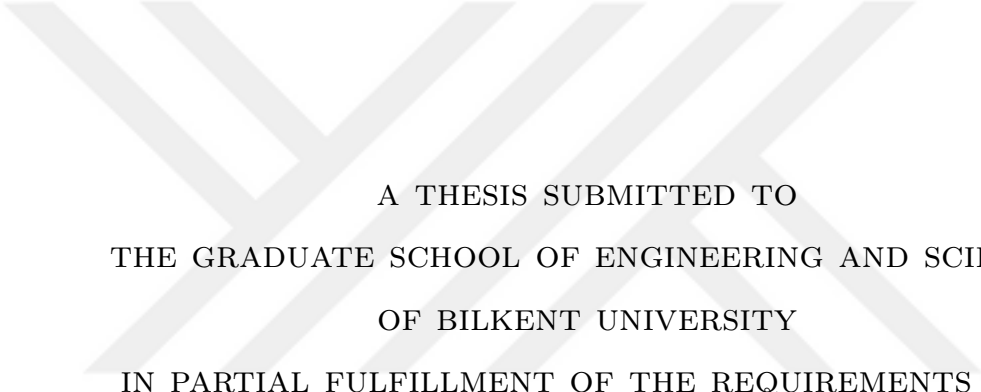


POLAR CODES FOR OPTICAL COMMUNICATIONS



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By
Tufail Ahmad
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Polar Codes for Optical Communications

By Tufail Ahmad

May 2016

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.



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ABSTRACT

POLAR CODES FOR OPTICAL COMMUNICATIONS

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May 2016

Optical communication systems have become the backbone of long distance communication networks due to their ability to transport data at high rates. A typical modern optical communication system should be capable to achieve data rates of 100 Gb/s or beyond. At such a high data rate, it is not feasible to retransmit the corrupted data. For reliable communication at improved power efficiency, communication systems use forward error correction (FEC) schemes. FEC schemes should have low latency to provide high throughput and good error performance to achieve an output bit error rate (BER) of 10^{-15} or lower in optical channels. Moreover, implementation schemes of these FEC codes should be simple, as telecommunications equipments in optical access networks can only accommodate restricted hardware complexity.

In this contribution, we study existing ITU-T G.975.1 recommended FEC schemes and recently proposed state-of-the-art FEC codes for optical networks. Next, we analyze polar codes, a recently proposed class of error-correcting codes with advantageous properties in terms of error performance, structure, latency and design method. Throughout our analysis we assume that optical channels can be modeled as additive white Gaussian noise (AWGN) channels. We investigate whether polar codes can compete with the above mentioned FEC schemes in the arena of optical communications. We conclude that polar codes outdo all G.975.1 recommended FEC codes in terms of error performance with the same overhead and relatively shorter block lengths. We also highlight some of the issues/aspects which need to be addressed to enhance the error performance of polar codes at finite block lengths so that they can catch up with (or surpass) recently proposed third generation FEC codes.

Most of the proposed FEC codes for next generation optical networks are based on LDPC and turbo codes. Unfortunately, these codes have error floors at very

low BER. Post-processing algorithms along with special construction techniques for these codes are proposed in literature to suppress their error floors. These special designs improve their error performance at the cost of extra complexity. Luckily, polar codes do not suffer from error floor problem in low BER regions. Moreover, polar codes have regular structure which makes its hardware implementation simple. There are various polar decoders proposed in literature with desirable properties in terms of error performance and complexity. These features make polar code an attractive candidate to be thoroughly analyzed for application in optical communications. Our analysis of polar codes in this thesis is restricted to its successive cancellation (SC) decoding as it provides a nice balance between complexity and error performance.

Error performance of polar codes with larger and moderate block lengths cannot be determined explicitly by Monte Carlo (MC) simulations for optical communication systems operating at high signal to noise ratio (SNR) due to prohibitive simulations time. To make sure that polar codes perform well in low BER regions, we use analytical methods to find bounds on their error rate. We use density evolution (DE) with Gaussian approximation (GA) for the construction and error performance estimation of polar codes. We conclude that DE-GA is a reliable algorithm for construction and error performance evaluation of polar codes by observing that our results obtained with simulations and DE-GA algorithm in low SNR region agree close enough to expect that there will not be too much deviation in high SNR region.

The performance of a code in optical communications is usually described by its net coding gain (NCG). Using Shannon's performance limits, maximum value of NCG that a FEC can achieve asymptotically can be calculated. But comparing the performance of a finite length code to the asymptotic performance of a FEC is not fair. Therefore in this thesis, we calculate the maximum NCG that can be achieved by a FEC with finite block length to make our performance comparisons more meaningful.

Keywords: Optical communications, error performance, very low BER, high throughput networks, optical fiber communications, polar codes, Gaussian approximation for code construction, density evolution.

ÖZET

OPTİK HABERLEŞMELER İÇİN KUTUPSAL KODLAR

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Optik haberleşme sistemleri sağladıkları yüksek hızlı veri iletim imkanı ile uzak mesafeli haberleşme altyapılarının omurgasını oluşturmaktadır. Günümüz için güncel olan bir optik haberleşme sisteminin erişebildiği veri iletim hızı 100 Gb/s veya üzerinde olması beklenir. Bu gibi yüksek iletim hızlarında gönderilmekte olan verinin bozulması durumunda verinin tekrar gönderimi uygun bir çözüm olmamaktadır. Haberleşme sistemlerinde güç verimliliği iyileştirilmiş daha güvenilir bir haberleşme için ileri yönde hata düzeltme (FEC) yöntemleri kullanılır. Optik kanallarda FEC yöntemlerinin yüksek veri iletim hızları için düşük gecikmeye ve bit hata oranı (BER) 10^{-15} veya daha düşük seviyelerde, oldukça yüksek bir hata başarımına sahip olmaları istenir. Ayrıca, optik erişim altyapısının donanım karmaşıklığına ilişkin kısıtları nedeniyle ilgili haberleşme cihazlarında sözü edilen FEC yöntemleri için donanımsal gerçeklemelerin basit olması gerekmektedir.

Bu çalışmada, var olan ITU-T G.975.1 ile tavsiye edilen FEC yöntemleri ve optik ağlar için önerilmiş en güncel FEC kodları araştırılmıştır. Sonrasında güncel olarak önerilen ve hata başarımı, yapısal özellikleri, gecikmesi ve tasarım yöntemi gibi çeşitli açılardan avantajlara sahip bir hata düzeltme kodlaması sınıfı olan kutupsal kodlar bu amaç için araştırılmıştır. Araştırmamız doğrultusunda ele aldığımız optik kanalların toplamsal beyaz Gauss gürültülü (AWGN) kanalları ile modellenebileceği varsayılmıştır. Optik haberleşme için kutupsal kodlamanın sözü edilen FEC yöntemlerine rakip olup olmayacağı araştırılmıştır. Hata başarımı açısından aynı eklenti oranı için G.975.1 ile tavsiye edilen tüm FEC yöntemlerinin göreceli olarak daha kısa blok uzunluklu kutupsal kodlar tarafından geçildiği sonucuna ulaşılmıştır. Ayrıca, hata başarımı açısından güncel olarak önerilen üçüncü nesil FEC kodları yakalamak (veya geçmek) için sonlu blok uzunluklarında kutupsal kodların hata başarımını iyileştirmek amacıyla gerekli olan veya sağlanması gereken kriterler belirlenmiştir.

Yeni nesil optik ağlar için önerilen FEC kodların çoğu LDPC ve turbo kodlara dayanmaktadır. Bu kodlar maalesef çok düşük BER seviyelerinde hata tabanına sahiptirler. Bu kodlar için hata tabanı etkisini ortadan kaldırmak amacıyla literatürde sonradan-işleme algoritmalarıyla özel tasarım teknikleri önerilmiştir. Bu özel tasarımlar hata başarımlarını ek bir karmaşıklık bedeli karşılığında iyileştirmektedir. Kutupsal kodlar düşük BER seviyelerinde hata tabanı probleminden etkilenmemektedir. Dahası, kutupsal kodlar donanım gerçeklemelerini basitleştiren düzenli bir yapısal özelliğe sahiptir. Ayrıca, hata başarımları ve karmaşıklık açısından birçok farklı istenilen özelliklere sahip kutupsal kod çözücüler literatürde önerilmiştir. Bu gibi özellikler kutupsal kodları optik haberleşme uygulamalarında araştırılabilircek ilgi çekici bir aday yapmaktadır. Bu tezde kutupsal kodlara ilişkin yaptığımız araştırmalar karmaşıklık ve hata başarımları arasında bir denge sağlayan yinelemeli eksiltme (SC) kod çözme ile sınırlanmıştır.

Monte Carlo benzetimi ile kutupsal kodların hata başarımlarını optik haberleşme sistemleri için belirlemek yüksek işaret gürültü oranında (SNR) aşırı uzun benzetim zamanı gerektirdiğinden mümkün değildir. Kutupsal kodların düşük BER seviyelerindeki başarımlarının iyiliğinden emin olabilmek için analitik yöntemler kullanarak hata oranlarına ilişkin sınırlar belirlenebilmektedir. Bu çalışmada kutupsal kodların kod tasarımı ve hata başarımları kestirimi için Gauss yaklaşımı (GA) altında yoğunluk evrimi (DE) yöntemini kullanıyoruz. Düşük SNR seviyesinde DE-GA ile benzetim sonuçlarının yeteri kadar yakın ve tutarlıdır. Bu nedenle, kutupsal kodların tasarımı ve hata başarımları kestirimi için kullandığımız DE-GA algoritmasının güvenilir sonuçlar ürettiği ve yüksek SNR seviyelerinde de elde edilen sonuçların çok büyük sapma göstermeyeceği değerlendirilmiştir.

Optik haberleşmelerde bir kodun başarımları genellikle net kodlama kazancı (NCG) ile tanımlanır. Bir FEC yönteminin asimtotik olarak erişebileceği en büyük NCG değeri Shannon'un başarımları limitleriyle hesaplanabilmektedir. Ancak sonlu blok uzunluklu bir kodun başarımlarının asimtotik bir FEC başarımlarıyla karşılaştırılması adil değildir. Bu nedenle, karşılaştırmaların daha anlamlı olabilmesi için bu tezde en büyük NCG değeri hesaplamaları sonlu blok uzunluklu FEC başarımları ile karşılaştırılmasına dayanmaktadır.

Anahtar sözcükler: Optik haberleşme, hata başarımları, çok düşük BER, yüksek

iletim hızlı aęlar, fiber optik haberleşme, kutupsal kodlar, kod tasarımı için Gauss yaklaşımı, yoğunluk evrimi.



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List of Abbreviations

AWGN additive white Gaussian noise.

B-DMC binary discrete memoryless channel.

BCH Bose-Chaudhuri-Hocquenghem.

BEC binary erasure channel.

BER bit error rate.

BP belief propagation.

BPSK binary phase shift keying.

BSC binary symmetric channel.

CG coding gain.

CRC cyclic redundancy check.

CSOC convolutional self-orthogonal code.

DE density evolution.

DWDM dense wave division multiplexing.

E_b/N_0 energy per bit to noise power spectral density ratio.

FEC forward error correction.

FER frame error rate.

FPGA field-programmable gate array.

GA Gaussian approximation.

GF(2) binary galois field.

HD hard decision.

IMDD intensity modulation direct detection.

ITU-T international telecommunication union- telecommunication.

LDPC low density parity check.

LL log-likelihood.

LLR log-likelihood ratio.

LR likelihood ratio.

MAP maximum a-posteriori.

MC Monte Carlo.

ML maximum likelihood.

MSA min-sum approximation.

NCG net coding gain.

ODU optical data unit.

OH overhead.

OSNR optical SNR.

OTN optical transport network.

OTU optical transport unit.

PPA post processing algorithm.

QC-LDPC quasi cyclic LDPC.

QPSK quaternary phase shift keying.

RS reed solomon.

SC successive cancellation.

SCL successive cancellation list.

SD soft decision.

SNR signal to noise ratio.

SPC single parity check.

TTE terminal transmission equipment.

List of Symbols

Gb/s giga bits per second.

I symmetric capacity.

K payload length in bits.

L list size.

n stages of encoding and decoding.

N block length of the code.

R code rate.

W binary symmetric channel.

Z Bhattacharyya parameter.

Chapter 1

Introduction

In order to transport data from one point to another, many types of communications systems have been developed with different features. In order to safeguard data against channel noise, most of the communication systems rely on FEC. Selection of FEC for a particular communication system depends on the type of its application.

1.1 Optical Communications

Optical communications have revolutionized communications the industry. The major reason for widespread popularity of optical communications is the capability of optical-fiber to transport data at very high rates of 100 Gb/s or even 1 Tb/s. Typically, optical communication systems do not use retransmission schemes in case of transmission errors, instead they rely on FEC schemes [9]. Due to high volume of traffic on optical links, the BER requirement for FEC schemes in optical communication systems is usually very stringent, such as 10^{-12} to 10^{-15} . Moreover, FEC encoders and decoders must be very fast to support data rates of 100 Gb/s. Additionally, a FEC scheme for optical communications should have very low complexity. Some FEC codes may have very good error performance,

but since their implementation complexity is huge, they can not be deployed in practical communication systems.

A number of state-of-the-art FEC schemes are proposed in the literature having very good error correction capabilities to achieve the above mentioned BER thresholds in optical channels. ITU-T (international telecommunication union-telecommunication standardization sector) have also recommended a number of FEC schemes for optical communication systems. G.975 and G.975.1 are two such recommendations by ITU-T which standardize FEC codes for optical networks.

Important parameters to be considered in choosing a FEC code for an optical communication system are summarized in Table 1.1.

Table 1.1: FEC codes design parameters and typical requirements in optical communications

BER	10^{-12} to 10^{-15}
Latency	of order $2N$ to $3N$, N is block length
Complexity	1 Gbps/mm ²
Energy consumption	10 pJ/bit

Hard decision (HD) codes are widely applied in optical communication because they can outdo soft decision (SD) codes due to their low complexity. In optical communications, it is normally desirable that BER of the system should be well below 10^{-12} . SD codes can outperform HD codes in this arena at the cost of extra complexity.

1.2 Three Generations of FEC Codes in Optical Communications

FEC codes in optical communications are generally classified in three generations and NCG (defined later) of approximately 6 dB, 8 dB and greater than 10 dB characterize them respectively [1]. Table 1.2 and Figure 1.1 give a synopsis of

these FECs generations.

Table 1.2: Three generations of FECs in optical communications [1][6]

FEC Generation	Data Rate (Gb/s)	NCG (dB) at BER= 10^{-15}	Standard
1G	2.5	6	G.975
2G	10-40	8	G.975.1
3G	100 and beyond	≥ 10	-

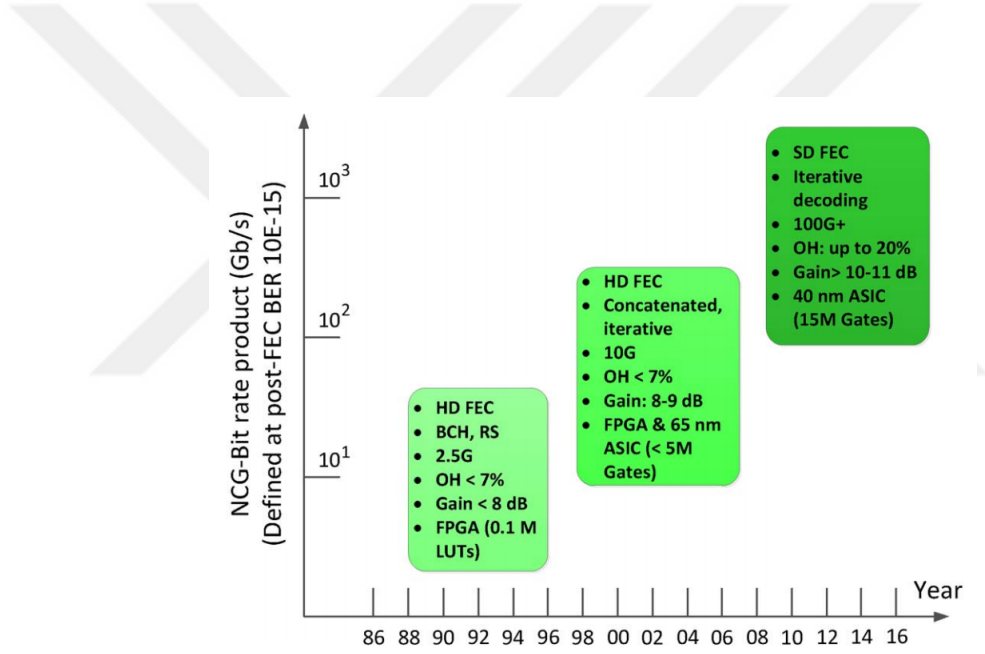


Figure 1.1: Evolution of FEC codes for optical communications [1]

First generation uses conventional HD block codes, such as BCH and RS codes for optical submarine communications. These codes work with an overhead of approximately 7%. Beside HD based algorithms, second-generation FEC codes achieve better coding gains with the use of concatenated codes and interleaving, iterative and convolutional decoding techniques. Third generation FECs mostly include SD FECs to achieve higher coding gains. Thanks to coherent detection, SD FECs can be easily implemented in next generation optical networks. These FECs are mostly based on turbo and LDPC codes with iterative decoding. There are some HD FECs which can be suitable for 100G systems and may be classified

as 2.5 generation. A typical example of 2.5 generation FEC code is BCH-two-iteration HD FEC scheme for a throughput beyond 100 Gb/s [10].

1.3 ITU-T FEC Recommendations for Optical Communications

ITU-T coordinates standards for telecommunications. Like many other telecommunication systems, elements of optical communication systems and their functions are also standardized under various ITU-T recommendations. For example, ITU-T recommendation G.709 is a fundamental format specification for optical transport networks (OTNs). It standardizes interfaces for the OTNs between submarine cable and inland networks [11]. These interfaces are called terminal transmission equipments (TTE).

Table 1.3 lists two ITU-T recommendations which analyze various FEC codes for their application in optical communications. Moreover, these recommendations give a guideline to determine whether a given FEC code can be used in state-of-the-art optical networks by specifying the essential features that a FEC code must meet in order to be practically employed in OTNs.

Table 1.3: ITU-T FEC recommendations for optical communications

G.975	Forward error correction for submarine systems
G.975.1	Forward error correction for high bit-rate DWDM submarine systems

1.3.1 ITU-T FEC Recommendation G.975

Recommendation G.975 is primarily concerned with implementation of a FEC function (comprises FEC encoder and decoder at TTE) in the multigigabit-per-second optical-fiber submarine cable systems [2]. The transmission data rates under consideration in G.975 are integer multiples of 2.5 Gb/s. G.975 standardizes RS (255, 239) code as FEC scheme for optical fiber submarine cable system.

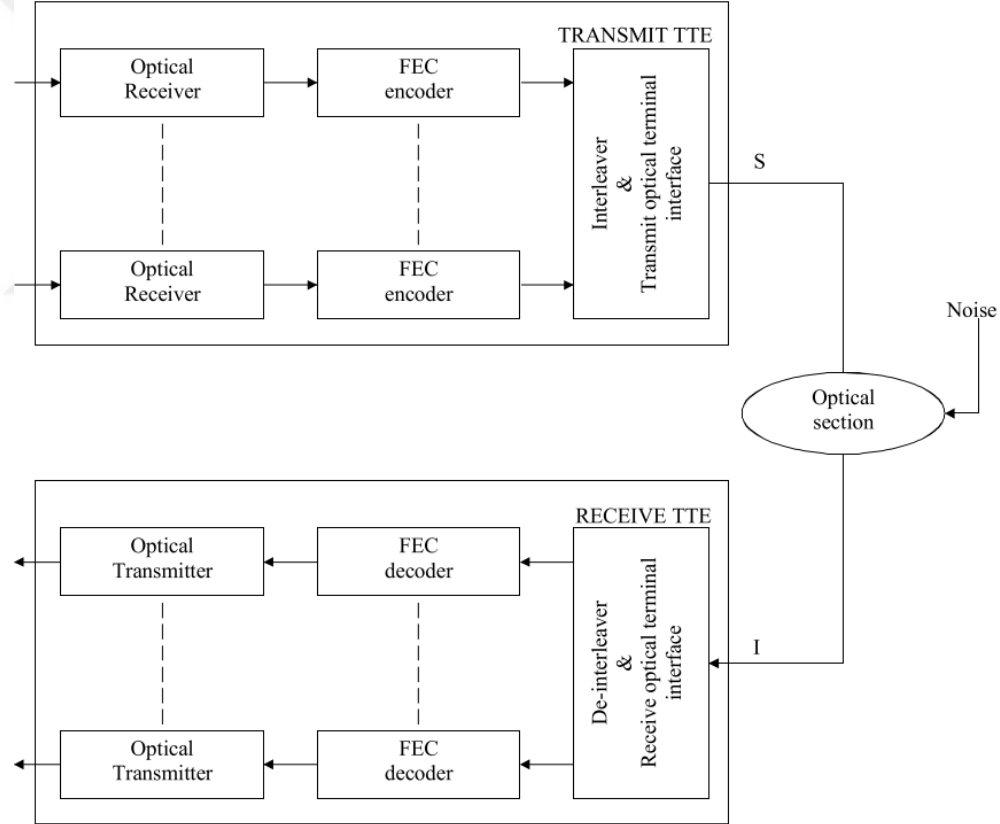


Figure 1.2: G.975 - Block diagram of a submarine optical-fiber system [2]

Figure 1.2 outlines the fact that the encoding and decoding procedures are performed at the TTE level only, on electrical signals, and benefit the overall optical-fiber submarine cable system, which comprises the optical fiber and possibly optical modules such as optical amplifiers.

1.3.2 ITU-T FEC Recommendation G.975.1

Recommendation G.975.1 primarily describes the FEC functions and Super-FEC schemes for high bit-rate DWDM (dense wave division multiplexing) submarine cable systems that have higher correction capability than RS (255, 239) FEC code defined in ITU-T recommendation G.975.

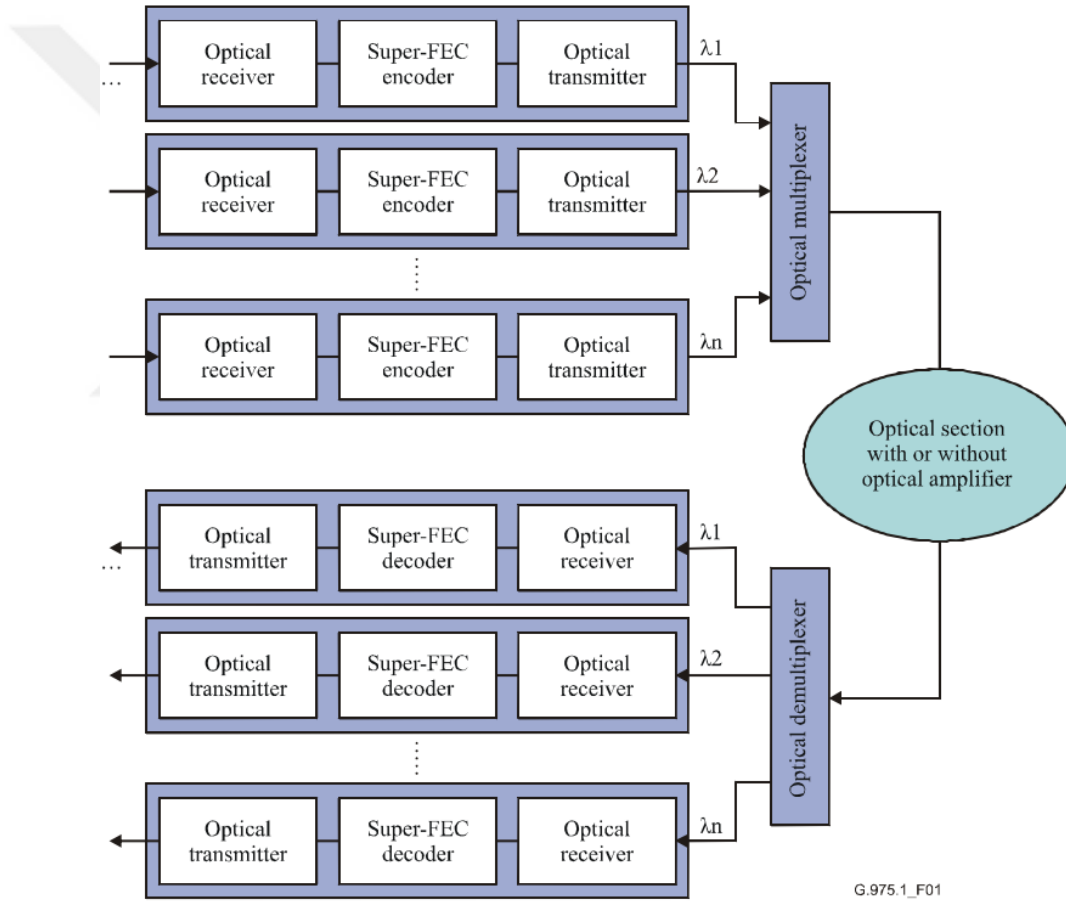


Figure 1.3: G.975.1 - Block diagram of DWDM submarine optical-fiber system [3]

Figure 1.3 is block diagram of a DWDM submarine optical-fiber system which uses a FEC function. The principle of operation of DWDM is illustrated by the figure: many wavelengths are multiplexed to transport data in parallel in a single optical-fiber but each wavelength acts as an independent carrier of a separate FEC packet.

G.975 has standardized RS (255, 239) code as FEC scheme for optical-fiber submarine cable system. To achieve higher correction abilities, G.975.1 has recommended several super FEC schemes. A list of super FEC codes recommended in G.975.1 is given below.

1. RS (255, 239)/CSOC ($n_0/k_0 = 7/6$, $J = 8$) super FEC
2. Concatenated BCH super FEC
3. RS (1023, 1007)/BCH (2047, 1952) super FEC
4. Concatenated RS/Product-code super FEC
5. LDPC super FEC
6. Two orthogonally concatenated BCH super FEC
7. RS (2720, 2550) super FEC code
8. Two interleaved extended BCH (1020, 988) super FEC

1.4 Polar Codes

Polar codes were introduced by Arikan in [5], and depend on a very natural phenomenon called “channel polarization”. Polar codes are the first provably capacity achieving codes for any symmetric binary-input discrete memoryless channel (B-DMC). They have very low encoding and decoding complexity, *i.e.*, $\mathcal{O}(N \log N)$, where N is the block length of the code. These codes have a regular structure and an explicit construction. Polar codes allow very flexible selection of the code rate with $\frac{1}{N}$ precision with the additional feature that a code rate can be selected without reconstructing the code from the scratch. Polar codes do not suffer from error floor problem in low BER regions.

1.5 Contributions of the Thesis

Polar codes can achieve symmetric capacity of a channel asymptotically. The larger the block length, the better the error performance, but unfortunately more latency and complexity. We cannot rely solely on block length to improve performance of polar codes. There are various decoders available for polar codes in literature with trade-off between error performance, latency and complexity. List decoder can achieve error performance of a maximum likelihood (ML) decoder, but unfortunately it has very high latency and cannot be used for high data rates. Moreover, its complexity does not make it a good candidate for inexpensive hardware implementations. SC decoder is a viable option to be used in optical communications guaranteeing good error performance with affordable latency and complexity. Variants of SC decoders are proposed in literature to reduce its complexity and latency for cheaper hardware implementations and high throughput communications, respectively [12][13][14][15]. We focus on SC decoder in this thesis for optical communications. We assume that optical channels can be modeled as AWGN channels.

The contribution of this thesis is twofold:

1. A number of ITU-T recommended FEC codes are studied in the thesis to use them as a reference for evaluating the relative performance of polar codes. Then, we analyze polar codes mainly in terms of their error performance for application in optical networks. We investigate whether polar codes can compete with existing ITU-T G.975/G.975.1 recommended FEC schemes and more modern third generation FEC schemes proposed recently in literature for optical communication systems. Third generation FEC codes provide higher coding gains using larger overheads, iterative decoding and post-processing algorithms. A brief analysis of polar codes with larger overheads is included to highlight their error correction capabilities in comparison to very strong third generation FEC schemes. We have also identified the challenges which need to be tackled for implementation of polar codes in optical networks, *i.e.*, latency and complexity.

2. Unlike the LDPC codes, polar codes have no error floors and can be used in optical communications which require a BER of as low as 10^{-15} [34]. At such low error rates, simulations can be used to evaluate error performance of short polar codes only. Unfortunately, performance of polar codes is not comparable to that of the other state-of-the-art codes for very short block lengths. G.975 and G.975.1 recommendations use large enough block lengths for FEC schemes such that performance of polar codes becomes very promising at these lengths. Moreover, in order to be compatible with the recommended FEC schemes, we (have to) use the same block lengths. At these block lengths, it is not possible to use simulations to estimate error performance of polar codes at very low BERs. We use density evolution with Gaussian approximation to evaluate performance of polar codes at very low BERs [16].

1.6 Outline of the Thesis

Next generation optical communications require error correction codes of very good performance. Many FEC schemes are proposed recently for high data rate optical communications. In this thesis, we want to determine where do polar codes stand in optical communications as compared to other FEC codes. We study state-of-the-art FEC schemes proposed for optical networks in Chapter 2. We analyze error performances of ITU-T recommended FECs and recently proposed third generation FECs. In Chapter 3, we focus on density evolution with Gaussian approximation for construction and error performance evaluation of polar codes. This is required since MC simulations can not be used for analysis of polar codes in low BER regions. In Chapter 4, we propose polar codes of specific blocks length which are compatible with the ITU-T recommended FECs in terms of code rates and block lengths. We analyze their error performances and compare them against other state-of-the-art codes. Conclusion of our study is presented in Chapter 5.

Chapter 2

State-of-the-art FEC Codes in Optical Communications

Optical data transport system has become the backbone of long distance communication networks due to its ability to transport data at very high rate. A typical OTN, which is a set of optical network elements connected by optical fiber links, is able to transport client signals at data rates as high as 100 Gb/s and beyond. Optical communication systems rely on FEC schemes for correction of transmission errors. A number of state-of-the-art FEC schemes are proposed in literature and ITU-T G.975 and G.975.1 recommendations for optical transport capable of achieving BERs of 10^{-12} or lower.

In this chapter, we conduct a survey of various FEC schemes proposed for optical communications and analyze their error performances after defining the parameters which are used to measure the error performance of FEC codes.

2.1 Error Performance Measures of FEC Codes

FEC codes are used in communication systems to correct the errors in the received data eliminating the need for retransmissions. Error correction ability of a code is indicated by various parameters, such as BER characteristics, NCG, coding gain (CG), *etc.* These parameters are closely related, nevertheless each parameter is conducive in highlighting a specific aspect of a FEC code.

2.1.1 BER Characteristics

BER characteristics for a FEC code shows relationship between BER of decoder input data and BER of corrected output data. The improvement in BER due to FEC scheme is the most interesting characteristics and indicates the correction capability of the FEC code explicitly. Some other variants of BER characteristics of FEC codes are also very common, *e.g.* a relationship between E_b/N_0 and BER of corrected output data is mostly used in literature.

2.1.2 Net Coding Gain

BER of received data can be improved by increasing signal power or equivalently E_b/N_0 . But it is not always feasible to increase power of the signal beyond some limits due to power constraints. A better solution for decreasing BER of the signal is to introduce FEC scheme in communication systems instead of increasing signal power. Line BER (due to random line noise) of the uncoded signal can be reduced by encoding the signal and then decoding it by a FEC decoder. The use of FEC in communication systems is producing the same effect as that of increasing the signal power to reduce BER, and this is done at the cost of redundant data of the FEC code. We explain this with the help of an example after defining some parameters as follows

$B_{in(uncoded)}$: Line BER of an uncoded signal.

It is an explicit function of E_b/N_0 . For BPSK modulation, relation between E_b/N_0 and $B_{in(uncoded)}$ is given as follows

$$B_{in(uncoded)} = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{E_b}{N_0}} \right) \quad (2.1)$$

$B_{in(coded)}$: Line BER of an encoded signal “input” to the decoder.

It is an explicit function of E_b/N_0 and code rate R . For BPSK modulation, relationship between E_b/N_0 and $B_{in(uncoded)}$ for a code rate of R is given as follows

$$B_{in(coded)} = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{RE_b}{N_0}} \right) \quad (2.2)$$

It should be noted that for $R < 1$, $B_{in(uncoded)} < B_{in(coded)}$ for same $\frac{E_b}{N_0}$ as shown in Figure 2.1. This is because of the fact that we are adding redundant data bits but have not utilized it for decoding.

B_{out} : BER of the coded signal “output” from the FEC decoder.

It should be noted that for a good FEC scheme, $B_{out} \ll B_{in(coded)}$ for the same $\frac{E_b}{N_0}$ as shown in Figure 2.1.

Figure 2.1 graphically explains the concept of CG and NCG. For example, to achieve a target BER of 10^{-12} , Figure 2.1 shows that for an uncoded system, $\frac{E_b}{N_0}$ should be equal to 13.9 dB. On the other hand, by using a specific code, BER of 10^{-12} can be achieved at a much smaller value of $\frac{E_b}{N_0}$, *i.e.*, 8.3 dB. This difference in values of $\frac{E_b}{N_0}$ to achieve the same BER with a coded and an uncoded system is referred to as NCG, which has a value of 5.6 dB for a particular FEC code in our example.

NCG is the true gain of a FEC code as it considers redundancy in the signal due to the code. Unlike NCG, CG assumes redundant data due to FEC scheme as a part of the payload (informative data). If redundant data due to FEC code is considered as a part of the payload and not as extra data, Figure 2.1 shows that for achieving a line BER of 10^{-12} , $\frac{E_b}{N_0}$ should be equal to 14.2 dB as compared to a value of 8.3 dB by using a FEC scheme. This difference is referred to as CG

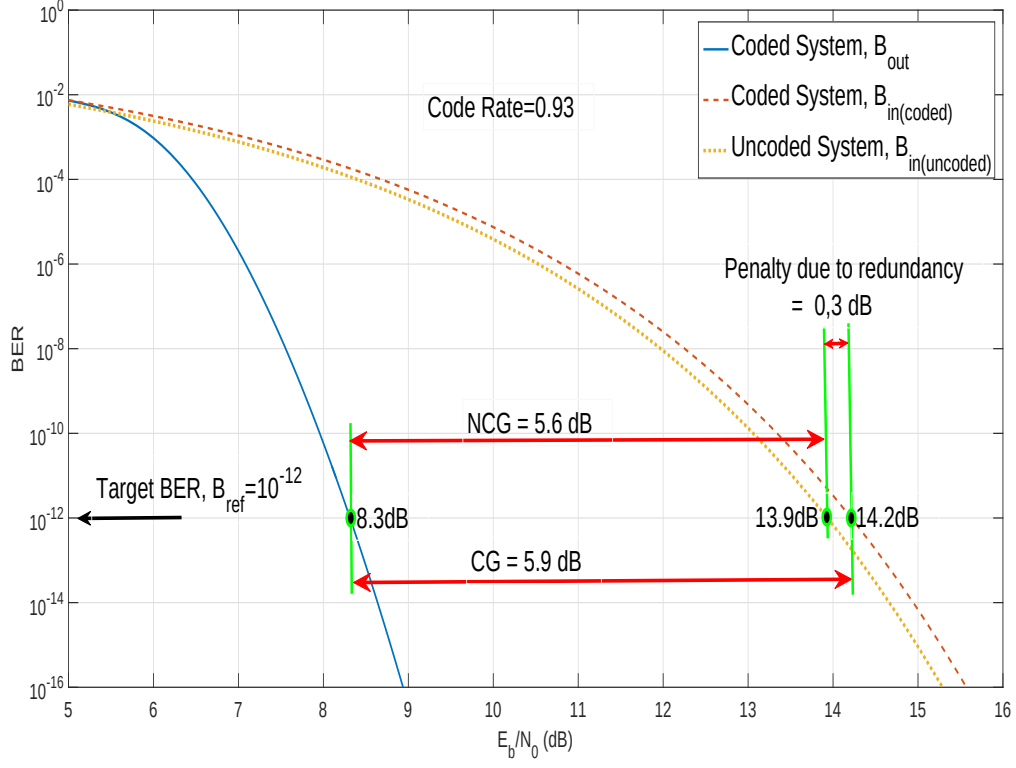


Figure 2.1: Graphical illustration of the notion of NCG and CG

and it has a values of 5.9 dB for a particular code in our example. Relationship between NCG and CG is given as

$$NCG = CG + 10 \log_{10} R \quad (2.3)$$

For $R=0.93$ we have

$$10 \log_{10} R = -0.3 \quad (2.4)$$

which is exactly equal to the penalty due to the redundant data of the FEC code as shown in Figure 2.1. For high rate codes, CG and NCG are almost equal.

To define CG and NCG more formally, first we define a function, “erfc” as below

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad (2.5)$$

CG in units of dB is defined as

$$CG = 20 \log_{10}(\text{erfc}^{-1}(2B_{ref})) - 20 \log_{10}(\text{erfc}^{-1}(2B_{in(coded)})) \quad (2.6)$$

NCG in units of dB is defined mathematically as

$$NCG = 20 \log_{10}(\text{erfc}^{-1}(2B_{ref})) - 20 \log_{10}(\text{erfc}^{-1}(2B_{in(uncoded)})) \quad (2.7)$$

From Equation 2.1 we get the following relationship for a BPSK system

$$\Rightarrow 20 \log_{10} \text{erfc}^{-1}(2B_{in(uncoded)}) = 10 \log_{10} \left(\frac{E_b}{N_0} \right) \quad (2.8)$$

From Equation 2.2 we get

$$\Rightarrow 20 \log_{10} \text{erfc}^{-1}(2B_{in(coded)}) = 10 \log_{10} \left(\frac{E_b}{N_0} \right) + 10 \log_{10} R \quad (2.9)$$

From Equations 2.8 and 2.9 we get

$$\Rightarrow 20 \log_{10} \text{erfc}^{-1}(2B_{in(uncoded)}) = 20 \log_{10} \text{erfc}^{-1}(2B_{in(coded)}) - 10 \log_{10} R \quad (2.10)$$

From equations 2.7 and 2.10 we get the expression for NCG as

$$NCG = 20 \log_{10}(\text{erfc}^{-1}(2B_{ref})) - 20 \log_{10}(\text{erfc}^{-1}(2B_{in(coded)})) + 10 \log_{10} R \quad (2.11)$$

2.1.3 Q limit and SNR

Q factor is the signal-to-noise ratio at the “receiver” decision circuit in voltage, and for an uncoded BPSK ($0 \mapsto +\mu$ and $1 \mapsto -\mu$) system with AWGN, it is given as

$$Q = \frac{\mu}{\sigma} \quad (2.12)$$

where

σ : Standard deviation of mark or space voltage at the receiver. For AWGN channel, it is equal to standard deviation of the white Gaussian noise.

Q limit is defined as the minimum Q factor of the input signal that is required for the receiver decision circuit to achieve a reference/target BER.

Another quantity which is very common in communications literature is SNR, sometimes denoted by $\frac{E_b}{N_0}$. It is given in units of dB as

$$\frac{E_b}{N_0}(dB) = 10 \log_{10} \left(\frac{E_b}{N_0} \right) \quad (2.13)$$

where

E_b : Energy per bit (J/bit).

N_0 : Noise energy per two dimensions (J/2D).

We have following important relationships for a coded communication system with a code rate, R

$$\frac{Q^2}{2} = \frac{RE_b}{N_0} \quad (2.14)$$

$$BER_{in(coded)} = \frac{1}{2} \operatorname{erfc} \left(\frac{Q}{\sqrt{2}} \right) \quad (2.15)$$

$$BER_{in(uncoded)} = \frac{1}{2} \operatorname{erfc} \left(\frac{Q}{\sqrt{2R}} \right) \quad (2.16)$$

$$\frac{E_b}{N_0}(dB) = 20 \log_{10}(\operatorname{erfc}^{-1}(2BER_{in})) \quad (2.17)$$

2.2 A FEC Coding Primer

Some of the most commonly used FEC codes in optical communications are reviewed here to facilitate the discussion about ITU-T recommended FEC schemes.

2.2.1 Reed Solomon Codes

We briefly explain Reed Solomon (RS) codes here and discuss their error performance [17]. Systematic RS codes are discussed in both ITU-T FEC recommendations, *i.e.*, G.975 and G.975.1 [2][3].

RS codes are linear block codes. They are a special case of BCH codes and work on symbols which may be represented as binary m -tuples. An RS code is described as (N, K) code. Parameters of an RS code are defined as

m : Number of bit per symbol

N : Block length in symbols

K : Uncoded message length in symbols

t : Number of correctable symbol errors

$$t = \lfloor \frac{N-K}{2} \rfloor$$

$$N \leq 2^m - 1$$

$$R = K/N \text{ Code rate}$$

RS (255, 239) code, recommended by G.975 works on 8 bit symbols and can correct up to 8 symbol errors. RS (2720, 2550) code, recommended by G.975.1 works on 12 bit symbols and can correct up to 85 symbols. Increasing the block length of the code improves error performance but increases complexity and latency. Next, we give formulas for estimating the error performance of RS codes.

2.2.1.1 Analytical Error Performance of RS codes

Any pattern of up to t symbol errors can be corrected by an RS code with $N - K \geq 2t + 1$. An “uncorrectable block error” occurs when more than t symbols are in error in a message block, resulting in one of the two actions at the decoder [2]. The message block is either recognized as being uncorrectable (called a recognized error) or the error pattern is assumed as correctable by the decoder, and the entire message block is mistakenly decoded to the wrong message (called a decoding error).

Here for our analysis, we assume that errors occur independently of each other. Under the above stated assumption an expression for BER is given by [2] as

$$BER_{in} = 1 - (1 - P_{SE})^{\frac{1}{m}} \quad (2.18)$$

$$BER_{out} = 1 - (1 - P_{UE})^{\frac{1}{m}} \quad (2.19)$$

where

P_{SE} : Probability of symbol error

P_{UE} : Probability of an uncorrectable symbol error, which can be calculated by using the following expression

$$P_{UE} = \sum_{i=t+1}^N \frac{i}{N} \binom{N}{i} \cdot (P_{SE})^i \cdot (1 - P_{SE})^{N-i} \quad (2.20)$$

Using these formulas, BER characteristics for RS (255, 239) and RS (2720, 2550) are plotted in Figures 2.2 and 2.3.

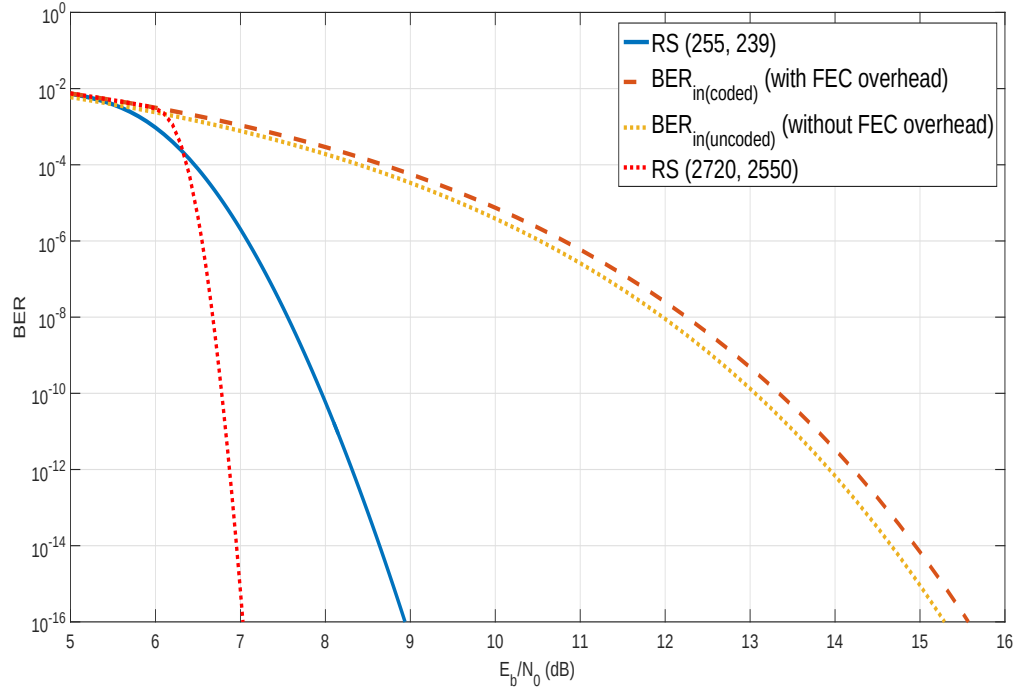


Figure 2.2: BER plots for RS (255, 239) and RS (2720, 2550) codes

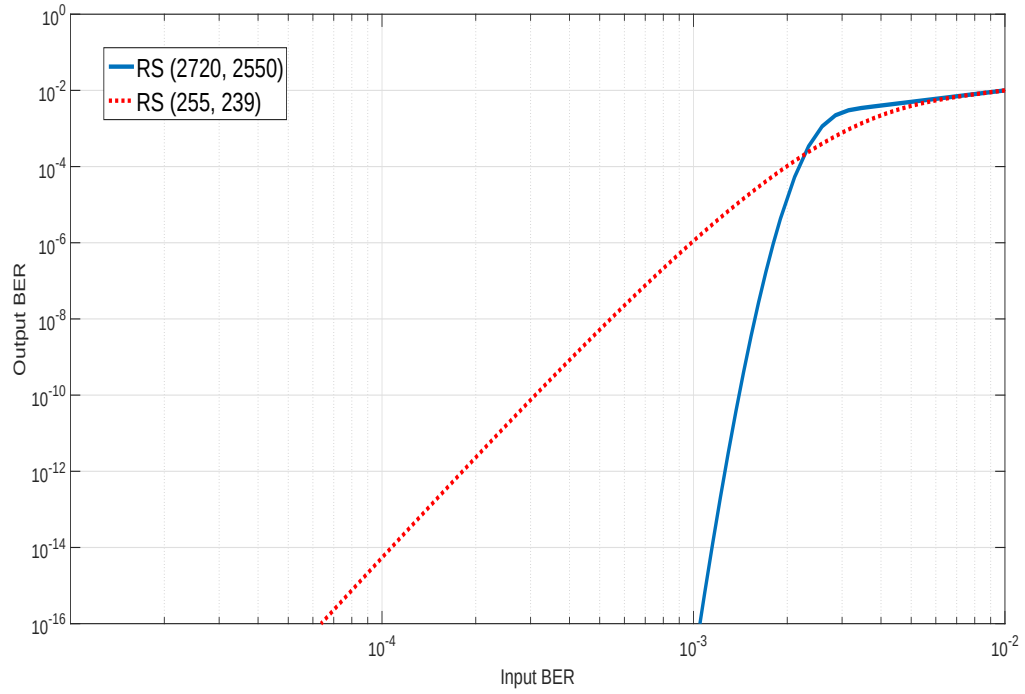


Figure 2.3: BER_{out} vs. BER_{in} for RS (255, 239) and RS (2720, 2550) codes

2.2.2 LDPC Codes

We briefly discuss low density parity check (LDPC) codes here and analyze their error performance [18]. G.975.1 recommendation has used LDPC (32640, 30592) for the design of a super FEC scheme providing larger coding gain [3]. Recently, LDPC codes are studied extensively for deployment in optical networks [19][20][21][22].

LDPC codes are SD linear block codes. Specially designed LDPC codes can provide performance very close to the capacity for a number of different channels [23]. Moreover, they have linear time complex algorithms for decoding, and they are suited for implementations that exploit parallelism [24].

Figure 2.4 shows error performance of G.975.1 recommended LDPC code. It shows error correction capabilities of LDPC code by comparing it with an uncoded system.

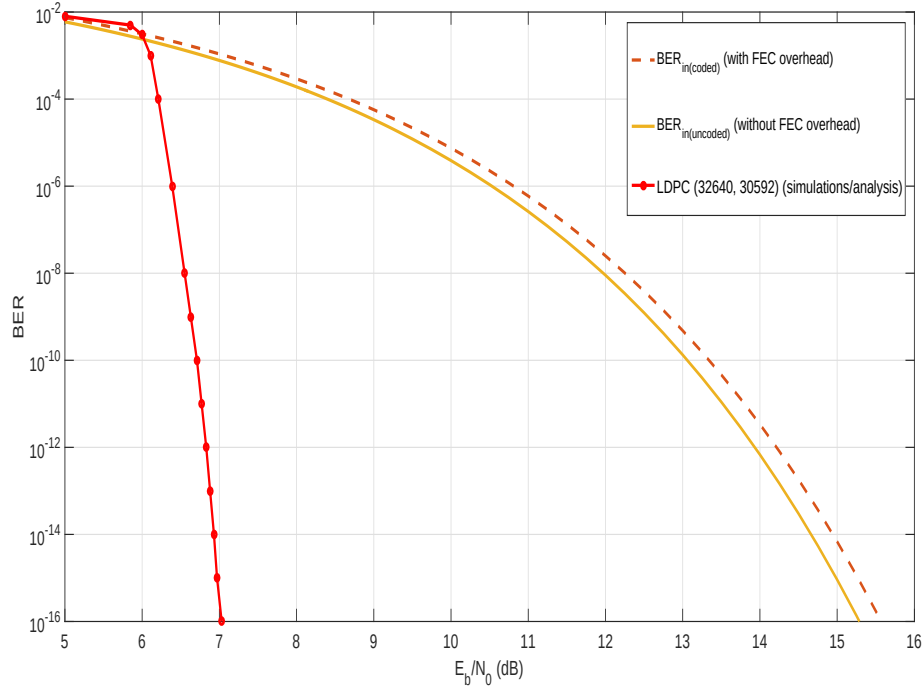


Figure 2.4: BER plot for LDPC (32640, 30592) code [3]

2.2.3 BCH Codes

We briefly explain BCH codes here and discuss their error performance [17]. Several super FEC schemes based on BCH codes are recommended by ITU-T G.975.1 [3].

The bose, chaudhuri, and hocquenghem codes, commonly known as BCH codes, form a large class of powerful random error correcting cyclic codes. BCH code is a generalization of the hamming code for multiple error corrections. It is a binary code and its non binary version is called RS codes [17]. A BCH (N, K, t) code is defined as below:

For any positive integer $m \geq 3$ and $t < 2^{m-1}$, there exists a binary BCH code with the following parameters

Block length: $N = 2^m - 1$

Number of parity check bits: $N - K \leq mt$

Minimum distance: $d_{min} \geq 2t + 1$

It is called a t error-correcting BCH code, as it can correct up to t binary errors.

Following three BCH codes are considered in G.975.1 recommendation.

1. BCH (3860, 3824, 3)
2. BCH (2040, 1930, 10)
3. BCH (2040, 1952, 8)

2.2.3.1 Analytical Error Performance of BCH Codes

Any pattern of up to t bit errors can be corrected by a BCH (N, K, t) code. An uncorrectable block error occurs when more than t bits are in error in a message

block, resulting in one of the two actions at the decoder [2]. The message block is either recognized as being uncorrectable (decoder failure) or the error pattern is assumed as correctable by the decoder, and the entire message block is mistakenly decoded to the wrong message (decoder error).

Here for our analysis, we assume that errors occur independently of each other. Under the above stated assumption an expression for BER is given by [17] as

$$BER_{output} = \sum_{i=t+1}^N \frac{i}{N} \binom{N}{i} \cdot (BER_{input})^i \cdot (1 - BER_{input})^{N-i} \quad (2.21)$$

where

BER_{input} : Input BER

BER_{output} : Output BER

Using Equation 2.21, we obtained following error performance results for the three BCH codes discussed above.

Table 2.1: Error performance parameters for BCH (3860, 3824, 3) code

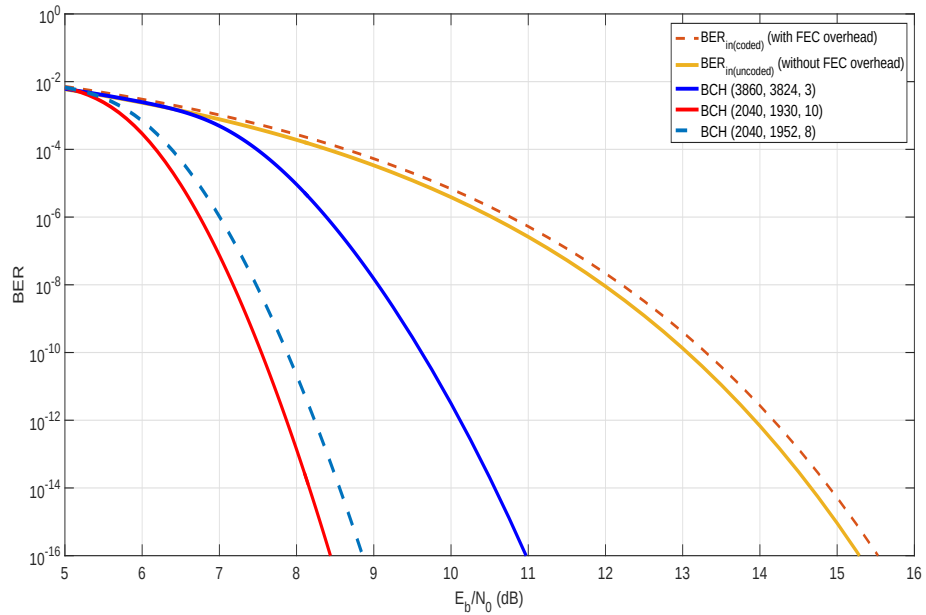
Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
1.82E-05	1.00E-09	3.20	3.24	12.32	9.35
1.02E-05	1.00E-10	3.44	3.48	12.59	9.62
5.72E-06	1.01E-11	3.64	3.68	12.85	9.88
3.21E-06	1.01E-12	3.82	3.86	13.09	10.12
1.80E-06	1.00E-13	3.97	4.01	13.32	10.35
1.01E-06	1.00E-14	4.10	4.14	13.54	10.57
5.69E-07	1.00E-15	4.21	4.25	13.74	10.77

Table 2.2: Error performance parameters for BCH (2040, 1930, 10) code

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
6.58E-04	1.00E-09	5.18	5.42	10.14	7.37
5.22E-04	1.00E-10	5.52	5.76	10.31	7.54
4.16E-04	1.01E-11	5.81	6.05	10.48	7.71
3.33E-04	1.00E-12	6.07	6.31	10.64	7.87
2.67E-04	1.00E-13	6.29	6.53	10.79	8.02
2.15E-04	1.00E-14	6.50	6.74	10.93	8.17
1.73E-04	1.01E-15	6.68	6.92	11.07	8.30

Table 2.3: Error performance parameters for BCH (2040, 1952, 8) code

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
4.04E-04	1.00E-09	4.87	5.06	10.50	7.68
3.07E-04	1.01E-10	5.18	5.37	10.69	7.88
2.34E-04	1.00E-11	5.46	5.65	10.88	8.06
1.79E-04	1.01E-12	5.70	5.89	11.05	8.23
1.37E-04	1.00E-13	5.92	6.11	11.22	8.40
1.06E-04	1.01E-14	6.11	6.30	11.37	8.56
8.16E-05	1.01E-15	6.28	6.47	11.53	8.71

**Figure 2.5:** BER plots for BCH codes

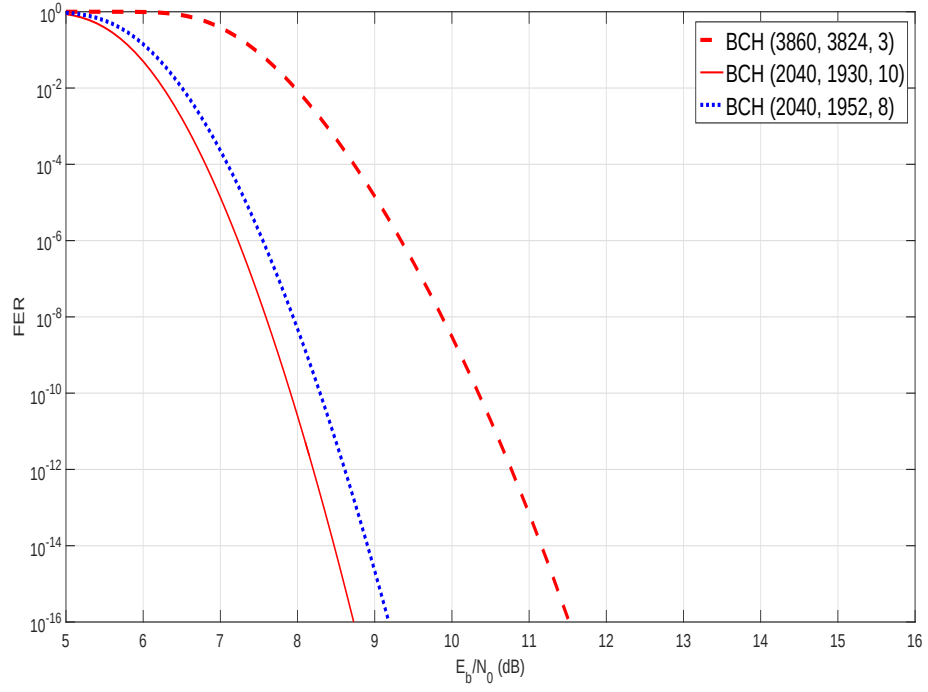


Figure 2.6: FER plots for BCH codes

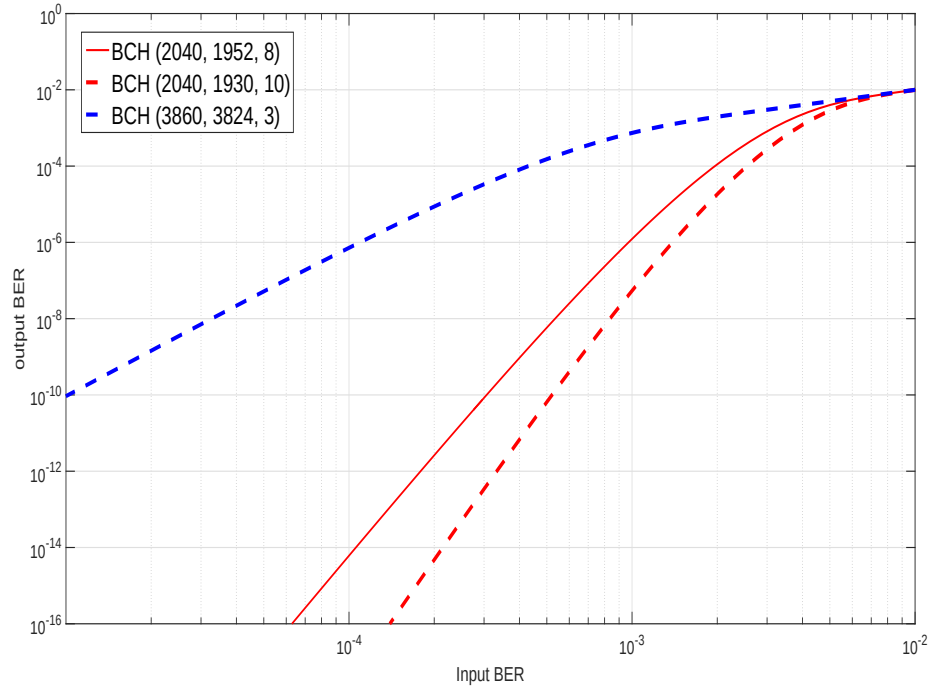


Figure 2.7: BER_{out} vs. BER_{in} for BCH codes

2.2.4 Concatenated Codes

A single error correction code with moderate block length does not always provide enough error protection for optical communications with reasonable complexity. Code concatenation improves error performance at the cost of more complexity. It is common to concatenate two codes, one used as inner code and the other one as outer code. Triple concatenated codes are also possible and available in literature [1]. There are different concatenated codes possible based on the selection of codes for inner and outer codes, see Figure 2.8. In this thesis, we consider some concatenated codes recommended in G.975.1. Each of these concatenated FEC schemes is compared with standard G.975 FEC code, *i.e.*, RS (255, 239) in terms of error performance, throughput and hardware complexity.

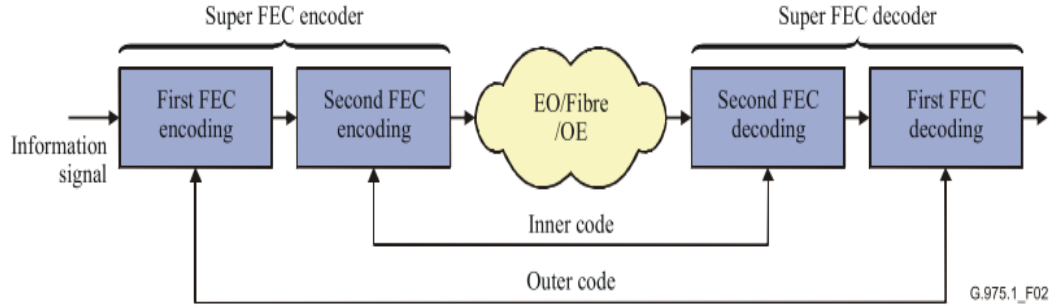


Figure 2.8: Descriptive block diagram for code concatenation [3]

For good error performance and relatively lesser complexity it is typical to use soft FEC codes as inner codes and hard FEC codes as outer codes [3].

2.3 ITU-T Recommended FECs for Optical Communications

FEC codes used in optical communications need to have very good error performance. For OTNs, we analyze FEC codes in very low decoded BER regions,

typically from 10^{-9} to 10^{-15} . We compare error performance of various FEC schemes by comparing their NCG values at output BER of 10^{-12} and sometimes 10^{-15} .

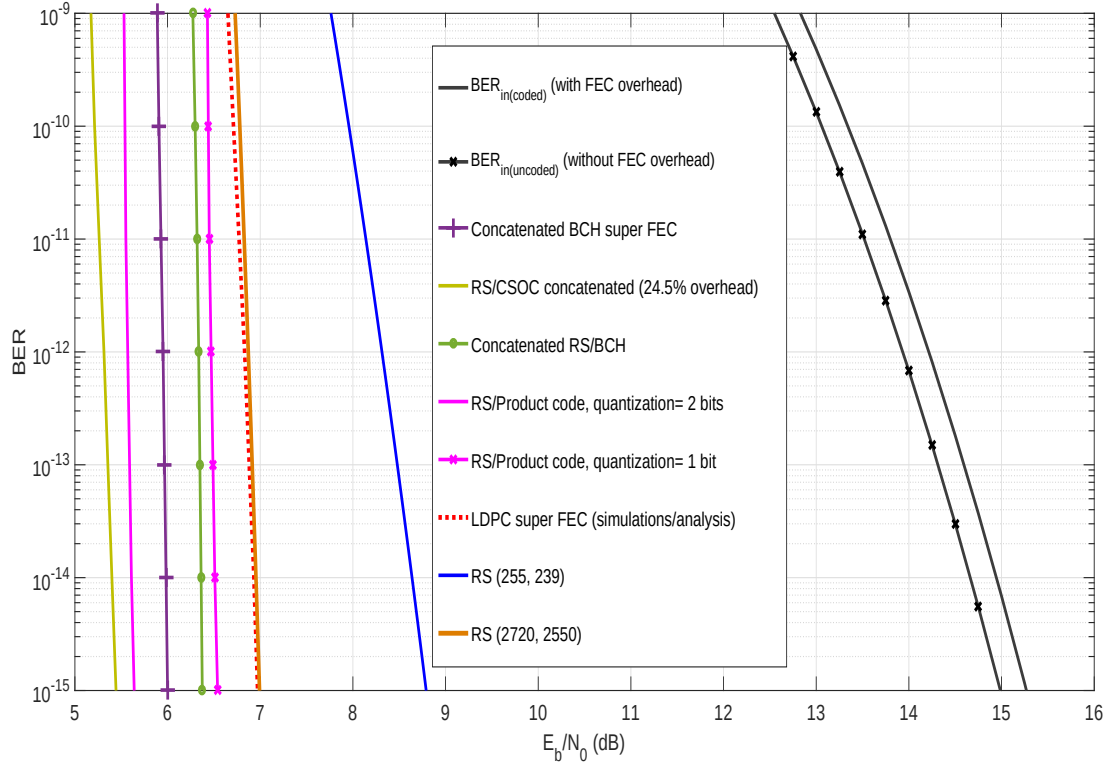


Figure 2.9: BER plots for ITU-T recommended FECs in optical communications [3]

Figure 2.9 shows a complete picture of the error performance of the FECs recommended in G.975 and G.975.1. BER performance of uncoded communication system is also plotted for reference. It is clear from Tables 2.4 and 2.5 that these FEC codes improve the reliability of communication system as compared to uncoded systems. This increase in reliability is at the cost of data redundancy, latency and complexity.

Table 2.4: Error performance parameters for ITU-T recommended FEC codes, with 7% overhead and $BER_{out} = 10^{-12}$ [3]

FEC Code	Input BER	NCG	CG	Q(dB)	$\frac{Eb}{No}$ (dB)
RS (255, 239)	1.82E-04	5.62	5.90	11.04	8.31
LDPC Super FEC code	1.33E-03	7.10	7.39	9.56	6.83
RS (2720, 2550)	1.26E-03	7.06	7.34	9.60	6.87
Concatenated RS/CSOC code (24.5% OH)	5.80E-03	7.95	8.90	8.04	5.31
Concatenated BCH code	3.30E-03	7.98	8.26	8.68	5.95
Concatenated RS/BCH code	2.26E-03	7.63	7.91	9.06	6.34
Concatenated RS/Product code	4.60E-03	8.40	8.70	8.30	5.57

Table 2.5: Error performance parameters for ITU-T recommended FEC codes, with 7% overhead and $BER_{out} = 10^{-15}$ [3]

FEC Code	Input BER	NCG	CG	Q(dB)	$\frac{Eb}{No}$ (dB)
RS (255, 239)	8.28E-05	6.20	6.48	11.52	8.79
LDPC Super FEC code	1.12E-03	8.02	8.30	9.70	6.97
RS (2720, 2550)	1.10E-03	7.99	8.27	9.72	6.99
Concatenated RS/CSOC code (24.5% OH)	5.20E-03	8.88	9.83	8.17	5.44
Concatenated BCH code	3.15E-03	8.99	9.27	8.73	6.00
Concatenated RS/BCH code	2.17E-03	8.67	8.95	9.10	6.37
Concatenated RS/Product code	4.50E-03	9.40	9.70	8.30	5.64

2.3.1 G.975 Recommended RS (255, 239) FEC Code

The FEC code recommended by ITU-T G.975 to safeguard information against in-line errors in the optical-fiber submarine cable system is a RS code: the RS (255, 239). RS (255, 239) code is a non binary code (it operates on 8-bit symbols) and belongs to a family of systematic linear cyclic block codes. It performs well under burst noise as well as under random noise with interleaving. RS (255, 239) code can correct up to 8 symbol errors in a block. RS (255, 239) code achieves an NCG of 5.62 dB at an output BER of 10^{-12} . We will use RS (255, 239) code as a standard for evaluating error performances of other FEC codes recommended for optical communications.

To be effective against all types of errors, 16 RS (255, 239) codewords are interleaved. A total of 30592 bits ($16 * 8 * 239$) are input to the G.975 FEC

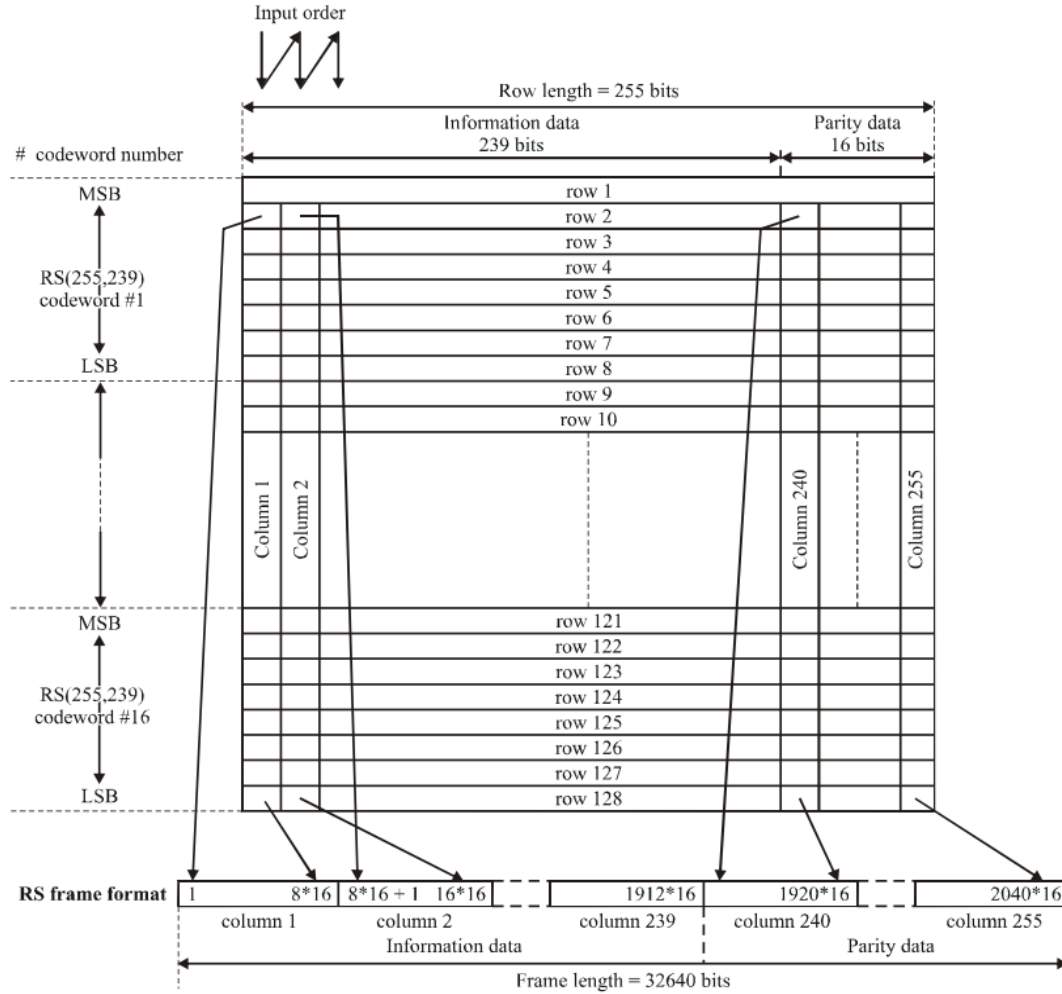


Figure 2.10: RS (255, 239) frame format input to interleaver [2]

encoder as one frame. It is important to know that the length of one row of an ODU2 (optical channel data unit) frame is also equal to 30592 ($3824 * 8$). After adding FEC redundancy to this frame, we get a total of 32640 ($255 * 8 * 16$) bits per frame. This number is equal to the number of bits in one row of OTU2 (optical channel transport unit) frame. This sequence of bits is given to interleaver, which interleaves the data bits as shown in Figure 2.10.

With this frame format it can correct up to 1024 bits long burst errors. RS (255, 239) code has a redundancy of 6.69%. RS codes are among the most efficient codes from implementation point of view in state-of-the-art hardware technology with good latency.

Table 2.6: Error performance parameters for RS (255, 239) code

Input BER	Output BER	NCG	CG	Q (dB)	E_b/N_0 (dB)
4.09E-04	1.01E-09	4.79	5.07	10.49	7.76
3.11E-04	1.00E-10	5.10	5.38	10.69	7.96
2.37E-04	1.00E-11	5.38	5.66	10.87	8.14
1.82E-04	1.01E-12	5.62	5.90	11.04	8.31
1.39E-04	1.01E-13	5.83	6.12	11.21	8.48
1.07E-04	1.00E-14	6.03	6.31	11.37	8.64
8.28E-05	1.01E-15	6.20	6.48	11.52	8.79

The given analytical results for RS (255, 239) in Figures 2.11, 2.12, and 2.13 are in close agreement with those we obtained through simulations (for low SNR values) in Matlab.

2.3.2 G.975.1 Recommended RS (2720, 2550) Super FEC Code

The RS (2720, 2550) super FEC code gives an NCG of 7.06 dB as compared to 5.62 dB for RS (255, 239) at an output BER of 10^{-12} with the same overhead of 6.69%. RS (2720, 2550) code works on 12-bit symbols and can correct up to 85 symbol errors. It has a block length of 32640 bits (2720*12) and a payload of 30592 bits (2550*12). It is shown to have low latency, almost of the same order as that of the order 16 interleaved RS (255, 239) FEC code [3]. The encoder and decoder for this code can be implemented efficiently in the current chip technologies for 2.5G, 10G and 40G applications [3].

RS (2720, 2550) super FEC code can correct up to 1020 bits long burst errors. The length of the input data frame to the RS (2720, 2550) super FEC encoder is equal to one fourth of an ODU2 frame. The length of the output frame from the FEC encoder is equal to one fourth of the OTU2 frame. Error Performance of this FEC is shown in Table 2.7, Figures 2.11, 2.12, and 2.13.

The analytical results for RS (2720, 2550) shown in Figures 2.11, 2.12, and

2.13 are in close agreement with those we obtained through simulations (for low SNR values) in Matlab.

Table 2.7: Error performance parameters for RS (2720, 2550) code

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
1.48E-03	1.02E-09	5.82	6.10	9.46	6.73
1.40E-03	1.04E-10	6.27	6.55	9.51	6.78
1.33E-03	1.05E-11	6.68	6.96	9.56	6.83
1.26E-03	1.02E-12	7.06	7.34	9.60	6.87
1.20E-03	1.04E-13	7.40	7.68	9.64	6.91
1.15E-03	1.00E-14	7.71	7.99	9.68	6.95
1.10E-03	1.04E-15	7.99	8.27	9.72	6.99

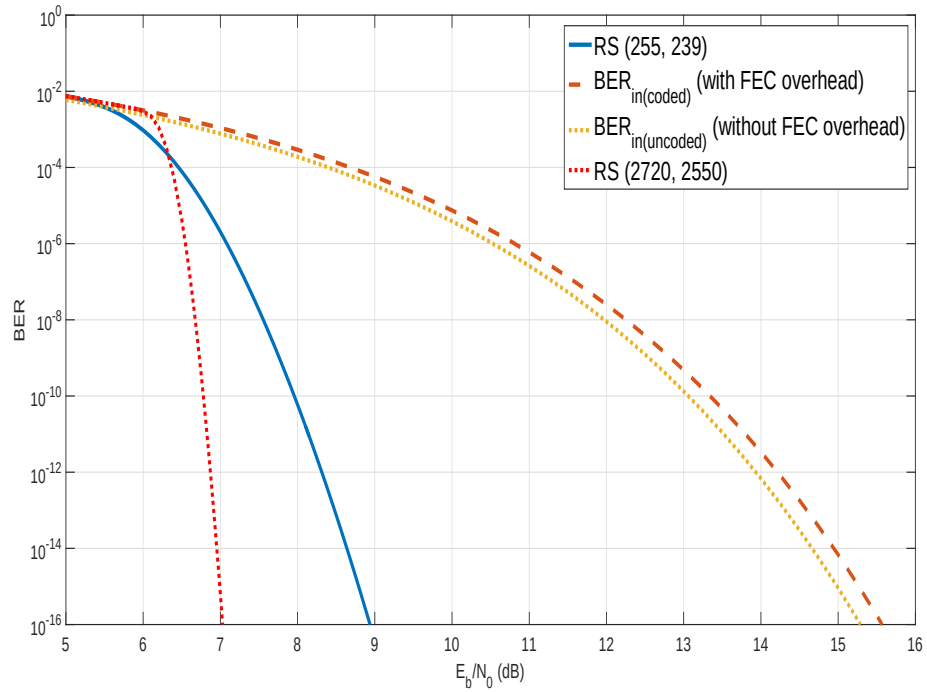


Figure 2.11: BER plots for RS (255, 239) and RS (2720, 2550) codes

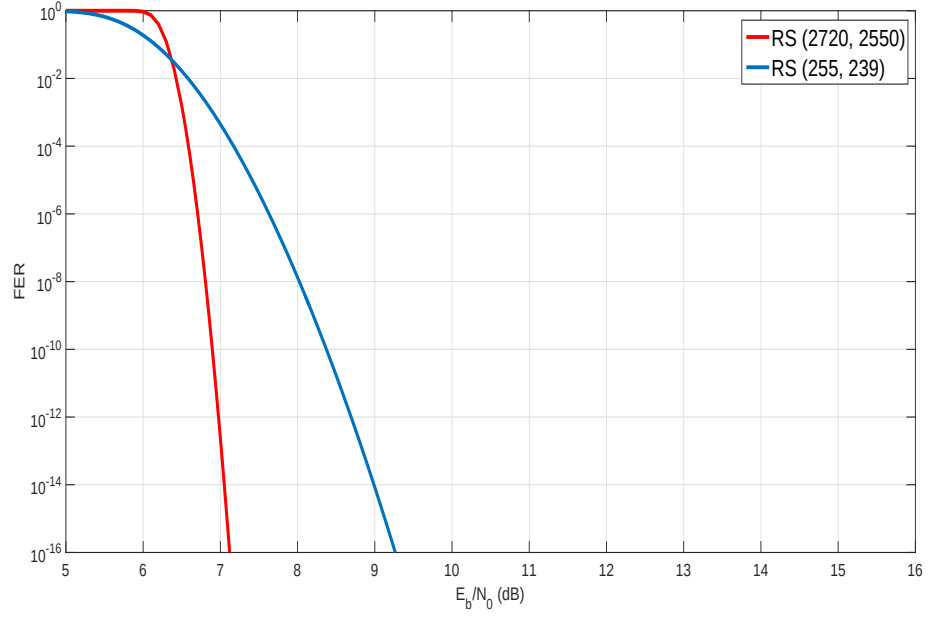


Figure 2.12: FER plots for RS (255, 239) and RS (2720, 2550) codes

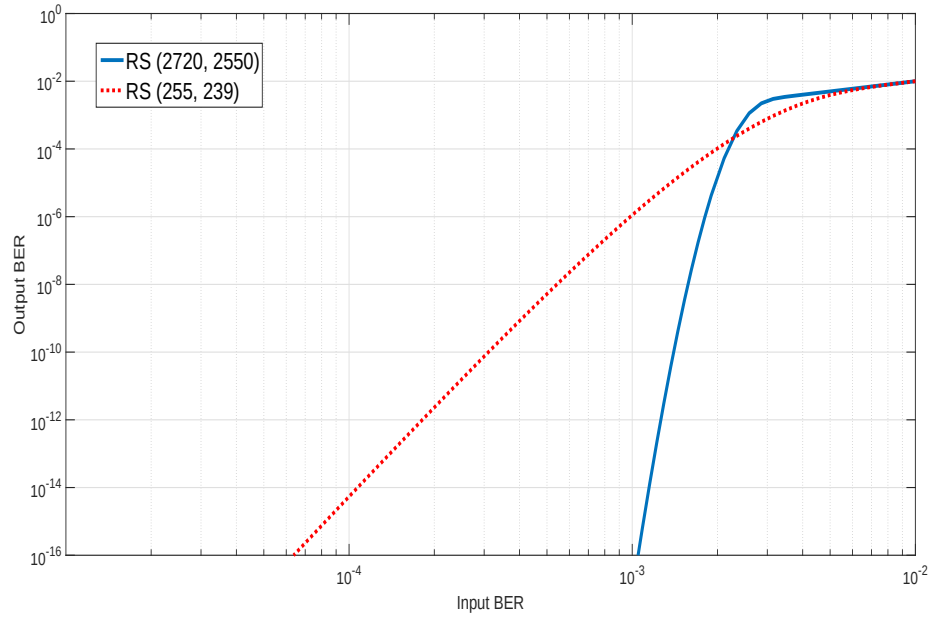


Figure 2.13: BER_{out} vs. BER_{in} for RS (255, 239) and RS (2720, 2550) codes

2.3.3 G.975.1 Recommended LDPC Super FEC Code

The G.975.1 recommended LDPC super FEC code is using systematic binary LDPC (32640, 30592). It is compatible with RS (2720, 2550) super FEC code, *i.e.*, it uses same ODU2 and OTU2 frames. It has the same redundancy as that of RS (255, 239) but improved error performance, see Figure 2.14. It achieves an NCG of 7.10 dB as compared to 5.62 dB for RS (255, 239) at an output BER of 10^{-12} as shown in Table 2.8.

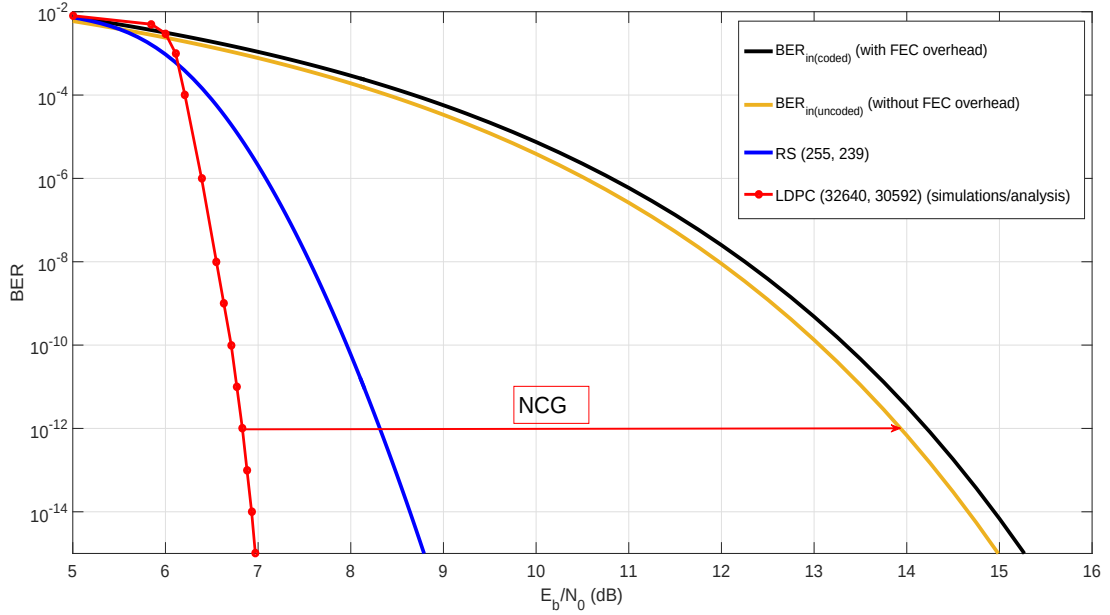


Figure 2.14: BER plot for G.975.1 recommended LDPC super FEC code [3]

Table 2.8: Error performance parameters for LDPC super FEC code [3]

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
1.61E-03	1.00E-09	5.90	6.18	9.38	6.65
1.51E-03	1.00E-10	6.35	6.63	9.44	6.71
1.42E-03	1.01E-11	6.75	7.03	9.50	6.77
1.33E-03	1.01E-12	7.10	7.39	9.56	6.83
1.25E-03	1.00E-13	7.43	7.72	9.61	6.88
1.18E-03	1.00E-14	7.73	8.02	9.66	6.93
1.12E-03	1.00E-15	8.02	8.30	9.70	6.97

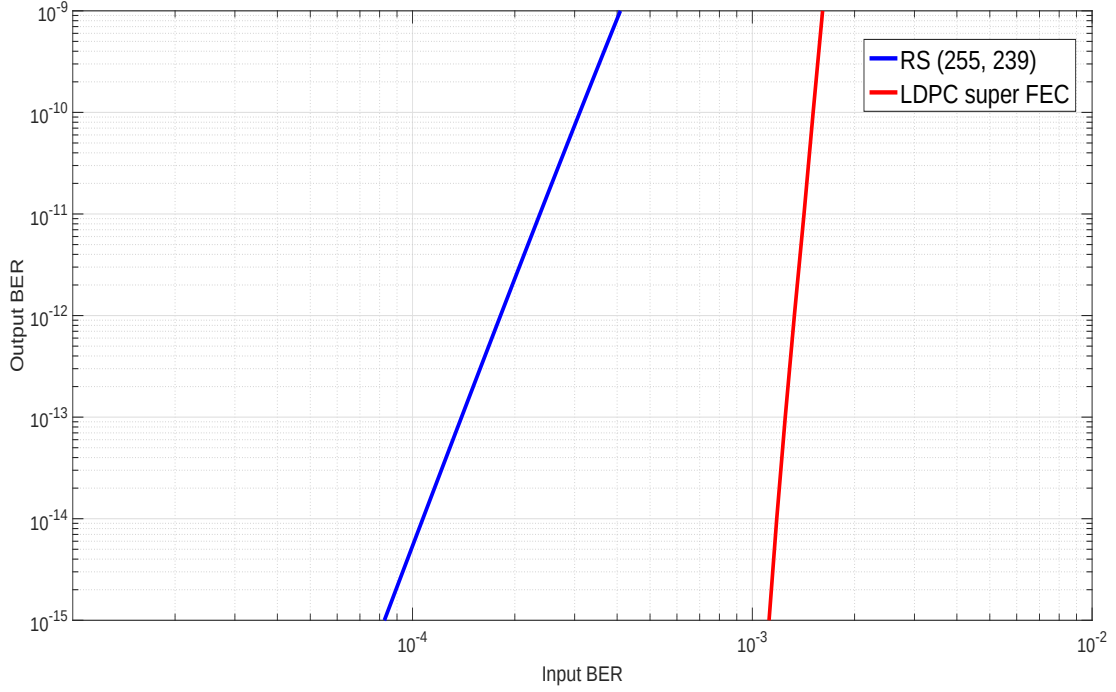


Figure 2.15: BER_{out} vs. BER_{in} for LDPC super FEC code [3]

Figure 2.15 shows that LDPC super FEC code can achieve a given target BER in a worse channel as compared to RS (255, 239) code which needs a better channel to achieve the same output BER.

G.975.1 recommended LDPC super FEC code is suitable for implementation in current chip technologies for 10G and 40G optical systems and has been shown to provide a significantly higher coding gain than the standardized RS code. Moreover, it can be implemented with low latency. Typical values of latency are of the order of 3 us for 10.7 Gb/s [3].

2.3.4 RS (255, 239)/CSOC ($n_0/k_0 = 7/6$, $J = 8$) Super FEC Code

This super FEC scheme concatenates non-iterative RS (255, 239) as an outer code and convolutional self-orthogonal code CSOC ($n_0/k_0 = 7/6$, $J=8$) as an inner code. An interleaver is used between inner and outer codes as shown in the Figure 2.16. It has a high redundancy of 24.48% as compared to other G.975.1 FEC codes. It can provide 7.95 dB of NCG at an output BER of 10^{-12} with three iterations of independent CSOC decoding, see Table 2.9. It is shown in Figure 2.16 that the concatenated scheme can also use iterative decoding beside independent CSOC iterative decoding to improve error performance.

Table 2.9: Error performance parameters for concatenated RS/CSOC super FEC code with 24.5% overhead [3]

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
6.50E-03	1.00E-09	6.70	7.66	7.90	5.17
6.30E-03	1.00E-10	7.19	8.13	7.94	5.21
6.05E-03	1.01E-11	7.59	8.54	7.99	5.26
5.80E-03	1.01E-12	7.95	8.90	8.04	5.31
5.60E-03	1.00E-13	8.29	9.24	8.08	5.36
5.40E-03	1.00E-14	8.60	9.55	8.13	5.40
5.20E-03	1.00E-15	8.88	9.83	8.17	5.44

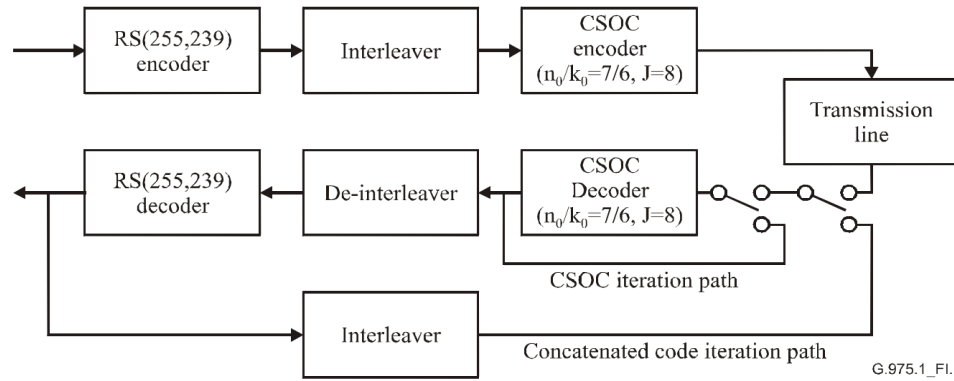


Figure 2.16: Block diagram of G.975.1 recommended concatenated RS/CSOC super FEC scheme [3]

RS (255, 239) code is already discussed in this chapter. Interleaver of order 16 is used in this super FEC scheme. There are 16 RS encoders, so a block of 30592 bits ($239 \times 8 \times 16$) is input to this FEC scheme. Figure 2.17 shows format of the frame that is fed to interleaver by 16 parallel RS encoders. It is important to know that the length of one row of ODU2 frame is also 30592 (3824×8). Therefore 4 blocks of data input to this super FEC encoder makes one ODU2 frame. Interleaver outputs a frame of size 32640 bits which is equal to one row of OTU2 frame.

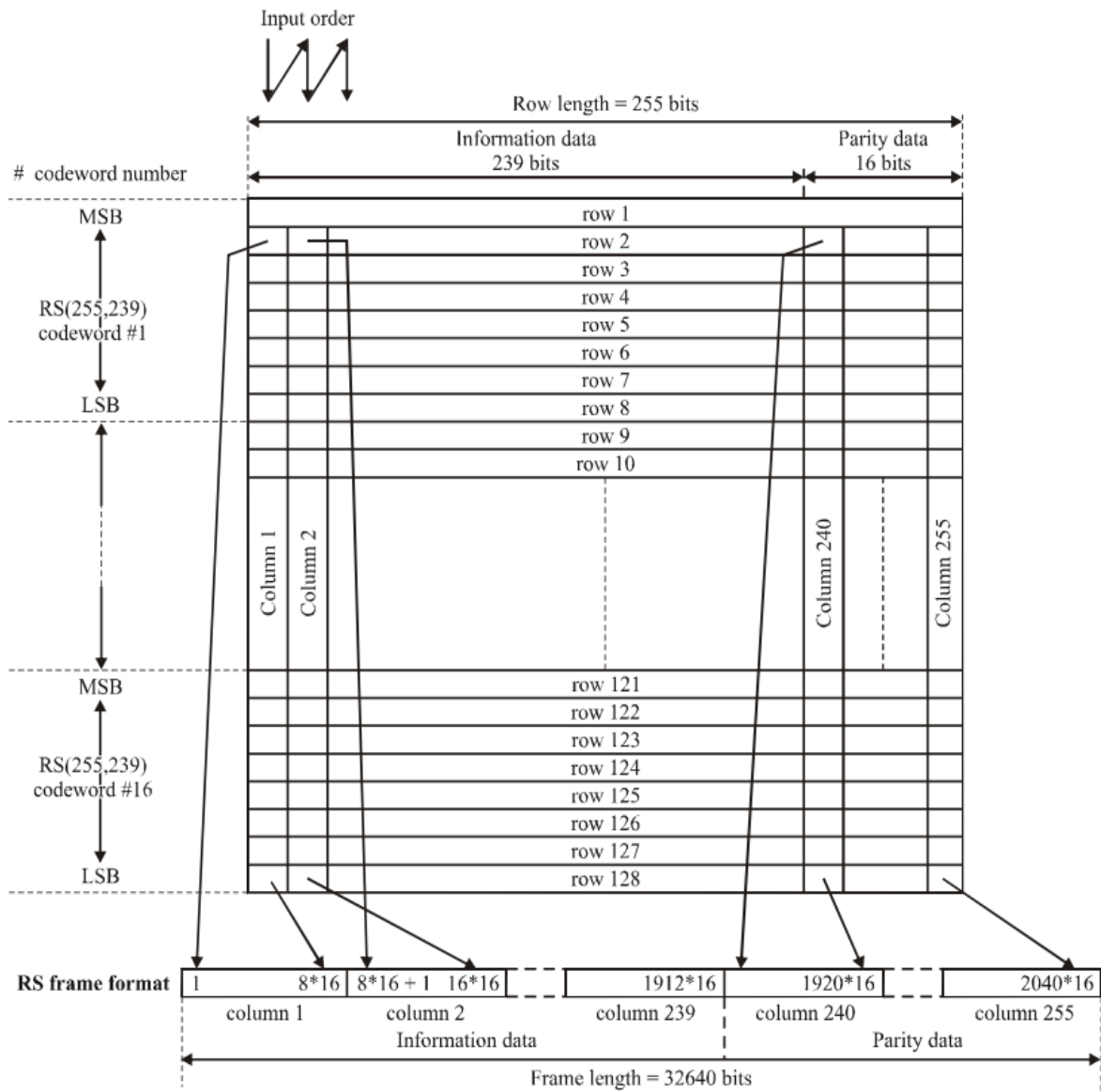


Figure 2.17: Frame format output from RS encoders and input to interleaver [3]

CSOC is binary convolutional self orthogonal code and uses iterative decoding with three iterations in G.975.1 recommended scheme. CSOC ($n_0/k_0 = 7/6$, $J=8$) encodes 6 input bits and calculates parity bit according to some preselected generator polynomials given in G.975.1 recommendation. J is total number of orthogonal check sets.

Error correction performance of this super FEC depends on number of iterations of CSOC and threshold values used for decoding in each iteration. G.975.1 considers 3 iterations with threshold values of 7, 6 and 5 in each iteration, respectively. Figures 2.18 and 2.19 show its error performance in comparison to RS (255, 239).

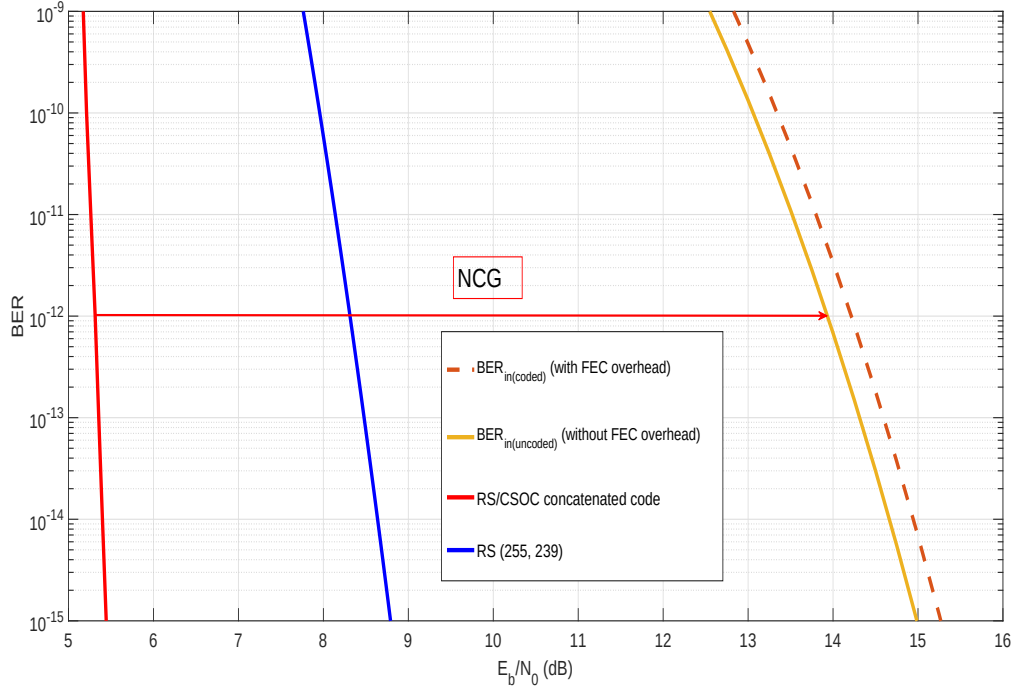


Figure 2.18: BER plot for concatenated RS/CSOC super FEC scheme [3]

This super FEC scheme has a latency of around 25 us for 10 Gb/s payload throughput. Moreover, it has less complexity and has very simple implementation in hardware [3].

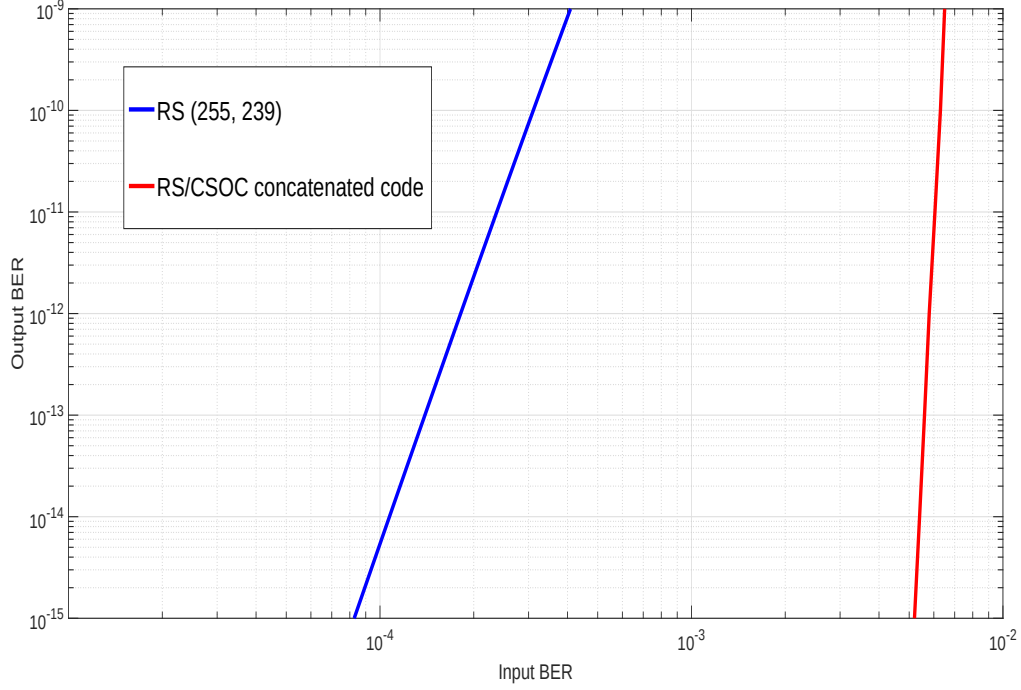


Figure 2.19: BER_{out} vs. BER_{in} for concatenated RS/CSOC super FEC scheme [3]

2.3.5 Concatenated BCH Super FEC Code

This super FEC concatenates BCH codes of two different block lengths and gives better error correction performance as compared to RS (255, 239) code. G.975.1 recommends concatenation of BCH (3860, 3824, 3) as an outer code and BCH (2040, 1930, 10) as an inner code. These codes when working separately, have already been discussed in this chapter and have not very good performance, but when they are concatenated, their performance is much better than G.975 standardized RS (255, 239) code for the same overhead of 6.69%.

As clear from the Figure 2.20, it uses iterative decoding. To keep latency low, it uses only 3 iterations and gives an NCG of 7.98 dB at an output BER of 10^{-12} as compared to 5.62 dB for RS (255, 239) code, see Table 2.10. Moreover, interleaver is used between inner and outer codes to improve random error performance. Figures 2.21 and 2.22 show its error performance in comparison to RS (255, 239).

Table 2.10: Error performance parameters for concatenated BCH super FEC code [3]

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
3.50E-03	1.00E-09	6.66	6.94	8.62	5.89
3.44E-03	1.00E-10	7.15	7.44	8.64	5.91
3.37E-03	1.01E-11	7.59	7.87	8.66	5.93
3.30E-03	1.01E-12	7.98	8.26	8.68	5.95
3.25E-03	1.00E-13	8.35	8.63	8.70	5.97
3.20E-03	1.00E-14	8.68	8.96	8.71	5.98
3.15E-03	1.00E-15	8.99	9.27	8.73	6.00

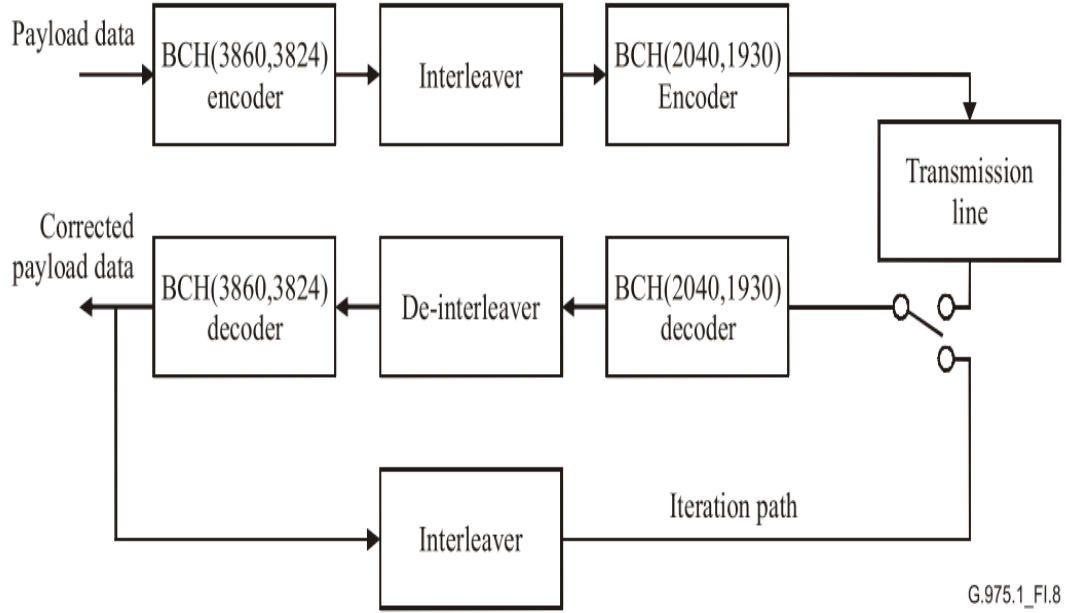


Figure 2.20: Block diagram of G.975.1 recommended concatenated BCH super FEC scheme [3]

This super FEC code has a latency of approximately 100 us for 10 Gb/s payload throughput [3].

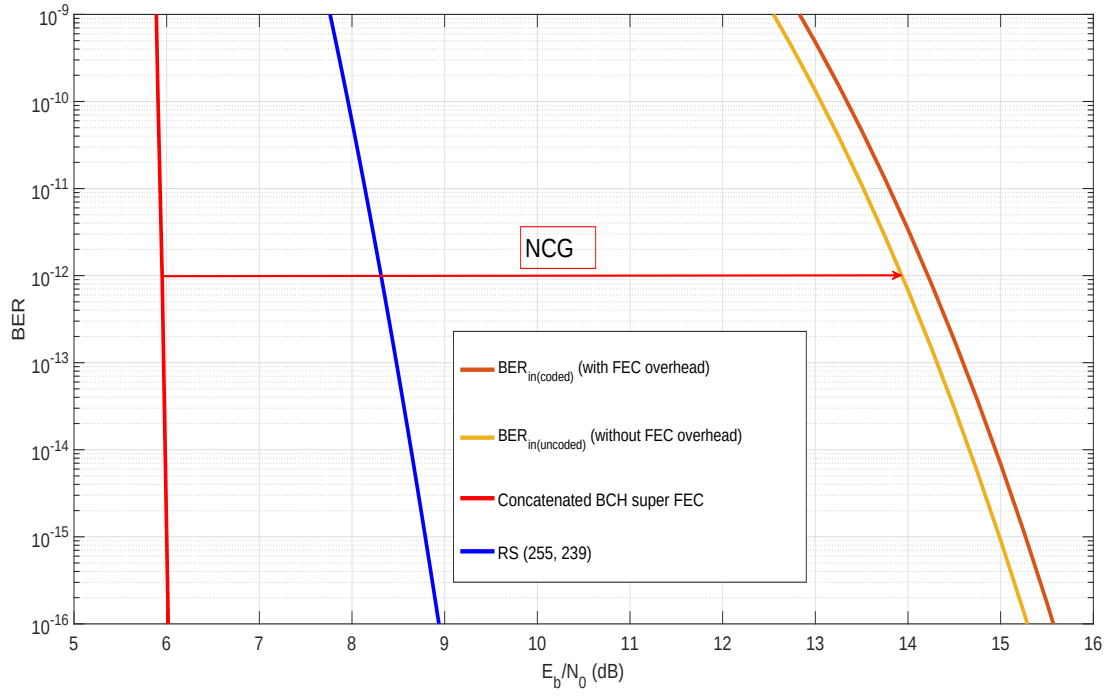


Figure 2.21: BER plot for G.975.1 recommended concatenated BCH super FEC [3]

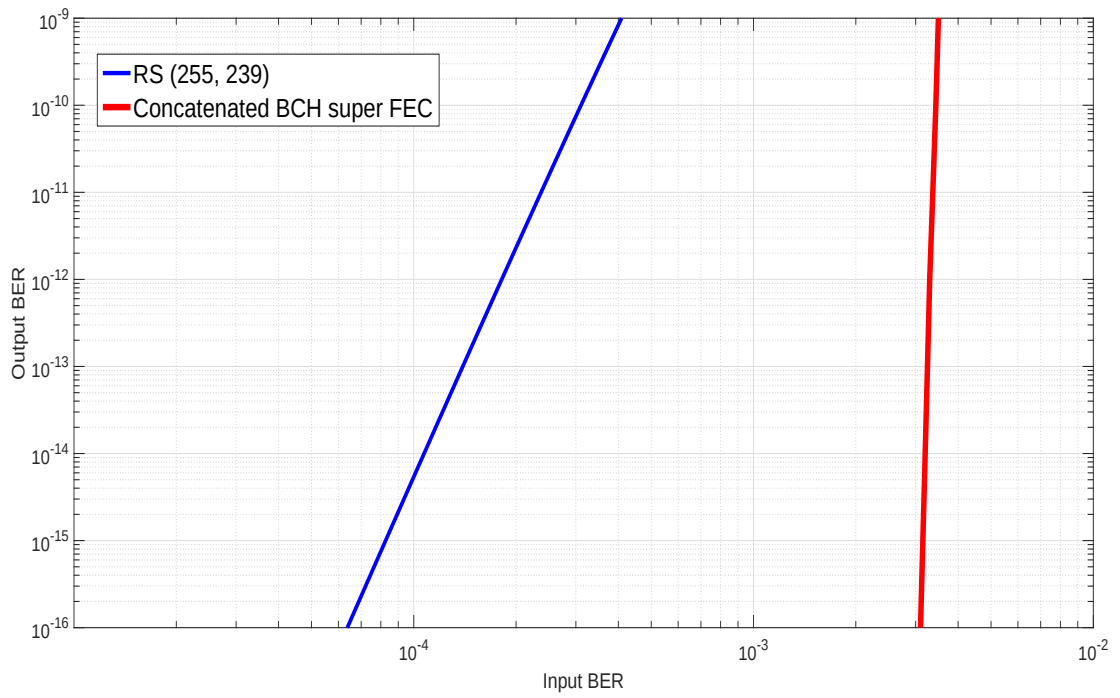


Figure 2.22: BER_{out} vs. BER_{in} for concatenated BCH super FEC code [3]

2.3.6 RS (1023, 1007)/BCH (2040, 1952, 8) Super FEC Code

This is a G.975.1 recommended concatenated super FEC code with interleaver to improve random error performance. RS (1023, 1007) is used as an outer code, which can correct up to 8 symbols each of 10 bits. BCH (2040, 1952) is used as an inner code, which can correct up to 8 bits. These interleaved codes achieve greater net coding gain (7.63 dB at an output BER of 10^{-12}) as compared to G.975 FEC code RS (255, 239) (which can achieve an NCG of 5.62 dB for the same output BER) with the same overhead of 6.69%, see Table 2.11.

Table 2.11: Error performance parameters for concatenated RS/BCH super FEC code [3]

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
2.41E-03	1.00E-09	6.31	6.59	9.00	6.27
2.35E-03	1.00E-10	6.83	7.11	9.03	6.30
2.30E-03	1.01E-11	7.30	7.58	9.05	6.32
2.26E-03	1.01E-12	7.63	7.91	9.06	6.34
2.23E-03	1.00E-13	8.03	8.31	9.08	6.35
2.20E-03	1.00E-14	8.34	8.62	9.09	6.36
2.17E-03	1.00E-15	8.67	8.95	9.10	6.37

Latency is implementation dependent but current implementations have been proven to have low latency [3]. Moreover this super FEC uses iterative decoding with 1 or 2 iterations and achieves better performance than RS (255, 239) FEC code. Figures 2.23 and 2.24 show its error performance in comparison to RS (255, 239) code.

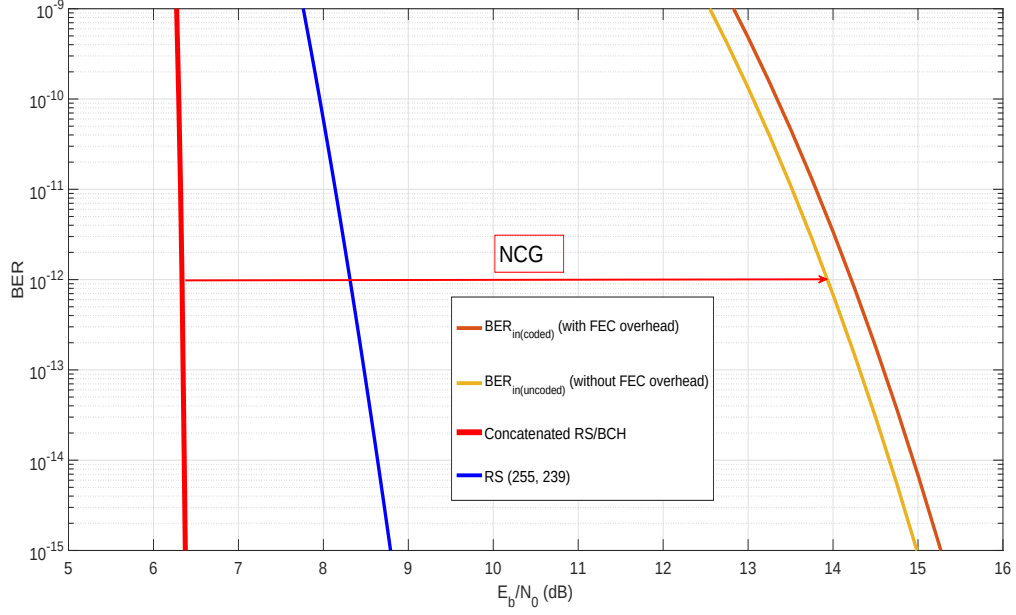


Figure 2.23: BER plot for concatenated RS/ BCH super FEC code [3]

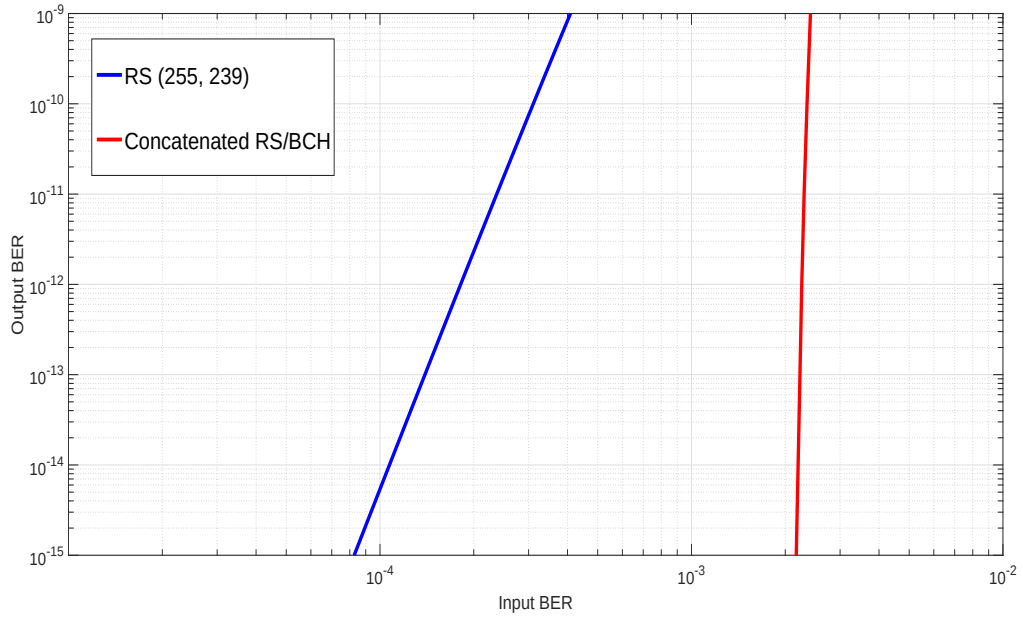


Figure 2.24: BER_{out} vs. BER_{in} for concatenated RS/BCH super FEC code [3]

2.3.7 Concatenated RS/Product-code Super FEC Scheme

This G.975.1 recommended super FEC concatenates RS (1901, 1855) as an outer code and SD extended hamming (512, 502) $\times$ (510, 500) product-code as an inner code. This concatenation provides higher error correction capability as compared to RS (255, 239) with exactly the same amount of redundancy (6.69%). It provides an NCG of at least 7.5 dB at an output BER of 10^{-12} as compared to 5.62 dB provided by G.975 FEC code for the same output BER. The outline of this super FEC scheme is shown in Figure 2.25.

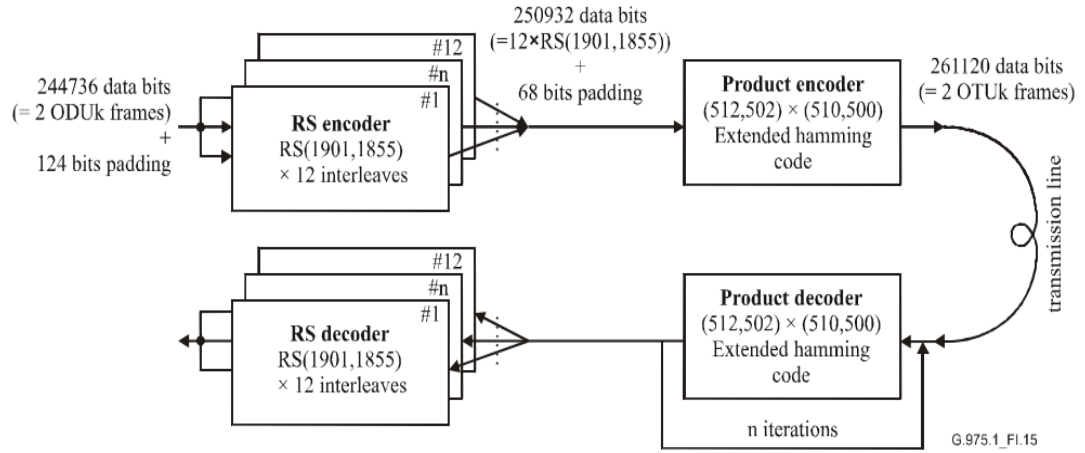


Figure 2.25: Block diagram of G.975.1 recommended concatenated RS/Extended-Hamming-Product super FEC scheme [3]

Inner code, *i.e.*, product-code is using soft decoding. It can also use soft in soft out (SISO) iterative decoding. Complexity increases with the number of iterations and the number of quantization levels for receiver. This super FEC can correct at least 124 bit long burst error. With single bit and two bits quantizer and 8 SISO iterations of product code decoder, error correction performance of this super FEC is shown in Tables 2.12 , 2.13 and Figures 2.26, 2.27.

Latency of encoder and decoder for this super FEC code is implementation dependent.

Table 2.12: Error performance parameters for concatenated RS/Product super FEC code (Quantization=1 bit) [3]

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
2.00E-03	1.00E-09	6.10	6.40	9.20	6.43
2.00E-03	1.00E-10	6.60	6.90	9.20	6.44
2.00E-03	1.01E-11	7.00	7.30	9.20	6.45
1.90E-03	1.01E-12	7.50	7.70	9.20	6.47
1.90E-03	1.00E-13	7.80	8.10	9.20	6.49
1.90E-03	1.00E-14	8.20	8.40	9.20	6.51
1.90E-03	1.00E-15	8.50	8.80	9.20	6.54

Table 2.13: Error performance parameters for concatenated RS/Product super FEC code (Quantization=2 bits) [3]

Input BER	Output BER	NCG	CG	Q (dB)	Eb/No (dB)
4.70E-03	1.00E-09	7.00	7.30	8.30	5.53
4.70E-03	1.00E-10	7.50	7.80	8.30	5.54
4.70E-03	1.01E-11	7.90	8.20	8.30	5.55
4.60E-03	1.01E-12	8.40	8.60	8.30	5.57
4.60E-03	1.00E-13	8.70	9.00	8.30	5.59
4.60E-03	1.00E-14	9.10	9.30	8.30	5.61
4.50E-03	1.00E-15	9.40	9.70	8.30	5.64

RS (1901, 1855) code is used as an outer code. It can correct up to 23 symbols each of 11 bits. It uses 46 parity symbols. It is a shortened RS code in GF (2^{11}). In full code N can be 2047. Shortening is possible by appending 0s in the remaining positions. We do not need to transmit these extra 0s, we can insert them in before decoding. Twelve RS (1901, 1855) encoders are used in parallel. A total of 244860 bits (244736 data+ 124 padding bits) are input to these twelve encoders. It is important to know that two ODU2 frames are also equal to 244736 bits, which means that two ODU2 frames can be input to this super FEC encoder at a time [11]. 244860 bits are equal to 22260 symbols of 11 bits. These symbols are divided into chunks of 1855 symbols ($1855 \times 12 = 22260$) for each encoder. Then the encoded symbols are interleaved in round robin fashion. Extra 68 bits are appended to this stream of data, making total number of bits equal to 251000 ($12 \times 11 \times 1901 + 68$). This stream of bits is input to product encoder.

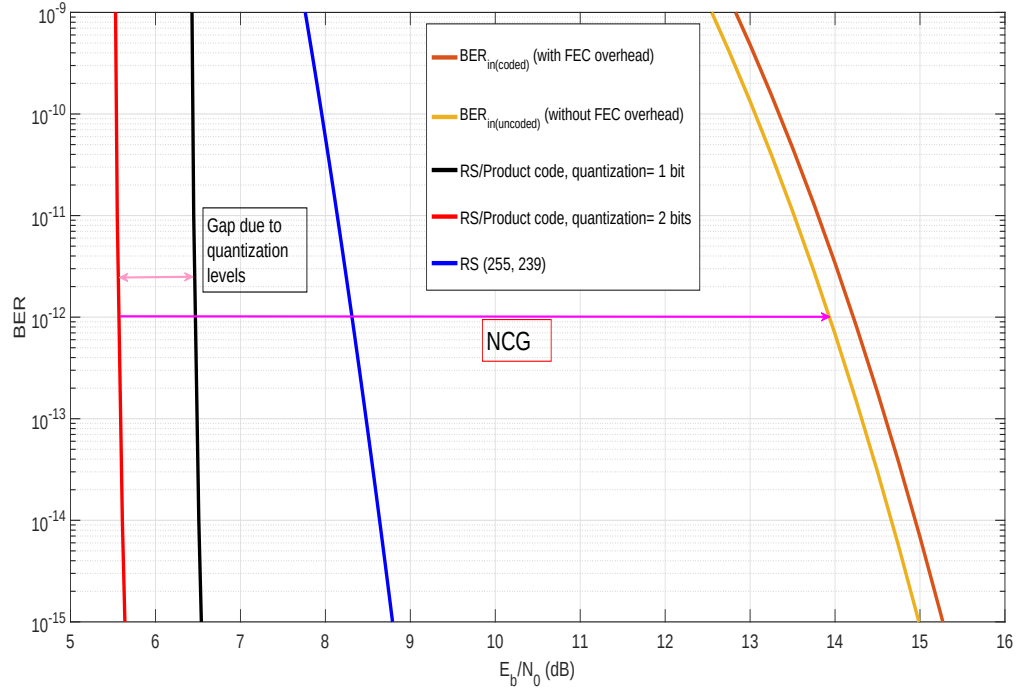


Figure 2.26: BER plots for concatenated RS/Product-code super FEC [3]

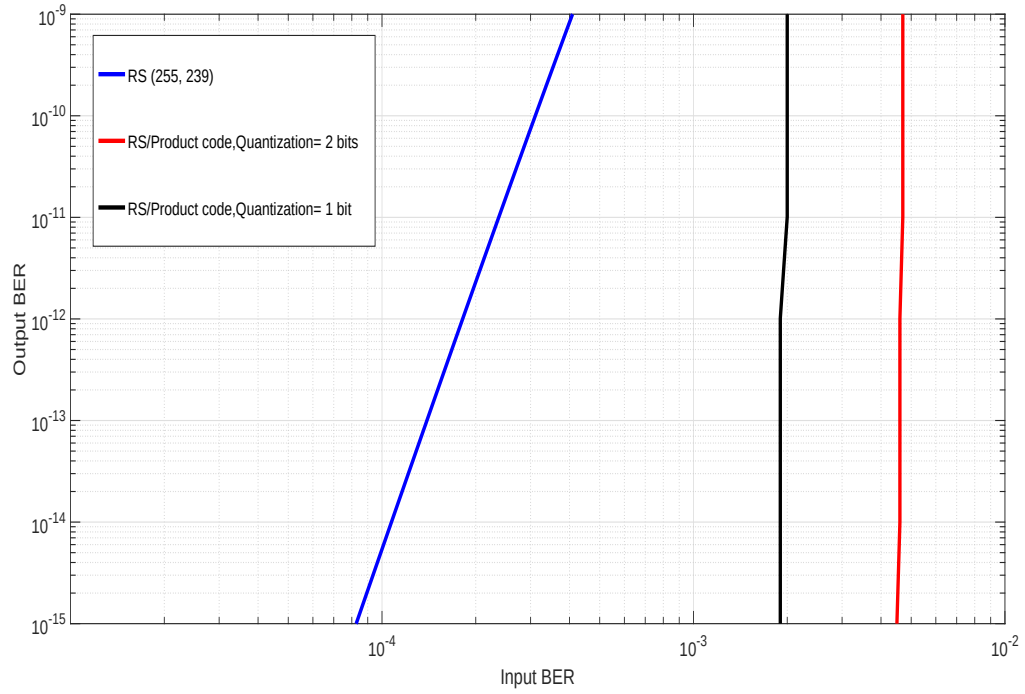


Figure 2.27: BER_{out} vs. BER_{in} for RS/Product-code super FEC [3]

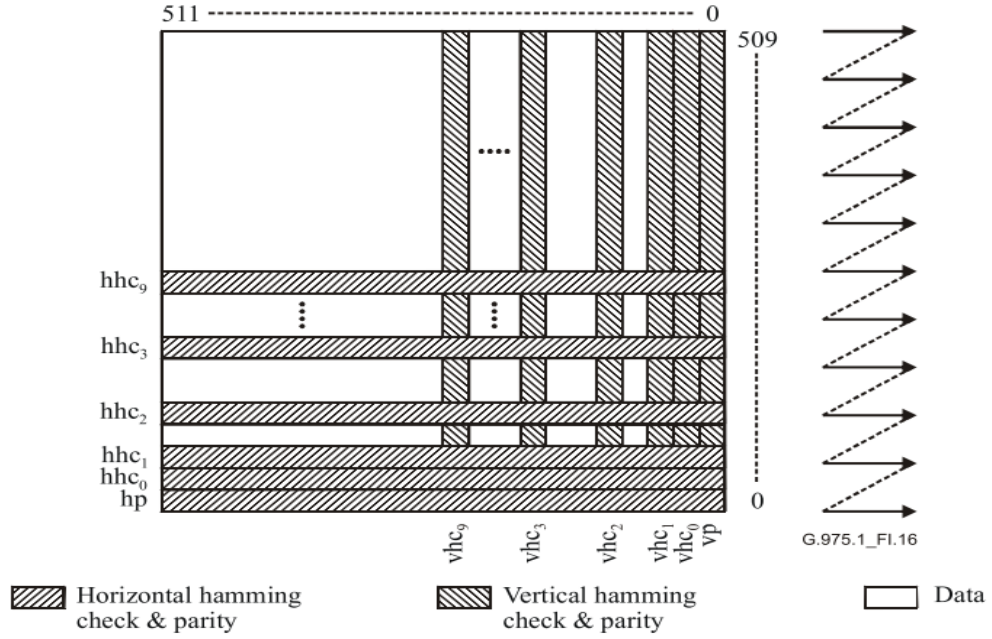


Figure 2.28: Product code matrix [3]

Figure 2.28 gives a pictorial description of the product-code matrix. Product-code in this super FEC uses extended hamming codes. Extended hamming code is derived from hamming code by appending an overall parity check bit which is calculated as parity bit of the hamming check bits. This super FEC uses extended hamming code (512, 502) for the 510 rows of the product-code. It means in each row out of 500 (10 rows are for horizontal checks), there are 502 data bits, 9 hamming check bits and 1 overall parity check bit. Similarly, it uses extended hamming code (510, 500) for a total of 502 columns (10 columns are for vertical checks) of the product-code, which means that every column consist of 500 data bits, 9 hamming check bits and 1 overall parity check bit. Total number of bits input to this product encoder is 251000 and it adds a total of 10120 check bits ($512 \times 10 + 510 \times 10 - 10 \times 10$). So total number of output bits is 261120 ($512 \times 510 = 10120 + 251000$). It is important to know that this number is equal to the length of two OTU2 frames [11]. Rate is calculated as $244736 / 261120 = 0.9372$ (ODU2/OTU2).

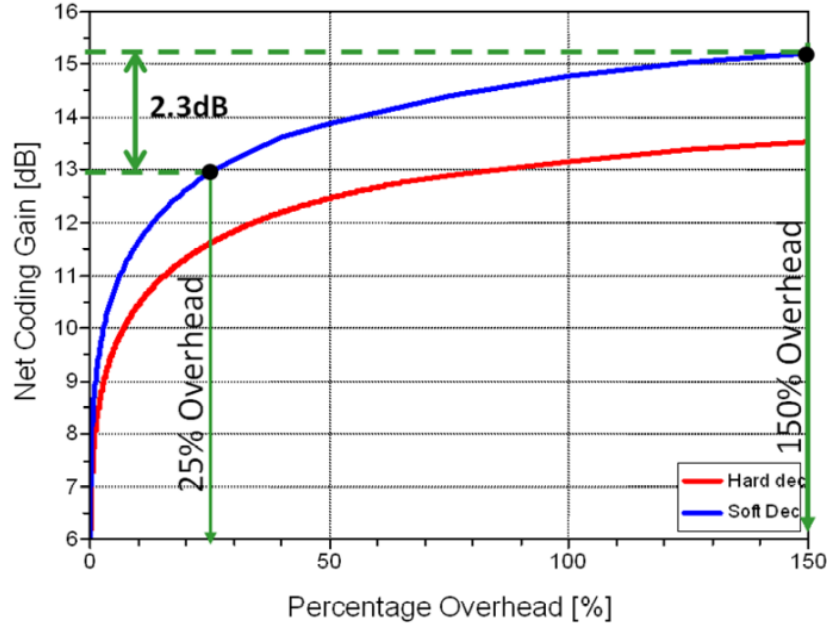


Figure 2.29: Maximum theoretical asymptotic NCG for a target BER of 10^{-15} with SD and HD FECs [4]

2.4 Third Generation FEC Codes in Optical Communications

FEC codes designed and proposed for next generation optical communications are normally referred to as third generation FEC codes in the literature. Third generation FEC schemes mostly use SD decoding and guarantee an NCG greater than 10 dB with 20 % overhead at a post-FEC BER of 10^{-15} . Coherent detection is used in modern optical networks for achieving data rate of 100 Gb/s or beyond. Some of the proposed FECs combine codes with modulation schemes to perform better in so-called coded-modulation schemes [1]. Although high order modulation provides high data rates, it brings with it new channel impairments such as uncompensated chromatic dispersion, polarization mode dispersion, and nonlinear effects which need to be considered.

Most of the state-of-the-art schemes are based on SD codes like Turbo or specially constructed LDPC codes with iterative decoding and interleaved concatenated coding [1]. However, since codes are usually judged by error performances based on their NCG values, there are some HD-FECs which are suitable for 100G systems but their error correcting performance is slightly inferior to SD-FECs.

Figure 2.29 explains the reason why most of the third generation FECs use 20% overhead. It can be seen that as the overhead increases for 0 to 25%, NCG increases by 12.9 dB. On the contrary, when overhead increases from 25 to 150%, the increase in NCG is only 2.3 dB. 20% overhead provides a nice balance between NCG and redundancy.

2.4.1 Achievable NCG Limits for FECs with Finite Block Lengths

Table 2.14 provides the Shannon's asymptotic limits for NCG which can be achieved by SD and HD FECs with different overheads. It is clear that SD perform better than HD and the difference in performance increases with overhead. That is the reason that third generation FECs mostly use SD algorithms.

Table 2.14: Shannon's theoretical limits of NCG for HD and SD decoding algorithms for a target BER of 10^{-15} [7]

Overhead (%)	HD (dB)	SD (dB)	Difference (dB)
7	10	11.10	1.10
15	10.95	12.20	1.25
20	11.33	12.60	1.27
25	11.6	12.90	1.30

Shannon's limits for NCG assumes infinite block lengths. In practice, it is not possible to have very long block lengths due to latency issues. In order to assess codes against more practical limits, we calculate NCG limits for FECs that can be achieved with finite block lengths [25]. Table 2.15 lists achievable NCG limits

for FECs corresponding to different block lengths. It also lists the difference between Shannon's limit and finite length limit of NCG for FECs with a given block length and overhead. We see that the difference decreases as the block length increases.

Table 2.15: Finite block lengths limits of NCG for SD decoding algorithms for a target BER of 10^{-15}

Overhead (%)	Block Length (N)	Finite length NCG limit (dB)	Difference (dB)	Shannon's NCG limit (dB)
7	2040	9.75	1.35	11.10
7	32640	10.76	0.34	11.10
7	130560	10.93	0.17	11.10
7	261120	10.98	0.12	11.10
20	2040	11.2	1.40	12.60
20	32640	12.24	0.36	12.60
20	130560	12.42	0.18	12.60
20	261120	12.47	0.13	12.60
20	522240	12.52	0.08	12.60

As we studied in previous sections, none of the codes recommended in G.975.1 achieves an NCG which is close to the finite length limits. Both the Shannon's limits and finite length limits for NCG hints that there is still a room for better codes with higher NCG values. Third generation of FECs focuses on this and proposes new codes whose performance is very close to these limits.

2.4.2 Intensity Modulated/Direct Detection Optical Communications

First and second generation FEC codes are based on optical communications that uses intensity modulation at transmitter and direct detection at receiver (IMDD). In these systems, intensity of lasers is modulated, and the intensity/strength of the optical signal transmitted through an optical channel is detected by a photo-diode, which simply acts as a square-law detector. This is widely employed

in optical communications, even today. The IMDD scheme has a significant advantage in that the receiver sensitivity is independent of both the carrier phase and the state of polarization of the signal, which randomly fluctuates in real systems.

Table 2.16: Comparison between coherent and IMDD schemes [8]

	Coherent	IMDD
Modulation parameters	I and Q (amp. and phase)	Intensity
Detection method	Heterodyne detection	Direct detection
Sensitivity to carrier phase	Yes	No
Sensitivity to polarization	Yes	No

2.4.3 Coherent Detection

Modern optical communications use coherent detection for high data rates of 100 Gb/s and beyond. In a coherent receiver the incoming signal is mixed with a local oscillator signal to extract phase information. Heterodyne and homodyne receivers are typical examples of coherent receivers. Reawakening of interest in coherent OTNs after mid 90s is due to practical demonstration of frequency stabilization of semiconductor lasers [8]. But it was in 2005, when advanced digital signal processing (DSP) techniques for digital carrier-phase estimation in coherent receivers caused revolutionary progress in coherent OTNs [8].

Coherent detection enables one to communicate at high data rates without increasing symbol rates as it can use a variety of spectrally efficient high order modulation schemes such as M-ary PSK and QAM, which rely upon carrier-phase estimation. 100 Gb/s transmission systems with DP-QPSK (dual polarization-QPSK) modulation and high speed DSP at a symbol rate of 25 GBd, have been developed [26]. 200 Gb/s optical transceivers with DP-16-QAM and 32 GBd symbol rates are commercially available [27].

2.4.4 Modulation Gain Due to Coherent Detection

Coherent detection in communication systems using high order spectrally efficient modulation schemes increases the data rate without increasing the symbol rate. DP-QPSK is a popular modulation scheme for 100 Gb/s optical networks considering the OSNR (optical SNR) requirement and spectral efficiency [28].

Symbol power should be two times for QPSK as compared to symbol power for BPSK (with Gray encoding) for an uncoded system to achieve the same BER (to make E_b/N_0 the same in both cases). But the advantage is that we can get double bit rate with QPSK as compared to BPSK for the same symbol rate. This increase in data rate can be used to save signal power and can be expressed in terms of modulation gain (dB). Figure 2.30 shows constellation points for BPSK and QPSK lying on a unit circle. BER for QPSK is more as compared to BPSK since E_b/N_0 is smaller for QPSK but we can transmit at higher data rate with QPSK. Can we use this higher bit rate of QPSK to save signal power and at the same time achieve same data throughput as that we can obtain with BPSK for the same BER? The answer is yes. Consider a BPSK system with 25% FEC overhead (code rate= 0.8). Figure 2.29 shows that an SD-FEC can achieve an NCG of 12.9 dB with 25% overhead. If we use QPSK and use an overhead of 150%, Figure 2.29 shows that we can achieve an NCG of 15.2 dB for the same BER. Moreover, QPSK with an overhead of 150% is capable to achieve same data throughput as that of BPSK with 25% overhead. Hence we can have a saving of 2.3 dB in symbol power for a QPSK system to transport the same amount of data.

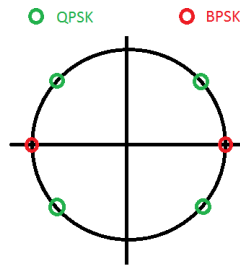


Figure 2.30: Constellation of QPSK and BPSK

2.4.5 LDPC based Third Generation FEC Codes

A brief survey of good error performance third generation FECs for optical communications based on LDPC codes is conducted here. LDPC codes, and more especially quasi-cyclic LDPCs (QC-LDPC), seem to be the most potential candidates for next generation OTNs, due to their desired features of low complexity implementation and efficient parallelization (required for $\geq 100\text{G}$ optical networks) [1]. Major problem associated with LDPC codes is their error floor at very low BER (approx. 10^{-10}). Figure 2.31 shows three possible ways (LDPC-only, double concatenation and triple concatenation) of using LDPC codes for the construction of FEC frames suitable for 100 Gb/s optical communication coherent systems, with a 20% overhead such that they can address the error floor problem [1].

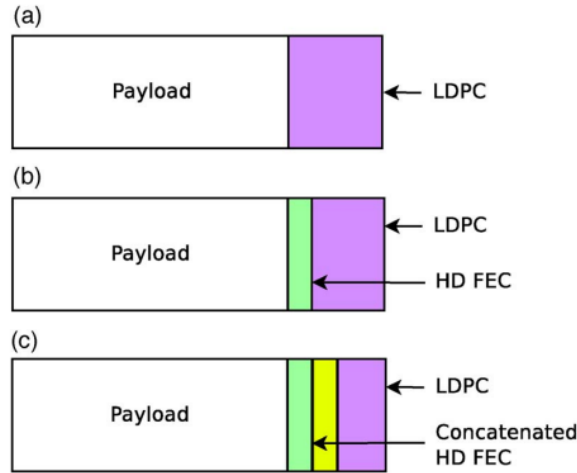


Figure 2.31: Types of FEC frame structures, including LDPC codes, for 100 Gb/s optical networks [1]

Non-Concatenated LDPC-only: Figure 2.31a depicts a single non-concatenated LDPC implementation with 20% FEC overhead. It is a general perception that LDPC-only codes can out-perform concatenated codes with much lower complexity. Morero et al. [29] have developed a (24576, 20482) (20% overhead) QC-LDPC-only code with a 5 bit min-sum algorithm (MSA) decoder for 100 Gb/s optical communications. It achieves an NCG of 10.70 dB at a BER

of 10^{-13} with a 13-iteration MSA-decoder. Moreover the code is found to be free of error-floor down to 10^{-13} . The main problem they have focused on is the evaluation and reduction of very low error floors (*i.e.*, $\leq 10^{-13}$).

Table 2.17: A summary of high performance LDPC based third generation FEC codes

FEC code	SD bits	OH (%)	NCG (dB)	No. of iterations	Target BER	Ref.
Concatenated QC-LDPC/SPC	4	20.5	11.3	15	10^{-15}	[23]
Single QC-LDPC (18360, 15300)	4	20	11.3	12	10^{-15}	[19]
LDPC-CC (10032, 4, 24)	4	20	11.5	12	10^{-15}	[20]
Concatenated LDPC/RS	2	20.5	9	4	10^{-13}	[30]
Concatenated SC-LDPC/BCH	4	25.5	12	32	10^{-15}	[22]
Triple-concatenated LDPC	3	20.5	10.80	16	10^{-15}	[31]

They have concluded in their work that low error floors below 10^{-13} are mainly due to the FP (fixed-point) implementation of MSA. They have developed a low complexity post-processing algorithm (PPA) to address the error floor problem in MSA decoders. With application of PPA and iterative decoding, error floors are lowered significantly, giving an NCG ≥ 11.30 dB at a BER of 10^{-15} . This value is consistent with the one that is achieved by [31] (will be discussed shortly) using concatenated code. But the LDPC code proposed in [29] is far superior as it does not use code concatenation and moreover block length is three times lesser than that in [31].

Chang et al. have proposed another such non-concatenated LDPC-only code [19]. Girth-8 QC-LDPC (18360, 15300) code with 20 % overhead achieves an NCG up to 11.3 dB at a BER of 10^{-15} with 12 iterations of decoding. Yang et al. [32] showed free of error floor beyond 1 Tb/s girth-12 QC-LDPC-coded 16-QAM OFDM communication with 50 % overhead. LDPC-CCs (convolutional codes) are more suitable for pipelined implementation to guarantee less complexity. LDPC-CC (10032, 4, 24) can achieve an NCG of 11.5 db at a post-FEC BER of 10^{-15} , as proved by FPGA emulation (maximum number of iterations set to 12) [20].

Concatenated LDPC-HD: Figure 2.31b shows another approach to suppress the error floor by concatenating LDPC code with an HD-FEC code. Mizuochi et al. [30] proposed the concatenation of LDPC (9216, 7936) and RS (992, 956) codes. It can achieve an NCG of 9 dB at a BER of 10^{-13} with only 2-bit SD, 4 iterations and 20.5% overhead. It can achieve data rates up to 31.3 Gb/s. Yet another high performance and low-complexity FEC scheme based on LDPC is proposed by Kamiya et al. [23]. The proposed QC-LDPC and SPC concatenated codes, with 20.5% overhead can give an NCG of 10.4 dB at a BER of 10^{-12} (NCG = 11.3 dB at a BER of 10^{-15}), with 4 SD bits and 15 iterations. Another class of LDPCs, called spatially coupled codes, can be derived from QC-LDPC design [33]. MC simulation in an AWGN channel with QPSK modulation gives an NCG of 12.0 dB at a BER of 10^{-15} by the code concatenation of irregular SC-LDPC (38400, 30832) and BCH (30832, 30592), when number of iterations is set to 32 [22]. This code has a 25.5% redundancy. Another LDPC based concatenated FEC scheme for 100 Gb/s OTN is the concatenated non-binary LDPC and HD-FEC code proposed by [21]. The concatenated NB-LDPC (2304, 2048) over GF (2^4) and RS (255, 239) can provide an NCG over 10.3 dB at a BER 10^{-15} .

Triple-Concatenated LDPC: Another approach to improve the NCG of LDPC codes at BER of 10^{-15} is to concatenate it with two other HD-FEC codes [31]. The triple concatenated FEC is designed for 100 Gb/s optical networks and uses QC-LDPC (4680, 4080) as inner code and two other HD block codes as outer codes. The overall overhead is 25.5%. It is demonstrated that it can achieve an NCG of 10.8 dB. Overhead of LDPC code is 12.94% and HD codes have a total redundancy of 7%. It uses 16 iterations for decoding and 3 bits for SD.

2.5 Summary

In this chapter, after defining the parameters for measurement of error performance of FEC codes, we analyzed a number of ITU-T recommended codes. These codes work with an overhead of 6.69% and provide coding gains around 8 dB. Among ITU-T recommended FECs, RS/Produce concatenated code provides the

highest coding gain of 8.4 db for an output BER of 10^{-12} . Third generation FECs were also summarized in this chapter, mainly focusing on LDPC codes. These FEC schemes provides coding gains above 10 dB at an overhead of 20% for next generation optical networks. Non-concatenated QC-LDPC code provides very good performance at a block length of around 18K and achieves an NCG of 11.3 dB for an output BER of 10^{-15} .



Chapter 3

Polar Codes

Polar codes introduced by Arikan in [5], are the result of a natural phenomenon called “channel polarization”, which will be explained later in this chapter. Here we list some of the features of polar codes which make it a special FEC code.

Polar codes are the first provably capacity achieving codes for any symmetric binary-input discrete memoryless channel (B-DMC). These codes have regular structure and have explicit construction. They have very low encoding and decoding complexity, *i.e.*, $\mathcal{O}(N \log N)$, where N is the block length of the code. Polar codes allow very flexible selection of the code rate with $\frac{1}{N}$ precision with the additional feature that a code rate can be selected without reconstructing the code from the scratch. Polar codes are proven to have no error floors [34].

Polar codes are channel specific, *i.e.*, a polar code constructed for a particular channel is not optimum for other channels in terms of error performance. Polar codes achieve symmetric capacity asymptotically under SC decoding. Its finite length performance can be improved under SC list decoding [35].

3.1 Preliminaries

Definition 1. (*Binary input discrete memoryless channel*). A memoryless channel W with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$ and transition probabilities $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ is denoted as $W : \mathcal{X} \rightarrow \mathcal{Y}$. The input alphabet $\mathcal{X} \in \{0, 1\}$, whereas output alphabet and transition probabilities can have any arbitrary valid values. Such a channel is referred to as a binary input discrete memoryless channel.

Definition 2. (*Symmetric B-DMC*). A B-DMC, W , is said to be symmetric channel if there exists a permutation π such that

$$\pi^{-1} = \pi \quad (3.1a)$$

$$\text{and} \quad W(y|1) = W(\pi(y)|0), \quad \forall y \in \mathcal{Y}. \quad (3.1b)$$

Definition 3. (*Memoryless channel*). A memoryless channel W with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$ and transition probabilities $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ is denoted as $W : \mathcal{X} \rightarrow \mathcal{Y}$. A vector memoryless channel $W^N(y^N|x^N)$ corresponding to N independent uses of W satisfies

$$W^N(y^N|x^N) = \prod_{\forall i} W(y_i|x_i). \quad (3.2)$$

Note that BSC(p), BEC(ϵ) and B-AWGNC(σ) are memoryless as well as symmetric channels.

Definition 4. (*Symmetric capacity of a B-DMC*). Symmetric capacity of a B-DMC, W is defined as

$$I(W) \triangleq \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}. \quad (3.3)$$

Note that $I(W)$ is the highest rate of reliable communication over W when the input alphabet is uniformly distributed. For symmetric channels, $I(W)$ is equal to the Shannon capacity of the channel, which provides upper bound on the code rate to guarantee reliable communication.

Definition 5. (*Bhattacharyya parameter of a B-DMC*). *Bhattacharyya parameter of a B-DMC, W , is defined as*

$$Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}. \quad (3.4)$$

Bhattacharyya parameter is the measure of reliability of W and it is an upper bound on the probability of ML decision error per channel use.

It is important to observe that whenever the transmission rate is close to 1, the reliability of channel is high (*i.e.*, $Z(W) \approx 0$ when $I(W) \approx 1$), which means that the B-DMC is a “good” channel. In contrast, when the maximum possible rate is close to zero, the reliability of the channel is low (*i.e.*, $Z(W) \approx 1$ when $I(W) \approx 0$), which means the channel is “bad” to transport any data over it.

Definition 6. (*Kronecker product*). *The Kronecker product of matrix $A_{m \times n}$ and matrix $B_{q \times p}$ is matrix $C_{(mp) \times (nq)}$ defined as*

$$C \triangleq A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}. \quad (3.5)$$

And n^{th} Kronecker power of a matrix A is defined as $A^{\otimes n} \triangleq A \otimes A^{\otimes (n-1)}$.

3.2 Channel Polarization

Polarization of a set of channels is in fact a “transformation of channels” that converts a set of non-zero capacity channels to extreme channels, *i.e.*, total capacity of all these independent channels is redistributed among newly synthesized channels so that some of the new channels get more portion of the total capacity while others get lesser portion.

The single step polar transform whose recursive application to a set of independent channels causes channel polarization is shown graphically in Figure 3.1.

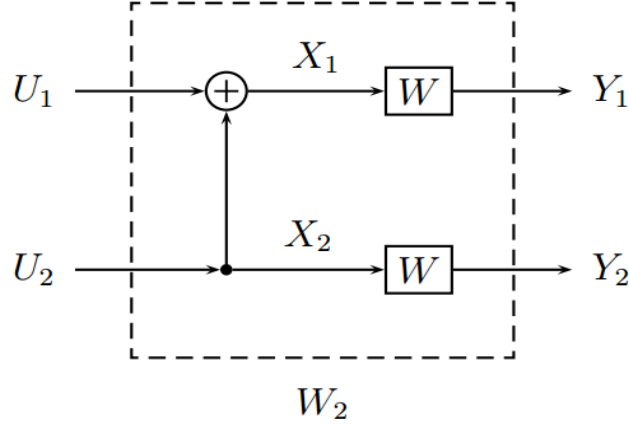


Figure 3.1: Single step polar transform [5]

Channel Combining: Consider a B-DMC W . We take two independent copies of this channel and “combine” them through the transform shown in Figure 3.1. Instead of feeding the two data inputs to the two raw/original channels directly, we add them up and feed the result (modulo-2 sum) to the first channel and feed the second channel with the second data input. Hence, we have designed a ”super channel” $W_2 : \mathcal{X}^2 = \{0, 1\}^2 \rightarrow \mathcal{Y}^2$ as

$$W_2(y_1, y_2 | u_1, u_2) = W(y_1 | u_1 \oplus u_2) W(y_2 | u_2). \quad (3.6)$$

It is easy to see that if random variables U_1 and U_2 are chosen independently and uniformly from $\mathcal{X} = \mathbb{F}_2$, X_1 and X_2 will also have uniform distribution in $\mathbb{F}_2$ and moreover they will be independent as well. Hence, we can derive the following straightforward equalities.

$$I(X_1 X_2; Y_1 Y_2) = 2I(W) \quad (3.7a)$$

$$I(W_2) = I(U_1 U_2; Y_1 Y_2) \quad (3.7b)$$

$$I(U_1 U_2; Y_1 Y_2) = I(X_1 X_2; Y_1 Y_2) \quad (3.7c)$$

$$I(W_2) = 2I(W) \quad (3.7d)$$

Above stated equalities use the fact that 1-1 polar transform is conserving the symmetric capacity.

Channel Splitting: Going one step further, we expand $I(U_1U_2; Y_1Y_2)$ by chain rule of mutual information as

$$\begin{aligned} I(U_1U_2; Y_1Y_2) &= I(U_1; Y_1Y_2) + I(U_2; Y_1Y_2|U_1) \\ &= I(U_1; Y_1Y_2) + I(U_2; Y_1Y_2U_1). \end{aligned} \quad (3.8)$$

Note that the last equality is because of the independence of U_1 and U_2 . Interpreting the above equation, we can say that we have formed two synthetic subchannels from the “super channel” W_2 , which are $\mathcal{U}_1 \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$ and $\mathcal{U}_2 \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{U}_1$ (whereas $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$ and $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}$). We are interested in their symmetric capacities. We manipulate second term, $I(U_2; Y_1Y_2U_1)$ as

$$\begin{aligned} I(U_2; Y_1Y_2U_1) &= H(U_2) - H(U_2|Y_1Y_2U_1) \\ &\geq H(U_2) - H(U_2|Y_2) \\ &= I(W). \end{aligned} \quad (3.9)$$

Combining Relation (3.9) with the mutual information chain rule Expansion (3.8) and the mutual information preserving Property (3.7a), we get polarized channels as shown below.

$$I(U_1; Y_1Y_2) \leq I(W) \leq I(U_2; Y_1Y_2U_1) \quad (3.10)$$

Inequalities (3.10) hint that the new channel $\mathcal{U}_1 \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$ is “worse” than the original channel W (in terms of symmetric capacity) while the channel $\mathcal{U}_2 \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{U}_1$ is a “better” channel than the original channel W . Hence, we can “split” the super channel into two sub channels and define the bad and good B-DMCs as

$$W^-(y_1, y_2|u_1) = \sum_{u_2 \in \mathcal{X}} \frac{1}{2} W_2(y_1, y_2|u_1, u_2), \quad (3.11)$$

$$W^+(y_1, y_2, u_1|u_2) = \frac{1}{2} W_2(y_1, y_2|u_1, u_2), \quad (3.12)$$

with the property (polarization property) that

$$I(W^-) \leq I(W) \leq I(W^+). \quad (3.13)$$

The transformation discussed above (including both channel combining and channel splitting) is the basis for channel polarization. In fact, if we apply the

transform once again on W^+ and W^- (which is equivalent to using the original channel W four times), we get sub channels that are “better than good” or ”worse than bad” in the following sense

$$I(W^{+-}) \leq I(W^+) \leq I(W^{++}) \quad (3.14)$$

$$I(W^{--}) \leq I(W^-) \leq I(W^{-+}). \quad (3.15)$$

At this point, it is not obvious, but we can expect after repeating this transform many times, we will end up with a set of ”extremal” channels which are either close to perfect (noiseless) or close to useless (pure noisy).

Below we formalize what we have discussed so far.

Definition 7. (*Single step polar transform*). Having a B-DMC W , the single step polar transform $(W, W) \rightarrow (W^-, W^+)$ “transforms” two identical copies of W to two B-DMCs, $W^- : \mathcal{X} \rightarrow \mathcal{Y}^2$ and $W^+ : \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{X}$ as

$$W^-(y_1, y_2|u_1) = \sum_{u_2 \in \mathcal{X}} \frac{1}{2} W(y_1|u_1 + u_2) W(y_2|u_2), \quad (3.16a)$$

$$W^+(y_1, y_2, u_1|u_2) = \frac{1}{2} W(y_1|u_1 + u_2) W(y_2|u_2). \quad (3.16b)$$

Lemma 1. (*Transformation of rate*). Suppose we apply the polar transform to a B-DMC W , $(W, W) \rightarrow (W^-, W^+)$, then

$$I(W^+) + I(W^-) = 2I(W), \quad (3.17a)$$

$$I(W^-) \leq I(W^+), \quad (3.17b)$$

with equality iff $I(W) = 1$ or $I(W) = 0$ (i.e., W is an “extremal” channel itself).

Lemma 2. (*Transformation of reliability*). Suppose we apply the polar transform to a B-DMC W , $(W, W) \rightarrow (W^-, W^+)$, then

$$Z(W^+) = Z(W)^2, \quad (3.18a)$$

$$Z(W^-) \leq 2Z(W) - Z(W)^2, \quad (3.18b)$$

$$Z(W^-) \geq Z(W) \geq Z(W^+), \quad (3.18c)$$

with equality in second sub-equation iff W is a BEC.

Since $Z(W^-) + Z(W^+) \leq 2Z(W)$, we conclude that the single step polar transform improves the reliability of the channels.

3.3 Encoding and Decoding of Polar Codes

Polar coding is in fact “polarizing” a set of B-DMC, $\mathbf{W}$, by iterative application of single step polar transform. The resulting sub channels are polarized so we need to select good channels to communicate over them at $rate = 1$ and freeze bad channels to zero. Moreover, frozen channels are known both to encoder and decoder.

3.3.1 Non-Systematic Polar Encoder

Polar code is a subset of a broader class of linear block codes, for which the block length is a power of ‘2’ ($N = 2^n$) and the encoding ($u^N \rightarrow x^N$) is done as

$$x^N = u^N G_N, \quad (3.19)$$

where G_N is the generator matrix, and u^N is data row vector of N binary elements. G_N is obtained as

$$G_N = B_N F^{\otimes n}, \quad (3.20)$$

where B_N is a bit-reversal matrix and $F^{\otimes n}$ is the n^{th} Kronecker power of matrix $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. It is to be noted that matrix F is mathematical equivalent of the single step polar transform shown in Figure 3.1. Let $\mathcal{I}$ be an arbitrary subset of $\{0, 1, \dots, N - 1\}$. Later we will show how to select this subset. Next, we define another subset of $\{0, 1, \dots, N - 1\}$, called the complementary set $\mathcal{I}^C \triangleq \{0, 1, \dots, N - 1\} \setminus \mathcal{I}$. Mapping 3.19 can then be rewritten as

$$x^N = u_{\mathcal{I}} G_N[\mathcal{I}] + u_{\mathcal{I}^C} G_N[\mathcal{I}^C], \quad (3.21)$$

where by sub-matrix $G_N[\mathcal{I}]$ we mean a matrix formed by keeping the rows of G_N whose indices are in $\mathcal{I}$. We fix $\mathcal{I}$ and by setting $u_{\mathcal{I}^C}$ to zeros, we obtain a linear mapping from the binary data vectors $u_{\mathcal{I}}$ of length $K \triangleq |\mathcal{I}|$ to codewords of length N . This mapping is generally called G_N -coset-code. The rate of this code is $R = \frac{K}{N}$. $\mathcal{I}$ is referred to as “information set” and it consist of indices of

good channels and the bits $u_{\mathcal{I}^c} \in \mathbb{F}_2^{N-K}$ are called frozen bits. A particular G_N -coset-code is identified by a set of parameters $(N, K, \mathcal{I}, u_{\mathcal{I}^c})$. For a polar code, $\mathcal{I}$ consist of all those subchannels which have lowest $Z(W)$ for a given channel SNR and code rate.

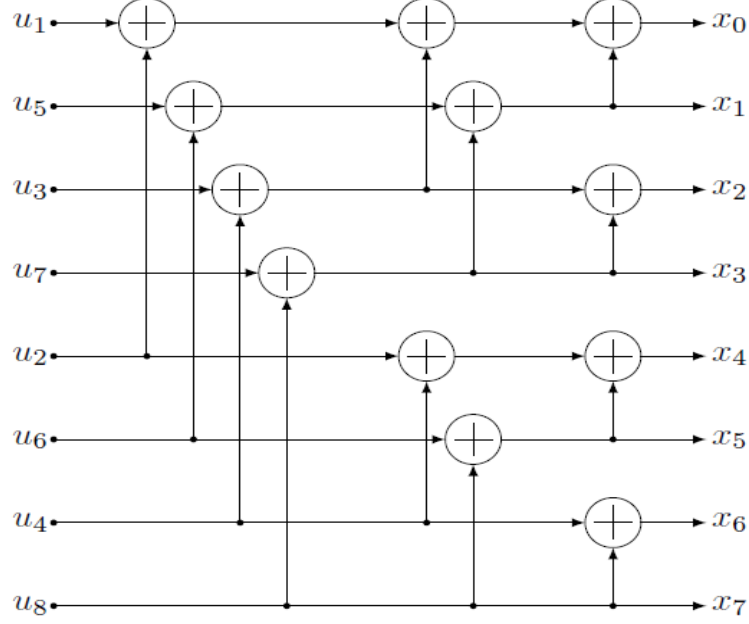


Figure 3.2: Factor graph representation of an 8-bit polar encoder

Factor graph representation of an 8-bit polar encoder is shown in Figure 3.2, where $\oplus$ symbol is binary XOR operation. The complexity of polar encoder is clearly $\mathcal{O}(N \log N)$.

3.3.2 Polar Decoders

Polar codes are soft codes, *i.e.*, they use channel likelihoods to decode the data. Surely they are more complex than hard decision codes. Main feature of polar codes unlike other random codes is their regular and recursive structure which can help in reducing hardware complexity. Error performance can further be improved by using ML decoding but it has higher complexity for large block lengths. With the addition of CRC, polar codes are very robust against channel

impairments. A few polar decoders are discussed here with their merits and demerits.

A polar encoder with parameters $(N, K, I, u_{\mathcal{I}c})$, encodes a data vector (binary vector $u_{\mathcal{I}}^K$) to the code vector x^N as we described in the previous sections. After passing through the channel this codeword is received as a channel output vector y^N , which is related to the transmitted codeword x^N by transition probability $W_N(y^N|x^N)$ of the channel. Decoder reproduces the transmitted codeword x^N and it already knows the code parameters. Due to 1 – 1 correspondence between code vector x^N and data vector u^N , decoder's task is equivalent to reproduce a best estimate of u^N , which is denoted by $\hat{u}^N$. Since the decoder knows $u_{\mathcal{I}c}$, the problem is reduced to estimating $u_{\mathcal{I}}^K$.

3.3.3 Maximum Likelihood Decoder

Decoding of codes is a search problem such that the goal is to reconstruct data vector u^N from noisy channel output vector y^N . For polar codes, the search space is $\mathcal{C}(N, 2^K)$ and it consists of all possible codewords. Frozen bits are the cause of relatively smaller cardinality of this code space.

Calculating likelihoods of all possible codewords and finding the most probable codeword can be the optimum method that minimizes the block error probability defined as $P_e = P\{\hat{u}_{\mathcal{I}} \neq u_{\mathcal{I}}\}$. This method is called ML decoding. Although ML decoding provides best error correction performance, exponential complexity makes this method impractical to implement. This method can be used only for short polar codes [36]. Other decoding methods are variants of the optimal ML decoding, hence naturally they are suboptimal.

3.3.4 Successive Cancellation Decoder

Successive cancellation decoder proposed by [5], is a sub-optimal ML decoder and is mathematically formalized as

$$\hat{u}_i(y^N, \hat{u}^{i-1}) = \begin{cases} u_i, & \text{if } i \in \mathcal{I}^c \\ 0, & \text{if } i \in \mathcal{I} \text{ and } \log \left(\frac{W_N^{(i)}(y, \hat{u}^{i-1}|0)}{W_N^{(i)}(y, \hat{u}^{i-1}|1)} \right) \geq 0 \\ 1, & \text{otherwise.} \end{cases} \quad (3.22)$$

Following algorithm describes SC decoding.

Algorithm 1: Successive cancellation decoding

Input: $y^N, (N, K, \mathcal{I}, u_{\mathcal{I}^c})$
Output: estimated information bits, $\hat{u}_{\mathcal{I}}$

```

1 begin
2   for  $i \leftarrow 1$  to  $N$  do
3     if  $i \in \mathcal{I}^c$  then
4        $\hat{u}_i \leftarrow u_i$ 
5     else
6       if  $\log \left( \frac{W_N^{(i)}(y^N, \hat{u}^{i-1}|0)}{W_N^{(i)}(y^N, \hat{u}^{i-1}|1)} \right) \geq 0$  then
7          $\hat{u}_i \leftarrow 0$ 
8       else
9          $\hat{u}_i \leftarrow 1$ 
10  return  $\hat{u}_{\mathcal{I}}$ 

```

Here we briefly explain why SC decoding is a sub-optimal version of ML decoding. We will analyze ML decoding and SC decoding as search path algorithms. SC decoding, instead of searching for all possible codewords as is done in ML decoding, searches for the current best decoding path. At each i^{th} decision step, which decides a new branch of the path, there are two decision candidates: $\hat{u}_i = 0$ or $\hat{u}_i = 1$. The decision between these two is made on the basis of the channel observation vector and all previously estimated information bits. Unlike ML decoding, in SC decoding the decoder at i^{th} step is not allowed to change its previous decisions to maximize the likelihood of the codeword. This restriction reduces

error correction performance of the SC decoding especially in low SNR regions, but the decoder has relatively manageable complexity, $\mathcal{O}(N \log N)$.

3.3.5 LLR based SC Decoder for AWGN Channels

Our study of polar codes in this thesis is restricted to performance analysis of SC decoders in optical communication systems. We assume that optical channels are Gaussian, therefore in this section, we specifically discuss SC decoding in AWGN environment. Figure 3.3 graphically shows how log likelihood ratios (LLRs) are calculated at decision nodes, starting from the channel LLRs, through “f” and “g” functions in a recursive manner. Following steps summarize a polar code communication system under SC decoding for an AWGN channel.

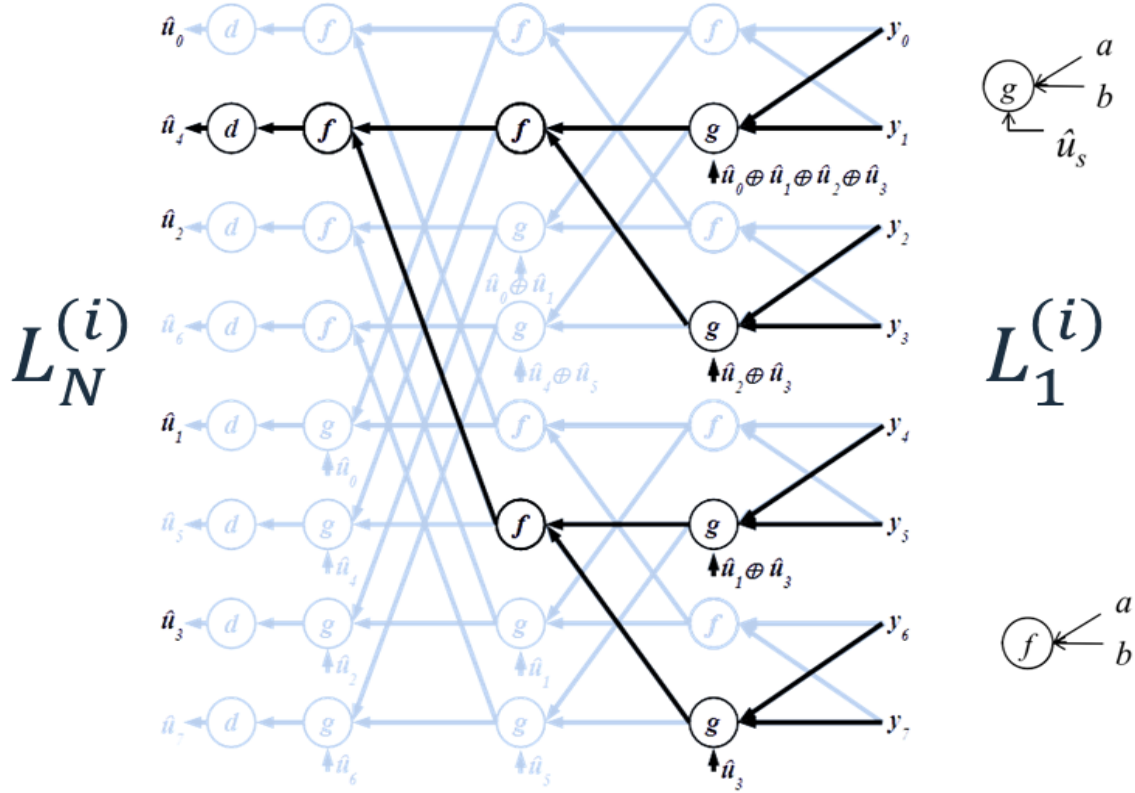


Figure 3.3: Decoding bit $\hat{u}_4$ by a successive cancellation decoder

1. Polar encoding: $x^N = u^N G_N$, where G_N : polar code generator matrix.
2. BPSK modulation of x^N : $\{0 \rightarrow +1, 1 \rightarrow -1\}$.
3. AWGN channel: $y^N = x^N + z^N$, where each $z_i \sim \mathcal{N}(0, \sigma^2)$.
4. Calculating channel LLR values: $L_1^{(i)} = a_W^{(i)} = \frac{2y_i}{\sigma^2}$.
5. Calculating LLR values at intermediate and decision nodes : Starting from the right, LLR values at different nodes are calculated using functions

$$f : f(a, b) = 2 \tanh^{-1}((\tanh(\frac{a}{2}) \tanh(\frac{b}{2})), \quad (3.23)$$

$$g : g(a, b, \hat{u}_s) = (-1)^{(\hat{u}_s)} a + b. \quad (3.24)$$

6. Hard decision rule at decision nodes:

$$\hat{u}_i(y^N, \hat{u}^{i-1}) = \begin{cases} u_i, & \text{if } i \in \mathcal{I}^c \\ 0, & \text{if } i \in \mathcal{I} \text{ and } L_N^{(i)}(y_1^N, \hat{u}_1^{i-1}) \geq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Deciding $\mathcal{I}$ (the set of good channels) is the problem of Polar code construction.

3.3.6 List Decoder based on SC

Tal and Vardy have proposed the SC list decoding algorithm to enhance the performance of SC decoding [35]. SC list (SCL) decoding enables tracking L best decoding paths in parallel at each decoding stage, unlike SC decoder which tracks single current best decoding path and hence it is not able to change previous estimates. List decoding is more closer to ML decoding and can change a limited number of the previous decisions, depending on the size of the list, to maximize the likelihood of the codeword. If L is large enough, ML decoding performance can be achieved, as sufficient number of decoding paths are visited. Its time complexity is $\mathcal{O}(L N \log N)$, which limits its application to moderate block lengths only.

A high level description of the SCL algorithm is shown in Algorithm 2. The SCL algorithm takes the received channel output vector y^N , code block length N , information set $\mathcal{I}$, the frozen bit vector $u_{\mathcal{I}^c}$ and the maximum list size L as inputs and decides the estimated information bits $\hat{u}_{\mathcal{I}}$ as output. The “current list size” variable, L_c is set to 1 at the initialization of the algorithm. If the i^{th} decision (hard estimate of bit u_i) belongs to frozen set $\mathcal{I}^c$, the i^{th} hard decision of all L lists are updated with the frozen decision, u_i . If it is a free decision, the decoder first checks whether the current list size L_c is equal to the maximum list size L . If not equal, the current list size is doubled and decoder can track likelihoods of both possible decisions (0 or 1). In case of all lists are exhausted, the decoder sorts $2L$ likelihoods and continue with the best L decoding paths. At the last decision stage, the decoder outputs free bits from the best list (path) as $\hat{u}_{\mathcal{I}}$. More detailed explanation of the algorithm is listed in [35].

Algorithm 2: Successive cancellation list decoding

Input: $y^N, (N, K, \mathcal{I}, u_{\mathcal{I}^c})$
Input: maximum list size, L
Output: estimated information bits, $\hat{u}_{\mathcal{I}}$
Variable: $L_c \leftarrow 1$ //current list size

```

1 begin
2   for  $i \leftarrow 1$  to  $N$  do
3     if  $i \in \mathcal{I}^c$  then
4       for  $l \leftarrow 1$  to  $L_c$  do
5          $\hat{u}_{l,i} \leftarrow u_i$ 
6     else
7       if  $L_c \neq L$  then
8         for  $l \leftarrow 1$  to  $L_c$  do
9            $\hat{u}_{l,i} \leftarrow 0$ 
10           $\hat{u}_{(l+L_c),i} \leftarrow 1$ 
11           $L_c \leftarrow 2L_c$ 
12       else
13          $s \leftarrow \text{sort } (W_N^{(i)}(y_1^N, \hat{u}^{i-1} | \hat{u}_i^L))$ 
14         for  $l \leftarrow 1$  to  $L_c$  do
15            $\hat{u}_{l,i} \leftarrow s_l$ 
16   return  $\hat{u}_{\mathcal{I}}$ 

```

3.3.7 Systematic Polar Codes

Originally, polar codes were introduced as non-systematic linear block codes. Since any linear code can be converted into a systematic code, polar code is no exception. Arikan [37] gives methods for encoding and decoding systematic polar codes that guarantee the FER of non-systematic polar codes while preserving the same low-complexity. In systematic codes, data vector $u_{\mathcal{I}}^K$ appear verbatim in codeword x^N . systematic Polar codes are described in detail in [37].

3.3.8 Systematic Polar Encoder

A two phase method of polar encoding is discussed in [9]. It is described as below.

1. Initialize data vector u^N such that u_i contain the data bits to be encoded for $i \in \mathcal{I}$, and $u_i = 0$ for $i \in \mathcal{I}^c$.
2. Obtain the intermediate vector as $d^N = u^N G_N$.
3. Set all d_i for $i \in \mathcal{I}^c$ to 0. Call this modified vector v^N .
4. Obtain the systematic code vector as $x^N = v^N G_N$.

3.3.9 Systematic Polar Decoder

In case of both systematic and non-systematic polar decoding, the decoder takes in $(y^N, \mathcal{I}, u_{\mathcal{I}^c})$ as input and produces $\hat{u}^N$ as an estimate to u^N . For non-systematic decoding the decoder stops after putting out $\hat{u}_{\mathcal{I}}$. For systematic decoding, the decoder has to perform an extra step of computing $x^N = u^N G_N$ as an estimate of x^N , and produces $\hat{x}_{\mathcal{I}}$ as output. In intermediate step of systematic decoding we can use any decoder like SC, ML or list decoder *etc.*

3.4 Polar Code Construction

Polar codes are channel specific [5]. As we know that channel polarization synthesizes, out of N independent copies of a given B-DMC $\mathbf{W}$, a second set of N binary-input channels $\{W_N^{(i)} : 1 \leq i \leq N\}$, such that as N becomes large, the fraction of indices i for which $I(W_N^{(i)})$ is near 1 approaches $I(W)$ and the fraction for which $I(W_N^{(i)})$ is near 0 approaches $1 - I(W)$. Hence some synthetic channels have arbitrarily low probability of error ($\hat{u}_i \neq u_i$), while others have higher probability of error. We need to send data at rate 1 through those with capacity near 1 (good channels) and at rate 0 through the remaining (bad channels). Identifying good and bad channels in the synthesized channels is a problem associated with code construction. Set of indices corresponding to good channels is represented by $\mathcal{I}$ and its cardinality is K , while set of frozen channels is represented by $\mathcal{F}$ or $\mathcal{I}^c$.

Construction of polar codes is dependent on the type of the decoder used. Polar code construction in this thesis is based on SC decoder. In context of polar codes, two parameters are very useful in indicating the quality of a channel, one is Bhattacharyya parameter, denoted by $Z(W)$ and the other one is probability of error of the channel. In fact both these parameters are closely related and will be discussed in the following sections.

3.4.1 Construction using BEC Approximation

Bhattacharyya parameter is helpful in indicating whether a given channel W is good or bad for communication. Bhattacharyya parameter is denoted by $Z(W)$ for a channel W . $Z(W)$ is defined in this chapter by Equation 3.4. Z parameter is related to quality/reliability of the channel, as it will become clear shortly.

We define an event that first bit error occurs at i^{th} subchannel as

$$\mathcal{B}_i := \{u_1^N, y_1^N, c_1^N | \hat{u}_1^{i-1} = u_1^{i-1}, \hat{U}_i(y_1^N, \hat{u}_1^{i-1}) \neq u_i\}, \quad (3.25)$$

where $c_1^N \in \{0, 1\}^N$ denote N independent fair coin flips, with c_i being used as

the decoding result of u_i if the two likelihood values for u_i are equal. Since $\{\mathcal{B}_i\}$ represents first bit error at i^{th} position for SC decoding, hence all previous bits are decoded correctly each with a probability between 0 and 1 inclusively while non-information bits are set to zeros, hence we can write

$$\mathcal{B}_i = \{u_1^N, y_1^N, c_1^N | \hat{u}_1^{i-1} = u_1^{i-1}, \hat{U}_i(y_1^N, u_1^{i-1}) \neq u_i\}. \quad (3.26)$$

We define another bit error event

$$\mathcal{A}_i := \{u_1^N, y_1^N, c_1^N | \hat{U}_i(y_1^N, u_1^{i-1}) \neq u_i\}. \quad (3.27)$$

The data bits are decoded in a sequential manner in the ascending order of their indices, via ML decoding of the channel $W_N^{(i)}$ such that the probability of correct decoding at stage i is maximized, hence the decoding is ML, not SC. In a nutshell, non-information bits are not set to 0 by brute force in ML decoding unlike SC decoding. We define the probability $P(\mathcal{A}_i)$ as the error probability of the ML decoding over the synthesized subchannel $W_N^{(i)}$. Similarly $P(\mathcal{B}_i)$ gives probability of error for a given synthetic subchannel for SC decoding. We have following obvious relationship [38]

$$\mathcal{B}_i \subseteq \mathcal{A}_i. \quad (3.28)$$

Hence we can write

$$P(\mathcal{B}_i) \leq P(\mathcal{A}_i). \quad (3.29)$$

Block error rate for a SC decoder is given as

$$P(\mathcal{E}) = \sum_{i \in \mathcal{I}} P(\mathcal{B}_i). \quad (3.30)$$

We have not discussed yet how to construct set $\mathcal{I}$. The most obvious and optimal strategy it is to design $\mathcal{I}$ such that it minimizes $\sum_{i \in \mathcal{I}} P(\mathcal{B}_i)$ under SC decoding. $P(\mathcal{B}_i)$ is a true measure of the quality of i^{th} subchannel under SC decoding. It represents the BER of that subchannel for successive cancellation decoder. Having calculated $P(\mathcal{B}_i)$ we can construct the code by selecting those subchannels with lowest $P(\mathcal{B}_i)$. But unfortunately calculation of $P(\mathcal{B}_i)$ is difficult analytically.

For ML decoding we need to select $\mathcal{I}$ such that it minimizes $\sum_{i \in \mathcal{I}} P(\mathcal{A}_i)$. $P(\mathcal{A}_i)$ is a true measure of the quality of i^{th} subchannel under ML decoding. Having

calculated $P(\mathcal{A}_i)$ we can construct the code by selecting those subchannels with lowest $P(\mathcal{A}_i)$ for maximum likelihood decoding.

It happens that calculation of $P(\mathcal{A}_i)$ is very easy as compared to $P(\mathcal{B}_i)$ analytically. Moreover, $P(\mathcal{A}_i)$ provides a tighter upper bound for $P(\mathcal{B}_i)$, *i.e.*, $P(\mathcal{B}_i) \leq P(\mathcal{A}_i)$. But the question one need to ask is whether selection of $\mathcal{I}$ for minimization of $\sum_{i \in \mathcal{I}} P(\mathcal{A}_i)$ for ML decoding is equivalent to selection problem of $\mathcal{I}$ for minimization of $\sum_{i \in \mathcal{I}} P(\mathcal{B}_i)$ for SC decoding. The answer is yes. If optimum polar code is constructed for ML decoding based on low $P(\mathcal{A}_i)$, it is also an optimum code for SC decoding which needs to be constructed on the basis of low $P(\mathcal{B}_i)$.

For SC decoder

$$P(\mathcal{E}) = \sum_{i \in \mathcal{I}} P(\mathcal{B}_i) \leq \sum_{i \in \mathcal{I}} P(\mathcal{A}_i). \quad (3.31)$$

For BSC, Arıkan has provided yet another bound on the error probability of the ML decoding $P(\mathcal{A}_i)$, which is simpler to evaluate but has its own limitations for arbitrary symmetric channels. Bhattacharyya parameter, an upper bound on $P(\mathcal{A}_i)$, coincides with $P(\mathcal{A}_i)$ only for BEC [5]. We have following important relationship [38]

$$P(\mathcal{B}_i) \leq P(\mathcal{A}_i) \leq Z(W_N^{(i)}). \quad (3.32)$$

$Z(W_N^{(i)})$ is an upper bound on the probability of ML decision error when synthetic subchannel $W_N^{(i)}$ is used only once to transmit a 0 or 1. In case of BSC, Z parameter for synthetic channel for an SC decoder is calculated as [5]

$$Z(W_N^{(i)}) = 2^{N-1} \cdot \sum_{y_1^N} \sqrt{W_N^{(i)}(y_1^N, 0_1^{i-1} | 0) W_N^{(i)}(y_1^N, 0_1^{i-1} | 1)}. \quad (3.33)$$

Since the output alphabet of $W_N^{(i)}$ is $\mathcal{Y}^N \times \{0, 1\}^i$, the cardinality of the output alphabet of the channels at the level n of the binary tree is doubly exponential in n or is exponential in the block-length [39]. So computing the exact transition probabilities of these channels seems to be intractable. Also we have the relationships

$$Z(W_{2N}^{(2i-1)}) \leq 2Z(W_N^{(i)}) - (Z(W_N^{(i)}))^2 \quad (3.34)$$

$$Z(W_{2N}^{(2i)}) = (Z(W_N^{(i)}))^2. \quad (3.35)$$

The above inequality becomes equality for BEC with $Z(W_1^1) = \epsilon$, erasure probability. Only for BEC above relations are recursive equations and can be easily computed in time $\mathcal{O}(N)$. For arbitrary BSC, the above inequality is indeterminate and hence recursive formulas cannot be used.

Polar codes are channel specific. One code may be good for one channel but not for the other. But as we have seen above, exact code calculation is very tough for channels other than BEC. Therefore the code construction method we consider for other BSC is actually approximating that channel as a BEC with erasure rate $1/2$, and it happens to work well [40].

Yet another approximate method suggested by Arikan called heuristic method: given an arbitrary binary-input channel with capacity C bits, use the polar code that is matched to the BEC with erasure rate $\epsilon = 1 - C$, *i.e.*, the BEC that has the same capacity as the given original channel. This rule has yielded good results in experiments [5].

3.4.2 Construction with Monte Carlo Simulations

Perhaps Monte Carlo simulations for code construction is the technique which gives the best polar code in terms of error performance. But unfortunately this technique is not an efficient one. It is viable only for low SNR and smaller block lengths. As block length and SNR increase, computer simulations take a lot of time, hence we have to switch to analytical methods of code construction as they are practically more feasible and less time consuming. Nevertheless, we cannot avoid numerical simulations for code construction as it gives us a reference to determine the accuracy of any analytical method of code construction. If the results of an analytical method are in agreement with that of simulations (at least in low SNR regions), we can infer that analytical method works well.

Basically, a MC simulation of any error correction code consists of generating the samples of noise and source information in a number of trials and measuring

the probability of an event of interest (event of bit error, $\mathcal{E}$) by counting the number of times that event occurred and dividing it by the total number of trials, M . Usually, the event of interest is a rare event. Hence, we must have sufficiently large number of trials in order to observe a certain number of its occurrences throughout the trials. This implies very long simulations especially when the event of interest is very rare, *e.g.*, in the case of optical channels where we have high SNR.

Numerical simulations, more specifically MC simulations consist of a software test bench properly designed according to the system model. For example in our case, we need a random binary data generator which produces a vector of N data bits in one iteration. Next we need polar encoder which inputs data vector and produces a code vector of N bits. Fortunately in our case, we do not need data generator and polar encoder as we can transmit all zero codeword for all zero data word. The reason for this is that we assume noise is symmetric and the fact that polar code is linear [38]. We apply BPSK modulation to our codeword, *i.e.*, simply to transmit bit 0 as +1 volt signal. Since we are dealing with Gaussian channels only, next we generate independent noise vector samples in each iteration according to Gaussian distribution $P(\cdot)$ for a given SNR. We add noise to BPSK codeword and give this signal to decoder.

Decoder implementation for the purpose of simulations is extremely simple. Since we have used all zero data word, so we know in advance that all partial sums used in the decoder will be zero in case of correct decoding. Hence we do not need to calculate partial sums at variable nodes of decoder. So our decoder can work in parallel and will be much faster. As it is apparent from our decoder structure, the decoding of bit i requires that we already know the correct bits up to $i - 1$. So if error occurs at i^{th} bit position, it is only due to the quality of this subchannel and not due to error propagation from previous decoded bits.

During one iteration of the above mentioned process we can find which bit indices (channels) results in error, *i.e.*, an error event $\mathcal{E}_i$ occurs for subchannel i . We can repeat this many times and for each bit index we need to record the number of times it resulted in error. Then simply by division of the number of

errors for a given bit index and the total number of iterations, M , we get $P(\mathcal{C}_i)$

$$P(\mathcal{C}_i) = \frac{1}{M} \sum_{j=1}^M 1_{\mathcal{E}_i}, \quad (3.36)$$

where $\mathcal{C}_i$ is an event that first bit error occurs at i^{th} subchannel while all previous bits are not obtained by decoding, instead they are provided correctly by a genie. Then for a given code rate R , we need to select all those subchannels with lowest $P(\mathcal{C}_i)$.

We have another important relationship as

$$P(\mathcal{B}_i) = P(\mathcal{C}_i) \prod_{j=1}^{i-1} (1 - P(\mathcal{C}_j)), \quad (3.37)$$

where we set $P(\mathcal{C}_i) = 0$ by brute force for $i \in \mathcal{I}^c$.

3.4.3 Construction using Density Evolution

The code construction method using density evolution algorithm was proposed in [41] for LDPC codes under message passing decoding (belief propagation) algorithm. This algorithm iteratively calculates message densities. With some approximations, density evolution algorithm can be implemented very efficiently. In this section we will see whether this algorithm can be applied to SC decoding of polar codes or not. We start with repeating the following definition of a bit error event under ML decoding

$$\mathcal{A}_i := \{u_1^N, y_1^N, c_1^N | \hat{U}_i(y_1^N, u_1^{i-1}) \neq u_i\}. \quad (3.38)$$

As discussed earlier, the probability $P(\mathcal{A}_i)$ is the error probability of the ML decoding over the synthesized subchannel $W_N^{(i)}$.

Density evolution is a concept from LDPC codes, which can be represented on a tree graph [41]. Error probability of a root node of a tree graph after message passing decoding is calculated via density evolution. we need to show that $P(\mathcal{A}_i)$ can be regarded as decoding error probabilities of belief propagation

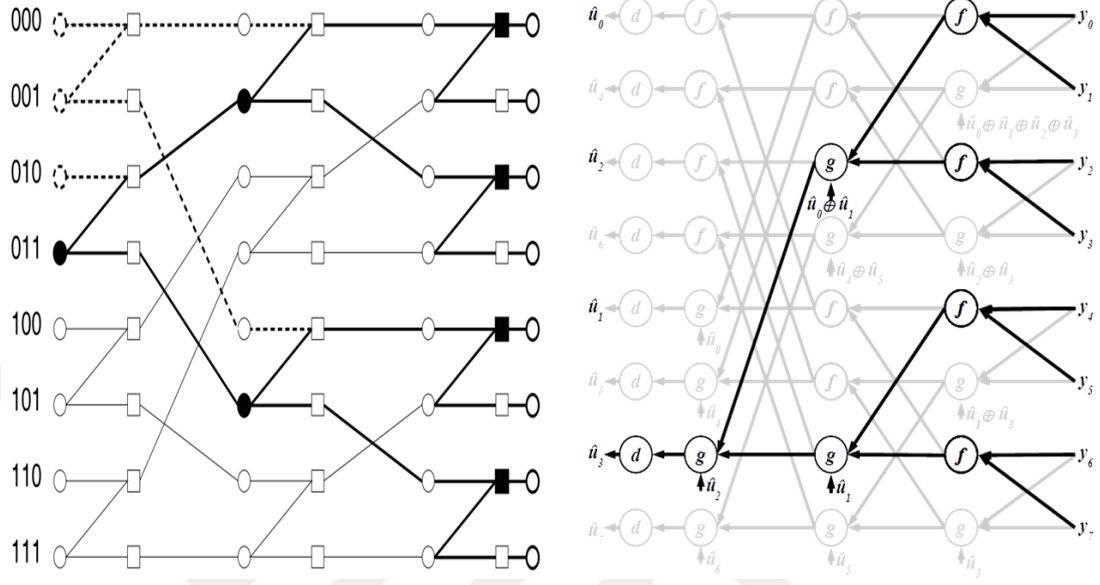


Figure 3.4: Decoding tree for $n = 3$, $i = 4$

(BP) decoding. Fortunately, SC decoding can be represented on a tree graph, see Figure 3.4. It shows the decoding tree for $n = 3$ and $i = 4$, where “f” and “g” functions are transformations at check and variable nodes, respectively [37]. Mori et al. have shown that $P(\mathcal{A}_i)$ can be considered as decoding error probabilities of BP decoding on a tree graph by arguing that the transformations performed by the SC decoding algorithm are essentially the same as those in the case of BP decoding [38]. Hence we can directly evaluate $P(\mathcal{A}_i)$ via density evolution. Once having obtained $P(\mathcal{A}_i)$ for all subchannels, one can select subchannels $W_N^{(i)}$ with lowest $P(\mathcal{A}_i)$. Since the Bhattacharyya parameter coincides with $P(\mathcal{A}_i)$ if and only if the channel is the BEC, the following result of density evolution is the same as the result for the BEC, shown by Arıkan.

Density evolution algorithm works as follows. To calculate $P(\mathcal{A}_i)$, we assume all-zero codeword transmitted as we have assume that channel $\mathbf{W}$ is symmetric. Let $a_N^{(i)}$ represent $L_N^{(i)}(Y_1^N, 0_1^{i-1})$, the following recursive equations are used to update the message densities

$$a_{2N}^{(2i)} = a_N^{(i)} \star a_N^{(i)} \quad (3.39)$$

$$a_{2N}^{(2i-1)} = a_N^{(i)} \boxtimes a_N^{(i)} \quad (3.40)$$

$$a_1^{(1)} = a_W. \quad (3.41)$$

where a_W is the probability density function of LLR of the original channel W when 0 is transmitted, and where $\star$ and $\boxtimes$ are the convolution operations in a variable node domain and a check node domain, respectively as given in [42]. After a_N^i are computed, $P(\mathcal{A}_i)$ is calculated via integration of a_N^i over interval $(-\infty, 0)$. Number of convolutions for calculating $P(\mathcal{A}_i)$ is $\mathcal{O}(N)$ [43]. The main problem with this construction is that it is not clear how one would implement such convolutions to be sufficiently precise on one hand while being tractable on the other hand. Therefore its very tough to use this method without any approximation as computation complexity is enormous. In the following section, we simplify the density evolution method by using Gaussian approximation.

3.4.4 Gaussian Approximation for Density Evolution

Density evolution is computationally intensive for channels other than BEC. For BEC DE is one dimensional problem. For more interesting channels like AWGN, DE becomes too complicated because of the implementation of convolutions. For AWGN channels, Chung et al. have used Gaussian approximation for DE for the construction of LDPC codes [43]. This approximation makes DE extremely tractable for very little sacrifice in accuracy. This method makes an infinite dimensional problem of iteratively calculating message densities a one dimensional problem by approximating message densities as Gaussians and one need to update only the “means” of these Gaussian densities.

GA for DE can be used for the construction of polar codes under SC decoding for AWGN channels [16][38]. Without loss of generality, for BSC it is assumed that all-zero codeword is transmitted with BPSK modulation ($0 \rightarrow +1, 1 \rightarrow -1$). The likelihood ratio $L_1^{(i)}$ for AWGN corrupted observation y_i is approximated to have Gaussian density

$$L_1^{(i)}(y_i) = \left(\frac{2y_i}{\sigma^2} \right) \sim \mathcal{N} \left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2} \right), \quad (3.42)$$

where σ^2 is the variance of AWGN.

This one dimensional problem due to GA requires us to update the means of

the intermediate message densities through the following recursive equations [16]

$$\mathbf{E}[L_n^{(2i-1)}] = \phi^{-1} \left(1 - \left(1 - \phi \left(\mathbf{E}[L_{n/2}^{(i)}] \right) \right)^2 \right) \quad (3.43)$$

$$\mathbf{E}[L_n^{(2i)}] = 2\mathbf{E}[L_{n/2}^{(i)}], \quad (3.44)$$

where

$$\phi(x) = \begin{cases} 1 - \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} \tanh \frac{u}{2} e^{-\frac{(u-x)^2}{4x}} du, & x > 0 \\ 1, & x = 0. \end{cases}$$

The error probability for each synthetic subchannel is given as

$$\pi_i = Q \left(\sqrt{\frac{\mathbf{E}[L_n^{(i)}]}{2}} \right), \quad 1 \leq i \leq N. \quad (3.45)$$

Having calculated π_i , we can easily decide about good channels and bad channels.

Even though GA has made DE simpler but $\phi(x)$ and $\phi^{-1}(x)$ are computationally inefficient functions. There are some efficient approximations for their implementations.

3.4.4.1 Chung Approximation:

An efficient approximation to the above mentioned $\phi(x)$ function is given in [43], we call it “Chung” approximation and is given as.

$$\phi_{\text{Chung}}(x) = \begin{cases} e^{-0.4527x^{0.86}+0.0218}, & 0 < x < 10 \\ \sqrt{\frac{\pi}{x}} e^{-\frac{x}{4}} \left(1 - \frac{10}{7x} \right), & x \geq 10. \end{cases}$$

This is a two-segment approximation to $\phi(x)$ and works well for polar codes with moderate block lengths.

3.4.4.2 Niu Approximation:

There is an intrinsic error in Chung approximation due to following discontinuity in it and may result in catastrophic loss of performance in construction of polar

codes for large block lengths [44]

$$\lim_{x \rightarrow 0} \phi_{\text{Chung}}(x) = e^{0.218} > \phi(0) = 1. \quad (3.46)$$

This discontinuity results in a difference between $\phi_{\text{Chung}}(x)$ and $\phi(x)$ which produces calculation error in π_i .

Another approximation for $\phi(x)$ is proposed in [44]. It is a three-segment approximation and is efficiently implementable, we call it “Niu” approximation. It is claimed to work well for polar codes with larger block lengths.

$$\phi_{\text{Niu}}(x) = \begin{cases} e^{0.06725x^2 - 0.4908x}, & 0 < x \leq 0.6357 \\ e^{-0.4527x^{0.86} + 0.0218}, & 0.6357 < x \leq 9.2254 \\ e^{-0.2832x - 0.4254}, & 0.6357 < x. \end{cases}$$

Figure 3.5 shows BER characteristic for polar codes with different block lengths. DE-GA is used to construct these polar codes and estimated the error performance. As polar codes are channel specific, hence construction is optimized for all SNR points to get the best performance over all SNR values. Practically, we construct polar codes for a fixed given SNR and use the same code even if line SNR changes, which results in performance degradation. The reason for using fixed code is the complexity associated with updating code construction for varying line SNR. In optical communications, we want to operate at a BER of around 10^{-12} . For this target BER, our optimized code construction method, can help to determine the SNR at which code should be constructed. Table 3.1 lists these SNR values for polar codes of three different block lengths for a target BER of 10^{-12} .

Table 3.1: SNR values for polar code construction for a target BER of 10^{-12}

Polar code	SNR(dB) with Chung	SNR(dB) with Niu
Polar (2040, 1912)	10.75	10.64
Polar (32640, 30592)	8.93	8.83
Polar (130560, 122368)	8.31	8.22

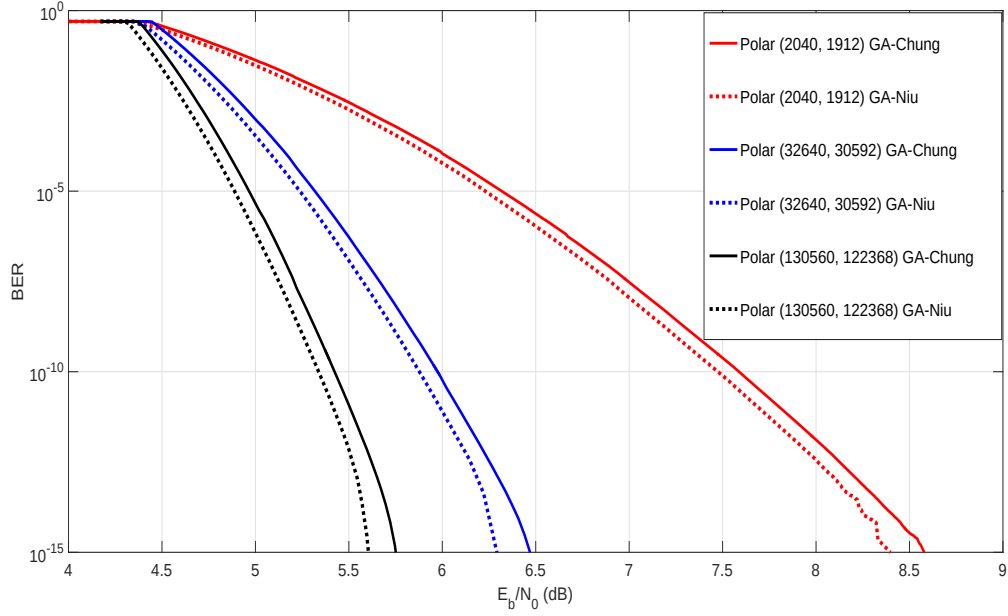


Figure 3.5: Error performance of polar codes estimated using DE with GA (Chung vs. Niu approximations for $\phi(x)$)

Figure 3.5 shows that there is a performance difference of almost 0.10 dB, 0.11 dB and 0.11 dB with Chung and Niu approximations for Polar (2040, 1912), Polar (32640, 30592) and Polar (130560, 122368) respectively at a BER of 10^{-12} . It confirms the claim by [44] that with increase in block length of the polar codes, Niu and Chung approximations will yield more difference in performance. But we are yet to determine which one is more accurate.

Figure 3.6 compares simulation and DE-GA results for Polar (2040, 1912). Code is optimized for an SNR of 10.5 dB. Figure 3.6 shows that simulation results are in agreement with those obtained through DE. Both Niu and Chung approximations give close results, but the plot is not convincing enough to decide which one is better. In this thesis, we mostly rely on Chung approximation as it gives good upper bound.

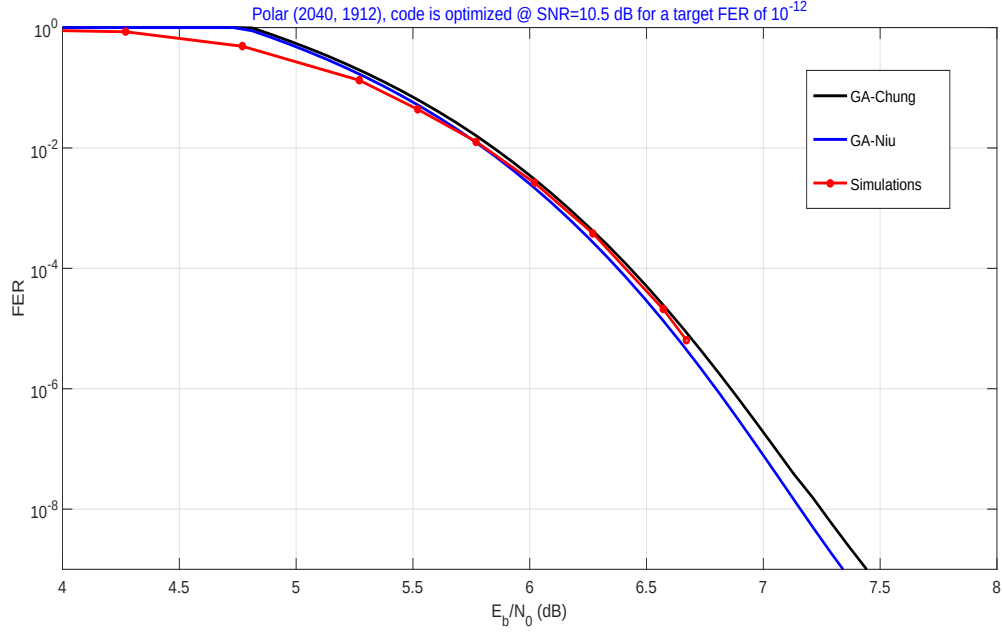


Figure 3.6: Comparison of Gaussian approximation and simulation results

3.5 Error Performance of Polar Codes

Polar codes can achieve a code rate equal to capacity of a BS channel asymptotically. It means that all subchannels either become perfectly good or perfectly bad. Hence we can transmit on good channels and freeze bad channels. At finite block lengths, some synthesized subchannels are very good and some are very bad, yet there are some channels which are in between good and bad. Hence capacity cannot be achieved at finite lengths. In order to keep the rate of the code sufficiently higher, we have to use some of these “in between” channels, which result in error. Therefore, we need to find a bound on error for polar codes as error bounds are always useful for any FEC code to be deployed in a practical communication system because we need to be sure about error rate beforehand. In this section we give upper bounds on the error performance of polar codes. We will be concerned mainly about upper bounds of FER/BLER for polar codes with a given construction. A decoder block error means vector $\hat{u}_{\mathcal{I}} \neq u_{\mathcal{I}}$, where $\mathcal{I}$ indicates set of information subchannels.

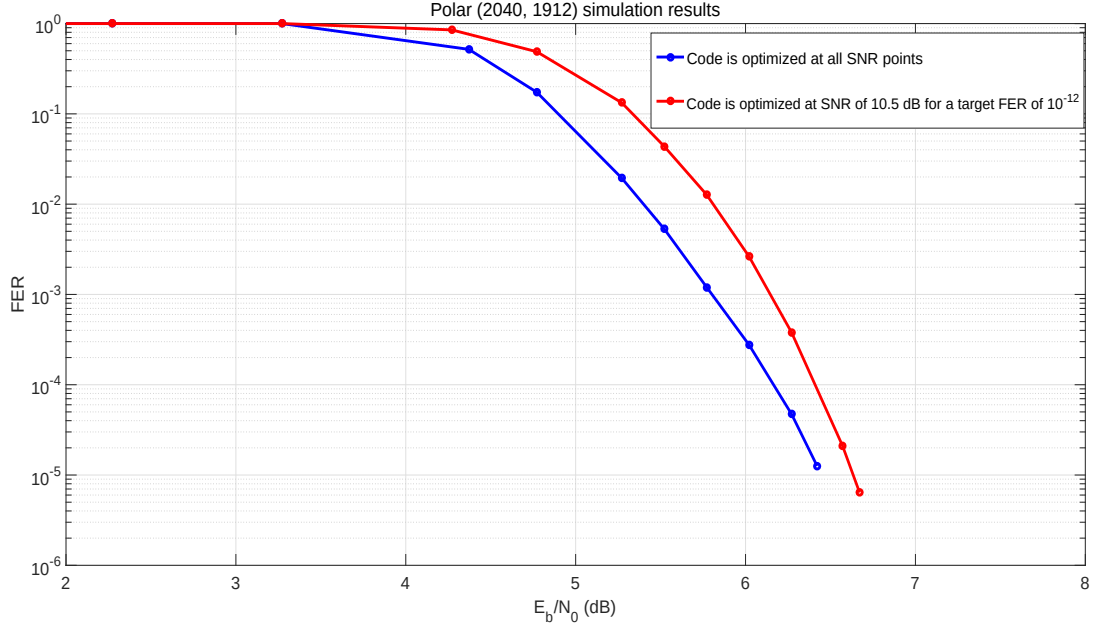


Figure 3.7: FER plots for Polar (2040, 1912) obtained through MC simulations

3.5.1 Error Performance using Monte Carlo Simulations

MC simulations are always beneficial as they give some idea about the performance of a code. Hence it gives a benchmark to determine the accuracy of analytical methods by the comparison of both results. Setup for MC simulations of polar codes is discussed in Section 3.4.2. Its merits and demerits are discussed in detail over there. Using the mentioned setup, we get $P(\mathcal{B}_i)$ for all subchannels. After deciding $\mathcal{I}$, we can calculate FER, $P(\mathcal{E})$ by using the summation

$$P(\mathcal{E}) = \sum_{i \in \mathcal{I}} P(\mathcal{B}_i). \quad (3.47)$$

Figure 3.7 shows simulation results for Polar (2040, 1912). These results are in agreement with those in [9].

3.5.2 Error Bound using Density Evolution

For SC decoding, the FER/BLER is upper bounded by the sum of the individual error probabilities of the ML decoding over synthesized subchannels $\{W_N^{(i)}\}$ such that the sum is over information bits only. An event of block error is expressed as a union of mutually-exclusive events $\{\mathcal{B}_i\}$. For a set of information bits $\mathcal{I}$, block error probability $P(\mathcal{E})$ is given as

$$P(\mathcal{E}) = \sum_{i \in \mathcal{I}} P(\mathcal{B}_i). \quad (3.48)$$

It is the exact block error rate for a given construction, but it is very difficult to calculate. We can obtain the upper bound as

$$P(\mathcal{E}) \leq \sum_{i \in \mathcal{I}} P(\mathcal{A}_i). \quad (3.49)$$

Procedure for calculation of $P(\mathcal{A}_i)$ is given in Section 3.4.3 using DE. Using DE-GA, BLER can be upper bounded as

$$P(\mathcal{E}) \leq \sum_{i \in \mathcal{I}} \pi_i = \sum_{i \in \mathcal{I}} Q \left(\sqrt{\frac{\mathbf{E}[L_n^{(i)}]}{2}} \right). \quad (3.50)$$

3.5.3 Error Bound using BEC Approximation

Arıkan [5] has proved in his seminal work that the sum of Bhattacharyya parameters for all information carrying subchannels provides an upper bound on the block error probability for a SC decoder, *i.e.*,

$$P(\mathcal{E}) \leq \sum_{i \in \mathcal{I}} Z(W_N^{(i)}). \quad (3.51)$$

Having calculated $Z(W_N^{(i)})$ for all information carrying subchannels, the upper bound can be easily found. As we know from the construction of polar codes with BEC approximation, calculation of $Z(W_N^{(i)})$ is easy only for BEC. For arbitrary BS channels, we have to use Arıkan's heuristic method, where polar codes are optimized for a hypothetical BEC with the capacity equal to that of the original

symmetric channel. By using this approximation we can use recursive equations for BEC to calculate $Z(W_N^{(i)})$ easily. Furthermore we have following important relationship

$$P(\mathcal{E}) = \sum_{i \in \mathcal{I}} P(\mathcal{B}_i) \leq \sum_{i \in \mathcal{I}} P(\mathcal{A}_i) \leq \sum_{i \in \mathcal{I}} Z(W_N^{(i)}), \quad (3.52)$$

which indicates that density evolution provides a tighter upper bound for FER as compared to BEC approximation.

3.6 Summary

Polar codes are the first provably capacity achieving codes under successive cancellation decoding. Their low complexity encoding and decoding algorithms make them very attractive for application in high speed optical communications. Successive cancellation decoder for polar codes provides a nice balance between error performance and complexity. In order to analyze its performance in very low BER region, we studied density evolution in this chapter for the construction of polar codes. We used Gaussian approximation to make density evolution simpler to implement. We found that Gaussian approximation can be used for error performance measurement of polar codes with sufficient accuracy.

Chapter 4

Polar Codes for Optical Communications

In this chapter, we analyze whether polar codes satisfy the requirements of next generation optical networks and compare its error performance with other state-of-the-art FEC codes proposed for optical communication systems.

4.1 Error Performance of Proposed Polar Codes

In this section, we study polar codes of specific block lengths which are compatible (both in rate and block length) with FEC schemes recommended by G.975 and G.975.1 for optical communications. We choose polar codes of four different block lengths for OTNs and analyze their error performances in comparison to standard FEC codes recommended for optical networks. All these polar codes have a fixed code rate of 0.9373 (overhead of 6.69%), which is the same as that of RS (255, 239) specified by G.975. We also analyze these polar codes as third generation FECs by using higher overhead of 20%.

In order to analyze error performance of these polar codes, we focus mainly

on their NCG values and compare it to the NCG value of standard RS (255, 239) code which can achieve an NCG of 5.62 dB at a BER of 10^{-12} . For all these polar codes, we specify a “minimum” NCG at an output BER of 10^{-12} and 10^{-15} . Since it is not possible to simulate performance of long polar codes at an output BER of 10^{-12} , we use density evolution with GA to evaluate error performances in very low BER regions. Density evolution facilitates to evaluate upper bound on FER of polar codes, but not BER. To calculate BER from FER, we assume worst case scenario such that $BER = \frac{FER}{2}$. That is the reason we refer to our calculated NCG value as the minimum NCG.

4.1.1 Polar (2040, 1912)

Polar (2040, 1912) is shortened form of Polar (2048, 1920), ($2048=2^{11}$). Shortening is achieved by freezing the last 8 most reliable bits of the block to zero. Polar (2040, 1912) is completely compatible with RS (255, 239) in terms of both code rate and frame length ($255*8=2040$ and $239*8=1912$). Hence it can use same ODU0 and OTU0 frames as standardized by G.975 for RS (255, 239) [2].

As shown in the Table 4.1, Polar (2040, 1912) can achieve a minimum NCG of 5.91 dB at a BER of 10^{-12} as compared to 5.62 dB for RS (255, 239). Our calculated value of NCG is very much in agreement with the one given by [9], which calculates an NCG of 6 dB by using a semi analytic approach.

Table 4.1: Error performance parameters for Polar (2040, 1912) code using Chung approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
2.81E-04	1.00E-12	5.91	6.19	10.75	8.03	10.75
1.18E-04	1.00E-15	6.41	6.69	11.31	8.58	11.31

Figure 4.1 shows improvement in error performance obtained by Polar (2040, 1912) as compared to RS (255, 239). Polar code is constructed/optimized for a target BER of 10^{-12} .

Table 4.2: Error performance parameters for Polar (2040, 1912) code using Niu approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
3.34E-04	1.00E-12	6.03	6.31	10.64	7.91	10.64
1.58E-04	1.00E-15	6.59	6.87	11.13	8.40	11.13

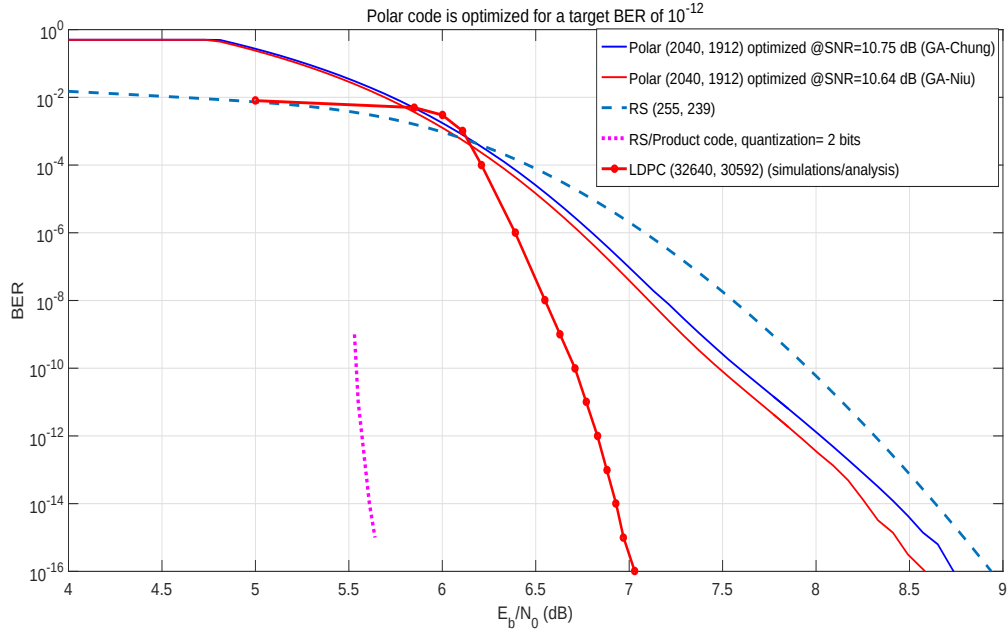


Figure 4.1: BER plot for Polar (2040, 1912) using density evolution, (Chung and Niu approximations)

All the performance parameters listed in Tables 4.1 and 4.2 are based on performance results of polar codes evaluated using density evolution with Gaussian approximation. To make sure that density evolution generates reliable results, we compare error performance of Polar (2040, 1912) evaluated with MC simulations and density evolution as shown in Figure 4.2. It is clear that density evolution results (both with Chung and Niu approximations) are in close agreement with the simulation results. These simulation results can be extrapolated to estimate the NCG value at a BER of 10^{-12} . Using this method, an NCG of 6.70 dB is estimated by [9]. Note that in Figure 4.2 polar code is optimized for every SNR point unlike Figure 4.1, where code is optimized only for one SNR point corresponding to a target BER of 10^{-12} .

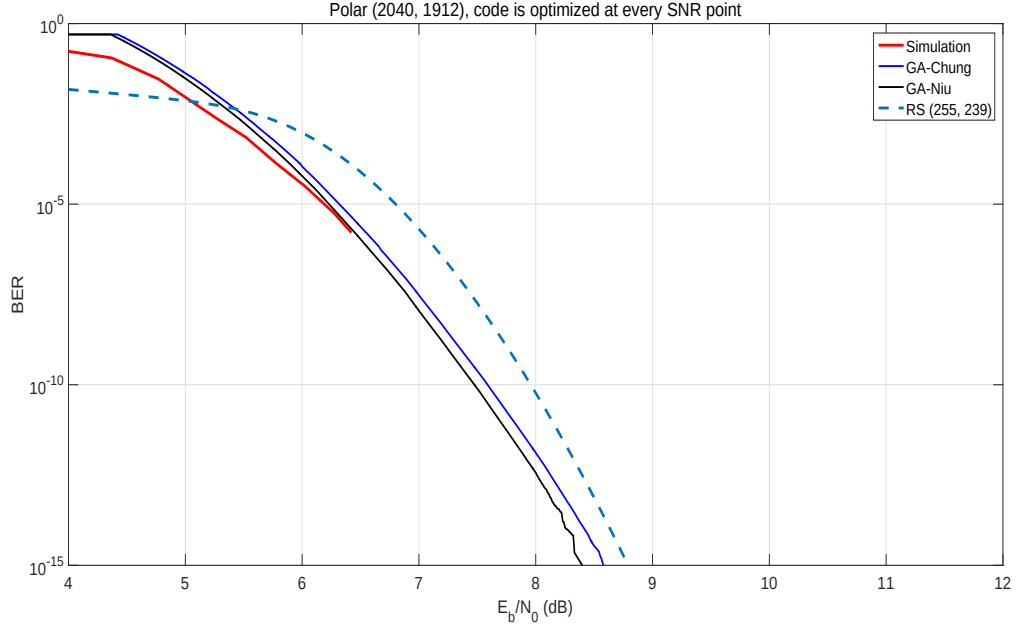


Figure 4.2: BER plot for Polar (2040, 1912) code, simulation vs. density evolution with Chung and Niu approximations

4.1.2 Polar (32640, 30592)

Polar (32640, 30592) is shortened form of Polar (32768, 30720), ($32768=2^{15}$). Shortening is achieved by freezing the last 128 most reliable bits of the block to zero. Polar (32640, 30592) is completely compatible with RS (2720, 2550) in terms of both code rate and frame length ($2720 \times 12 = 32640$ and $2550 \times 12 = 30592$). Hence it can use same ODU2 and OTU2 frames as standardized by G.975.1 for RS (2720, 2550) [3]. It is also compatible with LDPC super FEC.

As shown in Table 4.3, Polar (32640, 30592) can achieve a minimum NCG of 7.74 dB at a BER of 10^{-12} as compared to 5.62 dB for RS (255, 239), 7.06 dB for RS (2720, 2550) and 7.10 dB for LDPC super FEC. Our calculated value of NCG is very much in agreement with the one calculated by [9], which calculates an NCG of 7.5 dB by using a semi analytic approach.

Table 4.3: Error performance parameters for Polar (32640, 30592) code using Chung approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
2.60E-03	1.00E-12	7.74	8.02	8.93	6.20	8.93
1.96E-03	1.00E-15	8.52	8.80	9.20	6.47	9.20

Table 4.4: Error performance parameters for Polar (32640, 30592) code using Niu approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
2.87E-03	1.00E-12	7.84	8.12	8.83	6.10	8.83
2.37E-03	1.00E-15	8.70	8.98	9.02	6.29	9.02

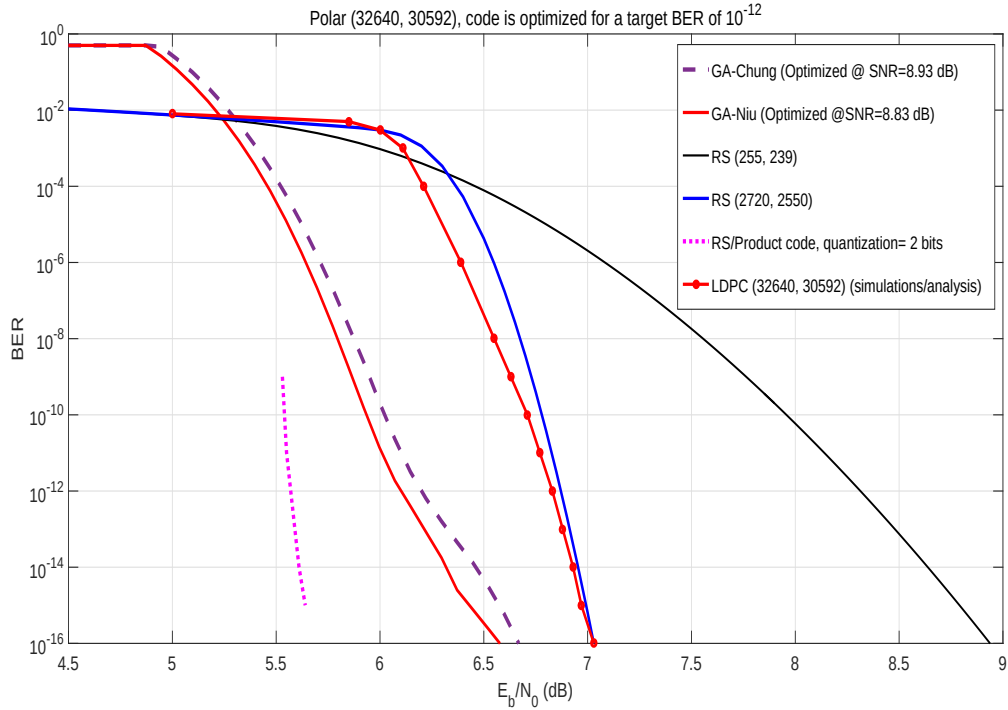


Figure 4.3: BER plot for Polar (32640, 30592) code using density evolution, (Chung and Niu approximations)

Figure 4.3 shows improvement in error performance obtained by Polar (32640, 30592) as compared to RS (255, 239) and RS (2720, 2550). Polar code is constructed/optimized for a target BER of 10^{-12} .

4.1.3 Polar (130560, 122368)

Polar (130560, 122368) is shortened form of Polar (131072, 122880), (131072= 2^{17}). Shortening is achieved by freezing the last 512 most reliable bits of the block to zero. Regarding block length compatibility of Polar (130560, 122368), we recall that concatenated RS/Product super FEC code specified in G.975.1 uses two ODU2 and two OTU2 frames per block and its length is twice as that of Polar (130560, 122368). Fortunately Polar (130560, 122368) is compatible with G.709 recommended frames and it can use one ODU2 and one OTU2 frame per block [3].

As shown in Table 4.5, Polar (130560, 122368) can achieve a minimum NCG of 8.35 dB at a BER of 10^{-12} as compared to 5.62 dB for RS (255, 239) and 7.06 dB for RS (2720, 2550). Only one FEC scheme from G.975.1 achieves higher NCG than Polar (130560, 122368), *i.e.*, concatenated RS/Product super FEC, which achieves an NCG of 8.40 dB but its block length is twice as that of Polar (130560, 122368). Our calculated value of NCG is very much in agreement with the one calculated by [9], which gives an NCG of 8.40 dB by using extrapolation of the simulation results.

Table 4.5: Error performance parameters for Polar (130560, 122368) code using Chung approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
4.61E-03	1.00E-12	8.35	8.63	8.31	5.58	8.31
3.97E-03	1.00E-15	9.24	9.52	8.48	5.75	8.48

Figure 4.4 shows improvement in error performance obtained by Polar (130560, 122368) as compared to RS (255, 239). Polar code is constructed/optimized for

a target BER of 10^{-12} .

Table 4.6: Error performance parameters for Polar (130560, 122368) code using Niu approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
5.00E-03	1.00E-12	8.44	8.73	8.22	5.49	8.22
4.54E-03	1.00E-15	9.39	9.67	8.33	5.6	8.33

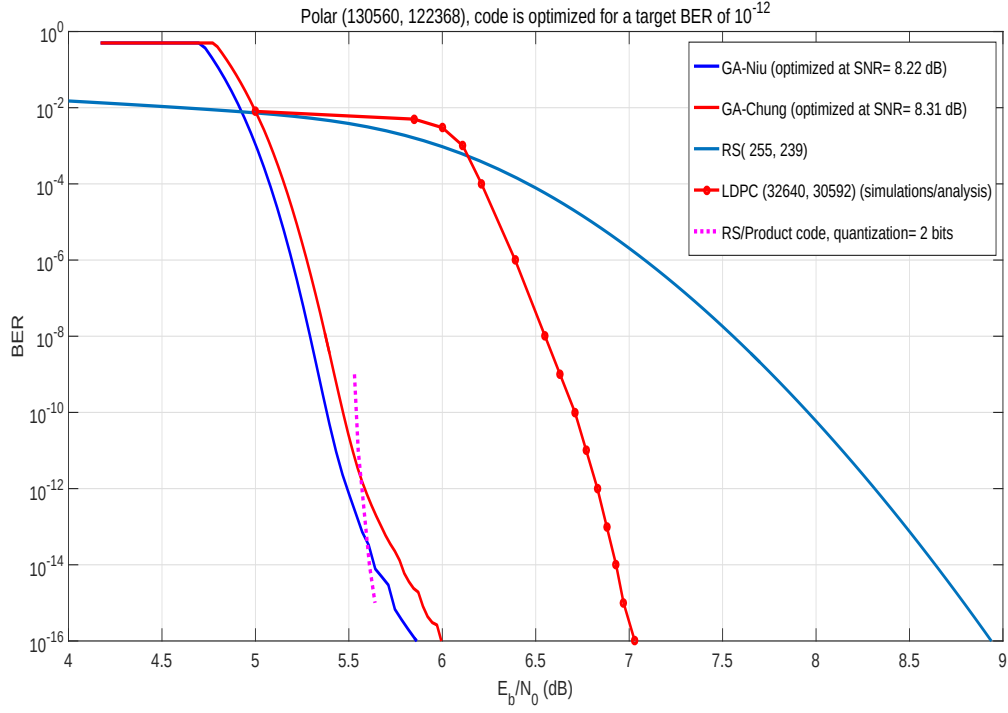


Figure 4.4: BER plot for Polar (130560, 122368) code using density evolution, (Chung and Niu approximations)

4.1.4 Polar (261120, 244736)

Polar (261120, 244736) is shortened form of Polar (262144, 245760), ($262144 = 2^{18}$). Shortening is achieved by freezing the last 1024 most reliable bits of the block to zero. Polar (261120, 244736) is completely compatible with G.975.1 recommended concatenated RS/Product super FEC code both in rate (0.9373)

and block length [3]. Hence, it can use two ODU2 and two OTU2 frames like concatenated RS/Product code.

As shown in Table 4.7, Polar (261120, 244736) can achieve a minimum NCG of 8.60 dB at a BER of 10^{-12} as compared to 5.62 dB for RS (255, 239), 7.06 dB for RS (2720, 2550) and 8.40 dB for concatenated RS/Product super FEC code.

Table 4.7: Error performance parameters for Polar (261120, 244736) code using Chung approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
5.72E-03	1.00E-12	8.60	8.89	8.06	5.33	8.06
5.11E-03	1.00E-15	9.53	9.81	8.19	5.46	8.19

Table 4.8: Error performance parameters for Polar (261120, 244736) code using Niu approximation

Input BER	Output BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	SNR (dB)
6.15E-03	1.00E-12	8.69	8.97	7.97	5.24	7.97
5.72E-03	1.00E-15	9.66	9.94	8.06	5.33	8.06

Figure 4.5 shows improvement in error performance obtained by Polar (261120, 244736) as compared to concatenated RS/Product code. Polar code is constructed/optimized for a target BER of 10^{-12} .

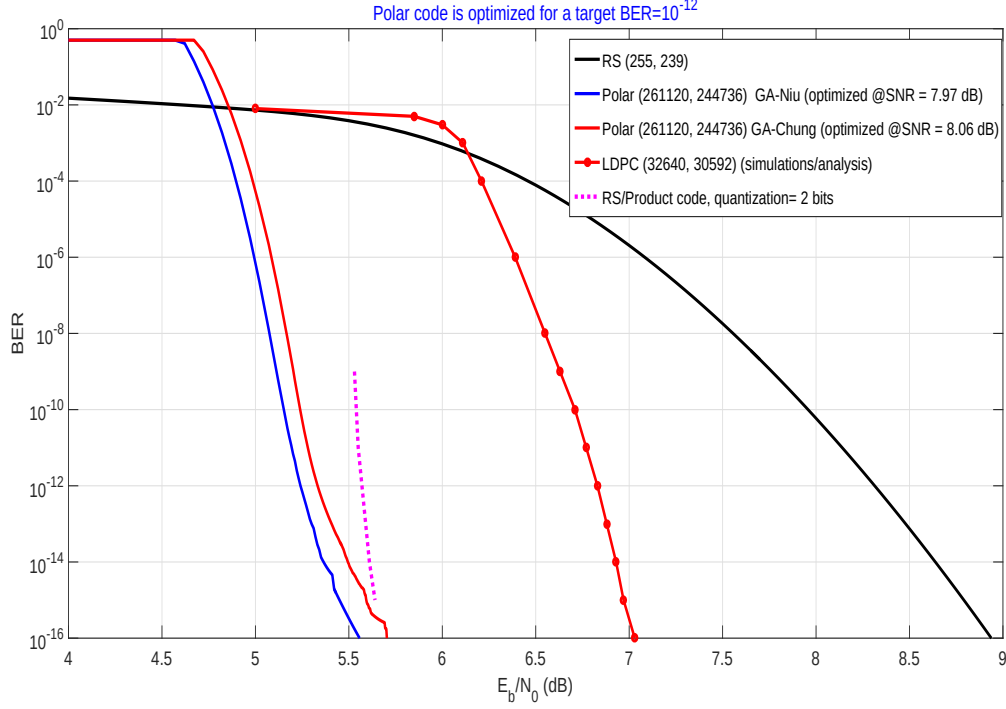


Figure 4.5: BER plot for Polar (261120, 244736) code using density evolution, (Chung and Niu approximations)

Table 4.9: Error performance parameters for the proposed polar codes and G.975/G.975.1 recommended FEC codes, with 7% overhead and $BER_{out} = 10^{-12}$ [3]

FEC Code	BER_{in}	NCG (dB)	CG (dB)	Q (dB)	$\frac{Eb}{No}$ (dB)
RS (255, 239)	1.82E-04	5.62	5.90	11.04	8.31
LDPC super FEC code	1.33E-03	7.10	7.39	9.56	6.83
RS (2720, 2550)	1.26E-03	7.06	7.34	9.60	6.87
Conc. RS/CSOC code(24.5%OH)	5.80E-03	7.95	8.90	8.04	5.31
Concatenated BCH code	3.30E-03	7.98	8.26	8.68	5.95
Conc. RS/BCH code	2.26E-03	7.63	7.91	9.06	6.34
Conc. RS/Product code	4.60E-03	8.40	8.68	8.30	5.57
Polar (2040, 1912)	2.81E-04	5.91	6.19	10.75	8.02
Polar (32640, 30592)	2.60E-03	7.74	8.02	8.92	6.20
Polar (130560, 122368)	4.61E-03	8.35	8.63	8.31	5.58
Polar (261120, 244736)	5.72E-03	8.60	8.89	8.06	5.33

4.2 Error Performance Comparison of Polar Codes and ITU-T Recommended FECs

Figure 4.6 and Table 4.9 compare the proposed polar codes for optical communications, discussed above, with G.975/G.975.1 recommended FEC codes. Only one code from G.975.1 achieves an NCG which is very close to that achieved by Polar (261120, 244736), *i.e.*, concatenated RS/Product super-FEC achieves an NCG of 8.40 dB as compared to 8.60 dB achieved by Polar (261120, 244736) for a target BER of 10^{-12} . It is important to know that both these codes have exactly the same block length but polar code has an advantage of 0.20 dB. We see that polar codes perform very good as compared to other codes even without any concatenation scheme. We can predict that with code concatenation performance of polar codes can be improved further.

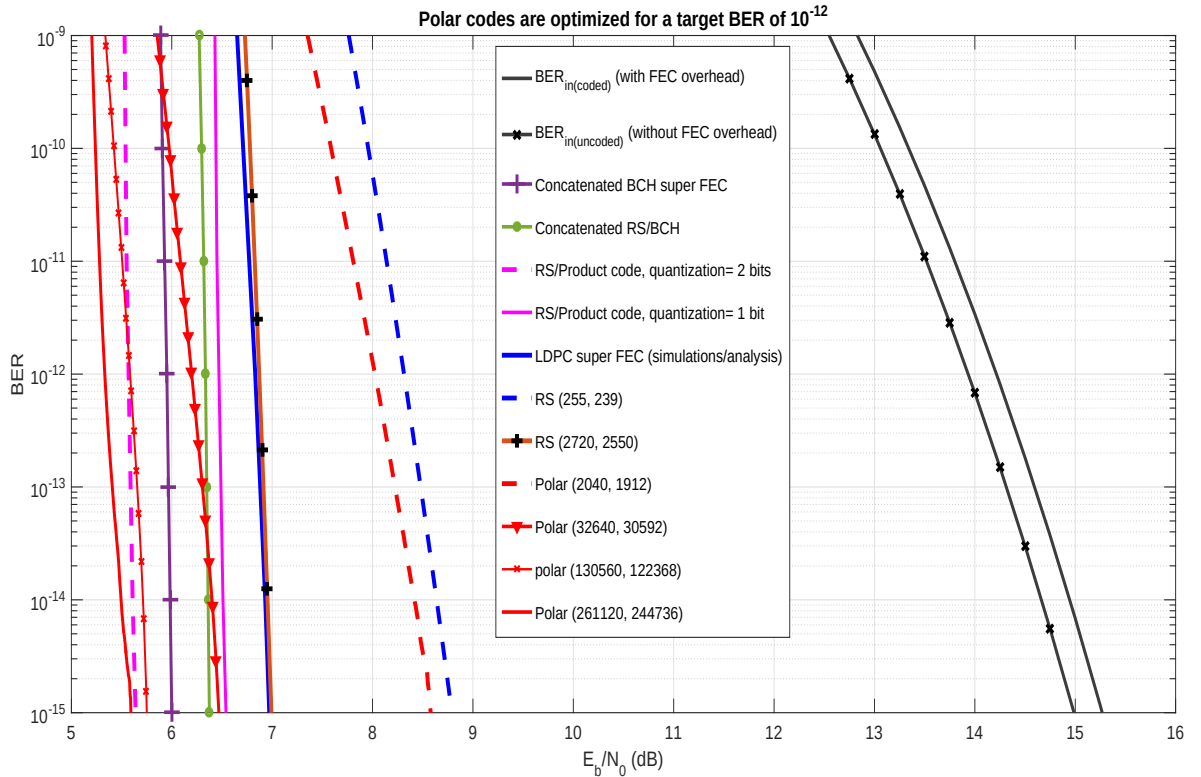


Figure 4.6: BER plot for comparison of polar codes and G.975/G.975.1 recommended FEC codes

4.3 Polar Codes as Third Generation FEC Codes

In this section, we study feasibility of polar codes for next generation optical networks in terms of their error performance and throughput capabilities. Modern optical communication systems require FEC codes which can work with flexible code rates [1]. Here we analyze polar codes with different overheads and compare their error performances. Polar codes have an advantage that they can easily be adjusted to variable code rates without the need of their reconstruction from the scratch. We also conduct a survey in this section to determine throughput capabilities of hardware polar decoders. We mainly focus on SC and BP decoders as they can provide high throughput required for optical communications.

4.3.1 Error Performance Analysis

Tables 4.10, 4.11 and 4.12 list error performance parameters for polar codes of three different block lengths, proposed above. We are interested mainly in their NCG values for different overheads.

Table 4.10 shows that the polar code of block length 32640 can provide an NCG of 10.07 dB with 20% overhead for a target BER of 10^{-15} . This value of NCG is not comparable with an NCG of 11.3 dB which is achieved by LDPC based non-concatenated codes proposed in [19][29] with the same overhead. Moreover, these LDPC based FEC codes have smaller block lengths of almost 24k and 18k respectively as compared to 32k for polar codes. One problem associated with these LDPC based codes is that they use many iterations, *i.e.*, 12 and 13 respectively along with post processing to suppress the error floors. Luckily, there is no error floor problem in polar codes and no post processing or iterative decoding is required for much simpler SC decoder.

Table 4.10: Error performance parameters for Polar (32640, K) code for a target BER of 10^{-15}

OH (%)	K	Input BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	Rate
6.69	30592	1.97E-03	8.52	8.80	9.20	6.47	0.937
15	28383	7.47E-03	9.67	10.27	7.73	5.322	0.87
20	27200	1.15E-02	10.07	10.86	7.14	4.917	0.83
25	26112	1.60E-02	10.41	11.38	6.63	4.584	0.80
40	23314	2.94E-02	11.01	12.47	5.53	3.981	0.71
100	16320	7.75E-02	11.93	14.94	3.06	3.06	0.50

Table 4.11: Error performance parameters for Polar (130560, K) code for a target BER of 10^{-15}

OH (%)	K	Input BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	Rate
6.69	122368	3.97E-03	9.24	9.52	8.48	5.75	0.937
15	113530	1.23E-02	10.36	10.96	7.04	4.63	0.87
20	108800	1.82E-02	10.79	11.58	6.42	4.19	0.83
25	104448	2.38E-02	11.10	12.07	5.94	3.89	0.80
40	93257	4.06E-02	11.71	13.17	4.83	3.28	0.71
100	65280	9.50E-02	12.67	15.68	2.32	2.32	0.50

Table 4.12: Error performance parameters for Polar (261120, K) code for a target BER of 10^{-15}

OH (%)	K	Input BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	Rate
6.69	244736	5.13E-03	9.53	9.81	8.19	5.46	0.937
15	227060	1.23E-02	10.36	10.96	7.04	4.634	0.87
20	217600	2.13E-02	11.07	11.86	6.14	3.924	0.83
25	208896	2.75E-02	11.37	12.34	5.66	3.62	0.80
40	186514	4.56E-02	11.99	13.45	4.55	3.002	0.71
100	130560	1.04E-01	12.97	15.98	2.02	2.02	0.50

Table 4.11 shows that the polar code of block length 130560 can achieve an NCG of 10.79 dB with 20% overhead. Table 4.12 shows that polar code of block length 261120 can achieve an NCG of 11.07 dB with an overhead of 20% for a target BER of 10^{-12} . It is a reasonable value but the problem is that block length is too large. With the increase in block length, latency of the decoder also increases. Hence polar codes with SC decoding and without any code concatenation may not be able to challenge LDPC based codes in the next generation optical networks, see Table 4.13 for a brief comparison.

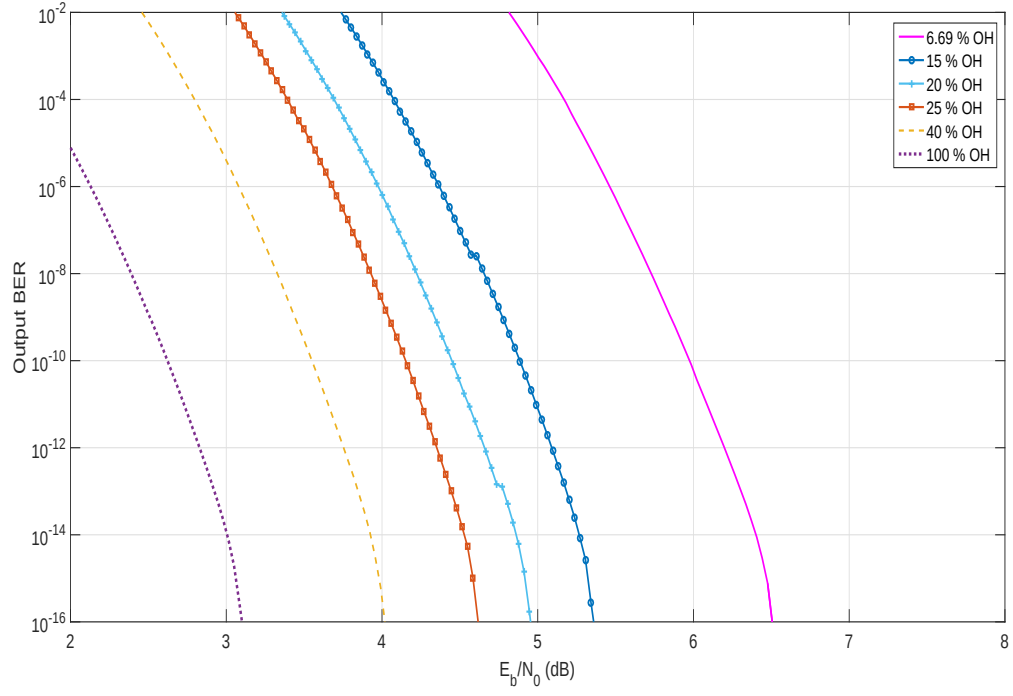


Figure 4.7: BER vs. E_b/N_0 for Polar (32640, K) with different overheads

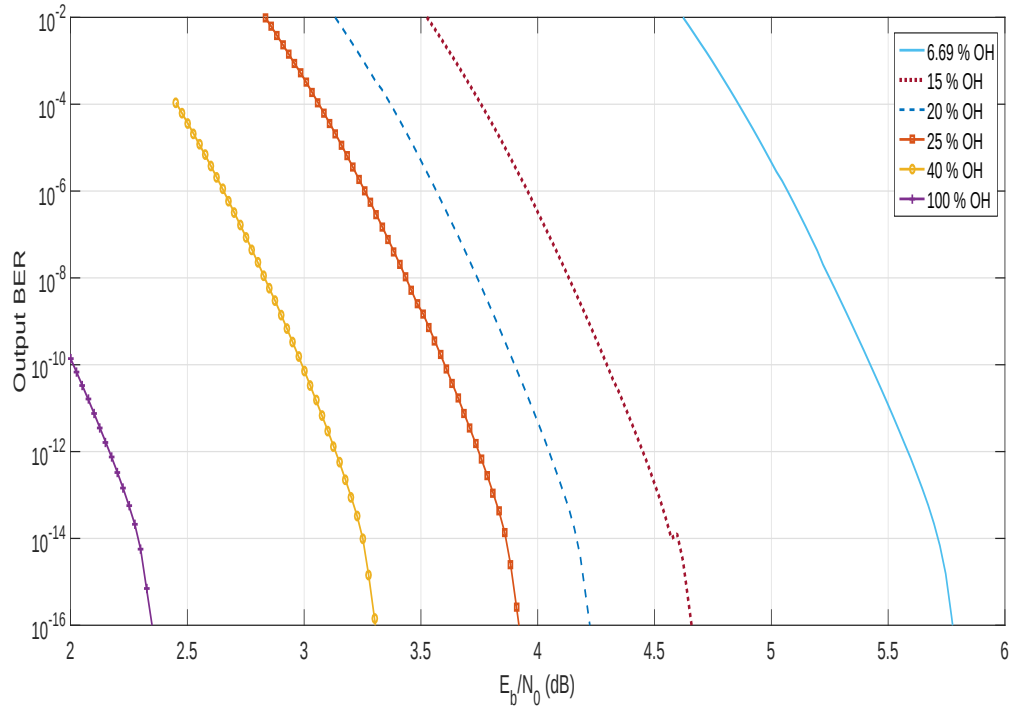


Figure 4.8: BER vs. E_b/N_0 for Polar (130560, K) with different overheads

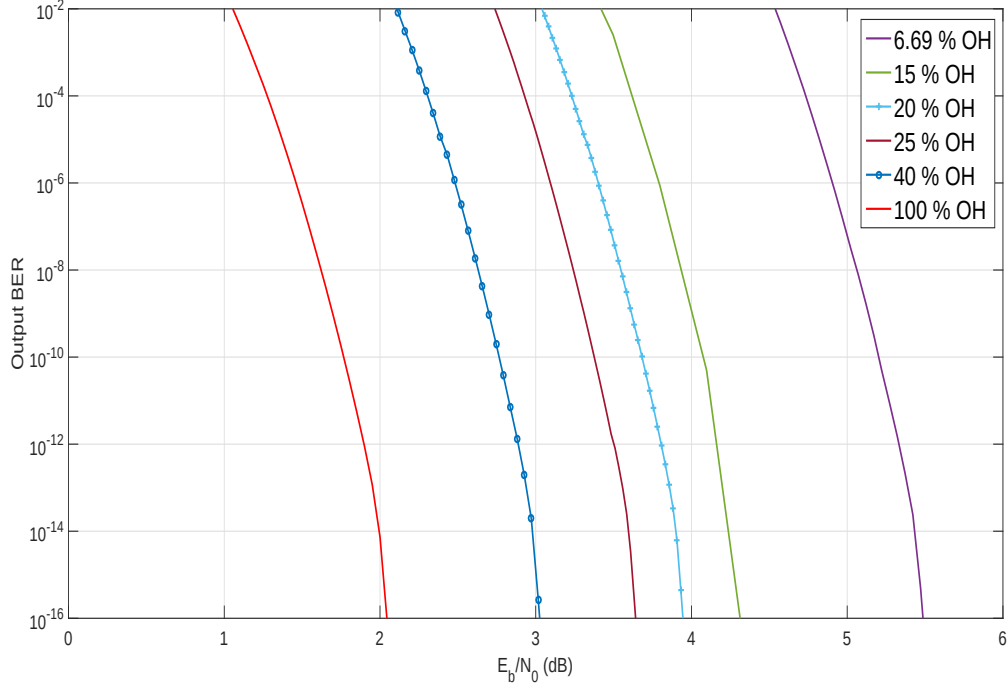


Figure 4.9: BER vs. E_b/N_0 for Polar (261120, K) with different overheads

Table 4.13: A summary of NCG values for 3G FEC codes proposed for optical communications for a target BER of 10^{-15} with 20% overhead

FEC code	NCG (dB)	Comments
Polar (32640, 27200)	10.07	Proposed as 3G FEC code
Polar (130560, 108800)	10.79	Proposed as 3G FEC code
Polar (261120, 217600)	11.07	Proposed as 3G FEC code
CC-LDPC (10032, 4, 24)	11.50	3G FEC, uses 12 iterations of decoding [20]
QC-LDPC (18360, 15300)	11.30	3G FEC, uses 12 iterations of decoding [19]

Table 4.13 shows that QC-LDPC (18360, 15300) block code can achieve an NCG of 11.3 dB. To achieve the same coding gain, we increase block length of the polar codes, and we find that for a block length of 522240, polar code is capable to achieve exactly the same NCG, see Table 4.14 . It is important to know that the length of four OTU2 frames is also equal to 522240. Block length of Polar (522240, 435200) is almost 29 times more than that of QC-LDPC (18360, 15300).

Table 4.14: Error performance parameters for Polar (522240, 435200) code for a target BER of 10^{-15}

OH (%)	Input BER	NCG (dB)	CG (dB)	Q (dB)	Eb/No (dB)	Rate
20	2.42E-02	11.30	12.09	5.91	3.69	0.83

4.3.2 Throughput Analysis

Due to its good recursive structure, encoding and decoding of polar codes is possible *within* complexity $\mathcal{O}(N \log N)$ for a code of block length N . But unfortunately this is not good enough for multi-gigabit communications systems. There are many proposals on implementations of polar codes in state-of-the-art hardware, which mainly focus on increasing throughput at the cost of hardware complexity, see Table 4.15. All the proposed architectures exploit the fact that some of the decoding steps can be performed in parallel to decrease the latency and increase the throughput.

Table 4.15: A summary of high throughput hardware polar decoders in literature with their specifications

Decoder type	Rate	Throughput (Gb/s)	Block Length	f_{clk} (MHz)	Reference
SCD	1/2	415.7	2048	203	[12]
	1/2	256	1024	250	
	5/6	255.7	1024	250	
SCD	9/10	1.091	16384	106	[13]
BPD	1/2	4.68	1024	300	[47]
BPD	1/2	2	1024	500	[48]

4.3.2.1 Throughput with SC Decoder

SC decoding algorithm is the most efficient sequential decoding algorithm for polar codes which provides a good balance between latency, error performance and

hardware complexity unlike other decoding algorithms which are biased towards one of the features.

Parallelization. SC decoder has a time complexity of $\mathcal{O}(N \log N)$ for a code of block length N but many steps can be performed in parallel and latency can be reduced. In a pipelined architecture for SCD, latency can be reduced to $2N - 2$ and hence throughput of synchronous SCD is limited to $\frac{f_c}{2}$ for clock frequency f_c [49]. Moreover, due to idle cycles in pipelined architecture of SCD, decoding of ‘P’ codewords can overlap and throughput can be increased to $\frac{Pf_c}{2}$ [49]. Zhang and Parhi have shown that latency can be reduced to N by doubling the number of adders in an SC decoder circuit and using pre-computation-look-ahead architecture [50]. Using semi parallel architecture, Mishra et al. have implemented an SCD in ASIC which provides a throughput of 49 Mb/s with 150 MHz clock frequency for a rate-1/2 code [51].

Recursive structure. Another approach to address the throughput problem of SCD is to identify the fact that polar codes are a class of generalized concatenated codes. In other words, a polar code of block length N is constructed from two smaller subcodes each of length $N/2$, using the Plotkin $|\mathbf{u}|\mathbf{u} + \mathbf{v}|$ code combining technique. Using this technique and some other properties of polar codes, throughput of 1 Gb/s is reported by using designs tailored for codes of specific rates and block lengths [13][14]. Fully-unrolled and deeply-pipelined SC decoder, using component code properties, achieving 254 Gb/s throughput for a rate-1/2 code is reported in [12].

In [15], for the first time a combinational architecture is proposed for SC decoding to obtain a high throughput decoder with low power consumption. The proposed combinational SC decoder decodes a codeword in one long clock cycle and operates at much lower clock frequencies compared to synchronous SC decoders. Post-synthesis results shows that the proposed combinational architectures are capable of providing a throughput of approximately 2.5 Gb/s.

4.3.2.2 Throughput with Belief Propagation Decoder

One method that can be used to reduce latency of polar codes is belief propagation decoding, introduced by [40] for polar codes. It is an iterative decoding algorithm and works well for moderate lengths polar codes. In BP decoding algorithm, the decoder has the capability to make multiple bit decisions/estimations in parallel. Recently, throughputs of 2 Gb/s (with clock frequency 500 MHz) and 4.6 Gb/s (with clock frequency 300 MHz) with BP decoding are reported in [47] and [48], respectively. It should be noted that the high-throughput of BP decoding is attainable only at high SNR values, where decoding can be terminated after a few number of iterations to reach the correct decision. This advantage of BPD over SCD diminishes as the SNR of the channel decreases.

4.3.3 A Few Remarks for Further Analysis of Polar Codes for Optical Communications

Our study in this thesis shows that finite length polar codes under SC decoding are not able to challenge the state-of-the-art third generation FEC codes in terms of error performance. In this section, we discuss the prospects of polar codes in optical communications and focus on a few issues which need to be tackled in order to improve their error performance while guaranteeing high throughput.

To improve error performance at finite block lengths, list decoding for polar codes may be considered in future to compare its performance to that of the third generation codes in optical communications. SC list decoder can achieve error performance close to ML performance for polar codes [35]. One problem associated with SC list decoding is that its complexity is higher, *i.e.*, $\mathcal{O}(LN \log N)$, which may decrease throughput drastically. This problem may be handled by using small list sizes as there is a significant SNR gain even with a list size of 2. But our conjecture is that list decoder performance may not be better than that of SC decoders at very high SNRs, typical of optical channels, see Figure 2 in [35]. Even spherical decoder [54] for polar codes, which is a type of ML

decoder, may not help to achieve better performance than SC decoder for high SNRs. This problem is not due to the type of decoder, instead it is because of the strategy used for code construction of polar codes. Polar code construction is not considering the optimization of the minimum hamming distance of the code. Therefore, in low noise environments, *i.e.*, high SNR, performance of SC list decoder degrades due to smaller hamming distances of polar codes. To increase the hamming distance, addition of CRC is helpful which improves performance in high SNR regimes. Some analytical method for error performance estimation of SC list decoder with CRC needs to be found for very low BERs.

Polar code concatenation may be another option to be considered to assess error performance of polar codes in comparison to third generations FEC schemes. Decay rate of error probability of polar codes can be improved with code concatenation as shown by [46]. It uses short polar codes as inner codes and high rate RS codes as outer codes. Eslami and Pishro [45] have shown that concatenating polar codes with LDPC codes results in better error performance compared to polar codes alone. They have used polar codes with modified belief propagation decoding algorithm as outer codes and LDPC codes as inner codes. Polar codes play an important role in this concatenation by suppressing the intrinsic error floor of LDPC codes at very low BER.

Polar coded-modulation is another area with joint encoding and modulation which needs to be considered for next generation coherent optical networks [52][53]. Like other coded-modulation schemes, polar coded-modulation can achieve achieve larger gains by exploiting both coding gains and modulation gains.

4.4 Summary

In this chapter, we proposed polar codes of four different block lengths which are compatible with ITU-T recommended FECs in terms of code rate and block lengths. We analyzed their error performances with 7% overhead and found that

they perform as good as any other ITU-T recommended FEC with the same overhead and relatively shorter block lengths. We also analyzed polar codes with 20% overhead to compare them against recently proposed third generation FECs for optical communications. We found that specially designed QC-LDPC codes using iterative decoding perform better than polar codes in terms of error performance for the same block lengths. QC-LDPC (18360, 15300) block code can achieve an NCG of 11.3 dB with 12 iterations of decoding. To achieve exactly the same NCG, polar codes require almost 29 times longer block length with SC decoding.

Chapter 5

Conclusion

In this thesis, we evaluated polar codes from the perspective of their suitability as FEC codes for optical-fiber communications by comparing them with other state-of-the-art FEC schemes proposed for optical networks. We analyzed moderate length polar codes in terms of their error correction capabilities under SC decoding assuming AWGN optical channels. In modern optical networks, BER of 10^{-15} or lower is required since high volume of data traffic makes retransmission of corrupted data almost impossible.

We analyzed error performances of polar codes for four different block lengths such that they are compatible (both in rate and block lengths) with FEC codes recommended by G.975.1. We found that polar codes can achieve error performances as good as any other FEC code recommended by G.975.1 for the same overhead and relatively smaller block length, see Table 5.1. Polar (261120, 217600) can achieve a minimum NCG of 8.60 dB for a post-FEC BER of 10^{-12} with an overhead of 6.69%. The best FEC scheme in G.975.1 recommendation is concatenated RS/Product code which achieves an NCG of 8.40 dB with the same block length and code rate [3]. Polar (32640, 30592) outperforms its compatible RS (2720, 2550) code by achieving a minimum NCG of 7.74 dB as compared to an NCG of 7.06 dB for the same post-FEC BER of 10^{-12} . We analyzed Polar (2040, 1912), which achieves a minimum NCG of 5.91 dB as compared to an NCG

of 5.62 dB achieved by a compatible FEC recommended by G.975, *i.e.*, RS (255, 239) for an output BER of 10^{-12} .

Table 5.1: A summary of FEC codes proposed for optical communications

OH (%)	FEC code	Target BER	NCG (dB)	Comments
6.69	Polar (2040, 1912)	10^{-12}	5.91	compatible to RS (255, 239)
		10^{-15}	6.41	
6.69	RS (255, 239)	10^{-12}	5.62	compatible to Polar (2040, 1912)
		10^{-15}	6.20	
6.69	Polar (32640, 30592)	10^{-12}	7.74	compatible to RS (2720, 2550)
		10^{-15}	8.52	
6.69	RS (2720, 2550)	10^{-12}	7.06	compatible to Polar (32640, 30592)
		10^{-15}	7.99	
6.69	Polar (130560, 122368)	10^{-12}	8.35	block length is half of that of RS/Product code
		10^{-15}	9.24	
6.69	Polar (261120, 244736)	10^{-12}	8.60	compatible to RS/Product code
		10^{-15}	9.53	
6.69	RS/Product code	10^{-12}	8.40	twice long as Polar (130560, 122368) compatible to Polar (261120, 244736)
		10^{-15}	9.40	
20	Polar (32640, 27200)	10^{-15}	10.07	proposed as 3G FEC code
20	Polar (130560, 108800)	10^{-15}	10.79	proposed as 3G FEC code
20	Polar (261120, 217600)	10^{-15}	11.07	proposed as 3G FEC code
20	Polar (522240, 435200)	10^{-15}	11.30	proposed as 3G FEC code
20	CC-LDPC (10032, 4, 24)	10^{-15}	11.5	3G FEC, uses 12 iterations of decoding [20]
20	QC-LDPC (18360, 15300)	10^{-15}	11.3	3G FEC, uses 12 iterations of decoding [19]

Recently, many high performance FEC codes have been proposed in literature for next generation optical networks, referred to as third generation FECs. These 3G FECs work at higher overheads to provide better error performance for optical communication systems operating at data rates beyond 100 Gb/s. Most of the modern FECs with high NCG values are based on LDPC and turbo codes. Up to date, specially designed CC-LDPC (10032, 4, 24) provides the highest NCG of 11.5 dB with 20% overhead for a target BER of 10^{-15} [20]. Specially constructed non-concatenated QC-LDPC (18360, 15300) code with 20% overhead gives an NCG of 11.3 dB with 12 iterations of decoding [19]. Polar (130560, 108800) can provide a minimum NCG of 10.79 dB with an overhead of 20%, while Polar (261120, 217600) can achieve a minimum NCG of 11.07 dB for a post-FEC BER of 10^{-15} . These values are satisfactory for a polar code with non-iterative and relatively lesser complex SC decoder. We analyzed another polar code of larger

block length, *i.e.*, Polar (522240, 435200) with 20% overhead, in order to find the minimum block length of polar codes to achieve a performance close to that of the best third generation codes. Polar (522240, 435200) achieves the same NCG as that achieved by QC-LDPC (18360, 15300) with a block length which is almost 29 times more.

Table 5.2: FECs gap to finite-length and Shannon’s asymptotic NCG limits for a target BER = 10^{-15}

FEC Code	OH (%)	NCG Limit finite-length	Gap to finite limit	NCG achieved (dB)	Gap to Shannon’s limit	NCG Limit Shannon’s
Polar(2040,1912)	7	9.75	3.34	6.41	4.69	11.10
Polar (32640, 30592)	7	10.76	3.02	7.74	3.36	11.10
Polar (130560, 122368)	7	10.93	2.58	8.35	2.75	11.10
Polar (261120, 244736)	7	10.98	1.45	9.53	1.57	11.10
Polar (32640, 27200)	20	12.24	2.17	10.07	2.53	12.60
Polar (130560, 108800)	20	12.42	1.63	10.79	1.81	12.60
Polar (261120, 217600)	20	12.47	1.40	11.07	1.53	12.60
Polar (522240, 435200)	20	12.52	1.22	11.30	1.30	12.60
QC-LDPC (18360, 15300)	20	12.11	0.81	11.30	1.30	12.60

In order to assess the true error correction capability of finite length codes, we calculated finite block length limits of NCGs for various code lengths and overheads as compared to Shannon’s asymptotic limits. Table 5.2 lists these finite-length and asymptotic limits for NCGs. It shows that QC-LDPC (18360, 15300) has a gap of only 0.81 dB to the maximum NCG possible with this code length. Polar (522240, 435200) achieves the same NCG as that of QC-LDPC (18360, 15300), but its gap to the maximum NCG possible with this code length is 1.22 dB. We see from the table that as block length of the code increases, the difference between finite length and Shannon’s limit for NCG decreases.

In our work, we applied density evolution algorithm to polar codes for their construction and error performance analysis under SC decoding. We used Gaussian approximation for density evolution as it makes it more tractable and at the same time guarantees almost the same accuracy. Implementation of Gaussian approximation was further simplified by using Chung and Niu approximations [44]. The accuracy of this approximate analytical method was confirmed through Monte Carlo simulations.

It is the regular structure, quasi-linear complexity and capacity achieving property of polar code which make it an attractive candidate for high data rate optical communications. To improve error performance at finite block lengths, list decoding for polar codes with CRC may be considered in future to compare its performance to that of the state-of-the-art codes in optical communications. List successive cancellation decoder can achieve error performance close to ML performance for polar codes at relatively larger complexity [35]. Polar code concatenation may be another option to be considered to assess its error performance in comparison to third generations FEC schemes. Decay rate of error probability of polar codes can be improved with code concatenation [46]. Polar coded-modulation is another area with joint encoding and modulation which may be considered to achieve large coding gains in next generation coherent optical networks [52][53].

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