geq 0$. Therefore*

$$y(t) \leq \int_a^t y(\tau)p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq 0 \text{ for all } t \in \mathbb{T}.$$

Corollary 1.2 *[38, Corollary 5.2.] Suppose y is rd-continuous and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > 0$. Therefore*

$$y(t) \leq \alpha(t - a) + \beta + \gamma \int_a^t y(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq \left(\frac{\alpha}{\gamma} + \beta\right) e_\gamma(t, a) - \frac{\alpha}{\gamma}.$$

We conclude this section with a Bihari type inequality from [45]. To prove this inequality, we need the following comparison result, whose proof can be found in [45, Theorem 3.1].

Theorem 1.31 *[38, Theorem 5.7.] Suppose $h : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in the second variable. Suppose v and w satisfy the dynamic inequalities*

$$v^{\Delta} \leq h(t, v) \text{ and } w^{\Delta} \geq h(t, w).$$

Therefore $v(t_0) \leq w(t_0)$ for some $t_0 \in \mathbb{T}$ implies $v(t) \leq w(t)$ for all $t \in \mathbb{T}$.

Then Bihari type inequality, see [45, Theorem 4.2], now reads as follows.

Theorem 1.32 *[38, Theorem 5.8.](Bihari's Inequality) Assume that g is continuous and nondecreasing, p is rd-continuous and nonnegative, and y is rd-continuous. Suppose w is the solution of*

$$w^{\Delta} = p(t)g(w), \; w(a) = \beta$$

and suppose there is a bijective function G with $(G \circ w)^{\Delta} = p$. Therefore

$$y(t) \leq \beta + \int_a^t p(\tau) g\left(y(\tau)\right) \Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq G^{-1}\left[G(\beta) + \int_a^t p(\tau)\Delta\tau\right] \text{ for all } t \in \mathbb{T}.$$

CHAPTER 2

GENERALIZATIONS OF OPIAL'S INEQUALITY

Over the past decades, generalizations of Opial's inequality in various directions have been given, and Opial type inequalities have become a subject in its own right. The purpose of this chapter is to systematically review these results.

2.1 OPIAL INEQUALITIES

2.1.1 Beesack's Generalization

Opial's inequality (1.6) can be generalized by using a different measure on the right side as follows: Assume that $-\infty \leq \alpha < \beta \leq \infty$, and $p(t)$, $r(t)$ are functions which are positive and continuous on (α, β). Further, let $(p(t)r(t))'$ be continuous on (α, β).

Theorem 2.1 *[14] Assume that $p(t)$ is positive and continuous on the interval $-\infty \leq \alpha < t < \tau < \infty$ with $\int_\alpha^\tau \frac{dt}{p(t)} < \infty$. Moreover, let $f(t)$ be absolutely continuous on $[\alpha, \tau]$ with $f(t) = \int_\alpha^t f'(s)ds$, $t \in [\alpha, \tau]$, and the conditions*

$$\int_\alpha^\tau p(t)\left(f'(t)\right)^2 dt < \infty,\ \ f^2(t) = O\left(\int_\alpha^t \frac{ds}{p(s)}\right)\ \text{ as } t \to \alpha^+ \tag{2.1}$$

are valid. Then, $f(t)$ satisfies the inequality

$$\int_\alpha^\tau |f(t)f'(t)|\,dt \leq \frac{1}{2}\int_\alpha^\tau \frac{dt}{p(t)} \int_\alpha^\tau p(t)\left(f'(t)\right)^2 dt. \tag{2.2}$$

In addition, in (2.2), equality holds only if $f(t) = c\int_\alpha^t \frac{ds}{p(s)}$.

Theorem 2.2 *[14] Assume that $p(t)$ is positive and continuous on the interval $\tau < t < \beta \leq \infty$ with $\int_\tau^\beta \frac{dt}{p(t)} < \infty$. In addition, let $f(t)$ be absolutely continuous on $[\tau, \beta]$ with $f(t) = -\int_t^\beta f'(t)dt$, $t \in [\tau, \beta]$, and the conditions*

$$\int_\tau^\beta p(t)\left(f'(t)\right)^2 dt < \infty,\ f^2(t) = O\left(\int_t^\beta \frac{ds}{p(s)}\right)\ \text{ as } t \to \beta^- \tag{2.3}$$

is valid. Then, $f(t)$ satisfies the inequality

$$\int_{\tau}^{\beta} |f(t)f'(t)|\, dt \leq \frac{1}{2}\int_{\tau}^{\beta} \frac{dt}{p(t)} \int_{\tau}^{\beta} p(t)\left(f'(t)\right)^2 dt. \tag{2.4}$$

Furthermore, in (2.4), equality holds only if $f(t) = d\int_t^{\beta} \frac{ds}{p(s)}$.

On combining Theorems 2.1 and 2.2, we obtain the following theorem.

Theorem 2.3 *[14] Assume that $p(t)$ is positive and continuous on (α, β) with $\int_{\alpha}^{\beta} \frac{dt}{p(t)} < \infty$. Furthermore, suppose $f(t)$ is absolutely continuous on each of the subintervals $[\alpha, \tau]$, $[\tau, \beta]$ with*

$$f(t) = \int_{\alpha}^{t} f'(s)ds,\ t \in [\alpha, \tau]\,,\ f(t) = -\int_{t}^{\beta} f'(s)ds,\ t \in [\tau, \beta]\,,$$

and such that the conditions (2.1) and (2.3) are satisfied.
Then, $f(t)$ satisfies the inequality

$$\int_{\alpha}^{\beta} |f(t)f'(t)|\, dt \leq \frac{1}{2}\kappa \int_{\alpha}^{\beta} p(t)\left(f'(t)\right)^2 dt, \tag{2.5}$$

where κ and τ are such that

$$\int_{\alpha}^{\tau} \frac{dt}{p(t)} = \int_{\tau}^{\beta} \frac{dt}{p(t)} = \kappa. \tag{2.6}$$

Further, in (2.5), equality is valid only if

$$f(t) = c\int_{\alpha}^{t} \frac{ds}{p(s)},\ t \in [\alpha, \tau) \tag{2.7}$$

and

$$f(t) = d\int_{t}^{\beta} \frac{ds}{p(s)},\ t \in (\tau, \beta]. \tag{2.8}$$

2.1.2 Hua's Generalization

Another non-trivial generalization of Theorem 1.9 is given as follows:

Theorem 2.4 *[46] Suppose that $f(t)$ is absolutely continuous on $[0, a]$, and $f(0) = 0$. Moreover, suppose that ℓ is a positive integer. Then, the following inequality is valid*

$$\int_{0}^{a} \left|f^{\ell}(t)f'(t)\right| dt \leq \frac{a^{\ell}}{\ell+1}\int_{0}^{a} |f'(t)|^{\ell+1}\, dt. \tag{2.9}$$

Further, in (2.9), equality holds if and only if $f(t) = ct$.

2.1.3 He and Wang's Generalization

As an application of Hua's inequality (2.9), we present the following interesting theorem.

Theorem 2.5 *[47] Suppose that $f(t)$ is absolutely continuous on $[0,a]$, and $f(0) = 0$. Moreover, let $\ell \geq 0$, $0 \leq \alpha < \beta \leq a$ be arbitrary numbers. Therefore, the following inequality is valid*

$$\int_{\alpha}^{\beta} \left|f^{\ell}(t)f'(t)\right| dt \leq \frac{\beta^{\ell}}{\ell+1}\int_{0}^{\beta} |f'(t)|^{\ell+1}\, dt - \frac{\alpha^{\ell}}{\ell+1}\int_{0}^{\alpha} |f'(t)|^{\ell+1}\, dt. \tag{2.10}$$

Furthermore, in (2.10), equality holds if and only if $f(t) = ct$.

2.1.4 Yang's Generalization

Theorem 2.6 *[48] Suppose that $p(t)$ is positive and continuous on $[\alpha, \tau]$ with $\int_{\alpha}^{\tau} \frac{dt}{p(t)} < \infty$, and let $q(t)$ be positive, bounded and non-increasing on $[\alpha, \tau]$. Furthermore, let $f(t)$ be absolutely continuous on $[\alpha, \tau]$, and $f(\alpha) = 0$. Then, the inequality*

$$\int_{\alpha}^{\tau} q(t)\,|f(t)f'(t)|\, dt \leq \frac{1}{2}\int_{\alpha}^{\tau} \frac{dt}{p(t)}\int_{\alpha}^{\tau} p(t)q(t)\,(f'(t))^{2}\, dt \tag{2.11}$$

holds. Moreover, equality is valid in (2.11) if and only if $q(t)$ =constant, $f(t) = c\int_{\alpha}^{t} \frac{ds}{p(s)}$.

Theorem 2.7 *[48] Suppose $p(t)$ is positive and continuous on $[\tau, \beta]$ with $\int_{\tau}^{\beta} \frac{dt}{p(t)} < \infty$, and let $q(t)$ be positive, bounded and non-decreasing on $[\tau, \beta]$. Furthermore, let $f(t)$ be absolutely continuous on $[\tau, \beta]$, and $f(\beta) = 0$. Therefore, the following inequality is valid*

$$\int_{\tau}^{\beta} q(t)\,|f(t)f'(t)|\, dt \leq \frac{1}{2}\int_{\tau}^{\beta} \frac{dt}{p(t)}\int_{\tau}^{\beta} p(t)q(t)\,(f'(t))^{2}\, dt. \tag{2.12}$$

Moreover, equality is valid in (2.12) if and only if $q(t) =$ constant, $f(t) = c\int_{t}^{\beta} \frac{ds}{p(s)}$.

Corollary 2.1 *Suppose that $q(t)$ is positive, bounded and monotonic on $[\alpha, \beta]$. Moreover, let $f(t)$ be absolutely continuous on $[\alpha, \beta]$, and $f(\alpha) = f(\beta) = 0$. Then, the following inequality is valid*

$$\int_{\alpha}^{\beta} q(t)\,|f(t)f'(t)|\, dt \leq \frac{\beta-\alpha}{2}\int_{\alpha}^{\beta} q(t)\,(f'(t))^{2}\, dt. \tag{2.13}$$

A generalization of Hua's inequality (2.9) is given in the following theorem.

Theorem 2.8 *[48] Suppose that $f(t)$ is absolutely continuous on $[0, a]$, and $f(0) = 0$. Then, for ℓ, $m \geq 1$, the following inequality is valid*

$$\int_0^a |f(t)|^\ell |f'(t)|^m \, dt \leq \frac{m}{\ell + m} a^\ell \int_0^a |f'(t)|^{\ell+m} \, dt. \tag{2.14}$$

Theorem 2.9 *[49] Suppose that $q(t)$ and $f(t)$ are as in Theorem 2.6. Then, for $\ell \geq 0$, $m \geq 1$, the following inequality is valid*

$$\int_\alpha^\tau q(t) |f(t)|^\ell |f'(t)|^m \, dt \leq \frac{m}{\ell + m} (\tau - \alpha)^\ell \int_\alpha^\tau q(t) |f'(t)|^{\ell+m} \, dt. \tag{2.15}$$

Theorem 2.10 *[49] Suppose that $q(t)$ and $f(t)$ are as in Theorem 2.7. Then, for $\ell \geq 0$, $m \geq 1$, the following inequality is valid*

$$\int_\tau^\beta q(t) |f(t)|^\ell |f'(t)|^m \, dt \leq \frac{m}{\ell + m} (\beta - \tau)^\ell \int_\tau^\beta q(t) |f'(t)|^{\ell+m} \, dt. \tag{2.16}$$

Theorem 2.11 *[49] Suppose that $q(t)$ and $f(t)$ are as in Corollary 2.1. Then, for $\ell \geq 0$, $m \geq 1$, the following inequality is valid*

$$\int_\alpha^\beta q(t) |f(t)|^\ell |f'(t)|^m \, dt \leq \frac{m}{\ell + m} (\beta - \alpha)^\ell \int_\alpha^\beta q(t) |f'(t)|^{\ell+m} \, dt. \tag{2.17}$$

2.1.5 Maroni's Generalization

Another extension of Theorem 2.1 is given as follows:

Theorem 2.12 *[50] Suppose $p(t)$ is positive and continuous on $[\alpha, \tau]$ with $\int_\alpha^\tau p^{1-\mu}(t)dt < \infty$, where $\mu > 1$. Furthermore, let $f(t)$ be absolutely continuous on $[\alpha, \tau]$, and $f(\alpha) = 0$. Then, the following inequality is valid*

$$\int_\alpha^\tau |f(t)f'(t)| \, dt \leq \frac{1}{2} \left(\int_\alpha^\tau p^{1-\mu}(t)dt \right)^{2/\mu} \left(\int_\alpha^\tau p(t) |f'(t)|^\nu \, dt \right)^{2/\nu}, \tag{2.18}$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Moreover, equality holds in (2.18) if and only if $f(t) = c \int_\alpha^t p^{1-\mu}(s)ds$.

Next, we present two results which generalize Theorems 2.2 and 2.3, respectively.

Theorem 2.13 *[50] Suppose $p(t)$ is positive and continuous on $[\tau, \beta]$ with $\int_\tau^\beta p^{1-\mu}(t)dt < \infty$, where $\mu > 1$. In addition, let $f(t)$ be absolutely continuous on $[\tau, \beta]$, and $f(\beta) = 0$. Then, the following inequality is valid*

$$\int_\tau^\beta |f(t)f'(t)| \, dt \leq \frac{1}{2} \left(\int_\tau^\beta p^{1-\mu}(t)dt \right)^{2/\mu} \left(\int_\tau^\beta p(t) |f'(t)|^\nu \, dt \right)^{2/\nu}, \tag{2.19}$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Furthermore, equality holds in (2.19) if and only if $f(t) = c \int_t^\beta p^{1-\mu}(s)ds$.

Theorem 2.14 *[50] Suppose $p(t)$ is positive and continuous on $[\alpha, \beta]$ with $\int_\alpha^\beta p^{1-\mu}(t)dt < \infty$, where $\mu \geq 2$. Furthermore, let $f(t)$ be absolutely continuous on $[\alpha, \beta]$, and $f(\alpha) = f(\beta) = 0$. Then, the following inequality is valid*

$$\int_\alpha^\beta |f(t)f'(t)|\, dt \leq \frac{1}{2}\kappa \left(\int_\alpha^\beta p(t) \left|f'(t)\right|^\nu dt \right)^{2/\nu}, \tag{2.20}$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$, and κ and τ are such that

$$\left(\int_\alpha^\tau p^{1-\mu}(t)dt \right)^{2/\mu} = \left(\int_\tau^\beta p^{1-\mu}(t)dt \right)^{2/\mu} = \kappa. \tag{2.21}$$

Furthermore, equality holds in (2.20) if and only if

$$f(t) = c \int_\alpha^t p^{1-\mu}(s)ds,\ t \in [\alpha, \tau] \tag{2.22}$$

and

$$f(t) = d \int_t^\beta p^{1-\mu}(s)ds,\ t \in [\tau, \beta]. \tag{2.23}$$

2.1.6 Calvert's Generalization

Using the method of Mallows' proof of (1.11), in the year 1967, Calvert [51] obtained three interesting generalizations of Opial's inequality. We present these generalizations here.

Theorem 2.15 *[51] Suppose that*

(i) *$f(t)$ is absolutely continuous in $[\alpha, \tau]$, and $f(\alpha) = 0$,*

(ii) *$g(t)$ is continuous, complex-valued, defined in the range of $f(t)$ and for all real t of the form $t(s) = \int_\alpha^s |f'(u)|\, du$; $|g(t)| \leq g(|t|)$ for all t and that $g(t)$ is real for $t > 0$ and is non-decreasing there,*

(iii) *$p(t)$ is positive, continuous and $\int_\alpha^\tau p^{1-\mu}(t)dt < \infty$, where $\frac{1}{\mu} + \frac{1}{\nu} = 1$, $\mu > 1$.*

Therefore, the following inequality is valid

$$\int_\alpha^\tau |g(f(t))\, f'(t)|\, dt \leq F\left[\left(\int_\alpha^\tau p^{1-\mu}(t)dt \right)^{1/\mu} \left(\int_\alpha^\tau p(t) \left|f'(t)\right|^\nu dt \right)^{1/\nu} \right], \tag{2.24}$$

where $F(t) = \int_0^t g(s)ds$, $t > 0$. Further, in (2.24), equality holds only if $f(t) = c\int_\alpha^t p^{1-\mu}(s)ds$.

Theorem 2.16 *[12, Theorem 2.7.2.] Suppose that*

(i) *$f(t)$ is absolutely continuous on $[\tau, \beta]$, and $f(\beta) = 0$,*

(ii) *$g(t)$ is continuous, complex-valued, defined on the range of $f(t)$ and for all real t of the form $t(s) = \int_s^\beta |f'(u)|\, du$; $|g(t)| \leq g(|t|)$ for all t and that $g(t)$ is real for $t > 0$ and is non-decreasing there,*

(iii) *$p(t)$ is positive, continuous and $\int_\tau^\beta p^{1-\mu}(t)dt < \infty$, where $\frac{1}{\mu} + \frac{1}{\nu} = 1$, $\mu > 1$.*

Then, the following inequality is valid

$$\int_\tau^\beta |g(f(t))\, f'(t)|\, dt \leq F\left[\left(\int_\tau^\beta p^{1-\mu}(t)dt\right)^{1/\mu}\left(\int_\tau^\beta p(t)\, |f'(t)|^\nu\, dt\right)^{1/\nu}\right], \tag{2.25}$$

where $F(t) = \int_0^t g(s)ds$, $t > 0$. Furthermore, in (2.25), equality is valid only if $f(t) = c\int_t^\beta p^{1-\mu}(s)ds$.

Theorem 2.17 *[51] Suppose that $f(t)$, $g(t)$ and $p(t)$ are as in Theorem 2.15, but $\nu < 1$, $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Then, the following inequality is valid*

$$\int_\alpha^\tau \left|\frac{f'(t)}{g(f(t))}\right| dt \geq G\left[\left(\int_\alpha^\tau p^{1-\mu}(t)dt\right)^{1/\mu}\left(\int_\alpha^\tau p(t)\, |f'(t)|^\nu\, dt\right)^{1/\nu}\right], \tag{2.26}$$

where $G(t) = \int_0^t \frac{ds}{g(s)}$, $t > 0$. Furthermore, in (2.26), equality holds only if $f(t) = c\int_\alpha^t p^{1-\mu}(s)ds$.

Theorem 2.18 *[12, Theorem 2.7.4.] Suppose that $f(t)$, $g(t)$ and $p(t)$ are as in Theorem 2.16, but $\nu < 1$, $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Therefore, the following inequality holds*

$$\int_\tau^\beta \left|\frac{f'(t)}{g(f(t))}\right| dt \geq G\left[\left(\int_\tau^\beta p^{1-\mu}(t)dt\right)^{1/\mu}\left(\int_\tau^\beta p(t)\, |f'(t)|^\nu\, dt\right)^{1/\nu}\right], \tag{2.27}$$

where $G(t) = \int_0^t \frac{ds}{g(s)}$, $t > 0$. Furthermore, in (2.27), equality is valid only if $f(t) = c\int_t^\beta p^{1-\mu}(s)ds$.

Theorem 2.19 *[51] Suppose that for $i = 1, 2$*

(i) *functions $f_i(t)$ are absolutely continuous on $[\alpha, \tau]$, and $f_i(\alpha) = 0$,*

(ii) *functions $p_i(t)$ are positive, continuous and $\int_\alpha^\tau (p_i(t))^{-2}\, dt < \infty$.*

Then, the following inequality is valid

$$\int_{\alpha}^{\tau}\left(|f_1(t)f_2'(t)|+|f_1'(t)f_2(t)|\right)dt \le \left[\int_{\alpha}^{\tau}(p_1(t))^{-2}\,dt\int_{\alpha}^{\tau}(p_2(t))^{-2}\,dt \times\int_{\alpha}^{\tau}(p_1(t))^2|f_1'(t)|^2\,dt\int_{\alpha}^{\tau}(p_2(t))^2|f_2'(t)|^2\,dt\right]^{\frac{1}{2}}. \tag{2.28}$$

2.1.7 Redheffer's Generalization

The following theorem replaces the product ff' in Opial's inequality (1.11) by a weighted sum of the factors.

Theorem 2.20 *[52] On (α,β), let f, v, w and y be absolutely continuous, with*

$$\frac{w'}{v}y^2+\frac{w}{v'}(y')^2\le 0,\quad wv'\ge 0,\quad v'\ne 0,\quad v>0.$$

Hence, the following inequality is valid

$$\int_{\alpha}^{\beta}\frac{w}{v'}\left[y^2(f')^2+\frac{1}{2}(y^2)'(f^2)'\right]dt\ge B-A, \tag{2.29}$$

where

$$A=\liminf_{t\to\alpha^+}f^2y^2wv^{-1},\ B=\limsup_{t\to\beta^-}f^2y^2wv^{-1}.$$

When $w\ne 0$, equality is valid in (2.29) if and only if $fy=cv$.

2.1.8 Beesack and Das' Generalization

Theorem 2.21 *[53] Assume that ℓ, m are real numbers such that $\ell m>0$ and either $\ell+m>1$, or $\ell+m<0$. Furthermore, suppose $p(t)$, $q(t)$ are non-negative, measurable functions on (α,τ) such that*

$$\int_{\alpha}^{\tau}(p(t))^{-1/(\ell+m-1)}\,dt<\infty \tag{2.30}$$

and

$$\begin{aligned}K_1(\tau,\ell,m) &= \left(\frac{m}{\ell+m}\right)^{m/(\ell+m)}\\ &\quad\times\left[\int_{\alpha}^{\tau}(q(t))^{(\ell+m)/\ell}(p(t))^{-m/\ell}\left(\int_{\alpha}^{t}(p(s))^{-1/(\ell+m-1)}\,ds\right)^{\ell+m-1}dt\right]^{\ell/(\ell+m)}\\ &< \infty.\end{aligned} \tag{2.31}$$

If $f(t)$ is absolutely continuous on $[\alpha, \tau]$, $f(\alpha) = 0$ and $f'(t)$ does not change sign in (α, τ), then the following inequality is valid

$$\int_\alpha^\tau q(t)\,|f(t)|^\ell\,|f'(t)|^m\,dt \le K_1(\tau,\ell,m)\int_\alpha^\tau p(t)\,|f'(t)|^{\ell+m}\,dt. \tag{2.32}$$

In addition, in (2.32), equality is valid if and only if either $m > 0$ and $f(t) \equiv 0$; or

$$q(t) = c\,(p(t))^{(m-1)/(\ell+m-1)}\left(\int_\alpha^t (p(s))^{-1/(\ell+m-1)}\,ds\right)^{\ell(1-m)/m} \quad (c \ge 0) \tag{2.33}$$

and

$$f(t) = d\int_\alpha^t (p(s))^{-1/(\ell+m-1)}\,ds\ (d\ \text{real}). \tag{2.34}$$

2.2 OPIAL INEQUALITIES INVOLVING HIGHER ORDER DERIVATIVES

2.2.1 Willett's Extension

Theorem 2.22 *[54] Suppose that $f(t) \in C^{(n)}[0,a]$ is such that $f^{(i)}(0) = 0$, $0 \le i \le n-1$ $(n \ge 1)$. Then, the following inequality is valid*

$$\int_0^a \left|f(t)f^{(n)}(t)\right|dt \le \frac{1}{2}a^n\int_0^a \left|f^{(n)}(t)\right|^2 dt. \tag{2.35}$$

2.2.2 Das' Extension

Theorem 2.23 *[19] Suppose that $f(t) \in C^{(n-1)}[0,a]$ is such that $f^{(i)}(0) = 0$, $0 \le i \le n-1$ $(n \ge 1)$. Furthermore, let $f^{(n-1)}(t)$ be absolutely continuous, and $\int_0^a \left|f^{(n)}(t)\right|^2 dt < \infty$. So, the following inequality is valid*

$$\int_0^a \left|f(t)f^{(n)}(t)\right|dt \le c_n a^n\int_0^a \left|f^{(n)}(t)\right|^2 dt, \tag{2.36}$$

where

$$c_n = \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}}. \tag{2.37}$$

The following Theorem is a generalization of Theorem 2.23.

Theorem 2.24 *[19] Assume ℓ and m are positive numbers satisfying $\ell + m > 1$. Furthermore, let $f(t) \in C^{(n-1)}[0,a]$ be such that $f^{(i)}(0) = 0$, $0 \le i \le n-1$ $(n \ge 1)$, $f^{(n-1)}(t)$ absolutely continuous, and $\int_0^a \left|f^{(n)}(t)\right|^{\ell+m} dt < \infty$. Hence, the following inequality is valid*

$$\int_0^a |f(t)|^\ell\left|f^{(n)}(t)\right|^m dt \le c_n^* a^{n\ell}\int_0^a \left|f^{(n)}(t)\right|^{\ell+m} dt, \tag{2.38}$$

where

$$c_n^* = \xi m^{m\xi} \left[\frac{n(1-\xi)}{(n-\xi)}\right]^{\ell(1-\xi)} (n!)^{-\ell},\ \xi = (\ell+m)^{-1}. \tag{2.39}$$

Theorem 2.25 *[19] Assume ℓ and m are positive numbers satisfying $\ell + m = 1$. Furthermore, let $f(t) \in C^{(n-1)}[0,a]$ be such that $f^{(i)}(0) = 0,\ 0 \le i \le n-1\ (n \ge 1)$, $f^{(n-1)}(t)$ absolutely continuous, and $\int_0^a \left|f^{(n)}(t)\right| dt < \infty$. Therefore, the following inequality is valid*

$$\int_0^a |f(t)|^\ell \left|f^{(n)}(t)\right|^m dt \le \frac{m^m}{(n!)^\ell} a^{n\ell} \int_0^a \left|f^{(n)}(t)\right| dt. \tag{2.40}$$

In (2.40), integrals are interpreted as $\lim_{\epsilon \to 0^+} \int_\epsilon^a$.

2.2.3 Yang's Extension

The inequality

$$\int_0^a \left|f^{(k)}(t)\right|^\ell \left|f^{(n)}(t)\right|^m dt \le c_{n-k}^* a^{(n-k)\ell} \int_0^a \left|f^{(n)}(t)\right|^{\ell+m} dt \tag{2.41}$$

has been obatined in [55]. A result which generalizes the inequality (2.41) the following:

Theorem 2.26 *[56] Let $\ell \ge 1$, $m > 0$, $r_k \ge 0$, $0 \le k \le n-1$, with $\sum_{k=0}^{n-1} r_k = 1$. Furthermore, let $f(t)$ be as in Theorem 2.24. Therefore, the following inequality is valid*

$$\int_0^a \left(\prod_{k=0}^{n-1} \left|f^{(k)}(t)\right|^{r_k}\right)^\ell \left|f^{(n)}(t)\right|^m dt \le \sum_{k=0}^{n-1} c_{n-k}^* r_k a^{(n-k)\ell} \int_0^a \left|f^{(n)}(t)\right|^{\ell+m} dt, \tag{2.42}$$

where c_n^* is given defined in (2.39).

2.2.4 Cheung's Extension

Lemma 2.27 *[12, Lemma 3.5.1.] Assume ℓ, m and $f(t)$ are as in Theorem 2.24. In addition, suppose $q(t)$ is positive, bounded and non-increasing on $[0,a]$. Therefore, the following inequality is valid*

$$\int_0^a q(t) |f(t)|^\ell \left|f^{(n)}(t)\right|^m dt \le c_n^* a^{n\ell} \int_0^a q(t) \left|f^{(n)}(t)\right|^{\ell+m} dt, \tag{2.43}$$

where c_n^* is given defined in (2.39).

Corollary 2.2 *Assume that ℓ, m and $f(t)$ are as in Theorem 2.24. Furthermore, let $0 < \zeta_1 \le q(t) \le \zeta_2$ be continuous on $[0,a]$. Then, the following inequality is valid*

$$\int_0^a q(t) |f(t)|^\ell \left|f^{(n)}(t)\right|^m dt \le \left(\frac{\zeta_2}{\zeta_1}\right)^{\ell/(\ell+m)} c_n^* a^{n\ell} \int_0^a q(t) \left|f^{(n)}(t)\right|^{\ell+m} dt. \tag{2.44}$$

Theorem 2.28 *[57] Assume that $\ell \geq 1$, $m > 0$, $r_k \geq 0$, $0 \leq k \leq n-1$ with $\sum_{k=0}^{n-1} r_k = 1$. Furthermore, suppose $f(t)$ and $q(t)$ are as in Theorem 2.24 and Lemma 2.27, respectively. Then, the following inequality is valid*

$$\int_0^a q(t) \left(\prod_{k=0}^{n-1} \left| f^{(k)}(t) \right|^{r_k} \right)^{\ell} \left| f^{(n)}(t) \right|^m dt \leq \sum_{k=0}^{n-1} c_{n-k}^* r_k a^{(n-k)\ell} \int_0^a q(t) \left| f^{(n)}(t) \right|^{\ell+m} dt. \quad (2.45)$$

Theorem 2.29 *[57] Assume that $\ell \geq 1$, $m > 0$, $r_k \geq 0$, $0 \leq k \leq n-1$ with $\sum_{k=0}^{n-1} r_k = 1$. Furthermore, suppose $f(t)$ and $q(t)$ are as in Theorem 2.24 and Corollary 2.2, respectively. Then, the inequality (2.45) with the right hand side multiplied by $(\zeta_2/\zeta_1)^{\ell/(\ell+m)}$ is valid.*

2.2.5 Li's Extension

An extension of Theorem 2.21 is the following:

Theorem 2.30 *[58] Suppose that*

(i) *r_k, $0 \leq k \leq n$ are non-negative numbers and satisfy $r = \sum_{k=0}^{n} r_k > r_n > 0$, $r > 1$,*

(ii) *$p(t)$ and $q(t)$ are positive functions on $[0, a]$ such that*

$$P_k(t) = \int_0^t (t-s)^{(n-k-1)r/(r-1)} (p(s))^{-1/(r-1)} ds < \infty,$$

and

$$K(p,q) = \left[\int_0^a \left(q^r(t) p^{-rn}(t) \right)^{1/(r-r_n)} \prod_{k=0}^{n-1} (P_k(t))^{r_k(r-1)/(r-r_n)} dt \right]^{(r-r_n)/r} < \infty,$$

(iii) *$f(t)$ is as in Theorem 2.22.*

In addition, the following inequality is valid

$$\int_0^a q(t) \prod_{k=0}^{n-1} \left| f^{(k)}(t) \right|^{r_k} \left| f^{(n)}(t) \right|^{r_n} dt \leq \left(\frac{r_n}{r} \right)^{r_n/r} \frac{K(p,q)}{\Omega} \int_0^a p(t) \left| f^n(t) \right|^r dt, \quad (2.46)$$

where $\Omega = \prod_{k=0}^{n-1} [(n-k-1)!]^{r_k}$.

2.2.6 Fink's Extension

Theorem 2.31 *[59] Suppose* $0 \leq k < r < n$ $(n \geq 2)$*, but fixed, and let* $f(t) \in C^{(n-1)}[0,a]$*,* $f^{(i)}(0) = 0$*,* $0 \leq i \leq n-1$*,* $f^{(n-1)}(t)$ *absolutely continuous, and* $\int_0^a \left|f^{(n)}(t)\right|^\mu dt < \infty$*, where* $\mu \geq 1$ *and* $\frac{1}{\mu} + \frac{1}{\nu} = 1$*. Thus, the following inequality is valid*

$$\int_0^a \left|f^{(k)}(t) f^{(r)}(t)\right| dt \leq C(n,k,r,\mu)\, a^{2n-k-r+1-2/\mu} \left[\int_0^a \left|f^{(n)}(t)\right|^\mu dt\right]^{2/\mu}, \tag{2.47}$$

where

$$C(n,k,k+1,\mu) = \frac{1}{2\left((n-k-1)!\right)^2 \left[(n-k-1)\nu+1\right]^{2/\nu}}, \tag{2.48}$$

and

$$C(n,k,r,\mu) \leq \frac{2^{-1/\mu}}{(n-k-1)!\,(n-r)!\,\left[(n-r)\nu+1\right]^{1/\nu}\left[(2n-k-r-1)\nu+2\right]^{1/\nu}}. \tag{2.49}$$

2.2.7 Pachpatte's Extension

A theorem involving two functions and their higher order derivatives is given as embodied in the following theorem:

Theorem 2.32 *[22] For* $j = 1,2$ *suppose* $f_j(t) \in C^{(n-1)}[0,a]$ *is such that* $f_j^{(i)}(0) = 0$*,* $0 \leq i \leq n-1$ $(n \geq 1)$*. Furthermore, suppose* $f_j^{(n-1)}(t)$ *is absolutely continuous, and* $\int_0^a \left|f_j^{(n)}(t)\right|^2 dt < \infty$*. Therefore, the following inequality is valid*

$$\int_0^a \left(\left|f_1(t) f_2^{(n)}(t)\right| + \left|f_1^{(n)}(t) f_2(t)\right|\right) dt \leq c_n a^n \int_0^a \left(\left|f_1^{(n)}(t)\right|^2 + \left|f_2^{(n)}(t)\right|^2\right) dt, \tag{2.50}$$

where c_n *is defined in (2.37). Furthermore, in (2.50), equality holds if and only if* $n = 1$ *and* $f_1^{(n)}(t) = f_2^{(n)}(t) = c$*.*

CHAPTER 3

DISCRETE OPIAL INEQUALITIES

1955 paper of Fan, Taussky and Todd [60] has brought about a lively interest in discrete inequalities. Further, the prominence discrete analysis has gained over the past decades makes the study of discrete inequalities even more important. Many discrete inequalities involving functions and their sums and diferences have been obtained. Although some results in the discrete case are similar to those already known in the continuous case, the adaptation from continuous to discrete is not always direct, but often requires special devices.

Discrete analogues of Opial type inequalities commenced in 1967-69 with the papers of Lasota [61], Wong [62], Lee [63] and Beesack [64], which provided discrete versions of Opial's original inequality, and the generalizations due to Hua and Yang, respectively.

The large volume of works on the subject (see for example [65]) demonstrates the vitality of the subject.

In this chapter, we use extensively [12].

3.1 LASOTA'S INEQUALITY

A discrete analogue of Theorem 1.6 is given as follows :

Theorem 3.1 *[61] Suppose* $\{f_i\}_{i=0}^{N}$ *is a sequence of numbers, and* $f_0 = f_N = 0$*. Then, the following inequality is valid*

$$\sum_{i=1}^{N-1} |f_i \Delta f_i| \leq \frac{1}{2}\left[\frac{N+1}{2}\right] \sum_{i=0}^{N-1} |\Delta f_i|^2 \,, \tag{3.1}$$

where Δ *is the forward difference operator, i.e.,* $\Delta f_i = f_{i+1} - f_i$*, and* $[.]$ *is the greatest integer function.*

If N is even, then the inequality (3.1) is sharp.

Theorem 3.2

(i) *[12, Theorem 5.2.2.](Discrete Opial inequality) Suppose* $\{f_i\}_{i=0}^{\tau}$ *is a sequence of numbers, and* $f_0 = 0$*. Then, the following inequality is valid*

$$\sum_{i=1}^{\tau-1} |f_i \Delta f_i| \leq \frac{\tau - 1}{2} \sum_{i=0}^{\tau-1} |\Delta f_i|^2 . \tag{3.2}$$

(ii) *[12, Theorem 5.2.3.] Suppose that* $\{f_i\}_{i=\tau}^{N}$ *is a sequence of numbers, and* $f_N = 0$*. Then, the following inequality is valid*

$$\sum_{i=\tau}^{N-1} |f_i \Delta f_i| \leq \frac{N - \tau + 1}{2} \sum_{i=\tau}^{N-1} |\Delta f_i|^2 . \tag{3.3}$$

(iii) *[12, Theorem 5.2.4.] Suppose that* $\{f_i\}_{i=0}^{\tau}$ *is a sequence of numbers, and* $f_0 = 0$*. Then, the following inequality is valid*

$$\sum_{i=1}^{\tau} |f_i \nabla f_i| \leq \frac{\tau + 1}{2} \sum_{i=1}^{\tau} |\nabla f_i|^2 , \tag{3.4}$$

where ∇ *is the backward difference operator, i.e.,* $\nabla f_i = f_i - f_{i-1}$.

3.2 WONG'S INEQUALITY

The following theorem is the discrete analogue of the inequality (2.9).

Theorem 3.3 *[62] Assume* $\{f_i\}_{i=0}^{\tau}$ *is a non-decreasing sequence of non-negative numbers, and* $f_0 = 0$*. Then, for* $\ell \geq 1$*, the following inequality is valid*

$$\sum_{i=1}^{\tau} f_i^{\ell} \nabla f_i \leq \frac{(\tau + 1)^{\ell}}{\ell + 1} \sum_{i=1}^{\tau} (\nabla f_i)^{\ell+1}. \tag{3.5}$$

where ∇ *is the backward difference operator, that is* $\nabla f_i = f_i - f_{i-1}$.

Remark 3.1 *In terms of the forward difference operator,* Δf_i*, above Wong's inequality (3.5) can be restated as follows;*
$\{f_i\}_{i=0}^{\tau}$ *is a non-decreasing sequence of non-negative numbers with* $f_0 = 0$*, for* $\ell \geq 1$*, the inequality*

$$\sum_{i=0}^{\tau-1} f_{i+1}^{\ell} \Delta f_i \leq \frac{(\sigma(\tau))^{\ell}}{\ell + 1} \sum_{i=0}^{\tau-1} (\Delta f_i)^{\ell+1}$$

is valid where Δ *is the forward difference operator.*

3.3 LEE'S INEQUALITY

A discrete analogue of the inequality (2.14) which in particular reduces to (3.5) is the following:

Theorem 3.4 *[63] Suppose* $\{f_i\}_{i=0}^{\tau}$ *is a non-decreasing sequence of non-negative numbers, and* $f_0 = 0$. *Then,*

(i) *if* $\ell > 0,\ m > 0,\ \ell + m \geq 1$ *or* $\ell < 0,\ m < 0$,

$$\sum_{i=1}^{\tau} f_i^{\ell} (\nabla f_i)^m \leq K_\tau \sum_{i=1}^{\tau} (\nabla f_i)^{\ell+m}, \tag{3.6}$$

where $K_0 = m\alpha,\ \alpha = (\ell + m)^{-1}$, *and for* $n = 1, 2, \ldots$

$$K_n = \max\left\{K_{n-1} + \ell\alpha n^{\ell-1},\ m\alpha\,(n+1)^{\ell}\right\};$$

(ii) *if* $\ell > 0,\ m < 0,\ \ell + m \leq 1,\ \ell + m \neq 0$ *or* $\ell < 0,\ m > 0,\ \ell + m \geq 1$,

$$\sum_{i=1}^{\tau} f_i^{\ell} (\nabla f_i)^m \geq C_\tau \sum_{i=1}^{\tau} (\nabla f_i)^{\ell+m}, \tag{3.7}$$

where $C_0 = m\alpha$, *and for* $n = 1, 2, \ldots$

$$C_n = \min\left\{C_{n-1} + \ell\alpha n^{\ell-1},\ m\alpha\,(n+1)^{\ell}\right\}.$$

Furthermore, in particular,

(iii) *if* $\ell \geq 1,\ m \geq 1$, *then (3.6) is valid with* K_τ *replaced by* $\tilde{K}_\tau$, *where* $\tilde{K}_\tau = m\alpha\,(\tau + 1)^{\ell}$;

(iv) *if* $\ell \leq 0,\ m < 0$, *then (3.6) is valid with* K_τ *replaced by* K'_τ, *where* $K'_0 = m\alpha$ *and for* $n = 1, 2, \ldots,\ K'_n = 1 + \ell\alpha \sum_{i=2}^{n} i^{\ell-1}$;

(v) *if* $\ell \geq 1,\ \ell + m < 0$, *then (3.7) is valid with* C_τ *replaced by* K'_τ.

Theorem 3.5 *[12, Theorem 5.4.2.] Suppose* $\{f_i\}_{i=0}^{\tau}$ *is a sequence of numbers, and* $f_0 = 0$. *Then,*

(i) *if* $\ell > 0$, $m > 0$, $\ell + m \geq 1$,

$$\sum_{i=1}^{\tau} |f_i|^{\ell} |\nabla f_i|^{m} \leq K_\tau \sum_{i=1}^{\tau} |\nabla f_i|^{\ell+m}, \tag{3.8}$$

(ii) *if* $\ell \geq 1$, $m \geq 1$, *(3.8) is valid with* K_τ *replaced by* $\tilde{K}_\tau$.

Theorem 3.6 *[12, Theorem 5.4.3.] Suppose* $\{f_i\}_{i=\tau+1}^{N}$ *is a sequence of numbers, and* $f_N = 0$. *Then,*

(i) *if* $\ell > 0$, $m > 0$, $\ell + m \geq 1$,

$$\sum_{i=\tau+1}^{N-1} |f_i|^{\ell} |\nabla f_i|^{m} \leq K_{N-\tau-1} \sum_{i=\tau+1}^{N-1} |\nabla f_i|^{\ell+m}, \tag{3.9}$$

(ii) $\ell \geq 1$, $m \geq 1$, *(3.9) is valid with* $K_{N-\tau-1}$ *replaced by* $\tilde{K}_{N-\tau-1}$.

$$\sum_{i=1}^{N-1} |f_i|^{\ell} |\nabla f_i|^{m} \leq K_{\left[\frac{N}{2}\right]} \sum_{i=1}^{N-1} |\nabla f_i|^{\ell+m},$$

Theorem 3.7 *[12, Theorem 5.4.4.] Assume* $\{f_i\}_{i=0}^{N}$ *is a sequence of numbers, and* $f_0 = f_N = 0$. *Then,*

(i) *if* $\ell > 0$, $m > 0$, $\ell + m \geq 1$,

$$\sum_{i=1}^{N-1} |f_i|^{\ell} |\nabla f_i|^{m} \leq K_{\left[\frac{N}{2}\right]} \sum_{i=1}^{N-1} |\nabla f_i|^{\ell+m}, \tag{3.10}$$

(ii) *if* $\ell \geq 1$, $m \geq 1$, *(3.10) is valid with* $K_{\left[\frac{N}{2}\right]}$ *replaced by* $\tilde{K}_{\left[\frac{N}{2}\right]}$.

3.4 PACHPATTE'S INEQUALITY

An inequality involving two sequences is embodied in the following:

Theorem 3.8 *[66] Assume* $\{f_i\}_{i=0}^{\tau}$ *and* $\{g_i\}_{i=0}^{\tau}$ *are non-decreasing sequences of non-negative numbers, and* $f_0 = g_0 = 0$. *Then, the following inequality is valid*

$$\sum_{i=0}^{\tau-1} \left[f_i \Delta g_i + g_{i+1} \Delta f_i\right] \leq \frac{\tau}{2} \sum_{i=0}^{\tau-1} \left[(\Delta f_i)^2 + (\Delta g_i)^2\right]. \tag{3.11}$$

Corollary 3.1 *[12, Corollary 5.5.2.] Assume* $\{f_i\}_{i=0}^{\tau}$ *is a non-decreasing sequence of non-negative numbers, and* $f_0 = 0$*. Then, the following inequality is valid*

$$\sum_{i=0}^{\tau-1} f_{i+1}\Delta f_i \leq \frac{\tau+1}{2}\sum_{i=0}^{\tau-1}(\Delta f_i)^2. \tag{3.12}$$

Theorem 3.9 *[12, Theorem 5.5.3.] Assume* $\{f_i\}_{i=0}^{\tau}$ *and* $\{g_i\}_{i=0}^{\tau}$ *are sequences of numbers, and* $f_0 = g_0 = 0$*. Then, the following inequality is valid*

$$\sum_{i=0}^{\tau-1} |f_i\Delta g_i + g_{i+1}\Delta f_i| \leq \frac{\tau}{2}\sum_{i=0}^{\tau-1}\left[(\Delta f_i)^2 + (\Delta g_i)^2\right]. \tag{3.13}$$

Theorem 3.10 *[67] Assume* $\{f_i\}_{i=0}^{\tau}$ *and* $\{g_i\}_{i=0}^{\tau}$ *are sequences of numbers, and* $f_0 = g_0 = 0$*. Then, the following inequality is valid*

$$\sum_{i=1}^{\tau} |f_i\nabla g_i + g_i\nabla f_i| \leq \frac{\tau+1}{2}\sum_{i=1}^{\tau}\left[(\nabla f_i)^2 + (\nabla g_i)^2\right]. \tag{3.14}$$

Theorem 3.11 *[67] Assume that* $\{f_i\}_{i=0}^{\tau}$ *be a sequence of numbers, and* $f_0 = 0$*. Furthermore, let* $x(t)$ *be defined for all* $t = f_i$*; and for all* t *of the form* $t(j) = \sum_{k=1}^{j}|\nabla f_k|$*;* $x(t) \leq x(|t|)$ *for all* t *and that* $x(t)$ *is non-decreasing for* $t \geq 0$*. Then, the following inequality is valid*

$$\sum_{i=1}^{\tau} |x(f_i)\nabla f_i| \leq F\left(\sum_{i=1}^{\tau}|\nabla f_i|\right) + \sum_{i=1}^{\tau}\left[x\left(\sum_{j=1}^{i}|\nabla f_j|\right) - x\left(\sum_{j=1}^{i-1}|\nabla f_j|\right)\right]|\nabla f_i|, \tag{3.15}$$

where $F(t) = \int_0^t x(s)ds$*,* $t \geq 0$*.*

Theorem 3.12 *[67] Assume* $\{f_i\}_{i=0}^{\tau}$ *and* $f(t)$ *are as in Theorem 3.11. Then, the following inequality is valid*

$$\begin{aligned}\sum_{i=1}^{\tau}\left|\frac{\nabla f_i}{x(f_i)}\right| \quad &\geq \quad G\left(\sum_{i=1}^{\tau}|\nabla f_i|\right) \\ &-\sum_{i=1}^{\tau}\left[\left(x\left(\sum_{j=1}^{i-1}|\nabla f_j|\right)\right)^{-1} - \left(x\left(\sum_{j=1}^{i}|\nabla f_j|\right)\right)^{-1}\right]|\nabla f_i|,\end{aligned} \tag{3.16}$$

where $G(t) = \int_0^t \frac{ds}{x(s)}$*,* $t \geq 0$*.*

3.5 G. MILOVANOVIC I. MILOVANOVIC'S INEQUALITY

The main theorem of this section which has been ascribed to Opial type inequalities is given as follows:

Theorem 3.13 *[68] Suppose that* $\{r_i\}_{i=1}^{\tau+1}$ *and* $\{p_i\}_{i=1}^{\tau+1}$ *are sequences of positive numbers, and suppose* $\{Q_i(t)\}_{i=0}^{\tau}$ *is polynomials defined by*

$$\frac{r_{i+1}}{2\sqrt{p_i p_{i+1}}} Q_i(t) = \left(\frac{r_i}{p_i} - t\right) Q_{i-1}(t) - \frac{r_i}{2\sqrt{p_{i-1} p_i}} Q_{i-2}(t),\ i = 1, 2, ..., \tau \tag{3.17}$$

$Q_0(t) = Q_0 \neq 0,\ Q_{-1}(t) = 0.$

In addition, suppose $\{f_i\}_{i=0}^{\tau}$ *is a sequence of numbers with* $f_0 = 0$*. Then, the following inequalities is valid*

$$\lambda_1 \sum_{i=1}^{\tau} p_i f_i^2 \leq \sum_{i=1}^{\tau} r_i f_i \nabla f_i \leq \lambda_\tau \sum_{i=1}^{\tau} p_i f_i^2, \tag{3.18}$$

where λ_1 *and* λ_τ *are the minimal and maximal zeros of the polynomial* $Q_\tau(t)$*. Furthermore, in the left (right) of (3.18), equality is valid if and only if* $f_i = \left(c/\sqrt{p_i}\right) Q_{i-1}(\lambda)$*,* $i = 1, ..., \tau$ *where* $\lambda = \lambda_1(\lambda_\tau)$ *and* c *is an arbitrary constant.*

3.6 BEESACK'S INEQUALITY

Theorem 3.14 *The following inequalities are true:*

$$\sum_{i=1}^{\tau} g_i^\alpha \left(\sum_{j=1}^{i} g_j\right)^\beta \leq A_\tau(\alpha, \beta) \left(\sum_{i=1}^{\tau} g_i\right)^{\alpha+\beta}, \tag{3.19}$$

or

$$\sum_{i=1}^{\tau} g_i^\alpha \left(\sum_{j=1}^{i} g_j\right)^\beta \geq a_\tau(\alpha, \beta) \left(\sum_{i=1}^{\tau} g_i\right)^{\alpha+\beta}, \tag{3.20}$$

for all values of the parameters α*,* β *and all* $g_i \geq 0$*.*

3.7 AGARWAL AND PANG'S INEQUALITY

In this section for the convenience we shall use the following notation: For $a < b$, a, $b \in \mathbb{N}$, $[a, b]$ represents the discrete interval $[a,\ a+1,\ \cdots, b]$, and for $c \in \mathbb{R}$, $(c)^{[r]}$ is the usual factorial notation, i.e., $(c)^{[r]} = c(c-1)\cdots(c-r+1)$. For the discrete function $f(i)$, $i \in [a, b]$ the first order forward difference operator Δ is defined as $\Delta f(i) = f(i+1) - f(i)$, $i \in [a,\ b-1]$, whereas the n-th order difference operator Δ^n is recursively defined as $\Delta^n = \Delta^{n-1}(\Delta)$, $n \geq 1$.

Theorem 3.15 *[69] Suppose that*

(i) $f(i)$, $i \in [0,\ \tau+n-1]$, *is such that* $\Delta^j f(0)=0$, $0 \le j \le n-1$,

(ii) $p(i)>0$ *and* $q(i)\ge 0$ *are defined on* $[0,\ \tau-1]$,

(iii) $r>1$, $r_j \ge 0$, $0\le j\le n-1$, $r>r_n>0$ *are given numbers.*

Then we have the following:

$$\sum_{i=0}^{\tau-1} q(i) \prod_{k=0}^{n} \left|\Delta^k f(i)\right|^{r_k} \le K_1 \left[\sum_{i=0}^{\tau-1} p(i) \left|\Delta^n f(i)\right|^{r}\right]^{(\sigma+r_n)/r}, \tag{3.21}$$

where

$$K_1 = K_1\left(p,q,\{r_j\},r_n,r\right) = \left(\frac{r_n}{\sigma+r_n}\right)^{r_n/r} K_0.$$

CHAPTER 4

OPIAL DYNAMIC INEQUALITIES ON TIME SCALES

In this chapter, we use extensively [70].

In 1960 Opial proved that if f is absolutely continuous on $[a, b]$ with $f(a) = f(b) = 0$, then

$$\int_a^b |f(t)|\,|f'(t)|\,dt \le \frac{(b-a)}{4}\int_a^b |f'(t)|^2\,dt. \tag{4.1}$$

We also note if f is absolutely continuous on $(0, b)$ with $f(0) = 0$, then

$$\int_0^b |f(t)|\,|f'(t)|\,dt \le \frac{b}{2}\int_0^b |f'(t)|^2\,dt. \tag{4.2}$$

The discrete version of (4.1) was presented by Lasota [61] and is given by

$$\sum_{i=1}^{h-1} |f_i \Delta f_i| \le \frac{1}{2}\left[\frac{h+1}{2}\right]\sum_{i=1}^{h-1} |\Delta f_i|^2, \tag{4.3}$$

where $\{f_i\}_{0\le i\le h}$ is a sequence of real numbers with $f_0 = f_h = 0$ and $[f]$ is the greatest integer function. For a real sequence $\{f_i\}_{0\le i\le h}$ with $f_0 = 0$, we obtain

$$\sum_{i=1}^{h-1} |f_i \Delta f_i| \le \frac{h-1}{2}\sum_{i=0}^{h-1} |\Delta f_i|^2. \tag{4.4}$$

The chapter is organized as follows. In Section 4.1 we establish some first order Opial type inequalities and in Section 4.2 we establish some generalizations. Section 4.3 discusses inequalities with two different weight functions and in Section 4.4 we present some Opial type inequalities involving higher order derivatives.

Throughout this chapter (usually without mentioning) the integrals in the statements of the theorems are supposed to exist.

4.1 OPIAL TYPE INEQUALITIES I

In this section, we will present some inequalities of Opial's type on time scales with first order derivatives. The results in this section are taken from [70] which were adapted from [33, 71- 75].

Theorem 4.1 *Suppose $\mathbb{T}$ is a time scale with $0,\ h \in \mathbb{T}$. For a delta differentiable $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ with $f(0) = 0$, therefore*

$$\int\limits_0^h |f(t) + f^\sigma(t)| \left|f^\Delta(t)\right| \Delta t \le h \int_0^h \left|f^\Delta(t)\right|^2 \Delta t, \tag{4.5}$$

with equality when $f(t) = ct$.

The following theorem gives the nabla Opial inequality on time scales.

Theorem 4.2 *Suppose $\mathbb{T}$ is a time scale with $0,\ h \in \mathbb{T}$. For a nabla differentiable $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ with $f^\nabla(t)$ ld-continuous and $f(0) = 0$, therefore*

$$\int\limits_0^h |f(t) + f^\rho(t)| \left|f^\nabla(t)\right| \nabla t \le h \int_0^h \left|f^\nabla(t)\right|^2 \nabla t, \tag{4.6}$$

with equality when $f(t) = ct$.

We next give a generalization of Theorem 4.1 when $f(0)$ need not be equal to 0.

Theorem 4.3 *Suppose $\mathbb{T}$ is a time scale with $0,\ h \in \mathbb{T}$. For a delta differentiable $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$, therefore*

$$\int\limits_0^h |f(t) + f^\sigma(t)| \left|f^\Delta(t)\right| \Delta t \le \alpha \int_0^h \left|f^\Delta(t)\right|^2 \Delta t + 2\beta \int_0^h \left|f^\Delta(t)\right| \Delta t, \tag{4.7}$$

where

$$\alpha \in \mathbb{T} \text{ with } \operatorname{dist}\left(\frac{h}{2}, \alpha\right) = \operatorname{dist}\left(\frac{h}{2}, \mathbb{T}\right), \tag{4.8}$$

and $\beta = \max\{f(0),\ f(h)\}$.

The proof of the following theorem follows from the proof of Theorem 4.3 when $\beta = 0$.

Theorem 4.4 *Suppose $\mathbb{T}$ is a time scale with $0,\ h \in \mathbb{T}$. For a delta differentiable $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$, with $f(0) = f(h) = 0$, therefore*

$$\int\limits_0^h |f(t) + f^\sigma(t)| \left|f^\Delta(t)\right| \Delta t \le \alpha \int_0^h \left|f^\Delta(t)\right|^2 \Delta t, \tag{4.9}$$

where α is given as in (4.8).

In the following, we present some generalizations of the above inequalities which lead to Opial type inequalities with weight functions.

Theorem 4.5 *Suppose* $\mathbb{T}$ *is a time scale with* $0,\ h \in \mathbb{T}$ *and* $w(t)$ *be a positive and rd-continuous function on* $[0,h]_{\mathbb{T}}$ *such that* $\int_0^h w^{1-q}(t)\Delta t < \infty$, $q > 1$. *For a delta differentiable* $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ *with* $f(0) = 0$, *therefore*

$$\int_0^h |f(t) + f^{\sigma}(t)| \left|f^{\Delta}(t)\right| \Delta t \leq \left(\int_0^h w^{1-q}(t)\Delta t\right)^{\frac{2}{q}} \left(\int_0^h w(t) \left|f^{\Delta}(t)\right|^p \Delta t\right)^{\frac{2}{p}}, \tag{4.10}$$

where $p > 1$ *and* $1/p + 1/q = 1$ *and with equality when* $f(t) = c\int_0^t w^{1-q}(s)\Delta s$.

Theorem 4.6 *Suppose* $\mathbb{T}$ *is a time scale with* $0,\ h \in \mathbb{T}$ *and* $w(t)$ *be a positive and rd-continuous function on* $[0,h]_{\mathbb{T}}$ *such that* $\int_0^h w^{1-q}(t)\Delta t < \infty$, $q > 1$. *For a delta differentiable* $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$, *therefore*

$$\int_0^h |f(t) + f^{\sigma}(t)| \left|f^{\Delta}(t)\right| \Delta t \leq v^{2/q} \left(\int_0^h w \left|f^{\Delta}(t)\right|^p \Delta t\right)^{2/p} + 2\beta \int_0^h \left|f^{\Delta}(t)\right| \Delta t,$$

where $p > 1$ *and* $1/p + 1/q = 1$, $\beta = \max\{f(0),\ f(h)\}$, *and*

$$v = \max\left\{\int_0^{\alpha} w^{1-q}(t)\Delta t,\ \int_{\alpha}^h w^{1-q}(t)\Delta t\right\},$$

and

$$\alpha \in \mathbb{T} \text{ with } \operatorname{dist}\left(\frac{h}{2}, \alpha\right) = \operatorname{dist}\left(\frac{h}{2}, \mathbb{T}\right).$$

Corollary 4.1 *Assume* $\mathbb{T}$ *is a time scale with* $0,\ h \in \mathbb{T}$ *and* $w(t)$ *be a positive and rd-continuous function on* $[0,h]_{\mathbb{T}}$ *such that* $\int_0^h w^{1-q}(t)\Delta t < \infty$, $q > 1$. *For a delta differentiable* $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ *with* $f(0) = f(h) = 0$, *thus*

$$\int_0^h |f(t) + f^{\sigma}(t)| \left|f^{\Delta}(t)\right| \Delta t \leq v^{2/q} \left(\int_0^h w \left|f^{\Delta}(t)\right|^p \Delta t\right)^{2/p}, \tag{4.11}$$

where $p > 1$ *and* $1/p + 1/q = 1$ *and*

$$v = \max\left\{\int_0^{\alpha} w^{1-q}(t)\Delta t,\ \int_{\alpha}^h w^{1-q}(t)\Delta t\right\},$$

and

$$\alpha \in \mathbb{T} \text{ with } \operatorname{dist}\left(\frac{h}{2}, \alpha\right) = \operatorname{dist}\left(\frac{h}{2}, \mathbb{T}\right). \tag{4.12}$$

Theorem 4.7 *Assume that $a, b \in \mathbb{T}$ and $q,\ f \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $f(a) = 0$. Therefore*

$$\int_a^b q(\eta) \left| [f(\eta) + f^\sigma(\eta)]\, f^\Delta(\eta) \right| \Delta\eta \le K_q(b,a) \int_a^b \left(f^\Delta(\eta)\right)^2 \Delta\eta, \tag{4.13}$$

where

$$K_q(b,a) := \left(2 \int_a^b q^2(t)\, [\sigma(t) - a]\, \Delta t\right)^{1/2}, \text{ for } t \in [a,b]_{\mathbb{T}}. \tag{4.14}$$

The following theorem is complementary to Theorem 4.7.

Theorem 4.8 *Suppose that $a, b \in \mathbb{T}$ and $q,\ f \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $f(b) = 0$. Therefore*

$$\int_a^b q(\eta) \left| (f(\eta) + f^\sigma(\eta))\, f^\Delta(\eta) \right| \Delta\eta \le L_q(b,a) \int_a^b \left(f^\Delta(\eta)\right)^2 \Delta\eta, \tag{4.15}$$

where

$$L_q(b,a) := \left(2 \int_a^b q^2(t)\, [b - t]\, \Delta t\right)^{1/2}, \text{ for } t \in [a,b]_{\mathbb{T}}. \tag{4.16}$$

The following result combines Theorem 4.7 and Theorem 4.8 on the segments $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, respectively.

Theorem 4.9 *Suppose that $a, b \in \mathbb{T}$ and $q,\ f \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $f(a) = f(b) = 0$. Therefore*

$$\int_a^b q(\eta) \left| [f(\eta) + f^\sigma(\eta)]\, f^\Delta(\eta) \right| \Delta\eta \le \max\left\{K_q(b,c), L_q(c,a)\right\} \int_a^b \left(f^\Delta(\eta)\right)^2 \Delta\eta,$$

is valid for any $c \in [a,b]_{\mathbb{T}}$, where K_q, L_q are as defined in (4.14) and (4.16), respectively.

Corollary 4.2 *Suppose that $a, b \in \mathbb{T}$ and $q,\ f \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $f(a) = f(b) = 0$. Therefore we obtain*

$$\begin{aligned} &\int_a^b q(\eta) \left| [f(\eta) + f^\sigma(\eta)]\, f^\Delta(\eta) \right| \Delta\eta \\ \le\ &\min_{c \in [a,b]_{\mathbb{T}}} \left\{\max\left\{K_q(b,c), L_q(c,a)\right\}\right\} \int_a^b \left(f^\Delta(\eta)\right)^2 \Delta\eta, \end{aligned}$$

where K_q, L_q are as defined in (4.14) and (4.16), respectively.

In the following, we give some Opial dynamic inequalities with two different weight functions.

Theorem 4.10 *If r and q are positive rd-continuous functions on $[0,h]_{\mathbb{T}}$, $\int_0^h (\Delta t/r(t)) < \infty$, q nonincreasing and $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $f(0)=0$, therefore*

$$\int_0^h q^\sigma(t) \left|(f(t)+f^\sigma(t)) f^\Delta(t)\right| \Delta t \leq \int_0^h \frac{\Delta t}{r(t)} \int_0^h r(t)q(t) \left|f^\Delta(t)\right|^2 \Delta t. \tag{4.17}$$

Theorem 4.11 *Suppose $\mathbb{T}$ is a time scale with $a,\ \tau \in \mathbb{T}$. Suppose that $s \in C_{rd}([a,\tau]_{\mathbb{T}}, \mathbb{R})$ and r are a positive rd-continuous function on $(a,\tau)_{\mathbb{T}}$ such that $\int_a^\tau r^{-1}(t)\Delta t < \infty$. If $y : [a,\tau]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$ (and y^Δ does not change sign in $(a,\tau)_{\mathbb{T}}$) then we get*

$$\int_a^\tau s(x) \left|y(x)+y^\sigma(x)\right| \left|y^\Delta(x)\right| \Delta x \leq K_1(a,\tau) \int_a^\tau r(x) \left|y^\Delta(x)\right|^2 \Delta x, \tag{4.18}$$

where

$$K_1(a,\tau) = \sqrt{2}\left(\int_a^\tau \frac{s^2(x)}{r(x)} \left(\int_a^x \frac{\Delta t}{r(t)}\right) \Delta x\right)^{\frac{1}{2}} + \sup_{a \leq x \leq \tau} \left(\mu(x)\frac{|s(x)|}{r(x)}\right). \tag{4.19}$$

Theorem 4.12 *Assume $\mathbb{T}$ is a time scale with $\tau, b \in \mathbb{T}$. Let $s \in C_{rd}([\tau,b]_{\mathbb{T}}, \mathbb{R})$ and r be a positive rd-continuous function on $(\tau,b)_{\mathbb{T}}$ such that $\int_\tau^b r^{-1}(t)\Delta t < \infty$. If $y : [\tau,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(b) = 0$ (and y^Δ does not change sign in $(\tau,b)_{\mathbb{T}}$), then we get*

$$\int_\tau^b s(x) \left|y(x)+y^\sigma(x)\right| \left|y^\Delta(x)\right| \Delta x \leq K_2(\tau,b) \int_\tau^b r(x) \left|y^\Delta(x)\right|^2 \Delta x, \tag{4.20}$$

where

$$K_2(\tau,b) = \sqrt{2}\left(\int_\tau^b \frac{s^2(x)}{r(x)} \left(\int_x^b \frac{\Delta t}{r(t)}\right) \Delta x\right)^{\frac{1}{2}} + \sup_{\tau \leq x \leq b} \left(\mu(x)\frac{|s(x)|}{r(x)}\right). \tag{4.21}$$

In the following, we assume that there exists $\tau \in (a,b)$ which is the unique solution of the equation

$$K(a,b) = K_1(a,\tau) = K_2(\tau,b) < \infty. \tag{4.22}$$

where $K_1(a,\tau)$ and $K_2(\tau,b)$ are defined as in Theorem 4.11 and Theorem 4.12 and we establish an inequality when $y(a) = 0 = y(b)$.

Theorem 4.13 *Suppose that $\mathbb{T}$ is a time scale with $a, b \in \mathbb{T}$. Assume that $s \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ and r be a positive rd-continuous function on $[a,b]_{\mathbb{T}}$ such that $\int_a^b r^{-1}(t)\Delta t < \infty$. If*

$y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$ (and y^{Δ} does not change sign in $(a,b)_{\mathbb{T}}$), then we obtain

$$\int_a^b s(x)\,|y(x) + y^{\sigma}(x)|\,\left|y^{\Delta}(x)\right|\Delta x \le K(a,b)\int_a^b r(x)\left|y^{\Delta}(x)\right|^2 \Delta x, \tag{4.23}$$

where $K(a,b)$ is given as in 4.22.

4.2 OPIAL TYPE INEQUALITIES II

In this section we give some other Opial type inequalities on time scales. The results in this section are taken from [70] which were adapted from [73, 74, 76, 77].

Theorem 4.14 *Assume that $a,\ \tau \in \mathbb{T}$ and $r \in C_{rd}([a,\tau]_{\mathbb{T}}, \mathbb{R}^+)$ be such that $r(t)$ is nonincreasing on $[a,\tau]_{\mathbb{T}}$ and $p \ge 0$ and $q \ge 1$. Assume that $f : [a,\tau]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $f(a) = 0$. Therefore*

$$\int_a^{\tau} r(t)\,|f(t)|^p \left|f^{\Delta}(t)\right|^q \Delta t \le \frac{q(\tau - a)^p}{p+q}\int_a^{\tau} r(t)\left|f^{\Delta}(t)\right|^{p+q}\Delta t. \tag{4.24}$$

Theorem 4.15 *Assume that $a,\ b \in \mathbb{T}$ and $r \in C_{rd}([\tau,b]_{\mathbb{T}}, \mathbb{R}^+)$ be such that $r(t)$ is nonincreasing on $[\tau,b]_{\mathbb{T}}$ and $p \ge 0$ and $q \ge 1$. Assume that $f : [\tau,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $f(b) = 0$. Therefore*

$$\int_{\tau}^{b} r(t)\,|f(t)|^p \left|f^{\Delta}(t)\right|^q \Delta t \le \frac{q}{p+q}(b-\tau)^p\int_{\tau}^{b} r(t)\left|f^{\Delta}(t)\right|^{p+q}\Delta t. \tag{4.25}$$

The combination of Theorems 4.14 and 4.15 by choosing $\tau = (a+b)/2$ gives the following theorem.

Theorem 4.16 *Assume that $a,\ b \in \mathbb{T}$ and $r \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ be such that $r(t)$ is nonincreasing on $[a,b]_{\mathbb{T}}$ and nonincreasing on $[a,b]_{\mathbb{T}}$ and $p \ge 0$ and $q \ge 1$. Assume that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $f(a) = f(b) = 0$. Therefore*

$$\int_a^b r(t)\,|f(t)|^p \left|f^{\Delta}(t)\right|^q \Delta t \le \frac{q}{p+q}\left(\frac{b-a}{2}\right)^p\int_a^b r(t)\left|f^{\Delta}(t)\right|^{p+q}\Delta t. \tag{4.26}$$

Theorem 4.17 *Suppose $\mathbb{T}$ is a time scale with $a, \tau \in \mathbb{T}$ and $p \in C_{rd}([a,\tau], \mathbb{R})$ with $\int_a^{\tau} (p(t))^{1-\alpha}\,\Delta t < \infty$, $\alpha > 1$. Suppose the function $q(t)$ is positive, bounded, and nonincreasing on $[a,\tau]_{\mathbb{T}}$. If $f : [a,\tau] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $f(a) = 0$, thus for*

$\gamma > 0$,

$$\int_a^\tau q(t)\,|f(t)|^\gamma \left|f^\Delta(t)\right| \Delta t \;\leq\; \frac{1}{\gamma+1}\left(\int_a^\tau \frac{1}{p^{\alpha-1}(t)}\Delta t\right)^{\frac{1+\gamma}{\alpha}} \times \left(\int_a^\tau p(t) q^{\frac{\nu}{1+\gamma}}(t) \left|f^\Delta(t)\right|^\nu \Delta t\right)^{\frac{1+\gamma}{\nu}} \tag{4.27}$$

where $\frac{1}{\alpha}+\frac{1}{\nu}=1$.

A slight modification of the argument above gives the following result.

Theorem 4.18 *Suppose that $\mathbb{T}$ is a time scale with $a,\tau \in \mathbb{T}$ and $p \in C_{rd}([\tau,b),\mathbb{R})$ with $\int_\tau^b (p(t))^{1-\alpha}\Delta t < \infty$, $\alpha > 1$. Suppose the function $q(t)$ is positive, bounded, and nondecreasing on $[\tau,b]_{\mathbb{T}}$. If $f : [\tau,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $f(b)=0$, thus for $\gamma > 0$,*

$$\int_\tau^b q(t)\,|f(t)|^\gamma \left|f^\Delta(t)\right| \Delta t \;\leq\; \frac{1}{\gamma+1}\left(\int_\tau^b \frac{1}{p^{\alpha-1}(t)}\Delta t\right)^{\frac{1+\gamma}{\alpha}} \times \left(\int_\tau^b p(t) q^{\frac{\nu}{1+\gamma}}(t) \left|f^\Delta(t)\right|^\nu \Delta t\right)^{\frac{1+\gamma}{\nu}} \tag{4.28}$$

where $\frac{1}{\alpha}+\frac{1}{\nu}=1$.

Theorem 4.19 *Assume $\mathbb{T}$ is a time scale with $a,b \in \mathbb{T}$ and $\tau \in [a,b]_{\mathbb{T}}$. Let $p \in C_{rd}([a,\tau),\mathbb{R})$ with $\int_a^\tau (p(t))^{1-\alpha}\Delta t < \infty$, and $\int_\tau^b (p(t))^{1-\alpha}\Delta t < \infty$, $\alpha \geq 1+\gamma$. Assume the function $q(t)$ is positive, bounded, and nonincreasing on $[a,\tau]_{\mathbb{T}}$ and nondecreasing in $[\tau,b]_{\mathbb{T}}$. Assume that*

$$\chi := \left(\int_a^\tau \frac{1}{p^{\alpha-1}(t)}\Delta t\right)^{\frac{1+\gamma}{\alpha}} = \left(\int_\tau^b \frac{1}{p^{\alpha-1}(t)}\Delta t\right)^{\frac{1+\gamma}{\alpha}}. \tag{4.29}$$

If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $f(a)=f(b)=0$, thus for $\gamma > 0$,

$$\int_a^b q(t)\,|f(t)|^\gamma \left|f^\Delta(t)\right| \Delta t \leq \frac{\chi}{\gamma+1}\left(\int_a^b p(t) q^{\frac{\nu}{1+\gamma}}(t) \left|f^\Delta(t)\right|^\nu \Delta t\right)^{\frac{1+\gamma}{\nu}}, \tag{4.30}$$

where $\frac{1}{\alpha}+\frac{1}{\nu}=1$.

In the following, we establish an inequality of Opial type which depends on the smallest eigenvalue of a boundary value problems on time scales. We will assume that the boundary value problem

$$\left.\begin{aligned} \left(r(t)\left(u^\Delta(t)\right)^p\right)^\Delta &= \beta s^\Delta(t) u^p(t), \\ u(0)=0,\ r(b)\left(u^\Delta(b)\right)^p &= \beta s(b) u^p(b), \end{aligned}\right\} \tag{4.31}$$

has a solution $u(t)$ such that $u^{\Delta}(t) \geq 0$ on the interval $[a,b]_{\mathbb{T}}$, where r, s are nonnegative rd-continuous functions on $(0,b)_{\mathbb{T}}$. Let

$$w(t) = \left(\frac{u^{\Delta}(t)}{u(t)}\right)^{p}.$$

Theorem 4.20 *Suppose $\mathbb{T}$ is a time scale with $0, b \in \mathbb{T}$ and suppose r, s are non-negative rd-continuous functions on $(0,b)_{\mathbb{T}}$ such that 4.31 has a solution for some $\beta > 0$. If $f : [0,b] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $f(0) = 0$, thus for $p > 0$,*

$$\begin{aligned} \int_0^b s(t)\,|f(t)|^p \left|f^{\Delta}(t)\right| \Delta t \;\; &\leq \;\; \frac{1}{(p+1)\beta}\int_0^b r(t)\left|f^{\Delta}(t)\right|^{p+1}\Delta t + \frac{p}{(p+1)\beta} \\ &\times \int_0^b \left[r(t)w^{\frac{p+1}{p}}(t) - r^{\sigma}(t)w^{\sigma}(t)w^{\frac{1}{p}}(t)\right]|f^{\sigma}(t)|^{p+1}\Delta t. \end{aligned} \tag{4.32}$$

4.3 OPIAL TYPE INEQUALITIES III

The results in this section are taken from [70] which were adapted from [78].

Theorem 4.21 *Suppose $\mathbb{T}$ is a time scale with a, $\tau \in \mathbb{T}$ and p, q is positive real numbers such that $p \geq 1$, and let r, s be nonnegative rd-continuous functions on $(a,\tau)_{\mathbb{T}}$ such that $\int_a^{\tau} r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. If $y : [a,\tau] \cap \mathbb{T} \to \mathbb{R}^{+}$ is delta differentiable with $y(a) = 0$, (and y^{Δ} does not change sign in $(a,\tau)_{\mathbb{T}}$), therefore we get*

$$\int_a^{\tau} s(x)\,|y(x)+y^{\sigma}(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x \leq K_1(a,\tau,p,q)\int_a^{\tau} r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{4.33}$$

where

$$\begin{aligned} K_1(a,\tau,p,q) \;\; &= \;\; 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\times \left(\int_a^{\tau} \frac{(s(x))^{\frac{p+q}{p}}}{(r(x))^{\frac{q}{p}}}\left(\int_a^{x} r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}} \\ &+ 2^{p-1}\sup_{a\leq x\leq \tau}\left(\mu^{p}(x)\frac{s(x)}{r(x)}\right). \end{aligned} \tag{4.34}$$

With $[a,\tau]_{\mathbb{T}}$ replaced by $[\tau,b]_{\mathbb{T}}$ and $|y(x)| = \int_x^b \left|y^{\Delta}(t)\right|\Delta t$ in Theorem 4.21, we have the following result.

Theorem 4.22 *Assume $\mathbb{T}$ is a time scale with τ, $b \in \mathbb{T}$ and p, q are positive real numbers such that $p \geq 1$, and let r, s be nonnegative rd-continuous functions on $(\tau,b)_{\mathbb{T}}$ such that*

$\int_\tau^b r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. *If* $y : [\tau, b] \cap \mathbb{T} \to \mathbb{R}^+$ *is delta differentiable with* $y(b) = 0$, *(and* y^Δ *does not change sign in* $(\tau, b)_\mathbb{T}$*), therefore we get*

$$\int_\tau^b s(x) |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x \leq K_2(\tau, b, p, q) \int_\tau^b r(x) \left|y^\Delta(x)\right|^{p+q} \Delta x, \tag{4.35}$$

where

$$\begin{aligned} K_2(\tau, b, p, q) &= 2^{2p-1} \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\quad \times \left(\int_\tau^b \frac{(s(x))^{\frac{p+q}{p}}}{(r(x))^{\frac{q}{p}}} \left(\int_x^b r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)} \Delta x\right)^{\frac{p}{p+q}} \\ &\quad +2^{p-1} \sup_{\tau \leq x \leq b} \left(\mu^p(x)\frac{s(x)}{r(x)}\right). \end{aligned} \tag{4.36}$$

In the following, we suppose that

$$K(p, q) = K_1(a, \tau, p, q) = K_2(\tau, b, p, q) < \infty,$$

where $K_1(a, \tau, p, q)$ and $K_2(\tau, b, p, q)$ are defined as in Theorems 4.21 and 4.22 and τ is the unique solution of the equation $K_1(a, \tau, p, q) = K_2(\tau, b, p, q)$. Note that,

$$\begin{aligned} &\int_a^b s(x) |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x \\ = &\int_a^\tau s(x) |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x + \int_\tau^b s(x) |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x, \end{aligned}$$

so combining Theorems 4.21 and 4.22 gives us the following theorem.

Theorem 4.23 *Assume that* $\mathbb{T}$ *is a time scale with* a, $b \in \mathbb{T}$ *and* p, q *are positive real numbers such that* $p \geq 1$, *and let* r, s *be nonnegative rd-continuous functions on* $(a, b)_\mathbb{T}$ *such that* $\int_a^b r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. *If* $y : [a, b] \cap \mathbb{T} \to \mathbb{R}^+$ *is delta differentiable with* $y(a) = 0 = y(b)$, *(and* y^Δ *does not change sign in* $(a, b)_\mathbb{T}$*), therefore we obtain*

$$\int_a^b s(x) |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x \leq K(p, q) \int_a^b r(x) \left|y^\Delta(x)\right|^{p+q} \Delta x. \tag{4.37}$$

Theorem 4.24 *Suppose* $\mathbb{T}$ *is a time scale with* a, $\tau \in \mathbb{T}$ *and* p, q *are positive real numbers such that* $p \leq 1$, $p + q > 1$ *and let* r, s *be nonnegative rd-continuous functions on* $(a, \tau)_\mathbb{T}$ *such that* $\int_a^\tau r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. *If* $y : [a, \tau] \cap \mathbb{T} \to \mathbb{R}^+$ *is delta differentiable with* $y(a) = 0$, *(and* y^Δ *does not change sign in* $(a, \tau)_\mathbb{T}$*), therefore we obtain*

$$\int_a^\tau s(x) |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x \leq K_1(a, \tau, p, q) \int_a^\tau r(x) \left|y^\Delta(x)\right|^{p+q} \Delta x, \tag{4.38}$$

where

$$K_1(a,\tau,p,q) = \sup_{a\le x\le\tau}\left(\mu^p(x)\frac{s(x)}{r(x)}\right)+2^p\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \left(\int_a^\tau \frac{s^{\frac{p+q}{p}}(x)}{r^{\frac{q}{p}}(x)}\left(\int_a^x r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{p+q-1}\Delta x\right)^{\frac{p}{p+q}}. \tag{4.39}$$

4.4 HIGHER ORDER OPIAL TYPE INEQUALITIES

In this section, we state some Opial type inequalities involving higher order derivatives. The results in this section are taken from [70] which were adapted from [33, 35, 74, 78-81]. We note that the definition of the generalized Taylor Monomials which is given as follows:

$$h_k(t,s) := \begin{cases} 1 & , \quad k=0, \\ \int_s^t h_{k-1}(\xi,s)\Delta\xi & , \quad k\in\mathbb{N}, \end{cases}$$

for all s, $t\in\mathbb{T}$. For simplicity, for y^{Δ^n} we mean the (nth) delta derivative of y which is equivalent to $\left(y^{\Delta^{n-1}}\right)^{\Delta}$ for $n\in\mathbb{N}$.

Theorem 4.25 *Suppose $\mathbb{T}$ is a time scale with a, $b\in\mathbb{T}$ and $y\in C_{rd}^{(n)}([a,b]\cap\mathbb{T})$. If $y^{\Delta^i}(a)=0$, for $i=0,1,...,n-1$, therefore*

$$\int_a^b |y(t)|\left|y^{\Delta^n}(t)\right|\Delta t \le \sqrt{\frac{1}{2}}\left(\int_a^b\left(\int_a^t |h_{n-1}(t,\sigma(s))|^2\Delta s\right)\Delta t\right)^{\frac{1}{2}} \times\int_a^b \left|y^{\Delta^n}(t)\right|^2\Delta t. \tag{4.40}$$

Theorem 4.26 *Assume $\mathbb{T}$ is a time scale with a, $b\in\mathbb{T}$, p and q are real numbers such that $1/p+1/q=1$ and $y\in C_{rd}^{(n)}([a,b]\cap\mathbb{T})$. If $y^{\Delta^i}(a)=0$, for $i=0,1,...,n-1$, therefore*

$$\int_a^b |y(t)|\left|y^{\Delta^n}(t)\right|\Delta t \le \left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\int_a^b\left(\int_a^t |h_{n-1}(t,\sigma(s))|^p\Delta s\right)\Delta t\right)^{\frac{1}{p}} \times\int_a^b \left|y^{\Delta^n}(t)\right|^q\Delta t.$$

Theorem 4.27 *Assume $\mathbb{T}$ is a time scale with a, $b\in\mathbb{T}$ and l, m are positive real numbers such that $l+m>1$, and $y\in C_{rd}^{(n)}([a,b]\cap\mathbb{T})$. If $y^{\Delta^i}(a)=0$, for $i=0,1,...,n-1$, therefore*

$$\int_a^b |y(t)|^l\left|y^{\Delta^n}(t)\right|^m\Delta t \le \left(\frac{m}{l+m}\right)^{\frac{m}{l+m}}\left(\int_a^b H^{(l+m-1)}(t,s)\Delta t\right)^{\frac{l}{l+m}} \times\int_a^b \left|y^{\Delta^n}(t)\right|^{l+m}\Delta t, \tag{4.41}$$

where

$$H(t,s) := \int_a^t (h_{n-1}(t,\sigma(s)))^{\frac{l+m}{l+m-1}} \Delta s.$$

Note that Theorem 4.27 cannot be applied when $l+m=1$. In the following theorem we present an inequality which can be applied in this case.

Theorem 4.28 *Assume $\mathbb{T}$ is a time scale with $a, b \in \mathbb{T}$ and l, m are positive real numbers such that $l+m=1$ and $y \in C_{rd}^{(n)}([a,b] \cap \mathbb{T})$. If $y^{\Delta^i}(a)=0$, for $i=0,1,...,n-1$, therefore*

$$\int_a^b |y(t)|^l \left|y^{\Delta^n}(t)\right|^m \Delta t \le m^m \left(\int_a^b h_{n-1}(t,a)\Delta t\right)^l \int_a^b \left|y^{\Delta^n}(t)\right| \Delta t. \tag{4.42}$$

Theorem 4.29 *Assume $\mathbb{T}$ is a time scale with $a, b \in \mathbb{T}$ and p, q are positive real numbers such that $p \ge 0$ and $q \ge 1$. Suppose that h is a positive rd-continuous function and is nonincreasing on $[a,\tau]_{\mathbb{T}}$ and $y \in C_{rd}^{(n)}([a,b] \cap \mathbb{T})$. If $y^{\Delta^i}(a)=0$, for $i=0,1,...,n-1$, therefore*

$$\int_a^b h(t)\,|y(t)|^p \left|y^{\Delta^n}(t)\right|^q \Delta t \le \frac{q(b-a)^{pn}}{p+q} \int_a^b h(t) \left|y^{\Delta^n}(t)\right|^{p+q} \Delta t. \tag{4.43}$$

Theorem 4.30 *Assume $\mathbb{T}$ is a time scale with $a, b \in \mathbb{T}$ and p, q are positive real numbers such that $p \ge 0$ and $q \ge 1$. Assume that h is a positive rd-continuous function and is nonincreasing on $[a,\tau]_{\mathbb{T}}$ and $x, y \in C_{rd}^{(n)}([a,b] \cap \mathbb{T})$. If $x^{\Delta^i}(a) = y^{\Delta^i}(a) = 0$, for $i=0,1,...,n-1$, therefore*

$$\begin{aligned} &\int_a^b h(t) \left[|x(t)|^p \left|y^{\Delta^n}(t)\right|^q + |y(t)|^p \left|x^{\Delta^n}(t)\right|^q\right] \Delta t \\ \le\ &\frac{2q(b-a)^{pn}}{p+q} \int_a^b h(t) \left[\left|x^{\Delta^n}(t)\right|^{p+q} + \left|y^{\Delta^n}(t)\right|^{p+q}\right] \Delta t. \end{aligned} \tag{4.44}$$

Theorem 4.31 *Assume $\mathbb{T}$ is a time scale with $a, b \in \mathbb{T}$ and p, q are positive real numbers such that $p \ge 0$ and $q \ge 1$. Assume that h be a positive rd-continuous function and is nonincreasing on $[a,\tau]_{\mathbb{T}}$ and $x, y \in C_{rd}^{(n)}([a,b] \cap \mathbb{T})$. If $x^{\Delta^i}(a) = y^{\Delta^i}(a) = 0$ and $x^{\Delta^i}(b) = y^{\Delta^i}(b) = 0$, for $i=0,1,...,n-1$, therefore*

$$\begin{aligned} &\int_a^b h(t) \left[|x(t)|^p \left|y^{\Delta^n}(t)\right|^q + |y(t)|^p \left|x^{\Delta^n}(t)\right|^q\right] \Delta t \\ \le\ &\frac{2q}{p+q} \left(\frac{b-a}{2}\right)^{pn} \int_a^b h(t) \left[\left|x^{\Delta^n}(t)\right|^{p+q} + \left|y^{\Delta^n}(t)\right|^{p+q}\right] \Delta t. \end{aligned} \tag{4.45}$$

In the following, we present some inequalities with two different weight functions.

Theorem 4.32 *Suppose $\mathbb{T}$ is a time scale with $a, \tau \in \mathbb{T}$, let $p \in C_{rd}([a,\tau]_{\mathbb{T}}, \mathbb{R})$ with*

$$\int_a^\tau (p(t))^{1-\alpha} \Delta t < \infty, \ (\alpha > 1).$$

Assume q is a positive bounded rd-continuous function and is nonincreasing on $[a,\tau]_{\mathbb{T}}$. Let $y \in C_{rd}^{(n)}([a,\tau]_{\mathbb{T}})$. If $y^{\Delta^i}(a) = 0$, for $i = 0, 1, ..., n-1$, thus for $r > 0$, we obtain

$$\int_a^\tau q(t)\left|y(t)\right|^r \left|y^{\Delta^n}(t)\right| \Delta t \leq \Lambda_1(r,\alpha)\left(\int_a^\tau p(t) q^{\frac{r}{r+1}}(t)\left|y^{\Delta^n}(t)\right|^\nu \Delta t\right)^{\frac{1+r}{\nu}}, \tag{4.46}$$

where

$$\Lambda_1(r,\alpha) := \frac{(\tau-a)^{r(n-1)}}{r+1}\left(\int_a^\tau p^{1-\alpha}(t)\Delta t\right)^{\frac{1+r}{\alpha}}, \ \frac{1}{\alpha}+\frac{1}{\nu} = 1. \tag{4.47}$$

Theorem 4.33 *Assume $\mathbb{T}$ is a time scale with $a,\ b \in \mathbb{T}$ and l,m,r are positive real numbers such that $l+m > 1$ and $r > 1$. In addition, let p and q be positive rd-continuous function defined on $[a,b] \cap \mathbb{T}$ and $y \in C_{rd}^{(n)}([a,b] \cap \mathbb{T})$. If $y^{\Delta^i}(a) = 0$, for $i = 0, 1, ..., n-1$, therefore*

$$\int_a^b q(t)\left|y(t)\right|^l \left|y^{\Delta^n}(t)\right|^m \Delta t \leq \Lambda_1(l,m,r,p,q)\left(\int_a^b p(s)\left|y^{\Delta^n}(s)\right|^r \Delta s\right)^{\frac{l+m}{r}}, \tag{4.48}$$

where

$$\Lambda_1(l,m,p,q,r) := \left(\frac{m}{l+m}\right)^{m/r}\left(\int_a^b q^{\frac{r}{r-m}}(t) p^{\frac{-m}{r-m}}(t)\left(P(t)\right)^{l\left(\frac{r-1}{r-m}\right)} \Delta t\right)^{\frac{r-m}{r}}, \tag{4.49}$$

and

$$P(t) := \int_a^t p^{\frac{-1}{r-1}}(s)\left(h_{n-1}(t,\sigma(s))\right)^{\frac{r}{r-1}} \Delta s.$$

Theorem 4.34 *Assume $\mathbb{T}$ is a time scale with $a,\ b \in \mathbb{T}$ and $l,\ m$ are positive real numbers such that $l+m > 1$ and $r > 1$. In addition, let p and q be positive rd-continuous functions defined on $[a,b] \cap \mathbb{T}$ and $y \in C_{rd}^{(n)}([a,b] \cap \mathbb{T})$. If $y^{\Delta^i}(a) = 0$, $k \leq i \leq n-1$, therefore*

$$\int_a^b q(t)\left|y^{\Delta^k}(t)\right|^l \left|y^{\Delta^n}(t)\right|^m \Delta t \leq \Lambda_2(l,m,r,p,q)\left(\int_a^b p(s)\left|y^{\Delta^n}(s)\right|^r \Delta s\right)^{\frac{l+m}{r}},$$

where

$$\Lambda_2(l,m,p,q,r) := \left(\frac{m}{l+m}\right)^{m/r}\left(\int_a^b q^{\frac{r}{r-m}}(t) p^{\frac{-m}{r-m}}(t)\left(P(t)\right)^{l\left(\frac{r-1}{r-m}\right)} \Delta t\right)^{\frac{r-m}{r}},$$

and

$$P(t) := \int_a^t p^{\frac{-1}{r-1}}(s) h_{n-k-1}^{\frac{r}{r-1}}(t,\sigma(s)) \Delta s.$$

Theorem 4.35 *Assume $\mathbb{T}$ is a time scale with $a, b \in \mathbb{T}$ and α, β are positive real numbers such that $\alpha+\beta>1$, and let p, q be nonnegative rd-continuous functions on $(a,b)_{\mathbb{T}}$ and $y \in C_{rd}^{(n)}([a,b]\cap\mathbb{T})$. If $y^{\Delta^i}(a)=0$, for $i=0,1,...,n-1$, therefore*

$$\int_a^b q(t)\,|y(t)|^{\alpha}\left|y^{\Delta^n}(t)\right|^{\beta}\Delta t \le \Lambda_3(a,b,\alpha,\beta)\int_a^b p(t)\left|y^{\Delta^n}(t)\right|^{\alpha+\beta}\Delta t, \tag{4.50}$$

where

$$\Lambda_3(a,b,\alpha,\beta)=\left(\frac{\beta}{\alpha+\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left(\int_a^b \frac{q^{\frac{\alpha+\beta}{\alpha}}(t)}{p^{\frac{\beta}{\alpha}}(t)}\left(\int_a^t \frac{h_{n-1}^{\frac{\alpha+\beta}{\alpha+\beta-1}}(t,\sigma(s))}{p^{\frac{1}{\alpha+\beta-1}}(s)}\Delta s\right)^{\alpha+\beta-1}\Delta t\right)^{\frac{\alpha}{\alpha+\beta}}. \tag{4.51}$$

CHAPTER 5

SOME WEIGHTED OPIAL TYPE INEQUALITIES ON TIME SCALES

In this chapter we use [73]. We establish some generalized weighted Opial type inequalities on time scales.

Theorem 5.1 *Suppose that the function $f(t)$ given by*

$f : [a,b] \cap \mathbb{T} \to \mathbb{R} \quad (b > a \geqq 0)$

be delta differentiable on $[a,b] \cap \mathbb{T}$. Also let

$p \geqq 0$ *and* $q \geqq 1$,

and suppose that $h(t)$ is a right-continuous, positive and non-increasing function on $[a,\tau] \cap \mathbb{T}$. Then the Opial type inequality

$$\int_a^b h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \leqq \left(\frac{q}{p+q}\right)(b-a)^p \int_a^b h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t$$

is valid true.

Proof. Assume that the function $g(t)$ is given by

$$g(t) = \int_a^t [h(s)]^{\frac{q}{p+q}} \left|f^\Delta(s)\right|^q \Delta s \quad (t \in [a,\tau] \cap \mathbb{T}),$$

so that

$$g(a) = 0 \text{ and } g^\Delta(t) = [h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q. \tag{5.1}$$

When $q > 1$, by using Hölder's inequality with indices

q and $\dfrac{q}{q-1}$,

we obtain

$$\begin{aligned}
|f(t)| &\leqq \int_a^t \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_a^t [h(s)]^{-\frac{1}{p+q}} [h(s)]^{\frac{1}{p+q}} \left|f^{\Delta}(s)\right| \Delta s \\
&\leqq \left[\int_a^t \left([h(s)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \Delta s\right]^{\frac{q-1}{q}} \left[\int_a^t \left([h(s)]^{\frac{1}{p+q}} \left|f^{\Delta}(s)\right|\right)^{q} \Delta s\right]^{\frac{1}{q}} \\
&\leqq \left[\left([h(t)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \left(\int_a^t 1.\Delta s\right)\right]^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}} \\
&= [h(t)]^{-\frac{1}{p+q}} (t-a)^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}},
\end{aligned}$$

which readily gives

$$[h(t)]^{\frac{p}{p+q}} |f(t)|^p \leqq (t-a)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}}. \tag{5.2}$$

When $q = 1$, we find that

$$\begin{aligned}
|f(t)| &\leqq \int_a^t \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_a^t [h(s)]^{-\frac{1}{p+1}} [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&\leqq [h(t)]^{-\frac{1}{p+1}} \int_a^t [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&= [h(t)]^{-\frac{1}{p+1}} g(t),
\end{aligned}$$

which shows that the inequality (5.2) is valid true also when $q = 1$. Therefore, by means of (5.1) and (5.2), we have that

$$\begin{aligned}
&\int_a^\tau h(t) |f(t)|^p \left|f^{\Delta}(t)\right|^q \Delta t \\
&= \int_a^\tau [h(t)]^{\frac{p}{p+q}} |f(t)|^p [h(t)]^{\frac{q}{p+q}} \left|f^{\Delta}(t)\right|^q \Delta t \\
&\leqq \int_{t=a}^\tau (t-a)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}} g^{\Delta}(t) \Delta t \\
&\leqq (\tau-a)^{\frac{p(q-1)}{q}} \int_a^\tau [g(t)]^{\frac{p}{q}} \Delta g(t) \\
&\leqq (\tau-a)^{\frac{p(q-1)}{q}} \left(\frac{q}{p+q}\right) [g(\tau)]^{\frac{p+q}{q}},
\end{aligned}$$

since (by definition) $g(a) = 0$ as given by (5.1).

On the other hand, from Hölder's inequality with indices

$\dfrac{p+q}{p}$ and $\dfrac{p+q}{q}$,

we obtain

$$
\begin{aligned}
g(\tau) &= \int_a^\tau [h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q \Delta t \\
&\leqq \left(\int_a^\tau 1^{\frac{p+q}{p}} .\Delta t\right)^{\frac{p}{p+q}} \left[\int_a^\tau \left([h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q\right)^{\frac{p+q}{q}} \Delta t\right]^{\frac{q}{p+q}} \\
&\leqq (\tau-a)^{\frac{p}{p+q}} \left(\int_a^\tau h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t\right)^{\frac{q}{p+q}}.
\end{aligned}
$$

Hence, we finally find

$$
\int_a^\tau h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \leqq \left(\frac{q}{p+q}\right)(\tau-a)^p \int_a^\tau h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t,
$$

which evidently completes the proof of Theorem 5.1. ■

Theorem 5.2 *Assume that the function* $f : [a,b] \cap \mathbb{T} \to \mathbb{R}$ *is delta differentiable with* $f(b) = 0$. *Also let*

$p \geqq 0$ *and* $q \geqq 1$,

and assume that $h(t)$ *is a right-continuous, positive and non-increasing function on* $[\tau, b] \cap \mathbb{T}$. *Then*

$$
\begin{aligned}
&\int_\tau^b h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \\
\leqq\ &\left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t. \qquad (5.3)
\end{aligned}
$$

Proof. We consider a function $g(t)$ given by

$$
g(t) = \int_t^b [h(s)]^{\frac{q}{p+q}} \left|f^\Delta(s)\right|^q \Delta s \quad (t \in [\tau, b] \cap \mathbb{T}),
$$

so that

$$
g(b) = 0 \text{ and } g^\Delta(t) = -[h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q. \qquad (5.4)
$$

When $q > 1$, by using Hölder's inequality with indices

q and $\dfrac{q}{q-1}$,

we obtain

$$\begin{aligned}
|f(t)| &\leqq \int_t^b \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_t^b [h(s)]^{-\frac{1}{p+q}} [h(s)]^{\frac{1}{p+q}} \left|f^{\Delta}(s)\right| \Delta s \\
&\leqq \left[\int_t^b \left([h(s)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \Delta s\right]^{\frac{q-1}{q}} \left[\int_t^b \left([h(s)]^{\frac{1}{p+q}} \left|f^{\Delta}(s)\right|\right)^q \Delta s\right]^{\frac{1}{q}} \\
&\leqq \left[\left([h(t)]^{-\frac{1}{p+q}}\right)^{\frac{q}{q-1}} \left(\int_t^b 1.\Delta s\right)\right]^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}} \\
&= [h(t)]^{-\frac{1}{p+q}} (b-t)^{\frac{q-1}{q}} [g(t)]^{\frac{1}{q}},
\end{aligned}$$

which implies the following inequality:

$$[h(t)]^{\frac{p}{p+q}} |f(t)|^p \leqq (b-t)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}}. \tag{5.5}$$

When $q = 1$, we obtain

$$\begin{aligned}
|f(t)| &\leqq \int_t^b \left|f^{\Delta}(s)\right| \Delta s \\
&= \int_t^b [h(s)]^{-\frac{1}{p+1}} [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&= [h(t)]^{-\frac{1}{p+1}} \int_t^b [h(s)]^{\frac{1}{p+1}} \left|f^{\Delta}(s)\right| \Delta s \\
&= [h(t)]^{-\frac{1}{p+1}} g(t),
\end{aligned}$$

which shows that the inequality (5.5) is valid true also when $q = 1$. Thus, by (5.4) and (5.5), we obtain that

$$\begin{aligned}
\int_\tau^b h(t) |f(t)|^p \left|f^{\Delta}(t)\right|^q \Delta t &= \int_\tau^b [h(t)]^{\frac{p}{p+q}} |f(t)|^p [h(t)]^{\frac{q}{p+q}} \left|f^{\Delta}(t)\right|^q \Delta t \\
&\leqq \int_\tau^b (b-t)^{\frac{p(q-1)}{q}} [g(t)]^{\frac{p}{q}} \left[-g^{\Delta}(t)\right] \Delta t \\
&\leqq (b-\tau)^{\frac{p(q-1)}{q}} \int_{t=\tau}^b \left(-[g(t)]^{\frac{p}{q}}\right) \Delta g(t) \\
&\leqq (b-\tau)^{\frac{p(q-1)}{q}} \left(\frac{q}{p+q}\right) \left(-[g(t)]^{\frac{p+q}{q}}\right)\Big|_{t=\tau}^b \\
&\leqq (b-\tau)^{\frac{p(q-1)}{q}} \left(\frac{q}{p+q}\right) [g(\tau)]^{\frac{p+q}{q}},
\end{aligned}$$

since (by definition) $g(b) = 0$ as given by (5.4).

On the other hand, in view of Hölder's inequality with indices

$$\frac{p+q}{p} \text{ and } \frac{p+q}{q},$$

we obtain

$$\begin{aligned}
g(\tau) &= \int_\tau^b [h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q \Delta t \\
&\leqq \left(\int_\tau^b 1^{\frac{p+q}{p}}.\Delta t\right)^{\frac{p}{p+q}} \left[\int_\tau^b \left([h(t)]^{\frac{q}{p+q}} \left|f^\Delta(t)\right|^q\right)^{\frac{p+q}{q}} \Delta t\right]^{\frac{q}{p+q}} \\
&\leqq (b-\tau)^{\frac{p}{p+q}} \left(\int_\tau^b h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t\right)^{\frac{q}{p+q}}.
\end{aligned}$$

Hence, we finally find

$$\begin{aligned}
&\int_\tau^b h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \\
\leqq &\left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t,
\end{aligned}$$

which is precisely the inequality (5.3) asserted by Theorem 5.2. ■

Theorem 5.3 *Let the function* $f : [a,b] \cap \mathbb{T} \to \mathbb{R}$ *be delta differentiable with*

$f(a) = f(b) = 0.$

Also let

$p \geqq 0, \quad q \geqq 1 \quad$ *and* $\quad \tau \in [a,b] \cap \mathbb{T}.$

Moreover, suppose that the function $h(t)$ *is positive and non-increasing on* $[a,\tau] \cap \mathbb{T}$ *and non-increasing on* $[\tau,b] \cap \mathbb{T}$. *Then*

$$\begin{aligned}
&\int_a^b h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \\
\leqq &\left(\frac{q}{p+q}\right)(\tau-a)^p \int_a^\tau h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t \\
&+\left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t. \qquad (5.6)
\end{aligned}$$

Proof. By Theorem 5.1 and 5.2, we obtain

$$\begin{aligned}
\int_a^b h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t &= \int_a^\tau h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \\
&\quad + \int_\tau^b h(t) \left|f(t)\right|^p \left|f^\Delta(t)\right|^q \Delta t \\
&\leqq \left(\frac{q}{p+q}\right)(\tau-a)^p \int_a^\tau h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t \\
&\quad +\left(\frac{q}{p+q}\right)(b-\tau)^p \int_\tau^b h(t) \left|f^\Delta(t)\right|^{p+q} \Delta t,
\end{aligned}$$

which obviously completes the proof of Theorem 5.3. ■

By setting

$$\tau = \frac{a+b}{2}$$

in Theorem 5.3, we arrive at an interesting special case as given below.

Corollary 5.1 *The following weighted Opial type inequality is valid true on time scales:*

$$\begin{aligned} & \int_a^b h(t) \left| f(t) \right|^p \left| f^{\Delta}(t) \right|^q \Delta t \\ \leqq & \left(\frac{q}{p+q} \right) \left(\frac{b-a}{2} \right)^p \int_a^b h(t) \left| f^{\Delta}(t) \right|^{p+q} \Delta t. \end{aligned} \tag{5.7}$$

CHAPTER 6

OPIAL TYPE INEQUALITIES FOR DIAMOND-ALPHA DERIVATIVES AND INTEGRALS ON TIME SCALES

In this chapter we use [82]. We establish some Opial type inequalities on time scales via the notion of the diamond-alpha derivative which is a general concept covering both delta and nabla derivatives on time scales.

The main goal of the present chapter is to construct some Opial type inequalities via the concepts of $\Diamond_\alpha$-differentiability and $\Diamond_\alpha$-integrability. Therefore, we not only unify the continuous and discrete versions of the well-known Opial type inequalities but also combine and extend Theorem 1.13 and Theorem 1.14.

We suppose throughout this chapter that 0 is contained in the time scale.

6.1 FIRST-ORDER OPIAL INEQUALITIES

In this section, we introduce a sequence of Opial type inequalities for first-order diamond alpha derivatives on time scales.

Theorem 6.1 *Suppose that* $\alpha \in [0,1]$ *and* $h \in \mathbb{T}$ *with* $h > 0$. *For any* $f \in C^1_{\Diamond_\alpha}$ *with* $f(0) = 0$ *and* $\alpha(1-\alpha) f^\Delta f^\nabla \geq 0$, *we get*

$$\alpha^3 \int_0^h \left|\left(f + f^\sigma\right) f^\Delta\right|(t)\Delta t + (1-\alpha)^3 \int_0^h \left|\left(f + f^\rho\right) f^\nabla\right|(t)\nabla t \leq h \int_0^h \left(f^{\Diamond_\alpha}\right)^2(t)\Diamond_\alpha t. \quad (6.1)$$

Proof. By Theorem 1.13, we may write that

$$\alpha^3 \int_0^h \left|\left(f + f^\sigma\right) f^\Delta\right|(t)\Delta t \leq \alpha^3 h \int_0^h \left(f^\Delta\right)^2(t)\Delta t,$$

and Theorem 1.14 gives

$$(1-\alpha)^3 \int_0^h \left|\left(f + f^\rho\right) f^\nabla\right|(t)\nabla t \leq (1-\alpha)^3 h \int_0^h \left(f^\nabla\right)^2(t)\nabla t.$$

Also, since

$$\left(f^{\Diamond_\alpha}\right)^2 = \alpha^2 \left(f^\Delta\right)^2 + 2\alpha\left(1-\alpha\right) f^\Delta f^\nabla + (1-\alpha)^2 \left(f^\nabla\right)^2$$

and

$f^{\Delta} f^{\nabla} \geq 0,$

we obtain

$$\alpha^2 \left(f^{\Delta}\right)^2 \leq \left(f^{\Diamond_\alpha}\right)^2 \text{ and } (1-\alpha)^2 \left(f^{\nabla}\right)^2 \leq \left(f^{\Diamond_\alpha}\right)^2. \tag{6.2}$$

Combining the above inequalities, we have

$$\begin{aligned}
&\alpha^3 \int_0^h \left|\left(f+f^{\sigma}\right) f^{\Delta}\right|(t)\Delta t + (1-\alpha)^3 \int_0^h \left|\left(f+f^{\rho}\right) f^{\nabla}\right|(t)\nabla t \\
&\leq h\left(\alpha^3 \int_0^h \left(f^{\Delta}\right)^2 (t)\Delta t + (1-\alpha)^3 \int_0^h \left(f^{\nabla}\right)^2 (t)\nabla t\right) \\
&\overset{(6.2)}{\leq} h\left(\alpha \int_0^h \left(f^{\Diamond_\alpha}\right)^2 (t)\Delta t + (1-\alpha) \int_0^h \left(f^{\Diamond_\alpha}\right)^2 (t)\nabla t\right) \\
&= h \int_0^h \left(f^{\Diamond_\alpha}\right)^2 (t)\Diamond_\alpha t.
\end{aligned}$$

This completes the proof. ■

Note that if we take $\alpha = 1$ or $\alpha = 0$, then Theorem 6.1 reduces to Theorem 1.13 and Theorem 1.14, respectively.

Theorem 6.2 *Assume that $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\Diamond_\alpha}$ with $\alpha f^{\Delta} \geq 0$ and $(1-\alpha) f^{\nabla} \geq 0$, we get*

$$\begin{aligned}
&\alpha^3 \int_0^h \left|\left(f+f^{\sigma}\right) f^{\Delta}\right|(t)\Delta t + (1-\alpha)^3 \int_0^h \left|\left(f+f^{\rho}\right) f^{\nabla}\right|(t)\nabla t \\
\leq\ &\beta \int_0^h \left(f^{\Diamond_\alpha}\right)^2 (t)\Diamond_\alpha t + 2\gamma\left(1-3\alpha+3\alpha^2\right)\left(f(h)-f(0)\right),
\end{aligned}$$

where

$$\beta := \min_{u \in [0,h]\cap\mathbb{T}} \max\{u, h-u\} \text{ and } \gamma := \max\{|f(0)|, |f(h)|\}.$$

Proof. Since $\alpha f^{\Delta} \geq 0$, it follows from [2, Theorem 6.27] that

$$\begin{aligned}
\alpha^3 \int_0^h \left|\left(f+f^{\sigma}\right) f^{\Delta}\right|(t)\Delta t &\leq \beta\alpha^3 \int_0^h \left(f^{\Delta}\right)^2 (t)\Delta t + 2\gamma\alpha^3 \int_0^h f^{\Delta}(t)\Delta t \\
&= \beta\alpha^3 \int_0^h \left(f^{\Delta}\right)^2 (t)\Delta t + 2\gamma\alpha^3 \left(f(h)-f(0)\right).
\end{aligned}$$

Similarly, using the fact $(1-\alpha) f^{\nabla} \geq 0$, one can obtain

$$(1-\alpha)^3 \int_0^h \left|\left(f+f^{\rho}\right) f^{\nabla}\right|(t)\nabla t \leq \beta (1-\alpha)^3 \int_0^h \left(f^{\nabla}\right)^2 (t)\nabla t + 2\gamma (1-\alpha)^3 \left(f(h)-f(0)\right).$$

Now combining these inequalities and also using (6.2), we get that

$$\begin{aligned}
&\alpha^3 \int_0^h \left|\left(f+f^\sigma\right) f^\Delta\right|(t)\Delta t+(1-\alpha)^3 \int_0^h \left|\left(f+f^\rho\right) f^\nabla\right|(t)\nabla t \\
&\leq \beta\left\{\alpha^3 \int_0^h \left(f^\Delta\right)^2(t)\Delta t+(1-\alpha)^3 \int_0^h \left(f^\nabla\right)^2(t)\nabla t\right\} \\
&+2\gamma\left[\alpha^3+(1-\alpha)^3\right](f(h)-f(0)) \\
&\overset{(6.2)}{\leq} \beta\left\{\alpha \int_0^h \left(f^{\Diamond_\alpha}\right)^2(t)\Delta t+(1-\alpha) \int_0^h \left(f^{\Diamond_\alpha}\right)^2(t)\nabla t\right\} \\
&+2\gamma\left[\alpha^3+(1-\alpha)^3\right](f(h)-f(0)) \\
&=\beta \int_0^h \left(f^{\Diamond_\alpha}\right)^2(t)\Diamond_\alpha t+2\gamma\left(1-3\alpha+3\alpha^2\right)(f(h)-f(0)),
\end{aligned}$$

which completes the proof. ■

Corollary 6.1 *Assume $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h>0$. For any $f \in C^1_{\Diamond_\alpha}$ with $\alpha f^\Delta \geq 0$, $(1-\alpha)f^\nabla \geq 0$, and $f(0)=f(h)=0$, we get*

$$\alpha^3 \int_0^h \left|\left(f+f^\sigma\right) f^\Delta\right|(t)\Delta t+(1-\alpha)^3 \int_0^h \left|\left(f+f^\rho\right) f^\nabla\right|(t)\nabla t \leq \beta \int_0^h \left(f^{\Diamond_\alpha}\right)^2(t)\Diamond_\alpha t,$$

where β is as in Theorem 6.2.

Proof. Since $f(h)-f(0)=0$, the proof follows immediately from Theorem 6.1. ■

Theorem 6.3 *Suppose $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h>0$. Suppose that $g:[0,h] \to \mathbb{R}^+$ is a nonincreasing continuous function. For any $f \in C^1_{\Diamond_\alpha}$ with $\alpha(1-\alpha)f^\Delta f^\nabla \geq 0$, we get*

$$\begin{aligned}
&\alpha^3 \int_0^h \left[g^\sigma\left|\left(f+f^\sigma\right) f^\Delta\right|\right](t)\Delta t+(1-\alpha)^3 \int_0^h \left[g^\rho\left|\left(f+f^\rho\right) f^\nabla\right|\right](t)\nabla t \\
\leq\ & h \int_0^h g(t)\left(f^{\Diamond_\alpha}\right)^2(t)\Diamond_\alpha t.
\end{aligned}$$

Proof. By [2, Theorem 6.29], we have

$$\alpha^3 \int_0^h \left[g^\sigma\left|\left(f+f^\sigma\right) f^\Delta\right|\right](t)\Delta t \leq \alpha^3 h \int_0^h g(t)\left(f^\Delta\right)^2(t)\Delta t.$$

Similarly, we can obtain that

$$(1-\alpha)^3 \int_0^h \left[g^\rho\left|\left(f+f^\rho\right) f^\nabla\right|\right](t)\nabla t \leq (1-\alpha)^3 h \int_0^h g(t)\left(f^\nabla\right)^2(t)\nabla t.$$

Combining these inequalities and using (6.2), we obtain at

$$\begin{aligned}
&\alpha^3 \int_0^h \left[g^\sigma \left|\left(f + f^\sigma\right) f^\Delta\right|\right](t)\Delta t + (1-\alpha)^3 \int_0^h \left[g^\rho \left|\left(f + f^\rho\right) f^\nabla\right|\right](t)\nabla t \\
&\leq h\left\{\alpha^3 \int_0^h g(t)\left(f^\Delta\right)^2(t)\Delta t + (1-\alpha)^3 \int_0^h g(t)\left(f^\nabla\right)^2(t)\nabla t\right\} \\
&\overset{(6.2)}{\leq} h\left\{\alpha \int_0^h g(t)\left(f^{\Diamond_\alpha}\right)^2(t)\Delta t + (1-\alpha) \int_0^h g(t)\left(f^{\Diamond_\alpha}\right)^2(t)\nabla t\right\} \\
&= \int_0^h g(t)\left(f^{\Diamond_\alpha}\right)^2(t)\Diamond_\alpha t,
\end{aligned}$$

which concludes the proof. ■

6.2 HIGHER-ORDER OPIAL INEQUALITIES

In this section, we get a sequence of Opial type inequalities for first-order diamond alpha derivatives on time scales. Throughout this section, we say that a function $f : [0, h] \to \mathbb{R}$ is in the class $C^n_{\Diamond_\alpha}$ if f is n times $\Diamond_\alpha$-differentiable such that αf^{Δ^n} is rd-continuous, $(1-\alpha)f^{\nabla^n}$ is ld-continuous, and $\alpha(1-\alpha)f^{\Diamond^n_\alpha}$ is continuous. This guarantees the existence of all occurring integrals.

Lemma 6.4 *Assume $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$. Suppose f is $(i+j)$ times $\Diamond_\alpha$-differentiable. If $(1-\alpha)f^{\Diamond^{i-1}_\alpha \nabla \Delta^j} \geq 0$, then*

$$f^{\Diamond^i_\alpha \Delta^j} \geq \alpha f^{\Diamond^{i-1}_\alpha \Delta^{j+1}},$$

and if $\alpha f^{\Diamond^{i-1}_\alpha \Delta \nabla^j} \geq 0$, then

$$f^{\Diamond^i_\alpha \nabla^j} \geq (1-\alpha) f^{\Diamond^{i-1}_\alpha \nabla^{j+1}}.$$

Proof. We get

$$f^{\Diamond^i_\alpha} = \left(f^{\Diamond^{i-1}_\alpha}\right)^{\Diamond_\alpha} = \alpha f^{\Diamond^{i-1}_\alpha \Delta} + (1-\alpha) f^{\Diamond^{i-1}_\alpha \nabla}$$

so that

$$f^{\Diamond^i_\alpha \Delta^j} = \alpha f^{\Diamond^{i-1}_\alpha \Delta^{j+1}} + (1-\alpha) f^{\Diamond^{i-1}_\alpha \nabla \Delta^j} \geq \alpha f^{\Diamond^{i-1}_\alpha \Delta^{j+1}}$$

and

$$f^{\Diamond^i_\alpha \nabla^j} = \alpha f^{\Diamond^{i-1}_\alpha \Delta \nabla^j} + (1-\alpha) f^{\Diamond^{i-1}_\alpha \nabla^{j+1}} \geq (1-\alpha) f^{\Diamond^{i-1}_\alpha \nabla^{j+1}},$$

where we get used the above assumptions. ■

Lemma 6.5 *Suppose that $n \in \mathbb{N}$. Assume that f is n times $\Diamond_\alpha$-differentiable. If $(1-\alpha)f^{\Diamond_\alpha^{n-j-1}\nabla\Delta^j} \geq 0$ for all $j \in \{0, ..., n-1\}$, then*

$$f^{\Diamond_\alpha^n} \geq \alpha^n f^{\Delta^n}$$

and if $\alpha f^{\Diamond_\alpha^{n-j-1}\Delta\nabla^j} \geq 0$ for all $j \in \{0, ..., n-1\}$, then

$$f^{\Diamond_\alpha^n} \geq (1-\alpha)^n f^{\nabla^n}.$$

Proof. We use Lemma 6.4 for $i = n - j$, where $j \in \{0, ..., n-1\}$, to obtain

$$f^{\Diamond_\alpha^n} = f^{\Diamond_\alpha^n \Delta^0} \geq \alpha f^{\Diamond_\alpha^{n-1}\Delta^1} \geq ... \geq \alpha^{n-1} f^{\Diamond_\alpha^1 \Delta^{n-1}} \geq \alpha^n f^{\Diamond_\alpha^0 \Delta^n} = \alpha^n f^{\Delta^n}$$

and similarly

$$f^{\Diamond_\alpha^n} = f^{\Diamond_\alpha^n \nabla^0} \geq (1-\alpha) f^{\Diamond_\alpha^{n-1}\nabla^1} \geq ... \geq (1-\alpha)^n f^{\Diamond_\alpha^0 \nabla^n} = (1-\alpha)^n f^{\nabla^n},$$

where we get used the above assumptions. ■

Theorem 6.6 *Suppose that $m, n \in \mathbb{N}$, $\alpha \in [0,1]$, and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^n_{\Diamond_\alpha}$ with $\alpha f^{\Delta^j}(0) = (1-\alpha) f^{\nabla^j}(0) = 0$, $\alpha f^{\Diamond_\alpha^{n-j-1}\nabla\Delta^j} \geq 0$, and $(1-\alpha) f^{\Diamond_\alpha^{n-j-1}\Delta\nabla^j} \geq 0$ for all $j \in \{0, ..., n-1\}$ and $\alpha f^{\Delta^n} \geq 0$ and $(1-\alpha) f^{\nabla^n} \geq 0$, we get*

$$\begin{aligned}
& \alpha^{n(m+1)+1} \int_0^h \left| \left(\sum_{k=0}^m f^k (f^\sigma)^{m-k} \right) f^{\Delta^n} \right| (t) \Delta t \\
& + (1-\alpha)^{n(m+1)+1} \int_0^h \left| \left(\sum_{k=0}^m f^k (f^\rho)^{m-k} \right) f^{\nabla^n} \right| (t) \nabla t \\
\leq \; & h^{nm} \int_0^h \left(f^{\Diamond_\alpha^n} \right)^{m+1} (t) \Diamond_\alpha t.
\end{aligned}$$

Proof. It follows from [2, Theorem 6.30] that

$$\alpha^{n(m+1)+1} \int_0^h \left| \left(\sum_{k=0}^m f^k (f^\sigma)^{m-k} \right) f^{\Delta^n} \right| (t) \Delta t \leq \alpha^{n(m+1)+1} h^{nm} \int_0^h \left(f^{\Delta^n} \right)^{m+1} (t) \Delta t.$$

Similarly, we obtain

$$\begin{aligned}
& (1-\alpha)^{n(m+1)+1} \int_0^h \left| \left(\sum_{k=0}^m f^k (f^\rho)^{m-k} \right) f^{\nabla^n} \right| (t) \nabla t \\
\leq \; & (1-\alpha)^{n(m+1)+1} h^{nm} \int_0^h \left(f^{\nabla^n} \right)^{m+1} (t) \nabla t.
\end{aligned}$$

Using the assumptions and raising the inequalities from Lemma 6.5 to the $(m+1)$st power, we get

$$\alpha^{n(m+1)}\left(f^{\Delta^n}\right)^{m+1} \le \left(f^{\Diamond_\alpha^n}\right)^{m+1} \text{ and } (1-\alpha)^{n(m+1)}\left(f^{\nabla^n}\right)^{m+1} \le \left(f^{\Diamond_\alpha^n}\right)^{m+1}.$$

So, combining everything, we get

$$\begin{aligned} & \alpha^{n(m+1)+1}\int_0^h \left|\left(\sum_{k=0}^{m} f^k (f^\sigma)^{m-k}\right) f^{\Delta^n}\right|(t)\Delta t \\ & + (1-\alpha)^{n(m+1)+1}\int_0^h \left|\left(\sum_{k=0}^{m} f^k (f^\rho)^{m-k}\right) f^{\nabla^n}\right|(t)\nabla t \\ \le\ & h^{nm}\left\{\alpha\int_0^h \left(f^{\Diamond_\alpha^n}\right)^{m+1}(t)\Delta t + (1-\alpha)\int_0^h \left(f^{\Diamond_\alpha^n}\right)^{m+1}(t)\nabla t\right\} \\ \le\ & h^{nm}\int_0^h \left(f^{\Diamond_\alpha^n}\right)^{m+1}(t)\Diamond_\alpha t, \end{aligned}$$

which completes the proof. ■

We note that if $m = n = 1$, then the conclusion of Theorem 6.6 reduces to the conclusion of Theorem 6.1.

Corollary 6.2 *Assume $n \in \mathbb{N}$, $\alpha \in [0,1]$, and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^n_{\Diamond_\alpha}$ with $\alpha f^{\Delta^j}(0) = (1-\alpha) f^{\nabla^j}(0) = 0$, $\alpha f^{\Diamond_\alpha^{n-j-1}\nabla\Delta^j} \ge 0$, and $(1-\alpha) f^{\Diamond_\alpha^{n-j-1}\Delta\nabla^j} \ge 0$ for all $j \in \{0,...,n-1\}$ and $\alpha f^{\Delta^n} \ge 0$ and $(1-\alpha) f^{\nabla^n} \ge 0$, we get*

$$\begin{aligned} & \alpha^{2n+1}\int_0^h \left|(f+f^\sigma) f^{\Delta^n}\right|(t)\Delta t + (1-\alpha)^{2n+1}\int_0^h \left|(f+f^\rho) f^{\nabla^n}\right|(t)\nabla t \\ \le\ & h^n \int_0^h \left(f^{\Diamond_\alpha^n}\right)^2(t)\Diamond_\alpha t. \end{aligned}$$

Proof. Take $m = 1$ in Theorem 6.6. ■

Corollary 6.3 *Assume that $m \in \mathbb{N}$, $\alpha \in [0,1]$, and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\Diamond_\alpha}$ with $f(0) = 0$ $f^\nabla \ge 0$, and $f^\Delta \ge 0$, we get*

$$\begin{aligned} & \alpha^{m+2}\int_0^h \left|\left(\sum_{k=0}^{m} f^k (f^\sigma)^{m-k}\right) f^{\Delta}\right|(t)\Delta t \\ & +(1-\alpha)^{m+2}\int_0^h \left|\left(\sum_{k=0}^{m} f^k (f^\rho)^{m-k}\right) f^{\nabla}\right|(t)\nabla t \\ \le\ & h^m \int_0^h \left(f^{\Diamond_\alpha}\right)^{m+1}(t)\Diamond_\alpha t. \end{aligned}$$

Proof. Take $n = 1$ in Theorem 6.6. ■

CHAPTER 7

GENERALIZED DIAMOND-ALPHA DYNAMIC OPIAL INEQUALITIES

In this chapter we use [83]. We establish some dynamic Opial type diamond-alpha inequalities in time scales.

In this chapter, the above given Theorem 6.1 is improved in the sense of removing the restriction given by the condition $\alpha(1-\alpha)f^{\Delta}f^{\nabla} \geq 0$ as well as the left hand side of inequality (6.1) refined to a compact form, being composed of a single diamond-alpha integral. In this sense, the next theorem, Theorem 7.1, and its consequences extend and unify the previously obtained delta and nabla Opial dynamic inequalities in a more accurate way than that given by Theorem 6.1.

7.1 IMPROVED DIAMOND-ALPHA OPIAL INEQUALITIES

Theorem 7.1 *Assume $\mathbb{T}$ is a time scale. For $\Diamond_\alpha$-differentiable $f : [\rho(0), \sigma(h)]_{\mathbb{T}} \to \mathbb{R}$ with $f \in C^1$ and $f(0) = 0$ we get*

$$\int_0^h \left|\left(f^2\right)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t \leq h\int_0^h \left|f^{\Diamond_\alpha}\right|^2(t)\Diamond_\alpha t, \tag{7.1}$$

with equality when $f(t) = ct$.

Proof. Starting with the left side of (7.1) and using the fact that

$$ff^{\Diamond_\alpha} = \alpha ff^{\Delta} + (1-\alpha)ff^{\nabla}$$

we obtain,

$$\begin{aligned}\int_0^h \left|\left(f^2\right)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t &= \int_0^h \left|ff^{\Diamond_\alpha} + \alpha f^{\sigma}f^{\Delta} + (1-\alpha)f^{\rho}f^{\nabla}\right|(t)\Diamond_\alpha t \\ &= \alpha\int_0^h \left|ff^{\Diamond_\alpha} + \alpha f^{\sigma}f^{\Delta} + (1-\alpha)f^{\rho}f^{\nabla}\right|(t)\Delta t \\ &\quad +(1-\alpha)\int_0^h \left|ff^{\Diamond_\alpha} + \alpha f^{\sigma}f^{\Delta} + (1-\alpha)f^{\rho}f^{\nabla}\right|(t)\nabla t\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha\int_0^h \left|ff^{\Diamond_\alpha}\right|(t)\Delta t+\alpha^2\int_0^h \left|f^\sigma f^\Delta\right|(t)\Delta t\\
&\quad+\alpha(1-\alpha)\int_0^h \left|f^\rho f^\nabla\right|(t)\Delta t+(1-\alpha)\int_0^h \left|ff^{\Diamond_\alpha}\right|(t)\nabla t\\
&\quad+\alpha(1-\alpha)\int_0^h \left|f^\sigma f^\Delta\right|(t)\nabla t+(1-\alpha)^2\int_0^h \left|f^\rho f^\nabla\right|(t)\nabla t\\
&= \alpha\int_0^h \left|\alpha ff^\Delta+(1-\alpha)ff^\nabla\right|(t)\Delta t+\alpha^2\int_0^h \left|f^\sigma f^\Delta\right|(t)\Delta t\\
&\quad+\alpha(1-\alpha)\int_0^h \left|f^\rho f^\nabla\right|(t)\Delta t\\
&\quad+(1-\alpha)\int_0^h \left|\alpha ff^\Delta+(1-\alpha)ff^\nabla\right|(t)\nabla t\\
&\quad+\alpha(1-\alpha)\int_0^h \left|f^\sigma f^\Delta\right|(t)\nabla t+(1-\alpha)^2\int_0^h \left|f^\rho f^\nabla\right|(t)\nabla t\\
&= \alpha^2\int_0^h \left(|f|+|f^\sigma|\right)\left|f^\Delta\right|(t)\Delta t\\
&\quad+\alpha(1-\alpha)\int_0^h \left(|f|+|f^\rho|\right)\left|f^\nabla\right|(t)\Delta t\\
&\quad+\alpha(1-\alpha)\int_0^h \left(|f|+|f^\sigma|\right)f^\Delta(t)\Delta t\\
&\quad+(1-\alpha)^2\int_0^h \left(|f|+|f^\rho|\right)\left|f^\nabla\right|(t)\Delta t\\
&= \alpha^2\int_0^h \left[\left(|f|+|f^\sigma|\right)\left|f^\Delta\right|\right](t)\Delta t\\
&\quad+(1-\alpha)^2\int_0^h \left[\left(|f|+|f^\rho|\right)\left|f^\nabla\right|\right](t)\nabla t\\
&\quad+\alpha(1-\alpha)\int_0^h \left[\left(|f|+|f^\sigma|\right)\left|f^\Delta\right|\right](t)\nabla t\\
&\quad+\alpha(1-\alpha)\int_0^h \left[\left(|f|+|f^\rho|\right)\left|f^\nabla\right|\right](t)\Delta t
\end{aligned}
$$

Letting $\gamma(t)=\int_0^t \left|f^{\Diamond_\alpha}(s)\right|\Diamond_\alpha s$, we then have $y^\Delta=\left|f^\Delta\right|$, $y^\nabla=\left|f^\nabla\right|$ and since

$$
\gamma(t)=\int_0^t \left|f^{\Diamond_\alpha}(s)\right|\Diamond_\alpha s\geq\left|\int_0^t \left|f^{\Diamond_\alpha}(s)\right|\Diamond_\alpha s\right|=|f(t)-f(0)|=|f(t)|
$$

giving $|f|\leq y$ as long as $\Diamond_\alpha$-anti derivative of f exists. So

$$
\begin{aligned}
&\leq \alpha^2\int_0^h \left[\left(\gamma+\gamma^\sigma\right)\gamma^\Delta\right](t)\Delta t+(1-\alpha)^2\int_0^h \left[\left(\gamma+\gamma^\rho\right)\gamma^\nabla\right](t)\nabla t\\
&\quad+\alpha(1-\alpha)\int_0^h \left[\left(\gamma+\gamma^\sigma\right)\gamma^\Delta\right](t)\nabla t+\alpha(1-\alpha)\int_0^h \left[\left(\gamma+\gamma^\rho\right)\gamma^\nabla\right](t)\Delta t
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^h \left(\gamma^2\right)^{\Diamond_\alpha}(t)\Diamond_\alpha t \\
&= \gamma^2(h) - \gamma^2(0) \\
&= \left[\int_0^h \left|f^{\Diamond_\alpha}(s)\right| \Diamond_\alpha s\right]^2 \\
&\leq h \int_0^h \left|f^{\Diamond_\alpha}(s)\right|^2 \Diamond_\alpha s
\end{aligned}
$$

where we get used Hölder's inequality for $h(x) = 1$ and $p = 2$. ■

We illustrate the result obtained in Theorem 7.1 with an example.

Example 7.1 *Suppose $\mathbb{T} = [-1, 4] \cap \mathbb{Z}$ is the time scale and $f : \mathbb{T} \to \mathbb{R}$ with $\alpha = \frac{1}{2}$,*

$$
f(t) = \begin{cases} -1 & , \; if \quad t = \pm 1 \\ 0 & , \; if \quad t = 0 \\ 5 & , \; if \quad t \geq 2. \end{cases}
$$

Then,

$$
\begin{aligned}
\int_0^h \left|\left(f^2\right)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t &= \frac{1}{2}\int_0^3 \left|(f^2)^{\Delta}(t) + (f^2)\nabla(t)\right| \Diamond_\alpha t \\
&= \frac{1}{4}\left[\int_0^3 \left|(f^2)^{\Delta}(t) + (f^2)\nabla(t)\right| \Delta(t)\right. \\
&\quad \left. + \int_0^3 \left|(f^2)^{\Delta}(t) + (f^2)\nabla(t)\right| \nabla(t)\right] \\
&= \frac{1}{4}[\sum_{t=0}^{2} |(f(t+1) + f(t))\,(f(t+1) - f(t)) \\
&\quad + (f(t) + f(t-1))\,(f(t) - f(t-1))| \\
&\quad + \sum_{t=1}^{3} |(f(t+1) + f(t))\,(f(t+1) - f(t)) \\
&\quad + (f(t) + f(t-1))\,(f(t) - f(t-1))|] \\
&= \frac{1}{4}\left[\sum_{t=0}^{2} \left|\left(f^2(t+1) - f^2(t-1)\right)\right| + \sum_{t=1}^{3} \left|\left(f^2(t+1) - f^2(t-1)\right)\right|\right] \\
&= \frac{49}{2}
\end{aligned}
$$

and similarly,

$$
\int_0^h \left|\left(f^{\Diamond_\alpha}\right)^2\right|(t)\Diamond_\alpha t = \frac{1}{4}\int_0^3 \left(f^{\Delta}(t) + \left(f^{\nabla}(t)\right)\right)^2 \Diamond_\alpha t
$$

$$
\begin{aligned}
&= \frac{1}{8}\left[\int_0^3 \left(f^{\Delta}(t)+\left(f^{\nabla}(t)\right)\right)^2 \Delta t+\int_0^3 \left(f^{\Delta}(t)+\left(f^{\nabla}(t)\right)\right)^2 \nabla t\right]\\
&= \frac{1}{8}\left[\sum_{t=0}^{2}\left(f(t+1)-\left(f(t-1)\right)\right)^2+\sum_{t=1}^{3}\left(f(t+1)-\left(f(t-1)\right)\right)^2\right]\\
&= \frac{63}{4}
\end{aligned}
$$

and so we get $3.\frac{63}{4}=\frac{189}{4}$.

Hence the inequality in Theorem 7.1 is valid.

The next two theorems are extensions of Theorem 7.1.

Theorem 7.2 *Assume* $f:[\rho(0),\sigma(h)]_{\mathbb{T}}\to\mathbb{R}$ *is* $\Diamond_\alpha$*-differentiable. Then*

$$\int_0^h \left|\left(f^2\right)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t \le \gamma \int_0^h \left|f^{\Diamond_\alpha}\right|^2 \Diamond_\alpha + 2\beta \int_0^h \left|f^{\Diamond_\alpha}\right| \Diamond_\alpha t,$$

where

$$\beta=\max\{|f(0)|,|f(h)|\},\ \gamma=\min_{u\in[0,h]\cap\mathbb{T}}\max\{u,h-u\}.$$

Proof. We consider $\gamma(t)=\int_0^t\left|f^{\Diamond_\alpha}(s)\right|\Diamond_\alpha s$ and $z(t)=\int_t^h\left|f^{\Diamond_\alpha}(s)\right|\Diamond_\alpha s$. Then we get, $z^{\Diamond_\alpha}=-\left|f^{\Diamond_\alpha}\right|$, $z^{\Diamond_\alpha}=-\left|f^{\Diamond_\alpha}\right|$, $y^{\Delta}=\left|f^{\Delta}\right|$, $y^{\Delta}=\left|f^{\Delta}\right|$, $y^{\nabla}=\left|f^{\nabla}\right|$, $y^{\nabla}=\left|f^{\nabla}\right|$ yielding

$$
\begin{aligned}
|f(t)| &\le |f(t)-f(0)|+|f(0)|\\
&= \left|\int_0^t f^{\Diamond_\alpha}(s)\Diamond_\alpha s\right|+|f(0)|\\
&\le \int_0^t \left|f^{\Diamond_\alpha}(s)\right|\Diamond_\alpha s+|f(0)|\\
&= \gamma(t)+|f(0)|
\end{aligned}
$$

and similarly, $|f(t)|\le z(t)+|f(h)|$.

Assume that $u\in[\rho(0),\sigma(h)]_{\mathbb{T}}$. Then

$$
\begin{aligned}
\int_0^h \left|(f^2)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t &\le \int_0^u \left[\left(\gamma+|f(0)|\right)\gamma^{\Diamond_\alpha}+\alpha\left(\gamma^{\sigma}+|f(0)|\right)\gamma^{\Delta}\right.\\
&\quad \left.+(1-\alpha)\left(\gamma^{\rho}+|f(0)|\right)\gamma^{\nabla}\right]\Diamond_\alpha t\\
&= \gamma^2(u)-\gamma^2(0)+2\,|f(0)|\,\gamma(u)\\
&\le u\int_0^u \left|f^{\Diamond_\alpha}\right|^2\Diamond_\alpha t+2\,|f(0)|\int_0^u\left|f^{\Diamond_\alpha}(t)\right|\Diamond_\alpha t,
\end{aligned}
$$

where we have used Hölder's inequality for $h(x)=1$ and $p=2$. Therefore we have

$$
\begin{aligned}
\int_u^h \left|(f^2)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t &\le z^2(u)-z^2(h)+2\,|f(h)|\,z(u)\\
&\le (h-u)\int_u^h\left|f^{\Diamond_\alpha}\right|^2\Diamond_\alpha t+2\,|f(h)|\int_u^h\left|f^{\Diamond_\alpha}\right|\Diamond_\alpha t.
\end{aligned}
$$

By putting $\gamma = \min\limits_{u\in[0,h]\cap\mathbb{T}} \max\{u, h-u\}$, $\beta = \max\{|f(0)|,\ |f(h)|\}$ and adding the above two inequalities, we obtain

$$\int_0^h \left|(f^2)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t \le \gamma \int_0^h \left|f^{\Diamond_\alpha}\right|^2 \Diamond_\alpha + 2\beta \int_0^h \left|f^{\Diamond_\alpha}\right| \Diamond_\alpha t. \blacksquare$$

Theorem 7.3 *Suppose that $f : [\rho(0), \sigma(h)]_{\mathbb{T}} \to \mathbb{R}$ is $\Diamond_\alpha$-differentiable with $f(0) = f(h) = 0$. Then*

$$\int_0^h \left|\left(f^2\right)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t \le \gamma \int_0^h \left|f^{\Diamond_\alpha}\right|^2 \Diamond_\alpha,$$

where γ is given in Theorem 7.2.

Proof. We consider $\gamma(t) = \int_0^t \left|f^{\Diamond_\alpha}(s)\right| \Diamond_\alpha s$ and $z(t) = \int_t^h \left|f^{\Diamond_\alpha}(s)\right| \Diamond_\alpha s$. Then,

$$\gamma^{\Diamond_\alpha} = \left|f^{\Diamond_\alpha}\right|,\ z^{\Diamond_\alpha} = -\left|f^{\Diamond_\alpha}\right|,\ y^{\Delta} = \left|f^{\Delta}\right|,\ y^{\nabla} = \left|f^{\nabla}\right|$$

and

$$\begin{aligned} |f(t)| &\le |f(t) - f(0)| + |f(0)| = \left|\int_0^t f^{\Diamond_\alpha}(s)\right| \Diamond_\alpha s + |f(0)| \\ &\le \int_0^t \left|f^{\Diamond_\alpha}(s)\right| \Diamond_\alpha s + |f(0)| = \gamma(t) + |f(0)| \end{aligned}$$

and similarly

$$|f(t)| \le z(t) + |f(h)|.$$

Assume that $u \in [\rho(0), \sigma(h)]_{\mathbb{T}}$, then

$$\begin{aligned} \int_0^h \left|\left(f^2\right)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t &\le \int_0^u \left[(\gamma + |f(0)|)\gamma^{\Diamond_\alpha} + \alpha\left(\gamma^{\sigma} + |f(0)|\right)\gamma^{\Delta}\right. \\ &\quad \left.+(1-\alpha)\left(\gamma^{\rho} + |f(0)|\right)\gamma^{\nabla}\right]\Diamond_\alpha \\ &= \gamma^2(u) - \gamma^2(0) + 2|f(0)|\gamma(u) \\ &\le u\int_0^u \left|f^{\Diamond_\alpha}\right|^2 \Diamond_\alpha t + 2|f(0)| \int_0^u \left|f^{\Diamond_\alpha}(t)\right| \Diamond_\alpha t \end{aligned}$$

where we have used Hölder's inequality for $h(x) = 1$ and $p = 2$. Therefore we have

$$\begin{aligned} \int_u^h \left|(f^2)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t &\le z^2(u) - z^2(h) + 2|f(h)|z(u) \\ &\le (h-u)\int_u^h \left|f^{\Diamond_\alpha}\right|^2 \Diamond_\alpha t + 2|f(h)| \int_u^h \left|f^{\Diamond_\alpha}\right| \Diamond_\alpha t. \end{aligned}$$

By putting $\gamma = \min\limits_{u\in[0,h]\cap\mathbb{T}} \max\{u, h-u\}$ and adding the above two inequalities, we obtain

$$\int_0^h \left|(f^2)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t \le \gamma \int_0^h \left|f^{\Diamond_\alpha}\right|^2 \Diamond_\alpha. \blacksquare$$

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