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FIRAT UNIVERSITY
THE INSTITUTE OF NATURAL AND APPLIED SCIENCES

OPTICAL AND OTHER SOLITON SOLUTIONS, LIE POINT SYMMETRIES, CONSERVATION LAWS AND MODULATION INSTABILITY ANALYSIS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Aliyu Isa Aliyu
(151121201)

Doctorate Thesis
Department of Mathematics
Thesis Supervisor: Prof. Dr. Mustafa İNÇ

OCTOBER-2018

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Thesis Supervisor	**Prof. Dr. Mustafa İNÇ (FÜ)**	
Jury Member	**Prof. Dr. Hasan BULUT**	
Jury Member	**Doç. Dr. Reşat YILMAZER**	
Jury Member	**Doç. Dr. ALİ AKGÜL**	
Jury Member	**Dr. Öğr. Üyesi Zeliha KÖRPINAR**	

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DEDICATION

This thesis is dedicated to my late father Alh. Isa Aliyu Danzabuwa, my mother Haj. Hafsat Abdullahi and Senator Engr. Rabiu Musa Kwankwaso.

Aliyu Isa Aliyu
Elazığ-2018

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ABSTRACT

OPTICAL AND OTHER SOLITON SOLUTIONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND SYSTEMS

This thesis deals with the investigation of Lie point symmetries, optical and other solitons, conservation laws (Cls) and modulation instability analysis (MI) of some of nonlinear Schrodinger equations (NLSEs). We present a detailed analysis for the existence of dark, bright, dark-bright or combined and singular soliton solutions of some NLSEs. Furthermore, the soliton solutions and Cls of some nonlinear partial differential equations (NLPDEs) are investigated. For some of the NLPDEs, we present a detailed analysis for the existence of topological, non-topological, hyperbolic, function, singular and periodic soliton solutions. Seven different integration schemes are used to study the nonlinear models, namely; the complex envelope ansatz, sine-Gordon expansion method (SGEM), the Riccati Bernoulli sub-ODE (RBSO), modified F-expansion, the generalized tanh, generalized projective Riccati equation, Jaccobi elliptic function ansatz and the undetermined coefficient methods. The problem on nonlinear self-adjointness of several nonlinear models has not been studied in previous time. In this thesis, we solve this problem for several nonlinear models and find an explicit form of the differential substitution satisfying the nonlinear self-adjoint condition. Then we use this fact to construct a set of conserved vectors using the classical symmetries admitted by the models and by invoking the general Cls theorem due to Ibragimov. Moreover, we investigate MI of some NLSEs by using the concept of linear stability analysis. Some figures are plotted to show the physical interpretations of the obtained results.

Keywords: Solitons, Modulation Instability, Conservation Laws, Lie Symmetries, optical soliton.

ÖZET

BAZI LİNEER OLMAYAN KISMİ DİFERANSİYEL DENKLEMLERİNİN OPTİK VE DİĞER SOLİTONLARI, LİE NOKTA SİMETRELERİ, KORUNUM KANUNLARI VE MODÜLASYON KARARSIZLIK ANALİZİ

Bu tez, bazı lineer olmayan Schrödinger denklemlerinin (NLSE) Lie simetrileri, optik ve diğer solitonları, korunum kanunları (Cls) ve modülasyon kararsızlık analizi (MI) araştırması ile ilgilidir. Bazı NLSE'lerin koyu, parlak, koyu parlak veya kombine ve tekil soliton çözümlerinin varlığı için detaylı bir analiz sunuyoruz. Ayrıca, bazı lineer olmayan kısmi diferansiyel denklemlerin (NLPDE) soliton çözümleri ve korunum kanunları incelenmiştir. Bazı NLPDE'ler için, topolojik, topolojik olmayan, hiperbolik, fonksiyon, tekil ve periyodik soliton çözümlerinin varlığı için detaylı bir analiz sunuyoruz. Doğrusal olmayan modelleri incelemek için yedi farklı entegrasyon şeması yani; karmaşık zarf tahmin yürütme hesaplaması, sinüs-Gordon açılım yöntemi (SGEM), Riccati Bernoulli alt-ODE (RBSO), değiştirilmiş F-genişlemesi, genelleştirilmiş tanh, genelleştirilmiş projektif Riccati denklemi, Jaccobi eliptik fonksiyonu tahmin yürütme hesaplaması ve belirsiz katsayı yöntemleri kullanılır. Bazı lineer olmayan modellerin lineer olmayan kendiliğinden eşleşmesi (self-adjointliği) ile ilgili problem önceki yıllarda incelenmemiştir. Bu tezde, bazı lineer olmayan model için bu problemi çözüyoruz ve lineer olmayan kendi kendine eşleşme koşulunu sağlayan diferansiyel yerine koyma metodunun açık bir formunu buluyoruz. Sonra bu gerçeği, modeller tarafından kabul edilen klasik simetrileri kullanarak ve Ibragimov'un oluşturduğu genel Cls teoremini kullanarak bir dizi korunmuş vektörler inşa etmek için kullanırız. Ayrıca, lineer kararlılık analizi kavramını kullanarak bazı NLSE'lerin MI'larını araştırdık. Elde edilen sonuçların fiziksel yorumlarını göstermek için bazı şekiller çizilmiştir.

Anahtar Kelimeler: Solitonlar, Modülasyon Kararsızlığı, Korunum Kanunları, Lie Simetrileri.

LIST OF FIGURES

SYMBOLS LIST

α	**:** Alpha
β	**:** Beta
γ	**:** Gamma
θ	**:** Theta
ξ	**:** Xi
I	**:** imaginary
δ	**:** delta
ϕ	**:** phi
Γ	**:** sigma
P_0	**:** nu
η	**:** eta
μ	**:** mu
τ	**:** tau
L	**:** Lagrangian
ρ	**:** rho
λ	**:** lambda
ω	**:** omega
Ω	**:** Omega
Γ	**:** Gamma

1. INTRODUCTION

A system is said to be nonlinear if its output is not linearly proportional to its input; on the basis of this definition, one can say that most of the systems in this universe qualify to be nonlinear [1]. The science which deals with nonlinear systems is known as nonlinear science. In the past few decades, nonlinear science has emerged as a tool to study all those complex natural Phenomeno which cannot be studied completely by linear science [2].

The study of nonlinear system means to study the nonlinearity present in it. Nonlinearity plays a role in dynamics of several physical phenomena like in laser physics, electronic circuits nonlinear mechanical vibrations, population dynamics, astrophysics, plasma physics, chemical reactions, nonlinear wave motions, heart beat, nonlinear diffusion, time-delay processes etc. Nonlinearity in any system make the system more complex and it become very difficult to study. Hence a nonlinear system exhibits a sensitive dependence on initial conditions [3]. However, linear systems are generally gradual, smooth and regular, common example of linear system are slowly flowing streams, engines working at low power, slowly reacting chemicals, etc. Any system with large input generally show nonlinear behavior. For example, the behavior of a spring is linear for small displacement, but if the initial displacement is large the spring shows nonlinear behavior. In similar way, for small initial displacement simple pendulum behaves as linear system however as the initial displacement become large enough, its motion become nonlinear [4].

In every first encounter of methods of solving ordinary differential equations, one understands them as tools that are designed to derive the solutions of some equations, which includes the variable separable equations, the exact analytic solutions, homogenous or non-homogenous equations [1].

To obtain the solution of NLPDEs is a challenging aspect of nonlinear dynamics. In recent times, the research in the field of finding exact solution of NLPDEs have reached an advanced stage due to development of several mathematical software and due to advancement in high speed computing [2]. Similarly, NLSE plays a significant role in nonlinear fiber optics, condensed matter physics, plasma physics, etc [4, 5]. The study of exact solutions of NLPDEs such as solitary waves and periodic solution plays a significant role in illustration of several natural phenomenon. This thesis involves the study of NLSE and its variants in contexts of nonlinear optics.

In the 19th century, the great scientist Sophus Lie made a great discovery on some special cases of a general technique of symmetries that is based on invariance of DEs that are under a continuous set of group of symmetries [6]. These set of groups now generally known as the Lie Groups have now gone a long way in making an impact on all fields of sciences, especially mathematics both applied and pure, as well as engineering and physics. Some of its applications include the field of algebraic topology, differential geometry, invariant theory, classical mechanics and so on.

In 1870, he developed the theory of ”finite and continuous groups”. Lie devoted his mathematical career to the development and application of the theory of continuous groups. These groups are now known as ”Lie groups” [6]. Lie gave an algorithm to find all infinitesimals generators of point transformations and, more generally, contact transformations admitted by a given differential equation. Importantly, for a given DE, the basic applications of Lie groups only require knowledge of the admitted infinitesimal generators [7].

Generally, a symmetry group of a differential equations can be seen as a group which tends to transform the solutions of the systems to other solutions without actually changing or affecting the original solutions [6]. Typical examples includes the group translations and rotations and the group of scaling symmetries. The search for these group symmetries can be done using recent computational methods.

Once the group symmetries of an equation are determined, several applications are available to utilize it. Such properties can be used to derive new solutions of the model. Therefore, symmetry group provides a means of classifying different classes of solutions. Another good application of these groups is that they can be used to determine which type of differential equation admits some uniqueness of its characteristics. This is achieved with the infinitesimal methods using the theory of invariants of differential equations [6]. In the study of symmetries of ODEs, invariance under one parameter symmetry group implies that the order of the equation can be reduced by a step of one, which give rise to recovering the solutions of the original equation. For systems of NLPDEs, the acquired symmetry groups generally is of no importance in establishing the general solution of the system. But rather one can employ the general symmetry groups to determine the some special types of solutions that are invariant under some subgroups. These group of invariant solutions are established by finding the solution of the reduced ODEs that involves a fewer number of independent variables which is easier to solve. For example, a solution of NLPDEs under some parameter symmetry group, are obtained by solving systems of ODEs using any of the available methods which include the power series method, the classical symmetry, travelling waves solutions etc [7]. As such, a solution of the obtained differential equations will satisfy the original partial differential equation.

It is a very well known fact that several differential equations have a number of features, that is, their symmetries and Cls. Cls are relevant in the analysis of NLPDEs particularly in the studies of the stability and uniqueness solutions [6].

In 1918, E. Noether proved two very remarkable theorems that relates symmetry groups of variational integrals with its associated Euler-Lagrange equations. In her first theorem, she shows how each single parameter variational symmetry group give rise to a conservation laws of the Euler-Lagrange equation [6].

Many NLPDEs admit infinitely many Cls. Cls are significant in the solution and reductions of NLPDEs [8]. In the last few decades, many powerful techniques have been put forward to derive the Cls of NLPDEs such as the multiplier approach a technique [9, 10] and the new conservation theorem [11- 14].

There are several applications of Cls. Some Cls are fundamental laws of physics (e.g., conservation of momentum, mass and energy) while others facilitate the analysis of the PDEs [6]. They assist in obtaining reductions and solutions of NLPDEs. A large number of Cls for a nonlinear model is a predictor of integrability of an equation. Without these conserved vectors (integrals of motion), an understanding of the problem would be incomplete [8].

In nonlinear dynamics, much attention has been given to the study of MI of nonlinear Schrodinger equations [15].

1.1. Solitary waves and solitons

The solitary wave was first observed in 1834 by J. Scott Russel during his experiment on efficient design of canal boat [16]. During experiment he saw a long water wave propagating without change in shape. He named this wave as ”Great Wave” of translation or ”Solitary Wave” and performed further investigations to study the nature of this wave. The speed of such wave is given by

$$v = \sqrt{g(h + a)}, \tag{1.1}$$

where g is acceleration due to gravity, h is depth of water channel, a is maximum amplitude of solitary wave. He published this work in the British Association in 1844 with the name Report on Waves. His description of solitary waves contradicted the theories of water waves according to G. B. Airy and G. G. Stokes; they raised questions on the existence of Russells solitary waves and conjectured that such waves cannot propagate in a liquid medium without a change of form. Despite the mathematical theory, the experimental evidence in favor of solitary waves was convincing. It was until the 1870s that Russells prediction was finally and independently confirmed by both J. Boussinesq in 1871 and Lord Rayleigh in 1876.

In 1895, D. J. Korteweg and G. de Veries formulated a mathematical model for solitary wave known as KdV equation, which can be express as

$$q_t + qq_x + \delta q_{xxx} = 0, \tag{1.2}$$

where q is amplitude of wave having functional dependence on space x and time t coordinate, second term is nonlinear term and δ is the coefficient of nonlinear term Dispersive term is responsible for the broadning of pulse while nonlinear term leads to the steepening effect. In 1965, Zabusky and Kruskal [17] solved the KdV equation numerically found that solitary wave solutions interacted with each other elastically. As a result of this particle-like property, which they termed these solitary wave solutions as solitons. When nonlinearity is balanced by dispersion then waveform take the shape of permanent form known as soliton. This research work attracted the attention of large number of research groups all over the world and hence soliton concept was widely accepted. With the acceptance of concept of solitons, a lot of research work was undergone to obtain the soliton solutions which leads to development of several mathematical methods. Figure. 1.1 shows the modern day re-creation of the soliton observed by Russell in 1834.

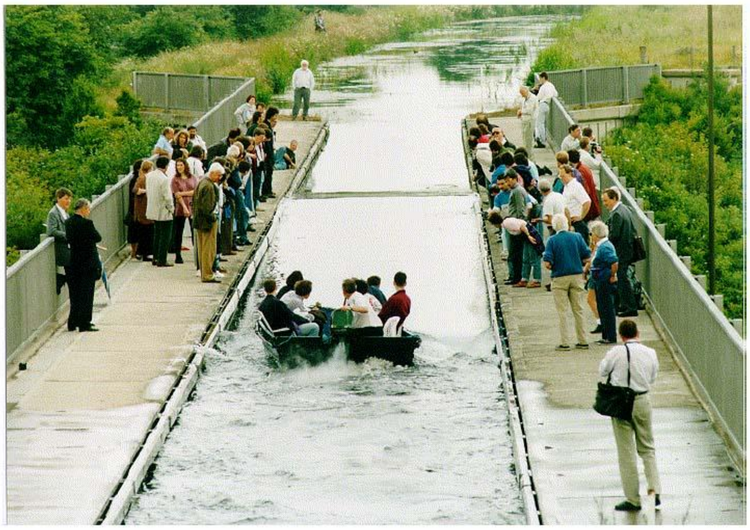

Figure 1.1: Modern day re-creation of the soliton observed by Russell in 1834. [Union Canal near Edinburgh, Scotland, July 1995, at a conference on nonlinear waves at Heriot-Watt University

1.1.1. Solitary wave

In some media, such as layer of water or an optical fiber, under suitable conditions the widening of wave packet due to dispersion could be balanced exactly by narrowing effect of nonlinearity of medium [17]. In these cases, it is possible to have localized waves, propagating with constant velocity and undistorted shape, often known as solitary waves.

1.1.2. Solitons

Solitons are those solitary waves which retain their individuality under collisions and eventually travel with their original shapes and speeds. This property looks like interaction between free particles. Due to this particle nature of solitary waves, they are named as 'Solitons' [18]. The solitons are mainly of many such as the bright, dark, dark-bright also called combined, dark-singular and singular solitons. The solitons pulse, intensity of which is larger than the background is known as bright solitons. It's physical appearance is shown in Figure. 1.2. Soliton with lower intensity than background is known as dark solitons. The evolution of dark soliton is shown in Figure. 1.3. While the soliton combining the properties of dark and bright solitons called the dark-bright soliton is shown in Figure. 1.4.

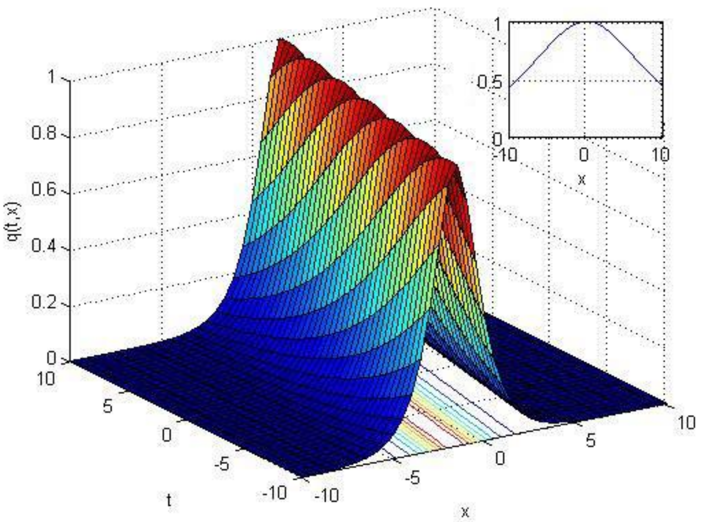

Figure 1.2: Evolution of bright soliton

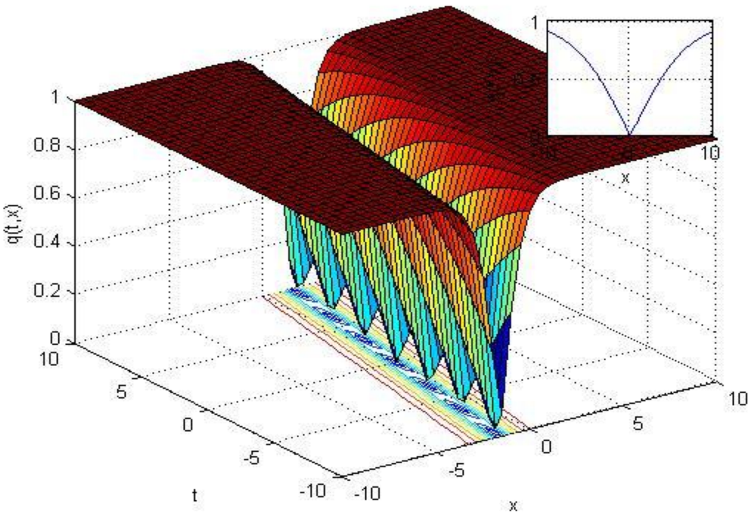

Figure 1.3: Evolution of dark soliton

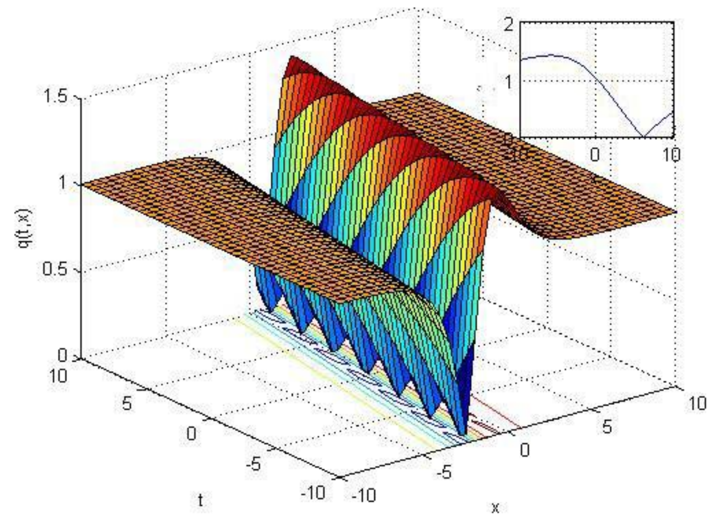

Figure 1.4: Evolution of dark-bright soliton

1.1.3. Properties of solitary waves

• Solitary waves do not survive collisions.

• The speed of the wave depends on the size of the wave, and its width depends on the depth of channel.

• The waves can travel over large distances without change in size and shape , and are stable.

1.1.4. Applications of solitary wave and solitons

(a) Fluid dynamics

Solitary wave was first noticed and studied by Russell. Solitary waves also arise in deep water Vladimir Zakharov [19] in 1968. Hence solitons find application in the study of various problems of fluid dynamics.

(b) Fiber optics

An optical soliton can propagate without distortion over long distances. This feature of optical soliton helpful in high speed communication through an optical fiber. Apart from the field of communications, solitons also find application soliton photonic switches, logic gates, fiber laser, timing jitter , pulse compression and pulse amplification [20, 21].

(c) Plasma Physics

Plasma consists of large number of charged ions. In perturbation of charge density, the local ion density can be studied by using KdV equation and Kadomtsev- Petviashvili (KP) equation. These equations admits soliton solutions. Soliton concept penetrated into plasma physics in the late 50s. Later on number of researchers showed interest in study of soliton solutions for plasmas [16, 17].

1.2. Outline of Thesis

The organization of this thesis is outlined as follows. Part 2 includes the full description and discussion of the techniques that will be applied to study the nonlinear models. Some definitions and theorems are also given in this part. Firstly, several forms soliton solutions dark, dark-singular, bright, dark-bright and singular solutions will be investigated. Secondly, we aim to give some concepts of the Lie point symmetries and the a detail of its mathematical formulation. In the third section, we provide the description of the Cls method due to Ibragimov. Part 2 concludes by giving a brief introduction of the concept of MI.

In part 3, we will study the optical solitons and Cls of three NLSE. In chapter 4, we will study the optical solitons and MI of three different NLSEs. In part 5, we aim to investigate the solitons and conservation laws of some NLEEs. In part 6, the general discussions, conclusions are given.

1.3. Papers included in the thesis

1. Journal of modern optics **64,** 2273-2280 (2017).
2. Modern Physics Letters B **31,** 1750163 (2017).
3. Superlattices and Microstructures **111,** 546-555 (2017).
4. Optik **155,** 257-266 (2017).
5. Superlattices and Microstructures **113,** 745-753 (2018).
6. Superlattices and Microstructures **113,** 319-327 (2018).
7. Superlattices and Microstructures **112,** 628-638 (2017).
8. Modern Physics Letters B **32** , 1850014 (2018).
9. Waves in Complex and Random Media.(2018). Article in press.
10. Optik **142,** 665-673 (2017)

2. FUNDAMENTAL CONCEPTS AND METHODOLOGY

In this part, we present the fundamental mathematical concepts that will be used throughout this thesis. The techniques that will be applied in subsequent chapters to study the nonlinear models will be briefly described.

2.1. Complex Envelope Ansatz Method

This method was put forward by Li. et al. [22] to investigate the solitary wave solutions of for the higher order nonlinear Shrödinger (HNLS). The method also provide an expression for computing nonlinear phase shift. Since its implementation, it has attracted the attention of several researchers [23, 24, 25]. To describe the technique, we consider a NLPDE given by

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{tx}, \ldots) = 0, \tag{2.1}$$

where P is a polynomial in $q(x,t)$. We consider the following transformation

$$q(x,t) = A(x,t)e^{i\phi(x,t)}, \quad \phi(x,t) = -kx + \omega t + \theta, \tag{2.2}$$

where $A(x,t)$ is a function. ϕ represents the linear phase shift, k represents the wave number and ω is the wave frequency, while θ represents the phase constant. The complex envelope ansatz assumes the following form

$$A(x,t) = \mathrm{i}\beta + \lambda \tanh[\eta(x - vt)] + \mathrm{i}\rho \operatorname{sech}[\eta(x - vt)], \tag{2.3}$$

where η is the pulse width and v is the velocity. If $\beta = \lambda = 0$ in (2.3), a bright soliton can be derived. But if $\rho = 0$, the solution given in (2.3) decompose to a dark soliton. The amplitude of $A(t,x)$ is given by

$$|A(x,t)| = (\lambda^2 + \beta^2 + 2\beta\rho \operatorname{sech}[\eta(x - vt)] + (\rho^2 - \lambda^2)\operatorname{sech}^2[\eta(x - vt)])^{\frac{1}{2}}, \tag{2.4}$$

while the nonlinear phase shift ϕ_{NL} is represented by

$$\phi_{NL} = \arctan\left[\frac{\beta + \rho \operatorname{sech}[\eta(x - vt)]}{\lambda \tanh[\eta(x - vt)]}\right]. \tag{2.5}$$

Substituting the ansatz (2.3) into the nonlinear equation (2.1), one can acquire the solutions based on the acquired coefficients.

2.2. Sine-Gordon expansion method (SGEM)

Here, we will describe the [25, 26]. Consider the following equation

$$q_{xt} = \alpha \sin(q), \tag{2.6}$$

with α been a constant. Applying

$$q(x,t) = u(\xi), \quad \xi = \eta(x + vt), \tag{2.7}$$

Putting (2.7) into (2.6) yields

$$u'' = \frac{\alpha}{v\eta^2} \sin(u(\xi)). \tag{2.8}$$

(2.8) is simplified to

$$[(\frac{u}{2})']^2 = \frac{\alpha}{v\eta^2} \sin^2[\frac{u(\xi)}{2}] + C, \tag{2.9}$$

with C being an integration constant. Setting $C = 0$, $w(\xi) = \frac{u(\xi)}{2}$ and $f^2 = \frac{\alpha}{v\eta^2}$, (2.10) reduces to

$$w'(\xi)^2 = f^2 \sin^2(w(\xi)), \tag{2.11}$$

which can be written as

$$w'(\xi) = f \sin(w(\xi)). \tag{2.12}$$

Letting $f = 1$ in (2.12), we get

$$w'(\xi) = \sin(w(\xi)). \tag{2.13}$$

(2.13) possess the solutions

$$\sin(w(\xi)) = \text{sech}(\xi) \quad \text{or} \cos(w(\xi)) = \tanh(\xi), \tag{2.14}$$

$$\sin(w(\xi)) = i\text{csch}(\xi) \quad \text{or} \cos(w(\xi)) = \coth(\xi). \tag{2.15}$$

Consider the NLPDE given by

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{tx}, \dots) = 0, \tag{2.16}$$

we apply the following series given by

$$u(w) = \sum_{j=1}^{n} \cos^{j-1}(w) \times [B_j \sin(w) + A_j \cos(w)] + A_0. \tag{2.17}$$

From (2.14) and (2.15), the solution of (2.16) can be written as:

$$u_1(\xi) = \sum_{j=1}^{n} \tanh^{j-1}(\xi) \times [B_j \text{sech}(\xi) + A_j \tanh(\xi)] + A_0, \qquad ((2.18)$$

and

$$u_2(\xi) = \sum_{j=1}^{n} \coth^{j-1}(\xi) \times [B_j \text{csch}(\xi) + A_j \coth(\xi)] + A_0. \tag{2.19}$$

The value of n can be acquired using the balancing principle. Substituting n into (2.17) and substituting the result into the ODE using (2.13) and carrying out all the required algebraic calculations, the solutions of (2.16) can be derived.

2.3. Riccati Bernoulli Sub-ODE method

This integration technique was introduced by Yang et.al [28] to investigate the solutions of nonlinear equations. One good advantage of this scheme is that it integrates a differential equation without the need of a balancing term. The technique begin by supposing a NLPDE given by

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{tx}, \dots) = 0, \tag{2.20}$$

Step1: Let

$$q(x,t) = U(\xi), \xi = \lambda(x \pm vt), \tag{2.21}$$

to get

$$P(U, U', U'', \dots) = 0, \tag{2.22}$$

where $U'(\xi) = \frac{dU}{d\xi}$.

Step 2: Assuming the solution of (2.22) is a solution of the RE given by

$$U' = bU + aU^{2-m} + cU^{m}, \tag{2.23}$$

where a, b, c, and m are constants. Differentiating (2.23), we get

$$U'' = U^{-1-2m}(aU^2 + cU^{2m} + bU^{1+m})(-a(-2+m)U^2 + cmU^{2m} + bU^{1+m}), \tag{2.24}$$

$$U''' = U^{-2(1+m)}(bu + aU^{2-m} + cU^{m})(a^2(-2+m)(-3+2m)U^4 + c^2m(-1+2m)U^{4m} + ab(-3+m)(-2+m)U^{3+m} + (b^2+2ac)U^{2+2m} + bcm(1+m)U^{1+3m}). \tag{2.25}$$

Remark 2.4 *(2.23) is a Riccati equation if* $ac \neq 0$ *and* $m = 0$.*When* $a \neq 0$, $c = 0$ *and* $m \neq 1$, *(2.23) is a Bernoulli equation. For the above reasons, we say RBE in order to avoid the introduction of new terminologies.*

(2.23) has solutions as follows:

Case 1: If $m = 1$, (2.23) has the solution given by:

$$U(\xi) = Ce^{(b+a+c)\xi}. \tag{2.26}$$

Case 2: If $m \neq 1$, $b = 0$, and $c = 0$, the solution of (2.23) is given as:

$$U(\xi) = (a(m-1)(\xi + C))^{\frac{1}{m-1}}. \tag{2.27}$$

Case 3: If $m \neq 1$, $b \neq 0$, and $c = 0$ (2.23) has the solution given by:

$$U(\xi) = \left(Ce^{(b(m-1)\xi)} - \frac{a}{b}\right)^{\frac{1}{m-1}}. \tag{2.28}$$

Case 4: If $m \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, (2.23) has the solution given by

$$U(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi + C)\right]\right)^{\frac{1}{1-m}}, \tag{2.29}$$

and

$$U(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi + C)\right]\right)^{\frac{1}{1-m}}. \tag{2.30}$$

Case 5: If $m \neq 1$, $a \neq 0$ and $b^2 - 4ac > 0$, (2.23) has the solution given by

$$U(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\tanh\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi + C)\right]\right)^{\frac{1}{1-m}}, \tag{2.31}$$

and

$$U(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\coth\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi + C)\right]\right)^{\frac{1}{1-m}}. \tag{2.32}$$

Case 6: If $m \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, (2.23) has the solution given by

$$U(\xi) = \left(\frac{1}{a(m-1)(\xi+C)} - \frac{a}{b}\right)^{\frac{1}{1-m}}, \tag{2.33}$$

where C is a constant.

Step 3: Putting the derivatives of U into (2.22) gives a set of algebraic equation in U. Setting the value of m and comparing the coefficients of u^i, gives a system algebraic eexpressions in b, a, c, v. Solving the system, one can derive the solutions of (2.20).

2.4. B$\ddot{a}$cklund Transformation

If $U_n(\xi)$ and $U_{n-1}(\xi)$ are solutions of (2.23), then we can have

$$\frac{dU_n(\xi)}{d\xi} = \frac{dU_n(\xi)}{dU_{n-1}(\xi)\xi}\frac{dU_{n-1}(\xi)}{d\xi} = \frac{dU_n(\xi)}{dU_{n-1}\xi}(aU_{n-1}^{2-m} + bU_{n-1} + cU_{n-1}^{m}), \tag{2.34}$$

namely

$$\frac{dU_n(\xi)}{aU_n^{2-m}+bU_n+cU_n^m} = \frac{du_{n-1}(\xi)}{aU_{n-1}^{2-m}+bU_{n-1}+cU_{n-1}^m}. \tag{2.35}$$

Integration of (2.35) with respect to ξ, we get

$$U_n(\xi) = \left(\frac{-cA_1+aA_2(U_{n-1}(\xi))^{1-m}}{bA_1+aA_2+aA_1(U_{n-1}(\xi))^{1-m}}\right)^{\frac{1}{1-m}}, \tag{2.36}$$

where A_1 and A_2 are any constants. (2.36) is a B$\ddot{a}$cklund transformation. Using (2.36), one can construct many solutions of (2.20).

2.5 Modified F-Expansion method

In this part, we will give the description of the above mentioned method [29, 30]. Considet a NLPDE given by

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, \dots) = 0, \tag{2.37}$$

Step 1: Let

$$q(x,t) = U(\xi), \quad \xi = (x \pm vt), \tag{2.38}$$

to obtain the ODE given by

$$P(U, U', U'', \dots) = 0, \tag{2.39}$$

We seek for the following solutions

$u(\xi) = \sum_{i=-n}^{n} a_i F(\xi)^i, \quad (a_n \neq 0),$ (2.40)

where $a_i(i = -n, \dots, n)$ are arbitrary constants and $F(\xi)$ satisfies the Riccati equation

$F'(\xi) = A + BF(\xi) + CF(\xi)^2 (C \neq 0).$ (2.41)

A, B, C are all parameters. To find $U(\xi)$ explicitly, we consider the following steps:

Step 2: Finding the integer n using balancing principle in (2.37).

Step 3: Substituting (2.40) into (2.39), using (2.41), one can get a finite series in $F(\xi)^i$ $(i = -n, \dots, n)$. Equating each coefficients of $F(\xi)^i$ to zero yields a system of algebraic equations in $a_i(= -n, \dots, n)$ and v.

Step 4: We Solve the resulting system of algebraic equations for $a_i(= -n, \dots, n)$ and v, then substitute the results into (2.40).

Step 5: Using the cases described in the table below, one can get a series of soliton-like solutions, trigonometric function solutions and rational solutions of (2.37).

Relation between values of A, B, C and corresponding $F(\xi)$ in Riccati equation $F'(\xi) = A + BF(\xi) + CF(\xi)^2 \quad (C \neq 0)$. Table.2.1 shows the structures of solutions of th method

Table. 2.1: Solution structure for solitons

Case	A	B	C	F
1	0	1	-1	$\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2}\xi\right)$
2	0	-1	1	$\frac{1}{2} - \frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$
3	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\tanh(\xi) \pm i\text{sech}(\xi), \coth(\xi) \pm \text{csch}(\xi)$
4	1	0	-1	$\tanh(\xi), \coth(\xi)$
5	$\frac{1}{2}$	0	$\frac{1}{2}$	$\tan(\xi) + \sec(\xi), \csc(\xi) - \cot(\xi)$
6	$\frac{-1}{2}$	0	$\frac{-1}{2}$	$\csc(\xi) + \cot(\xi), \sec(\xi) - \tan(\xi)$
7	$1(-1)$	0	$1(-1)$	$\tan(\xi), \cot\xi)$
8	0	0	$\neq 0$	$\frac{-1}{C\xi + \lambda}$
9	constant	0	0	$A\xi$
10	constant	$\neq 0$	0	$\frac{e^{B\xi} - A}{B}$

2.6. Generalized tanh method

In this part, we will give the description of the the general generalized tanh method (GTH) [31- 34]. Considet a NLPDE given by

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, \ldots) = 0. \tag{2.42}$$

Letting

$$q(x,t) = U(\xi), \tag{2.43}$$

where $\xi = x - vt$. (2.6.1) can be transformed into the ODE

$$P(U, U', U'', \ldots) = 0. \tag{2.44}$$

Next we introduce the function $\phi = \phi(\xi)$ which is a solution of the following equation

$$\phi' = K + \phi^2. \tag{2.45}$$

Suppose:

$$U(\xi) = \sum_{i=0}^{n} a_i \phi^i. \tag{2.46}$$

(2.45) admits the following solutions

$$\phi = \begin{cases} -\sqrt{-K} tanh[\sqrt{-K}\xi], \\ -\sqrt{-K} coth[\sqrt{-K}\xi], \end{cases} \text{for} \quad K < 0, \tag{2.47}$$

$$\phi = -\frac{1}{\xi}, \text{for} \quad K = 0, \tag{2.48}$$

$$\phi = \begin{cases} -\sqrt{-K}\tanh[\sqrt{-K}\xi], \\ -\sqrt{-K}\coth[\sqrt{-K}\xi], \end{cases} \quad \text{for} \quad K > 0. \tag{2.49}$$

The positive integer n in (2.46) is derived by the balancing principle. By putting (2.46) along with its derivatives into (2.44), yields a a set of algebraic expressions in terms of $\phi(\xi)$. After making some mathematical computations and simplification, one can get the solution of (2.42).

2.7. Generalized projective Riccati equations method

In this part, we will give the description of the above mentioned method

[35 - 37]. The method can be described below:

Suppose we have a NLPDE given by by

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, \ldots) = 0, \tag{2.50}$$

Step 1: We apply the travelling wave transformation given by

$$q(x,t) = U(\xi), \quad \xi = (x \pm vt), \tag{2.51}$$

to get the following ODE

$$P(U, U', U'', \ldots) = 0, \tag{2.52}$$

where$U'(\xi) = \frac{du}{d\xi}$.

Step 2: Suppose that (2.52) has the following solution

$$u(\xi) = A_0 + \sum_{i=1}^{N} \sigma(\xi)^{i-1}[A_i\sigma(\xi) + B_i\tau(\xi)], \tag{2.53}$$

where A_0 A_i and B_i are constants and $\sigma(\xi)$ and $\tau(\xi)$ satisfy the ODE:

$$\sigma'(\xi) = \varepsilon\sigma(\xi)\tau(\xi), \tag{2.54}$$

$$\tau'(\xi) = R + \varepsilon\tau(\xi)^2 - \mu\sigma(\xi), \quad \varepsilon = \pm 1, \tag{2.55}$$

where

$$\tau(\xi)^2 = -\varepsilon\left(R - 2\mu\sigma(\xi) + \frac{\mu^2+r}{R}\sigma(\xi)^2\right), \tag{2.56}$$

$r = \pm 1$, R and μ are non zero constants.

If $R = \mu = 0$, (2.52) has the following solution:

$$U(\xi) = \sum_{i=1}^{n} A_i\tau^i, \tag{2.57}$$

with $\tau(\xi)$ satisfying:

$$\tau'(\xi) = \tau^2. \tag{2.58}$$

Step 3: The integer n in (2.7.4) can be derived by the balancing principle equation.

Step 4: Substituting (2.57) along with (2.54), (2.55), (2.56) or (2.58) into (2.52), collecting all terms of the same order of $\sigma^j(\xi)\tau^i(\xi)(j = 0,1,\ldots; i = 0,1)$ together and setting each to zero, we get a system of algebraic expressions whose solution gives the values of A_0, A_i, B_i, μ, v, R.

Step 5: (2.54) and (2.55) the solutions represented by the following cases:

Case 1: If $\varepsilon = -1, r = -1, R > 0$, we get

$$\sigma_1(\xi) = \frac{R\text{sech}[\sqrt{R}\xi]}{\mu\text{sech}[\sqrt{R}\xi]+1}, \tau_1(\xi) = \frac{\sqrt{R}\tanh[\sqrt{R}\xi]}{\mu\text{sech}[\sqrt{R}\xi]+1}. \tag{2.59}$$

Case 2: If $\varepsilon = -1, r = 1, R > 0$, we obtain

$$\sigma_1(\xi) = \frac{R\text{csch}[\sqrt{R}\xi]}{\mu\text{csch}[\sqrt{R}\xi]+1}, \tau_1(\xi) = \frac{\sqrt{R}\coth[\sqrt{R}\xi]}{\mu\text{csch}[\sqrt{R}\xi]+1}. \tag{2.60}$$

Case 3: If $\varepsilon = 1, r = -1, R > 0$, we attain

$$\begin{aligned} \sigma_3(\xi) &= \frac{R\sec[\sqrt{R}\xi]}{\mu\sec[\sqrt{R}\xi]+1}, \tau_3(\xi) = \frac{\sqrt{R}\tan[\sqrt{R}\xi]}{\mu\text{Sec}[\sqrt{R}\xi]+1}, \\ \sigma_4(\xi) &= \frac{R\csc[\sqrt{R}\xi]}{\mu\csc[\sqrt{R}\xi]+1}, \tau_4(\xi) = -\frac{\sqrt{R}\cot[\sqrt{R}\xi]}{\mu\csc[\sqrt{R}\xi]+1}. \end{aligned} \tag{2.61}$$

Case 4: If $R = \mu = 0$, we have

$$\sigma_5(\xi) = \frac{C}{\xi}, \tau_5(\xi) = \frac{1}{\varepsilon\xi}, \tag{2.62}$$

where C is a constant.

Case 5: Putting the values of A_0, A_i, B_i, μ, v, R and using (2.59)-(2.62), the solutions of (2.50) can be obtained.

2.8. Undetermined Coefficients Method

The method of undetermined coefficients [38, 39, 40, 41] uses the ansatz approach. Consider a NLPDE of the form

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, \dots) = 0. \tag{2.63}$$

The solutions of (2.8.1) can be derived by apply solitary wave ansatz. Applying the transformation

$$q(x,t) = U(\xi), \tag{2.64}$$

where $\xi = (x \pm vt)$. (2.8.1) can be transformed into an ODE given by:

$$P(U, U', U'', \dots) = 0. \tag{2.65}$$

We will consider the following six ansatz to integrate:

$$w(\xi) = \sigma \tanh^p(\mu\xi), \tag{2.66}$$

$$w(\xi) = \sigma \operatorname{sech}^p(\mu\xi), \tag{2.67}$$

$$w(\xi) = \sigma \operatorname{csch}^p(\mu\xi), \tag{2.68}$$

$$w(\xi) = \sigma \coth^p(\mu\xi), \tag{2.69}$$

where σ, p and μ are constants. Substituting any of (2.66)-(2.69) into the reduced ODE (2.65) gives a algebraic equation of sech, csch, tanh, coth, cosine or sine terms among the unknowns σ, p and μ. We then collect the coefficients of the resulting hyperbolic and trigonometric functions and set each to zeros to acquire a set of algebraic expressions among the unknowns. Determining the constants σ, p and μ, the solutions proposed solutions of (2.63) follow immediately.

2.9. Concept of Lie point Symmetry Analysis

In this part, we give the description of the Lie point symmetry technique [6].

Symmetry can be seen as a transformation that tends to leaves an object unchanged or invariant. It is used in our daily life activities [7]. In its literal meaning, it denotes the property of an object that can be measured simultaneously. We will consider some examples to illustrate the concept of symmetries.

$$x = r\cos\theta, \quad \overline{x} = r\cos(\theta + \varepsilon), \quad y = r\sin\theta, \quad \overline{y} = r\sin(\theta + \varepsilon) \tag{2.70}$$

which after eliminating θ becomes

$$\overline{x} = x\cos\varepsilon - y\sin\varepsilon, \quad \overline{y} = y\cos\varepsilon + x\sin\varepsilon. \tag{2.71}$$

One can easily see that the circle is invariant under the transformation (2.9.2) since

$$\overline{x}^2 + \overline{y}^2 = r^2. \tag{2.72}$$

Another example showing the concept of symmetry involves the invariance of the line $y = \frac{1}{2}x$ under the transformation

$$\overline{x} = xe^{\varepsilon} \quad \overline{y} = ye^{\varepsilon}. \tag{2.73}$$

One can clearly see that

$$\overline{y} = \frac{1}{2}\overline{x}, \tag{2.74}$$

and then

$$ye^{\varepsilon} = \frac{1}{2}xe^{\varepsilon}, \tag{2.75}$$

implying that

$$y = \frac{1}{2}x, \tag{2.76}$$

and then the invariance followed. Another example is the invariance of a DE. We consider the ordinary differential equation.

$$\frac{dy}{dx} = xy^3, \tag{2.77}$$

under the transformations

$$\overline{x} = xe^{\varepsilon}, \quad \overline{y} = ye^{-\varepsilon}. \tag{2.78}$$

Then, we have

$$\frac{d\overline{y}}{d\overline{x}} = e^{-2\varepsilon}, \quad \overline{xy^3} = e^{-2\varepsilon}xy^3. \tag{2.79}$$

and From (2.79), we can clearly see that (2.77) is invariant under the transformation (2.78). Therefore, ordinary and partial differential equations possess quite a number of group transformations which make them invariant. In the subsequent sections, we shall see how to obtain these group of transformations that leave differential equations invariant.

Consider the PDE given by:

$$F(x,y,u,u_x,u_y)=0. \tag{2.80}$$

We intend to derive the general format of obtaining the Lie groups and infinitesimals [7].

We seek for the invariance of the PDE by assuming an arbitrary Lie groups of the form:

$$\begin{aligned}&\bar{x}=f_1(x,y,u,\varepsilon),\bar{y}=f_2(x,y,u,\varepsilon),\bar{u}=f_3(x,y,u,\varepsilon)\\&f_1(x,y,u,0)=x,f_2(x,y,u,0)=y,f_3(x,y,u,0)=u.\end{aligned} \tag{2.81}$$

If we assume ε is small, then we can establish the Taylor series expansion of (2.81) about $\varepsilon=0$:

$$\bar{x}=f_1(x,y,u,\varepsilon)+\varepsilon\frac{\partial f_1}{\partial\varepsilon}\bigg|_{\varepsilon=0}+O(\varepsilon^2), \tag{2.82}$$

$$\bar{y}=f_2(x,y,u,\varepsilon)+\varepsilon\frac{\partial f_2}{\partial\varepsilon}\bigg|_{\varepsilon=0}+O(\varepsilon^2), \tag{2.83}$$

$$\bar{u}=f_3(x,y,u,\varepsilon)+\varepsilon\frac{\partial f_3}{\partial\varepsilon}\bigg|_{\varepsilon=0}+O(\varepsilon^2). \tag{2.84}$$

Letting

$$\frac{\partial f_1}{\partial\varepsilon}\bigg|_{\varepsilon=0}=\xi^1(x,y,u),\quad \frac{\partial f_2}{\partial\varepsilon}\bigg|_{\varepsilon=0}=\xi^2(x,y,u),\quad \frac{\partial f_3}{\partial\varepsilon}\bigg|_{\varepsilon=0}=\eta(x,y,u), \tag{2.85}$$

and using (2.82)-(2.84), we have

$$\begin{aligned}&\bar{x}=x+\xi^1(x,y,u)+O(\varepsilon^2),\\&\bar{y}=y+\xi^2(x,y,u)+O(\varepsilon^2),\\&\bar{u}=u+\eta(x,y,u)+O(\varepsilon^2).\end{aligned} \tag{2.86}$$

The above equations are refereed to as Lie groups or vector fields. ξ^1, ξ^2 and η are called infinitesimals. Thus, if we have infinitesimals of a given NLPDE, we can recover the Lie group and vice versa.

In general, we construct the symmetries of n^{th} order NLPDE of the form:

$$F(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots,u_t(n),\ldots,u_t(n-i)x(i),\ldots u_x(n))=0. \tag{2.87}$$

Introducing the infinitesimal operator

$$\Gamma=\xi_i\frac{\partial}{\partial\overline{x}^i}+\eta_{\overline{\alpha}}\frac{\partial}{\partial u^{\overline{\alpha}}}, \tag{2.88}$$

where $\xi^1 = \xi^1(t,x,u)$, $\xi^2 = \xi^2(t,x,u)$ and $\eta = \eta(t,x,u)$ are to be determined. We therefore define nth extension of the operator Γ as Γ^n which is given in a recursive form as

$$\Gamma^n = \Gamma^{n-1} + \sum_{i=0}^{n} \eta_{[t(n-i)x(i)]} \frac{\partial^n}{\partial t^{n-i} \partial x^i}, \tag{2.89}$$

and the Lie's invariance condition given by

$$\Gamma^n(\Delta)|_{\Delta=0} = 0, \tag{2.90}$$

where

$$\Delta = F(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots,u_t(n),\ldots,u_t(n-i)x(i),\ldots u_x(n)) \tag{2.91}$$

and the extended transformations are given as:

$$\begin{aligned} \eta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \eta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2). \end{aligned} \tag{2.92}$$

The nth extension is

$$\begin{aligned} &\eta_{[t(n-i)x(i)]} = D_t(\eta_{[t(n-i-1)x(i)]}) - u_{[t(n-i)x(i)]} D_t(\xi^1) - \\ &u_{[t(n-i-1)x(i+1)]} D_t(\xi^2). \end{aligned} \tag{2.93}$$

The total derivative operators are given by the relations below,

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + \ldots + u_{t(i+1)x(j)} \frac{\partial^{i+j}}{\partial u_{t(i)x(j)}}, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \ldots + u_{t(i)x(j+1)} \frac{\partial^{i+j}}{\partial u_{t(i)x(j)}}. \end{aligned} \tag{2.94}$$

The Lie's invariance condition above give rise to a system of overdetermined equations as functions of ξ^1, ξ^2 and η, solving the overdetermined system of equations give rise to the solution of the infinitesimals ξ^1, ξ^2 and η. The infinitesimals can then be applied to reduce the original equation to a lower order ODE involving less parameters [6]. And the solution of these ODEs imply the solutions of the original NLPDE.

2.10. Concept of Nonlinear Self-Adjointness

Here, we give a brief preliminary of the concept nonlinear self-adjointness given by Ibragimov [12]. Many differential equations cannot be formulated as Euler-Lagrange equations since they have no Lagrangians. Therefore, it is impossible to apply Noether's theorem for calculating conservation laws. However, according to [12], it is possible to introduce a formal Lagrangian if any given system of equations is taken into consideration together with the adjoint system. In his recent paper [12], Ibragimov has proved that the adjoint system inherits symmetries of the given system and has suggested a new theorem on nonlocal conservation laws.

A locally analytic function of a certain finite number of the variables $(\overline{x}, u)$ and the differentials of the dependent variable u is called a differentiable function. And the highest order of derivatives in the differential function is named the order of the function.

Theorem 9.1 *The system of* $\overline{m}$ *differential equations*

$$F_{\overline{\alpha}}(\overline{x}, u, u_1, \dots, u_s) = 0, \quad \overline{\alpha} = 1, \dots, \overline{m}, \quad (2.95)$$

having m dependent variables $u = (u^1, \dots, u^m)$ has adjoint equation

$$F^*_{\overline{\alpha}}(\overline{x}, u, z, u_1, z_1, \dots, u_s, z_s) = \frac{\delta(z^{\overline{\beta}} F_{\overline{\beta}})}{\delta u^{\overline{\alpha}}}, \quad \overline{\alpha} = 1, \dots, m, \quad (2.96)$$

and Lagrangian represented by

$$\mathcal{L} = z^{\overline{\beta}} F_{\overline{\beta}}, \quad (2.97)$$

where $z^{\overline{\beta}} = z^{\overline{\beta}}(\overline{x}, t)$.

2.11. New Conservation Laws theorem

Ibragimov introduced the following new conservation theorem [11].

Theorem 10.1 *Let*

$$\Gamma = \xi_i(\overline{x}, u, u_1, \dots) \frac{\partial}{\partial \overline{x}^i} + \eta_{\overline{\alpha}}(\overline{x}, u, u_1, \dots) \frac{\partial}{\partial u^{\overline{\alpha}}}, \quad (2.98)$$

be a symmetry (point, B$\ddot{a}$cklund or contact) of (2.95) with an adjoint (2.96) and a Lagrangian (2.97). Then (2.10.1) satisfies the conservation equation as:

$$D_i(T^i)\big|_{(Eq.2.95=0)} = 0, \quad (2.99)$$

with

$$T^i = \xi_i \mathcal{L} + W^{\overline{\alpha}}[\frac{\partial \mathcal{L}}{\partial u_i^{\overline{\alpha}}} - D_j(\frac{\partial \mathcal{L}}{\partial u_{ij}^{\overline{\alpha}}}) + D_j D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^{\overline{\alpha}}} \dots] \\ + D_j(W^{\overline{\alpha}})[\frac{\partial \mathcal{L}}{\partial u_{ij}^{\overline{\alpha}}} - D_k(\frac{\partial \mathcal{L}}{\partial u_{ijk}^{\overline{\alpha}}}) + \dots] + D_j D_k(W^{\overline{\alpha}})[\frac{\partial \mathcal{L}}{\partial u_{ijk}^{\overline{\alpha}}}] + \dots, \tag{2.100}$$

and

$$W^{\alpha} = \eta_{\overline{\alpha}} - \xi_j u_j^{\overline{\alpha}}. \tag{2.101}$$

2.12. Modulation instability analysis

The relation between MI and solitons is best observed in the fact that the trains of pulses that emerge from the MI process are actually trains of almost ideal solitons. Hence it can be loosely considered as a precursor to soliton formation. MI can be classified into three main categories-spatial, temporal and spatiotemporal [42, 43, 44]. The spatial MI occurs due to the interaction between the nonlinearity and diffraction which results into the breaks up of homogeneous beam into numerous small filaments. The temporal MI occurs due to interplay between the group velocity dispersion(GVD) and nonlinearity and manifests itself as break-up of cw into a train of ultrashort pulses. In temporal MI the anomalous GVD plays the same role as is played by diffraction term in spatial MI. However in spatiotemporal MI all the three terms-nonlinearity, dispersion and diffraction are nonzero and it occurs due to the simultaneous presence of spatial and temporal MI in nonlinear medium. To study the MI of any nonlinear evolution equation (NLEE), following steps have to be followed:

• Steady state solution of the equation is considered

• The beginning of instability can be investigated by perturbing the steady state solution of NLEE, by applying some weak perturbation

• As perturbation is assumed as small so the resultant equation is linearized in perturbation term by neglecting the terms containing higher order perturbation.

• This equation can be easily solved in frequency domain. However, the equation contain the term having complex conjugate of perturbation hence the Fourier components at frequencies Ω and $-\Omega$ are coupled. Therefore perturbation term is splited into two parts.

• The dispersion relation for MI gain spectrum is obtained from resultant equation.

3. OPTICAL SOLITONS AND CONSERVATION LAWS OF SOME NONLINEAR SCHRODINGER'S EQUATION

A nonlinear Schrödinger's equation as previously described in part 2 is one of the equations describing wide class of nonlinear systems. It has been widely used to address the physical, biological and engineering systems. We will consider three different nonlinear Schrödinger type equations in this chapter. The solitons, point symmetries and Cls of the equations will be investigated by using the concepts discussed in chapter two.

3.1. The nonlinear Schrödinger equation with spatio-temporal dispersion (NLSE-STD)

In this section, we will study the (NLSE-STD) that arises in a propagation of light in nonlinear optical fibers, planar wave guides, Bose-Einstein condensate theory. We aim to investigate the soliton solutions of the equation for two types of nonlinearities, namely; Kerr and Parabolic law nonlinear fibers using the complex envelope ansatz and Riccatti Beroulli sub-Ode methods. Then, we construct the conservation laws using the vector fields of the equation. The results of this section have been published in [45, 46]. The model is given by [47, 48, 49, 50]

$$iq_t + sq_{tx} + jq_{xx} + lF(|q|^2)q = 0, \quad i = \sqrt{-1}, \tag{3.1}$$

where x represents the non-dimensional distance along the fiber and t is the temporal variable. The constants s and j are real valued. The term $q(x,t)$ represents the dependent variable. On the left of (3.1.1), the coefficient s is the STD while j is group velocity dispersion [47]. Finally, l represents the coefficient of the nonlinear term and the functional F is the non-Kerr law nonlinearity [5].

3.1.1. Application of the complex envelope ansatz method to the Kerr law nonlinearity

This is the instance where $F(u) = u$ [5]. For this case, (3.1) reduces to

$$iq_t + sq_{tx} + jq_{xx} + l(|q|^2)q = 0. \tag{3.2}$$

Substituting (2.2) into (3.2), we get

$$jA_{xx} + i(-2jk + s\omega)A_x - i(-1 + sk)A_t + sA_{tx} + A(-jk^2 - \omega + sk\omega + l|A|^2) = 0. \tag{3.3}$$

Substituting (2.3) into (3.3) and performing all the necessary algebraic computations, we acquired

Family 1: We have

$$\lambda = \pm\rho, j = \pm\frac{l\rho^2}{k^2}, s = \frac{1}{k}, v = \frac{l\rho^2}{k}, \beta = 0, \omega = 2l\rho^2. \tag{3.4}$$

Thus, we derive the following dark-bright soliton

$$q(x,t) = \{\pm\rho\tanh[\eta(x - vt)] + \mathrm{i}\rho\mathrm{sech}[\eta(x - vt)]\} \times e^{i(-kx+\theta+t\omega)}, \tag{3.5}$$

with intensity

$$|q(x,t)|^2 = \rho^2. \tag{3.6}$$

The nonlinear phase shift is represented by

$$\psi_{NL} = \arctan\left[\frac{\mathrm{sech}[\eta(x-vt)]}{\pm\tanh[\eta(x-vt)]}\right] \tag{3.7}$$

Family 2 If

$$\lambda = \pm\rho, s = \pm\rho\sqrt{\frac{j}{l}}, v = \sqrt{\mathrm{j}l}\rho, \beta = 0, \omega = \sqrt{\mathrm{bc}}k\rho + l\rho^2. \tag{3.8}$$

Thus, we derive the following dark-bright soliton

$$q(x,t) = [\pm\rho\tanh[\eta(x - vt)] + \mathrm{i}\rho\mathrm{sech}[\eta(x - vt)]] \times e^{i(-kx+\theta+t\omega)}, \tag{3.9}$$

with intensity given by

$$|q(x,t)|^2 = \rho^2. \tag{3.10}$$

The nonlinear phase shift is represented by:

$$\psi_{NL} = \arctan\left[\frac{\mathrm{sech}[\eta(x-vt)]}{\pm\tanh[\eta(x-vt)]}\right]. \tag{3.11}$$

Family 3: If

$$\lambda = 0, j = \frac{l\rho^2}{2k^2}, s = \frac{1}{k}, v = \frac{c\left(1-\frac{k^2}{\eta^2}\right)\rho^2}{2k}, \beta = 0, \omega = l\rho^2. \tag{3.12}$$

Thus, we acquire the bright optical soliton

$$q(x,t) = [\mathrm{i}\rho\mathrm{sech}[\eta(x-vt)]] \times e^{i(-kx+\theta+t\omega)}, \tag{3.13}$$

The intensity is

$$q(x,t)|^2 = \rho^2\mathrm{sech}^2[\eta(x-vt)]. \tag{3.14}$$

Family 4: If

$$\rho = 0, s = \frac{1}{k}, v = \frac{k(2j\eta^2+l\lambda^2)}{2\eta^2}, \beta = -\sqrt{\frac{jk^2-l\lambda^2}{l}}, \omega = \frac{k(2jk\eta+l\beta\lambda)}{\eta}. \tag{3.15}$$

Thus, we acquire the dark soliton given by

$$q(x,t) = [\mathrm{i}\beta + \lambda\tanh[\eta(x-vt)]] \times e^{i(-kx+\theta+t\omega)}. \tag{3.16}$$

The intensity gives is

$$|q(x,t)|^2 = \beta^2 + \lambda^2 - \lambda^2\mathrm{sech}^2[\eta(x-vt)]. \tag{3.17}$$

The nonlinear phase shift is represented by

$$\psi_{NL} = \arctan\left[\frac{\beta}{\lambda\tanh[\eta(x-vt)]}\right]. \tag{3.18}$$

Family 5: If

$$\lambda = 0, s = \frac{lk\rho^2 \pm \sqrt{c\eta^2\rho^2(2j(k^2+\eta^2)-l\rho^2)}}{l(k^2+\eta^2)\rho^2},$$
$$v = \frac{-lk\rho^2+\sqrt{c\eta^2\rho^2(2j(k^2+\eta^2)-l\rho^2)}}{2\eta^2}, \beta = 0, \quad (3.19)$$
$$\omega = \frac{l\eta^2\rho^2+k\sqrt{c\eta^2\rho^2(2j(k^2+\eta^2)-l\rho^2)}}{2\eta^2},$$

Thus, we acquire the bright soliton given by

$$q(x,t) = [\mathrm{i}\rho\mathrm{sech}[\eta(x-vt)]] \times e^{i(-kx+\theta+t\omega)}, \tag{3.20}$$

The intensity is

$$|q(x,t)|^2 = \rho^2\mathrm{sech}^2[\eta(x-vt)]. \tag{3.21}$$

Family 6: If

$$\rho = 0, v = \frac{2j\eta^2 + l\lambda^2}{2s\eta^2},$$
$$\beta = \frac{l(\lambda - sk\lambda) \pm \sqrt{l(4j\eta^2 + l(3(-1+sk)^2 - 4s^2\eta^2)\lambda^2)}}{2sl\eta}, \tag{3.22}$$
$$\omega = \frac{2j(1+sk)\eta^2 + \lambda\left(-2l(-1+sk)\lambda + \sqrt{l(4j\eta^2 + l(3(-1+sk)^2 - 4s^2\eta^2)\lambda^2)}\right)}{2s^2\eta^2}.$$

Thus, we derive the following dark soliton

$$q(x,t) = [\mathrm{i}\beta + \lambda\tanh[\eta(x - vt)]] \times e^{i(-kx+\theta+t\omega)}. \tag{3.23}$$

With intensity given by

$$|q(x,t)|^2 = \beta^2 + \lambda^2 - \lambda^2\mathrm{sech}^2[\eta(x - vt)]. \tag{3.24}$$

The nonlinear phase shift is represented by

$$\psi_{NL} = \arctan\left[\frac{\beta}{\lambda\tanh[\eta(x-vt)]}\right]. \tag{3.25}$$

3.1.2. Application of the complex envelope ansatz method to the parabolic law nonlinearity

For this case, $F(s) = d_1 s + d_2 s^2$, where d_1 and d_2 are constants [6]. For this case, (3.1) reduces to

$$iq_t + sq_{tx} + jq_{xx} + l(d_1|q|^2 + d_2|q|^4)q = 0. \tag{3.26}$$

Substituting (2.2) into (3.26), we get

$$(A(-jk^2 - \omega + sk\omega + ld_1|A|^2 + ld_2|A|^4) + i(-2jk + s\omega)A_x = 0. \tag{3.27}$$

Substituting (2.3) into Eq.(3.27) and performing all necessary algebraic calculations, the following cases are acquired.

Family 1: If

$$\lambda = \pm\rho, b = \frac{c\rho^2(d_1 + \rho^2 d_2)}{k^2}, a = \frac{1}{k}, v = \frac{c\rho^2(d_1 + \rho^2 d_2)}{k}, \omega = 2c\rho^2(d_1 + \rho^2 d_2), \beta = 0, \tag{3.28}$$

we acquire the combined optical soliton

$$q(x,t) = [\pm\rho\tanh[\eta(x - vt)] + \mathrm{i}\rho\mathrm{sech}[\eta(x - vt)]] \times e^{i(-kx+\theta+t\omega)}, \tag{3.29}$$

and the intensity is

$$|q(x,t)|^2 = \rho^2. \tag{3.30}$$

The nonlinear phase shift is represented by:

$$\psi_{NL} = \arctan\left[\frac{\text{sech}[\eta(x-vt)]}{\pm\tanh[\eta(x-vt)]}\right] \tag{3.31}$$

Family 2: If

$$\lambda = \pm\rho, a = \pm\sqrt{\frac{b}{c\rho^2 d_1 + c\rho^4 d_2}}, v = \rho\sqrt{\text{bc}(d_1 + \rho^2 d_2)},$$
$$\beta = 0, \omega = c\rho^2 d_1 + c\rho^4 d_2 + k\sqrt{\text{bc}\rho^2(d_1 + \rho^2 d_2)}, \tag{3.32}$$

we obtain the combined soliton given by

$$q(x,t) = [\pm\rho\tanh[\eta(x - vt)] + \text{i}\rho\text{sech}[\eta(x - vt)]] \times e^{i(-kx+\theta+t\omega)}, \tag{3.33}$$

and the intensity is

$$|q(x,t)|^2 = \rho^2. \tag{3.34}$$

The nonlinear phase shift is represented by:

$$\psi_{NL} = \arctan\left[\frac{\text{sech}[\eta(x-vt)]}{\pm\tanh[\eta(x-vt)]}\right]. \tag{3.35}$$

3.2. Investigation of solitons to the (NLSE-STD) using RBSO method

To solve (3.1) using the above mentioned technique, we let

$$q(x,t) = U(\xi)e^{i\phi(x,t)}, \quad \xi = K(x + vt), \tag{3.36}$$

where

$$\phi = -kx + \omega t + \theta. \tag{3.37}$$

Putting (3.36) into (3.1), the imaginary part is being represented by

$$v = \frac{2jk - s\omega}{1 - sk}, \tag{3.38}$$

and the real part gives

$$(-jk^2 - \omega + ks\omega)u + F(|u|^2)u + \left(jK^2 + \frac{K^2 s(2jk - s\omega)}{1 - ks}\right)u'' = 0. \tag{3.39}$$

3.2.1 Application of RBSO method to Kerr law nonlinearity

For this nonlinearity , (3.39) simplifies to

$$(-jk^2-\omega+ks\omega)u+u^3+\left(jK^2+\frac{K^2s(2jk-s\omega)}{1-ks}\right)u''=0. \tag{3.40}$$

Putting (2.23) and (3.24) into (3.40), we get

$$ks\omega u+u^3+jK^2u^{-1-2m}(au^2+cu^{2m}+bu^{1+m})(-a(-2+m)u^2+cmu^{2m}+bu^{1+m})$$
$$\frac{1}{-1+ks}K^2s(-2jk+s\omega)u^{-1-2m}(au^2+cu^{2m}+bu^{1+m})(-a(-2+m)u^2+cmu^{2m}+bu^{1+m})-$$
$$(jk^2+\omega)u=0. \tag{3.41}$$

Substituting $m=0$ into (3.41) and performing all the necessary algebraic calculations, the following coefficients are obtained:

$$a=\frac{-2cK^2s^2\pm\sqrt{4c^2K^4s^4-8jK^2(1-2ks+k^2s^2)}}{4jK^2}, b=0, \omega=\frac{k^2-2acK^2-k^3s-2ackK^2s}{2a^2K^2}. \tag{3.42}$$

Subsequently, we acquire the singular and dark solitons given by

$$q(x,t)=[\sqrt{\frac{-2c(cK^2s^2+\sqrt{K^2(c^2K^2s^4-2j(-1+ks)^2)})}{jK^2}}\times$$
$$\tan[\sqrt{\frac{c(cK^2s^2-\sqrt{K^2(c^2K^2s^4-2j(-1+ks)^2)})}{2jK^2}}(C+\xi)]]\times e^{-i(kx-\theta-t\omega)}, \tag{3.43}$$

$$q(x,t)=[\sqrt{\frac{2c(cK^2s^2-\sqrt{K^2(c^2K^2s^4-2j(-1+ks)^2)})}{jK^2}}\times$$
$$\cot[\sqrt{\frac{c(cK^2s^2+\sqrt{K^2(c^2K^2s^4-2j(-1+ks)^2)})}{2jK^2}}(C+\xi)]]\times e^{-i(kx-\theta-t\omega)}, \tag{3.44}$$

where $A=\sqrt{-\frac{jk^2+\omega-ks\omega}{l}}$. With $(-jk^2+\omega-ks\omega)l<0$ for existence of soliton.

$$q(x,t)=[\sqrt{\frac{2c(cK^2s^2-\sqrt{K^2(c^2K^2s^4-2j(-1+ks)^2)})}{jK^2}}\times$$
$$\tanh[\sqrt{\frac{c(cK^2s^2-\sqrt{K^2(c^2K^2s^4-2j(-1+ks)^2)})}{2jK^2}}(C+\xi)]]\times e^{-i(kx-\theta-t\omega)}, \tag{3.45}$$

and

$$q(x,t) = [\sqrt{\frac{2c(cK^2s^2 - \sqrt{K^2(c^2K^2s^4 - 2j(-1+ks)^2)})}{jK^2}} \times$$

$$\coth[\sqrt{\frac{c(cK^2s^2 - \sqrt{K^2(c^2K^2s^4 - 2j(-1+ks)^2)})}{2jK^2}}(C+\xi)]] \times e^{-i(kx-\theta-t\omega)},$$

(3.46)

where $B = \sqrt{\frac{jk^2+\omega-ks\omega}{l}}$. $(jk^2 + \omega - ks\omega)l > 0$ for existence of soliton.

3.2.2. Application of RBSO method to Parabolic law

For this case, (3.39) simplifies to

$$(-jk^2 - \omega + ks\omega)U + d_1U^3 + d_2U^5 + \left(jK^2 + \frac{K^2s(2jk-s\omega)}{1-ks}\right)U'' = 0. \quad (3.47)$$

Setting

$$U = u^{\frac{1}{2}}, \quad (3.48)$$

(3.47) is transformed to:

$$-4(jk^2 + \omega - ks\omega)u^2 + 4d_1u^3 + 4d_2u^4 - K^2(j + \frac{s(2jk-s\omega)}{1-ks})(u')^2 + 2K^2(j + \frac{s(2jk-s\omega)}{1-ks})uu'' = 0. \quad (3.49)$$

Substituting (2.23) and (2.24) into (3.49), we get

$$4d_1u^3 + 4d_2u^4 + 2K^2\left(j + \frac{s(-2jk+s\omega)}{-1+ks}\right)u^{-2m}(au^2 + cu^{2m} + bu^{1+m})$$
$$(-a(-2+m)u^2 + cmu^{2m} + bu^{1+m}) - \left(4(jk^2 + \omega - ks\omega)u^2 + K^2\left(j + \frac{s(-2jk+s\omega)}{-1+ks}\right)\right.$$
$$\left.(bu + au^{2-m} + cu^m)^2\right) = 0.$$

(3.50)

Setting $m = 0$ in (3.50) and performing all the necessary algebraic calculations, the following coefficients are obtained:

$$b = \frac{3ad_1}{4d_2},$$
$$c = 0,$$
$$j = \frac{-3d_1^2 - 16d_2\omega + 16ksd_2\omega}{16k^2d_2}, \quad (3.51)$$
$$v = \frac{abjK^2 + d_1}{absK^2}.$$

Subsequently, we acquire the following solutions

$$q(x,t) = 3d_1\left(3e^{-\frac{3ad_1}{4\beta}}C\alpha - 4\beta\right)^{-1} e^{i(-kx+\theta+t\omega)}, \tag{3.52}$$

$$q(x,t) = -\frac{3d_1}{8d_2}\left(1 + \tanh\left[\frac{3ad_1}{8d_2}(C+\xi)\right]\right) e^{i(-kx+\theta+t\omega)}, \tag{3.53}$$

$$q(x,t) = -\frac{3d_1}{8d_2}\left(1 + \coth\left[\frac{3ad_1}{8d_2}(C+\xi)\right]\right) e^{i(-kx+\theta+t\omega)}, \tag{3.54}$$

where $A = -\frac{4d_2}{3a^2sK} + \frac{3d_1^2K}{16k^2sd_2} - \frac{K\omega}{k} + \frac{K\omega}{k^2s}$. With $ad_1d_2 > 0$ for the existence of the soliton.

3.3 Lie symmetry Analysis of NLSE-STD

In this part, we will apply the Lie point symmetry technique to study the Cls of the Kerr and parabolic laws media. To perform this task, we let

$$q(x,t) = u(x,t) + iv(x,t). \tag{3.55}$$

Putting (3.55) into (3.2),we obtain

$$\begin{aligned} v_t &= lu(u^2+v^2) + su_{tx} + ju_{xx}, \\ u_t &= -l(u^2+v^2)v - sv_{tx} - jv_{xx}. \end{aligned} \tag{3.56}$$

And the parabolic law (3.2) is transformed to the system

$$\begin{aligned} v_{,t} &= ld_1u^3 + ld_1d_2u^5 + ld_1uv^2 + 2ld_1d_2u^3v^2 + ld_1d_2uv^4 + ju_{xx} + su_{tx}, \\ u_t &= -ld_1u^2v - ld_1d_2u^4v - ld_1v^3 - 2l\alpha\beta u^2v^3 - ld_1d_2v^5 - jv_{xx} - sv_{tx}. \end{aligned} \tag{3.57}$$

The Lie symmetries of (3.56) and (3.57) are generated by the following infinitesimal generators

$$\xi_x = 2C_2tj - C_2sx + C_4, \tag{3.58}$$

$$\xi_t = -C_2st + C_3, \tag{3.59}$$

$$\eta_u = v(C_1 - xC_2), \tag{3.60}$$

$$\eta_v = -u(C_1 - xC_2), \tag{3.61}$$

with C_1, C_2, C_3 and C_4 being constants. The algebras of point symmetry of (3.2) are represented by the following generators

$$\Gamma_1 = \frac{\partial}{\partial_x}, \tag{3.62}$$

$$\Gamma_2 = \frac{\partial}{\partial_t}, \tag{3.63}$$

$$\Gamma_3 = -u\frac{\partial}{\partial_v} + v\frac{\partial}{\partial_u}, \tag{3.64}$$

$$\Gamma_4 = st\frac{\partial}{\partial_t} + ux\frac{\partial}{\partial_v} - vx\frac{\partial}{\partial_u} + (2jt - sx)\frac{\partial}{\partial_x}. \tag{3.65}$$

3.3.1. Non local conservation Laws for kerr law nonlinearity of NLSE-STD

Applying Theorems 2.10.1 and 2.11.1 on (3.2), the Lagrangian of (3.2) is obtained as:

$$\begin{aligned}\mathcal{L} = {} & p(-lu(u^2+v^2) + v_t - su_{tx} - ju_{xx}) + \\ & w(l(u^2+v^2)v + u_t + sv_{tx} + jv_{xx}),\end{aligned} \tag{3.66}$$

where $p = p(x,t)$ and $w = w(\mathrm{x},\mathrm{t})$ are dependent variables. The adjoint of (3.2) is presented as:

$$\begin{aligned}& -lp(u^2+v^2) - w_{,t} - sp_{tx} - jp_{xx} = 0, \\ & l(u^2+v^2) - p_t + sw_{tx} + jw_{xx} = 0.\end{aligned} \tag{3.67}$$

Using (2.100), we construct the conserved vectors for the Kerr law nonlinearity.

- $\Gamma_1 = \frac{\partial}{\partial x}$ gives:

$$T_1^x = \frac{1}{2}(-sp_tu_x - 2jp_xu_x + sw_tv_x + 2jw_xv_x - p(2lu^2u + 2luv^2 - 2v_t + su_{tx}) +$$

$$w(2lu^2v + 2lv^2v + 2u_t + sv_{tx})),$$

$$T_1^t = \frac{1}{2}(-(2w + sp_x)u_x - (2p - sw_x)v_x + spu_{xx} - swv_{xx}).$$

- $\Gamma_2 = \frac{\partial}{\partial t}$ yields the following conserved vector:

$$T_2^x = \frac{1}{2}(-sp_tu_t + sw_tv_t + spu_{tt} - swv_{tt} - 2ju_tp_x + 2jv_tw_x + 2jpu_{tx} - 2jwv_{tx}),$$

$$\begin{aligned}T_2^t = {} & \frac{1}{2}(-su_tp_x + sv_tw_x - p(2lu^2u + 2luv^2 + su_{tx} + 2ju_{xx}) + \\ & w(2lu^2v + 2lv^2v + sv_{tx} + 2jv_{xx})).\end{aligned}$$

• $\Gamma_3 = -u\frac{\partial}{\partial_v} + v\frac{\partial}{\partial_u}$ gives:

$$T_3^x = \frac{1}{2}(-swu_t - spv_t + v(sp_t + 2jp_x) + u(sw_t + 2jw_x) - 2jwu_x - 2jpv_x),$$

$$T_3^t = \frac{1}{2}\left(s(vp_x + uw_x) + w(2v - su_x) - p(2u + sv_x)\right).$$

• $\Gamma_4 = st\frac{\partial}{\partial_t} + ux\frac{\partial}{\partial_v} - vx\frac{\partial}{\partial_u} + (2jt - sx)\frac{\partial}{\partial_x}$ yields:

$$T_4^x = \frac{1}{2}\Big((sp_t + 2jp_x)(-xv - stu_t + (-2jt + sx)u_x) - (sw_t + 2jw_x)(xu - stv_t + (-2jt + sx)v_x) + sp(su_t + xv_t + stu_{tt} + 2ju_x + 2jtu_{tx} - sxu_{tx}) + sw(xu_t - sv_t - stv_{tt} - 2jv_x - (2jt - sx)v_{tx}) + 2jp(v - su_x + xv_x + stu_{tx} + 2jtu_{xx} - sxu_{xx}) + 2jw(u + xu_x + sv_x - stv_{tx} - (2jt - sx)v_{xx}) + 2(2jt - sx)(p(-lu(u^2 + v^2) + v_t - su_{tx} - ju_{xx}) + w(l(u^2 + v^2)v + u_t + sv_{tx} + jv_{xx}))\Big),$$

$$T_4^t = \frac{1}{2}\Big((2w + sp_x)(-xv - stu_t + (-2jt + sx)u_x) + 2\left(p - \frac{sw_x}{2}\right)(xu - stv_t + (-2jt + sx)v_x) - sp(-v + su_x - xv_x - stu_{tx} - (2jt - sx)u_{xx}) + sw(u + xu_x + sv_x - stv_{tx} - (2jt - sx)v_{xx}) + 2st(p(-lu(u^2 + v^2) + v_t - su_{tx} - u_{xx}) + w(l(u^2 + v^2)v + ju_t + sv_{tx} + jv_{xx}))\Big).$$

3.2. Non local Conservation Laws for Parabolic law nonlinearity of NLSE-STD

Applying Theorems 2.10.1 and 2.11.1 on (3.26), the Lagrangian of the (3.26) is obtained as:

$$\mathcal{L} = p(-ld_1u^3 - ld_1d_2u^5 - ld_1uv^2 - 2ld_1d_2u^3v^2 - ld_1d_2uv^4 - ju_{xx} + v_t - su_{tx}) + w(ld_1u^2v + ld_1d_2u^4v + ld_1v^3 + 2ld_1d_2u^2v^3 + ld_1d_2v^5 + jv_{xx} + u_t + sv_{tx}) \quad (3.68)$$

The adjoint to (3.26) is

$$\begin{aligned}&2ld_1wuv(1 + 2d_2u^2 + 2d_2v^2) - ld_1p\big(5d_2u^4 + v^2 + d_2v^4 + u^2(3 + 6d_2v^2)\big) - \\ &jp_{xx} - w_t - sp_{tx} = 0,\\ &-2ld_1puv(1 + 2d_2u^2 + 2d_2v^2) + ld_1w(d_2u^4 + \\ &v^2(3 + 5d_2v^2) + u^2(1 + 6\beta v^2)\big) + jw_{xx} - p_t + sw_{tx} = 0.\end{aligned} \quad (3.69)$$

Using (2.100), we construct the conserved vectors for the parabolic law nonlinearity:

- $\Gamma_1 = \frac{\partial}{\partial x}$ gives:

$$T_1^x = -p(ld_1d_2u^5 + ld_1uv^2(1 + d_2v^2) + ld_1u^3(1 + 2d_2v^2) + ju_{xx} - v_t + su_{tx}) +$$

$$w(ld_1d_2u^4v + ld_1v^3 + ld_1d_2v^5 + ld_1u^2v(1 + 2d_2v^2) + jv_{xx} + u_t + sv_{tx}),$$

$$T_1^t = 0.$$

- $\Gamma_2 = \frac{\partial}{\partial t}$ yields:

$$T_2^x = 0,$$

$$T_2^t = -p(ld_1d_2u^5 + ld_1uv^2(1 + \beta v^2) + l\alpha u^3(1 + 2\beta v^2) + ju_{xx} -$$

$$v_{,t} + su_{tx}\big) + w(l\alpha\beta u^4v + l\alpha v^3 + l\alpha\beta v^5 +$$

$$l\alpha u^2v(1 + 2\beta v^2) + jv_{xx} + u_t + sv_{tx}).$$

- For the point symmetry $\Gamma_3 = -u\frac{\partial}{\partial_v} + v\frac{\partial}{\partial_u}$, the conserved vector T_3^x, T_3^t vanishes. Thus, no conserved vector is obtained for this vector field.

• Applying the same procedure to the symmetry $\Gamma_4 = st\frac{\partial}{\partial_t} + ux\frac{\partial}{\partial_v} - vx\frac{\partial}{\partial_u} + (2jt - sx)\frac{\partial}{\partial_x}$, we get the conserved quantities given by:

$T_4^x = (2jt - sx)(-p(l\alpha\beta u^5 + l\alpha uv^2(1 + \beta v^2) + l\alpha u^3(1 + 2\beta v^2) + ju_{xx} - v_t + su_{tx}) + w(l\alpha\beta u^4 v + l\alpha v^3 + l\alpha\beta v^5 + l\alpha u^2 v(1 + 2\beta v^2) + jv_{xx} + u_t + sv_{tx})), T_4^t = st(-p(l\alpha\beta u^5 + l\alpha uv^2(1 + \beta v^2) + l\alpha u^3(1 + 2\beta v^2) + ju_{xx} - v_{,t} + su_{tx}) + w(l\alpha\beta u^4 v + l\alpha v^3 + l\alpha\beta v^5 + l\alpha u^2 \mathrm{v}1 + 2\beta v^2) + jv_{xx} + u_t + sv_{tx})$.

3.4. Cubic Nonlinear Schrödinger's equation with Repulsive Delta Potential (CNSE)

In this part, we will study the optical solitons to the NLSE having a Repulsive Delta Potential that arises in nonlinear optical fibers by utilizing using the envelope ansatz technique. Furthermore, the Lie point symmetries will be constructed and will be used to obtain the Cls of the model. Then, by applying the general Cls theorem, the Cls of the model will be explored. This δ −NLSE equation is given by [51]

$$iq_t + \frac{q_{xx}}{2} - \alpha\delta q - \gamma|q|^2 q = 0, \quad i = \sqrt{-1}, \tag{3.70}$$

where $\alpha \neq 0$ and $\gamma \neq 0$ are constants, δ represents the Dirac measure [51]. The delta potential is called attractive for $\alpha < 0$, and repulsive for $\alpha > 0$. The model was studied using different approaches in [52, 53, 54, 55]. The results of this section have been published in [56].

3.4.1. Application of envelope function ansatz to the CNSE

We start the analysis by substituting (2.2) into (3.70) to obtain

$$A(k^2 + 2\alpha\delta + 2\omega + 2\gamma|A|^2) + 2ikA_x - A_{xx} - 2iA_t = 0. \tag{3.71}$$

Substituting (2.3) into (3.71) and performing all the necessary algebraic calculations, we obtain the families given by

Family 1: If

$$\rho = 0, v = -k \pm \beta\sqrt{\gamma}, \omega = \frac{1}{2}(-k^2 - 2\beta^2\gamma - 2\alpha\delta - 2\gamma\lambda^2), \eta = \frac{\beta\gamma\lambda}{k+v}, \tag{3.72}$$

we get the dark soliton soliton represented by

$$q(x,t) = [\mathrm{i}\beta + \lambda \tanh[\eta(x - vt)]] \times e^{i(-kx+\theta+t\omega)}. \tag{3.73}$$

The intensity yields

$$|q(x,t)|^2 = \beta^2 + \lambda^2 - \lambda^2 \mathrm{sech}^2[\eta(x - vt)]. \tag{3.74}$$

The nonlinear phase shift is represented by

$$\psi_{NL} = \arctan\left[\frac{\beta}{\lambda \tanh[\eta(x-vt)]}\right]. \tag{3.75}$$

Family 2: If

$$\rho = 0, \beta = 0, v = -k, \omega = \frac{1}{2}(-k^2 - 2\alpha\delta - 2\gamma\lambda^2), \eta = \pm\sqrt{\gamma}\lambda, \tag{3.76}$$

we get the dark soliton soliton represented by

$$q(x,t) = [\mathrm{i}\beta + \lambda \tanh[\eta(x - vt)]] \times e^{i(-kx+\theta+t\omega)}. \tag{3.77}$$

Intensity yields

$$|q(x,t)|^2 = \beta^2 + \lambda^2 - \lambda^2 \mathrm{sech}^2[\eta(x - vt)]. \tag{3.78}$$

(3.70) admit dark and bright solitons. The profile views of the dark soliton equatıon (3.73) and the intensity equatıon (3.74) which appear as bright soliton are shown in Figure 3.1. Figure 3.2 describes the contour surfaces of solutions equatıons equatıons (3.73) and (3.74) by setting the values of $\beta = 0.5, \gamma = 0.5, K = 0.3, \theta = 0.4, \lambda = 0.2, k = 0.8, \delta = 0.5$,$(-10 \leq x \leq 10,\ -10 \leq t \leq 10)$. Figure 3.3 describe the the profile views of of the dark soliton equatıon (3.77) and the intensity equatıon (3.78) which appear as bright soliton. Figure 3.4 describes the contour surfaces of solutions equatıons (3.77) and (3.78) by setting the values of $\alpha = 0.2, \lambda = 0.6, \theta = 0.5, k = 0.3, \delta = 0.5$,$(-10 \leq x \leq 10, -10 \leq t \leq 10)$.

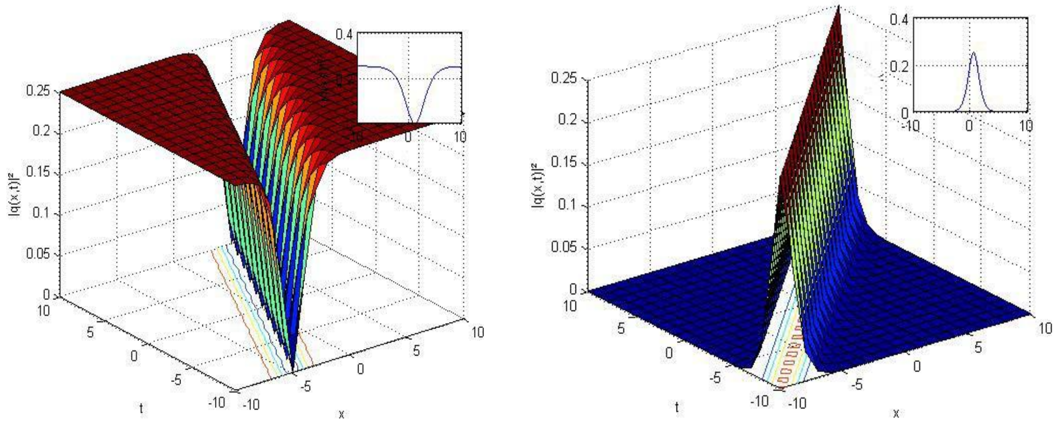

Figure 3.1: The 3D and 2D surfaces of equatıons (3.73) and (3.74)

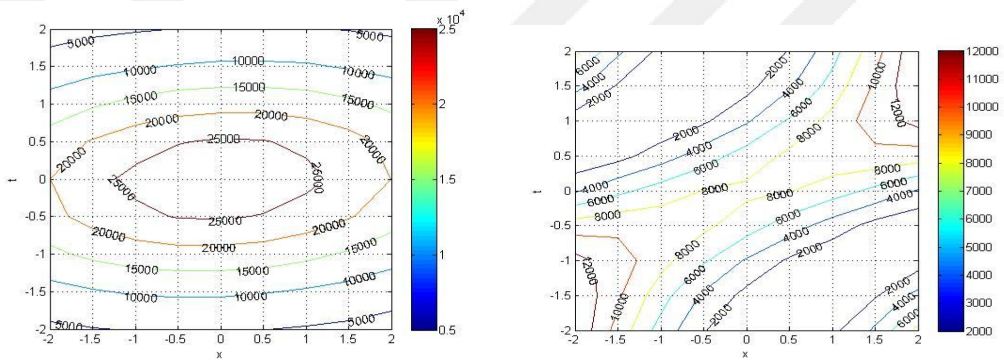

Figure 3.2: Contour surfaces of equatıons (3.73) and (3.74)

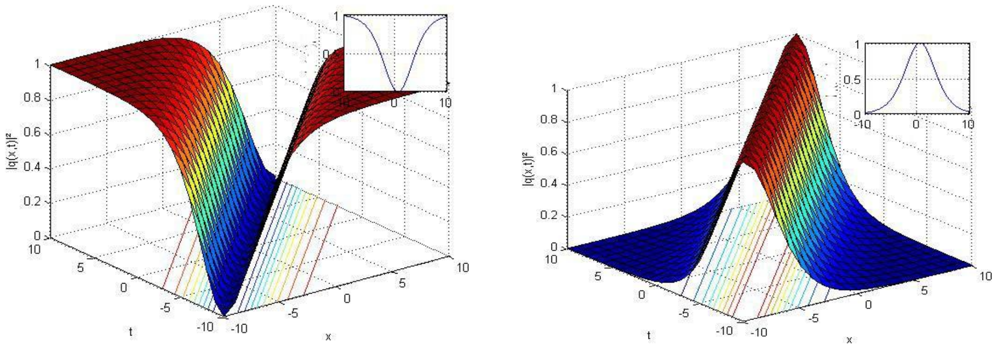

Figure 3.3: The 3D and 2D surfaces of equatıons (3.77) and (3.78).

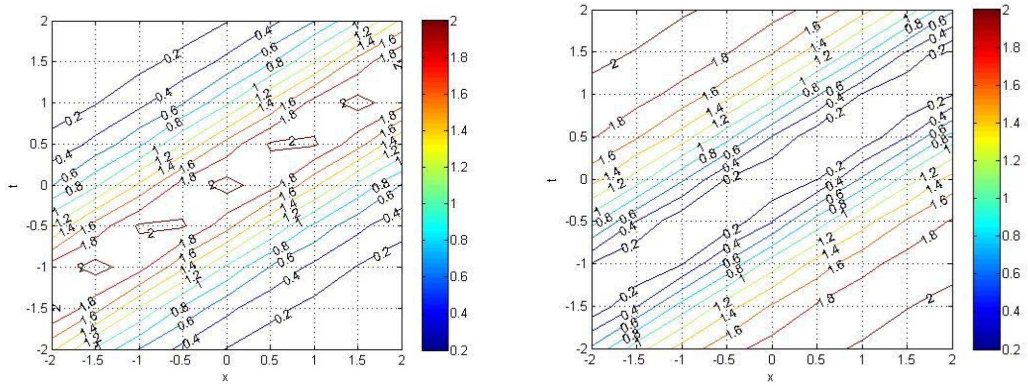

Figure 3.4: Contour surfaces of equations (3.77) and (3.78)

3.5. Lie Symmetry Analysis of the CNSE

In this part, we aim to apply the Lie symmetry technique to (3.4.1). To begin, we let

$$q(x,t) = u(x,t) + iv(x,t), \tag{3.79}$$

we obtain

$$\begin{cases} 2u_t = 2\alpha\delta v + 2\gamma v(u^2+v^2) - v_{xx}, \\ 2v_t = -2\alpha\delta u - 2\gamma u(u^2+v^2) + u_{xx}. \end{cases} \tag{3.80}$$

The Lie point symmetries of (3.70) is generated by a vector field of the form

$$\Gamma = \xi_1(x,t,u,v)\frac{\partial}{\partial x} + \xi_2(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}. \tag{3.81}$$

(3.70) has the infinitesimals given by

$$\xi_1(x,t,u,v) = \frac{xC_2}{2} + tC_3 + C_4, \tag{3.82}$$

$$\xi_2(x,t,u,v) = tC_2 + C_1, \tag{3.83}$$

$$\eta_1(x,t,u,v) = \frac{-uC_2}{2} - v(-t\alpha\delta C_2 + xC_3 + C_5), \tag{3.84}$$

$$\eta_2(x,t,u,v) = \frac{-vC_2}{2} + u(-t\alpha\delta C_2 + xC_3 + C_5), \tag{3.85}$$

C_1, C_2, C_3, C_4 and C_5 are constants. The symmetries of (3.70) are generated

$$\Gamma_1 = \frac{\partial}{\partial x}, \tag{3.86}$$

$$\Gamma_2 = \frac{\partial}{\partial t}, \tag{3.87}$$

$$\Gamma_3 = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \tag{3.88}$$

$$\Gamma_4 = -vx\frac{\partial}{\partial u} + ux\frac{\partial}{\partial v} + t\frac{\partial}{\partial x}, \tag{3.89}$$

$$\Gamma_5 = 2t\frac{\partial}{\partial t} + (-u + 2tv\alpha\delta)\frac{\partial}{\partial u} - (v + 2tu\alpha\delta)\frac{\partial}{\partial v} + x\frac{\partial}{\partial x}. \tag{3.90}$$

3.6. Nonlinear Self-Adjointness of the CNSE

In this part, we study the nonlinear self-adjointness of (3.70). Considering (3.70), we obtain its Lagrangian as follows:

$$\mathcal{L} = r(2\alpha\delta u + 2\gamma u^3 + 2\gamma uv^2 + 2v_t - u_{xx}) + z(-2\alpha\delta v - 2\gamma u^2 v - 2\gamma v^3 + 2u_t + v_{xx}). \tag{3.91}$$

The adjoint system can be obtained by

$$F_1^* = \frac{\delta\mathcal{L}}{\delta u} = 0, F_2^* = \frac{\delta\mathcal{L}}{\delta v} = 0, \tag{3.92}$$

where

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta u} &= \frac{\partial\mathcal{L}}{\partial u} - D_t\frac{\partial\mathcal{L}}{\partial u_t} - D_x\frac{\partial\mathcal{L}}{\partial u_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial u_{xx}},\\ \frac{\delta\mathcal{L}}{\delta v} &= \frac{\partial\mathcal{L}}{\partial v} - D_t\frac{\partial\mathcal{L}}{\partial v_t} - D_x\frac{\partial\mathcal{L}}{\partial v_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial v_{xx}}.\end{aligned} \tag{3.93}$$

On the basis of Lagrangian (3.91), we obtain the adjoint equation as:

$$\begin{cases} F_1^* = 2r(\alpha\delta + 3\gamma u^2 + \gamma v^2) - (4\gamma uvz + 2z_t + r_{xx}) = 0,\\ F_2^* = 4\gamma ruv + z_{xx} - \Big(2\big((\alpha\delta + \gamma u^2 + 3\gamma v^2)z + r_t\big)\Big) = 0. \end{cases} \tag{3.94}$$

Theorem 6.1 *(3.70) is nonlinear self-adjoint whenever z and r in (3.94) are given by*

$$\begin{aligned} z &= uc_1,\\ r &= vc_1. \end{aligned} \tag{3.95}$$

Proof:

Let

$$\begin{aligned} z &= \phi(x,t,u,v),\\ r &= \psi(x,t,u,v). \end{aligned} \tag{3.96}$$

Putting (3.96) into the adjoint (3.94), we get

$$\begin{aligned}&\phi_{vv}=0,\phi_{uu}=0,\psi_{vv}=0,\phi_{uv}=0,\psi_{uv}=0,\psi_{uu}=0,\phi_{xv}=0,\\&\psi_{xv}=0,\phi_{xu}=0,\psi_{xu}=0,\phi_v+\psi_u=0,-\psi_v+\phi_u=0,-\phi_v-\psi_u=0,\\&-2(u^2\gamma+3v^2\gamma+\alpha\delta)\phi+2u^3\gamma\psi_v+2u(2v\gamma\psi+v^2\gamma\psi_v+\alpha\delta\psi_v)-\\&2u^2v\gamma\psi_u-2v^3\gamma\psi_u-2v\alpha\delta\psi_u-2\psi_t+\phi_{xx}=0,\\&2(v^2\gamma+\alpha\delta)\psi+2u^3\gamma\phi_v+2u(-2v\gamma\phi+v^2\gamma\phi_v+\alpha\delta\phi_v)-\\&2v^3\gamma\phi_u-2v\alpha\delta\phi_u+u^2(6\gamma\psi-2v\gamma\phi_u)-2\phi_t-\psi_{xx}=0.\end{aligned}\tag{3.97}$$

Solving (3.97) by SYM [63], we obtain

$$\begin{aligned}&\phi=uc_1,\\&\psi=vc_1.\end{aligned}\tag{3.98}$$

Substituting (3.98) into (3.95) gives the required result.

3.7. Conservation Laws of the CNSE

We can consider the case when $c_1=1$ in (3.95), we obtain

$$z=u,\quad r=v.\tag{3.99}$$

Putting (3.99) into the Lagrangian (3.91), we obtain

$$\mathcal{L}^*=v(2\alpha\delta u+2\gamma u^3+2\gamma uv^2+2v_t-u_{xx})+u(-2\alpha\delta v-2\gamma u^2v-2\gamma v^3+2u_t+v_{xx}).\tag{3.100}$$

The conserved vectors are given by:

• $\Gamma_1=\partial_x$ yields

$$T_1^x=2(uu_t+vv_t),$$

$$T_1^t=-2(uu_x+vv_x).$$

• The generator $\Gamma_2=\partial_t$ gives

$$T_2^x=v_tu_x-u_tv_x+vu_{xt}-uv_{xt},$$

$$T_2^t=-vu_{xx}+uv_{xx}.$$

• The generator $\Gamma_4=-vx\,\partial_u+ux\,\partial_v+t\,\partial_x$ gives

$$T_4^x = u^2 + 2tuu_t + v(v + 2tv_t),$$

$$T_4^t = -2t(uu_x + vv_x).$$

• The generator $\Gamma_5 = 2t\,\partial_t + (-u + 2tv\alpha\delta)\,\partial_u - (v + 2tu\alpha\delta)\,\partial_v + x\,\partial_x$ gives

$$T_5^x = 2t(v_t u_x - u_t v_x) + v(2xv_t + 3u_x + 2tu_{xt}) + u(2xu_t - 3v_x - 2tv_{xt}),$$

$$T_5^t = -2\big(u^2 + v(v + xv_x + tu_{xx}) + u(xu_x - tv_{xx})\big).$$

3.8. Nonlinear Schrödinger's Equation in Compressional Dispersive Alv$e$n Waves (NLSE-CAW)

In this section, the CDA waves are considered. Two experiments have been carried out on the CDA waves. The first experiment [57] studied the relationship between the perturbation and the CDA waves within a low plasma. The second experiment [58] considered the amplitude of the waves in a magnetic electron-positron plasma. In all the two experiments, it was finalized that the system of equations obtained can be written as single wave equation represented by

$$\phi_{tt} - (3a^2 + c^2)\phi_{xx} - \delta^2\phi_{xxxx} - \delta^2\phi_{xxtt} = 0. \tag{3.101}$$

When the envelope of the CDA evolves, the following NLSE is obtained [57]

$$iq_t + \mu|q|^2 q + i\gamma q_x - \delta q_{xx} = 0, \tag{3.102}$$

where x is the distance along the fiber, t represents the temporal variable, δ is the coefficient of group-velocity dispersion (GVD), μ is the self-phase modulation (SPM) and γ is the intermodal dispersion (IMD). The model was studied using different approaches in [57 -60].

In the remaining part of this section, we report some solutions of (3.102) by applying the SGEM. Then, we construct the Cls by invoking the new conservation theorem. The results of this section have been published in [62].

3.9. Application of sine-Gordon equation method to the NLSE-CAW

In order to solve (3.102), we let

$$q(x,t) = U(\xi)e^{i\phi(x,t)}, \quad \xi = K(x - vt), \tag{3.103}$$

where

$$\phi = -kx + \omega t + \theta. \tag{3.104}$$

Putting (3.103) into (3.102), we acquire

$$v = \gamma + 2k\delta, \tag{3.105}$$

and

$$(-k\gamma - k^2\delta + \omega)U - \mu U^3 + K^2\delta U'' = 0.$$

(3.106)

Balancing the terms U^3 and U'' in (3.106) gives $n = 1$. Putting $n = 1$ into (2.17), we obtain

$$U(w) = iB_1\sin(w) + A_1\cos(w) + iA_0. \tag{3.107}$$

Differentiating (3.107), we acquire

$U'' = -2\cos(w)\sin(w)^2A_1 + i\cos(w)^2\sin(w)B_1 - i\sin(w)^3B_1$. (3.108).

Putting (3.107) and (3.108) into (3.106) using (2.12) and applying trigonometric identities, gives

$$\begin{gathered}-k\gamma A_0 - k^2\delta A_0 + \omega A_0 - \mu A_0^3 - k\gamma\cos(w)A_1 - k^2\delta\cos(w)A_1 + \\ \omega\cos(w)A_1 - 2K^2\delta\cos(w)\sin(w)^2A_1 - 3\mu\cos(w)A_0^2A_1 \\ -3\mu A_0A_1^2 + 3\mu\sin(w)^2A_0A_1^2 - \mu\cos(w)A_1^3 + \mu\cos(w)\sin(w)^2 \\ \sin(w)A_0A_1B_1 - 3\mu\cos(w)^2\sin(w)A_1^2B_1 - 3\mu\sin(w)^2A_0B_1^2 - \\ 3\mu\cos(w)\sin(w)^2A_1B_1^2 - \mu\sin(w)B_1^3 + \mu\cos(w)^2\sin(w)B_1^3 = 0.\end{gathered} \tag{3.109}$$

Collecting terms and performing all the necessary algebraic calculations, we obtain:

$$-A_1(k\gamma + k^2\delta - \omega + 3\mu A_0^2 + \mu A_1^2) = 0, \tag{3.110}$$

$$-A_0(k\gamma + k^2\delta - \omega + \mu A_0^2 + 3\mu A_1^2) = 0, \tag{3.111}$$

$$-6\mu A_0 A_1 B_1 = 0, \tag{3.112}$$

$$3\mu A_0(A_1^2 - B_1^2) = 0, \tag{3.113}$$

$$-B_1(k\gamma + k^2\delta + K^2\delta - \omega + 3\mu A_0^2 + \mu B_1^2) = 0, \tag{3.114}$$

$$B_1(2K^2\delta - 3\mu A_1^2 + \mu B_1^2) = 0, \tag{3.115}$$

$$A_1(-2K^2\delta + \mu A_1^2 - 3\mu B_1^2) = 0. \tag{3.116}$$

Solving (3.110)-(3.116), we acquire the sets of solutions given by:

Family 1: When

$$A_0 = 0, B_1 = 0, A_1 = \pm K\sqrt{\frac{2\delta}{\mu}}, \omega = k\gamma + k^2\delta + 2K^2\delta, \tag{3.117}$$

we obtain the dark soliton given by

$$q(x,t) = [\pm K\sqrt{\frac{2\delta}{\mu}}\tanh[x - (\gamma + 2k\delta)t]] \times e^{i(-kx+(k\gamma+k^2\delta+2K^2\delta)t+\theta)}, \tag{3.118}$$

along with the dark-singular soliton

$$q(x,t) = [\pm K\sqrt{\frac{2\delta}{\mu}}\coth[x - (\gamma + 2k\delta)t]] \times e^{i(-kx+(k\gamma+k^2\delta+2K^2\delta)t+\theta)}. \tag{3.119}$$

Family 2: When

$$A_0 = 0, A_1 = 0, B_1 = \pm\sqrt{\frac{2\delta}{\delta}}, \omega = k\gamma + k^2\delta - K^2\delta, \tag{3.120}$$

we obtain the bright soliton given by

$$q(x,t) = [\pm K\sqrt{\frac{2\delta}{\mu}}\operatorname{sech}[x-(\gamma+2k\delta)t]] \times e^{i(-kx+(k\gamma+k^2\delta-K^2)t+\theta)}, \tag{3.121}$$

along with the singular soliton

$$q(x,t) = [\pm iK\sqrt{\frac{2\delta}{\mu}}\operatorname{csch}[x-(\gamma+2k\delta)t]] \times e^{i(-kx+(k\gamma+k^2\delta-K^2)t+\theta)}. \tag{3.122}$$

Family 3: When

$$A_0 = 0, B_1 = \pm iK\sqrt{\frac{\delta}{2\mu}}, A_1 = \pm K\sqrt{\frac{\delta}{2\mu}}, \omega = \frac{1}{2}(2k\gamma + 2k^2\delta + K^2\delta), \tag{3.123}$$

we obtain the dark-bright soliton given by

$$q(x,t) = \left[\pm iK\sqrt{\frac{\delta}{2\mu}}\operatorname{sech}[x-(\gamma+2k\delta)t] \pm K\sqrt{\frac{\delta}{2\mu}}\tanh[x-(\gamma+2k\delta)t]\right] \times \tag{3.124}$$

$$e^{i(-kx+(\frac{1}{2}(2k\gamma+2k^2\delta+K^2\delta))t+\theta)},$$

and the combined singular soliton

$$q(x,t) = \left[-K\sqrt{\frac{\delta}{2\mu}}\operatorname{csch}[x-(\gamma+2k\delta)t] \pm K\sqrt{\frac{\delta}{2\mu}}\coth[x-(\gamma+2k\delta)t]\right] \times$$

$$e^{i(-kx+(\frac{1}{2}(2k\gamma+2k^2\delta+K^2\delta))t+\theta)}. \tag{3.125}$$

(3.102) admit dark and bright solitons. The profile view of the dark soliton equatıon (3.118) and the bright soliton equatıon (3.121) are shown in Figure 3.5 for the values of $K = 1, \mu = 0.1, \gamma = 0.5, k = 0.3, \delta = 0.3, \theta = 1$. Figure 3.6 describes the surface view of the dark- bright soliton of solutions equatıon (3.124) by setting the values of $\beta = 0.5, \gamma = 0.5, K = 0.3, \theta = 0.4, \lambda = 0.2, k = 0.8, \delta = 0.5$,($-10 \leq x \leq 10$, $-10 \leq t \leq 10$).

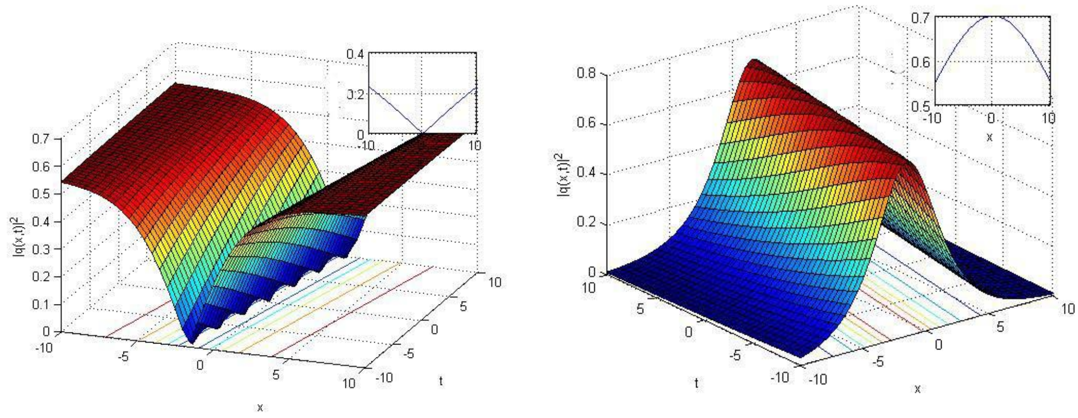

Figure 3.5: The 3D and 2D views of equatıons (3.118) and (3.121)

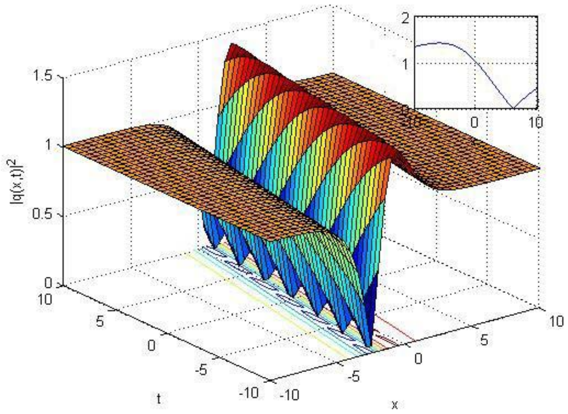

Figure 3.6: The 3D and 2D views of equatıon (3.124)

3.10. Lie point symmetry analysis of the NLSE-CAW

Here, we will study the Lie symmetry of (3.102). To begin, we let

$$q(x,t) = u(x,t) + iv(x,t). \tag{3.126}$$

Putting (3.126) into (3.102), we acquire

$$\begin{aligned} u_t &= -v\mu(u^2+v^2) - \gamma u_x + \delta v_{xx}, \\ v_t &= u\mu(u^2+v^2) - \gamma v_x - \delta u_{xx}. \end{aligned} \tag{3.127}$$

(3.127) has the following Lie point symmetries

$$\Gamma_1 = \frac{\partial}{\partial x}, \tag{3.128}$$

$$\Gamma_2 = \frac{\partial}{\partial t}, \tag{3.129}$$

$$\Gamma_3 = -u\frac{\partial}{\partial v} + v\frac{\partial}{\partial u}, \tag{3.130}$$

$$\Gamma_4 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + (\gamma t + x)\frac{\partial}{\partial x}, \tag{3.131}$$

$$\Gamma_5 = -2t\delta\frac{\partial}{\partial x} + (\gamma tv - vx)\frac{\partial}{\partial u} + (-\gamma tu + ux)\frac{\partial}{\partial v}. \tag{3.132}$$

3.11 Nonlinear Self-Adjointness of the NLSE-CAW

Considering (3.127), we compute its formal Lagrangian as

$$\mathcal{L} = r(-\mu u^3 - \mu uv^2 + v_t + \gamma v_x + \delta u_{xx}) + z(\mu u^2 v + \mu v^3 + u_t + \gamma u_x - \delta v_{xx}). \tag{3.133}$$

The adjoint system can be derived using:

$$F_1^* = \frac{\delta\mathcal{L}}{\delta u} = 0, F_2^* = \frac{\delta\mathcal{L}}{\delta v} = 0, \tag{3.134}$$

where

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta u} &= \frac{\partial\mathcal{L}}{\partial u} - D_t\frac{\partial\mathcal{L}}{\partial u_t} - D_x\frac{\partial\mathcal{L}}{\partial u_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial u_{xx}},\\ \frac{\delta\mathcal{L}}{\delta v} &= \frac{\partial\mathcal{L}}{\partial v} - D_t\frac{\partial\mathcal{L}}{\partial v_t} - D_x\frac{\partial\mathcal{L}}{\partial v_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial v_{xx}}.\end{aligned} \tag{3.135}$$

On the basis of Lagrangian of (3.133), one can get the adjoint system as

$$\begin{aligned}F_1^* &= 2\mu uzv - \mu r(3u^2 + v^2) - z_t - \gamma z_x + \delta r_{xx} = 0,\\ F_2^* &= -2\mu ruv + \mu z(u^2 + 3v^2) - r_t - \gamma r_x - \delta z_{xx} = 0.\end{aligned} \tag{3.136}$$

Theorem 11.1 *System of (3.127) is nonlinear self-adjoint if z and r in (3.136) are given by*

$$\begin{aligned}z &= uc_1\\ r &= vc_1.\end{aligned} \tag{3.137}$$

Proof:

Suppose

$$\begin{aligned}z &= \phi(x,t,u,v),\\ r &= \psi(x,t,u,v).\end{aligned} \tag{3.138}$$

Substituting (3.138) into the adjoint (3.136), we obtain

$$\delta\psi_{vv} = 0, \delta\psi_{uu} = 0, \delta\phi_{vv} = 0, \delta\phi_{uv} = 0, \delta\psi_{uv} = 0, \delta\phi_{uu} = 0,$$
$$\delta\phi_{xv} = 0, \delta\psi_{xv} = 0, \delta\phi_{xu} = 0, \delta\psi_{xu} = 0, \delta(\phi_v + \psi_u) = 0, \delta(\phi_v + \psi_u) = 0,$$
$$\delta\psi_v = \delta\phi_u, 3\mu v^2\phi - u^3\mu\psi_v - u\mu v(2\psi + v\psi_v) + \mu v^3\psi_u + u^2\mu(\phi + v\psi_u) -$$
$$\psi_t - \gamma\psi_x - \delta\phi_{xx} = 0, -\mu v^2\psi - u^3\mu\phi_v + u\mu v(2\phi - v\phi_v) +$$
$$\mu v^3\phi_u + u^2(-3\mu\psi + \mu v\phi_u) - \phi_t - \gamma\phi_x + \delta\psi_{xx} = 0. \tag{3.139}$$

Solving (3.139) by SYM [63] gives

$$\begin{aligned} \phi &= uc_1 \\ \psi &= vc_1. \end{aligned} \tag{3.140}$$

Substituting (3.140) into (3.137) gives the required result.

3.12. Conservation laws of NLSE-CAW

(3.127) is nonlinear self-adjoint with the substitution (3.137), this fact along with the Lie point symmetries (3.128)-(3.132) will be used to construct the Cls. We consider the case when $c_1 = 1$ in (3.137) which reduces to the following

$$z = u(x,t), r = v(x,t). \tag{3.141}$$

Substituting (3.141) into the Lagrangian (3.133), we obtain

$$\mathcal{L}^* = u(-\mu u^3 - \mu u v^2 + v_t + \gamma v_x + \delta u_{xx}) + u(\mu u^2 v + \mu v^3 + u_t + \gamma u_x - \delta v_{xx}). \tag{3.142}$$

The conserved vectors are given by.

• The generator $\Gamma_1 = \partial_x$ gives the conserved vector given by

$$T_1^x = -\mu u^4 + \mu u^3 v - \mu u^2 v^2 + u(\mu v^3 + v_t) + \delta u_x(u_x - v_x),$$

$$T_1^t = -u v_x.$$

• The generator $\Gamma_2 = \partial_t$ gives the conserved vector given by

$$T_2^x = \delta(u_t - v_t)u_x - u\big(\gamma v_t + \delta(u_{xt} - v_{xt})\big),$$

$$T_2^t = -u(\mu u^3 - \mu u^2 v + \mu u v^2 - \mu v^3 - \gamma v_x - \delta u_{xx} + \delta v_{xx}).$$

• The generator $\Gamma_3 = -u\,\partial_v + v\,\partial_u$ gives

$$T_3^x = -\gamma u^2 - \delta v u_x + u(\gamma v + \delta v_x),$$

$$T_3^t = u(-u + v).$$

• The generator $\Gamma_4 = 2t\,\partial_t - u\,\partial_u - v\,\partial_v + (\gamma t + x)\,\partial_x$ gives

$$T_4^x = -(x + t\gamma)\mu u^4 + (x + t\gamma)\mu u^3 v + \delta u_x\big(2tvv_t + (x + t\gamma)(u_x - v_x)\big) - u^2\big((x + t\gamma)\mu v^2 - 2t(\gamma u_t + \delta u_{xt})\big) + u((x + t\gamma)\mu v^3 + (x + t\gamma)u_t + \delta(-u_x + v_x) + v_t(x + t\gamma - 2t\delta v_x) + 2tv(\gamma v_t - \delta v_{xt})\big),$$

$$T_4^t = u\big(2tuu_t + 2tvv_t - (x + t\gamma)(u_x + v_x)\big).$$

• The generator $\Gamma_5 = -2t\delta\,\partial_x + (\gamma tv - vx)\,\partial_u + (-\gamma tu + ux)\,\partial_v$ gives

$$T_5^x = 2t\delta\mu u^4 - 2t\delta\mu u^3 v + 2t\delta\mu u^2 v^2 - u((\gamma(x - t\gamma) + \delta)v + 2t\delta\mu v^3 + 2t\delta(u_t + v_t) + (x - t\gamma)\delta v_x) + \delta u_x\big((x - t\gamma)v + 2t\delta(-u_x + v_x)\big),$$

$$T_5^t = u\big((-x + t\gamma)v + 2t\delta(u_x + v_x)\big).$$

4. OPTICAL SOLITONS AND MODULATION INSTABILITY ANALYSIS TO SOME NONLINEAR SCHR$\ddot{o}$DINGER'S EQUATIONS

The relation between MI and solitons is best observed in the fact that the trains of pulses that emerge from the MI process are actually trains of almost ideal solitons. Hence it can be loosely considered as a precursor to soliton formation. In this chapter, we will discuss three different nonlinear schr$\ddot{o}$dinger equations. We beging by finding the soliton solutions and thereafter study the MI analysis of each of the equations.

4.1. Coupled Nonlinear Schr$\ddot{o}$dinger Equation in Monomode Step-Index Optical Fibers (CNLSE)

This section addresses the CNLSE in monomode step-index in optical fibers describing the nonlinear modulations of two monochromatic waves. Dark, bright and combined optical solitary wave the model are derived using the envelope function ansatz. The MI analysis of the equation. The results of this section have been published in [67].

4.1.1. Theoretical model

The CNLSE equation is given by [68- 70]:

$$\begin{cases} i\dfrac{\partial A_1}{\partial t}+\dfrac{\partial^2 A_1}{\partial x^2}+(a|A_1|^2+|A_2|^2)A_1=0, \\ \dfrac{\partial A_2}{\partial t}+\dfrac{\partial^2 A_2}{\partial x^2}+(a|A_2|^2+|A_1|^2)A_2=0, \end{cases} \tag{4.1}$$

where A_1 and A_2 are complex functions, a is a real valued constant. The equation is an extension of the NLSE given by [69]

$$i\frac{\partial A_1}{\partial t}+\frac{\partial^2 A_1}{\partial x^2}+A_1|A_1|^2=0. \tag{4.2}$$

4.1.2. Optical Solitary Waves

We let

$$A_1(x,t) = B(x,t)e^{i\phi(x,t)}, \tag{4.3}$$

$$A_2(x,t) = S(x,t)e^{i\phi(x,t)}, \tag{4.4}$$

where

$$\phi(x,t) = -kx + \omega t + \theta. \tag{4.5}$$

In (4.3) and (4.4), $B(x,t)$ and $S(x,t)$ are functions and ϕ is as described in (2.2).

The ansatz [22] requires a modification in the form

$$B(x,t) = \mathrm{i}\beta_1 + \lambda_1 \tanh[\eta(x - vt)] + \mathrm{i}\rho_1 \operatorname{sech}[\eta(x - vt)], \tag{4.6}$$

$$S(x,t) = \mathrm{i}\beta_2 + \lambda_2 \tanh[\eta(x - vt)] + \mathrm{i}\rho_2 \operatorname{sech}[\eta(x - vt)], \tag{4.7}$$

η is the pulse width and v represents the velocity. The functions $B(x,t)$ and $S(x,t)$ can be represented as: as

$$|B(x,t)| = \{\lambda_1^2 + \beta_1^2 + 2\beta_1\rho_1 \operatorname{sech}[\eta(x - vt)] + (\rho_1^2 - \lambda_1^2)\operatorname{sech}^2[\eta(x - vt)]\}^{\frac{1}{2}}, \tag{4.8}$$

$$|S(x,t)| = \{\lambda_2^2 + \beta_2^2 + 2\beta_2\rho_2 \operatorname{sech}[\eta(x - vt)] + (\rho_2^2 - \lambda_2^2)\operatorname{sech}^2[\eta(x - vt)]\}^{\frac{1}{2}}. \tag{4.9}$$

The nonlinear phase shifts are represented by

$$\phi_{NL_1} = \arctan\left[\frac{\beta_1 + \rho_1 \operatorname{sech}[\eta(x - vt)]}{\lambda_1 \tanh[\eta(x - vt)]}\right], \tag{4.10}$$

$$\phi_{NL_2} = \arctan\left[\frac{\beta_2 + \rho_2 \operatorname{sech}[\eta(x - vt)]}{\lambda_2 \tanh[\eta(x - vt)]}\right]. \tag{4.11}$$

Putting (4.3) and (4.4) into (4.1) gives:

$$\begin{cases} B(-k^2 - \omega + |S|^2 + a|B|^2) - 2ikB_x + B_{xx} + iB_t = 0, \\ S(-k^2 - \omega + a|S|^2 + |B|^2) - 2ikS_x + S_{xx} + iS_t = 0. \end{cases} \tag{4.12}$$

Substituting (4.6) and (4.7) into Eq.(4.12), we acquire

$$-2\eta^2\mathrm{sech}(\tau)^2\lambda_1\tanh(\tau) + (-k^2 - \omega + \beta_2^2 + \lambda_2^2 + a(\beta_1^2 + \lambda_1^2 + 2\mathrm{sech}(\tau)\beta_1\rho_1 + \mathrm{sech}(\tau)^2(-\lambda_1^2 + \rho_1^2)) + 2\mathrm{sech}(\tau)\beta_2\rho_2 + \mathrm{sech}(\tau)^2(-\lambda_2^2 + \rho_2^2)\Big)(i\beta_1 + i\mathrm{sech}(\tau)\rho_1 + \lambda_1\tanh(\tau)) - 2ik(\eta\mathrm{sech}(\tau)^2\lambda_1 - i\eta\mathrm{sech}(\tau)\rho_1\tanh(\tau)) + i(-v\eta\mathrm{sech}(\tau)^2\lambda_1 + iv\eta\mathrm{sech}(\tau)\rho_1\tanh(\tau)) + i\rho_1(-\eta^2\mathrm{sech}(\tau)^3 + \eta^2\mathrm{sech}(\tau)\tanh(\tau)^2) = 0, \tag{4.13}$$

$$-2\eta^2\mathrm{sech}(\tau)^2\lambda_2\tanh(\tau) + (-k^2 - \omega + \beta_1^2 + \lambda_1^2 + 2\mathrm{sech}(\tau)\beta_1\rho_1 + \mathrm{sech}(\tau)^2(-\lambda_1^2 + \rho_1^2) + a\big(\beta_2^2 + \lambda_2^2 + 2\mathrm{sech}(\tau)\beta_2\rho_2 + \mathrm{sech}(\tau)^2(-\lambda_2^2 + \rho_2^2)\big)\Big)(i\beta_2 + i\mathrm{sech}(\tau)\rho_2 + \lambda_2\tanh(\tau)) - 2ik(\eta\mathrm{sech}(\tau)^2\lambda_2 - i\eta\mathrm{sech}(\tau)\rho_2\tanh(\tau)) + i(-v\eta\mathrm{sech}(\tau)^2\lambda_2 + iv\eta\mathrm{sech}(\tau)\rho_2\tanh(\tau)) + i\rho_2(-\eta^2\mathrm{sech}(\tau)^3 + \eta^2\mathrm{sech}(\tau)\tanh(\tau)^2) = 0, \tag{4.14}$$

where $\tau = \eta(x - vt)$. After making some computations, we obtain:

$$\begin{aligned} &i\beta_1(-k^2 - \omega + \beta_2^2 + a(\beta_1^2 + \lambda_1^2) + \lambda_2^2) = 0,\\ &i\beta_2\big(-k^2 - \omega + \beta_1^2 + \lambda_1^2 + a(\beta_2^2 + \lambda_2^2)\big) = 0. \end{aligned} \tag{4.15}$$

$$\begin{aligned} &i\big((-k^2 + \eta^2 - \omega + 3a\beta_1^2 + \beta_2^2 + a\lambda_1^2 + \lambda_2^2)\rho_1 + 2\beta_1\beta_2\rho_2\big) = 0,\\ &i(2\beta_1\beta_2\rho_1 + (-k^2 + \eta^2 - \omega + \beta_1^2 + 3a\beta_2^2 + \lambda_1^2 + a\lambda_2^2)\rho_2) = 0. \end{aligned} \tag{4.16}$$

$$\begin{aligned} &-i\big((2k + v)\eta\lambda_1 + a\beta_1\lambda_1^2 - 2\beta_2\rho_1\rho_2 + \beta_1(\lambda_2^2 - 3a\rho_1^2 - \rho_2^2)\big) = 0,\\ &-i\big((2k + v)\eta\lambda_2 - 2\beta_1\rho_1\rho_2 + \beta_2(\lambda_1^2 + a\lambda_2^2 - \rho_1^2 - 3a\rho_2^2)\big) = 0. \end{aligned} \tag{4.17}$$

$$\begin{aligned} &i\rho_1(-2\eta^2 - a\lambda_1^2 - \lambda_2^2 + a\rho_1^2 + \rho_2^2) = 0,\\ &i\rho_2(-2\eta^2 - \lambda_1^2 - a\lambda_2^2 + \rho_1^2 + a\rho_2^2) = 0. \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\lambda_1(-k^2 - \omega + \beta_2^2 + a(\beta_1^2 + \lambda_1^2) + \lambda_2^2) = 0,\\ &\lambda_2\big(-k^2 - \omega + \beta_1^2 + \lambda_1^2 + a(\beta_2^2 + \lambda_2^2)\big) = 0. \end{aligned} \tag{4.19}$$

$$(-((2k+v)\eta - 2a\beta_1\lambda_1)\rho_1 + 2\beta_2\lambda_1\rho_2) = 0,$$
$$\left(-(2k+v)\eta\rho_2 + 2\lambda_2(\beta_1\rho_1 + a\beta_2\rho_2)\right) = 0. \tag{4.20}$$

$$\lambda_1(-2\eta^2 - a\lambda_1^2 - \lambda_2^2 + a\rho_1^2 + \rho_2^2) = 0,$$
$$\lambda_2(-2\eta^2 - \lambda_1^2 - a\lambda_2^2 + \rho_1^2 + a\rho_2^2) = 0. \tag{4.21}$$

The solution of (4.15)-(4.21) gives the following sets and solutions.

(a) Bright Optical Solitary Waves

If

$$\lambda_2 = 0, \lambda_1 = 0, \beta_2 = 0, \beta_1 = 0, a = 1, v = -2k,$$
$$\eta = \pm\sqrt{\frac{\rho_1^2+\rho_2^2}{2}}, \omega = \frac{1}{2}(-2k^2 + \rho_1^2 + \rho_2^2), \tag{4.22}$$

the following bright optical solitary wave is retrieved

$$A_1(x,t) = \mathrm{i}\rho_1 \mathrm{sech}[\eta(x - vt)] \times e^{i(-kx+\omega t+\theta)},$$
$$A_2(x,t) = \mathrm{i}\rho_2 \mathrm{sech}[\eta(x - vt)] \times e^{i(-kx+\omega t+\theta)}. \tag{4.23}$$

The intensities are given by

$$|A_1(x,t)|^2 = \rho_1 \mathrm{sech}^2[\eta(x - vt)],$$
$$|A_2(x,t)|^2 = \rho_2 \mathrm{sech}^2[\eta(x - vt)]. \tag{4.24}$$

The 2D and 3D surfaces of the bright solitary wave (4.23) are given in Figure 4.1 for some selected values of the parameters written in the figures:

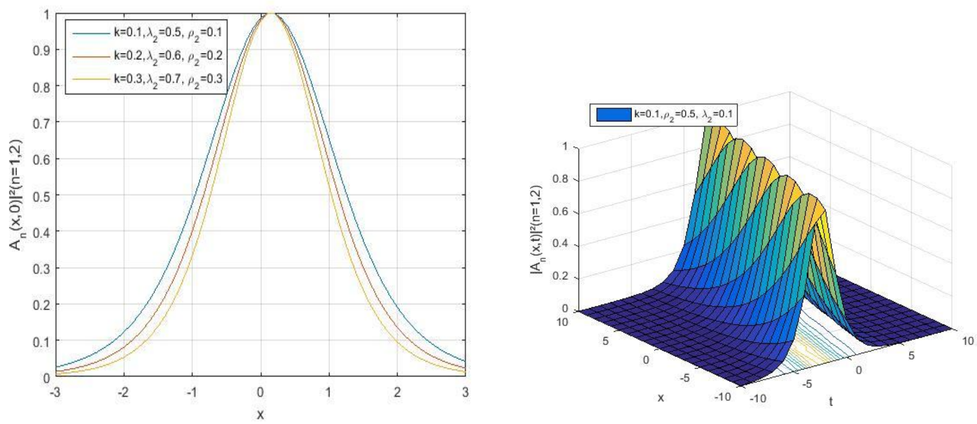

Figure 4.1: The 3D and 2D views of equation (4.23).

(b) **Dark Optical Solitary Waves**

If

$$\rho_2 = 0, \rho_1 = 0, \beta_2 = 0, \beta_1 = 0, a = 1, v = -2k,$$
$$\eta = \pm\sqrt{\frac{-\lambda_1^2-\lambda_2^2}{2}}, \omega = -k^2 + \lambda_1^2 + \lambda_2^2, \tag{4.25}$$

the following dark optical solitary wave is retrieved

$$A_1(x,t) = \lambda_1 \tanh[\eta(x - vt)] \times e^{i(-kx+\omega t+\theta)},$$
$$A_2(x,t) = \lambda_2 \tanh[\eta(x - vt)] \times e^{i(-kx+\omega t+\theta)}. \tag{4.26}$$

The intensities are given by

$$|A_1(x,t)|^2 = \{\lambda_1^2 - \lambda_1^2 \mathrm{sech}^2[\eta(x - vt)]\},$$
$$|A_2(x,t)|^2 = \{\lambda_2^2 - \lambda_2^2 \mathrm{sech}^2[\eta(x - vt)]\}. \tag{4.27}$$

The solitary waves are exist provided $(-\lambda_1^2 - \lambda_2^2) < 0$.

The 2D and 3D surfaces of the bright solitary wave (4.26) are given in Figure 4.2 for some selected values of the parameters written in the figures:

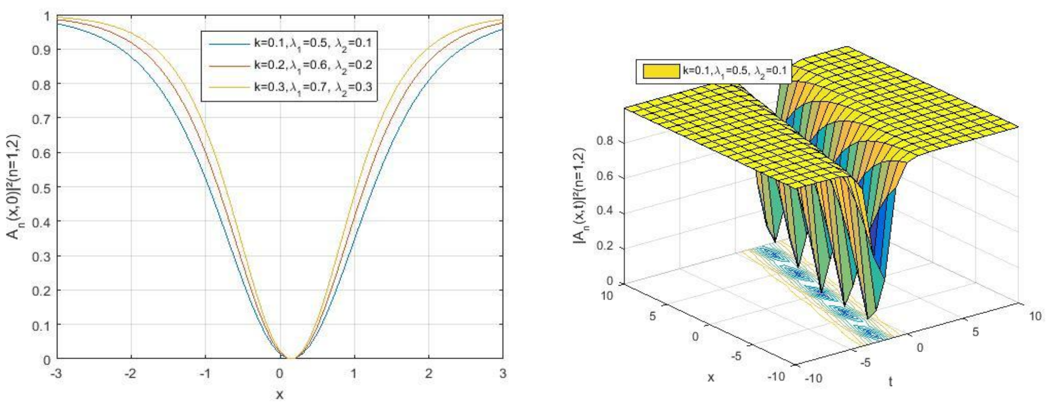

Figure 4.2: The 3D and 2D views of (4.26)

(c) Dark-bright Optical Solitary Waves

If

$$\begin{aligned} &\rho_1 = \pm\rho_2, \lambda_1 = \pm\lambda_2, \beta_1 = \pm\sqrt{\frac{-\lambda_2^2+\rho_2^2}{2}}, \beta_2 = \pm\sqrt{\frac{-\lambda_2^2+\rho_2^2}{2}}, \\ &a = -1, v = -2k - \frac{2\lambda_2}{\sqrt{3}}, \eta = \sqrt{\frac{2(-\lambda_2^2+\rho_2^2)}{3}}, \omega = -k^2, \end{aligned} \tag{4.28}$$

the following dark-bright optical solitary wave is retrieved

$$\begin{aligned} &A_1(x,t) = \{\mathrm{i}\beta_1 + \lambda_1 \tanh[\eta(x-vt)] + \mathrm{i}\rho_1 \operatorname{sech}[\eta(x-vt)]\} \times e^{i(-kx+\omega t+\theta)}, \\ &A_2(x,t) = \{\mathrm{i}\beta_2 + \lambda_2 \tanh[\eta(x-vt)] + \mathrm{i}\rho_2 \operatorname{sech}[\eta(x-vt)]\} \times e^{i(-kx+\omega t+\theta)}. \end{aligned} \tag{4.29}$$

The intensities are given by

$$\begin{aligned} &|A_1(x,t)|^2 = \{\lambda_1^2 + \beta_1^2 + 2\beta_1\rho_1 \operatorname{sech}[\eta(x-vt)] + (\rho_1^2 - \lambda_1^2)\operatorname{sech}^2[\eta(x-vt)]\}, \\ &|A_2(x,t)|^2 = \{\lambda_2^2 + \beta_2^2 + 2\beta_2\rho_2 \operatorname{sech}[\eta(x-vt)] + (\rho_2^2 - \lambda_2^2)\operatorname{sech}^2[\eta(x-vt)]\}. \end{aligned} \tag{4.30}$$

The solitary waves exist provided $(-\lambda_2^2 + \rho_2^2) > 0$.

Physically, (4.30) describes dual pulse solutions based on the signs of $\beta_1, \rho_1, \beta_2, \rho_2$. If $(\beta_1\rho_1) > 0$ and $\beta_2\rho_2 > 0$, the intensities (4.30) are bright solitary waves, whereas the dark solitary waves are obtained whenever $\beta_1\rho_1 < 0$ and $\beta_2\rho_2 < 0$.

Thus, the nonlinear phase shifts are represented by

$$\begin{cases} \phi_{NL_1} = \arctan\left[\dfrac{\beta_1+\rho_1 \operatorname{sech}[\eta(x-vt)]}{\lambda_1 \tanh[\eta(x-vt)]}\right], \\ \phi_{NL_2} = \arctan\left[\dfrac{\beta_2+\rho_2 \operatorname{sech}[\eta(x-vt)]}{\lambda_2 \tanh[\eta(x-vt)]}\right]. \end{cases} \tag{4.31}$$

The 2D and 3D surfaces of the dark-bright solitary wave (4.29) are given in Figure 4.3 for some selected values of the parameters written in the figures:

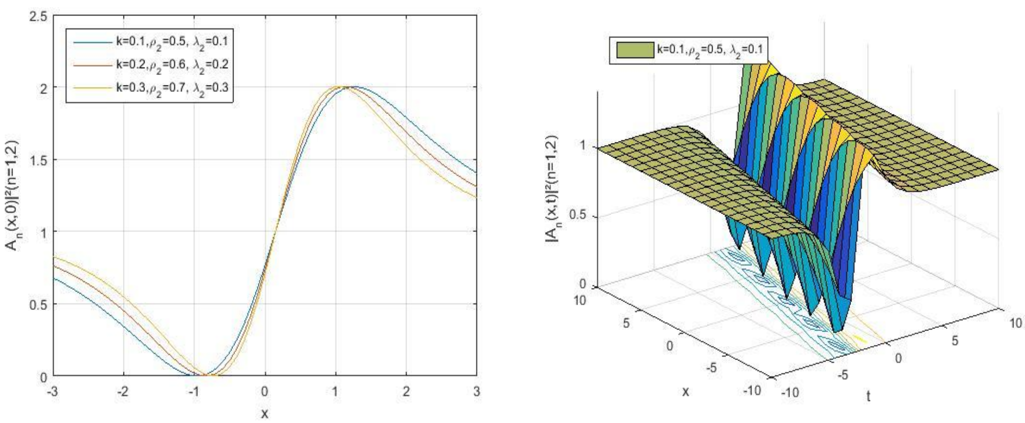

Figure 4.3: The 3D and 2D views of equation (4.29).

4.1.3. Modulation instability analysis of the CNLSE

Suppose (4.1) has steady-state solution

$$A_j(x,t) = \{\sqrt{P_j} + F_j(x,t)\} \times e^{i\phi_j}, \tag{4.32}$$

where $(F_j << P_j, P_j)$ is incident power,

$$\phi_j(x) = a(P_j + 2P_{3-j})x \tag{4.33}$$

is the phase component [82]. Substituting (4.32) into (4.1) and linearizing, the perturbations $F_1(x,t)$ and $F_2(x,t)$ satisfy system given by

$$\begin{cases} i\frac{\partial F_1}{\partial t} + \frac{\partial^2 F_1}{\partial x^2} + P_1 a(F_1 + F_1^*) + 2(P_1P_2)^{\frac{1}{2}}(F_2 + F_2^*) = 0, \\ i\frac{\partial a_2}{\partial t} + \frac{\partial^2 a_2}{\partial x^2} + P_2 a(F_2 + F_2^*) + 2(P_1P_2)^{\frac{1}{2}}(F_1 + F_1^*) = 0. \end{cases} \tag{4.34}$$

Because of F_j^*, the frequencies Ω and $-\Omega$ are coupled. So,

$$F_j(x,t) = \beta_j \times e^{i(Kx-\Omega t)} + \alpha_j \times e^{-i(Kx-\Omega t)}, \quad j = 1,2. \tag{4.35}$$

In (4.35), Ω represents frequency and K represents the wave number. (4.34) and (4.35) give a set of two sets of equations. Putting (4.35) into (4.34), we acquire the following:

$$\begin{aligned} &-(K^2 - a + \Omega)\alpha_1 + a\beta_1 + 2\sqrt{P_1P_2}(\alpha_2 + \beta_2) = 0, \\ &a\alpha_1 + (-K^2 + a + \Omega)\beta_1 + 2\sqrt{P_1P_2}(\alpha_2 + \beta_2) = 0, \\ &-(K^2 - a + \Omega)\alpha_2 + 2\sqrt{P_1P_2}(\alpha_1 + \beta_1) + a\beta_2 = 0, \\ &a\alpha_2 + 2\sqrt{P_1P_2}(\alpha_1 + \beta_1) + (-K^2 + a + \Omega)\beta_2 = 0. \end{aligned} \tag{4.36}$$

From (4.36), we obtain the following matrix in α_1, α_2, β_1, and β_2

$$\begin{pmatrix} (-K^2 + a - \Omega) & 2\sqrt{P_1P_2} & a & 2\sqrt{P_1P_2} \\ a & 2\sqrt{P_1P_2} & (-K^2 + a + \Omega) & 2\sqrt{P_1P_2} \\ 2\sqrt{P_1P_2} & (-K^2 + a - \Omega) & 2\sqrt{P_1P_2} & a \\ 2\sqrt{P_1P_2} & a & 2\sqrt{P_1P_2} & (-K^2 + a + \Omega) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.37}$$

Expanding the determinant of (4.37), we acquire the dispersion relation given by

$$-K^8 + 4K^6 a - 4K^4 a^2 + 2K^4\Omega^2 - 4K^2 a\Omega^2 - \Omega^4 + 16K^4 P_1P_2 = 0. \tag{4.38}$$

The dispersion relation of (4.38) has the solution

$$\Omega = \pm\sqrt{K^4 - 2K^2 a \pm 4K^2\sqrt{P_1 P_2}}. \tag{4.39}$$

If Ω is imaginary, the steady-state is unstable. On the other hand, if Ω is real, then it is stable. Therefore, the condition of existence of MI from (4.39) is

$$P_1 P_2 < 0, \tag{4.40}$$

or is

$$(K^4 - 2K^2 a \pm 4K^2\sqrt{P_1 P_2}) < 0. \tag{4.41}$$

We acquire the gain as

$$g(K) = 2Im(\Omega) = 2\sqrt{K^4 - 2K^2 a \pm 4K^2\sqrt{P_1 P_2}}. \tag{4.42}$$

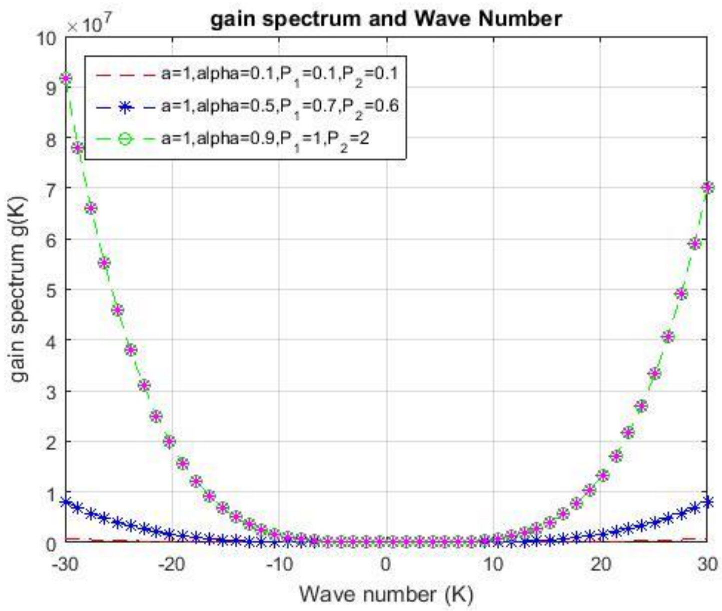

Figure 4.4: Regions of modulation instability gain spectrum (4.42).

The modulation-instability gain is significantly affected by the incident power P_1 and P_2 as can be seen from Figure 4.4.

4.2. Nonlinear Schrödinger-Hirota Equation with Spatio-Temporal Dispersion and Kerr Law Nonlinearity (NSHE)

Dispersive optical solitons are studied when group velocity dispersion (GVD) is small and needs to be supplemented with third order dispersion (3OD) [71]. The model that accurately describes this phenomena is the (NSHE) which originated from the nonlinear

Schrödinger equation (NLSE) through Lie transform. The results of this section have been published in [77]. The dimensionless form of the perturbed SHE with STD and Kerr law nonlinearity is given by

[72- 74]

$$i\psi_t + a\psi_{xx} + b\psi_{xt} + c\psi|\psi|^2 + i[\gamma\psi_{xxx} + \sigma|\psi|^2\psi_x] - (i[\alpha\psi_x + \lambda(\psi|\psi|^2)_x + \nu\psi(\psi|\psi|^2)_x]) = 0, \tag{4.43}$$

where a is the GVD and b represents the (STD) term. The inclusion of STD makes the model well-posed. In the perturbation terms, γ is the 3OD coefficient, σ and ν are the nonlinear dispersions, α is the inter-modal dispersion and λ represents the self steepening term.

The integration tool that will be applied to SHE is the sine-Gordon expansion method. This scheme will lead to the extraction of dark, bright, dark-bright and singular optical solitons. The MI of the equation will be studied. Then, we derive the Cls of the equation.

4.2.1. Application of sine-Gordon equation method to nonlinear the NSHE

To solve (4.43), we apply the transformation

$$\psi(x,t) = U(\xi)e^{i\phi(x,t)}, \quad \xi = K(x - vt), \tag{4.44}$$

where

$$\phi = -kx + \omega t + \theta. \tag{4.45}$$

Putting (4.44) into (4.43) and then separating the result into real and imaginary components, a pair of equation is acquired. The imaginary part yields

$$-K(2ak + v - bkv + \alpha + 3k^2\gamma - b\omega)U' + K(-3\lambda - 2\nu + \sigma)u^2U' + K^3\gamma U''' = 0, \tag{4.46}$$

real part gives

$$(-ak^2 - k\alpha - k^3\gamma - \omega + bk\omega)U + (c + k(-\lambda + \sigma))U^3 + K^2(a - bv + 3k\gamma)U'' = 0. \tag{4.47}$$

Integrating (4.46) once, we get

$$-K(2ak + v - bkv + \alpha + 3k^2\gamma - b\omega)U + \frac{1}{3}K(-3\lambda - 2\nu + \sigma)U^3 + K^3\gamma U'' = 0. \tag{4.48}$$

Balancing the terms of U^3 and U'' in (4.47) and (4.48) gives $n = 1$. Putting $n = 1$ into (2.17), we obtain

$$U = B_1\sin(w) + A_1\cos(w) + A_0. \tag{4.49}$$

Differentiating (4.49) twice, we acquire

$$U'' = -2\cos(w)\sin(w)^2 A_1 + \cos(w)^2\sin(w)B_1 - \sin(w)^3 B_1. \tag{4.50}$$

Substituting (4.49) and (4.50) into (4.47) and (4.48) using (2.12) and performing all necessary computations, we acquire the following:

$$\begin{gathered}-A_1(ak^2 + k\alpha + k^3\gamma + \omega - bk\omega - 3(c + k(-\lambda + \sigma))A_0^2 - (c - k\lambda + k\sigma)A_1^2) = 0,\\ -\frac{1}{3}KA_1(3(2ak + v - bkv + \alpha + 3k^2\gamma - b\omega) + (9\lambda + 6v - 3\sigma)A_0^2 + (3\lambda + 2v - \sigma)A_1^2) = 0.\end{gathered} \tag{4.51}$$

$$\begin{gathered}-A_0(ak^2 + k\alpha + k^3\gamma + \omega - bk\omega - (c + k(-\lambda + \sigma))A_0^2 - 3(c - k\lambda + k\sigma)A_1^2) = 0,\\ -\frac{1}{3}KA_0\big((3\lambda + 2v - \sigma)A_0^2 + 3(2ak + v - bkv + \alpha + 3k^2\gamma - b\omega + (3\lambda + 2v - \sigma)A_1^2)\big) = 0.\end{gathered} \tag{4.52}$$

$$\begin{gathered}6(c + k(-\lambda + \sigma))A_0A_1B_1 = 0,\\ 2K(-3\lambda - 2v + \sigma)A_0A_1B_1 = 0\end{gathered} \tag{4.53}$$

$$\begin{gathered}-3(c + k(-\lambda + \sigma))A_0(A_1^2 - B_1^2) = 0,\\ K(3\lambda + 2v - \sigma)A_0(A_1^2 - B_1^2) = 0.\end{gathered} \tag{4.54}$$

$$\begin{gathered}-B_1(ak^2 + aK^2 - bK^2v + k\alpha + k^3\gamma + 3kK^2\gamma + \omega - bk\omega -\\ 3(c + k(-\lambda + \sigma))A_0^2 - (c - k\lambda + k\sigma)B_1^2) = 0,\\ -\frac{1}{3}KB_1(3(2ak + v - bkv + \alpha + 3k^2\gamma + K^2\gamma - b\omega) +\\ (9\lambda + 6v - 3\sigma)A_0^2 + (3\lambda + 2v - \sigma)B_1^2) = 0.\end{gathered} \tag{4.55}$$

$$B_1(2K^2(a-bv+3k\gamma)+3(c+k(-\lambda+\sigma))A_1^2-(c-k\lambda+k\sigma)B_1^2)=0,$$
$$\frac{1}{3}KB_1(6K^2\gamma+(-9\lambda-6\nu+3\sigma)A_1^2+(3\lambda+2\nu-\sigma)B_1^2)=0. \tag{4.56}$$

$$A_1(-2K^2(a-bv+3k\gamma)-(c+k(-\lambda+\sigma))A_1^2+3(c-k\lambda+k\sigma)B_1^2)=0,$$
$$-\frac{1}{3}KA_1(6K^2\gamma+(-3\lambda-2\nu+\sigma)A_1^2+(9\lambda+6\nu-3\sigma)B_1^2)=0. \tag{4.57}$$

Solving (4.51)-(4.57), we obtain the following solutions

Family 1: When

$$A_0=0, A_1=\pm K\sqrt{\frac{6\gamma}{3\lambda+2\nu-\sigma}}, B_1=0,$$
$$a=\frac{1}{3\lambda+2\nu-\sigma}[-3c((-1+bk)^2-2b^2K^2)\gamma-6k\gamma(\lambda+\nu)-2b^2k(k^2-2K^2)\gamma(\nu+\sigma)+ b(\alpha(-3\lambda-2\nu+\sigma)+\gamma(2K^2(-3\lambda-2\nu+\sigma)+3k^2(\lambda+2\nu+\sigma)))],$$
$$v=\frac{1}{3\lambda+2\nu-\sigma}[\alpha(-3\lambda-2\nu+\sigma)+\gamma(c(6k-3bk^2+6bK^2)+2K^2(-3\lambda-2\nu+\sigma)- 2bk^3(\nu+\sigma)+4bkK^2(\nu+\sigma)+3k^2(\lambda+2\nu+\sigma))],$$
$$\omega=\frac{1}{3\lambda+2\nu-\sigma}[3c(k^2-bk^3+2K^2+2bkK^2)\gamma+k(\alpha(-3\lambda-2\nu+\sigma)+ \gamma(6K^2(-\lambda+\sigma)-2bk^3(\nu+\sigma)+4bkK^2(\nu+\sigma)+k^2(3\lambda+4\nu+\sigma)))], \tag{4.58}$$

we acquire the dark soliton given by:

$$\psi(x,t)=[\pm K\sqrt{\frac{6\gamma}{3\lambda+2\nu-\sigma}}\tanh[K(x-vt)]]\times e^{i(-kx+\omega t+\theta)}, \tag{4.59}$$

and the dark-singular soliton

$$\psi(x,t)=[\pm K\sqrt{\frac{6\gamma}{3\lambda+2\nu-\sigma}}\coth[K(x-vt)]]\times e^{i(-kx+\omega t+\theta)}. \tag{4.60}$$

Family 2: When

$$A_0 = 0, A_1 = 0, B_1 = \pm K\sqrt{\frac{6\gamma}{-3\lambda - 2\nu + \sigma}},$$

$$a = \frac{1}{3\lambda + 2\nu - \sigma}[-3c((-1 + bk)^2 + b^2K^2)\gamma - 6k\gamma(\lambda + \nu) - 2b^2k(k^2 + K^2)\gamma(\nu + \sigma) + b(\alpha(-3\lambda - 2\nu + \sigma) + \gamma(K^2(3\lambda + 2\nu - \sigma) + 3k^2(\lambda + 2\nu + \sigma)))],$$

$$v = \frac{1}{3\lambda + 2\nu - \sigma}[\alpha(-3\lambda - 2\nu + \sigma) - \gamma(3c(k(-2 + bk) + bK^2) + K^2(-3\lambda - 2\nu + \sigma) + 2bk^3(\nu + \sigma) + 2bkK^2(\nu + \sigma) - 3k^2(\lambda + 2\nu + \sigma))],$$

$$\omega = \frac{1}{3\lambda + 2\nu - \sigma}[-3c(k^2(-1 + bk) + (1 + bk)K^2)\gamma + k(\alpha(-3\lambda - 2\nu + \sigma) + \gamma(3K^2(\lambda - \sigma) - 2bk^3(\nu + \sigma) - 2bkK^2(\nu + \sigma) + k^2(3\lambda + 4\nu + \sigma)))],$$

(4.61)

we obtain the bright soliton given by:

$$\psi(x,t) = [\pm K\sqrt{\frac{6\gamma}{-3\lambda - 2\nu + \sigma}}\operatorname{sech}[K(x - vt)]] \times e^{i(-kx + \omega t + \theta)}, \tag{4.62}$$

and the singular soliton

$$\psi(x,t) = [\pm iK\sqrt{\frac{6\gamma}{-3\lambda - 2\nu + \sigma}}\operatorname{csch}[K(x - vt)]] \times e^{i(-kx + \omega t + \theta)}. \tag{4.63}$$

Family 3: When

$$A_0 = 0, A_1 = \pm K\sqrt{\frac{3\gamma}{2(3\lambda + 2\nu - \sigma)}}, B_1 = \pm iK\sqrt{\frac{3\gamma}{2(-3\lambda - 2\nu + \sigma)}},$$

$$a = \frac{1}{6\lambda + 4\nu - 2\sigma}[3c(-2(-1 + bk)^2 + b^2K^2)\gamma - 12k\gamma(\lambda + \nu) - 2b^2k(2k^2 - K^2)\gamma(\nu + \sigma) + b(2\alpha(-3\lambda - 2\nu + \sigma) + \gamma(K^2(-3\lambda - 2\nu + \sigma) + 6k^2(\lambda + 2\nu + \sigma)))],$$

$$v = \frac{1}{6\lambda + 4\nu - 2\sigma}[2\alpha(-3\lambda - 2\nu + \sigma) + \gamma(3c(4k - 2bk^2 + bK^2) + K^2(-3\lambda - 2\nu + \sigma) - 4bk^3(\nu + \sigma) + 2bkK^2(\nu + \sigma) + 6k^2(\lambda + 2\nu + \sigma))],$$

$$\omega = \frac{1}{6\lambda + 4\nu - 2\sigma}[3c(2k^2 - 2bk^3 + K^2 + bkK^2)\gamma + k(2\alpha(-3\lambda - 2\nu + \sigma) + \gamma(3K^2(-\lambda + \sigma) - 4bk^3(\nu + \sigma) + 2bkK^2(\nu + \sigma) + 2k^2(3\lambda + 4\nu + \sigma)))],$$

(4.64)

we obtain the combined soliton given by:

$$\psi(x,t) = [\pm iK\sqrt{\frac{3\gamma}{2(-3\lambda - 2\nu + \sigma)}}\operatorname{sech}[K(x - vt)] \pm K\sqrt{\frac{3\gamma}{2(3\lambda + 2\nu - \sigma)}}\tanh[K(x - vt)]] \times e^{i(-kx + \omega t + \theta)}, \tag{4.65}$$

and the combined singular soliton

$$\psi(x,t) = [\pm K\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}\operatorname{csch}[K(x-vt)] \pm K\sqrt{\frac{3\gamma}{2(3\lambda+2\nu-\sigma)}}\coth[K(x-vt)]] \times e^{i(-kx+\omega t+\theta)}. \tag{4.66}$$

Family 4: When

$$\begin{aligned}
&A_0 = 0, B_1 = \pm K\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}, A_1 = \pm iK\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}\\
&a = -\frac{2\alpha}{2k-\sqrt{2}K} - k\gamma + \frac{K\gamma}{\sqrt{2}}, b = \frac{2k+\sqrt{2}K}{2k^2-K^2},\\
&v = \frac{1}{6\lambda+4\nu-2\sigma}\big[2\alpha(-3\lambda-2\nu+\sigma) + \gamma\big(c\big(6k-3\sqrt{2}K\big) + K^2(-3\lambda-2\nu+\sigma) - 2\sqrt{2}kK(\nu+\sigma) + 2k^2(3\lambda+4\nu+\sigma)\big)\big],\\
&\omega = \frac{1}{6\lambda+4\nu-2\sigma}\big[2k(-\alpha+k^2\gamma)(3\lambda+2\nu-\sigma) + 3K^2\gamma(c+k(-\lambda+\sigma)) - \sqrt{2}kK\gamma(3c+2k(\nu+\sigma))\big],
\end{aligned} \tag{4.67}$$

we obtain the soliton given by:

$$\psi(x,t) = [\pm K\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}\operatorname{sech}[K(x-vt)] \pm iK\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}\tanh[K(x-vt)]] \times e^{i(-kx+\omega t+\theta)}, \tag{4.68}$$

$$\psi(x,t) = i[\pm K\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}\operatorname{csch}[K(x-vt)] \pm K\sqrt{\frac{3\gamma}{2(-3\lambda-2\nu+\sigma)}}\coth[K(x-vt)]] \times e^{i(-kx+\omega t+\theta)}. \tag{4.69}$$

(4.43) admit dark and bright solitons. The profile view of the dark soliton (4.59) are shown in Figure 4.5 for the sellected values of $K = -0.2, \omega = 0.5, b = 0.3, \theta = 0.5, k = 0.5$, $(-10 \le x \le 10,\ -10 \le t \le 10)$. Figure 4.6 describes the surfaces of the bright soliton solutions (4.62) by setting the values of $K = -0.2, \omega = 0.5, b = 0.3, \theta = 0.5, k = 0.8$, $(-10 \le x \le 10, -10 \le t \le 10)$. Finally,Figure 4.7 describe the the profile views of of the dark-bright soliton (4.65) by choosing the values of $\alpha = 0.2, \lambda = 0.6, \theta = 0.5, k = 0.3, \delta = 0.5$,$(-10 \le x \le 10, -10 \le t \le 10)$.

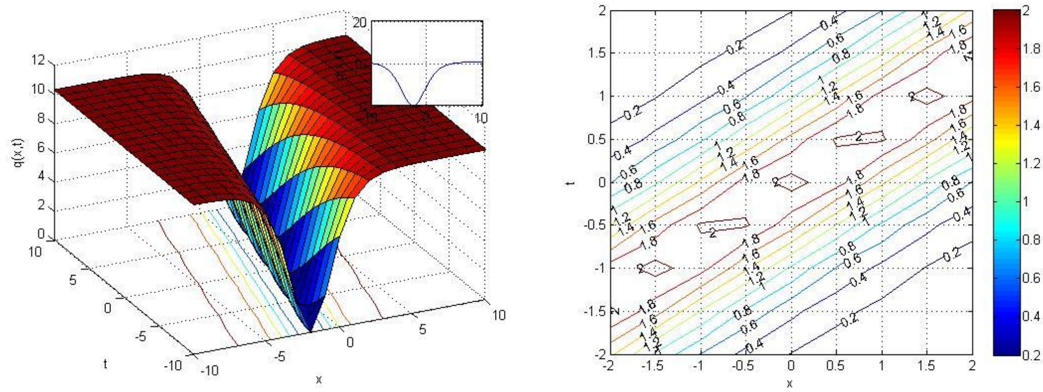

Figure 4.5: The 3D and 2D views of (4.59).

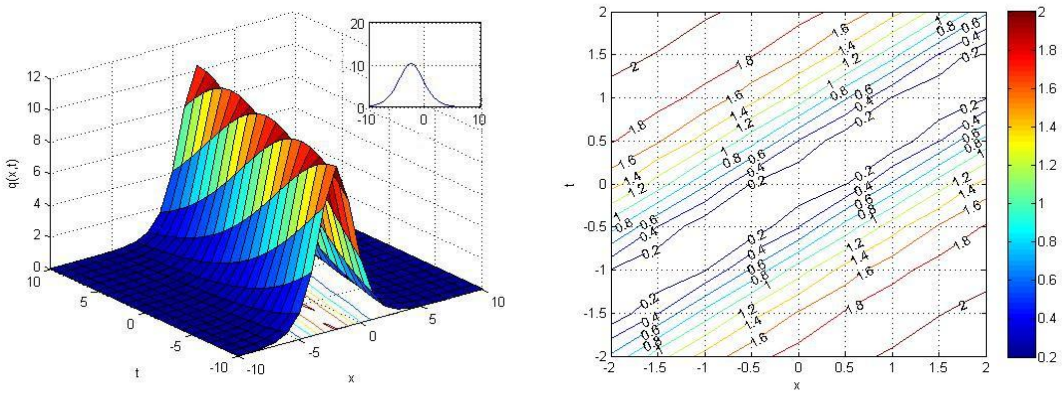

Figure 4.6: The 3D and 2D views of (4.62).

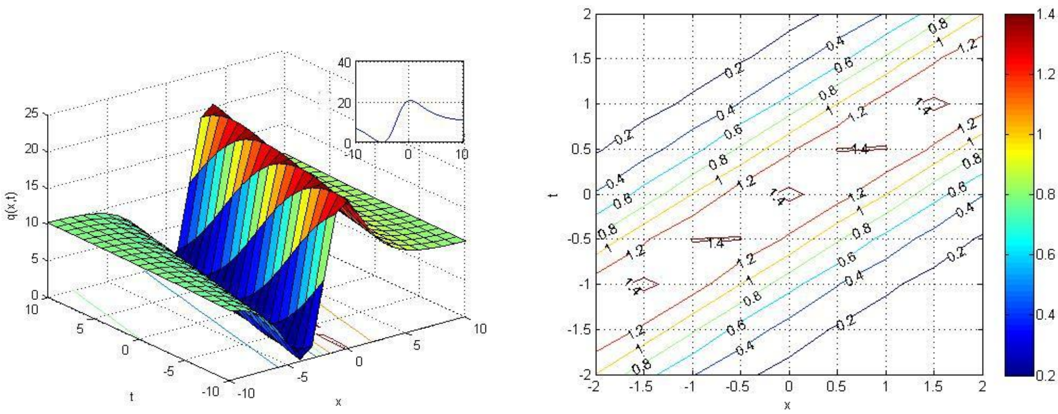

Figure 4.7: The 3D and 2D views of (4.65)

4.2.2. Modulation instability analysis of the NSHE

To study the MI of the NSHE. We apply the linear stability analysis technique discussed in section 2. We suppose that (4.2.1) has the perturbed steady-state solution of the form

$$\psi(x,t) = \left[\sqrt{P_0} + \rho(x,t)\right] \times e^{(i\phi_{NL})}, \quad \phi_{NL} = P_0 x, \tag{4.70}$$

Substituting (4.70) into (4.43) and linearizing the result in $\rho(x,t)$, we acquire

$$\rho_t + a\rho_{xx} + b\rho_{xt} + cP_0(\rho + \rho^*) + i(\gamma\rho_{xxx} + \sigma P_0\rho_x) = i\alpha\rho_x + i\lambda P_0(2\rho_x + \rho_x^*) + i\nu P_0(\rho_x + \rho_x^*). \tag{4.71}$$

We seek for

$$\rho(x,t) = a_1\cos(Kx - \Omega t) + ia_2\sin(Kx - \Omega t). \tag{4.72}$$

Substituting (4.72) into (4.71), separating the coefficients of $\cos(Kx - \Omega t)$ and $\sin(Kx - \Omega t)$ and solving the result, we get

$$\begin{aligned} K = {} & [\alpha^2 + 2\alpha\Omega + \Omega^2 - b^2\Omega^2 + \gamma^2 + 4\alpha\lambda P_0 + 2\alpha\nu P_0 - 2\alpha\sigma P_0 - \\ & 2bc\Omega P_0 + 4\lambda\Omega P_0 + 2\nu\Omega P_0 - 2\sigma\Omega P_0 + 3\lambda^2 P_0^2 + \\ & 2\lambda\nu P_0^2 - 4\lambda\sigma P_0^2 - 2\nu\sigma P_0^2 + \sigma^2 P_0^2 + (2ba\Omega + 2caP_0) + \\ & (-a^2 + 2\alpha\gamma + 2\gamma\Omega + 4\gamma\lambda P_0 + 2\gamma\nu P_0 - 2\gamma\sigma P_0)]^{\frac{1}{2}}. \end{aligned} \tag{4.73}$$

In the case when

$$\begin{aligned} & [\alpha^2 + 2\alpha\Omega + \Omega^2 - b^2\Omega^2 + \gamma^2 + 4\alpha\lambda P_0 + 2\alpha\nu P_0 - 2\alpha\sigma P_0 - \\ & 2bc\Omega P_0 + 4\lambda\Omega P_0 + 2\nu\Omega P_0 - 2\sigma\Omega P_0 + 3\lambda^2 P_0^2 + 2\lambda\nu P_0^2 - 4\lambda\sigma P_0^2 - \\ & 2\nu\sigma P_0^2 + \sigma^2 P_0^2 + (2ba\Omega + 2caP_0) + (-a^2 + 2\alpha\gamma + 2\gamma\Omega + 4\gamma\lambda P_0 + \\ & 2\gamma\nu P_0 - 2\gamma\sigma P_0)] > 0, \end{aligned}$$

the wave number K is real and the steady state is stable against small perturbations. However, the occurrence of modulation instability is

$$\begin{aligned} & [\alpha^2 + 2\alpha\Omega + \Omega^2 - b^2\Omega^2 + \gamma^2 + 4\alpha\lambda P_0 + 2\alpha\nu P_0 - 2\alpha\sigma P_0 - \\ & 2bc\Omega P_0 + 4\lambda\Omega P_0 + 2\nu\Omega P_0 - 2\sigma\Omega P_0 + 3\lambda^2 P_0^2 + 2\lambda\nu P_0^2 - 4\lambda\sigma P_0^2 - \\ & 2\nu\sigma P_0^2 + \sigma^2 P_0^2 + (2ba\Omega + 2caP_0) + (-a^2 + 2\alpha\gamma + 2\gamma\Omega + 4\gamma\lambda P_0 + \\ & 2\gamma\nu P_0 - 2\gamma\sigma P_0)] < 0, \end{aligned}$$

the wave number K is imaginary since the perturbation grows exponentially. Thus, the growth rate of the modulation stability gains spectrum $g(\Omega)$ can be expressed as:

$$g(\Omega) = 2Im(K) = 2[[\alpha^2 + 2\alpha\Omega + \Omega^2 - b^2\Omega^2 + \gamma^2 + 4\alpha\lambda P_0 + 2\alpha\nu P_0 - 2\alpha\sigma P_0 - 2bc\Omega P_0 + 4\lambda\Omega P_0 + 2\nu\Omega P_0 - 2\sigma\Omega P_0 + 3\lambda^2 P_0^2 + 2\lambda\nu P_0^2 - 4\lambda\sigma P_0^2 - 2\nu\sigma P_0^2 + \sigma^2 P_0^2 + (2ba\Omega + 2caP_0) + (-a^2 + 2\alpha\gamma + 2\gamma\Omega + 4\gamma\lambda P_0 + 2\gamma\nu P_0 - 2\gamma\sigma P_0)]^{\frac{1}{2}}]. \tag{4.74}$$

The modulation-instability gain is significantly affected by the incident power P_0 as can be seen from Figure 4.8.

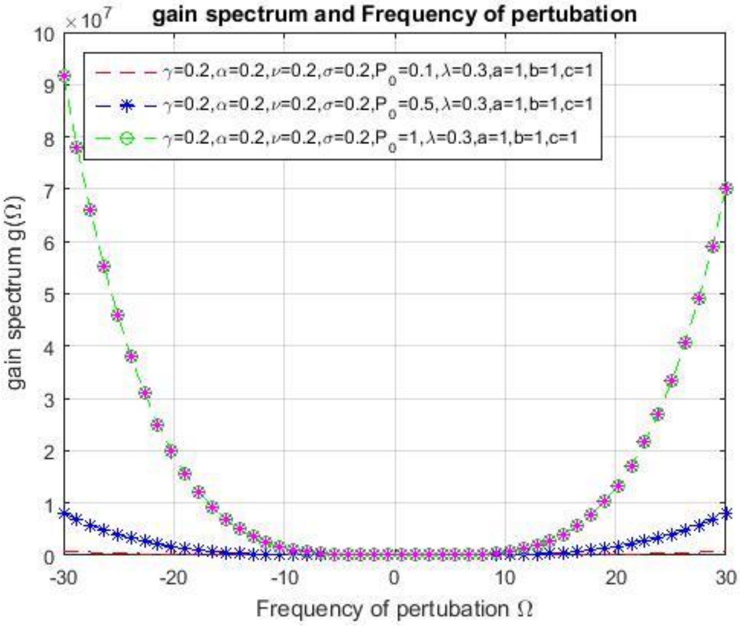

Figure 4.8: Regions of modulation instability gain spectrum (4.74).

4.3. An integrable model of (2+1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation (HFSC)

In this section, we study the integrable model of (2+1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation which describes the magnetic ordering in ferromagnetic materials. The optical soliton solutions of the HFSC will be studied by using the complex envelope function ansatz and the GTM. Moreover, we will study the MI with the aid of standard linear-stability analysis. Then, we construct the conservation laws using the Lie point symmetries of the equation. The results of this section have been published in [78]. The model is given by [79].

$$iq_t + \alpha q_{xx} + \mu q_{yy} + \delta q_{xy} - \gamma q|q|^2 = 0, \quad i = \sqrt{-1}, \tag{4.75}$$

where

$$\alpha = \sigma^4(J + J_2), \mu = \sigma^4(J_1 + J_2), \delta = 2\sigma^4 J_2, \gamma = 2\sigma^4 A. \tag{4.76}$$

The term σ is the lattice parameter, J and J_1 are the coefficients of bilinear exchange interactions along the X and Y axis. J_2 refers to the neighboring interaction on the diagonal,while A denote the uniaxial crystal field anisotropy parameter [81]. The model was studied using different approaches in [80, 81, 82, 83].

4.3.1. Application of the complex envelope function ansatz to the Heisenberg Ferromagnetic spin chain equation

Putting Eq.(2.2) into (4.75), we obtain

$$\begin{aligned}&A(\omega + \gamma|A|^2 + \alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2) + i(k_1(\delta A_y + 2\alpha A_x) + k_2(2\mu A_y + \delta A_x) + \\ &i(\mu A_{yy} + \delta A_{xy} + \alpha A_{xx}) - A_t) = 0.\end{aligned} \tag{4.77}$$

Substituting (2.3) into (4.77), we get

$$\begin{aligned}&i\beta^3\gamma + i\beta\gamma\lambda^2 + i\beta\omega + 3i\beta^2\gamma\rho\mathrm{sech}(\tau) + i\gamma\lambda^2\rho\mathrm{sech}(\tau) + i\rho\omega\mathrm{sech}(\tau) + iv\eta\lambda\mathrm{sech}(\tau)^2 \\ &-i\beta\gamma\lambda^2\mathrm{sech}(\tau)^2 + 3i\beta\gamma\rho^2\mathrm{sech}(\tau)^2 + i\alpha\eta^2\rho\mathrm{sech}(\tau)^3 + i\delta\eta^2\rho\mathrm{sech}(\tau)^3 - i\gamma\lambda^2\rho\mathrm{sech}(\tau)^3 + \\ &i\eta^2\mu\rho\mathrm{sech}(\tau)^3 + i\gamma\rho^3\mathrm{sech}(\tau)^3 + 2i\alpha\eta\lambda\mathrm{sech}(\tau)^2 k_1 + i\delta\eta\lambda\mathrm{sech}(\tau)^2 k_1 + i\alpha\beta k_1^2 + \\ &i\alpha\rho\mathrm{sech}(\tau)k_1^2 + i\delta\eta\lambda\mathrm{sech}(\tau)^2 k_2 + 2i\eta\lambda\mu\mathrm{sech}(\tau)^2 k_2 + i\beta\delta k_1 k_2 + i\delta\rho\mathrm{sech}(\tau)k_1 k_2 + \\ &i\beta\mu k_2^2 + i\mu\rho\mathrm{sech}(\tau)k_2^2 + \beta^2\gamma\lambda\tanh(\tau) + \gamma\lambda^3\tanh(\tau) + \lambda\omega\tanh(\tau) + v\eta\rho\mathrm{sech}(\tau)\tanh(\tau) + \\ &2\beta\gamma\lambda\rho\mathrm{sech}(\tau)\tanh(\tau) + 2\alpha\eta^2\lambda\mathrm{sech}(\tau)^2\tanh(\tau) + 2\delta\eta^2\lambda\mathrm{sech}(\tau)^2\tanh(\tau) - \gamma\lambda^3\mathrm{sech}(\tau)^2 \\ &\tanh(\tau) + 2\eta^2\lambda\mu\mathrm{sech}(\tau)^2\tanh(\tau) + \gamma\lambda\rho^2\mathrm{sech}(\tau)^2\tanh(\tau) + 2\alpha\eta\rho\mathrm{sech}(\tau)k_1\tanh(\tau) + \\ &\delta\eta\rho\mathrm{sech}(\tau)k_1\tanh(\tau) + \alpha\lambda k_1^2\tanh(\tau) + \delta\eta\rho\mathrm{sech}(\tau)k_2\tanh(\tau) + 2\eta\mu\rho\mathrm{sech}(\tau)k_2\tanh(\tau) + \\ &\delta\lambda k_1 k_2\tanh(\tau) + \lambda\mu k_2^2\tanh(\tau) - i\alpha\eta^2\rho\mathrm{sech}(\tau)\tanh(\tau)^2 - i\delta\eta^2\rho\mathrm{sech}(\tau)\tanh(\tau)^2 - \\ &i\eta^2\mu\rho\mathrm{sech}(\tau)\tanh(\tau)^2 = 0,\end{aligned} \tag{4.78}$$

where $\tau = \eta(x + y - vt)$. Performing some algebraic calculations, we obtain:

$$i\beta(\gamma(\beta^2 + \lambda^2) + \omega + \alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2) = 0, \tag{4.79}$$

$$i\rho(3\beta^2\gamma + \gamma\lambda^2 + \omega + \alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2) = 0, \tag{4.80}$$

$$i(v\eta\lambda-\beta\gamma\lambda^2+3\beta\gamma\rho^2+(2\alpha+\delta)\eta\lambda k_1+\eta\lambda(\delta+2\mu)k_2)=0, \tag{4.81}$$

$$i\rho\left(\eta^2(\alpha+\delta+\mu)+\gamma(-\lambda^2+\rho^2)\right)=0, \tag{4.82}$$

$$\lambda(\gamma(\beta^2+\lambda^2)+\omega+\alpha k_1^2+\delta k_1k_2+\mu k_2^2)=0, \tag{4.83}$$

$$\rho(v\eta+2\beta\gamma\lambda+(2\alpha+\delta)\eta k_1+\eta(\delta+2\mu)k_2)=0, \tag{4.84}$$

$$\lambda\left(2\eta^2(\alpha+\delta+\mu)+\gamma(-\lambda^2+\rho^2)\right)=0, \tag{4.85}$$

$$-i\eta^2(\alpha+\delta+\mu)\rho=0, \tag{4.86}$$

Solving (4.79)-(4.86), we acquire the families given by:

Family 1: If

$$\beta=0,\rho=\pm\lambda,v=-(\delta+2\mu)(-k_1+k_2),\omega=-\gamma\rho^2+\delta k_1^2+\mu k_1^2. \tag{4.87}$$

we acquire dark-bright soliton given by

$$q(x,t,y)=[\lambda\tanh[\eta(x+y-vt)]\pm i\lambda\mathrm{sech}[\eta(x+y-vt)]]\times e^{i(-k_1x-k_2y+\omega t+\theta)}. \tag{4.88}$$

$$|q(x,t,y)|^2=\lambda^2. \tag{4.89}$$

The nonlinear phase shift is given by as:

$$\psi_{NL}=\arctan\left[\frac{\mathrm{sech}[\eta(x+y-vt)]}{\tanh[\eta(x+y-vt)]}\right]. \tag{4.90}$$

Family 2: When

$$\begin{aligned}&\rho=0,v=\sqrt{2\beta^2\gamma(\alpha+\delta+\mu)}-(2\alpha+\delta)k_1-(\delta+2\mu)k_2,\\&\omega=-\gamma(\beta^2+\lambda^2)-\alpha k_1^2-\delta k_1k_2-\mu k_2^2,\\&\eta=\frac{\beta\gamma\lambda}{\sqrt{2\beta^2\gamma(\alpha+\delta+\mu)}},\end{aligned} \tag{4.91}$$

we obtain the dark soliton given by

$$q(x,t,y)=[\mathrm{i}\beta+\lambda\tanh[\eta(x+y-vt)]]\times e^{i(-k_1x-k_2y+\omega t+\theta)}. \tag{4.92}$$

The intensity is also given by

$$|q(x,t,y)|^2=\lambda^2+\beta^2-\lambda^2\mathrm{sech}^2[\eta(x+y-vt)]. \tag{4.93}$$

With $(2\beta^2\gamma(\alpha+\delta+\mu)>0$ for the soliton to exist.

Family 3: When

$$\rho=0,\beta=0,v=-2\alpha k_1-\delta k_1-\delta k_2-2\mu k_2,\omega=-\gamma\lambda^2-\alpha k_1^2-\delta k_1k_2-\mu k_2^2,$$
$$\eta=\lambda\sqrt{\frac{\gamma}{2(\alpha+\delta+\mu)}}, \tag{4.94}$$

we get the following dark optical soliton

$$q(x,t,y)=[\lambda\tanh[\eta(x+y-vt)]]\times e^{i(-k_1x-k_2y+\omega t+\theta)}. \tag{4.95}$$

The intensity is also given by

$$|q(x,t,y)|^2=[\lambda^2-\lambda^2\mathrm{sech}^2[\eta(x+y-vt)]], \tag{4.96}$$

with $\gamma(2(\alpha+\delta+\mu)>0$ for the soliton to exist.

4.3.2. Application of the generalized tanh method to the HFSC

We begin by applying the transformation given by

$$q(x,t,y)=U(\xi)\times e^{i\phi(x,t,y)}, \tag{4.97}$$

with

$$\xi=(x+y-vt). \tag{4.98}$$

Substituting (4.97) into (4.75) and then decomposing into real and imaginary components, the imaginary component gives

$$v=-(2\alpha k_1+\delta k_1+2\beta k_2+\delta k_2), \tag{4.99}$$

representing the velocity and the real component yields

$$(\omega+\alpha k_1^2+\delta k_1k_2+\beta k_2^2)U+\gamma U^3-\eta^2(\alpha+\beta+\delta)U''=0. \tag{4.100}$$

Balancing the terms of U^3 and U'' in (4.100) gives $m=1$. Substituting $m=1$ into (2.46), we obtain

$$U(\xi)=a_0+a_1\phi(\xi). \tag{4.101}$$

Putting (4.101) along with the necessary derivatives into (4.100) using (2.45) and performing all the necessary algebraic computations, we obtain

$$a_0=0,a_1=\pm\sqrt{\frac{2(\alpha+\delta+\mu)}{\gamma}},K=\frac{\omega+\alpha k_1^2+\delta k_1k_2+\mu k_2^2}{2(\alpha+\delta+\mu)}. \tag{4.102}$$

Thus, we acquire dark soliton given by

$$q(x,t,y) = -\sqrt{-\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{\gamma}}\tanh[-\sqrt{-\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{2(\alpha+\delta+\mu)}}\xi] \times e^{i(\theta+t\omega-xk_1-yk_2)}, \tag{4.103}$$

the dark-singular soliton

$$q(x,t,y) = -\sqrt{-\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{\gamma}}\coth[-\sqrt{-\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{2(\alpha+\delta+\mu)}}\xi] \times e^{i(\theta+t\omega-xk_1-yk_2)}, \tag{4.104}$$

and the periodic singular solutions

$$q(x,t,y) = \sqrt{\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{\gamma}}\tan[\sqrt{\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{2(\alpha+\delta+\mu)}}\xi] \times e^{i(\theta+t\omega-xk_1-yk_2)}, \tag{4.105}$$

$$q(x,t,y) = -\sqrt{\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{\gamma}}\cot[-\sqrt{\frac{\omega+\alpha k_1^2+\delta k_1 k_2+\mu k_2^2}{2(\alpha+\delta+\mu)}}\xi] \times e^{i(\theta+t\omega-xk_1-yk_2)}, \tag{4.106}$$

with $(\omega + \alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2)(2(\alpha + \delta + \mu)) < 0$ for the soliton to exist in (4.3.29), (4.3.30) and (4.3.32) and $(\omega + \alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2)(2(\alpha + \delta + \mu)) > 0$ for the soliton to exist in (4.3.31). Where $\xi = x + y + (2\alpha k_1 + \delta k_1 + \delta k_2 + 2\mu k_2)t$.

(4.75) admit dark and dark-bright solitons. The profile view of surfaces of the dark-bright soliton (4.88) are shown in Figure 4.9 for the sellected values of $\lambda = 0.4, k_1 = 0.8, k_2 = 0.4, \mu = 2, t = 1, \delta = 0.8, \gamma = 3, \theta = 0.4$, $(-10 \leq x \leq 10,\ -10 \leq y \leq 10)$. Figure 4.10 describes the surfaces of the dark soliton (4.92) by setting the values of $\omega = 0.4, k_1 = 0.8, k_2 = 0.4, \alpha = 2, t = 1, \delta = 0.8, \gamma = 0.1, \mu = 0.8, \theta = 0.1$, $(-10 \leq x \leq 10,\ -10 \leq y \leq 10)$.

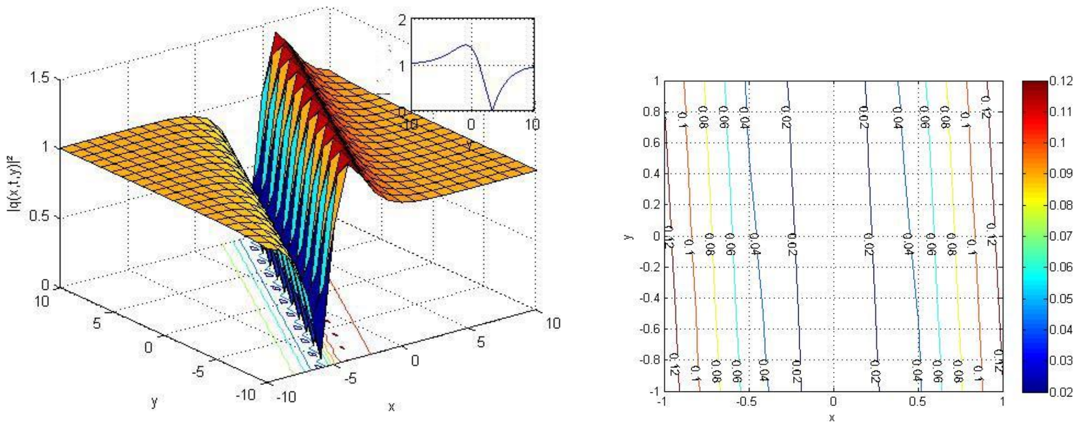

Figure 4.9: The 3D and contour views of (4.88).

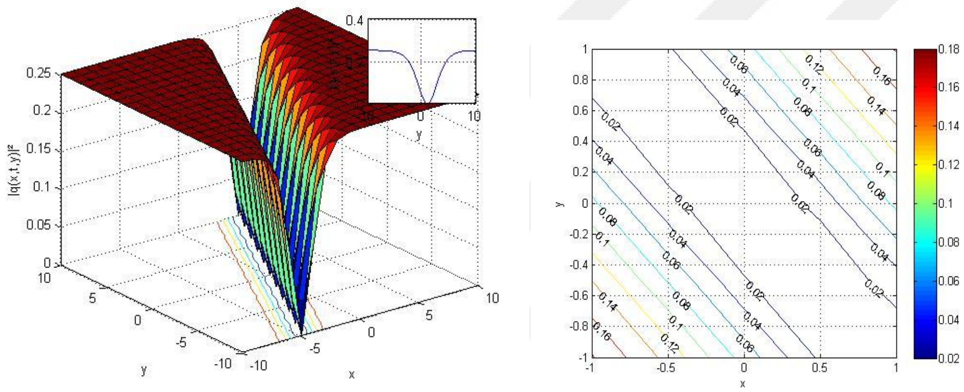

Figure 4.10: The 3D and contour views of (4.92) .

4.3.3. Modulation instability analysis of the HFSC

To analyze the MI of (4.75), we apply the linear- stability analysis. Assuming (4.75) has the steady-state solution given by

$$q(x,t,y) = \left[\sqrt{P_0} + a(x,t,y)\right] \times e^{(i\phi_{NL})}, \quad \phi_{NL} = \gamma P_0 x, \tag{4.107}$$

Substituting (4.107) into (4.75) and linearizing, we obtain

$$ia_t + \alpha a_{xx} + \mu a_{yy} + \delta a_{xy} - \gamma P_0(a + a^*) = 0. \tag{4.108}$$

We look for a solution of the form

$$a(x,t,y) = a_1 e^{i(Kx+k_1y-\Omega t)} + a_2 e^{-i(Kx+k_1y-\Omega t)}. \tag{4.109}$$

Substituting (4.109) into (4.108), separation the coefficients of $e^{i(Kx+k_1y-\Omega t)}$ and $e^{-i(Kx+k_1y-\Omega t)}$ and solving, we get

$$K = \frac{-\delta k_1 \pm \sqrt{\delta^2 k_1^2 - 4\alpha\mu k_1^2 - 4\alpha\gamma P_0 \pm 4\alpha\sqrt{\Omega^2 + \gamma^2 P_0^2}}}{2\alpha}. \quad (4.110)$$

In the case when

$$\{\delta^2 k_1^2 - 4\alpha\mu k_1^2 - 4\alpha\gamma P_0 \pm 4\alpha\sqrt{\Omega^2 + \gamma^2 P_0^2}\} > 0,$$

the wave number K is real, the steady state is stable. Modulation instability occurs when

$$\{\delta^2 k_1^2 - 4\alpha\mu k_1^2 - 4\alpha\gamma P_0 \pm 4\alpha\sqrt{\Omega^2 + \gamma^2 P_0^2}\} < 0,$$

the wave number K is imaginary. Thus, the growth rate of the MI gain spectrum $g(\Omega)$ can be expressed as

$$g(\Omega) = 2Im(K) = \frac{1}{\alpha}\sqrt{\delta^2 k_1^2 - 4\alpha\mu k_1^2 - 4\alpha\gamma P_0 \pm 4\alpha\sqrt{\Omega^2 + \gamma^2 P_0^2}}. \quad (4.111)$$

The modulation-instability gain is significantly affected by the incident power P_0 as can be seen from Figure 4.11.

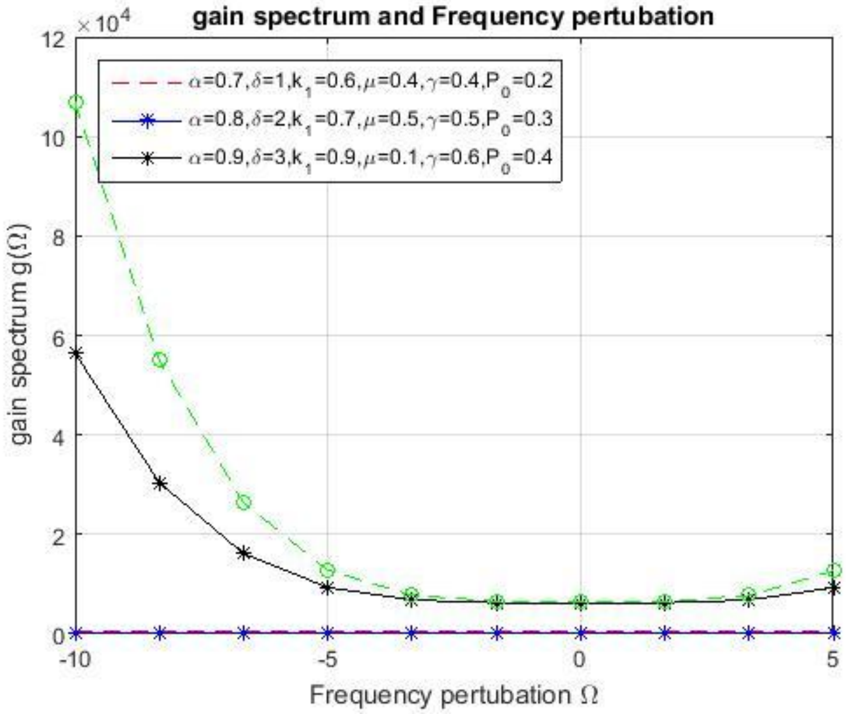

Figure 4.11: Regions of modulation instability gain spectrum (4.111).

5. DYNAMICS OF SOLITONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

This chapter addresses three different NPDEs, namely; Coupled Nonlinear Maccari's System (CNMS), the generalized Schrödinger-Boussinesq system (GSBE) and the the nonlinear Kudryashov-Sinelshchikov (NKSE). The solitons, point symmetries and conservation laws of NKSE and GSBE will be investigated using the concepts discussed in chapter two. For the CNMS, only the soliton solutions will be investigated.

5.1. Coupled Nonlinear Maccari's System

In this section, we will consider the CNMS system. The System is a kind of nonlinear equation often presented to describe the motion of an isolated waves which is localized in a small part of space and applied in many fields such as plasma physics, hydrodynamic, nonlinear optic, e.t.c [90]. We will investigate solitary wave solutions of the CNMS using the modified F-Expansion and projective Riccati equation methods. More precisely, we present some new solution structures, including topological, non topological, singular and trigonometric function and complexiton soliton solutions. These solutions may be useful to explain some physical phenomena in isolated waves. The results of this section have been published in [93]. The CNMS is given by [91, 92]:

$$\begin{cases} iQ_t + Q_{xx} + RQ = 0, \\ iS_t + S_{xx} + RS = 0, \\ iN_t + N_{xx} + RN = 0, \\ R_t + R_y + (|Q + S + N|^2)_x = 0. \end{cases} \tag{5.1}$$

(5.1) has been discussed in [90] using the (G'/G) −Expansion method, in [91] using the first integral method and in [92] using the Exp-function method.

5.1.1 Analytic studies

To derive the travelling wave solutions (5.1.1), we let

$$\begin{aligned} Q(x,t,y) &= u(x,t,y) \times e^{i(kx+\alpha y+\omega t+\gamma)}, \\ S(x,t,y) &= v(x,t,y) \times e^{i(kx+\alpha y+\omega t+\gamma)}, \\ N(x,t,y) &= w(x,t,y) \times e^{i(kx+\alpha y+\omega t+\gamma)}, \end{aligned} \tag{5.2}$$

where k, α, ω and γ are arbitrary constants. Putting (5.2) into (5.1), we acquire

$$\begin{aligned} &-(k^2+\omega-R)u+i(u_t+2ku_x)+u_{xx}=0,\\ &-(k^2+\omega-R)v+i(v_t+2kv_x)+v_{xx}=0,\\ &-(k^2+\omega-R)w+i(w_t+2kw_x)+w_{xx}=0,\\ &R_y+R_t+2(u+v+w)(u_x+v_x+w_x)=0. \end{aligned} \tag{5.3}$$

Applying the following transformations on (5.3)

$$u(x,t,y)=U(\xi),\quad v(x,t,y)=v(\xi),\quad R(x,t,y)=R(\xi),\quad w(x,t,y)=w(\xi), \tag{5.4}$$

where

$$\xi=x+y\beta-2kt. \tag{5.5}$$

We acquire the system of ODE given by

$$\begin{cases} (k^2+\omega-R)U-u''=0,\\ (k^2+\omega-R)v-v''=0,\\ (k^2+\omega-R)w-w''=0,\\ (2k-\beta)R'-2(U+v+w)(U'+v'+w')=0. \end{cases} \tag{5.6}$$

Integrating the fourth equation of (5.6), we get

$$R=\frac{(U+v+w)^2}{2k-\beta}. \tag{5.7}$$

Substituting (5.7) into the first three equations of (5.1.6), we obtain

$$\begin{cases} -(k^2+\omega-R)u+i(u_t+2ku_x)+u_{xx}=0,\\ -(k^2+\omega-R)v+i(v_t+2kv_x)+v_{xx}=0,\\ -(k^2+\omega-R)w+i(w_t+2kw_x)+w_{xx}=0,\\ R_y+R_t+2(u+v+w)(u_x+v_x+w_x)=0. \end{cases} \tag{5.8}$$

Giving a simple relation in u, v, w, we let

$$v=gU,\quad w=zU, \tag{5.9}$$

where g and z are any constants. Putting (5.9) into the last two equations of (5.8), the system reduces to the following ODE

$$-U''+U(k^2+\omega)-\frac{(1+g+z)^2}{2k-\beta}U^3=0. \tag{5.10}$$

5.1.2 Application of the modified F-expansion method to the CNMS

In this part, we find the solutions of (5.1). From (5.10), balancing the terms U^3 and U'', gives $n=1$. Therefore, (2.40) reduces to

$$U(\xi)=\frac{a_{-1}}{F}+a_0+Fa_1, \tag{5.11}$$

where $F=F(\xi)$. Substituting (5.11) into (5.10) yields

$$\left(\frac{a_{-1}}{F}+a_0+Fa_1\right)\left(k^2+\omega-\frac{(1+g+z)^2\left(\frac{a_{-1}}{F}+a_0+Fa_1\right)^2}{2k-\beta}\right)- \left(\frac{(A+F(B+CF))\left((2A+BF)a_{-1}+F^3(B+2CF)a_1\right)}{F^3}\right). \tag{5.12}$$

Case 1: Solving (5.12) with the values of $A=0, B=1, C=-1$, we obtain

$$k=\sqrt{\frac{-1-2\omega}{2}}, a_{-1}=0, a_0=\frac{\sqrt{\frac{\beta\pm\sqrt{-2-4\omega}}{2}}}{(1+g+z)}, a_1=-\frac{\sqrt{\frac{2\left(\beta\pm\sqrt{-2-4\omega}\right)}{2}}}{(1+g+z)}. \tag{5.13}$$

using $F(\xi)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{1}{2}\xi\right)$, we obtain the the topological solitons given by

$$Q(x,t,y)=\left(-\frac{\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\tanh\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]\right)\times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.14}$$

$$S(x,t,y)=\left(-\frac{g\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\tanh\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]\right)\times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.15}$$

$$N(x,t,y)=\left(-\frac{z\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\tanh\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]\right)\times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.16}$$

$$R(x,t,y)=-\frac{1}{2}\tanh\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]^2. \tag{5.17}$$

Case 2: Solving (5.12) with the values of $A=0, B=-1, C=1$, we acquire

$$k=\sqrt{\frac{-1-2\omega}{2}}, a_{-1}=0, a_0=\frac{\sqrt{\frac{\beta\pm\sqrt{-2-4\omega}}{2}}}{(1+g+z)}, a_1=-\frac{\sqrt{\frac{2\left(\beta\pm\sqrt{-2-4\omega}\right)}{2}}}{(1+g+z)}. \tag{5.18}$$

Using $F(\xi)=\frac{1}{2}-\frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$, we obtain the the singular solitons given by

$$Q(x,t,y)=\left(\frac{\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]\right)e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.19}$$

$$S(x,t,y)=\left(\frac{g\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]\right)\times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.20}$$

$$N(x,t,y)=\left(\frac{z\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]\right)\times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.21}$$

$$R(x,t,y)=-\frac{1}{2}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]^2. \tag{5.22}$$

Case 3: Solving (5.12) with the values of $A=\frac{1}{2}, B=0, C=\frac{-1}{2}$, we acquire the following family and solutions using $F(\xi)=\tanh(\xi)\pm i\text{sech}(\xi)$ and $F(\xi)=\coth(\xi)\pm\text{csch}(\xi)$.

Family 1: When

$$k=\pm\sqrt{\frac{-1-2\omega}{2}}, a_0=0, a_{-1}=0, a_1=\pm\frac{\sqrt{\frac{\beta\pm\sqrt{-2-4\omega}}{2}}}{(1+g+z)}, \tag{5.23}$$

we obtain the following complexiton solutions

$$Q(x,t,y)=\frac{\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}(i\text{sech}[x+y\beta-t\sqrt{-2-4\omega}]+\tanh[x+y\beta-t\sqrt{-2-4\omega}])\times e^{i(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega)}, \tag{5.24}$$

$$S(x,t,y)=\frac{g\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}(i\text{sech}[x+y\beta-t\sqrt{-2-4\omega}]+\tanh[x+y\beta-t\sqrt{-2-4\omega}])\times e^{i(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega)}, \tag{5.25}$$

$$N(x,t,y) = \frac{z\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}(i\mathrm{sech}[x+y\beta-t\sqrt{-2-4\omega}] + \tanh[x+y\beta-t\sqrt{-2-4\omega}]) \times e^{i(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega)}, \tag{5.26}$$

$$R(x,t,y) = \frac{1}{2}(\mathrm{sech}[x+y\beta-t\sqrt{-2-4\omega}] - i\tanh[x+y\beta-t\sqrt{-2-4\omega}])^2, \tag{5.27}$$

and the singular soliton solutions

$$Q(x,t,y) = \frac{\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.28}$$

$$S(x,t,y) = \frac{g\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.29}$$

$$N(x,t,y) = \frac{z\sqrt{\frac{\beta-\sqrt{-2-4\omega}}{2}}}{(1+g+z)}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-\frac{1}{2}-\omega}+t\omega\right)}, \tag{5.30}$$

$$R(x,t,y) = -\frac{1}{2}\coth\left[\frac{1}{2}\left(x+y\beta-t\sqrt{-2-4\omega}\right)\right]^2. \tag{5.31}$$

Family 2: When

$$\left(k=\pm\sqrt{-2-\omega}, k=\pm\sqrt{1-\omega}\right), a_0=0, a_1=\pm\sqrt{\frac{-2k+\beta}{1+2g+g^2+2z+2gz+z^2}}, \quad a_{-1}=\frac{1}{3}(-a_1-2k^2a_1-2\omega a_1), \tag{5.32}$$

we obtain the following complexiton solutions using $k=\sqrt{1-\omega}$

$$Q(x,t,y) = \frac{i\sqrt{2\left(\beta-2\sqrt{1-\omega}\right)}}{(1+g+z)}\mathrm{sech}\left[x+y\beta-2t\sqrt{1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{1-\omega}+t\omega)}, \tag{5.33}$$

$$S(x,t,y) = \frac{ig\sqrt{2\left(\beta-2\sqrt{1-\omega}\right)}}{(1+g+z)}\mathrm{sech}\left[x+y\beta-2t\sqrt{1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{1-\omega}+t\omega)}, \tag{5.34}$$

$$N(,t,y) = \frac{iz\sqrt{2(\beta-2\sqrt{1-\omega})}}{(1+g+z)}\operatorname{sech}\left[x+y\beta-2t\sqrt{1-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{1-\omega}+t\omega)}, \quad (5.35)$$

$R(x,t,y) = 2\operatorname{sech}\left[x+y\beta-2t\sqrt{1-\omega}\right]^2$, (5.36)

and singular solutions

$$Q(x,t,y) = \frac{i\sqrt{2(\beta-2\sqrt{1-\omega})}}{(1+g+z)}\operatorname{csch}\left[x+y\beta-2t\sqrt{1-\omega}\right]e^{i(y\alpha+\gamma+x\sqrt{1-\omega}+t\omega)}, \quad (5.37)$$

$$S(x,t,y) = \frac{ig\sqrt{2(\beta-2\sqrt{1-\omega})}}{(1+g+z)}\operatorname{csch}\left[x+y\beta-2t\sqrt{1-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{1-\omega}+t\omega)}, \quad (5.38)$$

$$N(x,t,y) = \frac{iz\sqrt{2(\beta-2\sqrt{1-\omega})}}{(1+g+z)}\operatorname{csch}\left[x+y\beta-2t\sqrt{1-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{1-\omega}+t\omega)}, \quad (5.39)$$

$R(x,t,y) = 2\operatorname{csch}\left[x+y\beta-2t\sqrt{1-\omega}\right]^2$,(5.40)

and using $k = \sqrt{-2-\omega}$, we acquire the following topological soliton solutions

$$Q(x,t,y) = \frac{\sqrt{2(\beta-2\sqrt{-2-\omega})}}{(1+g+z)}\tanh\left[x+y\beta-2t\sqrt{-2-\omega}\right]e^{i(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega)}, \quad (5.41)$$

$$S(x,t,y) = \frac{g\sqrt{2(\beta-2\sqrt{-2-\omega})}}{(1+g+z)}\tanh\left[x+y\beta-2t\sqrt{-2-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega)}, \quad (5.42)$$

$$N(x,t,y) = \frac{z\sqrt{2(\beta-2\sqrt{-2-\omega})}}{(1+g+z)}\tanh\left[x+y\beta-2t\sqrt{-2-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega)}, \quad (5.43)$$

$$R(x,t,y) = -2\tanh\left[x+y\beta-2t\sqrt{-2-\omega}\right]^2. \quad (5.44)$$

and the following singular soliton solutions

$$Q(x,t,y) = \frac{\sqrt{2(\beta-2\sqrt{-2-\omega})}}{(1+g+z)}\coth\left[x+y\beta-2t\sqrt{-2-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega)}, \quad (5.45)$$

$$S(x,t,y) = \frac{g\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\coth\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.46}$$

$$N(x,t,y) = \frac{z\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\coth\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.47}$$

$$R(x,t,y) = -2\coth\left[x + y\beta - 2t\sqrt{-2-\omega}\right]^2. \tag{5.48}$$

Case 4: Solving (5.12) with the values of $A = 1, B = 0, C = -1$, we acquire the following family and solutions with $F(\xi) = \tanh(\xi)$ and $F(\xi) = \coth(\xi)$.

Family 1: When

$$k = \pm\sqrt{-2-\omega}, a_0 = 0, a_1 = \pm\frac{\sqrt{2\left(\beta\pm2\sqrt{-2-\omega}\right)}}{(1+g+z)}, a_{-1} = 0, \tag{5.49}$$

we acquire the following topological soliton solutions

$$Q(x,t,y) = \frac{\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\tanh\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.50}$$

$$S(x,t,y) = \frac{g\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\tanh\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.51}$$

$$N(x,t,y) = \frac{z\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\tanh\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.52}$$

$$R(x,t,y) = -2\tanh\left[x + y\beta - 2t\sqrt{-2-\omega}\right]^2, \tag{5.53}$$

and

$$Q(x,t,y) = \frac{\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\tanh\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.54}$$

$$S(x,t,y) = \frac{g\sqrt{2\left(\beta-2\sqrt{-2-\omega}\right)}}{(1+g+z)}\tanh\left[x + y\beta - 2t\sqrt{-2-\omega}\right] \times e^{i\left(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega\right)}, \tag{5.55}$$

$$N(x,t,y) = \frac{z\sqrt{2(\beta-2\sqrt{-2-\omega})}}{(1+g+z)}\tanh\left[x+y\beta-2t\sqrt{-2-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-2-\omega}+t\omega)}, \tag{5.56}$$

$$R(x,t,y) = -2\tanh\left[x+y\beta-2t\sqrt{-2-\omega}\right]^2. \tag{5.57}$$

Family 2: When

$$(k=\pm\sqrt{-8-\omega}, k=\pm\sqrt{4-\omega}), a_0=0, a_1=\pm\sqrt{\frac{2(-2k+\beta)}{1+2g+g^2+2z+2gz+z^2}},$$
$$a_{-1} = -\frac{1}{6}(2+k^2+\omega)a_1, \tag{5.58}$$

we acquire the following solutions using $k=\sqrt{-8-\omega}$

$$Q(x,t,y) = \frac{\sqrt{2(\beta-2\sqrt{-8-\omega})}}{(1+g+z)}\left(1+\coth\left[x+y\beta-2t\sqrt{-8-\omega}\right]^2\right)\times$$
$$\tanh\left[x+y\beta-2t\sqrt{-8-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-8-\omega}+t\omega)}, \tag{5.59}$$

$$S(x,t,y) = \frac{g\sqrt{2(\beta-2\sqrt{-8-\omega})}}{(1+g+z)}\left(1+\coth\left[x+y\beta-2t\sqrt{-8-\omega}\right]^2\right)\times$$
$$\tanh\left[x+y\beta-2t\sqrt{-8-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-8-\omega}+t\omega)}, \tag{5.60}$$

$$N(x,t,y) = \frac{z\sqrt{2(\beta-2\sqrt{-8-\omega})}}{(1+g+z)}\left(1+\coth\left[x+y\beta-2t\sqrt{-8-\omega}\right]^2\right)\times$$
$$\tanh\left[x+y\beta-2t\sqrt{-8-\omega}\right]\times e^{i(y\alpha+\gamma+x\sqrt{-8-\omega}+t\omega)}, \tag{5.61}$$

$$R(x,t,y) = -2\left(\coth\left[x+y\beta-2t\sqrt{-8-\omega}\right]+\tanh\left[x+y\beta-2t\sqrt{-8-\omega}\right]\right)^2, \tag{5.62}$$

and the following singular solutions using $k=\pm\sqrt{4-\omega}$

$$Q(x,t,y) = \pm\frac{2\sqrt{2(\beta-2\sqrt{4-\omega})}}{(1+g+z)}\operatorname{csch}\left[2\left(x+y\beta-2t\sqrt{4-\omega}\right)\right]\times e^{i(y\alpha+\gamma+x\sqrt{4-\omega}+t\omega)}, \tag{5.63}$$

$$S(x,t,y) = \pm\frac{2g\sqrt{2(\beta-2\sqrt{4-\omega})}}{(1+g+z)}\operatorname{csch}\left[2\left(x+y\beta-2t\sqrt{4-\omega}\right)\right]\times e^{i(y\alpha+\gamma+x\sqrt{4-\omega}+t\omega)}, \tag{5.64}$$

$$N(x,t,y) = \pm\frac{2z\sqrt{2\left(\beta-2\sqrt{4-\omega}\right)}}{(1+g+z)}\operatorname{csch}\left[2\left(x+y\beta-2t\sqrt{4-\omega}\right)\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{4-\omega}+t\omega\right)}, \quad (5.65)$$

$$R(x,t,y) = -8\operatorname{csch}\left[2\left(x+y\beta-2t\sqrt{4-\omega}\right)\right]^2. \quad (5.66)$$

Case 5: Solving (5.12) with the values of $A = \frac{1}{2}, B = 0, C = \frac{1}{2}$, we acquire the following family and solutions using $F(\xi) = \csc(\xi) - \cot(\xi)$ and $F(\xi) = \tan(\xi) + \sec(\xi)$.

Family 1: When

$$k = \pm\sqrt{\frac{1-2\omega}{2}}, a_0 = 0, a_{-1} = 0, a_1 = \pm\frac{\sqrt{\frac{\beta\pm\sqrt{2-4\omega}}{2}}}{(1+g+z)}, \quad (5.67)$$

we acquire

$$Q(x,t,y) = \frac{\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}(\sec[x+y\beta-t\sqrt{2-4\omega}] + \tan[x+y\beta-t\sqrt{2-4\omega}])\times e^{\frac{1}{2}i\left(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega\right)}, \quad (5.68)$$

$$S(x,t,y) = \frac{g\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}(\sec[x+y\beta-t\sqrt{2-4\omega}] + \tan[x+y\beta-t\sqrt{2-4\omega}])\times e^{\frac{1}{2}i\left(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega\right)}, \quad (5.\ 69)$$

$$N(x,t,y) = \frac{z\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}(\sec[x+y\beta-t\sqrt{2-4\omega}] + \tan[x+y\beta-t\sqrt{2-4\omega}])\times e^{\frac{1}{2}i\left(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega\right)}, \quad (5.70)$$

$$R(x,t,y) = -\frac{1}{2}(\sec[x+y\beta-t\sqrt{2-4\omega}] + \tan[x+y\beta-t\sqrt{2-4\omega}])^2, \quad (5.71)$$

and

$$Q(x,t,y) = \frac{\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}\tan\left[\frac{1}{2}\left(x+y\beta-t\sqrt{2-4\omega}\right)\right]\times e^{\frac{1}{2}i\left(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega\right)}, \quad (5.72)$$

$$S(x,t,y) = \frac{g\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}\tan\left[\frac{1}{2}\left(x+y\beta-t\sqrt{2-4\omega}\right)\right]\times e^{\frac{1}{2}i\left(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega\right)}, \quad (5.73)$$

$$N(x,t,y)=\frac{z\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}\tan\left[\frac{1}{2}\left(x+y\beta-t\sqrt{2-4\omega}\right)\right]\times e^{\frac{1}{2}i\left(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega\right)}, \quad (5.74)$$

$$R(x,t,y)=-\frac{1}{2}\tan\left[\frac{1}{2}\left(x+y\beta-t\sqrt{2-4\omega}\right)\right]^2. \quad (5.75)$$

Family 2: When

$$\left(k=\pm\sqrt{2-\omega},k=\pm\sqrt{-1-\omega}\right),a_0=0,a_1=\pm\sqrt{\frac{-2k+\beta}{1+2g+g^2+2z+2gz+z^2}}, \\ a_{-1}=\frac{1}{3}(a_1-2k^2a_1-2\omega a_1), \quad (5.76)$$

we acquire the following trigonometric function solutions using $k=\sqrt{2-\omega}$

$$Q(x,t,y)=\frac{\sqrt{2\left(\beta-2\sqrt{2-\omega}\right)}}{(1+g+z)}\tan\left[x+y\beta-2t\sqrt{2-\omega}\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{2-\omega}+t\omega\right)}, \quad (5.77)$$

$$S(x,t,y)=\frac{g\sqrt{2\left(\beta-2\sqrt{2-\omega}\right)}}{(1+g+z)}\tan\left[x+y\beta-2t\sqrt{2-\omega}\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{2-\omega}+t\omega\right)}, \quad (5.78)$$

$$N(x,t,y)=\frac{z\sqrt{2\left(\beta-2\sqrt{2-\omega}\right)}}{(1+g+z)}\tan\left[x+y\beta-2t\sqrt{2-\omega}\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{2-\omega}+t\omega\right)}, \quad (5.79)$$

$$R(x,t,y)=-2\tan\left[x+y\beta-2t\sqrt{2-\omega}\right]^2, \quad (5.80)$$

and

$$Q(x,t,y)=-\frac{\sqrt{2\left(\beta-2\sqrt{2-\omega}\right)}}{(1+g+z)}\cot\left[x+y\beta-2t\sqrt{2-\omega}\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{2-\omega}+t\omega\right)}, \quad (5.81)$$

$$S(x,t,y)=-\frac{g\sqrt{2\left(\beta-2\sqrt{2-\omega}\right)}}{(1+g+z)}\cot\left[x+y\beta-2t\sqrt{2-\omega}\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{2-\omega}+t\omega\right)}, \quad (5.82)$$

$$N(x,t,y)=-\frac{z\sqrt{2\left(\beta-2\sqrt{2-\omega}\right)}}{(1+g+z)}\cot\left[x+y\beta-2t\sqrt{2-\omega}\right]\times e^{i\left(y\alpha+\gamma+x\sqrt{2-\omega}+t\omega\right)}, \quad (5.83)$$

$$R(x,t,y)=-2\cot\left[x+y\beta-2t\sqrt{2-\omega}\right]^2 \quad (5.84)$$

and the following trigonometric function solutions when $k = \sqrt{-1-\omega}$

$$Q(x,t,y) = \frac{\sqrt{2(\beta - 2\sqrt{-1-\omega})}}{(1+g+z)} \sec\left[x + y\beta - 2t\sqrt{-1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{-1-\omega}+t\omega)}, \quad (5.85)$$

$$S(x,t,y) = \frac{g\sqrt{2(\beta - 2\sqrt{-1-\omega})}}{(1+g+z)} \sec\left[x + y\beta - 2t\sqrt{-1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{-1-\omega}+t\omega)}, \quad (5.86)$$

$$N(x,t,y) = \frac{z\sqrt{2(\beta - 2\sqrt{-1-\omega})}}{(1+g+z)} \sec\left[x + y\beta - 2t\sqrt{-1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{-1-\omega}+t\omega)}, \quad (5.87)$$

$$R(x,t,y) = -2\sec\left[x + y\beta - 2t\sqrt{-1-\omega}\right]^2. \quad (5.88)$$

and

$$Q(x,t,y) = \frac{\sqrt{2(\beta - 2\sqrt{-1-\omega})}}{(1+g+z)} \csc\left[x + y\beta - 2t\sqrt{-1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{-1-\omega}+t\omega)}, \quad (5.89)$$

$$S(x,t,y) = \frac{g\sqrt{2(\beta - 2\sqrt{-1-\omega})}}{(1+g+z)} \csc\left[x + y\beta - 2t\sqrt{-1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{-1-\omega}+t\omega)}, \quad (5.90)$$

$$N(x,t,y) = \frac{z\sqrt{2(\beta - 2\sqrt{-1-\omega})}}{(1+g+z)} \csc\left[x + y\beta - 2t\sqrt{-1-\omega}\right] \times e^{i(y\alpha+\gamma+x\sqrt{-1-\omega}+t\omega)}, \quad (5.91)$$

$$R(x,t,y) = -2\csc\left[x + y\beta - 2t\sqrt{-1-\omega}\right]^2. \quad (5.92)$$

Case 6: Solving (5.12) with the values of $A = \frac{1}{2}, B = 0, C = \frac{1}{2}$, we acquire the following parameters

$$k = \pm\sqrt{\frac{1-2\omega}{2}}, a_0 = 0, a_{-1} = 0, a_1 = \pm\frac{\sqrt{\frac{\beta \pm \sqrt{2-4\omega}}{2}}}{(1+g+z)}, \quad (5.93)$$

and using $F(\xi) = \csc(\xi) + \cot(\xi)$ and $F(\xi) = \sec(\xi) - \tan(\xi)$.

$$Q(x,t,y) = \frac{\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}(\sec[x + y\beta - t\sqrt{2-4\omega}] - \tan[x + y\beta - t\sqrt{2-4\omega}]) \times e^{\frac{1}{2}i(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega)}, \tag{5.94}$$

$$S(x,t,y) = \frac{g\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}(\sec[x + y\beta - t\sqrt{2-4\omega}] - \tan[x + y\beta - t\sqrt{2-4\omega}]) \times e^{\frac{1}{2}i(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega)}, \tag{5.95}$$

$$N(x,t,y) = \frac{z\sqrt{\frac{\beta-\sqrt{2-4\omega}}{2}}}{(1+g+z)}(\sec[x + y\beta - t\sqrt{2-4\omega}] - \tan[x + y\beta - t\sqrt{2-4\omega}])e^{\frac{1}{2}i(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega)}, \tag{5.96}$$

$$R(x,t,y) = -\frac{1}{2}(\sec[x + y\beta - t\sqrt{2-4\omega}] - \tan[x + y\beta - t\sqrt{2-4\omega}])^2. \tag{5.97}$$

and

$$Q(x,t,y) = \frac{\sqrt{2(\beta-2\sqrt{2-4\omega})}}{(1+g+z)}\cot\left[x + y\beta - 2t\sqrt{2-4\omega}\right] \times e^{\frac{1}{2}i(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega)}, \tag{5.98}$$

$$S(x,t,y) = \frac{g\sqrt{2(\beta-2\sqrt{2-4\omega})}}{(1+g+z)}\cot\left[x + y\beta - 2t\sqrt{2-4\omega}\right] \times e^{\frac{1}{2}i(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega)}, \tag{5.99}$$

$$N(x,t,y) = \frac{z\sqrt{2(\beta-2\sqrt{2-4\omega})}}{(1+g+z)}\cot\left[x + y\beta - 2t\sqrt{2-4\omega}\right] \times e^{\frac{1}{2}i(2y\alpha+2\gamma+x\sqrt{2-4\omega}+2t\omega)}, \tag{5.100}$$

$R(x,t,y) = -\frac{1}{2}\cot\left[x + y\beta - 2t\sqrt{2-4\omega}\right]^2$.(5.101)

Case 7: Solving (5.1.12) with the values of $A = 0, B = 0, C \neq 0$, we acquire

$$k = \pm i\sqrt{\omega}, a_0 = 0,, a_{-1} = 0, a_1 = \pm\frac{\sqrt{2C^2(\beta-2i\sqrt{\omega})}}{(1+g+z)^2}, \tag{5.102}$$

and using $F(\xi) = \frac{-1}{C\xi+\lambda}$, we get

$$Q(x,t,y)=\left(-\frac{\sqrt{2C^2(\beta-2i\sqrt{\omega})}}{(1+g+z)\left(\lambda+C(x+y\beta-2it\sqrt{\omega})\right)}\right)\times e^{i(y\alpha+\gamma+ix\sqrt{\omega}+t\omega)}, \tag{5.103}$$

$$S(x,t,y)=\left(-\frac{g\sqrt{2C^2(\beta-2i\sqrt{\omega})}}{(1+g+z)\left(\lambda+C(x+y\beta-2it\sqrt{\omega})\right)}\right)\times e^{i(y\alpha+\gamma+ix\sqrt{\omega}+t\omega)}, \tag{5.104}$$

$$N(x,t,y)=\left(-\frac{z\sqrt{2C^2(\beta-2i\sqrt{\omega})}}{(1+g+z)\left(\lambda+C(x+y\beta-2it\sqrt{\omega})\right)}\right)\times e^{i(y\alpha+\gamma+ix\sqrt{\omega}+t\omega)}, \tag{5.105}$$

$$R(x,t,y)=-\frac{2C^2}{\left(\lambda+C(x+y\beta-2it\sqrt{\omega})\right)^2}. \tag{5.106}$$

5.1.3. Application of the generalized projective Riccati equation method to the CNMS

In this section, we find the solutions of (5.1) using the GPRE method. In (5.10), balancing U^3 and U'', gives $n=1$. Therefore, (2.53) reduces to

$$U(\xi)=A_0+A_1\sigma(\xi)+B_1\tau(\xi). \tag{5.107}$$

Substituting (5.107) into (5.10) and using (2.54)-(2.56) and performing some algebraic calculations, we obtain

$$-\frac{A_0\left(-(2k-\beta)(k^2+\omega)+(1+g+z)^2A_0^2-3R(1+g+z)^2\varepsilon B_1^2\right)}{2k-\beta}=0, \tag{5.108}$$

$$\frac{1}{2k-\beta}\left(((2k-\beta)(k^2-R\varepsilon+2R\varepsilon^3+\omega)-3(1+g+z)^2A_0^2)A_1+\right.$$
$$\left.3(1+g+z)^2\varepsilon(-2\mu A_0+RA_1)B_1^2\right)=0, \tag{5.109}$$

$$\frac{1}{R(2k-\beta)}(RA_1(-(2k-\beta)\varepsilon(-1+4\varepsilon^2)\mu-3(1+g+z)^2A_0A_1)+$$
$$3(1+g+z)^2\varepsilon\left((r+\mu^2)A_0-2R\mu A_1\right)B_1^2)=0, \tag{5.110}$$

$$\frac{A_1\left(-R(1+g+z)^2A_1^2+\varepsilon(r+\mu^2)\left(2(2k-\beta)\varepsilon^2+3(1+g+z)^2B_1^2\right)\right)}{R(2k-\beta)}=0, \tag{5.111}$$

$$\frac{B_1\left((2k-\beta)(k^2+\omega)-3(1+g+z)^2A_0^2+R(1+g+z)^2\varepsilon B_1^2\right)}{2k-\beta}=0, \tag{5.112}$$

$$-\frac{B_1\left(6(1+g+z)^2A_0A_1+\varepsilon\mu\left((2k-\beta)(-1+2\varepsilon^2)+2(1+g+z)^2B_1^2\right)\right)}{2k-\beta}=0, \tag{5.113}$$

$$\frac{B_1\left(-3R(1+g+z)^2A_1^2+\varepsilon(r+\mu^2)\left(2(2k-\beta)\varepsilon^2+(1+g+z)^2B_1^2\right)\right)}{R(2k-\beta)}=0. \tag{5.114}$$

Consider the following cases:

Case 1: Setting $\varepsilon=-1$ and $r=-1$ in (5.108)-(5.114) and solving the system of algebraic equations, we acquire

$$R=-2(k^2+\omega), A_0=0, A_1=\frac{\pm\sqrt{\frac{(2k-\beta)(-1+\mu^2)}{2(k^2+\omega)}}}{(1+g+z)}, B_1=\frac{\pm\sqrt{\frac{-2k+\beta)}{2}}}{(1+g+z)}, \tag{5.115}$$

which gives the hyperbolic solutions represented by

$$Q(x,t,y)=\left(-\sqrt{(2k-\beta)(-1+\mu^2)(k^2+\omega)}(1+g+z)+(1+g+z)\times \sqrt{(-2k+\beta)(-k^2-\omega)}\sinh\left[(-2kt+x+y\beta)\sqrt{2(-k^2-\omega)}\right]\right)((1+g+z)^2\times \left(\mu+\cosh\left[\sqrt{2(-k^2-\omega)(-2kt+x+y\beta)}\right]\right)\Big)^{-1})\times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.116}$$

$$S(x,t,y)=g\left(-\sqrt{(2k-\beta)(-1+\mu^2)(k^2+\omega)}(1+g+z)+(1+g+z)\times \sqrt{(-2k+\beta)(-k^2-\omega)}\sinh\left[(-2kt+x+y\beta)\sqrt{2(-k^2-\omega)}\right]\right)((1+g+z)^2\times \left(\mu+\cosh\left[\sqrt{2(-k^2-\omega)(-2kt+x+y\beta)}\right]\right)\Big)^{-1})\times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.117}$$

$$N(x,t,y)=z\left(-\sqrt{(2k-\beta)(-1+\mu^2)(k^2+\omega)}(1+g+z)+(1+g+z)\times \sqrt{(-2k+\beta)(-k^2-\omega)}\sinh\left[(-2kt+x+y\beta)\sqrt{2(-k^2-\omega)}\right]\right)((1+g+z)^2\times \left(\mu+\cosh\left[\sqrt{2(-k^2-\omega)(-2kt+x+y\beta)}\right]\right)\Big)^{-1})\times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.118}$$

$$R(x,t,y)=\Big((1+g+z)\sqrt{(2k-\beta)(-1+\mu^2)}(k^2+\omega)- \sqrt{(-2k+\beta)(-k^2-\omega)(k^2+\omega)}\times (1+g+z)\sinh\left[(-2kt+x+y\beta)\sqrt{2(-k^2-\omega)}\right]\Big)^2\times ((1+g+z)(2k-\beta)(k^2+\omega)\times \left(\mu+\cosh\left[(-2kt+x+y\beta)\sqrt{2(-k^2-\omega)}\right]\right)^2\Big)^{-1}. \tag{5.119}$$

Case 2: Setting $\varepsilon = -1$ and $r = 1$ in (5.108)-(5.114) and solving the results, we obtain

$$R = -2(k^2+\omega), A_0 = 0, A_1 = \frac{\pm\sqrt{\frac{-2k+\beta}{2}}}{(1+g+z)}, B_1 = \pm\sqrt{\frac{-2k+\beta}{2(1+2g+g^2+2z+2gz+z^2)}}, \tag{5.120}$$

which gives the hyperbolic solutions represented by

$$Q(x,t,y) = \Bigg(\Big(-\sqrt{(2k-\beta)(1+\mu^2)(k^2+\omega)}\,(1+g+z) + (1+g+z)\times \sqrt{(-2k+\beta)(-k^2-\omega)}\cosh\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{-k^2-\omega}\right]\Big)\times \left((1+g+z)^2\left(\mu+\sinh\left[\sqrt{2(-k^2-\omega)}(-2kt+x+y\beta)\right]\right)\right)^{-1}\Bigg)\times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.121}$$

$$S(x,t,y) = g\Bigg(\Big(-\sqrt{(2k-\beta)(1+\mu^2)(k^2+\omega)}\,(1+g+z) + (1+g+z)\times \sqrt{(-2k+\beta)(-k^2-\omega)}\cosh\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{-k^2-\omega}\right]\Big)\times \left((1+g+z)^2\left(\mu+\sinh\left[\sqrt{2(-k^2-\omega)}(-2kt+x+y\beta)\right]\right)\right)^{-1}\Bigg)\times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.122}$$

$$N(x,t,y) = z\Bigg(\Big(-\sqrt{(2k-\beta)(1+\mu^2)(k^2+\omega)}\,(1+g+z) + (1+g+z)\times \sqrt{(-2k+\beta)(-k^2-\omega)}\cosh\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{-k^2-\omega}\right]\Big)\times \left((1+g+z)^2\left(\mu+\sinh\left[\sqrt{2(-k^2-\omega)}(-2kt+x+y\beta)\right]\right)\right)^{-1}\Bigg)\times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.123}$$

$$R(x,t,y) = \Big((1+g+z)(k^2+\omega)\sqrt{(2k-\beta)(1+\mu^2)} - \sqrt{(-2k+\beta)(-k^2-\omega)\big((k^2+\omega)\big)}\times (1+g+z)\cosh\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{-k^2-\omega}\right]\Big)^2\times \left((1+g+z)^2(2k-\beta)(k^2+\omega)\left(\mu+\sinh\left[\sqrt{2(-k^2-\omega)}(-2kt+x+y\beta)\right]\right)^2\right)^{-1}. \tag{5.124}$$

Case 3: Setting $\varepsilon = 1$ and $r = -1$ in (5.108)-(5.114) and solving the system of algebraic equations, we obtain

$$R = 2(k^2+\omega), A_0 = 0, A_1 = \frac{\pm\sqrt{\frac{(2k-\beta)(-1+\mu^2)}{2(k^2+\omega)}}}{(1+g+z)}, B_1 = \frac{\pm\sqrt{\frac{-2k+\beta}{2}}}{(1+g+z)}, \tag{5.125}$$

and subsequently, we acquire the following trigonometric function solutions

$$Q(x,t,y) = \left(\sqrt{(2k-\beta)(k^2+\omega)(-1+\mu^2)}(1+g+z) + (1+g+z) \times \sqrt{(-2k+\beta)(k^2+\omega)}\sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right) \times \left((1+g+z)^2\left(\mu+\cos\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)^{-1} \times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.126}$$

$$S(x,t,y) = g\left(\sqrt{(2k-\beta)(k^2+\omega)(-1+\mu^2)}(1+g+z) + (1+g+z) \times \sqrt{(-2k+\beta)(k^2+\omega)}\sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right) \times \left((1+g+z)^2\left(\mu+\cos\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)^{-1} \times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.127}$$

$$N(x,t,y) = z\left(\sqrt{(2k-\beta)(k^2+\omega)(-1+\mu^2)}(1+g+z) + (1+g+z) \times \sqrt{(-2k+\beta)(k^2+\omega)}\sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right) \times \left((1+g+z)^2\left(\mu+\cos\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)^{-1} \times e^{i(kx+y\alpha+\gamma+t\omega)}, \tag{5.128}$$

$$R(x,t,y) = \left(\sec\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]^2(1+g+z) \times \left(\sqrt{(-1+\mu^2)(2k-\beta)}(k^2+\omega) + \sqrt{(-2k+\beta)(k^2+\omega)(k^2+\omega)} \times \sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)^2\right) \times \left((1+g+z)^2(2k-\beta)(k^2+\omega)\left(1+\mu\sec\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)^2\right)^{-1}. \tag{5.129}$$

and

$$Q(x,t,y)=\left((1+g+z)\left(\sqrt{(-1+\mu^2)(2k-\beta)(k^2+\omega)}-\sqrt{(-2k+\beta)(k^2+\omega)}\times\right.\right.$$
$$\left.\left.\cos\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)\times$$
$$\left((1+g+z)^2\left(\mu+\sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)^{-1}\times e^{i(kx+y\alpha+\gamma+t\omega)},$$
(5.130)

$$S(x,t,y)=g\left((1+g+z)\left(\sqrt{(-1+\mu^2)(2k-\beta)(k^2+\omega)}-\sqrt{(-2k+\beta)(k^2+\omega)}\times\right.\right.$$
$$\left.\left.\cos\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)\times$$
$$\left((1+g+z)^2\left(\mu+\sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)^{-1}\times e^{i(kx+y\alpha+\gamma+t\omega)},$$
(5. 131)

$$N(x,t,y)=z\left((1+g+z)\left(\sqrt{(-1+\mu^2)(2k-\beta)(k^2+\omega)}-\sqrt{(-2k+\beta)(k^2+\omega)}\times\right.\right.$$
$$\left.\left.\cos\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)\times$$
$$\left((1+g+z)^2\left(\mu+\sin\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)\right)^{-1}\times e^{i(kx+y\alpha+\gamma+t\omega)},$$
(5. 132)

$$R(x,t,y)=\left((1+g+z)(k^2+\omega)\left(\left(\sqrt{(-1+\mu^2)(2k-\beta)}\right.\right.\right.$$
$$\left.-\sqrt{(-2k+\beta)}\cos\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]\right)^2\times$$
$$\left.\left.\csc\left[(-2kt+x+y\beta)\sqrt{2(k^2+\omega)}\right]^2\right)\right)((1+g+z)^2\times$$
$$\left.(2k-\beta)(k^2+\omega)\left(1+\mu\csc\left[\sqrt{2}(-2kt+x+y\beta)\sqrt{k^2+\omega}\right]\right)^2\right)^{-1}.$$
(5. 133)

Figure.5.1 shows the surface profiles (a) topological solutions (5.14)-(5.17), and (b) topological solution (5.50)-(5.53) by choosing the values $\omega=0.4, z=1, \alpha=0.5, \beta=0.5, \gamma=0.2, t=0.5, g=1$, $(-10\leq x\leq 10,\ -10\leq y\leq 10)$. Figure.5.2 shows the surface profiles (a) complexiton solutions (5.24)-(5.26), and (b) complexiton solution (5.27) by choosing the values $\omega=0.5, z=1, \alpha=0.6, \beta=0.5, \gamma=2, g=1, t=0.1$,$(-10\leq x\leq 10, -10\leq y\leq 10)$ of (5.14)-(5.17).

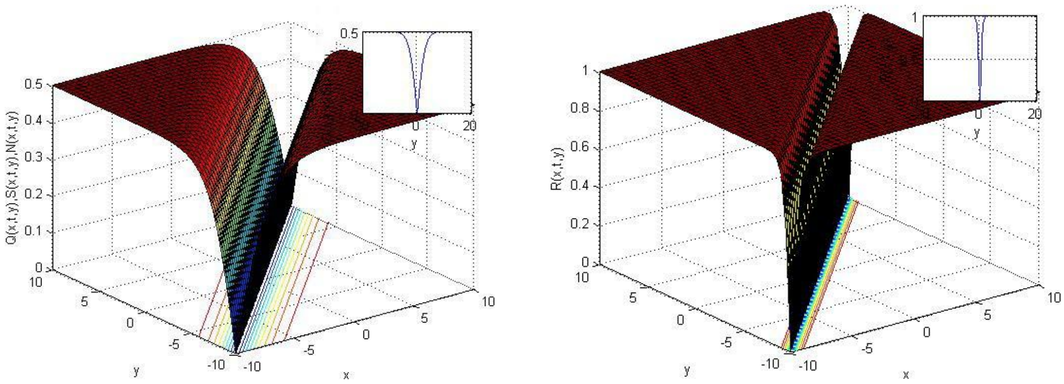

Figure 5.1: The 3D and 2D profiles of equation (5.1) .

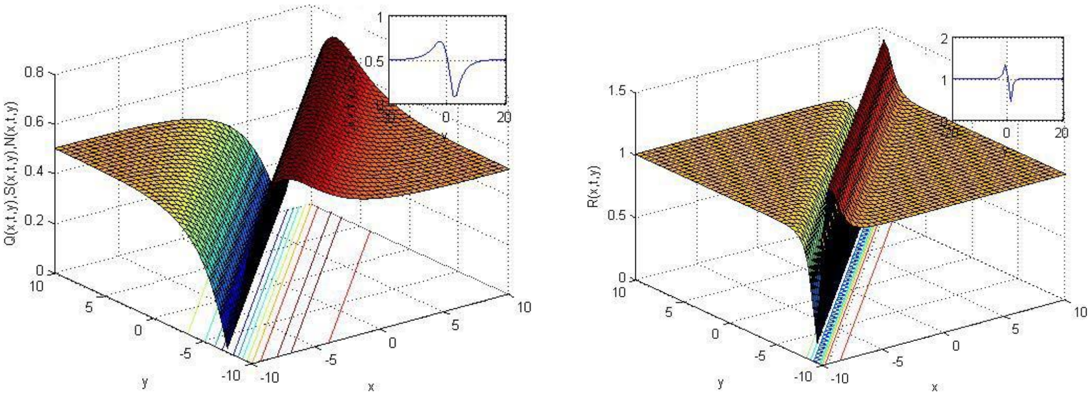

Figure 5.2: The 3D and 2D profiles of equation (5.1.1) .

5.2. Generalized Schrödinger-Boussinesq System (GSBE)

In this section, we will consider the generalized Schrödinger-Boussinesq system (GSBE). Solutions of the CGSBs based on the undetermined coefficients method will be studied in this section. Applying the new conservation theorem, the Cls will be established. The results of this section have been published in [94]. The CGSBs is given by [95, 96, 97]

$$\begin{cases} iq_t + q_{xx} + \alpha q = qw, \\ 3w_{tt} - w_{xxxx} + 3(w^2)_{xx} + \beta w_{xx} = (|q|^2)_{xx}. \end{cases} \tag{5.134}$$

The independent variables are x and t, while q, w represent the dependent variables. (5.2.1) have been solved by using different integration techniques. Biswas et.al [95] studied the equation using the tanh method. In [96], the $(G'/G, 1/G)$ was used to obtain the solutions of the equation. However, in [97], the extended simplest equation method was used to acquire the solutions of the equation.

5.2.1 Analytic study

In this Let

$$q(x,t) = U(\xi)e^{i\phi(x,t)},\\ w(x,t) = W(\xi), \xi = x + ct, \tag{5.135}$$

where $U(\xi)$ and $W(\xi)$ represents the pulse shapes and

$$\phi(x,t) = px + vt. \tag{5.136}$$

Substituting (5.135) into (5.134), we acquire the relation $c = -2p$ and the system of ODEs

$$(\alpha - v - p^2)U + U'' - UW = 0,\\ (\beta + 12p^2)W'' + (3W^2 - U^2)'' - W'''' = 0. \tag{5.137}$$

Integrating the second equation in (5.137) twice, taking the integration constant to be zero, we acquire

$$(\alpha - v - p^2)U + U'' - UW = 0,\\ (\beta + 12p^2)W + (3W^2 - U^2) - W'' = 0. \tag{5.138}$$

5.2.2. Application of the undetermined coefficients method to the GSBE

In this part, we will employ the technique of undetermined coefficients to acquire non topological and singular solitons for (5.134).

(a) Non topological Solitons

The non topological soliton of (5.134) can be written as:

$U(\xi) = \sigma_1 \mathrm{sech}(\xi)^{R_1}, W(\xi) = \sigma_2 \mathrm{sech}(\xi)^{R_2}$, (5.139)

where σ_1, σ_2, R_1 and R_2 are to be determined later. Inserting (5.139) into (5.138), we acquire

$$p^2\mathrm{sech}(\mu\xi)^{R_1}\sigma_1 + v\mathrm{sech}(\mu\xi)^{R_1}\sigma_1 - \alpha\mathrm{sech}(\mu\xi)^{R_1}\sigma_1 + \mu^2\mathrm{sech}(\mu\xi)^{2+R_1}R_1\sigma_1 -\\ \mu^2\mathrm{Sech}(\mu\xi)^{2+R_1}\mathrm{sech}(\mu\xi)^2R_1^2\sigma_1 + \mathrm{sech}(\mu\xi)^{R_1+R_2}\sigma_1\sigma_2 = 0,\\ \mathrm{sech}(\mu\xi)^{2R_1}\sigma_1^2 - 12p^2\mathrm{sech}(\mu\xi)^{R_2}\sigma_2 - \beta\mathrm{sech}(\mu\xi)^{R_2}\sigma_2 -\\ \mu^2\mathrm{sech}(\mu\xi)^{2+R_2}R_2\sigma_2 + \mu^2\mathrm{sech}(\mu\xi)^{2+R_2}\mathrm{sech}(\mu\xi)^2R_2^2\sigma_2 - 3\mathrm{sech}(\mu\xi)^{2R_2}\sigma_2^2 = 0. \tag{5.140}$$

After some algebraic computations, we obtain

$$2 + R_1 = 2R_1, \tag{5.141}$$

$$2 + R_2 = 2R_2, \tag{5.142}$$

from which one can obtain $R_1 = R_2 = 2$. Substituting into (5.140), we acquire

$$\begin{gathered}\operatorname{sech}(\mu\xi)\sigma_1\left(p^2 + v - \alpha - 4\mu^2 + \operatorname{sech}(\mu\xi)^2(6\mu^2 + \sigma_2)\right) = 0,\\ \operatorname{sech}(\mu\xi)^2\left(-(12p^2 + \beta - 4\mu^2)\sigma_2 + \operatorname{sech}(\mu\xi)^2\left(\sigma_1^2 - 3\sigma_2(2\mu^2 + \sigma_2)\right)\right) = 0.\end{gathered} \tag{5.143}$$

After performing some calculations, we obtain

$$p = \pm\frac{1}{2}\sqrt{\frac{-\beta+4\mu^2}{3}}, \sigma_2 = -6\mu^2, \sigma_1 = \pm 6\sqrt{2}\mu^2, v = \frac{1}{12}(12\alpha + \beta + 44\mu^2). \tag{5.144}$$

The non-topological soliton solutions read

$$\begin{gathered}q(x,t) = \left(6\sqrt{2}\mu^2\operatorname{sech}[\mu(x-2pt)]^2\right) \times e^{i(px+vt)},\\ w(x,t) = -6\mu^2\operatorname{sech}[\mu(x-2pt)]^2,\end{gathered} \tag{5.145}$$

where $\xi = x - 2pt$.

(b) Singular Solitons

The singular solitons of (5.134) can be derived using the following ansatz:

$U(\xi) = \sigma_1\operatorname{csch}(\xi)^{R_1}, W(\xi) = \sigma_2\operatorname{csch}(\xi)^{R_2}$, (5.146)

Inserting (5.146) into (5.138), we acquire

$$\begin{gathered}p^2\operatorname{csch}(\mu\xi)^{R_1}\sigma_1 + v\operatorname{csch}(\mu\xi)^{R_1}\sigma_1 - \alpha\operatorname{csch}(\mu\xi)^{R_1}\sigma_1 - \mu^2\operatorname{csch}(\mu\xi)^{2+R_1}R_1\sigma_1 -\\ \mu^2\operatorname{Csch}(\mu\xi)^{R_1}R_1^2\sigma_1 - \mu^2\operatorname{csch}(\mu\xi)^{2+R_1}R_1^2\sigma_1 + \operatorname{csch}(\mu\xi)^{R_1+R_2}\sigma_1\sigma_2 = 0,\\ \operatorname{csch}(\mu\xi)^{2R_1}\sigma_1^2 - 12p^2\operatorname{csch}(\mu\xi)^{R_2}\sigma_2 - \beta\operatorname{csch}(\mu\xi)^{R_2}\sigma_2 + \mu^2\operatorname{csch}(\mu\xi)^{2+R_2}R_2\sigma_2 +\\ \mu^2\operatorname{csch}(\mu\xi)^{R_2}R_2^2\sigma_2 + \mu^2\operatorname{csch}(\mu\xi)^{2+R_2}R_2^2\sigma_2 - 3\operatorname{csch}(\mu\xi)^{2R_2}\sigma_2^2 = 0.\end{gathered} \tag{5.147}$$

After some algebraic computations, we obtain

$$2 + R_1 = 2R_1, \tag{5.148}$$

$$2 + R_2 = 2R_2, \tag{5.149}$$

from which one can obtain $R_1 = R_2 = 2$. Substituting R_1 and R_2 into (5.147), we get

$$\operatorname{csch}(\mu\xi)\sigma_1\left(p^2+v-\alpha-4\mu^2+\operatorname{csch}(\mu\xi)^2(-6\mu^2+\sigma_2)\right)=0,$$
$$(12p^2+\beta-4\mu^2)\operatorname{csch}(\mu\xi)\sigma_2-\operatorname{csch}(\mu\xi)^3(\sigma_1^2+6\mu^2\sigma_2-3\sigma_2^2)=0. \tag{5.150}$$

After some algebraic computations, we obtain

$$p=\pm\frac{1}{2}\sqrt{\frac{-\beta+4\mu^2}{3}},\sigma_2=6\mu^2,\sigma_1=\pm6\sqrt{2}\mu^2,v=\frac{1}{12}(12\alpha+\beta+44\mu^2). \tag{5.151}$$

The singular soliton solutions read

$$q(x,t)=\left(6\sqrt{2}\mu^2\operatorname{csch}[\mu(x-2pt)]^2\right)\times e^{i(px+vt)},$$
$$w(x,t)=6\mu^2\operatorname{csch}[\mu(x-2pt)]^2. \tag{5.152}$$

Figure.5.3 shows the surface views of the non topological soliton solutions (5.145) by choosing the values $\beta=0.5,\mu=0.5,\alpha=0.2,(-10\le x\le 10,-10\le t\le 10)$.

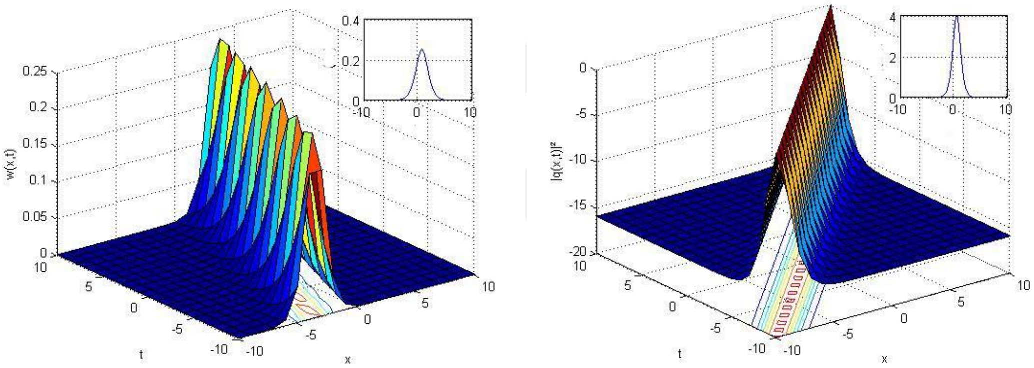

Figure 5.3: The 3D and 2D surfaces of the soliton solutions (5.145).

5.2.4. Lie Symmetry Analysis of the GSBE

Here, we aim to obtain the point symmetries of (5.134), we let

$$q(x,t)=u(x,t)+iv(x,t). \tag{5.153}$$

Putting (5.153) into (5.134), we acquire

$$F_1=u_t+\alpha v-wv+v_{xx}=0,$$
$$F_2=v_t-\alpha u+uw-u_{xx}=0,$$
$$F_3=3w_{tt}+6{w_x}^2+(\beta+6w)w_{xx}-2({u_x}^2+{v_x}^2+uu_{xx}+vv_{xx})-w_{xxxx}=0. \tag{5.154}$$

The algebras of (5.154) are generated by [30]

$$\Gamma = \xi_1(x,t,u,v,w)\frac{\partial}{\partial x} + \xi_2(x,t,u,v,w)\frac{\partial}{\partial t} + \eta_1(x,t,u,v,w)\frac{\partial}{\partial u} + \eta_2(x,t,u,v,w)\frac{\partial}{\partial v} + \eta_3(x,t,u,v,w)\frac{\partial}{\partial w}. \tag{5.155}$$

(5.154) has the infinitesimals given by

$$\xi_1(x,t,u,v,w) = C_1 - \frac{xC_2}{2}, \tag{5.156}$$

$$\xi_2(x,t,u,v,w) = -tC_2 + C_3, \tag{5.157}$$

$$\eta_1(x,t,u,v,w) = C_1 u + \frac{1}{6}v(t(6\alpha+\beta)C_2 - 6C_4), \tag{5.158}$$

$$\eta_2(x,t,u,v,w) = \frac{1}{6}(6w+\beta)C_2, \tag{5.159}$$

$$\eta_3(x,t,u,v,w) = C - \frac{1}{6}tuC_2(6\alpha+\beta) + vC_2 + uC_4, \tag{5.160}$$

(5.154) has the following vector fields

$$\Gamma_1 = \partial_x, \tag{5.161}$$

$$\Gamma_2 = \partial_t, \tag{5.162}$$

$$\Gamma_3 = -v\,\partial_u + u\,\partial_v, \tag{5.163}$$

$$\Gamma_4 = -6t\,\partial_t + (6u + 6t\alpha v + tv\beta)\,\partial_u + (6w+\beta)\,\partial_w - 3x\,\partial_x + (-6tu\alpha - tu\beta + 6v)\,\partial_v. \tag{5.164}$$

5.2.5. Nonlinear Self-Adjointness of the GSBE

In this section, we study the nonlinear self-adjointness of (5.2.21).The Lagrangian of (5.2.21) is represented by

$$\begin{aligned}&\mathcal{L} = r(-\alpha u + uw + v_t - u_{xx}) + z(\alpha v - wv + u_t + v_{xx}) + \\ &h(3w_{tt} + 6{w_x}^2 + (\beta + 6w)w_{xx} - 2({u_x}^2 + {v_x}^2 + uu_{xx} + vv_{xx}) - w_{xxxx}),\end{aligned} \tag{5.165}$$

The adjoint can be derived using the following relation

$$F_1^* = \frac{\delta\mathcal{L}}{\delta u} = 0, F_2^* = \frac{\delta\mathcal{L}}{\delta v} = 0, F_3^* = \frac{\delta\mathcal{L}}{\delta w} = 0, \quad (5.166)$$

with

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta u} &= \frac{\partial\mathcal{L}}{\partial u} - D_t\frac{\partial\mathcal{L}}{\partial u_t} - D_x\frac{\partial\mathcal{L}}{\partial u_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial u_{xx}} - (D_x)^3\frac{\partial\mathcal{L}}{\partial u_{xxx}} + (D_x)^4\frac{\partial\mathcal{L}}{\partial u_{xxxx}},\\ \frac{\delta\mathcal{L}}{\delta v} &= \frac{\partial\mathcal{L}}{\partial v} - D_t\frac{\partial\mathcal{L}}{\partial v_t} - D_x\frac{\partial\mathcal{L}}{\partial v_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial v_{xx}} - (D_x)^3\frac{\partial\mathcal{L}}{\partial v_{xxx}} + (D_x)^4\frac{\partial\mathcal{L}}{\partial v_{xxxx}},\\ \frac{\delta\mathcal{L}}{\delta w} &= \frac{\partial\mathcal{L}}{\partial w} - D_t\frac{\partial\mathcal{L}}{\partial w_t} - D_x\frac{\partial\mathcal{L}}{\partial w_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial w_{xx}} - (D_x)^3\frac{\partial\mathcal{L}}{\partial w_{xxx}} + (D_x)^4\frac{\partial\mathcal{L}}{\partial w_{xxxx}}.\end{aligned} \tag{5.167}$$

From the Lagrangian (5.165), one can get the adjoint system as

$$\begin{aligned}F_1^* &= r(-\alpha + w) - z_t - 2uh_{xx} - r_{xx} = 0,\\ F_2^* &= ru - zv + 3h_{tt} + (\beta + 6w)h_{xx} - h_{xxxx} = 0,\\ F_3^* &= (\alpha - w)z - r_t - 2vh_{xx} + z_{xx} = 0.\end{aligned} \tag{5.169}$$

Theorem 2.1 *A System of PDEs is nonlinearly self-adjoint on the condition of the adjoint system if it satisfies*

$$\begin{aligned}F_1^*|_{z=Z(x,t,u,v,w),r=R(x,u,v,w),h=H(x,u,v,w)} &= \lambda_{11}F_1 + \lambda_{12}F_2 + \lambda_{13}F_3 = 0,\\ F_2^*|_{z=Z(x,t,u,v,w),r=R(x,u,v,w),h=H(x,u,v,w)} &= \lambda_{21}F_1 + \lambda_{22}F_2 + \lambda_{23}F_3 = 0,\\ F_3^*|_{z=Z(x,t,u,v,w),r=R(x,u,v,w),h=H(x,u,v,w)} &= \lambda_{31}F_1 + \lambda_{32}F_2 + \lambda_{33}F_3 = 0,\end{aligned} \tag{5.169}$$

where not all $z = Z(x, t, u, v, w)$, $r = R(x, t, u, v, w)$ and $h = H(x, t, u, v, w)$ equal to zero. The coefficients $\lambda_{ij}(i, j = 1,2,3)$ are to be determined. From the coefficients of u_t, v_t w_t, we get

$$\begin{aligned}&\lambda_{11} = -Z_u, \quad \lambda_{12} = -Z_v, \quad \lambda_{13} = -Z_w,\\ &\lambda_{21} = -R_u, \quad \lambda_{22} = -R_v, \quad \lambda_{23} = -R_w,\\ &\lambda_{31} = -H_u, \quad \lambda_{32} = -H_v, \quad \lambda_{33} = -H_w.\end{aligned} \tag{5.170}$$

Substituting (5.170) into (5.169), we get a huge system of linear PDEs. Solving the system using SYM package [63], we obtain

$$z = u\mathbf{c}_1,$$
$$r = v\mathbf{c}_1, \qquad (5.171)$$
$$h = \mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5).$$

Theorem 2.2 *System (5.154) is nonlinear self-adjoint if* z, r *and* h *are given by (5.171).*

5.2.6. Conservation Laws of the GSBE

In accordance of Theorem 2.11.1, we construct the Cls for (5.134) using the Lie point symmetries (5.161)-(5.164). Substituting the nonlinear self-adjoint substitution (5.171) into the Lagrangian (5.165), we obtain

$$\mathcal{L}^* = \mathbf{c}_1 v(-\alpha u + uw + v_t - u_{xx}) + \mathbf{c}_1 u(\alpha v - wv + u_t + v_{xx}) + \big(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5)\big) \times (3w_{tt} + 6{w_x}^2 + (\beta + 6w)w_{xx} - 2({u_x}^2 + {v_x}^2 + uu_{xx} + vv_{xx}) - w_{xxxx}). \qquad (5.172)$$

The property (5.2172) will be used together with the point symmetries and conservation formular to construct the Cls. The only trivial conservation law is that generated by the point symmetry Γ_3. The conserved vectors are given as follows:

• The symmetry $\Gamma_1 = \partial x$ admits the conserved vector:

$$T_1^x = \mathbf{c}_1(uu_t + vv_t) + 3(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)w_{tt} - (\mathbf{c}_4 + t\mathbf{c}_5)(2uu_x - (\beta + 6w)w_x + 2vv_x + w_{xxx}),$$

$$T_1^t = 3(\mathbf{c}_3 + x\mathbf{c}_5)w_x - \mathbf{c}_1(uu_x + vv_x) - 3(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)w_{xt}.$$

• Lie point symmetry generator $\Gamma_2 = \partial t$ yields the conserved vector with components:

$$T_2^x = -w_t(-(\mathbf{c}_4 + t\mathbf{c}_5)(\beta + 6w) + 6(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5))w_x) - v_t(2(\mathbf{c}_4 + t\mathbf{c}_5)v - \mathbf{c}_1 u_x - 2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)v_x) - u_t(2(\mathbf{c}_4 + t\mathbf{c}_5)u - 2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)u_x + \mathbf{c}_1 v_x) - (-2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)u - \mathbf{c}_1 v)u_{xt} - (\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5))(\beta + 6w)w_{xt} - (\mathbf{c}_1 u - 2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)v)v_{xt} - (\mathbf{c}_4 + t\mathbf{c}_5)w_{xxt} + (\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5))w_{xxxt}$$

$$T_2^t = -\mathbf{c}_1 uu_t + 3(\mathbf{c}_3 + x\mathbf{c}_5)w_t - \mathbf{c}_1 vv_t - 3(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5))w_{tt} + \mathbf{c}_1 v(u(-\alpha + w) + v_t - u_{xx}) + \mathbf{c}_1 u((\alpha - w)v + u_t + v_{xx}) + (\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5) (3w_{tt} + 6{w_x}^2 + (\beta + 6w)w_{xx} - 2({u_x}^2 + {v_x}^2 + uu_{xx} + vv_{xx}) - w_{xxxx}).$$

• The Lie point symmetry $\Gamma_4 = -6t\,\partial_t + (6u + 6t\alpha v + tv\beta)\,\partial_u + (6w + \beta)\,\partial_w - 3x\,\partial_x + (-6tu\alpha - tu\beta + 6v)\,\partial_v$ determines the conserved vector:

$$T_4^x = (\beta + 6w + 6tw_t + 3xw_x)\big(-(\mathbf{c}_4 + t\mathbf{c}_5)(\beta + 6w) + 6\big(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5)\big)w_x\big) + (-t(6\alpha + \beta)u + 6v + 6tv_t + 3xv_x)(2(\mathbf{c}_4 + t\mathbf{c}_5)v - \mathbf{c}_1u_x - 2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)v_x) + (6u + t(6\alpha + \beta)v + 6tu_t + 3xu_x)(2(\mathbf{c}_4 + t\mathbf{c}_5)u - 2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)u_x + \mathbf{c}_1v_x) + (-2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)u - \mathbf{c}_1v)(9u_x + t(6\alpha + \beta)v_x + 6tu_{xt} + 3xu_{xx}) + 3(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5)\big)(\beta + 6w)(3w_x + 2tw_{xt} + xw_{xx}) + (\mathbf{c}_1u - 2(\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)v)(-t(6\alpha + \beta)u_x + 9v_x + 6tv_{xt} + 3xv_{xx}) + 3(\mathbf{c}_4 + t\mathbf{c}_5)(4w_{xx} + 2tw_{xxt} + xw_{xxx}) - 3x\big(\mathbf{c}_1\big(v(v_t - u_{xx}) + u(u_t + v_{xx})\big) + (\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)(3w_{tt} - 2{u_x}^2 + 6{w_x}^2 - 2{v_x}^2 - 2uu_{xx} + (\beta + 6w)w_{xx} - 2vv_{xx} - w_{xxxx})\big) - 3\big(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5)\big)(5w_{xxx} + 2tw_{xxxt} + xw_{xxxxx}),$$

$$T_4^t = \mathbf{c}_1u(6u + t(6\alpha + \beta)v + 6tu_t + 3xu_x) - 3(\mathbf{c}_3 + x\mathbf{c}_5)(\beta + 6w + 6tw_t + 3xw_x) + \mathbf{c}_1v(-t(6\alpha + \beta)u + 6v + 6tv_t + 3xv_x) + 9(\mathbf{c}_2 + x\mathbf{c}_4 + t(\mathbf{c}_3 + x\mathbf{c}_5))(4w_t + 2tw_{tt} + xw_{xt}) - 6t(\mathbf{c}_1(v(v_t - u_{xx}) + u(u_t + v_{xx})) + (\mathbf{c}_2 + t\mathbf{c}_3 + x\mathbf{c}_4 + tx\mathbf{c}_5)(3w_{tt} - 2{u_x}^2 + 6{w_x}^2 - 2{v_x}^2 - 2uu_{xx} + (\beta + 6w)w_{xx} - 2vv_{xx} - w_{xxxx})).$$

5.3. The Nonlinear Kudryashov-Sinelshchikov Equation (NKSE)

In 2010, Kudryashov and Sinelshchikov [98] derived a NPDE describing the pressure waves in a mixture liquid. We intend to investigate the solutions of this equation using the RBSO method. Furthermore, we will apply the new Cls theorem to derive the Cls of the equation. The results of this section have been published in [99]. The equation that will be studied in this section is given by [100, 101]:

$$u_t + \alpha uu_x + \beta u_{xxx} + \gamma(uu_{xx})_x + du_xu_{xx} = 0. \tag{5.173}$$

In (5.3.1), u is the density. α, β, γ and d are real constants. (5.173) was studied using different approaches in [102- 106].

5.3.1. Application of the Riccati Bernoulli sub-Ode method to the NKSE

In this part, we apply the Riccati-Bernoulli sub-ODE to (5.3.1). Let

$$u(x,t) = U(\xi), \quad \xi = K(x + vt). \tag{5.174}$$

Putting (5.174) into (5.173), we acquire:

$$KU'(v+\alpha U+K^2(d+\gamma)U'')+K^3(\beta+\gamma U)U'''=0. \tag{5.175}$$

Substituting (2.23), (2.24) and (2.25) into (5.175), we acquire

$$\begin{aligned}&k(bU+aU^{2-m}+cU^m)(v+\alpha U+k^2(d+\gamma)U^{-1-2m}(aU^2+\\&cU^{2m}+bU^{1+m})(-a(-2+m)U^2+cmU^{2m}+bU^{1+m})+\\&k^2U^{-2(1+m)}(\beta+\gamma U)(a^2(-2+m)(-3+2m)U^4+\\&c^2m(-1+2m)U^{4m}+ab(-3+m)(-2+m)U^{3+m}+\\&(b^2+2ac)U^{2+2m}+bcm(1+m)U^{1+3m}\big)\big)=0.\end{aligned} \tag{5.176}$$

Setting $m=0$ in (5.176), we obtain

$$\begin{aligned}&ck(v+b^2k^2\beta+2ack^2\beta+bck^2(d+\gamma))+k(b^3k^2\beta+b(v+8ack^2\beta)+\\&b^2ck^2(2d+3\gamma)+c(\alpha+2ack^2(d+2\gamma)))U+k(7ab^2k^2\beta+\\&a(v+8ack^2\beta)+b^3k^2(d+2\gamma)+b(\alpha+2ack^2(3d+7\gamma)))U^2+\\&ak(\alpha+12abk^2\beta+4ack^2(d+3\gamma)+b^2k^2(4d+11\gamma))U^3+\\&a^2k^3(5bd+6a\beta+17b\gamma)U^4+2a^3k^3(d+4\gamma)U^5=0.\end{aligned} \tag{5.177}$$

After performing some algebraic computations, we get

$$ck(v+b^2k^2\beta+2ack^2\beta+bck^2(d+\gamma))=0, \tag{5.178}$$

$$k(b^3k^2\beta+b(v+8ack^2\beta)+b^2ck^2(2d+3\gamma)+c(\alpha+2ack^2(d+2\gamma)))=0, \tag{5.179}$$

$$k(7ab^2k^2\beta+a(v+8ack^2\beta)+b^3k^2(d+2\gamma)+b(\alpha+2ack^2(3d+7\gamma)))=0, \tag{5.180}$$

$$b^2k^2(4d+11\gamma))=0, \tag{5.181}$$

$$a^2k^3(5bd+6a\beta+17b\gamma)=0, \tag{5.182}$$

$$2a^3k^3(d+4\gamma)=0. \tag{5.183}$$

Solving (5.178)-(5.183), we obtain the cases given by:

$$a = \frac{ck^2\gamma^2 \pm \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}}{2k^2\beta^2},$$
$$b = \frac{ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}}{k^2\beta\gamma}$$
$$d = -4\gamma,$$
$$v = k^2\beta\left(\frac{2c(ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4})}{k^2\beta^2} - \frac{(ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4})^2}{k^4\beta^2\gamma^2}\right). \quad (5.184)$$

Using the parameters obtained in (5.184), we obtain the following traveling wave solutions:

$$u(x,t) = -\frac{\beta}{\gamma} + \frac{k^2\beta^2\sqrt{A}k^4\beta^2\gamma^2\tan[\frac{1}{2}\sqrt{A}k^4\beta^2\gamma^2(C + k(x + k^2t\beta A]}{ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}}, \quad (5.185)$$

$$u(x,t) = -\frac{\beta}{\gamma} - \frac{k^2\beta^2\sqrt{A}k^4\beta^2\gamma^2\cot[\frac{1}{2}\sqrt{A}k^4\beta^2\gamma^2(C + k(x + k^2t\beta A]}{ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}}, \quad (5.186)$$

where $A = \sqrt{\frac{2c(ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4})}{k^2\beta^2} - \frac{(ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4})^2}{k^4\beta^2\gamma^2}},$

$$u(x,t) = -\frac{\beta}{\gamma} - \frac{k^2\beta^2\sqrt{B}k^4\beta^2\gamma^2\tanh[\frac{1}{2}\sqrt{B}k^4\beta^2\gamma^2(C + k(x + k^2t\beta B]}{ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}}, \quad (5.187)$$

$$u(x,t) = -\frac{\beta}{\gamma} - \frac{k^2\beta^2\sqrt{B}k^4\beta^2\gamma^2\coth[\frac{1}{2}\sqrt{B}k^4\beta^2\gamma^2(C + k(x + k^2t\beta b]}{ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}},$$

where $B = \sqrt{\frac{-2c(ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4})}{k^2\beta^2} - \frac{(ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4})^2}{k^4\beta^2\gamma^2}}.$

$$c = 0, a = \pm\frac{i\sqrt{\alpha\gamma}}{2k\beta}, b = i\frac{1}{k}\sqrt{\frac{\alpha}{\gamma}}, v = \frac{\alpha\beta}{\gamma}, d = -4\gamma. \quad (5.188)$$

Using the parameters obtained in (5.3.17), we obtain the following traveling wave solutions:

$$u(x,t) = \left(e^{i\sqrt{\frac{\alpha}{\gamma}}\left(x + \frac{t\alpha\beta}{\gamma}\right)}C - \frac{\gamma}{2\beta}\right)^{-1}, \quad (5.189)$$

$$u(x,t) = -\frac{\beta}{\gamma} - \frac{i\beta}{\gamma^2}\tan[\frac{1}{2k}\sqrt{\frac{\alpha}{\gamma}}(C + k(x + \frac{t\alpha\beta}{\gamma}))], \quad (5.190)$$

$$u(x,t) = -\frac{\beta}{\gamma} + \frac{i\beta}{\gamma^2}\cot[\frac{1}{2k}\sqrt{\frac{\alpha}{\gamma}}(J + k(x + \frac{t\alpha\beta}{\gamma}))], \tag{5.191}$$

$\alpha\gamma > 0$, $C, k > 0$ for the solitons to exist.

$$u(x,t) = -\frac{\beta}{\gamma} - \frac{\beta}{\gamma^2}\tanh[\frac{1}{2k}\sqrt{\frac{-\alpha}{\gamma}}(C + k(x + \frac{t\alpha\beta}{\gamma}))], \tag{5.192}$$

and the singular soliton solution

$$u(x,t) = -\frac{\beta}{\gamma} - \frac{\beta}{\gamma^2}\coth[\frac{1}{2k}\sqrt{\frac{-\alpha}{\gamma}}(C + k(x + \frac{t\alpha\beta}{\gamma}))], \tag{5.193}$$

$\alpha\gamma < 0$, $C, k > 0$ for the solitons to exist.

$$c = 0, a = 0, b = \pm\sqrt{\frac{\alpha}{-dk^2 - 2k^2\gamma}}, v = \frac{\alpha\beta}{d + 2\gamma}. \tag{5.194}$$

Using the parameters obtained in (5.3.23), we obtain the following traveling wave solutions:

$$u(x,t) = \frac{1}{J}e^{k\sqrt{\frac{\alpha}{-dk^2 - 2k^2\gamma}}\left(x - \left(\frac{k^2\alpha\beta}{-dk^2 - 2k^2\gamma}\right)t\right)}. \tag{5.195}$$

5.3.2. Lie symmetry Analysis of the NKSE

In this part, we give the Lie point symmetries of (5.173). The infinitesimals are given by the following

$$\xi^1 = abtC_1 + C_3, \tag{5.196}$$

$$\xi^2 = -\gamma t C_1 + C_2, \tag{5.197}$$

$$\eta = C_1(b + cu), \tag{5.198}$$

The point symmetries are represented by:

$$\Gamma_1 = \frac{\partial}{\partial x}, \tag{5.199}$$

$$\Gamma_2 = \frac{\partial}{\partial t}, \tag{5.200}$$

$$\Gamma_3 = (\beta + \gamma u)\frac{\partial}{\partial u} - \gamma t\frac{\partial}{\partial t} + \alpha\beta t\frac{\partial}{\partial x}. \tag{5.201}$$

5.3.3. Conservation Laws of the NKSE

The Lagrangian of (5.173) is represented by

$$\mathcal{L} = v(x,t)(u_t + \alpha u u_x + \beta u_{xxx} + \gamma(u u_{xx})_x + d u_x u_{xx}). \tag{5.202}$$

And the adjoint equation reads

$$F^*(x,u,v,u_1,v_1,u_2,v_2,u_3,v_3,u_4,v_4,u_5,v_5) = \frac{\delta(vF)}{\delta u} = \frac{\delta(v[u_t+\alpha u u_x+\beta u_{xxx}+\gamma(u u_{xx})_x+d u_x u_{xx}])}{\delta u} = 0. \tag{5.203}$$

After some calculations, the adjoint yields

$$F^* = (d-2\gamma)v_x u_{xx} + d u_x v_{xx} - 2\gamma u_x v_{xx} - \beta v_{xxx} - u(\alpha v_x + \gamma v_{xxx}) - v_t = 0. \tag{5.204}$$

Using the symmetries Eqs.(5.199)-(5.201) and utilizing the conservation formlar, we obtain the following fluxes

- $\Gamma_1 = \frac{\partial}{\partial x}$ gives

$$T_1^t = -v u_x,$$

$$T_1^x = v_x((d-\gamma)u_x{}^2 + (\beta+\gamma u)u_{xx}) - (\beta+\gamma u)u_x v_{xx} + v u_t.$$

- $\Gamma_2 = \frac{\partial}{\partial t}$ yields

$$T_2^t = v(u_x(\alpha u + (d+\gamma)u_{xx}) + (\beta+\gamma u)u_{xxx}), T_2^x = -((-d+\gamma)u_x v_x + \gamma v u_{xx} + \beta v_{xx} + u(\alpha v + \gamma v_{xx})).$$

- $\Gamma_3 = (\beta + \gamma u)\frac{\partial}{\partial u} - \gamma t\frac{\partial}{\partial t} + \alpha\beta t\frac{\partial}{\partial x}$ gives

$$T_3^t = -\upsilon(tu_x(\alpha\beta + \gamma(d+\gamma)u_{xx}) + \beta(-1 + t\gamma u_{xxx}) + \gamma u(-1 + t\alpha u_x + t\gamma u_{xxx})), T_3^x$$
$$= t\alpha\beta\upsilon(u_x(\alpha u + (d+\gamma)u_{xx}) + \beta u_{xxx} + \gamma u u_{xxx} + u_t)$$
$$+ ((-d+\gamma)u_x\upsilon_x + \gamma\upsilon u_{xx} + \beta\upsilon_{xx} + u(\alpha\upsilon + \gamma\upsilon_{xx}))(\beta + \gamma u - t\alpha\beta u_x$$
$$+ t\gamma u_t) + d\upsilon u_x - (\beta + \gamma u)\upsilon_x)(\gamma u_x - t\alpha\beta u_{xx} + t(\gamma u_{xt}) + (\beta$$
$$+ \gamma u)\upsilon(\gamma u_{xx} - t\alpha\beta u_{xxx} + t\gamma u_{txx}).$$

6. CONCLUSION

In the thesis, an emphasis on nonlinear Schrödinger and evolution equations was placed. Although these have been studied in the past, and continuing to be studied, it is our hope that the new results of the equations reported in this thesis will be of interest to the world of physics and mathematics.

In section 2 of the thesis, we gave detailed description of the methods used in the thesis. In section 3, we investigated the optical solitons and Cls for three different NLSEs. The optical solitons were investigated using several techniques including the complex envelope ansatz, sine-Gordon expansion method, the Riccati Bernoulli sub-ODE, modified F-expansion, the generalized tanh, generalized projective Riccati equation, Jaccobi elliptic function ansatz and the undetermined coefficient methods. The methods successfully derived the optical soliton solutions of some nonlinear Schrodinger equations. It is worth mentioning that some of the optical soliton solutions derived in this thesis are to the best of our knowledge new. The Cls of the equations in this chapter were investigated using the new conservation theorem introduced by Ibragimov. In section 4, we constructed the bright, dark and dark-bright optical solitons of three NLSEs. Singular solitons for the equations were also reported. Some of the solutions derived are also new. Furthermore, we studied the MI of the equations presented in this section by applying the concept of linear stability analysis, and reported the MI gain spectrum. Analysis shows that MI exist for all regions although there are some conditions which must be satisfied for the MI to exist. In section 5, we constructed the soliton solutions and Cls of three NLPDEs. Topological, non topological, hyperbolic and singular solitons were derived by applying the techniques described in section 2. The Cls of the equations studied in this part were derived using the new conservation theorem due to Ibragimov. Some figures have been provided to describe physical meaning of the acquired results.

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CURRICULUM VITAE

Aliyu Isa Aliyu
Firat University Turkey, Science Faculty, Department of Mathematics,
Elazig, Turkey.
(905) 398657155, aliyu.isa@fud.edu.ng

PERSONAL DETAILS

DATE OF BIRTH: Born on September 9, 1984. NATIONALITY: Nigerian

PLACE OF BIRTH: Kano Nigeria. MARITAL STATUS: Married.

NUMBER OF CHILDREN: 3.

RESEARCH INTEREST

Analytic studies, Lie Symmetry Analysis, Stability Analysis and Conservation Laws of Classical and Fractional Di erential Equations.

EDUCATION

Ph.D. candidate in Applied Mathematics, Firat University, Elazig, Turkey. GPA:89.86/100.

PhD Thesis title: Optical and Other Solitons, Lie Point Symmetries, Conservation Laws and Stability Analysis of some Nonlinear Partial Differential Equations.

Master of Science in Mathematics, Jordan University of science and Technology, Irbid, Jordan. Completed in 2014 GPA:80.8/100.

Msc Thesis title: Genetic algorithm for designing health care appointment system

Bachelor of Science in Mathematics, Bayero University Kano, Nigeria. Completed in 2010 GPA:3.71/5.

Problem Solving

Excellent analytical and logical reasoning skills. Can learn new skills quickly. Able to lead or work within a group environment.

Computer Languages

Some experience with maple package, Matlab, Mathematica, Reduce, Arena simulation software. Sound knowledge in LaTeX Typesetting. Cisco certified computer hardware and software mainte-nance technician.

Other

Creative, motivated, innovative, stunningly beautiful teaching skills.

EMPLOYMENT: Federal University Dutse, Nigeria

Lecturer II of Mathematics March 2013 to present

I was employed as a graduate assistant lecturer in 2013. After my masters degree, i was promoted to the position of Assistant lecturer in 2014. In October 2015, I was promoted to the position of Lecturer II. Since 2014, I've been engaged in research, preparation and presentation of lectures, supervision of group work, writing and grading tests and quizzes, grading projects and papers, preparing and grading the nal exam for the course. For some of the courses it was also necessary for me to create my own syllabus, as well as choose (or create) all of the homework problems.

PUBLICATIONS

1. Aliyu Isa Aliyu, Mustafa Inc, Abdullahi Yusuf, Dumitru Baleanu, 2018, Symmetry Analy-sis, Explicit Solutions, and Conservation Laws of a Sixth-Order Nonlinear Ramani Equation, Symmetry, volume 10, 341. (ISI: Science Citation Index Expanded.)
2. Aliyu Isa Aliyu, Mustafa Inc, Abdullahi Yusuf, Dumitru Baleanu, 2018, A fractional Model of Vertical Transmission and Cure of Vector-Borne Diseases Pertaining to the Atangana-Baleanu Fractional Derivatives, Chaos Solitons and Fractals volume 116, 268-277. (ISI: Science Citation Index Expanded.)
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5. Aliyu Isa Aliyu, Mustafa Inc, Abdullahi Yusuf, Dumitru Baleanu, 2018, Optical Solitons and Stability Analysis in Ring-Cavity Fiber System with Carbon Nanotube as Saturable Absorber. Communications in Theoretical Physics. Accepted article in press. (ISI: Science Citation Index)
6. Aliyu Isa Aliyu, Mustafa Inc, Abdullahi Yusuf, Dumitru Baleanu, 2018,Invariant Subspace and Lie Symmetry Analysis, Exact Solutions and Conservation Laws of a Nonlinear Reaction-Di usion Murray Equation Arising in Mathematical Biology, Journal of Advanced Physics. Volume 7, 1-7. (ISI: Science Citation Index Expanded.)
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8. Mustafa Inc, Aliyu Isa Aliyu, Abdullahi Yusuf and Dumitru Baleanu, 2017, Optical solitons and modulation instability analysis with (3+ 1)-dimensional nonlinear Schrodinger equation, Superlattices and Microstructures, volume 112, 296-302. (ISI: Science Citation Index).
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10. Mustafa Inc, Aliyu Isa Aliyu and Abdullahi Yusuf, 2017, Solitons and conservation laws to the resonance nonlinear Shrodingers equation with both spatio-temporal and inter-modal disper-sions, Optik, volume 142, 509-522. (ISI: Science Citation Index).
11. Mustafa Inc, Aliyu Isa Aliyu, Abdullahi Yusuf and Dumitru Baleanu, 2018, Darkbright opti-cal solitary waves and modulation instability analysis with (2 + 1)-dimensional cubic-quintic nonlinear Schrodinger equation Waves in Random and Complex Media Article in press. DOI: 10.1080/17455030.2018.1440096. (ISI: Science Citation Index).
12. Mustafa Inc, Aliyu Isa Aliyu, Abdullahi Yusuf and Dumitru Baleanu, 2018, Dark and combined optical solitons, and modulation instability analysis in dispersive metamaterial, Optik, volume 157, 484-491. (ISI: Science Citation Index).

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14. Fairouz Tchier, Aliyu Isa Aliyu, Abdullahi Yusuf and Mustafa Inc, 2017, Dynamics of solitons to the ill-posed Boussinesq equation, The European Physical Journal Plus, volume 132, 136.
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17. Dumitru Baleanu ,Mustafa Inc, Aliyu Isa Aliyu and Abdullahi Yusuf, 2018, Optical solitons, conservation laws and modulation instability analysis for the modi ed nonlinear Schrdingers equation for Davydov solitons, Journal of Electromagnetic Waves and Application, volume 32, 858-875. (ISI: Science Citation Index).
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25. Dumitru Baleanu , Mustafa Inc, Aliyu Isa Aliyu and Abdullahi Yusuf, 2017, Optical solitons, nonlinear self-adjointness and conservation laws for the cubic nonlinear Shrodinger's equation with repulsive delta potential, Superlattices and Microstructures, volume 111, 546-555. (ISI: Science Citation Index).
26. Mustafa Inc Aliyu Isa Aliyu, Abdullahi Yusuf and, 2017, Traveling wave solutions and con-servation laws of some fth-order nonlinear equations, The European Physical Journal Plus, volume 132, 224. (ISI: Science Citation Index).
27. Mustafa Inc, Aliyu Isa Aliyu, Abdullahi Yusuf , Dumitru Baleanu, 2018, Optical and singular solitary waves to the PNLSE with third order dispersion in Kerr media via two integration approaches, Optik, volume 163, 142-151. (ISI: Science Citation Index).
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31. Mustafa Inc Aliyu Isa Aliyu and Abdullahi Yusuf, 2017, Optical solitons, explicit solutions and modulation instability analysis with second-order spatio-temporal dispersion, The European Physical Journal Plus, volume 132, 528. (ISI: Science Citation Index).
32. Mustafa Inc, Aliyu Isa Aliyu and Abdullahi, 2017, Dark optical, singular solitons and conser-vation laws to the nonlinear Schrodinger's equation with spatio-temporal dispersion, Modern Physics Letters B, volume 31, 750163. (ISI: Science Citation Index).
33. Mustafa Inc, Aliyu Isa Aliyu, Abdullahi Yusuf and Dumitru Baleanu, 2018, Optical solitary waves, conservation laws and modulation instability analysis to the nonlinear Schrdingers equa-tion in compressional dispersive Alvn waves, Optik, volume 155, 257-673.

CITATION AND PERFORMANCE RECORDS

1. Google scholar record as at (01/10/2018): H-index (14) and citations (510). This is the same as the record in the web of science profile.
2. Researchgate points as at (01/010/2018): 29.91.

REVIEW ACTIVITIES

Reviewer for several international journals including the following:

1. Physica A: Statistical Mechanics and its Applications (Elsevier Journal), (ISI: Science Ci-tation Index).
2. Modern Physics Letters B (World Scienti c Publishers), (ISI: Science Citation Index).
3. Waves in Complex and Random Media (Taylors and Francis), (ISI: Science Citation In-dex).
4. Waves in Complex and Random Media (Taylors and Francis), (ISI: Science Citation In-dex).

5. Results in Physics (Elsevier Journal), (ISI: Science Citation Index Expanded).
6. Journal of the Operational Research Society (Springer Journal), (ISI: Science Citation Index).
7. Advances in Di erence Equations (Springer Journal), (ISI: Science Citation Index Ex-panded).
8. Journal of Ocean Engineering and Science (Elsevier Journal), (ISI: Emerging Sources Ci-tation Index).
9. Frontiers in physics (Frontiers Journal), (ISI: Emerging Sources Citation Index Ex-panded)
10. International Journal of Applied and Computational Mathematics (Springer), (ISI: Emerg-ing Sources Citation Index Expanded) .

REFERENCES

- Professor Mustafa Inc Department of mathematics, Firat University Turkey, E-mail:minc@firat.edu.tr.
- Professor Dumitru Baleanu Department of mathematics, Firat University Turkey, E-mail: dumitru@cankaya.edu.tr.
- Professor Mahmoud Alrefai Department of Mathematics, Cankaya University, Ankara, Turkey. E-mail:alrefaei@just.edu.jo.
- Professor Bashir Ali, Department of mathematics, Bayero University Kano, Nigeria. E-mail:bashiralik@yahoo.com.