

ESSAYS IN COLLECTIVE DECISION MAKING

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ABSTRACT

ESSAYS IN COLLECTIVE DECISION MAKING

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Four different problems in collective decision making are studied, all of which are either formulated directly in a game-theoretical context or are concerned with neighboring research areas.

The first two problems fall into the realm of cooperative game theory. In the first one, a decomposition of transferable utility games is introduced. Based on that decomposition, the structure of the set of all transferable utility games is analyzed. Using the decomposition and the notion of minimal balanced collections, a set of necessary and sufficient conditions for a transferable utility game to have a singleton core is given. Then, core selective allocation rules that, when confronted with a change in total cost, not only distribute the initial cost in the same manner as before, but also treat the remainder in a consistent way are studied. Core selective rules which own a particular kind of additivity that turns out to be relevant in this context are also characterized.

In the second problem, different notions of merge proofness for allocation rules pertaining to transferable utility games are introduced. Relations between these merge proofness notions are studied, and some impossibility as well as possibility results for allocation rules are established, which are also extended to allocation correspondences.

The third problem deals with networks. A characterization of the Myerson value with two axioms is provided. The first axiom considers a situation where there is a change in the value function at a network g along with all networks containing g . At such a situation, the axiom requires that this change is to be divided equally between all the players in g who are not isolated. The second axiom requires that if the value function assigns zero to each network, then each

player gets zero payoff at each network. Modifying the first axiom, along a characterization of the Myerson value, a characterization of the position value is also provided.

Finally, the fourth problem is concerned with social choice theory which deals with collective decision making in a society. A characterization of the Borda rule for a given set of alternatives with a variable number of voters is studied on the domain of weak preferences, where indifferences between alternatives are allowed at agents' preferences. A new property, which we refer to as degree equality, is introduced. A social choice rule satisfies degree equality if and only if, for any two profiles of two finite sets of voters, equality between the sums of the degrees of every alternative under these two profiles implies that the same alternatives get chosen at both of them. The Borda rule is characterized by the conjunction of faithfulness, reinforcement, and degree equality on the domain of weak preferences.

Keywords: Game theory, Cooperative game theory, Transferable utility games, Core, Allocation rules, Allocation correspondences, Additivity, Proportionality, Merge proofness, Shapley value, Networks, Myerson value, Position value, Social choice theory, Borda rule.

ÖZET

ORTAK KARAR ALMA ÜZERİNE MAKALELER

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Bu tezde, her biri ya doğrudan oyun kuramsal bir bağlamda formüle edilen ya da bu kuramın komşu araştırma alanları ile ilgili olan dört değişik problem incelenmektedir.

İlk iki problem, işbirlikli oyunlar kuramı kapsamına girmektedir. Birinci problemde aktarılabilir yarar oyunları için bir “ayrıştırma” tanımlanmaktadır. Bu ayrıştırma temelinde, aktarılabilir yarar oyunları kümesi çözümlenmektedir. Bu ayrıştırma ve “minimal dengeli topluluklar” kavramı kullanılarak, bir aktarılabilir yarar oyununun tek elemanlı bir çekirdeğe sahip olması için bir gerek ve yeter koşullar kümesi belirlenmektedir. Daha sonra toplam maliyette bir değişme olduğunda, yalnızca başlangıçtaki maliyeti eskisi gibi dağıtmakla kalmayıp, oluşan farkın dağıtımını da bununla uyumlu bir biçimde gerçekleştiren ve seçtiği elemanlar çekirdeğe ait olan dağıtım kuralları incelenmektedir. Bu durumun elemanları çekirdeğe ait dağıtım kurallarının bir tür toplamsallık özelliğine sahip olmalarıyla ilişkili olduğu anlaşıp, bu toplamsallık karakterize edilmektedir.

İkinci problemde, aktarılabilir yarar oyunlarına ilişkin dağıtım kuralları için çeşitli “kaynaşmaya dayanıklılık” kavramları tanımlanmaktadır. Farklı kaynaşmaya dayanıklılık kavramları arasındaki ilişkiler incelenip, bazı olanaksızlık ve olanaklılık sonuçları elde edildikten sonra, bunlar küme değerli dağıtım kurallarına genişletilmektedir.

Üçüncü problem ağılarla ilgilidir. Burada Myerson değerinin iki aksiyomlu bir karakterizasyonu verilmektedir. Birinci aksiyomda, değer fonksiyonunun bir g ağı ve g ağı içeren bütün ağlarda aldığı değerde bir değişikliğin olduğu durum ele alınmaktadır. Bu aksiyoma göre, böyle bir durumda değerde oluşan farkın g

ağında yalıtık olmayan bütün oyuncular arasında eşit paylaşılması gerekmektedir. İkinci aksiyoma göre, bütün ağlara sıfır değerini atayan değer fonksiyonu altında, her bir oyuncunun bütün ağlardaki getirisi sıfıra eşit olur. Birinci aksiyomda yapılan bir uyarlamayla, Myerson değerinin yanı sıra, “konumsal değer” için de bir karakterizasyon elde edilmektedir.

Dördüncü ve sonuncu problem, bir toplulukta ortaklaşa karar verme sorun-salını ele alan toplumsal seçim kuramına ilişkindir. Seçmen sayısı değişken olmak üzere verili bir seçenek kümesi için Borda kuralı, farklı seçenekler arasında kayıtsız kalmaya izin veren zayıf tercih sistemlerinin oluşturduğu tanım kümesi üstünde karakterize edilmektedir. Burada “derece eşitliği” adını verdiğimiz yeni bir özellik tanımlanmaktadır. Bir toplumsal seçme kuralının derece eşitliğini sağlaması demek, her bir seçeneğin sonlu sayıda seçmen içeren iki tercih sistemindeki toplam dereceleri eşitse, kuralın her iki tercih sistemi altında da aynı seçenekleri seçmesi demektir. Borda kuralı zayıf tercih sistemlerinden oluşan tanım bölgesi üstünde, “sadakat”, “pekiştirme” ve “derece eşitliği” aksiyomları ile karakterize edilmektedir.

Anahtar sözcükler: Oyunlar kuramı, İşbirlikli oyunlar kuramı, Aktarılabılır yarar oyunları, Çekirdek, Dağıtım kuralları, Küme değerli dağıtım kuralları, Toplam-sallık, Orantısallık, Kaynaşmaya dayanıklılık, Shapley değeri, Ağlar, Myerson değeri, Konumsal değer, Sosyal seçim kuramı, Borda kuralı.

Babama...

*This dissertation is dedicated to the memory of my father, Mehmet Derya.
His principles, his words of support and encouragement
in pursuit of fairness and excellence, still linger on.*

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Chapter 1

Introduction

Game theory is the formal study of decision making under conflictual situations between rational decision-makers, who may behave non-cooperatively or cooperatively contingent upon the context. There are different mathematical constructs used to model a game, each meant to fit best a particular context. In each type of a game, however, the “rational decision-makers” are referred to as players. The foundations of modern game theory date back to the first half of the 20th century. The grounds of game theory are first developed jointly by John von Neumann -a mathematician- and Oskar Morgenstern -an economist-. The point of departure of their book “Theory of Games and Economic Behavior” [1] is a prior research article written by von Neumann [2]. In this dissertation, we mainly deal with four different problems in “collective decision making”, all of which are either formulated directly in a game-theoretical context or are concerned with neighboring research areas.

The first two problems fall into the realm of cooperative game theory, a sub-field of game theory where players cooperate in order to optimize their payoffs. In both cases, we study problems in transferable utility games. A transferable utility game (also called a cooperative game in characteristic function form with side payments) with player set $N = \{1, \dots, n\}$ is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$, where $n \in \mathbb{N}$. The core of a transferable utility game, defined by Gillies [3], is the set of all feasible payoff allocations upon which no coalition can

improve. The core, which formally is an allocation correspondence, is accepted as the major stability notion in cooperative game theory and has been widely studied in the literature. An allocation rule for transferable utility games is a function that assigns a payoff vector to each game. An allocation rule that chooses a single point from the core of each game, whenever it is non-empty, is said to be core selective. A great amount of the cooperative game theory literature deals with core selective allocation rules.

We now turn to a summary of what we do in each of the following chapters separately. A more detailed introduction will be found at the beginning of each chapter.

In Chapter 2, we first introduce a decomposition of transferable utility games and analyze the structure of the set of all transferable utility games based on that decomposition. Using the decomposition and the notion of minimal balanced collections [4, 5], we give a set of necessary and sufficient conditions for a transferable utility game to have a singleton core. We then study core selective allocation rules that, when confronted with a change in total cost, not only distribute the initial cost in the same manner as before, but also treat the remainder in a “consistent” way. A particular kind of additivity of core selective allocation rules turns out to be relevant there, and we base our characterization on that notion. Moreover, we characterize both proportional core selective allocation rules and inverse proportional core selective allocation rules in a similar manner.

In Chapter 3, we define different notions of merge proofness for allocation rules pertaining to transferable utility games. Merging of a coalition into a single player is considered mainly in two different ways. One is where merging is allowed for only one coalition, i.e., the external players stay as they are. In the second way, on the other hand, the external players are also allowed to merge in any way they wish. We base our merge proofness notions for allocation rules on these two different merging structures, establish how different merging notions are related to each other and give our results. We adapt our merge proofness notions to allocation correspondences for transferable utility games and extend our results to such correspondences.

The third chapter deals with networks, which can also be regarded as a subfield of game theory in the broad sense where again players interact in order to optimize their gains. The main difference between transferable utility games and networks is due to the existence of links in networks, which are absent in transferable utility games. To be more specific, many of the social and economic problems that involve communication and cooperation among agents via forming coalitions can be modeled either as cooperative games or network games. Yet, in cooperative games, be the utility transferable or non-transferable, the formation of coalitions does not involve any structure among players further than that of just a set. The network structure, on the other hand, is endowed with a link architecture via which all interaction among the players taken place. Thus, how a member of a coalition is located within this architecture becomes now relevant and enables the analyst to take it into account. This makes network structures to become more complex than cooperative game structures, in general, and transferable utility games, in particular.

One of the main and earliest contributions to network literature is due to Myerson [6], and it constitutes our main line of interest in Chapter 4. Myerson adapts the Shapley value -which is an allocation rule for transferable utility games defined by Shapley [7]- to allocation rules for networks. The common basis of both the Shapley and the Myerson values is that the share of each individual from the total societal value is expressed as a weighted sum of his/her marginal contributions to each possible coalitions.

In Chapter 4, we provide a characterization of the Myerson value with two axioms. Different than Myerson's original characterization, ours is not restricted to component additive value functions, but covers all value functions. Our first axiom considers a situation where there is a change in the value function at a network g along with all networks containing g . At such a situation, our axiom requires that this change -be an increase or a decrease- is to be divided equally between all the players in g who are not isolated. Our second axiom concerns the form the value function takes when the value of each network is zero. Thus axiom requires that each player gets zero payoff at each network under such a

value function. By changing our first axiom slightly, we also obtain a characterization for the so-called “position value”, which is an allocation rule introduced by Meessen [8, 9]. Thus, our characterizations also allow us to compare the Myerson value and the position value.

Finally, the fourth problem is concerned with social choice theory which deals with collective decision making in a society. It considers the problem of aggregating the individual preferences of the members of a given society to a societal will. In case of a collective decision problem in social choice theory, the societal will can be represented either as a social preference or as a social choice. Namely, a bundle of agents’ preferences (preference profile) is either mapped into a social ordering of alternatives (social preference), which is called the social welfare function, or into a set of -selected- alternatives (social choice), by a so-called a social choice rule. Many different social welfare functions and social choice rules have been introduced to determine which alternative(s) should be chosen at a preference profile of a society. One rule that has received a great deal of attention in the literature is the Borda rule [10], which is a scoring rule, and constitutes our focus of interest in Chapter 5.

When agents (voters) have strict preferences over m alternatives, a vector $(s_1, \dots, s_m)$ with $s_1 \geq s_2 \geq \dots \geq s_m$ and $s_1 > s_m$ is called a score vector, where the score s_1 is assigned to each agents’ best (the first ranked) alternative, s_2 to each agents’ second best alternative, and in general s_k to each agents’ k^{th} most preferred alternative. The total score of an alternative under a society’s preference profile is then the sum of its assigned scores. A scoring rule is a social choice rule under which the alternative(s) with the highest total score are selected. Borda rule is a scoring rule where the differences between consecutive scores are same.

In Chapter 5, we study the Borda rule on the domain of weak preferences, i.e., on the domain where indifferences between alternatives are allowed at agents’ preferences. The definition of the Borda rule is adjusted accordingly. To be more specific, in an indifference class of alternatives of an agent, the average of the Borda scores is assigned to each alternative in that indifference class. Borda rule again selects the alternative(s) with the highest total score. In Chapter 5,

we give a characterization of the Borda rule on the domain of weak preferences. We introduce a new property which we refer to as degree equality. A social choice rule satisfies degree equality if, for any two profiles of two finite sets of voters, equality between the sums of the degrees of every alternative under the two profiles implies that the same alternatives get chosen by the social choice rule at these two profiles. We show that the Borda rule is characterized by the conjunction of faithfulness, reinforcement, and degree equality on the domain of weak preferences. In addition, we introduce a new cancellation property which is weaker than degree equality, and show that it characterizes the Borda rule among all scoring rules.

Chapter 2

A decomposition of transferable utility games

2.1 Introduction

During the 1930's, scientists working with the Tennessee Valley Authority (TVA) in the United States developed theories related to certain solution concepts of transferable utility games to solve the problem of fair allocation¹ of the costs of dams among consumers. Young [13] considered cost allocation in water resources development that TVA is concerned with. The core [3], core selective single valued solution concepts (i.e., core selective allocation rules), and the Shapley value [7] are well-known solution concepts that are considered in the literature.

The core of a transferable utility game is the set of all feasible payoff allocations upon which no coalition can improve. The core has been well accepted and widely studied in the literature.² An allocation rule that chooses a single point from the core of a game whenever it is non-empty is called core selective. In this chapter, we restrict ourselves to core selective allocation rules.

¹Fair division is widely studied in the literature, see for example [11] or [12] for a survey.

² [3, 11, 14, 15, 16, 17] are some examples.

A simple motivating example: Consider the problem of allocating the cost of a shared facility between two neighbors, such as building water pipelines for their houses. Suppose the total cost of the shared facility is 60 dollars and the neighbors have agreed that one pays 20 dollars and the other pays 40 dollars. Suppose further that as the construction proceeds, the total cost is increased to 66 dollars. Now, the problem is how to reallocate the cost that will depend on the rationale that the original is based upon. A natural way to solve this problem is to allocate 60 dollars of the new total cost as agreed before and *reduce the problem* to that of allocating the extra cost, i.e., to that of deciding how to allocate the $66 - 60 = 6$ dollar cost. Suppose instead that the total cost decreased to 54 dollars. In a similar way, we can reduce this problem to that of allocating a 6 dollar surplus.³

Depending on the problem and the agreement, commonly used practices include distributing an increase/decrease of k units of the grand coalition either proportionally, or inversely proportionally, or consistent with the allocation rule that is used before, or equally between the players. In this chapter, our main focus is in core selective allocation rules that, when confronted with a change in total cost, distributes the original cost in the same manner as before the change and treat the remainder in a consistent way. In that respect, the allocation rule turns out to be consistent with itself. Yet, we consider each of these four distributions mentioned above. There is a tool, a decomposition of transferable utility games, that is common for the analysis of each of these four distribution methods.

Given a transferable utility game, we define ‘the decomposition associated with the game’ to be based on shifting the value of the grand coalition so that the associated game has a non-empty core. We divide the set of all transferable

³In terms of transferable utility games, consider the following games with two players. $v(1) = 20$, $v(2) = 40$, $v(12) = 60$, and $w(1) = w(2) = 0$, $w(12) = 6$, and $\tilde{v}(1) = 20$, $\tilde{v}(2) = 40$, $\tilde{v}(12) = 66$. Note that $\tilde{v} = v + w$. Most of the allocation rules distribute the value of the grand coalition as 20 units to player 1 and 40 units to player 2 at the game v , and equally between the players at the game w , i.e., 3 units to each player at w . Both of which are well accepted and fair. However, another fair allocation for the game w may be 2 units for player 1 and 4 units for player 2, if one wants the allocation rule that is used for w to be proportional to the distribution that is done in v , for the distribution in $\tilde{v}$. Or the extra cost can be distributed according to the allocation rule that is used before for v , or inversely proportional to the distribution of v . Similar arguments are valid for the distribution of 54 units if the total cost decreased 6 units.

utility games into two groups based on this decomposition, games with empty cores and games with non-empty cores. Given a game, a new game, called ‘the minimal game associated with the game’, is obtained by minimizing (if the game has a non-empty core) or maximizing (if the game has an empty core) the value of the grand coalition so that the new game has a non-empty core.⁴ The core of a minimal game is either a singleton or multi-valued. We divide both the games with non-empty cores and the games with empty cores into two disjoint groups depending on the size of the core of their associated minimal games.

Now, we turn back to the four practices that are commonly used for distributing an increase/decrease of k units of the grand coalition. The decomposition that we define allows us to analyze core selective allocation rules in terms of each of these four different practices. Specifically, a rule that, when confronted with a change in total cost, distributes the original cost in the same manner as before the change and distributes any surplus proportionally -to the original cost- is called the proportional allocation rule. And, similarly, a rule that distributes surplus inversely proportional is called the inversely proportional allocation rule. With the aid of the decomposition of transferable utility games, we characterize the class of core selective allocation rules that are proportional, and the class of games that are inversely proportional. Besides these two methods of distributing any surplus, the surplus can also be distributed either equally or according to the allocation rule that is used before the change. For both of the latter methods, we see that a particle kind of additivity turns out to be relevant, which we characterize. Additivity of an allocation rule is defined as usual. Specifically, additivity is a property that is satisfied by the well-known Shapley value, yet it is known that Shapley value is not always core selective. Both core selectivity and additivity are widely used in the literature. This chapter contributes to the literature for the compatibility of these two properties as well. We give an axiomatization of core selective allocation rules that are additive on a specific domain pair. The result gives a relation between the property, additivity on a specific domain pair, and two other properties -namely zero independence and equality at equivalence

⁴If the associated minimal game of the game is itself, then we call that game a root game. The idea of root games is also used by [18] where the aggregate-monotonic core is introduced and characterized.

classes on some domains- of an allocation rule.

Besides answering the above problems about allocation rules, the decomposition idea allows us to understand the geometric structure of the set of all transferable utility games more precisely. Using the minimal game idea and minimal balanced collections⁵, we give a set of necessary and sufficient conditions for a game to have a singleton core. As far as we know, there is no result about this set of games in the literature in terms of balanced collections. Moreover, we compare cardinalities of different sets of games obtained by the classification of games via decomposition. For example, nearly each game that has a non-empty core has a single element in the core of its minimal game.

Lastly, we want to note that monotonicity with respect to the value of the grand coalition is a property that is also related to additivity of an allocation rule.⁶ Monotonicity with respect to the value of the grand coalition [24] states that if the worth of the grand coalition increases and the worth of all other coalitions remain same, then the payoff of each player should increase weakly. For the allocation rules that are concerned in this chapter, obviously, we want more than monotonicity with respect to the value of the grand coalition. Because if the worth of the grand coalition increases and the worth of all other coalitions remain same, we not only care that the payoff of each player should increase weakly, but also care *how* that is distributed. Finally, we give relations between allocation rules that are monotonic with respect to the value of the grand coalition and allocation rules that are additive on some specific domains.

The rest of this chapter is organized mainly in three parts as follows.

Part I (Section 2.2) consists of three sections. Basic notions and preliminaries for transferable utility games are given in Section 2.2.1 and our decomposition of transferable utility games is given in Section 2.2.2. Necessary and sufficient conditions for a transferable utility game to have a singleton core, and our results

⁵Minimal balanced collections are a result of balanced collections that are used for the characterization of the core by [4, 5].

⁶Monotonicity with respect to the value of the grand coalition is also called aggregate monotonicity in the literature. Aggregate monotonicity and several other monotonicity properties are widely studied in the literature, [17, 18, 19, 20, 21, 22, 23] are a few examples of them.

related to the geometric structure of games are given in Section 2.2.3.⁷

Part II (Section 2.3) consists of five sections. Basic notions and preliminaries for allocation rules and some well-known allocation rules are given in Section 2.3.1 and Section 2.3.2, respectively. Our characterizations of additivity and its dual, subtractivity on specific domain pairs are given in Section 2.3.3 and Section 2.3.4, respectively. Some new allocation rules are given in Section 2.3.5.

Part III (Section 2.4) consists of three sections. Proportional and inverse proportional allocation rules are studied in Section 2.4.1, monotonicity with respect to grand coalition is studied in Section 2.4.2, and finally a modification of the decomposition is given in Section 2.4.3.

2.2 Part I

2.2.1 Preliminaries for transferable utility games

For each $n \in \mathbb{N}$, let $N := \{1, \dots, n\}$ be the set of finite players. A *transferable utility game* (or simply a *game*), with player set N is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) := 0$. For each $T \subseteq N$, we refer to $v(T)$ as the *worth of coalition* T . Let G^N denote the set of all games with player set N and let $G := \bigcup_{N: n \in \mathbb{N}} G^N$.

Non empty subsets of the player set are called *coalitions*. The collection of all coalitions is denoted by Ω , that is $\Omega := \{S \subseteq N | S \neq \emptyset\}$.

A vector $x \in \mathbb{R}^N$ ⁸ assigning payoff $x_i \in \mathbb{R}$ to player $i \in N$ is called a *payoff vector*. For a payoff vector $x \in \mathbb{R}^N$ and $S \in \Omega$, the total payoff of the players in coalition S is $x(S) := \sum_{i \in S} x_i$.

A payoff vector x is *feasible* if $x(N) \leq v(N)$, and *stable* if for each $S \in \Omega$, $x(S) \geq v(S)$. The set of all feasible and stable payoff vectors is called the *core of*

⁷Results of Part I is accepted for publication, [25].

⁸Note that we use $\mathbb{R}^N$ instead of $\mathbb{R}^{|N|}$.

the game v , denoted by $C(v)$; i.e.,

$$C(v) := \{x \in \mathbb{R}^N : x(N) \leq v(N) \text{ and for each } S \in \Omega, x(S) \geq v(S)\}.$$

The set of all games with player set N and non-empty cores is denoted by G_c^N and set of all games with player set N and non-empty cores is denoted by $\overline{G}_c^N$, i.e., $\overline{G}_c^N \equiv G^N \setminus G_c^N$. Let $G_c := \bigcup_{N:n \in \mathbb{N}} G_c^N$ and $\overline{G}_c := \bigcup_{N:n \in \mathbb{N}} \overline{G}_c^N$.

A collection $\{S_1, \dots, S_k\}$ of coalitions of N is *balanced* if there exists a collection of real numbers $\lambda_1, \dots, \lambda_k \in [0, 1]$ such that for each $i \in N$, $\sum_{j \in \{1, \dots, k\} : i \in S_j} \lambda_j = 1$. The numbers $\lambda_1, \dots, \lambda_k$ are called *balancing coefficients*. A balanced collection $\{S_1, \dots, S_k\}$ is a *minimal balanced collection* if no proper subcollection is balanced.

2.2.2 A decomposition of games

In this section, we give our definition of the decomposition of games and classify the set of all games based on our decomposition of games.

Given a pair of games $v, \tilde{v} \in G^N$, for each $S \subseteq N$, $(v + \tilde{v})(S) := v(S) + \tilde{v}(S)$ and $(v - \tilde{v})(S) := v(S) - \tilde{v}(S)$.

Given $v \in G^N$, for each $r \in \mathbb{R}$, v_r is defined as follows:

$$v_r(S) := \begin{cases} v(S) & \text{if } S \subset N, \\ r & \text{if } S = N. \end{cases} \quad (2.2.1)$$

Let $M_v := \{r \in \mathbb{R} : C(v_r) \neq \emptyset\}$ and let $r^* := \min_{r \in M_v} r$.

We will briefly discuss the existence of the minimum of the set M_v . It is well-known that games with non-empty cores, that is G_c is characterized by the following theorem of Bondareva and Shapley [4, 5].

“For each player set N and $v \in G^N$, $C(v) \neq \emptyset$ if and only if for each minimal collection $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, inequality $\sum_{i=1}^k \lambda_i v(S_i) \leq v(N)$ holds.”

Let $\mathbb{B}$ be the set of all minimal balanced collections of the player set N , except the minimal balanced collection $\{N\}$. For each $\mathcal{B} \in \mathbb{B}$, say $\mathcal{B} = \{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, $\sum_{j=1}^k \lambda_j v(S_j) \leq \max_{\mathcal{B} \in \mathbb{B}} \sum_{S_j \in \mathcal{B}} \lambda_j v(S_j)$. By the Shapley-Bondareva theorem, one can easily check $r^* = \min_{r \in M_v} r = \max_{\mathcal{B} \in \mathbb{B}} \sum_{S_j \in \mathcal{B}} \lambda_j v(S_j)$. In other words, the value r^* is a ‘boundary value’ with the property that for each $r \geq r^*$, the game v_r has a non-empty core and for each $r < r^*$, the game v_r has an empty core.

We call the game v_{r^*} as the *minimal game associated with the game v* .

Note that for each game, there is a unique minimal game associated with that game, but not vice versa.

Given $v \in G^N$, we define a new game w as follows:

$$w(S) := \begin{cases} 0 & \text{if } S \subset N, \\ |v(N) - v_{r^*}(N)| & \text{if } S = N. \end{cases}$$

Given $v \in G^N$, $v := v_{r^*} \oplus |w|$ is called *the decomposition associated with the game v where*

$$v_{r^*} \oplus |w| = \begin{cases} v_{r^*} + w & \text{if } v(N) \geq v_{r^*}(N), \\ v_{r^*} - w & \text{if } v(N) < v_{r^*}(N). \end{cases}$$

If $v \in G_c^N$, then the decomposition associated with the game v , that is $v = v_{r^*} \oplus w = v_{r^*} + w$, is called *the decomposition of the game v* .⁹ The game v_{r^*} , that is the minimal game associated with v , is called the *root game associated with the game v* . We call it *root game of v* for short. Note that, the idea of root game of a game is also used by Calleja et al. [18] where they introduce and characterize the aggregate monotonic core. Our definition of ‘root game associated with the game v ’ is the same as their definition of ‘root game associated to the game v ’.

⁹Note that the definition of ‘the decomposition of the game v ’ depends on the set M_v and its minimum value r^* . By changing this set and its minimum value, the definition of ‘the decomposition of a game’ can be modified and can be used as a tool for different problems of cooperative games.

If the root game of a game v is itself, then it is called a *root game*, that is if v is a root game, then $v = v_{r^*} + w$ is the decomposition of v with $v_{r^*} \equiv v$ and for each $S \subseteq N$, $w(S) = 0$. The set of all root games with player set N is denoted by G_r^N and $G_r := \bigcup_{N:n \in \mathbb{N}} G_r^N$.

Remark 2.2.1. Given $v \in G$, via equation (2.2.1), one observes that G_r is a small subset of G . For comparing their sizes, we can formalize the class of all games and the class of all root games as follows: Consider any labeling $S_1, \dots, S_{2^N-2}, S_{2^N-1}$ of the non-empty subsets of the player set N such that S_{2^N-1} corresponds to N , i.e., $S_{2^N-1} = N$. Let f be a function that assigns the $(2^N - 1)$ -tuple $(v(S_1), \dots, v(S_{2^N-1})) \in \mathbb{R}^{2^N-1}$ to each $v \in G$. The function f shows that there is a one-to-one correspondence between the games in G and the elements in $\mathbb{R}^{2^N-1}$. Similarly, let f_r be a function that assigns the $(2^N - 2)$ -tuple $(v(S_1), \dots, v(S_{2^N-2})) \in \mathbb{R}^{2^N-2}$ to each $v_{r^*} \in G_r$. The function f_r shows that there is a one-to-one correspondence between the games in G_r and the elements in $\mathbb{R}^{2^N-2}$. Thus, the dimension of G_r is one less than the dimension of G .

While it is a small class, G_r allows us to understand the structure of G . For that, we classify G into groups with the help of the following classification of G_r .

G_r is divided into two disjoint groups depending on the size of their cores.

- (i) G_{sin} : The set of all root games with player set N each of which has a single vector in its core is denoted by G_{sin}^N , that is $G_{sin}^N := \{v \in G_r^N : |C(v)| = 1\}$. Let G_{sin} denote the set of all root games each of which has a singleton in its core, that is $G_{sin} := \bigcup_{N:n \in \mathbb{N}} G_{sin}^N$.
- (ii) G_{mul} : The set of all root games with player set N each of which has more than one vector in its core is denoted by G_{mul}^N , that is $G_{mul}^N := \{v \in G_r^N : |C(v)| > 1\}$. Let G_{mul} denote the set of all root games each of which has more than one vector in its core, that is $G_{mul} := \bigcup_{N:n \in \mathbb{N}} G_{mul}^N$.

Note that $G_r = G_{sin} \cup G_{mul}$ where $G_{sin} \cap G_{mul} = \emptyset$.

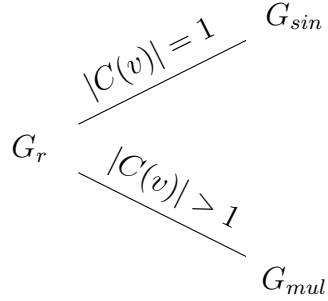


Figure 2.1: Summary of classification of G_r .

Now we classify the set of all games depending on their associated minimal games. First, G is divided into the two disjoint groups, G_c and $\overline{G}_c$. Next, G_c is divided into two disjoint groups depending on the size of cores of root games.

- (i) G_s : For each player set N , $G_s^N := \{v \in G_c^N : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{sin}\}$. The set of all games with non-empty cores each of which has a singleton in the core of its root game is denoted by G_s , that is $G_s := \bigcup_{N:n \in \mathbb{N}} G_s^N$.¹⁰
- (ii) G_m : For each player set N , $G_m^N := \{v \in G_c^N : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{mul}\}$. The set of all games with non-empty cores each of which has more than one element in the core of its root game is denoted by G_m , that is $G_m := \bigcup_{N:n \in \mathbb{N}} G_m^N$.¹¹

Lastly, the set of all games with empty cores, that is $\overline{G}_c$, is divided into two disjoint groups depending on the size of cores of their associated minimal games.

- (i) $\overline{G}_s$: For each player set N , $\overline{G}_s^N := \{v \in \overline{G}_c^N : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{sin}\}$. The set of all games with empty cores each of which has a singleton in the core of its associated minimal game is denoted by $\overline{G}_s$, that is $\overline{G}_s := \bigcup_{N:n \in \mathbb{N}} \overline{G}_s^N$.

¹⁰Note that $G_{sin}^N \subset G_s^N$ and $G_{sin} \subset G_s$.

¹¹Note that $G_{mul}^N \subset G_m^N$ and $G_{mul} \subset G_m$.

- (ii) $\overline{G}_m$: For each player set N , $\overline{G}_m^N := \{v \in \overline{G}_c^N : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{mul}\}$.
The set of all games with empty cores each of which has more than one element in the core of its associated minimal game is denoted by $\overline{G}_m$, that is $\overline{G}_m := \bigcup_{N:n \in \mathbb{N}} \overline{G}_m^N$.

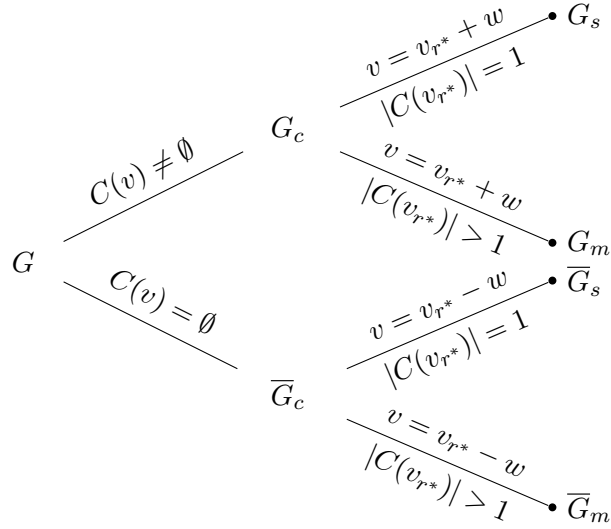


Figure 2.2: Summary of classification of G .

Since each $v \in G$ has a unique minimal game associated with v , the set of all games in G can be partitioned into equivalence classes according to this. For each player set N , for $v, \hat{v} \in G^N$, let the decompositions of v and $\hat{v}$ be $v = v_{r^*} \oplus |w|$ and $\hat{v} = \hat{v}_{r^*} \oplus |\hat{w}|$, respectively. We define an *equivalence relation between two games v and $\hat{v}$* , denoted by $v\mathcal{R}\hat{v}$, if $C(v_{r^*}) = C(\hat{v}_{r^*})$ and $v(N) = \hat{v}(N)$. Note, in that case $w \equiv \hat{w}$. *Two games v and $\hat{v}$ belong to the same equivalence class, if $v\mathcal{R}\hat{v}$.*¹²

Finally, we give some important subsets of games:

- (1) The set of all games in G^N where all the coalitions except the grand coalition have zero worth and the worth of the grand coalition is non-negative is

¹²A similar argument of partitioning the set of all games into equivalence is also used in [26], where they use their partitioning for to study core of combined games.

denoted by G_z^N , that is

$$G_z^N := \{v \in G^N : \forall S \subset N \ v(S) = 0 \text{ and } v(N) \in \mathbb{R}^+ \cup \{0\}\}.$$

Let G_z denote the set of all such games, that is $G_z := \bigcup_{N:n \in \mathbb{N}} G_z^N$.

One can observe that if $v \in G_z^N$, then the decomposition associated with v is obviously $v = v_{r^*} \oplus w = v_{r^*} + w$, where for each $S \subseteq N$, $v_{r^*}(S) = 0$ and $w \equiv v$, so $C(v_{r^*}) = \{(0, \dots, 0)\}$ and $v \in G_s^N$. Hence, $G_z^N \subset G_s^N \subset G_c^N$.

- (2) A game $v \in G^N$ is called *symmetric* if for each $S \subseteq N$, $v(S) = |S|$. Let G_{sym}^N denote the set of all symmetric games with player set N and $G_{sym} := \bigcup_{N:n \in \mathbb{N}} G_{sym}^N$.

One can observe that if $v \in G_{sym}^N$, then the decomposition associated with v is obviously $v = v_{r^*} \oplus w = v_{r^*} + w$, where $v_{r^*} \equiv v$ and for each $S \subseteq N$, $w(S) = 0$, so $v \in G_{sin}^N$. Hence, $G_{sym}^N \subset G_{sin}^N \subset G_s^N \subset G_c^N$.

Remark 2.2.2. Our classification of the set of all games is based on the core and the root game of a game. A similar argument can be used for other classifications of the set of all games by changing the worth of some other coalition instead of the grand coalition. In general, similar to equation (2.2.1), given $v \in G$ and $\emptyset \neq T \subseteq N$, for each $r \in \mathbb{R}$, let $v_{(r,T)}$ be defined as follows:

$$v_{(r,T)}(S) := \begin{cases} v(S) & \text{if } T \neq S \subseteq N, \\ r & \text{if } S = T. \end{cases}$$

Let $M_{(v,T)} := \{r \in \mathbb{R} : C(v_{(r,T)}) \neq \emptyset\}$, and $r_T^* := \min_{r \in M_{(v,T)}} r$. Note that $v_{r_N^*} = v_{r^*}$. Now, using $v_{r_T^*}$ instead of v_{r^*} for any $\emptyset \neq T \subset N$, other classifications of the set of all games can be done similar to our classification in Figure 2.2. Here, we are working with v_{r^*} , because geometrically, the change in the core is given by a hyperplane, while it will be given by a region bounded by a hyperplane otherwise.

2.2.3 Structure of games

In this section, we examine the class of games defined in Section 2.2.2 in terms of minimal balanced collections. First, we give the geometric intuition behind our theorems.

Geometrically, it is not hard to see that nearly all the games in G_c are in G_s . Let a game that has a non-empty core be given. Roughly, if one shifts/changes the worth of the grand coalition as much as possible to obtain the root game of the given game, then the probability of ending up with a single point is higher than ending up with a line segment (or a hyperplane segment). In other words, the probability of getting a root game in G_{sin} is higher than getting a root game in G_{mul} .

As an example, consider $N = \{1, 2, 3\}$ and the game $v(12) = 1$, $v(123) = 5$, and $v(S) = 0$ otherwise. Note that

$$C(v) = \{(a + b, (1 - a) + c, d) : 0 \leq a \leq 1, 0 \leq b, c, d \text{ and } b + c + d = 4\}.$$

Figure 2.3 below shows the cores of v_r for $r = \{1, 2, 3, 4, 5\}$. Note that v_1 corresponds to v_{r*} , and $C(v_{r*}) = \{(a, 1 - a, 0) : 0 \leq a \leq 1\}$ is a line segment, thus $v \in G_m$.

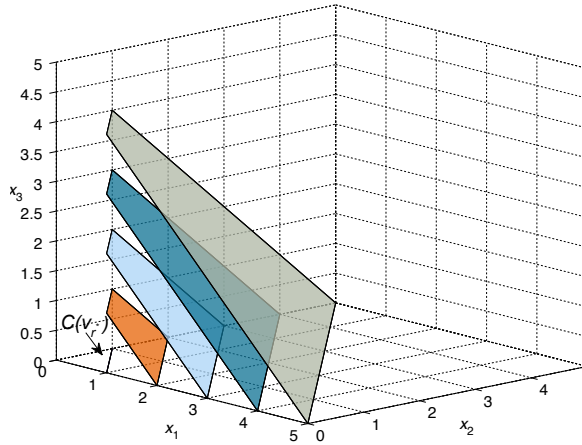


Figure 2.3: Left to right: $C(v_r)$ for $r = \{1, 2, 3, 4, 5\}$ for $v \in G_m$.

As another example, consider $N = \{1, 2, 3\}$ and the game $v(1) = v(2) = 0.5$,

$v(123) = 5$, and $v(S) = 0$ otherwise. Note that

$$C(v) = \{(0.5 + b, 0.5 + c, d) : 0 \leq b, c, d \text{ and } b + c + d = 4\}.$$

Figure 2.4 below shows the cores of v_r for $r = \{1, 2, 3, 4, 5\}$. Note that v_1 corresponds to v_{r^*} , and $C(v_{r^*}) = \{(0.5, 0.5, 0)\}$ is a singleton, thus $v \in G_s$.

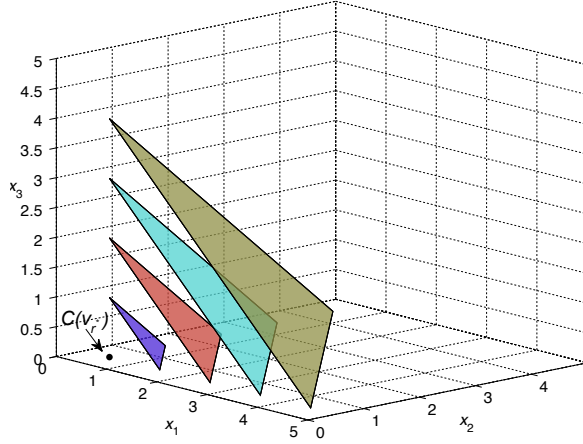


Figure 2.4: Left to right: $C(v_r)$ for $r = \{1, 2, 3, 4, 5\}$ for $v \in G_s$.

In general, given $v \in G_c^N$, let R be the region defined by the collection of the inequalities $(x(S) \geq v(S))_{\emptyset \neq S \subset N}$. For each $r \in \mathbb{R}$, let P_r denote the hyperplane $x(N) = r$. Note that for each $r \in \mathbb{R}$, the normal vector of P_r is $(1, \dots, 1)$. Remember that $C(v) = R \cap P_{v(N)}$. Also, in order to find the root game of v , one looks for minimum value of $r \in \mathbb{R}$ such that $R \cap P_r \neq \emptyset$; which in fact is denoted by r^* . Note that $v_{r^*} \in G_{mul}$ if there is a line segment (or a hyperplane segment) on the boundary of R with the normal vector $(1, \dots, 1)$, and $v_{r^*} \in G_{sin}$ otherwise. Geometrically, given $v \in G_c$, the probability of having a line segment (or a hyperplane segment) on the boundary of R with the normal vector $(1, \dots, 1)$ is nearly zero. Thus, probabilistic measure of the set G_{mul} is zero. Therefore, nearly all of the games in G_c are in G_s . Similar geometric results hold for the set $\overline{G_c}$.

Our next results in this section explain the above geometric reasonings more precisely by minimal balanced collections. They allow us to compare the cardinalities of the sets and understand the structure of games more precisely.

The first theorem concerns the games in G_r . It is pretty straightforward to drive this theorem by the (strong version of) Bondareva-Shapley theorem. Yet, it gives an obvious characterization of the games in G_r , and helps us to consider the latter theorems given in this section.

Theorem 2.2.3. *For each player set N , $v \in G_r^N$ if and only if the following conditions hold:*

- (i) *for each minimal balanced collection $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, inequality*

$$\sum_{j=1}^k \lambda_j v(S_j) \leq v(N) \quad (2.2.2)$$

holds,

- (ii) *there is at least one minimal balanced collection different than $\{N\}$, say $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, inequality (2.2.2) is an equality; that is $\sum_{j=1}^k \lambda_j v(S_j) = v(N)$.*

Proof. ($\Rightarrow$) Let $v \in G_r^N$, then the decomposition of v is $v = v_{r^*} \oplus w = v_{r^*} + w$, where $v_{r^*} \equiv v$ and for each $S \subseteq N$, $w(S) = 0$. Note, by the strong version of the Bondareva-Shapley theorem, (i) holds obviously. For to show (ii), suppose there is not any minimal balanced collection satisfying inequality (2.2.2) as an equality, except the collection $\{N\}$. Then, by (i) for each minimal balanced collection, say $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, $\sum_{j=1}^k \lambda_j v(S_j) < v(N)$. Let $\mathbb{B}$ be the set of all minimal balanced collections of the player set N , except the minimal balanced collection $\{N\}$. Define the number K_v ¹³ as follows:

$$K_v := \max_{\mathcal{B} \in \mathbb{B}} \sum_{S_j \in \mathcal{B}} \lambda_j v(S_j). \quad (2.2.3)$$

Obviously, for each minimal collection $\mathcal{B}$, $\mathcal{B} \neq \{N\}$, say $\mathcal{B} = \{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, $\sum_{j=1}^k \lambda_j v(S_j) \leq K_v < v(N)$. Now, for each $S \subset N$, define $\tilde{v}$ as $\tilde{v}(S) = v(S)$ and $\tilde{v}(N) = K_v$. By the Bondareva-Shapley theorem, $C(\tilde{v}) \neq \emptyset$, contradicting to the definition of r^* , since $K_v = \tilde{v}(N) < v(N) = v_{r^*}(N)$. Hence, (ii) holds.

¹³For each $v \in G$, indeed $r^* = K_v$.

($\Leftarrow$) Let (i) and (ii) hold for some $v \in G^N$. By the Bondareva-Shapley theorem, $C(v) \neq \emptyset$, so that $v \in G_c^N$. Then, $v = v_{r^*} \oplus w = v_{r^*} + w$. Moreover by (ii), there is at least one minimal balanced collection, say $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$ satisfying $\sum_{j=1}^k \lambda_j v(S_j) = v(N)$. Now, obviously $v_{r^*} \equiv v$, cause otherwise $v_{r^*}(N) < v(N)$, but then by definition of v_{r^*} , for the minimal balanced collection satisfying the inequality,

$$\sum_{j=1}^k \lambda_j v_{r^*}(S_j) = \sum_{j=1}^k \lambda_j v(S_j) = v(N) > v_{r^*}(N),$$

which contradicts to the fact that $C(v_{r^*}) \neq \emptyset$ and thus satisfies the conditions of the Bondareva-Shapley theorem. Hence, $v_{r^*} \equiv v$, that is $v \in G_r^N$. $\square$

Next, we give necessary and sufficient conditions for the set of games each of which has a single vector in it is core, i.e., for G_{sin} . The result leads also to sufficient conditions for G_{mul} . Moreover, using these results, we compare the cardinalities of the set of games given via decomposition.

We first analyze the special case $|N| = 3$.

For $N = \{1, 2, 3\}$, the minimal balanced collections that are different than $\{N\}$ are $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, $\{\{3\}, \{1, 2\}\}$, $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Now, we have the following characterization of G_{mul}^N :

Theorem 2.2.4. *Let the player set be $N = \{1, 2, 3\}$. $v \in G_{mul}^N$ if and only if the inequality (2.2.2) of Theorem 2.2.3 is a strict inequality at the minimal balanced collections $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and it is an equality at only one of the minimal balanced collections below:*

(i) $\{\{1\}, \{2, 3\}\}$, (ii) $\{\{2\}, \{1, 3\}\}$, (iii) $\{\{3\}, \{1, 2\}\}$.

Proof. We know $G_{mul}^N \subset G_r^N$, thus by Theorem 2.2.3, we only need to show that the inequality (2.2.2) of Theorem 2.2.3 is an equality at only one of the minimal balanced collections given in the theorem gives us the fact that $v \in G_{mul}^N$, and otherwise gives $v \in G_{sin}^N$. For $N = \{1, 2, 3\}$, the minimal balanced collections that are different than $\{N\}$ are $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, $\{\{3\}, \{1, 2\}\}$, $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

We consider the cases one by one.

Assume that the inequality (2.2.2) of Theorem 2.2.3 is an equality at $\{\{1\}, \{2\}, \{3\}\}$, then $v(1) + v(2) + v(3) = v(123)$. But then for $x \in C(v)$, obviously, for each $i \in N$, $x_i = v(i)$. Thus $v \in G_{sin}^N$. Therefore, for to have $v \in G_{mul}^N$, the inequality (2.2.2) of Theorem 2.2.3 can not be an equality at $\{\{1\}, \{2\}, \{3\}\}$.

Now, assume that the inequality (2.2.2) of Theorem 2.2.3 is an equality at $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, then we have $v(12) + v(13) + v(23) = 2v(123)$. For $x \in C(v)$, we know

$$x_1 + x_2 \geq v(12),$$

$$x_1 + x_3 \geq v(13),$$

$$x_2 + x_3 \geq v(23).$$

Adding them up, we get $2[x_1 + x_2 + x_3] \geq v(12) + v(13) + v(23) = 2v(123)$. But then, for each $i, j, k \in N$, we have $x_i + x_j = v(ij)$. Thus, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v(12) \\ v(13) \\ v(23) \end{bmatrix}$$

which has a unique solution, because the determinant of the 3×3 matrix on the left hand side is non-zero. Therefore, for to have $v \in G_{mul}$, the inequality (2.2.2) of Theorem 2.2.3 can not be an equality at $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Now assume that the inequality (2.2.2) of Theorem 2.2.3 is an equality at any two of the balanced collections, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$ and $\{\{3\}, \{1, 2\}\}$. Without loss in generality, let the inequality (2.2.2) of Theorem 2.2.3 be an equality at $\{\{1\}, \{2, 3\}\}$ and $\{\{2\}, \{1, 3\}\}$. Then we have $v(1) + v(23) = v(123) = v(2) + v(13)$. Now, for $x \in C(v)$, we have $x_1 \geq v(1)$ and $x_2 + x_3 \geq v(23)$, but adding them up and using the previous equality, we get $x_1 = v(1)$. Similarly, $x_2 = v(2)$. Then, since $x_1 + x_2 + x_3 = v(123)$, we have x_3 uniquely determined as well. Thus, $v \in G_{sin}^N$. Hence, the inequality (2.2.2) of Theorem 2.2.3 can not be an equality at any two (or three) of the balanced collections, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$ and $\{\{3\}, \{1, 2\}\}$. Thus, given G_r^N , to have a game $v \in G_{mul}$, due to symmetry, the only possibilities are to have an equality at only one of the

minimal balanced collections below (except $\{N\}$) :

(i) $\{1\}, \{2, 3\}$, (ii) $\{2\}, \{1, 3\}$, (iii) $\{3\}, \{1, 2\}$. □

By the negation of this theorem combined with Theorem 2.2.3, one can easily get a characterization of G_{sin}^N for $N = \{1, 2, 3\}$.

For the case $|N| = 3$, there are $2^5 - 1 = 31$ possible cases that the inequality (2.2.2) of Theorem 2.2.3 is an equality (since equality can hold at a unique minimal balanced collection or at multiple minimal balanced collections)¹⁴. Thus, by Theorem 2.2.4, given $v \in G_r^N$, the probability of v being in G_{mul}^N is $3/31 \approx 0.1$ and the probability of v being in G_{sin}^N is $28/31 \approx 0.9$. Thus in fact, given $v \in G_c^N$, the probability of v being in G_s^N is approximately 0.9.

Before giving our general result for any $|N| \geq 3$, we first study the special case $|N| = 4$, which provide insight to the general case.

Table 2.1: Minimal balanced collections for $N = \{1, 2, 3, 4\}$ (up to symmetries)

Type	Collection	Balancing coefficients	Number
1	$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$	$1/3, 1/3, 1/3, 1/3$	1
2	$\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}$	$2/3, 1/3, 1/3, 1/3$	4
3	$\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3\}$	$1/2, 1/2, 1/2, 1/2$	12
4	$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}$	$1/2, 1/2, 1/2, 1$	4
5	$\{1\}, \{2\}, \{3\}, \{4\}$	$1, 1, 1, 1$	1
6	$\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}$	$1/2, 1/2, 1/2$	6
7	$\{1, 2\}, \{3\}, \{4\}$	$1, 1, 1$	6
8	$\{1, 2, 3\}, \{4\}$	$1, 1$	4
9	$\{1, 2\}, \{3, 4\}$	$1, 1$	3
Total:			41

For $N = \{1, 2, 3, 4\}$, the minimal balanced collections different than $\{N\}$, up to symmetries are given by the above table.

The table shows all the minimal balanced collections different than $\{N\} = \{\{1, 2, 3, 4\}\}$, their corresponding balancing coefficients and the number

¹⁴For $N = \{1, 2, 3\}$, there are 5 different balanced collections different than $\{N\}$. Thus, there exist $2^5 - 1$ non-empty subsets of the set of balanced collections that may satisfy the equality.

of minimal balanced collections of that type considering its symmetries. Since there exists 41 different minimal balanced collections in total, different than $\{N\}$, there exists $2^{41} - 1$ possible cases that the inequality (2.2.2) of Theorem 2.2.3 can be an equality, thus can satisfy conditions of Theorem 2.2.3.

For $|N| = 4$, given a game $v \in G_r^N$, one can check that if the inequality (2.2.2) of Theorem 2.2.3 is an equality for *any* of the minimal balanced collections that is of type i , $i \in \{1, 2, 3, 4, 5\}$, then $|C(v)| = 1$, and thus $v \in G_{sin}^N$. Similarly, if the inequality (2.2.2) of Theorem 2.2.3 is an equality for *only one* minimal balanced collection that is either type 6 or 7 or 8 or 9, then $|C(v)| > 1$, thus $v \in G_{mul}^N$. For example, if *the only* equality is at $\{1, 2\}, \{3\}, \{4\}$, because of the dependance of the payoffs of player 1 and player 2, $|C(v)| > 1$.

The above information in the last two paragraphs gives us the following. If $|N| = 4$ and $v \in G_r^N$, then the probability of v being in G_{mul}^N is less than $(2^{19} - 1)/(2^{41} - 1) \approx 2.3 \times 10^{-7}$, and the probability of v being in G_{sin}^N is approximately $1 - (2.3 \times 10^{-7}) \approx 1$. Thus in fact, when $|N| = 4$, given $v \in G_c^N$, the probability of v being in G_s^N is approximately $1 - (2.3 \times 10^{-7}) \approx 1$. In other words, for $|N| = 4$, nearly all the games in G_c^N have the structure $v = v_{r*} + w$, where $v_{r*} \in G_{sin}^N$ and $w \in G_z^N$.

This gives us the information that given $v \in G_c^N$, it is more likely to have $v \in G_s^N$, thus by definition of the core, there is a unique payoff vector, say $x_v \in \mathbb{R}^4$ such that $C(v_{r*}) = \{x_v\}$, and

$$C(v) = \{x_v + (a_1, a_2, a_3, a_4) : 0 \leq a_1, a_2, a_3, a_4, \sum_{i=1}^4 a_i = v(N) - v_{r*}(N)\}.$$

In light of the case $N = \{1, 2, 3, 4\}$ discussed above, we have the following general result.

Theorem 2.2.5. *Let $v \in G_r^N$. If $v \in G_{sin}^N$, then for each pair $i, j \in N$, there is at least one minimal balanced collection different than $\{N\}$, say $\mathcal{P}_{ij} = \{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$ satisfying $\sum_{l=1}^k \lambda_l v(S_l) = v(N)$, at which there is at least one coalition $S \in \mathcal{P}_{ij}$ such that $i \in S$, but $j \notin S$.*

Proof. Let $v \in G_r^N$. Without loss in generality, let $C(v) = \{x\} = \{(x_1, \dots, x_n)\}$ such that for each pair $i, j \in N$, $x_i + x_j \neq 0$.¹⁵ For each pair $i, j \in N$, let

$$\mathcal{A}_{ij} := \{S \subset N : i \in S, j \notin S\},$$

$$\mathcal{B}_{ij} := \{S \subset N : i \notin S, j \in S\},$$

$$\mathcal{C}_{ij} := \{S \subset N : i, j \notin S\},$$

$$\mathcal{D}_{ij} := \{S \subseteq N : i, j \in S\}.$$

Note that any subset of N is in $\mathcal{A}_{ij} \cup \mathcal{B}_{ij} \cup \mathcal{C}_{ij} \cup \mathcal{D}_{ij}$, and $\mathcal{A}_{ij}, \mathcal{B}_{ij}, \mathcal{C}_{ij}, \mathcal{D}_{ij}$ are pairwise disjoint.

For each pair $i, j \in N$, define $w_{ij} \in G^N$ as follows:

$$w_{ij}(S) := \begin{cases} v(S) - x(S) + x_i & \text{if } S \in \mathcal{A}_{ij}, \\ v(S) - x(S) + x_j & \text{if } S \in \mathcal{B}_{ij}, \\ v(S) - x(S) & \text{if } S \in \mathcal{C}_{ij}, \\ v(S) - x(S) + x_i + x_j & \text{if } S \in \mathcal{D}_{ij}. \end{cases}$$

Note, since $x \in C(v)$, for each $S \in \mathcal{A}_{ij}$, $w_{ij}(S) \leq x_i$; for each $S \in \mathcal{B}_{ij}$, $w_{ij}(S) \leq x_j$; for each $S \in \mathcal{C}_{ij}$, $w_{ij}(S) \leq 0$; for each $S \in \mathcal{D}_{ij}$, $w_{ij}(S) \leq x_i + x_j$, and $w_{ij}(N) = x_i + x_j$.

For each pair $i, j \in N$, let $\tilde{x}_{ij} := (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)$. For each $S \subseteq N$, $v(S) = w_{ij}(S) + \tilde{x}_{ij}(S)$. Since core is a solution concept satisfying covariant under strategic equivalence property,¹⁶ $C(v) = C(w_{ij}) + \tilde{x}_{ij}$. Thus, $C(w_{ij}) = \{x - \tilde{x}_{ij}\}$. Now, one can easily see that there is at least one minimal balanced collection different then $\{N\}$, say $\tilde{\mathcal{P}}_{ij} = \{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$ satisfying $\sum_{l=1}^k \lambda_l w_{ij}(S_l) = w_{ij}(N)$, at which there is at least one coalition $S \in \tilde{\mathcal{P}}_{ij}$ such that $i \in S$, but $j \notin S$, cause otherwise $|C(w_{ij})| \neq 1$. We claim that $\tilde{\mathcal{P}}_{ij}$ satisfies the condition given in the conclusion of the theorem. For that, we prove the following lemma.

Lemma 2.2.6. *Let $v \in G_{sin}^N$, $i, j \in N$, w_{ij} be the game defined as above and*

¹⁵If $x_i + x_j = 0$, then nothing will change in the proof. In fact, we take $x_i + x_j \neq 0$ just for clarity of the proof.

¹⁶It is well-known that the core satisfies the following property, which is known as *covariant under strategic equivalence*: If $v, w \in G$, $\alpha > 0$, $\beta \in \mathbb{R}^N$ and $w = \alpha v + \beta$, then $C(w) = \alpha C(v) + \beta$. For the property ‘covariance under strategic equivalence’, see for example [16].

$\mathcal{P} = \{S_1, \dots, S_k\}$ be a minimal balanced collection with balancing coefficients $\lambda_1, \dots, \lambda_k$. Now, $\sum_{j=1}^k \lambda_j w_{ij}(S_j) = w(N)$ if and only if $\sum_{j=1}^k \lambda_j v(S_j) = v(N)$.

Proof. Let the hypothesis of the lemma hold. Note by definition of w_{ij} and definition of balancedness, we have

$$\begin{aligned}
\sum_{S_l \in \mathcal{P}_{ij}} \lambda_l w_{ij}(S_l) &= \sum_{S_l \in \mathcal{P}_{ij} \cap A_{ij}} \lambda_l w_{ij}(S_l) + \sum_{S_l \in \mathcal{P}_{ij} \cap B_{ij}} \lambda_l w_{ij}(S_l) + \sum_{S_l \in \mathcal{P}_{ij} \cap C_{ij}} \lambda_l w_{ij}(S_l) + \sum_{S_l \in \mathcal{P}_{ij} \cap D_{ij}} \lambda_l w_{ij}(S_l) \\
&= \sum_{S_l \in \mathcal{P}_{ij} \cap (A_{ij} \cup B_{ij} \cup C_{ij} \cup D_{ij})} \lambda_l (v(S_l) - x(S_l)) + x_i \sum_{S_l \in \mathcal{P}_{ij} \cap (A_{ij} \cup D_{ij})} \lambda_l + x_j \sum_{S_l \in \mathcal{P}_{ij} \cap (B_{ij} \cup D_{ij})} \lambda_l \\
&= \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l v(S_l) - \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l x(S_l) + x_i \sum_{S_l \in \mathcal{P}_{ij}: i \in S_l} \lambda_l + x_j \sum_{S_l \in \mathcal{P}_{ij}: j \in S_l} \lambda_l \\
&= \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l v(S_l) - x(N) + x_i + x_j \\
&= \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l v(S_l) - v(N) + w(N).
\end{aligned}$$

The above equality gives us the desired result of the lemma. $\square$

Finally, using the above lemma, for each pair $i, j \in N$, $\tilde{\mathcal{P}}_{ij}$ satisfies the necessary condition given in the theorem. $\square$

Theorem 2.2.5 provides a necessary condition for a singleton core in root games, but it is not a sufficient condition, as the next example shows.

First, we need some definitions. Let $v \in G_r^N$ be a game that satisfies the conclusion of Theorem 2.2.5. $\mathbb{BC}_v$ will denote the set of all minimal balanced collections that satisfy the condition given in the conclusion of Theorem 2.2.5. Formally, for each pair $i, j \in N$, $i \neq j$, define the set $\mathbb{B}_{ij}^v \subseteq \mathbb{B}$ as follows: $\mathcal{P} = \{S_1, \dots, S_k\} \in \mathbb{B}_{ij}^v$ (with balancing coefficients $\lambda_1, \dots, \lambda_k$) if $\sum_{l=1}^k \lambda_l v(S_l) = v(N)$, and if there is at least one coalition $S \in \mathcal{P}$ such that $i \in S$, but $j \notin S$. For each pair $i, j \in N$, $i \neq j$, we have $\mathbb{B}_{ij}^v \neq \emptyset$, because v satisfies the conclusion of Theorem 2.2.5. Now, let $\mathbb{BC}_v := \bigcup_{i,j \in N, i \neq j} \mathbb{B}_{ij}^v$. Note that $\mathbb{B}_{ij}^v = \mathbb{B}_{ji}^v$, and $\mathbb{BC}_v$ is well-defined only for v that satisfies the conclusion of Theorem 2.2.5.

Example 2.2.7. Consider $N = \{1, 2, 3, 4\}$ and the game $v(12) = v(13) = v(24) = v(34) = 1$, $v(1234) = 2$, and $v(S) = 0$ otherwise.

Consider the minimal balanced collections $\mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}\}$ and $\mathcal{P}_2 = \{\{1, 3\}, \{2, 4\}\}$ (both with balancing coefficients 1 and 1).

For $\mathcal{P}_1$, we have $v(12) + v(34) = v(1234)$, and for $\mathcal{P}_2$, we have $v(13) + v(24) = v(1234)$.

Note that $\mathbb{B}_{12} = \{\mathcal{P}_2\}$, $\mathbb{B}_{13} = \{\mathcal{P}_1\}$, $\mathbb{B}_{14} = \{\mathcal{P}_1, \mathcal{P}_2\}$, $\mathbb{B}_{23} = \{\mathcal{P}_1, \mathcal{P}_2\}$, $\mathbb{B}_{24} = \{\mathcal{P}_1\}$, $\mathbb{B}_{34} = \{\mathcal{P}_2\}$. Thus, $\mathbb{B}\mathbb{C}_v = \{\mathcal{P}_1, \mathcal{P}_2\}$.

Yet, $C(v) = \{(a, 1 - a, 1 - a, a) : 0 \leq a \leq 1\}$. Thus, $v \in G_{mul} \subset G_r$.

The example shows that the conclusion of Theorem 2.2.5 is not enough for a sufficient condition, yet it gives us the intuition for sufficiency. For a sufficient condition, let $v \in G_r^N$ satisfy the conclusion of Theorem 2.2.5. Let $E_v := \{\emptyset \neq S \subset N : S \in \mathcal{P}, \mathcal{P} \in \mathbb{B}\mathbb{C}_v\}$. Now, for each $S \in E_v$, let $\delta_S = (\delta_1, \dots, \delta_n)$ where $\delta_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$. Note that for each $\emptyset \neq T \subseteq N$, $x(S) = \delta_S \cdot x$. Without loss in generality, let $E_v = \{T_1, \dots, T_m\}$.

Now, define

$$A_v = \begin{bmatrix} \delta_{T_1} \\ \vdots \\ \delta_{T_m} \end{bmatrix}, \quad x^t = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad b_v = \begin{bmatrix} v(T_1) \\ \vdots \\ v(T_m) \end{bmatrix},$$

where A_v is a $m \times n$ matrix, x^t is the $n \times 1$ matrix formed by writing $x \in C(v)$ as a column matrix, and b_v is a $m \times 1$ matrix. Note that E_v , thus, A_v and b_v are well-defined, because v satisfies conclusion of Theorem 2.2.5. If $x \in C(v)$ and $\mathcal{P} = \{S_1, \dots, S_k\} \in \mathbb{B}\mathbb{C}_v$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, then we have

$$\begin{array}{rcl} \lambda_1 x(S_1) & \geq & \lambda_1 v(S_1) \\ & \vdots & \\ \lambda_l x(S_l) & \geq & \lambda_l v(S_l) \\ & \vdots & \\ \lambda_k x(S_k) & \geq & \lambda_k v(S_k) \\ \hline + & & \\ x(N) = \sum_{l=1}^k \lambda_l x(S_l) & \geq & \sum_{l=1}^k \lambda_l v(S_l) = v(N) = x(N). \end{array}$$

Thus, for each $T \in E_v$, we have $x(T) = v(T)$. Hence, $A_v x^t = b_v$. Thus, if the solution of the system of equations given by $A_v x^t = b_v$ is unique, then x is the only element in $C(v)$, i.e., $v \in G_{sin}$. Hence, we have shown the following.

Theorem 2.2.8. *Let $v \in G_r^N$. If for each pair $i, j \in N$, there is at least one minimal balanced collection different then $\{N\}$, say $\mathcal{P}_{ij} = \{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$ satisfying $\sum_{l=1}^k \lambda_l v(S_l) = v(N)$, at which there is at least one coalition $S \in \mathcal{P}_{ij}$ such that $i \in S$, but $j \notin S$ and if $A_v x^t = b_v$ has a unique solution, then $v \in G_{sin}^N$.*

In light of Theorem 2.2.5, for G_{mul} , we also have the following result.

Theorem 2.2.9. *Let $v \in G_r^N$. If the condition*

- *there is at least one pair $i, j \in N$, for each minimal balanced collection $\mathcal{P} = \{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$ satisfying $\sum_{l=1}^k \lambda_l v(S_l) = v(N)$, if $i \in S \in \mathcal{P}$, then $j \in S$,*

holds, then $v \in G_{mul}^N$.

The proof of the theorem is omitted, because the theorem is simply the contrapositive of Theorem 2.2.5 combined with the fact that $G_r = G_{sin} \cup G_{mul}$ where $G_{sin} \cap G_{mul} \neq \emptyset$.

In light of Theorem 2.2.5 and 2.2.9, similar to the case in $|N| = 4$, given any $|N| \geq 4$ and $v \in G_c^N$, the probability of v being in G_s^N is approximately 1. Also note that, as $|N|$ increases, this probability tends to 1 more rapidly. Thus, nearly all games that are in G_c are in G_s , and thus have the structure $v = v_{r^*} + w$, where $v_{r^*} \in G_{sin}$ and $w \in G_z$. Similar results hold for the set of games in $\overline{G}_c$, i.e., nearly all games that are in $\overline{G}_c$ are in $\overline{G}_s$, and thus have the structure $v = v_{r^*} - w$, where $v_{r^*} \in G_{sin}$ and $w \in G_z$.

There is obviously a relation between our results in this section and allocation rules. That is studied in the next part.

2.3 Part II

2.3.1 Preliminaries for allocation rules

An allocation rule for transferable utility games is a function that assigns a payoff vector to each game in G , formally an *allocation rule* is a function $\Gamma : G \rightarrow \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$ such that for each $n \in \mathbb{N}$ and each $v \in G^N$:

$$\Gamma(v) := (\Gamma_1(v), \dots, \Gamma_n(v)) \in \mathbb{R}^N \text{ and } \sum_{i=1}^n \Gamma_i(v) = v(N).$$

The followings are some well-known properties that are satisfied by some of the well-known allocation rules.

- For each $v \in G$, an allocation rule Γ is said to be *core selective* if $\Gamma(v) \in C(v)$ whenever $C(v) \neq \emptyset$.
- For each pair $v, w \in G$, an allocation rule Γ is said to be *core-dependent* if $\Gamma(v) = \Gamma(w)$ whenever $C(v) = C(w) \neq \emptyset$.
- An allocation rule Γ is said to be *zero independent*¹⁷ if for each player set N , each $v, w \in G^N$, and each $\beta \in \mathbb{R}^n$, one has

$$[\forall S \subseteq N : w(S) = v(S) + \sum_{i \in S} \beta_i] \Rightarrow \Gamma(w) = \Gamma(v) + \beta.$$

- An allocation rule Γ is called *additive* if for each player set N ,

$$[v \in G^N \text{ and } w \in G^N] \Rightarrow \Gamma(v + w) = \Gamma(v) + \Gamma(w).$$

- An allocation rule Γ is said to be *monotonic with respect to the value of the grand coalition* if for each N and each $v, w \in G^N$, one has

$$[w(N) > v(N) \text{ and } \forall S \subset N \ w(S) = v(S)] \Rightarrow \forall i \in N, \Gamma_i(w) \geq \Gamma_i(v).$$

Note that, our concerns is in allocation rules that are core selective. Next we define some new properties of allocation rules.

Let $D^N, E^N \subset G^N$ and $D := \bigcup_{N: n \in \mathbb{N}} D^N \subset G$ and $E := \bigcup_{N: n \in \mathbb{N}} E^N \subset G$.

¹⁷Also called translation invariant.

- An allocation rule Γ is said to be *egalitarian on G_z* if for each N , each $v \in G_z^N$ and each $i \in N$, $\Gamma_i(v) = v(N)/|N|$.¹⁸

- An allocation rule Γ is *additive on the domain pair D and E* if for each N ,

$$[v \in D^N \text{ and } w \in E^N] \Rightarrow \Gamma(v + w) = \Gamma(v) + \Gamma(w).$$

- An allocation rule Γ is *subtractive on the domain pair D and E* if for each N ,

$$[v \in D^N \text{ and } w \in E^N] \Rightarrow \Gamma(v - w) = \Gamma(v) - \Gamma(w).$$

- An allocation rule Γ is said to be *core faithful egalitarian on the domain pair D and G_z* if it is core selective, egalitarian on G_z , and additive on the domain pair D and G_z .

- We say that an allocation rule Γ satisfies *equality at the equivalence classes* if $\Gamma(v) = \Gamma(\hat{v})$, whenever $v \mathcal{R} \hat{v}$.

We say that an allocation rule Γ satisfies *equality at the equivalence classes on a domain D* if for each $v \in D$, $\Gamma(v) = \Gamma(\hat{v})$, whenever $v \mathcal{R} \hat{v}$.

Remember our concerns related to the consistency in the introduction. Note that, a natural way of solving the distribution of an increase/decrease of the grand coalition is either by distributing it consistent with the allocation rule or equally. For both in fact by additivity and subtractivity on the domain pair D and G_z is necessary. Thus, the importance of the definitions of additivity and subtractivity on the domain pair D and E is clear. Also, all the rest of the properties are pretty clear, maybe except equality at the equivalence classes. We will briefly discuss equality at the equivalence classes. Note by definition, equality at the equivalence classes on G_{sin} is equivalent to core-dependence on G_{sin} , and equality at the equivalence classes on G_{mul} is equivalent to core-dependence on G_{mul} . In fact, equality at equivalence classes is a generalization of core-dependence. Core-dependence requires for an allocation rule to choose the same point whenever two

¹⁸Note that for each $v \in G$, equal division (ED) is the allocation rule that distributes the value of the grand coalition equally between the players, that is for each $i \in N$, $ED_i(v) = v(N)/|N|$. Hence, if an allocation rule is egalitarian on G_z , then for each game in G_z , it is equal to the equal division (ED).

different games have the same core. Equality at the equivalence classes requires for an allocation rule to choose the same point whenever the root games of two different games have the same core. Yet, core-dependance does not say anything about the games that have an empty core, while equality at the equivalence classes does.¹⁹

Finally, in general, as usual, given any property of an allocation rule Γ , Γ satisfies ‘*the given property*’ on D , if that property is satisfied for the games in D , not necessarily for all the games in G .

2.3.2 Some well-known allocation rules

In this section, we remind some well-known rules, namely the nucleolus [27], the average lexicographic value²⁰ [28], the core-center [29] and the per-capita nucleolus [30].²¹ It is well-known that all of these allocation rules are core selective and egalitarian on G_z . Also, it is known that except the per-capita nucleolus, all of these are *not* monotonic with respect to the value of the grand coalition on G_c . We briefly show by examples, except the per-capita nucleolus all of these allocation rules are *not* additive on the domain pair G_{sin} and G_z .

2.3.2.1 The nucleolus

Let $v \in G^N$ and let $B_v := \{x \in R^N : \sum_{i \in N} x_i = v(N)\}$. For each player set N , each $v \in G^N$ and each $x \in B_v$, let $e(v, x) \in \mathbb{R}^{2^n-1}$ be such that for each $S \in \Omega$,

$$e_S(v, x) := \sum_{i \in S} x_i - v(S).$$

¹⁹Relation between the two properties, namely equality at the equivalence classes and core-dependence is also analyzed in Section 2.3.3, see Remark 2.3.13 for details.

²⁰Also called *AL-value* or *Alexia value* in the literature.

²¹In fact, the AL-value in [28] and the core-center in [29] are defined only for the games in G_c , but are called allocation rules. Thus, we also consider these rules as ‘allocation’ rules here, though how they act on $\overline{G_c}$ is not defined.

Let $\tilde{e}(v, x)$ be the vector obtained by rearranging the coordinates of $e(v, x)$ in increasing order. For each pair $u, v \in \mathbb{R}^n$, u is *leximin-preferred* to v , written as $u \succ_{lex} v$, if (i) $u_1 > v_1$, or (ii) there is $k \in \{1, \dots, n-1\}$ such that $u_{k+1} > v_{k+1}$ and for each $m \leq k$, $u_m = v_m$.

For each $v \in G$, the *nucleolus*, defined by Schmeidler [27] and denoted by Nu , is,

$$Nu(v) := \{x \in B_v : \text{for each } y \in B_v \setminus \{x\}, \tilde{e}(v, x) \succ_{lex} \tilde{e}(v, y)\}.$$

Schmeidler shows that Nu is single-valued and hence we write $Nu(v) = x$ instead of $Nu(v) = \{x\}$.

By definition, obviously, nucleolus is egalitarian on G_z . The next example shows that the nucleolus is not additive on the domain pair G_{sin} and G_z .

Example 2.3.1. Let $N = \{1, 2, 3, 4\}$ and $v \in G_c^N$ be defined by: $v(1) = v(2) = v(3) = v(4) = 0$, $v(12) = v(34) = 0$, $v(13) = v(14) = v(23) = v(24) = 1$, $v(123) = v(124) = 1$, $v(134) = v(234) = 2$ and $v(1234) = 3$.²²

$v = v_{r^*} + w$ is the decomposition of v , where

$$v_{r^*}(S) := \begin{cases} v(S) & \text{if } S \subset N, \\ 2 & \text{if } S = N \text{ and,} \end{cases}$$

$$w(S) := \begin{cases} 0 & \text{if } S \subset N, \\ 1 & \text{if } S = N. \end{cases}$$

Note that $Nu(v_{r^*}) = (0, 0, 1, 1)$ and $Nu(w) = (1/4, 1/4, 1/4, 1/4)$, but obviously $Nu(v_{r^*}) + Nu(w) = (1/4, 1/4, 5/4, 5/4) \neq Nu(v)$.

2.3.2.2 The per-capita nucleolus

Let $v \in G^N$. For each $S \in \Omega$, set $\bar{e}_S(v, x) = \frac{1}{|S|}e_S(v, x)$. Let $\hat{e}(v, x)$ be the vector obtained by rearranging the coordinates of $\bar{e}(v, x)$ in increasing order.

²²Core of v is the convex hull of the points $(1, 0, 1, 1)$, $(0, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 1)$, $(0, 0, 2, 1)$ and $(0, 0, 1, 2)$.

For each $v \in G$, the *per-capita nucleolus*, defined by Grotte [30] and denoted by $Nupc$, is,

$$Nupc(v) := \{x \in B_v : \text{for each } y \in B_v \setminus \{x\}, \hat{e}(v, x) \succ_{lex} \hat{e}(v, y)\}.$$

As in the case of Nu , $Nupc$ is also single-valued and hence we write $Nupc(v) = x$ instead of $Nupc(v) = \{x\}$.

By definition obviously, the per-capita nucleolus is egalitarian on G_z . In the literature, it is well-known that the per-capita nucleolus is monotonic with respect to the value of the grand coalition, and in fact it divides an increase in the value of the grand coalition equally between the players. Therefore the per-capita nucleolus is obviously additive on the domain pair G_r and G_z and hence core faithful egalitarian on the domain pair G_r and G_z . Since $v \in G_{sin} \subset G_r$ implies $|C(v)| = 1$, specifically, there is a unique allocation rule *defined on* G_s that is core faithful egalitarian on the domain G_{sin} and G_z , namely the per-capita nucleolus. In other words, if the allocation rule Γ is core faithful egalitarian on the domain G_{sin} and G_z , then for each $v \in G_s$, $\Gamma(v) = Nupc(v)$.

2.3.2.3 The average lexicographic value

For each $v \in G_c^N$ and any ordering $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ of the player set N , the *lexicographic maximum of the core* $C(v)$ with respect to σ is denoted by $S^\sigma(v)$, defined as the unique point of the core $C(v)$ with the properties: $(S^\sigma(v))_{\sigma(1)} = \max\{x_{\sigma(1)} : x \in C(v)\}$, $(S^\sigma(v))_{\sigma(2)} = \max\{x_{\sigma(2)} : x \in C(v) \text{ with } x_{\sigma(1)} = (S^\sigma(v))_{\sigma(1)}\}, \dots$, $(S^\sigma(v))_{\sigma(n)} = \max\{x_{\sigma(n)} : x \in C(v) \text{ with } (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n-1)}) = ((S^\sigma(v))_{\sigma(1)}, (S^\sigma(v))_{\sigma(2)}, \dots, (S^\sigma(v))_{\sigma(n-1)})\}$. Note that $S^\sigma(v)$ is an extreme point of the core for each σ .

Let $\Pi(N)$ denote the set of n orderings of N . The *average lexicographic value* of v , defined by Tijs [28] and denoted by $AL(v)$, is the average over all $S^\sigma(v)$,

that is,

$$AL(v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} S^\sigma(v).^{23}$$

By definition obviously, the AL -value is egalitarian on G_z . The next example shows that it is not additive on the domain pair G_{sin} and G_z .

Example 2.3.2. Let $N = \{1, 2, 3\}$ and $v \in G_c^N$ be defined by: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(123) = 4$.

$v = v_{r^*} + w$ is the decomposition of v , where

$$v_{r^*}(S) := \begin{cases} v(S) & \text{if } S \subset N, \\ 1 & \text{if } S = N \text{ and,} \end{cases}$$

$$w(S) := \begin{cases} 0 & \text{if } S \subset N, \\ 3 & \text{if } S = N. \end{cases}$$

Note that $AL(v_{r^*}) = (1, 0, 0)$ and $AL(w) = (1, 1, 1)$, but obviously $AL(v_{r^*}) + AL(w) = (2, 1, 1) \neq AL(v) = (\frac{10}{6}, \frac{7}{6}, \frac{7}{6})$.

2.3.2.4 The core-center

Given $v \in G_c^N$, suppose all core points are equally valuable. This can be formalized by endowing the core with the uniform distribution. ‘The core-center of v ’ is defined as the mathematical expectation of such a probability distribution. Let $U(A)$ denote the uniform distribution defined over the set A and $E(\mathbb{P})$ the expectation of the probability distribution $\mathbb{P}$. For each $v \in G_c^N$, the *core-center of v* [29], denoted by $CC(v)$, is defined by $CC(v) := E(U(C(v)))$.²⁴

By definition obviously, the core-center is egalitarian on G_z . The next example shows that it is not additive on the domain pair G_{sin} and G_z .

²³ The definition of the AL -value is taken from the paper [28].

²⁴ The definition of the core-center is taken from the paper [29].

Example 2.3.3. Consider the same game in Example 2.3.1. Note that $CC(v_{r*}) = (0, 0, 1, 1)$ and $CC(w) = (1/4, 1/4, 1/4, 1/4)$, but obviously $CC(v_{r*}) + CC(w) = (1/4, 1/4, 5/4, 5/4) \neq CC(v) = (1/2, 1/2, 1, 1)$.

2.3.3 Additivity on the domain pair G_{sin} and G_z for core selective allocation rules

An increase of k units of the grand coalition can be distributed either consistent with the allocation rule that is used before or equally between the players. For both of such distributions, an allocation rule should be additive on the domain pair G_{sin} and G_z . Moreover, in Section 2.2.3, we have shown that nearly all games in G_c are formed by the addition of a game from G_{sin} and a game from G_z . In this section, we give necessary and sufficient conditions for core selective allocation rules that keep the same addition structure. Thus, we analyze additivity on the domain pair G_{sin} and G_z for core selective allocation rules. Before that, we want to note the following proposition.

Proposition 2.3.4. An allocation rule is additive on the domain pair G_s and G_z if and only if it is additive on the domain pair G_{sin} and G_z and additive on the domain pair G_z and G_z .

Proof. ($\Rightarrow$) Directly follows from facts that $G_{sin} \subset G_s$ and $G_z \subset G_s$.

($\Leftarrow$) Let $v \in G_s$ and $\bar{w} \in G_z$. We want to show that $\Gamma(v + \bar{w}) = \Gamma(v) + \Gamma(w)$.

Note that $v \in G_s$ implies that there exist $v_{r*} \in G_{sin}$ and $w \in G_z$ such that $v = v_{r*} + w$. Then, by additivity on the domain pair G_{sin} and G_z , we have $\Gamma(v) = \Gamma(v_{r*}) + \Gamma(w)$.

Then, $\Gamma(v) + \Gamma(\bar{w}) = \Gamma(v_{r*}) + \Gamma(w) + \Gamma(\bar{w}) = \Gamma(v_{r*}) + \Gamma(w + \bar{w})$, where the last equality follows from additivity on the domain pair G_z and G_z . Note that $w + \bar{w} \in G_z$, thus using additivity on the domain pair G_{sin} and G_z one more time, we have $\Gamma(v_{r*}) + \Gamma(w + \bar{w}) = \Gamma(v_{r*} + w + \bar{w}) = \Gamma(v + \bar{w})$. Thus, $\Gamma(v + \bar{w}) = \Gamma(v) + \Gamma(w)$. $\square$

We first show that if a core selective allocation rule is additive on the domain pair G_{sin} and G_z , which is an algebraic property, then it is zero independent on G_s and core-dependent on G_s , which are both geometric properties.²⁵ First we show couple of lemmas.

Lemma 2.3.5. *Let $v \in G_c^N$ such that root game of v is itself, that is $v = v + w$ is the decomposition of v where $w \in G_z$ such that $w(N) = 0$. Then*

$$\forall \beta \in \mathbb{R}^N, \forall S \subseteq N, \hat{v}(S) := v(S) + \beta(S) \Rightarrow C(\hat{v}) = C(v) + \beta.$$

Lemma 2.3.5 is a special case of the covariant under strategic equivalence property of the core, yet we give its proof for clarity.

Proof. Let v satisfy the hypothesis of the lemma and let β be any vector in $\mathbb{R}^N$. For each $S \subseteq N$, define $\hat{v}(S) := v(S) + \beta(S)$.

First we prove that $C(v) + \beta \subseteq C(\hat{v})$. Let $x \in C(v) + \beta$. Then there exists $y \in C(v)$ such that $x = y + \beta$. By $y \in C(v)$, for each $S \subseteq N$, $y(S) \geq v(S)$, which implies $x(S) = y(S) + \beta \geq v(S) + \beta = \hat{v}(S)$. Also obviously, $x(N) = y(N) + \beta(N) = \hat{v}(N)$. Hence $x \in C(\hat{v})$. Hence, $C(v) + \beta \subseteq C(\hat{v})$.

Next we prove that $C(\hat{v}) \subseteq C(v) + \beta$. Suppose not, then there is $x \in C(\hat{v})$ such that $x \notin C(v) + \beta$. Then

$$y := x - \beta \notin C(v). \tag{2.3.1}$$

By $x \in C(\hat{v})$, for each $S \subseteq N$, $x(S) \geq \hat{v}(S)$ and hence we have $y(S) = x(S) - \beta(S) \geq \hat{v}(S) - \beta(S) = v(S)$. Also, $y(N) = x(N) - \beta(N) = v(N)$. Then $y \in C(v)$, contradicting to Equation (2.3.1). Hence for each $x \in C(\hat{v})$, $x \in C(v) + \beta$. So $C(\hat{v}) \subseteq C(v) + \beta$. Therefore $C(\hat{v}) = C(v) + \beta$. $\square$

²⁵Given the cores of games geometrically and the points on the space that are selected by an allocation rule acting on these games, one can check if the allocation rule satisfies core-dependence or not, without knowing what the actual games are; that is without knowing the worths of the coalitions. In a similar manner, one can check if an allocation rule satisfies zero independence or not by knowing the geometric structure of the given two games satisfying the hypothesis of the condition and the points that are selected by the allocation rule acting on them, without knowing what the actual games are. But that is not possible for the property additivity on the domain pair G_{sin} and G_z .

For each pair of sets A and B , $A + B := \{a + b : a \in A, b \in B\}$.

Lemma 2.3.6. *Let $v \in G_c^N$ and $v = v_{r*} \oplus w = v_{r*} + w$ be the decomposition of v . Then $C(v_{r*}) + C(w) \subseteq C(v)$.*

Proof. Let $v \in G_c^N$ and $v = v_{r*} + w$ be the decomposition of v . For each $x \in C(v_{r*}) + C(w)$, there is $y \in C(v_{r*})$ and $z \in C(w)$ such that $x = y + z$. Then obviously for each $S \subset N$, $x(S) = y(S) + z(S) \geq y(S) + 0 \geq v_{r*}(S) = v(S)$ and $v(N) = x(N) = y(N) + z(N)$, hence $x \in C(v)$. So $C(v_{r*}) + C(w) \subseteq C(v)$. $\square$

A more deep study of cores of combined games is considered by Bloch and Clippel [26]. Namely, in their paper, Bloch and de Clippel give under which conditions, $\tilde{v} = v + w$ implies $C(\tilde{v}) = C(v) + C(w)$ and discuss about $C(\tilde{v}) \subseteq C(v) + C(w)$ and $C(\tilde{v}) \supseteq C(v) + C(w)$. Lemma 2.3.6 is a very special case of cores of combined games, which is enough for our purposes.

Lemma 2.3.7. *Let $v, \hat{v} \in G_c^N$ such that $C(v) = C(\hat{v})$ and let the decompositions of v and $\hat{v}$ be $v = v_{r*} + w$ and $\hat{v} = \hat{v}_{r*} + \hat{w}$, respectively. Then $C(v_{r*}) = C(\hat{v}_{r*})$ and $w \equiv \hat{w}$.*

Proof. Let v and $\hat{v}$ satisfy the hypothesis of the lemma. If $v, \hat{v} \in G_r^N$, then the lemma holds obviously; thus suppose $v, \hat{v} \in G_c^N \setminus G_r^N$.

First we prove for each $x \in C(v_{r*})$ and $S \subset N$, $x(S) \geq \hat{v}_{r*}(S)$. Suppose not, then there is $x \in C(v_{r*})$ and $S_0 \subset N$ such that

$$x(S_0) < \hat{v}_{r*}(S_0). \quad (2.3.2)$$

Define $y \in \mathbb{R}^N$ as follows:

$$y_i = \begin{cases} 0 & \text{if } i \in S_0, \\ \frac{w(N)}{|N| - |S_0|} & \text{if } i \in N \setminus S_0. \end{cases}$$

Note that $S_0 \subset N$ implies $N \setminus S_0 \neq \emptyset$ and hence y is well-defined and for each $i \in N$, $y_i \geq 0$. Also observe that $y(N) = w(N)$ and hence $y \in C(w)$. Then by Lemma 2.3.6 and by the hypothesis of the lemma for each $x \in C(v_{r*})$,

$x + y \in C(v_{r^*}) + C(w) \subseteq C(v_{r^*} + w) = C(v) = C(\hat{v})$. But then $x(S_0) + y(S_0) = x(S_0) + 0 \geq \hat{v}(S_0) = \hat{v}_{r^*}(S_0)$, contradicting to Equation (2.3.2).

Hence,

$$\forall x \in C(v_{r^*}) \text{ and } \forall S \subset N, x(S) \geq \hat{v}_{r^*}(S). \quad (2.3.3)$$

Similarly,

$$\forall y \in C(\hat{v}_{r^*}) \text{ and } \forall S \subset N, y(S) \geq v_{r^*}(S). \quad (2.3.4)$$

Now, we prove that $v_{r^*}(N) = \hat{v}_{r^*}(N)$. Suppose $v_{r^*}(N) > \hat{v}_{r^*}(N)$. Then for each $y \in C(\hat{v}_{r^*})$, $y(N) = \hat{v}_{r^*}(N) < v_{r^*}(N)$. By the last inequality together with Equation (2.3.4), each $y \in C(\hat{v}_{r^*})$ is a vector in $\mathbb{R}^N$ such that $y(N) < v_{r^*}(N)$ and for each $S \subset N$, $y(S) \geq v_{r^*}(S)$, which contradicts to the minimality of $r^* = v_{r^*}(N)$. Hence $v_{r^*}(N) \leq \hat{v}_{r^*}(N)$.

Similarly, $\hat{v}_{r^*}(N) \leq v_{r^*}(N)$. Hence,

$$v_{r^*}(N) = \hat{v}_{r^*}(N). \quad (2.3.5)$$

Therefore by Equations (2.3.3), (2.3.4) and (2.3.5), $C(v_{r^*}) = C(\hat{v}_{r^*})$, and thus $w \equiv \hat{w}$. $\square$

Proposition 2.3.8. *Let Γ be a core selective allocation rule. If Γ is additive on the domain pair G_{sin} and G_z , then Γ is zero independent on G_s and core-dependent on G_s .*

Proof. Let Γ be a core selective allocation rule that is additive on the domain pair G_{sin} and G_z . First we prove that Γ is zero independent on G_s^N , for each player set N . Let v be a game in G_s^N , then there is a number $r^* \in \mathbb{R}$ and there exist two games v_{r^*} and w such that decomposition of v is $v = v_{r^*} + w$ where

$$v_{r^*}(S) = \begin{cases} v(S) & \text{if } S \subset N, \\ r^* & \text{if } S = N, \end{cases}$$

so that $|C(v_{r^*})| = 1$ and

$$w(S) = \begin{cases} 0 & \text{if } S \subset N, \\ v(N) - r^* & \text{if } S = N. \end{cases}$$

Since $v_{r^*} \in G_{sin}^N$ and $w \in G_z^N$, by additive on the domain pair G_{sin} and G_z ,

$$\Gamma(v) = \Gamma(v_{r^*}) + \Gamma(w). \quad (2.3.6)$$

For each $\beta \in \mathbb{R}^N$, let now $\hat{v}(S) = v(S) + \sum_{i \in S} \beta_i$ for each $S \subseteq N$. Define $\hat{v}_{r^*}(S) = v_{r^*}(S) + \sum_{i \in S} \beta_i$ for each $S \subseteq N$. Note that $\hat{v} = \hat{v}_{r^*} \oplus w = \hat{v}_{r^*} + w$, where $|C(\hat{v}_{r^*})| = 1$ so that $\hat{v}_{r^*} \in G_{sin}^N$. Since also $w \in G_z^N$, by additive on the domain pair G_{sin} and G_z ,

$$\Gamma(\hat{v}) = \Gamma(\hat{v}_{r^*}) + \Gamma(w). \quad (2.3.7)$$

Since the root game of v_{r^*} is itself, by Lemma 2.3.5, $C(\hat{v}_{r^*}) = C(v_{r^*}) + \beta$. Then, $C(v_{r^*}) := \{x\}$ implies $C(\hat{v}_{r^*}) = \{x + \beta\}$. Therefore, by the fact that Γ is a core selective allocation rule,

$$\Gamma(\hat{v}_{r^*}) = x + \beta = \Gamma(v_{r^*}) + \beta. \quad (2.3.8)$$

Then by Equations (2.3.6), (2.3.7) and (2.3.8), we have

$$\Gamma(\hat{v}) = \Gamma(\hat{v}_{r^*}) + \Gamma(w) = \Gamma(v_{r^*}) + \beta + \Gamma(w) = \Gamma(v) + \beta.$$

Hence Γ is zero independent on G_s^N .

Next we show that Γ is core-dependent on G_s^N , for each player set N . Let $v, \hat{v} \in G_s^N$ be two games such that $C(v) = C(\hat{v})$. Then, $v(N) = \hat{v}(N)$. Let $v = v_{r^*} + w$ and $\hat{v} = \hat{v}_{r^*} + \hat{w}$ be the decompositions of v and $\hat{v}$, respectively. Then by Lemma 2.3.7, $C(v_{r^*}) = C(\hat{v}_{r^*})$ and $w \equiv \hat{w}$. By $v, \hat{v} \in G_s^N$, we have $|C(v_{r^*})| = |C(\hat{v}_{r^*})| = 1$. Hence $\Gamma(v_{r^*}) = \Gamma(\hat{v}_{r^*})$. Therefore by additivity on the domain pair G_{sin} and G_z ,

$$\Gamma(v) = \Gamma(v_{r^*}) + \Gamma(w) = \Gamma(\hat{v}_{r^*}) + \Gamma(\hat{w}) = \Gamma(\hat{v}),$$

that is Γ is core-dependent on G_s^N .

Since Γ is both zero independent and core-dependent on G_s^N for each player set N , Γ is both zero independent and core-dependent on G_s . $\square$

Remark 2.3.9. The converse of Proposition 2.3.8 is not true in general, that is, if Γ be a core selective allocation rule that is zero independent on G_s and core-dependent on G_s , then Γ need not be additive on the domain pair G_{sin} and G_z . Here is an example.

Let $N = \{1, 2, 3\}$ and $\tilde{v} \in G_c^N$ be defined by: $\tilde{v}(1) = \tilde{v}(2) = \tilde{v}(3) = 0$, $\tilde{v}(12) = 2$, $\tilde{v}(13) = \tilde{v}(23) = 3$ and $\tilde{v}(123) = 7$. Let A be the set of all games with player set N whose core equal to $C(\tilde{v})$, that is $A := \{v \in G^N : C(v) = C(\tilde{v})\}$. For each $\beta \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, let

$$B_\beta := \{v \in G_c^N : \forall S \subseteq N, \hat{v} \in A \Rightarrow v(S) = \hat{v}(S) + \beta(S)\}.$$

Let $Nupc$ denote the per-capita nucleolus. Now let Γ be defined by:

$$\Gamma(v) = \begin{cases} (1, 1, 5) & \text{if } v \in A, \\ (1, 1, 5) + \beta & \text{if } \beta \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \text{ and } v \in B_\beta, \\ Nupc(v) & \text{otherwise.} \end{cases}$$

One can easily check that Γ be a core selective allocation rule that is zero independent on G_s and core-dependent on G_s , but Γ is not additive on the domain pair G_{sin} and G_z . In fact, if $\tilde{v} = \tilde{v}_{r*} + \tilde{w}$ is the decomposition of $\tilde{v}$, then one can easily check $(1, 1, 5) = \Gamma(\tilde{v}) \neq \Gamma(\tilde{v}_{r*}) + \Gamma(\tilde{w}) = (1, 1, 2) + (1, 1, 1) = (2, 2, 3)$.

By the above remark, we see that the conditions of Proposition 2.3.8 is not enough for an characterization of core selective allocation rules that are additive on the domain pair G_{sin} and G_z . Next, we give the characterization. In fact, given a core selective allocation rule, we relate the algebraic property, namely additivity on the domain pair G_{sin} and G_z , with two geometric properties, namely zero independence on G_s and equality at the equivalence classes on the domain G_s .

Theorem 2.3.10. *Let Γ be a core selective allocation rule. Γ is additive on the domain pair G_{sin} and G_z if and only if Γ satisfies zero independence on G_s and equality at the equivalence classes on the domain G_s .*

Proof. ($\Rightarrow$) Let Γ be additive on the domain pair G_{sin} and G_z . We already know that Γ satisfies zero independence on G_s by Proposition 2.3.8. Let $v \in G_s$, $v\mathcal{R}\hat{v}$

and let the decompositions of v and $\hat{v}$ be $v = v_{r*} + w$ and $\hat{v} = \hat{v}_{r*} + \hat{w}$, respectively. Then obviously, since $v\mathcal{R}\hat{v}$ and $v_{r*} \in G_{sin}^N$, $\Gamma(v_{r*}) = \Gamma(\hat{v}_{r*})$ and $w \equiv \hat{w}$. Then by additivity on the domain pair G_{sin} and G_z ,

$$\Gamma(v) = \Gamma(v_{r*}) + \Gamma(w) = \Gamma(\hat{v}_{r*}) + \Gamma(\hat{w}) = \Gamma(\hat{v}).$$

So Γ satisfies equality at the equivalence classes on the domain G_s .

($\Leftarrow$) Let Γ satisfy zero independence on G_s and equality at the equivalence classes on the domain G_s . For each player set N , let $v \in G_{sin}^N$ and $w \in G_z^N$ be any two games. Let $\beta = \Gamma(v)$. For each $S \subseteq N$, define $v'(S) := w(S) + \beta(S)$. Note that root game of v' is $v'_{r*}(\cdot) := \beta(\cdot)$. Then $C(v'_{r*}) = C(v)$ and $(v+w)(N) = v'(N)$ and hence $(v+w)\mathcal{R}v'$. Therefore by equality at the equivalence classes on the domain G_s , $\Gamma(v+w) = \Gamma(v')$. Note also that by zero independence on G_s , $\Gamma(v') = \Gamma(w) + \beta$. Therefore, $\Gamma(v+w) = \Gamma(v) + \Gamma(w)$, that is Γ is additive on the domain pair G_{sin} and G_z . $\square$

For a core selective allocation rule, the two conditions of the above theorem, that is, zero independence on G_s and equality at the equivalence classes on the domain G_s are two independent properties, given in the next remark.

Remark 2.3.11. Note that for a core selective allocation rule, zero independence on G_s and equality at the equivalence classes on the domain G_s are two independent properties. For to see this consider the following example:

$$\Gamma(v) = \begin{cases} (2, 2, 3) & \text{if } v\mathcal{R}\tilde{v}, \\ (5, 2, 3) & \text{if } v\mathcal{R}\hat{v}, \\ Nucp(v) & \text{otherwise,} \end{cases}$$

where $Nucp$ denotes the per capita nucleolus, $\tilde{v}$ and $\hat{v}$ are two games defined on $N = \{1, 2, 3\}$ as follows: $\tilde{v}(i) = 0$, $\tilde{v}(12) = 2$, $\tilde{v}(13) = \tilde{v}(23) = 3$, $\tilde{v}(123) = 7$, and $\hat{v}(i) = 1$, $\hat{v}(12) = 4$, $\hat{v}(13) = \hat{v}(23) = 5$ and $\hat{v}(123) = 10$.

One can easily see that Γ satisfies equality at the equivalence classes on the domain G_s . Note that $\tilde{v} \in G_s$ and for $\beta = (1, 1, 1)$, $\hat{v} = \tilde{v} + \beta$, but $(5, 2, 3) = \Gamma(\hat{v}) \neq \Gamma(\tilde{v}) + \beta = (2, 2, 3) + (1, 1, 1) = (3, 3, 4)$, therefore Γ does not satisfy zero independence on G_s .

By Proposition 2.3.4, we have the following corollary.

Corollary 2.3.12. *Let Γ be a core selective allocation rule. Γ is additive on the domain pair G_s and G_z if and only if Γ is additive on the domain pair G_z and G_z , Γ satisfies zero independence on G_s and equality at the equivalence classes on the domain G_s .*

Remark 2.3.13. Let Γ be a core selective allocation rule. Using Lemma 2.3.7, if Γ satisfies equality at the equivalence classes on $D \subseteq G_c$, then Γ is core-dependent on D . But the converse of this is not true in general.²⁶

Similar results hold for additivity on the domain pair G_{mul} and G_z , which we give below. We want to remind that the set of games that have this addition structure is very rare -as shown in Section 2.2.3. We do not know a much more general case, the necessary and sufficient conditions for additivity on the domain pair G_r and G_z , which is an open problem.

Proposition 2.3.14. *Let Γ be a core selective allocation rule. If Γ is additive on the domain pair G_{mul} and G_z and core-dependent on G_{mul} , then Γ satisfies equality at the equivalence classes on G_m .*

Proof. Let Γ be a core selective allocation rule such that Γ is additive on the domain pair G_{mul} and G_z and core-dependent on G_{mul} . Let $v \in G_m$ and let $v\mathcal{R}\hat{v}$ be such that the decompositions of v and $\hat{v}$ are $v = v_{r*} + w$ and $\hat{v} = \hat{v}_{r*} + \hat{w}$ respectively. Then $C(v_{r*}) = C(\hat{v}_{r*})$ and $w \equiv \hat{w}$. Then by core-dependance on G_{mul} , $\Gamma(v_{r*}) = \Gamma(\hat{v}_{r*})$ and hence by additivity on the domain pair G_{mul} and G_z , $\Gamma(v) = \Gamma(v_{r*}) + \Gamma(w) = \Gamma(\hat{v}_{r*}) + \Gamma(\hat{w}) = \Gamma(\hat{v})$. So, Γ satisfies equality at the equivalence classes on G_m . $\square$

Converse of the above proposition is not true in general, one can easily find a core selective allocation rule that satisfies equality at the equivalence classes on G_m , but is not additive on the domain pair G_{mul} and G_z .

²⁶For example, if Γ is core-dependent on G_s , then Γ need not satisfy equality at the equivalence classes on G_s . In fact, otherwise, by the Theorem 2.3.10 and Proposition 2.3.8, one has “If Γ is zero independent on G_s and core-dependent on G_s , then Γ is additive on the domain pair G_{sin} and G_z ”, which is not true in general as shown at Remark 2.3.9.

Using the above proposition, we have the following.

Corollary 2.3.15. *Let Γ be a core selective allocation rule. If Γ is additive on the domain pair G_{mul} and G_z and core-dependent on G_{mul} , then Γ is core-dependent on G_m .*

Proposition 2.3.16. *Let Γ be a core selective allocation rule. If Γ is additive on the domain pair G_{mul} and G_z and zero independent on G_{mul} , then Γ is zero independent on G_m .*

Proof. Let N be the set of players, and let Γ satisfy the hypothesis of the proposition.

Let v be a game in G_m^N . Then there is a number $\epsilon \in \mathbb{R}$ and there exist games v_{r^*}, w such that decomposition of v is $v = v_{r^*} + w$ and

$$v_{r^*}(S) = \begin{cases} v(S) & \text{if } S \subset N, \\ \epsilon & \text{if } S = N, \end{cases}$$

so that $|C(v_{r^*})| > 1$ and

$$w(S) = \begin{cases} 0 & \text{if } S \subset N, \\ v(N) - \epsilon & \text{if } S = N. \end{cases}$$

Note that $v_{r^*} \in G_{mul}^N$ and $w \in G_z^N$. By additive on the domain pair G_{mul} and G_z ,

$$\Gamma(v) = \Gamma(v_{r^*}) + \Gamma(w). \quad (2.3.9)$$

For each $\beta \in \mathbb{R}^N$, let now $\hat{v}(S) = v(S) + \sum_{i \in S} \beta_i$ for each $S \subseteq N$. Define $\hat{v}_{r^*}(S) = v_{r^*}(S) + \sum_{i \in S} \beta_i$ for each $S \subseteq N$. Note that $\hat{v} = \hat{v}_{r^*} + w$ is the decomposition of $\hat{v}$, where $|C(\hat{v}_{r^*})| > 1$ so that $\hat{v}_{r^*} \in G_{mul}^N$. Since also $w \in G_z^N$, by additive on the domain pair G_{mul} and G_z ,

$$\Gamma(\hat{v}) = \Gamma(\hat{v}_{r^*}) + \Gamma(w). \quad (2.3.10)$$

Since $v_{r^*} \in G_{mul}$, by zero independence on G_{mul} ,

$$\Gamma(\hat{v}_{r^*}) = \Gamma(v_{r^*}) + \beta. \quad (2.3.11)$$

Therefore, by Equations (2.3.9), (2.3.10) and (2.3.11), we have

$$\Gamma(\hat{v}) = \Gamma(\hat{v}_{r*}) + \Gamma(w) = \Gamma(v_{r*}) + \beta + \Gamma(w) = \Gamma(v) + \beta.$$

Hence Γ is zero independent on G_m^N . $\square$

Corollary 2.3.17. *Let Γ be a core selective allocation rule. If Γ is additive on the domain pair G_{mul} and G_z , zero independent on G_{mul} and core-dependent on G_{mul} , then Γ is zero independent on G_m and core-dependent on G_m .*

Remark 2.3.18. The converse of Corollary 2.3.17 is not true in general; hence the converse of Corollary 2.3.15 and the above proposition is not true in general. In fact, if Γ is a core selective allocation rule that is zero independent on G_m and core-dependent on G_m , then Γ need not be additive on the domain pair G_{mul} and G_z . Here is an example.

Let $N = \{1, \dots, 5\}$ and $\tilde{v} \in G_c^N$ be defined by: $\tilde{v}(12) = \tilde{v}(13) = \tilde{v}(45) = 1$, $\tilde{v}(12345) = 7$ and $\tilde{v}(S) = 0$ otherwise. Note that decomposition of $\tilde{v}$ is $\tilde{v} = \tilde{v}_{r*} + \tilde{w}$, where $\tilde{v}_{r*}(12) = \tilde{v}_{r*}(13) = 1 = \tilde{v}_{r*}(45) = 1$, $\tilde{v}_{r*}(12345) = 2$ and $\tilde{v}_{r*}(S) = 0$ otherwise, and $\tilde{w} \in G_z$ such that $\tilde{w}(N) = 5$. Let A be the set of all games with player set N whose core equal to $C(\tilde{v})$, i.e., $A := \{v \in G^N : C(v) = C(\tilde{v})\}$ and let B be the set of all games with player set N whose core equal to $C(\tilde{v}_{r*})$, i.e., $B := \{v \in G^N : C(v) = C(\tilde{v}_{r*})\}$. For each $\beta \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, let $A_\beta := \{v \in G_c^N : \forall S \subseteq N, \hat{v} \in A \Rightarrow v(S) = \hat{v}(S) + \beta(S)\}$ and let $B_\beta := \{v \in G_c^N : \forall S \subseteq N, \hat{v} \in B \Rightarrow v(S) = \hat{v}(S) + \beta(S)\}$.

Let $\Gamma_0(v)$ be a core selective allocation rule that is core-dependent G_m and zero independent on G_m .²⁷ Now let Γ be defined by:

$$\Gamma(v) = \begin{cases} (0, 3, 3, \frac{1}{2}, \frac{1}{2}) & \text{if } v \in A, \\ (1, 0, 0, \frac{1}{2}, \frac{1}{2}) & \text{if } v \in B, \\ (0, 3, 3, \frac{1}{2}, \frac{1}{2}) + \beta & \text{if } \beta \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \text{ and } v \in A_\beta, \\ (1, 0, 0, \frac{1}{2}, \frac{1}{2}) + \beta & \text{if } \beta \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \text{ and } v \in B_\beta, \\ (1, 1, 1, 1, 1) & \text{if } v = \tilde{w}, \\ \Gamma_0(v) & \text{otherwise.} \end{cases}$$

²⁷One can easily check that such an allocation rule exists.

One can easily check that Γ be a core selective allocation rule that is zero independent on G_m and core-dependent on G_m , but Γ is not additive on the domain pair G_{mul} and G_z . In fact, $(0, 3, 3, \frac{1}{2}, \frac{1}{2}) = \Gamma(\tilde{v}) \neq \Gamma(\tilde{v}_{r^*}) + \Gamma(\tilde{w}) = (1, 0, 0, \frac{1}{2}, \frac{1}{2}) + (1, 1, 1, 1, 1) = (2, 1, 1, \frac{3}{2}, \frac{3}{2})$.

2.3.4 Subtractive on the domain pair G_{sin} and G_z for core selective allocation rules

A decrease of k units of the grand coalition can be distributed either consistent with the allocation rule that is used before or equally between the players. For both of such distributions, an allocation rule should be subtractive on the domain pair G_{sin} and G_z . Moreover, in Section 2.2.3, we have shown that nearly all games in $\overline{G}_c$ are formed by the difference of a game from G_{sin} and a game from G_z . In this section, we give necessary and sufficient conditions for core selective allocation rules that keep the same subtraction structure. Thus, we analyze subtractivity on the domain pair G_{sin} and G_z for core selective allocation rules. The axiomatization is similar to the axiomatization of core selective allocation rules that are additive on the domain pair G_{sin} and G_z that is given in Section 2.3.3.

Theorem 2.3.19. *Let Γ be a core selective allocation rule. Γ is subtractive on the domain pair G_{sin} and G_z if and only if Γ satisfies zero independence on $\overline{G}_s$, equality at the equivalence classes on the domain $\overline{G}_s$, and for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$.*

Proof. ($\Rightarrow$) Let Γ be subtractive on the domain pair G_{sin} and G_z . First, we prove that Γ satisfies equality at the equivalence classes on the domain $\overline{G}_s$. Let $v \in \overline{G}_s$, $v \mathcal{R} \hat{v}$, and let the decompositions associated with v and $\hat{v}$ be $v = v_{r^*} - w$ and $\hat{v} = \hat{v}_{r^*} - \hat{w}$, respectively. Then obviously, since $v \mathcal{R} \hat{v}$ and $v_{r^*} \in G_{sin}^N$, we have $\Gamma(v_{r^*}) = \Gamma(\hat{v}_{r^*})$ and $w \equiv \hat{w}$. Thus, by subtractive on the domain pair G_{sin} and G_z ,

$$\Gamma(v) = \Gamma(v_{r^*}) - \Gamma(w) = \Gamma(\hat{v}_{r^*}) - \Gamma(\hat{w}) = \Gamma(\hat{v}),$$

hence, Γ satisfies equality at the equivalence classes on the domain $\overline{G}_s$.

Next, we show that Γ satisfies zero independence on $\overline{G}_s$. For each player set N , let $v \in \overline{G}_s^N$. For each $\beta \in \mathbb{R}^N$ and each $S \subseteq N$, let $\hat{v}(S) = v(S) + \beta(S)$. Let $v = v_{r^*} - w$ be the decomposition associated with v . Since $v \in \overline{G}_s^N$, $v_{r^*} \in G_{sin}^N$, by subtractive on the domain pair G_{sin} and G_z ,

$$\Gamma(v) = \Gamma(v_{r^*}) - \Gamma(w). \quad (2.3.12)$$

Obviously, $\hat{v} = \hat{v}_{r^*} - w$ is the decomposition of $\hat{v}$ such that for each $S \subseteq N$, $\hat{v}_{r^*}(S) = v_{r^*}(S) + \beta(S)$ and $\hat{v}_{r^*} \in G_{sin}^N$. In fact, since $v_{r^*} \in G_{sin}^N$, $C(\hat{v}_{r^*}) = C(v_{r^*}) + \beta$. Also note, by core selectivity of Γ ,

$$\Gamma(\hat{v}_{r^*}) = \Gamma(v_{r^*}) + \beta. \quad (2.3.13)$$

Now by subtractive on the domain pair G_{sin} and G_z , Equations (2.3.13) and (2.3.12) respectively,

$$\Gamma(\hat{v}) = \Gamma(\hat{v}_{r^*}) - \Gamma(w) = \Gamma(v_{r^*}) + \beta - \Gamma(w) = \Gamma(v) + \beta, \quad (2.3.14)$$

hence, Γ satisfies zero independence on $\overline{G}_s$.

Finally, we show that for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$. Note that $w \in G_z$ implies $(-w) \in \overline{G}_s$, cause the decomposition associated with $(-w)$ is $v_0 - w$, where for each player set N and for each $S \subseteq N$, $v_0(S) = 0$. Hence, by subtractive on the domain pair G_{sin} and G_z , $\Gamma(-w) = \Gamma(v_0) - \Gamma(w) = -\Gamma(w)$.

($\Leftarrow$) Let Γ satisfy the hypothesis of the theorem. For each player set N , each $v \in G_{sin}^N$ and each $w \in G_z^N$, we show $\Gamma(v - w) = \Gamma(v) - \Gamma(w)$.

Let $\beta = \Gamma(v)$. Note $\beta \in \mathbb{R}^N$. For each $S \subseteq N$, define $\hat{v}(S) = (-w)(S) + \beta(S)$, where $(-w)(S) = -w(S)$. The decomposition associated with $(-w)$ is $(-w) = v_0 - w$ such that for each $S \subseteq N$, $v_0(S) = 0$, that is $(-w) \in \overline{G}_s^N$. Since Γ satisfies zero independence on $\overline{G}_s$, $\Gamma(\hat{v}) = \Gamma(-w) + \beta$. Since for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$, $\Gamma(\hat{v}) = -\Gamma(w) + \beta = \Gamma(v) - \Gamma(w)$.

Finally, one can easily check that $(v - w)\mathcal{R}\hat{v}$ and $v - w \in \overline{G}_s$. Hence, by equality at the equivalence classes on the domain $\overline{G}_s$, we have $\Gamma(v - w) = \Gamma(\hat{v}) = \Gamma(v) - \Gamma(w)$. Therefore, Γ satisfies subtractive on the domain pair G_{sin} and G_z . $\square$

For a core selective allocation rule, the three conditions of the above theorem, that is, zero independence on $\overline{G}_s$, equality at the equivalence classes on the domain $\overline{G}_s$ and the condition that for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$ are three independent properties, given in the next remark.

Remark 2.3.20. Note that for a core selective allocation rule Γ , the three properties,

- (i) zero independence on $\overline{G}_s$,
- (ii) equality at the equivalence classes on the domain $\overline{G}_s$ and,
- (iii) for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$

are independent from each other. To see this consider the following examples.

$$((i) \text{ and } (ii) \not\Rightarrow (iii))$$

Let $Nupc$ denote the per-capita nucleolus. For each N and each $v \in G^N$, let $v = v_{r*} \oplus w$ is the decomposition associated with v .

Let $G_{(-z)}^N := \{-w \in \overline{G}_s^N : w \in G_z^N\}$. Now, for each N , define

$$\Gamma(v) = \begin{cases} (\frac{v(N)}{|N|}, \dots, \frac{v(N)}{|N|}) & \text{if } v \in G_z^N, \\ (-v(N), 0, \dots, 0) & \text{if } v \in G_{(-z)}^N, \\ Nupc(v_{r*}) + (\frac{w(N)}{|N|}, \dots, \frac{w(N)}{|N|}) & \text{if } v \in (G_s^N \cup G_m^N) \setminus G_z^N, \\ Nupc(v_{r*}) - (\frac{w(N)}{|N|}, \dots, \frac{w(N)}{|N|}) & \text{if } v \in (\overline{G}_s^N \cup \overline{G}_m^N) \setminus G_{(-z)}^N. \end{cases}$$

Γ is a core selective allocation rule that satisfies zero independence on $\overline{G}_s$ and equality at the equivalence classes on the domain $\overline{G}_s$, but for each $w \in G_z$, $\Gamma(-w) \neq -\Gamma(w)$, if $w(N) \neq 0$.

$$((i) \text{ and } (iii) \not\Rightarrow (ii))$$

For each N , let $G_{(-z)}^N := \{-w \in \overline{G}_s^N : w \in G_z^N\}$. For $N = \{1, 2, 3\}$, define

$$\tilde{v}(S) = \begin{cases} 1 & \text{if } S = \{1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{v}(S) = \begin{cases} 1 & \text{if } S = \{1, 2\}, \text{ or } S = \{1, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\tilde{v}, \hat{v} \in \overline{G}_s^N$. Let $A := \{v \in G : v = \tilde{v} + \beta \text{ for some } \beta \in \mathbb{R}^3\}$ and $B := \{v \in G : v = \hat{v} + \beta \text{ for some } \beta \in \mathbb{R}^3\}$.

Define

$$\Gamma(v) = \begin{cases} Nuc(v) & \text{if } v \in G \setminus (A \cup B), \\ (0, 1, 0) + \beta & \text{if } v \in A \text{ and } v = \tilde{v} + \beta, \\ (1, 0, 0) + \beta & \text{if } v \in B \text{ and } v = \hat{v} + \beta. \end{cases}$$

Γ is a core selective allocation rule that satisfies zero independence on $\overline{G}_s$ and for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$, but Γ does not satisfy equality at the equivalence classes on the domain $\overline{G}_s$, since $\tilde{v} \mathcal{R} \hat{v}$, but $\Gamma(\tilde{v}) \neq \Gamma(\hat{v})$.

((ii) and (iii) $\nRightarrow$ (i))

For $N = \{1, 2, 3\}$, define

$$\tilde{v}(S) = \begin{cases} 1 & \text{if } S = \{1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\hat{v}(S) := \tilde{v}(S) + (1, 1, 1)$. Note $\tilde{v}, \hat{v} \in \overline{G}_s^N$ and in fact $\tilde{v} = v_{r^*} \oplus w = v_{r^*} - w$ is the decomposition associated with $\tilde{v}$, where $v_{r^*}(123) = 1$ and for each $S \subset \{1, 2, 3\}$, $v_{r^*}(S) = \tilde{v}(S)$ and $w \in G_z$ such that $w(123) = 1$.

Let $A := \{v \in G : v \mathcal{R} \tilde{v}\}$ and $B := \{v \in G : v \mathcal{R} \hat{v}\}$.

Now given any $v \in G \setminus (G_c \cup A \cup B)$, let $v = v_{r^*} - w$ be the decomposition associated with v and define

$$\Gamma(v) = \begin{cases} Nuc(v) & \text{if } v \in G_c, \\ (\frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}) & \text{if } v \in A, \\ (0, 0, 0) & \text{if } v \in B, \\ Nuc(v_{r^*}) - Nuc(w) & \text{if } v \in G \setminus (G_c \cup A \cup B). \end{cases}$$

Γ is a core selective allocation rule that satisfy equality at the equivalence classes on the domain $\overline{G}_s$ and for each $w \in G_z$, $\Gamma(-w) = -\Gamma(w)$, but Γ does not satisfy zero independence on $\overline{G}_s$, since $\hat{v} = \tilde{v} + (1, 1, 1)$, but $\Gamma(\hat{v}) \neq \Gamma(\tilde{v}) + (1, 1, 1)$.

2.3.5 Some new allocation rules

In this section, we construct some new core selective allocation rules that consider distribution of an increase/decrease of k units either consistent with the distribution that is used before or equally between the players. These core selective allocation rules are basically obtained via the decomposition idea.

As given in Section 2.3.2, many of the well-known allocation rules do not satisfy additivity on the domain pair G_{sin} and G_z . Thus, many new allocation rules may be driven from the known ones.

Let Γ be a core selective allocation rule that is not additive on the domain pair G_{sin} and G_z . Given a game $v \in G_c$, let $v = v_{r^*} \oplus w = v_{r^*} + w$ be the decomposition of v .

For each $v \in G$, define Γ^* as:

$$\Gamma^*(v) := \begin{cases} \Gamma(v_{r^*}) + \Gamma(w) & \text{if } v \in G_c, \\ \Gamma(v) & \text{if } v \in \overline{G}_c. \end{cases}$$

Note that Γ^* is a core selective allocation rule that is additive on the domain pair G_{sin} and G_z , thus by Theorem 2.3.10, Γ^* satisfies zero independence on G_s and equality at the equivalence classes on the domain G_s .

In fact, by construction, Γ^* is a core selective allocation rule that is additive on the domain pair G_r and G_z . Note that Γ^* is not necessarily egalitarian on G_z . Next is the example, where the allocation rule is also egalitarian on G_z .

For each player set N and $v \in G^N$, define Γ^{**} as:

$$\Gamma^{**}(v) := \begin{cases} \Gamma(v_{r^*}) + (\frac{w(N)}{n}, \dots, \frac{w(N)}{n}) & \text{if } v \in G_c^N, \\ \Gamma(v) & \text{if } v \in \overline{G}_c. \end{cases}$$

Note that Γ^{**} is a core selective allocation rule that is additive on the domain pair G_r and G_z , egalitarian on G_z , i.e., core faithful egalitarian on the domain pair G_r and G_z .

Obviously, if Γ^* is egalitarian on G_z , then $\Gamma^* = \Gamma^{**}$.

Also, it is not hard to see Γ^{**} is monotonic with respect to the value of the grand coalition on G_c . Moreover for each $v \in G_s$, $\Gamma^{**}(v) = Nupc(v)$.

As a specific example to the ones above, since the nucleolus is not additive on the domain pair G_{sin} and G_z , one can drive a new allocation rule from the nucleolus. Since it is egalitarian on G_z , we have $Nu^* = Nu^{**}$.²⁸

Similarly, if Γ is a core selective allocation rule that is not additive on the domain pair G_{sin} and G_z and/or subtractive on the domain pair G_{sin} and G_z and/or additive on the domain pair G_{mul} and G_z and/or subtractive on the domain pair G_{mul} and G_z , then new operators can be constructed so that the new operators are still core selective, and satisfy some desired properties. The followings are some more examples of those. The following examples are also important in the sense that if a rule is *only* defined on G_c -such as the *AL*-value or the core-center-, then the followings extend the rule to an allocation rule.

Let Γ be a core selective allocation rule and $v = v_{r^*} \oplus w$ be the decomposition of $v \in G$. For each $v \in G$, define $\tilde{\Gamma}^*$ as:

$$\tilde{\Gamma}^*(v) := \begin{cases} \Gamma(v_{r^*}) + \Gamma(w) & \text{if } v \in G_c, \\ \Gamma(v_{r^*}) - \Gamma(w) & \text{if } v \in \overline{G}_c. \end{cases}$$

Note $\tilde{\Gamma}^*$ is both additive and subtractive on G_{sin} and G_z , thus, by Theorem 2.3.10, Γ satisfies zero independence on G_s and equality at the equivalence classes on the domain G_s , by Theorem 2.3.19, $\tilde{\Gamma}^*$ satisfies zero independence on $\overline{G}_s$, equality at the equivalence classes on the domain $\overline{G}_s$, and for each $w \in G_z$, $\tilde{\Gamma}^*(-w) = -\tilde{\Gamma}^*(w)$.

²⁸Similarly, since the *AL*-value and the core-center are not additive on the domain pair G_{sin} and G_z , but egalitarian on G_z , one could have defined new allocation rules from the *AL*-value and the core-center if *AL* and *CC* were also defined on $\overline{G}_c$. In that case, one would get $AL^* = AL^{**}$ and $CC^* = CC^{**}$.

In fact, by construction, $\tilde{\Gamma}^*$ is both additive and subtractive on G_r and G_z . Note that $\tilde{\Gamma}^*$ is not necessarily egalitarian on G_z . Next is an example, where the allocation rule is also egalitarian on G_z .

Define $\tilde{\Gamma}^{**}$ as:

$$\tilde{\Gamma}^{**}(v) := \begin{cases} \Gamma(v_{r^*}) + (\frac{w(N)}{n}, \dots, \frac{w(N)}{n}) & \text{if } v \in G_c, \\ \Gamma(v_{r^*}) - (\frac{w(N)}{n}, \dots, \frac{w(N)}{n}) & \text{if } v \in \overline{G}_c. \end{cases}$$

Note $\tilde{\Gamma}^{**}$ is both additive on the domain pair G_r and G_z , egalitarian on G_z , i.e., core faithful egalitarian on the domain pair G_r and G_z . Moreover, it is subtractive on the domain pair G_r and G_z and monotonic with respect to the value of the grand coalition. For each $v \in G_s \cup \overline{G}_s$, $\tilde{\Gamma}^{**}(v) = Nupc(v)$.

As specific examples, one can consider Γ as the nucleolus, the AL -value or the core center. Note, $\widetilde{AL}^*$, $\widetilde{AL}^{**}$, $\widetilde{CC}^*$, $\widetilde{CC}^{**}$ are well-defined.²⁹ Obviously, we have $\widetilde{Nu}^* = \widetilde{Nu}^{**}$, $\widetilde{AL}^* = \widetilde{AL}^{**}$ and $\widetilde{CC}^* = \widetilde{CC}^{**}$.

The above constructions show that once a core selective allocation rule is additive on the domain pair G_{sin} and G_z , then *sharing* problems, such as those discussed in the motivation, are reduced to the *sharing the extra cost* problems. Note, if one wants the allocation rule to be *consistent with itself* in distributing an increase and a decrease of the grand coalition at a game in G_s , or wants to distribute it *equally*, then allocation rules such as $\tilde{\Gamma}^*$ or $\tilde{\Gamma}^{**}$ can be used, respectively. Both of these rules satisfy the conditions of Theorem 2.3.10 and Theorem 2.3.19.

Finally, as another perspective, if one wants the *distribution of the extra cost and the distribution of the decrease of the cost* to be independent of the game, but dependent on the allocation rule itself, then allocations such as $\tilde{\Gamma}^*$ can be used. If one wants the *distribution of the extra cost and the distribution of the decrease of the cost* to be independent of both the game and the allocation itself, then allocations such as $\tilde{\Gamma}^{**}$ can be used.

²⁹In other words, different than the cases in footnote 28, no need of the assumption that AL -value and the core-center were also defined on $\overline{G}_c$.

2.4 Part III- Miscellaneous

2.4.1 Proportional and inversely proportional core selective allocation rules

We turn back to our simple motivating example in Section 2.1 and our concerns about an increase/decrease of k units of the grand coalition. Commonly, depending on the problem or scenario, an increase/decrease of k units of the grand coalition can be distributed either proportionally, or inversely proportionally, or consistent with the allocation rule that is used before, or equally between the players. In this section, we define and give simple characterizations of (directly) proportional and inversely proportional rules.

Let $Z^N := \{v \in G^N : \forall S \subset N, v(S) = 0\}$. In this section except this set of games, we assume $v(N) \neq 0$ and $v_{r^*}(N) \neq 0$, i.e., for each $n \in \mathbb{N}$ and each $v \in G^N \setminus Z^N$, we assume $v(N) \neq 0$ and $v_{r^*}(N) \neq 0$.³⁰ Also, we assume for each $S \subseteq N$, $v(S) \geq 0$.³¹ We say that an allocation rule Γ is ‘(directly) proportional’ if it divides an increase and/or decrease at the grand coalition proportionally. Formally, we say that an allocation rule Γ is *(directly) proportional* if the following two conditions hold:

- (i) $\forall N, \quad \forall v \in G^N \setminus Z^N, \quad (v(N) \neq 0) \quad \text{and} \quad \forall w \in Z^N,$
 $\Gamma(v + w) = \Gamma(v) + \left(\frac{\Gamma_1(v)w(N)}{v(N)}, \dots, \frac{\Gamma_n(v)w(N)}{v(N)}\right) = \left(1 + \frac{w(N)}{v(N)}\right)\Gamma(v),$
- (ii) $\forall v \in Z^N, \Gamma(v) = \left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right).$

Given a core selective allocation rule Γ , for each player set N and $v \in G^N$ ($v_{r^*}(N) \neq 0$), define $\Gamma_p(v)$ as:

$$\Gamma_p(v) := \begin{cases} \Gamma(v_{r^*}) + \left(\frac{\Gamma_1(v_{r^*})w(N)}{v_{r^*}(N)}, \dots, \frac{\Gamma_n(v_{r^*})w(N)}{v_{r^*}(N)}\right) & \text{if } v \in G_c^N \setminus Z^N, \\ \left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right) & \text{if } v \in Z^N, \\ \Gamma(v_{r^*}) - \left(\frac{\Gamma_1(v_{r^*})w(N)}{v_{r^*}(N)}, \dots, \frac{\Gamma_n(v_{r^*})w(N)}{v_{r^*}(N)}\right) & \text{if } v \in \overline{G}_c^N \setminus Z^N. \end{cases}$$

³⁰Note that the set of games either with $v(N) = 0$ or $v_{r^*}(N) = 0$ can be dealt with by appropriate translations.

³¹Any game with $v(S) < 0$ for some S can be dealt with by an appropriate translation.

Let $PA := \{\Gamma_p : \Gamma \text{ core selective allocation rule}\}$.

Theorem 2.4.1. *A core selective allocation rule, Γ is (directly) proportional if and only if $\Gamma \in PA$.*

Proof of the above theorem is straightforward, thus it is omitted.

By Section 2.2.3, we know that nearly all of the games in G_c are in G_s and nearly all of the games in $\overline{G}_c$ are in $\overline{G}_s$. We want to note that, if $v \in G_s \cup \overline{G}_s$ -which covers nearly all transferable utility games-, then there is a unique vector, say x , such that for all $\Gamma_p \in PA$, $\Gamma_p(v) = x$.

Next, in a similar way we define ‘inversely proportional allocation rules. We say that an allocation rule Γ is ‘inversely proportional’ if it divides an increase and/or decrease at the grand coalition inversely proportional. Formally, given a vector $x \in \mathbb{R}^N$, let $\tilde{x} \in \mathbb{R}^N$ be the vector by rearranging the elements of x according to the lexicographic order so that $\tilde{x}_1 \geq \tilde{x}_2 \geq \dots \geq \tilde{x}_n$. Let σ denote the permutation of the player set N such that for each $i \in N$, let $\tilde{x}_{\sigma(i)} = x_i$. An allocation rule Γ is *inversely proportional* if the following two conditions hold:

- (i) $\forall N, \forall v \in G^N \setminus Z^N, (v(N) \neq 0) \text{ and } \forall w \in Z^N,$
 $\Gamma(v + w) = \Gamma(v) + (\frac{\tilde{\Gamma}_{n+1-\sigma(1)}(v)w(N)}{v(N)}, \dots, \frac{\tilde{\Gamma}_{n+1-\sigma(n)}(v)w(N)}{v(N)}),$
- (ii) $\forall v \in Z^N, \Gamma(v) = (\frac{w(N)}{n}, \dots, \frac{w(N)}{n}).$

Given a core selective allocation rule Γ , for each player set N and $v \in G^N$ ($v_{r^*}(N) \neq 0$), define $\Gamma_{ip}(v)$ as:

$$\Gamma_{ip}(v) := \begin{cases} \Gamma(v_{r^*}) + (\dots, \frac{\Gamma_{n+1-\sigma(k)}(v_{r^*})w(N)}{v_{r^*}(N)}, \dots) & \text{if } v \in G_c^N \setminus Z^N, \\ (\frac{v(N)}{n}, \dots, \frac{v(N)}{n}) & \text{if } v \in Z^N, \\ \Gamma(v_{r^*}) - (\dots, \frac{\Gamma_{n+1-\sigma(k)}(v_{r^*})w(N)}{v_{r^*}(N)}, \dots) & \text{if } v \in \overline{G}_c^N \setminus Z^N. \end{cases}$$

Let $IPA := \{\Gamma_{ip} : \Gamma \text{ core selective allocation rule}\}$.

Theorem 2.4.2. *A core selective allocation rule, Γ is inversely proportional if and only if $\Gamma \in IPA$.*

Proof of the above theorem is straightforward, thus it is omitted.

As in the case of Theorem 2.4.1, we want to note that, if $v \in G_s \cup \overline{G}_s$, then there is a unique vector, say y , such that for all $\Gamma_p \in IPA$, $\Gamma_p(v) = y$.

2.4.2 Monotonicity with respect to the value of the grand coalition

Our concerns is related with the monotonicity with respect to the value of the grand coalition. In this section, we turn back to additivity and subtractivity on specific domains, and give some simple relations between these two and monotonicity with respect to grand coalition for an interested reader.

Lemma 2.4.3. *If an allocation rule Γ is additive on the domain pair G_r and G_z and egalitarian on G_z , then Γ is monotonic with respect to the value of the grand coalition on G_c .*

Proof. Let Γ be an allocation rule that satisfies the hypothesis of the lemma. Let N be the set of players and let $v \in G_c^N$ be any game with the decomposition $v = v_{r*} + w$. Then by additive on the domain pair G_r and G_z , and egalitarian on G_z , for each $i \in N$,

$$\Gamma_i(v) = \Gamma_i(v_{r*}) + \frac{w(N)}{n}. \quad (2.4.1)$$

Let $\hat{v}(N) > v(N)$ and for each $S \subset N$, $\hat{v}(S) = v(S)$. Note that the root game of $\hat{v}$ is also v_{r*} , so by additive on the domain pair G_r and G_z , and egalitarian on G_z , for each $i \in N$,

$$\Gamma_i(\hat{v}) = \Gamma_i(v_{r*}) + \frac{w(N) + (\hat{v}(N) - v(N))}{n}. \quad (2.4.2)$$

By Equations (2.4.1) and (2.4.2), for each $i \in N$,

$$\Gamma_i(\hat{v}) = \Gamma_i(v) + \frac{(\hat{v}(N) - v(N))}{n}.$$

Hence, Γ is monotonic with respect to the value of the grand coalition on G_c^N . Since this is true for each player set N , Γ is monotonic with respect to the value of the grand coalition on G_c . $\square$

Remark 2.4.4. The converse of Lemma 2.4.3 is not true in general, that is, if an allocation rule is monotonic with respect to the value of the grand coalition on G_c , Γ need not be an allocation rule that is additive on G_r and G_z and egalitarian on G_z . In fact, that is not true, even if Γ is a core selective allocation rule. Here is an example.

Let $N = \{1, 2, 3\}$ and $\tilde{v} \in G_c^N$ be defined by: $\tilde{v}(1) = \tilde{v}(2) = \tilde{v}(3) = 0$, $\tilde{v}(12) = \tilde{v}(13) = 1$, $\tilde{v}(23) = 0$ and $\tilde{v}(123) = 1$. Let A be the set of all games with player set N each of which has its root game as $\tilde{v}$. In other words, let $A := \{v \in G_c^N : v = v_{r^*} \oplus w \Rightarrow v_{r^*} = \tilde{v}\}$. Let $Nupc$ denote the per-capita nucleolus. Now let Γ be defined by:

$$\Gamma(v) = \begin{cases} (1, v(N) - 1, 0) & \text{if } v \in A, \\ Nupc(v) & \text{if } v \in G \setminus A. \end{cases}$$

One can easily check that Γ is a core selective allocation rule that is monotonic with respect to the value of the grand coalition on G_c and egalitarian on G_z , but it is not additive on the domain pair G_r and G_z and hence it is not additive on the domain pair G_r and G_z . In fact, define $v := \tilde{v} + w$ where $w \in G_z^N$ such that $w(N) > 0$, then one can easily check $(1, w(N), 0) = \Gamma(\tilde{v}) \neq \Gamma(\tilde{v}) + \Gamma(w) = (1, 0, 0) + (\frac{w(N)}{3}, \frac{w(N)}{3}, \frac{w(N)}{3}) = (1 + \frac{w(N)}{3}, \frac{w(N)}{3}, \frac{w(N)}{3})$. Yet, we want to note that Γ is a piecewise allocation rule and depends on a game, namely on $\tilde{v}$.

Remark 2.4.5. Since $G_c = G_s \cup G_m$ such that $G_s \cap G_m = \emptyset$ and $G_r = G_{sin} \cup G_{mul}$ such that $G_{sin} \cap G_{mul} = \emptyset$, similar to the proof of Lemma 2.4.3, one can easily prove the following as well:

If an allocation rule Γ is additive on the domain pair G_{mul} and G_z and egalitarian on G_z , then Γ is monotonic with respect to the value of the grand coalition on G_m .³²

Theorem 2.4.6. *If an allocation rule Γ is both additive and subtractive on the domain pair G_r and G_z , and egalitarian on G_z , then Γ is monotonic with respect to the value of the grand coalition.*

³²Similarly, if an allocation rule Γ is additive on the domain pair G_{sin} and G_z and egalitarian on G_z , then Γ is monotonic with respect to the value of the grand coalition on G_s . In fact, in that case, Γ on G_s is uniquely determined on G_s , and is equal to the per-capita nucleolus on G_s .

Proof. Let Γ be an allocation rule that satisfies the hypothesis of the theorem.

If $v \in G_c$, by Lemma 2.4.3, Γ is monotonic with respect to the value of the grand coalition on G_c .

If $v \in \overline{G}_c$, then for each player set N , let the decomposition associated with v be $v = v_{r^*} - w$. Then by subtractive on the domain pair G_r and G_z , and egalitarian on G_z , for each $i \in N$,

$$\Gamma_i(v) = \Gamma_i(v_{r^*}) - \frac{w(N)}{n}. \quad (2.4.3)$$

Let $\hat{v}(N) > v(N)$ and for each $S \subset N$, $\hat{v}(S) = v(S)$. Note that the associated minimal game of $\hat{v}$ is also v_{r^*} . There are two cases.

(i) $\hat{v} \in G_c$.

Then the decomposition of $\hat{v}$ is $\hat{v} = v_{r^*} + \hat{w}$, where v_{r^*} is the root game of v , and $\hat{w} \in G_z$ such that $\hat{w}(N) = \hat{v}(N) - v_{r^*}(N)$. By additive on the domain pair G_r and G_z , and egalitarian on G_z , for each $i \in N$,

$$\Gamma_i(\hat{v}) = \Gamma_i(v_{r^*}) + \frac{\hat{w}(N)}{n}. \quad (2.4.4)$$

Since $\hat{w}(N) = \hat{v}(N) - v_{r^*}(N) + v(N) - v(N) = \hat{v}(N) - w(N) - v(N)$, for each $i \in N$,

$$\frac{\hat{w}(N)}{n} = \frac{\hat{v}(N) - v(N)}{n} - \frac{w(N)}{n}. \quad (2.4.5)$$

By Equations (2.4.3), (2.4.4) and (2.4.5), for each $i \in N$,

$$\Gamma_i(\hat{v}) = \Gamma_i(v) + \frac{(\hat{v}(N) - v(N))}{n}. \quad (2.4.6)$$

(ii) $\hat{v} \in \overline{G}_c$.

Then the decomposition associated with $\hat{v}$ is $\hat{v} = v_{r^*} - \hat{w}$, where v_{r^*} is the minimal game associated with v , and $\hat{w}(N) = v_{r^*}(N) - \hat{v}(N)$. By subtractive on the domain pair G_r and G_z , and egalitarian on G_z , for each $i \in N$,

$$\Gamma_i(\hat{v}) = \Gamma_i(v_{r^*}) - \frac{\hat{w}(N)}{n}. \quad (2.4.7)$$

Since $\hat{w}(N) = v_{r^*}(N) - \hat{v}(N) + v(N) - v(N) = w(N) - (\hat{v}(N) - v(N))$, for each $i \in N$,

$$\frac{\hat{w}(N)}{n} = \frac{w(N)}{n} - \frac{\hat{v}(N) - v(N)}{n}. \quad (2.4.8)$$

By Equations (2.4.3), (2.4.7) and (2.4.8), for each $i \in N$,

$$\Gamma_i(\hat{v}) = \Gamma_i(v) + \frac{(\hat{v}(N) - v(N))}{n}. \quad (2.4.9)$$

By Equations (2.4.6) and (2.4.9), for each $v \in \overline{G}_c^N$ and $i \in N$, $\Gamma_i(\hat{v}) > \Gamma_i(v)$ and hence Γ is monotonic with respect to the value of the grand coalition on $\overline{G}_c^N$. Since this is true for each player set N , Γ is also monotonic with respect to the value of the grand coalition on $\overline{G}_c$. Therefore, Γ is monotonic with respect to the value of the grand coalition. $\square$

Remark 2.4.7. Converse of Theorem 2.4.6 is not true in general, that is, if an allocation rule Γ is monotonic with respect to the value of the grand coalition, then Γ need not be both additive and subtractive on the domain pair G_r and G_z , and egalitarian on G_z . In fact, the example in Remark 2.4.4 is an example for this. Yet, again we want to note that the example depends on a game.

2.4.3 Modification of the decomposition of a game

Lastly, we want to note that definition of ‘the decomposition of a game’ can be modified. Here we give a simple example to that. Modified version is defined only for convex games, based on the idea of *convexity*.

A game $v \in G$ is *convex* if for each pair $S, T \in \Omega$, $v(S) + v(T) \leq v(T \cup S) + v(T \cap S)$. Let G_{con}^N denote the set of all convex games with player set N and $\bigcup_{N:n \in \mathbb{N}} G_{con}^N$ is denoted by G_{con} . It is known that $G_{con} \subset G_c$.

Let $v \in G_{con}^N$. For each $r \in \mathbb{R}$, as before, v_r is defined as follows:

$$v_r(S) := \begin{cases} v(S) & \text{if } S \subset N, \\ r & \text{if } S = N. \end{cases}$$

Let $\widetilde{M}_v := \{r \in \mathbb{R} : v_r \in C_{con}^N\}$ and let $\widetilde{r}^* := \min_{r \in \widetilde{M}_v} r$.

Given a game $v \in G_{con}^N$, $v := v_{\widetilde{r}^*} + \widetilde{w}$ is called *the convex-decomposition associated with the game v* where

$$\widetilde{w}(S) = \begin{cases} 0 & \text{if } S \subset N, \\ v(N) - v_{\widetilde{r}^*}(N) & \text{if } S = N. \end{cases}$$

One of the well-known allocation rules that is in the core of a convex game is defined by Shapley. For each player set N and each $v \in G^N$, the *Shapley value* [7] $Sh(v) = (Sh_1(v), \dots, Sh_n(v))$ is the allocation rule that is defined by: for each $i \in N$,

$$Sh_i(N, v) := \sum_{S \subset N \setminus \{i\}} (v(S \cup i) - v(S)) \frac{|S|!(n-1-|S|)!}{n!}.$$

One can easily check that if $v := v_{\widetilde{r}^*} + \widetilde{w}$ is the convex-decomposition associated with the game $v \in G_{con}$, then $Sh(v) = Sh(v_{\widetilde{r}^*}) + Sh(\widetilde{w}) = Sh(v_{\widetilde{r}^*}) + (\frac{\widetilde{w}}{|N|}, \dots, \frac{\widetilde{w}}{|N|})$.

In fact, it is well-known that the Shapley value is egalitarian on G_z and additive, thus the result follows.

Chapter 3

Mergeproofness of allocation rules and allocation correspondences at transferable utility games

3.1 Introduction

The main purpose of this chapter is to study the concept of merging in the context of transferable utility games. In this context, mainly two types of merging of players is discussed in terms of number of players. One is the type where the number of players do not change after merging, as in [31, 32]¹, and the other is the type where the number of players are reduced after merging, as in [33, 34, 35, 36, 37].² Specifically, in one type, two players merge and enter the game with prior proxy and association agreements in order to strengthen their

¹Carreras [31] calls merging as partnership, yet again the number of players is fixed as in Haller's [32].

²In [37] based on the well-known solution concept, the core, disadvantages of merging - called syndicates in their paper- of players is discussed by some examples. Although it is not mentioned specifically in their paper, in fact the number of players is not fixed after merging. In the same manner, disadvantages of syndicates and the nucleolus is studied in [34].

position and a specific new game is formed from the original game with the same number of players, as in [32], and in the other, two players merge into a single player and number of players is reduced by one, as in [35]. In our work, we consider the latter type of merging in terms of number of players.

In the literature, merging of a coalition into a single player is allowed for only a single coalition. Thus, the players outside the merging coalition are assumed to stay as they are, which is the main concern of this chapter, and which is also addressed as a problem by Derks and Tijs [33]. To be more specific, consider the following example:

Suppose there are four firms at an auction, say firm A,B,C and D. Suppose firm A and B recognize that at the current situation, they get better off if they merge and form a single firm. Thus, they merge and form, say, the firm Union-AB. Being aware of each firms's power in the market, firm C and D were ready for such a merging movement of firm A and B, and were ready to merge, knowing the merged Union-CD firm will get the firm Union-AB worse off. Do the firms A and B really improve via merging? Similarly, for firms C and D.

The example above can be adapted to another examples, four banks at a district, etc. In fact, depending on the scenario, the merging movements can be done simultaneously or one at a time. In general, our concern is: when does an 'improvement' via merging occur for a coalition, no matter what the other opponents do in terms of merging.

In our study, given a partition, merging of players in each of the coalitions is allowed, so that each coalition is assumed to act as a single player and hence the number of players are reduced to the number of coalitions of the partition. Derks and Tijs point out the problem that merging players are unaware of the merging behaviors of the other players, so that there is an uncertainty of which game is actually played. Derks and Tijs proceed by assuming that the other players do not merge. In [33],³ a coalition makes a merging decision under the assumption

³Derks and Tijs also assume that the remaining players do not merge. They answer under which conditions, both on the game and the partition of the players set into merging coalitions, it is profitable to merge for a coalition if the Shapley value allocation rule is applied, assuming

that the other coalitions do not merge, while in our study a coalition takes the possible merging decisions of the other coalitions into account. We propose two definitions for to solve this addressed problem, by simply considering all the possibilities of the final game after considering all the possible merging behaviors of the players. In other words, we consider all possible partitions that a coalition belongs to and consider different profitability definitions, which leads us to two different definitions. In other words, we consider all possible partitions that a coalition belongs and consider different profitability definitions,⁴ which leads us to two different merge proof definitions.

In the first one, it is beneficial for a coalition to merge, and thus the coalition improves, if the merged coalition is at least as good as it was at the original game and if there is at least one possibility that it gets better off, no matter what the other players do in terms of merging. In the second one, which is stronger than the above, it is beneficial for a coalition to merge if it gets better off no matter what the other players do in terms of merging. Thus the second definition considers strict improvement of a coalition no matter how the other players merge between themselves.

We briefly discuss these different merge proof definitions that we propose for allocation rules and give relations between them. We establish some impossibility as well as some possibility results for allocation rules. Our possibility results follow from convex combinations of some allocation rules -mainly from convex combinations of two well-known allocation rules which are far from each other, the equal division and the dictatorship. Lastly, we point out how merging can be considered as a stability notion.

Though some attention is paid to merging of players in the context of transferable utility games in the literature, only allocation rules are used for evaluation

the players outside the merged coalition stay as they are.

⁴Profitability after merging can be defined in different ways. For example, in [36], merging is beneficial for a coalition, if the payoff of the merging coalition is greater than the sum of the payoffs of its players at the original game. Yet, in [33], the measure of profitability is different than the one in [36]. In fact, according to Derks and Tijs, if the Shapley value of a merging coalition is at least the sum of the Shapley value of its players in the original game, then the merging of these players is profitable.

of the games. As far as we know, there has not been any research where an allocation correspondence is used for evaluation of games. In this chapter we also try to take some attention to these problems that has not been addressed before. In the second part of this chapter, we give different possible merge proof definitions for allocation correspondences, and extend our results to allocation correspondences.

We want to note that merge proofness of allocation rules is also discussed in many different areas in the literature -besides in transferable utility games- such as in bargaining problems, in networks, in bankruptcy problems or in claim problems. In the context of bargaining problems, Harsanyi [38] introduces the joint-bargaining paradox of the Nash bargaining allocation correspondence. In claim problems, a division rule is merge proof, if no group of players can increase their total awards by merging their claims. The notion was first introduced by O'Neill [39]. Coalitional manipulability via claims merging and splitting is widely studied in the literature, some of the remarkable studies are [40, 41, 42].⁵ In bankruptcy problems, it is well known merge proofness have been used to characterize the proportional allocation rule. Non-manipulable division rules in claim problems and generalizations are studied by [42], where a general class of allocation problems including bankruptcy problems as a special case is introduced.⁶ For a good survey in general, we refer to [21, 23]. In networks, in [43], coalition manipulation on networks via reallocation merge proofness is considered, and recently in [44] some impossibility and possibility results is given on networks where collusion (merging) of only two players are allowed. In all the above problems, merging is allowed only to one group of players. In other words, the other players outside that group assumed to stay same. Thus, we want to note that the merging definitions that we propose in this chapter can be adapted in different areas in the literature, where it is appropriate.

The rest of this chapter is organized mainly in two parts as follows. Part I (3.2) consists of five sections.⁷ Section 3.2.1 consists of preliminaries for allocation

⁵de Frutos [40] gives a large family of division rules non-manipulable via merging and a large family of division rules non-manipulable via splitting. Ju [41] improves her results and provide axiomatic characterization theorems for merge proof rules and for split-proof rules.

⁶Ju et al. give an axiomatic characterization of reallocation-proof rules.

⁷Results in Part I can also be found in [45].

rules. In Section 3.2.2, we give different definitions of merge proofness of allocation rules and compare them. Section 3.2.3 consists of the impossibility results and Section 3.2.4 consists of the possibility results, and finally merge proofness as a stability notion is given in Section 3.2.5. Part II (3.3) consists of three sections. Section 3.3.1 consists of preliminaries for allocation correspondences. In Section 3.3.2, we give different definitions of merge proofness of allocation correspondences and compare them. Section 3.3.3 consists of our results for allocation correspondences. Finally, some final remarks -including open problems- is given in Section 3.4.

3.2 Part I-Allocation rules

3.2.1 Preliminaries for allocation rules

For each $n \in \mathbb{N}$, let N denote the set of players such that $|N| = n$.

Non empty subsets of N are called *coalitions*. A *partition* $\mathcal{P}$ of N is the set of pairwise disjoint coalitions whose union is equal to the set of players, i.e., there is $k \in \mathbb{N}$, $k \leq n$, such that $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ such that for each $P \in \mathcal{P}$, $P \in 2^N \setminus \{\emptyset\}$, and for each pair $l, l' \in \{1, \dots, k\}$, $l \neq l'$, $P_l \cap P_{l'} = \emptyset$, and $\bigcup_{i=1}^k P_i = N$. Let $\mathbb{P}(N)$ denote the set of all partitions of N . Given a partition $\mathcal{P}$, a generic element of $\mathcal{P}$ is denoted by P .⁸ For each $S \subseteq N$, $S \neq \emptyset$, let $\mathbb{P}_S := \{\mathcal{P} \in \mathbb{P}(N) : S \in \mathcal{P}\}$.

A *transferable utility game*, (*game* for short) is defined as in Section 2.2.1. A game $v \in G^N$ is *trivial*, if the worth of the grand coalition is positive and the worth of all other coalitions is zero. Let G_{trv}^N denote the set of all trivial games with player set N , and G_{trv} denote the set of all trivial games, i.e., $G_{trv} := \bigcup_{n \in \mathbb{N}} \{v \in G^N : \forall S \subset N, v(S) = 0, v(N) > 0\}$.

⁸We sometimes will abuse the notation. We will not use set-parenthesis of the set $\mathcal{P}$, and for the sets in $\mathcal{P}$ we use parenthesis instead of set-parenthesis; e.g. we will write (123) instead of $\{1, 2, 3\}$ or (12)(3) instead of $\{\{1, 2\}, \{3\}\}$.

A game $v \in G^N$ is *monotonic* if for each $i \in N$ and each $S \in 2^N$, $v(S \cup \{i\}) \geq v(S)$.

Let $v \in G^N$ and $\emptyset \neq S \subset N$ be given. The subgame of v relative to S is the game with player set S , denoted by $v|_S$, defined as follows: for each $T \in 2^S$, $v|_S(T) := v(T)$.

Definition 3.2.1. Let $v \in G^N$ and $\mathcal{P} \in \mathbb{P}(N)$ be given. The subgame of v with respect to partition $\mathcal{P}$ is the game with player set $\mathcal{P}$, denoted by $v|_{\mathcal{P}}$ defined as follows: for each $T \in 2^{\mathcal{P}}$, $v|_{\mathcal{P}}(T) := v(\bigcup_{P \in T} P)$.

Note by Definition 3.2.1, if $\mathcal{P} = \{P_1, \dots, P_k\}$, then $v|_{\mathcal{P}}$ is the game in $G^{\mathcal{P}}$ (i.e., it has k players) and $v|_{\mathcal{P}}(\mathcal{P}) = v(P_1 \cup P_2 \cup \dots \cup P_k) = v(N)$.

Example 3.2.2. Let $N = \{1, 2, 3, 4\}$ and $v \in G^N$ be defined by: $v(1) = v(2) = 0$, $v(3) = 1.5$, $v(4) = 2.5$, $v(12) = 1.5$, $v(34) = 4.5$, $v(123) = 5$, $v(1234) = 10$ and $v(S) = 0$ otherwise.

For $\mathcal{P} = (12)(3)(4)$, $v|_{\mathcal{P}}$ is the game in G^3 defined by setting,
 $v|_{\mathcal{P}}(\underline{12}) = 1.5$, $v|_{\mathcal{P}}(3) = 1.5$, $v|_{\mathcal{P}}(4) = 2.5$,
 $v|_{\mathcal{P}}(\underline{123}) = 5$, $v|_{\mathcal{P}}(\underline{124}) = 0$, $v|_{\mathcal{P}}(34) = 4.5$,
 $v|_{\mathcal{P}}(\underline{1234}) = 10$.

For $\mathcal{P} = (12)(34)$, $v|_{\mathcal{P}}$ is the game in G^2 defined by setting,
 $v|_{\mathcal{P}}(\underline{12}) = 1.5$, $v|_{\mathcal{P}}(\underline{34}) = 4.5$, $v|_{\mathcal{P}}(\underline{1234}) = 10$.

Note that $v|_{\mathcal{P}}(\{\{1, 2\}\})$ is denoted by $v|_{\mathcal{P}}(\underline{12})$, $v|_{\mathcal{P}}(\{\{1, 2\}, \{3\}\})$ is denoted by $v|_{\mathcal{P}}(\underline{123})$ and similarly for the rest. Similar notation is used throughout the chapter for the players at the subgames with respect to partitions.

For each vector $\beta \in \mathbb{R}^N$, $\emptyset \neq S \subseteq N$, we denote $\sum_{i \in S} \beta_i$ by $\beta(S)$, and $\beta(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^N$ is defined as in 2.2.1. Feasibility and stability of a payoff vector is defined as in 2.2.1. Let $F(v)$ be the set of all feasible payoff vectors for v and $S(v)$ be the set of all stable payoff vectors for v , respectively. Note that the set of all feasible and stable payoff vectors of a game v is the *core* of the game v , denoted by $C(v)$, i.e., $C(v) = F(v) \cap S(v)$.

For each $v \in G^N$, a payoff vector $x \in \mathbb{R}^N$ is called *individually rational for v* if for each $i \in N$, $x_i \geq v(i)$. Let $I(v)$ be the set of all individually rational payoff vectors for v .

An allocation rule is defined as in 2.3.1. Next, we state some well-known axioms defined for allocation rules, which we use in this chapter.

1. First is independence of the names of the players. Formally, let $\sigma : N \rightarrow N$ denote a permutation of N . Let $v \in G^N$. A new game induced by σ and v , denoted by σv , is defined by setting, for each $S \subseteq N$, $\sigma v(S) = v(\sigma(S))$. An allocation rule Φ is *anonymous* if for each $n \in \mathbb{N}$, each $v \in G^N$, each permutation σ , and each $i \in N$, $\Phi_{\sigma i}(\sigma v) = \Phi_i(v)$.
2. Let $v \in G^N$. A player $i \in N$ is a *null player in v* if for each $S \subseteq N$, $v(S \cup \{i\}) = v(S)$. Our next requirement is that each null player be assigned a zero payoff. Formally, for each $v \in G^N$, let $D(v)$ be the set of null players in v . An allocation rule Φ satisfies *the null player axiom* if for each $v \in G$, and each $i \in D(v)$, $\Phi_i(v) = 0$.⁹
3. An allocation rule Φ is *individually rational* if for each N and $v \in G^N$ with $v(N) \geq \sum_{i \in N} v(i)$, $\Phi(v) \in I(v)$.
4. Next is the requirement that an allocation rule treats in a similar way to the players that are in a similar situation at a game. Formally, an allocation rule Φ satisfies *equal treatment of equals* if for each N and each pair $i, j \in N$, if for each $S \subseteq N \setminus \{i, j\}$ with $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\Phi_i(v) = \Phi_j(v)$.
5. An allocation rule Φ is *core selective* if for each $v \in G$ with $C(v) \neq \emptyset$, $\Phi(v) \in C(v)$.

Next, we remind two well-known allocation rules, namely the Shapley value [46] and the Fujishige-Dutta Ray egalitarian solution [47, 48], and two trivial allocation rules, namely the equal division and the dictatorial rule.

⁹In the literature, a null player is sometimes called a dummy player, and the null player axiom is sometimes called the dummy player axiom.

1. The *Shapley value* $Sh(.) = (Sh_1(.), \dots, Sh_n(.))$ is the allocation rule that is defined by setting, for each $n \in \mathbb{N}$, each $v \in G^N$, and each $i \in N$,

$$Sh_i(v) := \sum_{0 \leq s \leq n-1} \left(\frac{s!(n-1-s)!}{n!} \sum_{S \subset N \setminus \{i\}, |S|=s} (v(S \cup i) - v(S)) \right).$$

It is well-known that the Shapley value is in the core of a convex game.

2. Another well-known allocation rule that is in the core of a convex game is defined in [47] and [48].

For convex games the *Fujishige-Dutta Ray egalitarian solution* (FDR) is defined by the following algorithm.¹⁰

Define first, for any game v' and any coalition $S \subseteq N$, $e(S, v') = v'(S)/|S|$ so that $e(S, v')$ is the average worth of S under v' . Given a game $v \in G_{con}^N$, define $v_1 = v$.

Step 1: Define S_1 to be the unique coalition such that (i) $e(S_1, v_1) \geq e(S, v_1)$ for each coalition S ; (ii) $|S_1| > |S|$ for each $S \neq S_1$ such that $e(S, v_1) = e(S_1, v_1)$; so that S_1 is the unique largest coalition having the highest average worth. Define for each $i \in S_1$, $FDR_i(v) := e(S_1, v_1)$.

Step k : Suppose that $S_1, \dots, S_{k-1}$ have been defined recursively and $S_1 \cup \dots \cup S_{k-1} \neq N$. Define a new game v_k with player set $N \setminus \{S_1 \cup \dots \cup S_{k-1}\}$ as follows. For all subcoalitions S of this new player set, $v_k(S) := v_{k-1}(S_{k-1} \cup S) - v_{k-1}(S_{k-1})$. Note that v_k is also a convex game. Just as in Step 1, define S_k to be the largest coalition with the highest average worth in this game. Define for each $i \in S_k$, $FDR_i(v) := e(S_k, v_k)$.

Clearly in m of these steps ($m \leq n$), there is a partition of N into sets $S_1, \dots, S_m$. Using the algorithm above, the allocation rule defined by $FDR(.) := (FDR_1(.), \dots, FDR_n(.))$ is called the *Fujishige-Dutta Ray egalitarian solution*.

3. Next is the allocation that threats all players equally. Formally, the *equal division allocation rule*, denoted by ED , is defined by setting, for each

¹⁰The algorithm is taken from [48].

$n \in \mathbb{N}$, each $i \in N$ and each $v \in G^N$,

$$ED_i(v) := \frac{v(N)}{|N|}.$$

4. Next is the case, whenever the player who is the dictator is in the game, no matter how the initial worths of coalitions are, the dictator gets the total value of the grand coalition and the rest of the players get nothing.

Let N be given and let $d \in N$.¹¹

The *dictatorial allocation rule*, denoted by Γ^{Dct} , is defined as follows:
for each $i \in N$ and $v \in G^N$,

$$\Gamma_i^{Dct}(v) := \begin{cases} v(N) & \text{if } i = d, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $\mathcal{P} \in \mathbb{P}(N)$, each $P \in \mathcal{P}$ and $v \in G^{\mathcal{P}}$,

$$\Gamma_P^{Dct}(v) := \begin{cases} v(N) & \text{if } d \in P, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, Γ^{Dct} is well-defined for fixed N with $d \in N$.

3.2.2 Merge proofness of allocation rules

In this section, we study merge proofness of allocation rules. We give three different definitions of merge proofness of an allocation rule, and study their relations between themselves.

First, we give the following definition, which is the basic tool at merging.

Definition 3.2.3. Given a game v , an allocation rule Φ , a coalition $S \subset N$ with $|S| > 1$, and a partition $\mathcal{P} \in \mathbb{P}_S$,

- (i) we say that S *improves weakly via merging in v relative to partition $\mathcal{P}$ under Φ* if $\Phi_S(v|_{\mathcal{P}}) \geq \sum_{i \in S} \Phi_i(v)$ and,

¹¹Player d is the dictator.

(ii) we say that S improves strictly via merging in v relative to partition $\mathcal{P}$ under Φ if $\Phi_S(v|\mathcal{P}) > \sum_{i \in S} \Phi_i(v)$.

The above definitions are pretty clear. Basically, there is a comparison between two cases in Definition 3.2.3, both in (i) and (ii). One is the total payoff the coalition S at (Φ, v) and the other is the payoff of the player S at $(\Phi, v|\mathcal{P})$.

First, we define ‘merge proofness’ of allocation rules, which is the same as the one defined in [36]. Basically, we only allow one coalition to merge and allow the coalition to compare its total payoff before and after merging, where the players outside the merger coalition stay same. Thus, it is the common ‘merging’ notion that is used in general in the literature. In other words, to check if an allocation rule Φ is ‘merge proof’, for each coalition $S \subset N$, $|S| > 1$, we only need to check if the total payoff of S under Φ at v is larger than the payoff of S under Φ at the partition $\bigcup_{i \in N \setminus S} \{\{i\}\} \cup \{S\}$. Thus, formally we define it as follows.

Let $v \in G^N$ and $\mathcal{P} = \bigcup_{i \in N \setminus S} \{\{i\}\} \cup \{S\}$. An allocation rule Φ is *merge proof* at v if no $S \subset N$ with $|S| > 1$ can improve strictly via merging in v relative to $\mathcal{P}$ under Φ .

Definition 3.2.4. An allocation rule Φ is *merge proof* if for each $v \in G$, Φ is merge proof at v .

Note that in [36], specifically, merge proofness of an allocation rule is also analyzed.

Now, we briefly discuss merge proofness of allocation rules. Note that, if an allocation rule is not merge proof at a game v , then it means that there is at least one coalition, say S , that improves strictly via merging in v , assuming that all the other players outside the coalition stay same. A rational behavior of the coalition S is to merge in that case. However, the knowledge of ‘ S improves strictly if it merges’ is a common knowledge, i.e., known by all the players. It is a common knowledge if both the game and the allocation rule -that will be used for evaluation- is known.¹² Thus, it is expected to have other players outside the

¹²Throughout this chapter, we assume both the game and the allocation rule is known by

coalition S to search for their own goods. In other words, searching for their own (strict) improvement would be a rational behavior for the other players outside the coalition S . Thus, one way of looking at the definition is that it assumes either ‘the players outside the coalition S will not search for their own (strict) improvement’ or ‘they do not have the right to merge’. In simple words one can say that the definition neglects external players.

Obviously, when external players are not neglected, then there is an uncertainty of which game will be the final game after merging behaviors of players, which is also pointed out as a problem in Derks and Tijs [33]. We propose two merge proof definitions where these external players are not neglected. In both proposed definitions, roughly speaking, we solve this problem by considering ‘all the possible merging behaviors of the players’ and ‘all the possibilities of final games’. The differences of the two definitions will be discussed briefly.

Firstly, we define how a coalition improves via merging at a game under some allocation rule no matter how the players outside the coalition behave in terms of merging. Thus, for each $\mathcal{P} \in \mathbb{P}_S$, we need to compare the total payoff of the coalition S at (Φ, v) with the payoff of the player S at each $(\Phi, v|_{\mathcal{P}})$ and ‘decide’ when it is profitable for the coalition S to merge. The two different approaches that we propose of an improvement of a coalition via merging are as follows.

Definition 3.2.5. Given a game $v \in G^N$, an allocation rule Φ and a coalition $S \subset N$ with $|S| > 1$,

1. we say that S *improves strictly-strongly via merging in v under Φ* if the following condition hold:
 - (i) for each partition $\mathcal{P} \in \mathbb{P}_S$, S improves strictly via merging in v relative to $\mathcal{P}$ under Φ and,
2. we say that S *improves strongly via merging in v under Φ* if the following two conditions hold:
 - (i) for each partition $\mathcal{P} \in \mathbb{P}_S$, S improves weakly via merging in v relative to $\mathcal{P}$ under Φ ,

each of the players.

- (ii) there exists at least one partition $\mathcal{P} \in \mathbb{P}_S$ such that S improves strictly via merging in v relative to $\mathcal{P}$ under Φ .

Note that if a coalition S improves strictly-strongly via merging in v under Φ (Definition 3.2.5-1), then by merging, coalition S gets always strictly better off, no matter how the other players act in terms of merging. In other words, even if the players outside of the coalition S have the right and the will to merge in anyway between themselves, then in any possible scenario, via merging, coalition S gets strictly better off, and thus improves. Thus, it is definitely beneficial for the coalition to merge and act as a single player. Yet, Definition 3.2.5-1 can be criticizable for being ‘too’ strong. In other words, it can be criticizable due to the cases similar to the one below.

Even if there exists a unique partition $\mathcal{P}' \in \mathbb{P}_S$ such that the payoff of S at $v|_{\mathcal{P}'}$ is equal to the total payoff of S at v (i.e., $\Phi_S(v|_{\mathcal{P}'}) = \sum_{i \in S} \Phi_i(v)$) and if for each partition $\mathcal{P} \in \mathbb{P}_S$, $\mathcal{P} \neq \mathcal{P}'$, S improves strictly via merging in v relative to $\mathcal{P}$ under Φ (i.e., $\Phi_S(v|_{\mathcal{P}}) > \sum_{i \in S} \Phi_i(v)$), then S does not improve according to Definition 3.2.5-1. Definition 3.2.5-2 takes care of such cases.

In fact, according to Definition 3.2.5-2, if the players outside of the coalition S have to right to merge in anyway between themselves, then in any possible scenario, via merging, coalition S either is at least as good as before or gets better off, and thus improves. Thus, it is beneficial for the coalition to merge and act as a single player. In other words, merging and acting as a single player is a *secure utility level* for the coalition S in Definition 3.2.5-2. Thus, obviously, if a coalition improves strictly-strongly via merging in v , then it improves strongly via merging in v , but the converse does not always hold.

Based on Definition 3.2.5, we propose two different definitions for merge proofness of an allocation rule at v as below.

Definition 3.2.6. Let $v \in G^N$.

1. An allocation rule Φ is *strongly merge proof at v* if there is no $S \subset N$ with $|S| > 1$ that can improve strictly-strongly via merging in v under Φ and,

2. an allocation rule Φ is *strictly-strongly merge proof at v* if there is no $S \subset N$ with $|S| > 1$ that can improve strongly via merging in v under Φ .

For the relation between the two definitions above, consider the following. Suppose that for each coalition $S \subset N$ with two or more players, there is a unique partition $\mathcal{P}' \in \mathbb{P}_S$ such that the payoff of S at $v|_{\mathcal{P}'}$ is equal to the total payoff of S at v and, for each $\mathcal{P} \in \mathbb{P}_S$, $\mathcal{P} \neq \mathcal{P}'$, S improves strictly via merging in v relative to $\mathcal{P}$ under Φ . Then, the allocation rule Φ is strongly merge proof at v . Yet, in the same case, Φ is not strictly-strongly merge proof at v .

Finally, we propose two definitions of ‘merge proofness’ of an allocation rule.

Definition 3.2.7. Given an allocation Φ ,

1. Φ is *strongly merge proof* if for each $v \in G$, Φ is strongly merge proof at v ;
2. Φ is *strictly-strongly merge proof* if for each $v \in G$, Φ is strictly-strongly merge proof at v .

Now, it is easy to see the following relationships between these three definitions.

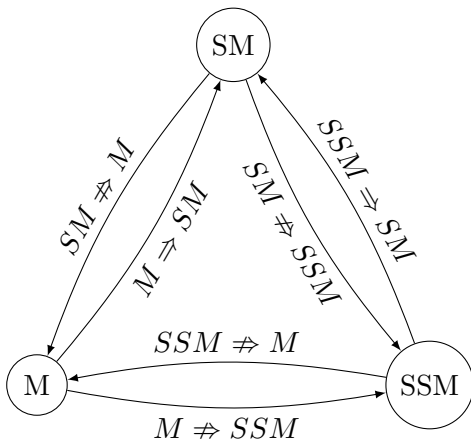


Figure 3.1: Relationships between different notions of merge proofness

Remark 3.2.8. As shown in the above figure, given an allocation rule Φ ,

- Φ is strictly-strongly merge proof $\Rightarrow \Phi$ is strongly merge proof, but the converse is not true in general, i.e., Φ is strongly merge proof $\nRightarrow \Phi$ is strictly-strongly merge proof.
- Φ is merge proof $\Rightarrow \Phi$ is strongly merge proof, but the converse is not true in general, i.e., Φ is strongly merge proof $\nRightarrow \Phi$ is merge proof.
- Φ is merge proof $\nRightarrow \Phi$ is strictly-strongly merge proof, and Φ is strictly-strongly merge proof $\nRightarrow \Phi$ is merge proof.

3.2.3 Impossibility results

In this section, we give our impossibility results.

Theorem 3.2.9. *No core selective allocation rule is anonymous and strongly merge proof.*

Proof. Consider the following game with the player set $N = \{1, 2, 3\}$: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, $v(123) = 1$. Note that $C(v) = \{(1, 0, 0)\}$. Thus, each core selective allocation rule selects $(1, 0, 0)$.

Now, let $\mathcal{P} = (1)(23)$. Note that the game $v|_{\mathcal{P}}$ is defined by the setting $v|_{\mathcal{P}}(\underline{23}) = v|_{\mathcal{P}}(1) = 0$, $v|_{\mathcal{P}}(\underline{123}) = 1$. By anonymity, each core selective allocation rule selects $(1/2, 1/2)$. Therefore, for each allocation rule Φ , $\Phi_{\underline{23}}(v|_{\mathcal{P}}) = 1/2 > 0 = \Phi_2(v) + \Phi_3(v)$, i.e., $\{2, 3\} \subset N$ improves strictly via merging in v relative to $\mathcal{P}$ under Φ . Thus, no core selective allocation rule is strongly merge proof. $\square$

Note that although the example in proof of Theorem 3.2.9 is with three players, it can easily be extended to n players. Specifically, consider the example for any N : Let $S_1, S_2 \subset N$ such that $S_1 \cap S_2 \neq \emptyset$, $S_1 \setminus S_2 \neq \emptyset$, $S_2 \setminus S_1 \neq \emptyset$, $|(S_1 \cup S_2) \setminus (S_1 \cap S_2)| > 2$, and let $v(S_1) = v(S_2) = v(N) = 1$, $v(T) = 0$ otherwise. Now, let $R = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$. One can check that for each $\mathcal{P} \in \mathbb{P}_R$, $\Phi_R(v|_{\mathcal{P}}) > \sum_{i \in R} \Phi_i(v)$.

Theorem 3.2.9 combined with Remark 3.2.8 implies that no core selective rule is anonymous and strictly-strongly merge proof, and no core selective rule is anonymous and merge proof. It is well-known that there are core selective allocation rules that are anonymous, and we do not know if there are core selective allocation rules that are strongly merge proof, but we know that anonymity in Theorem 3.2.9 can be relaxed. For example, following the proof anonymity can be replaced with equal treatment of equals.

Next, we give a more general result than Theorem 3.2.9.

Theorem 3.2.10. *No allocation rule is anonymous, individually rational and strongly merge proof.*

Proof. Suppose, by contradiction, Φ be an allocation rule satisfying the hypothesis of the theorem. Consider the game v with the player set $N = \{1, 2, 3\}$, defined by: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, $v(123) = 1$.

For each $i \in N$, let $\mathcal{P}_i := (i)(jk)$.

By individually rationality, $\Phi_{12}(v|_{\mathcal{P}_3}) = \Phi_{13}(v|_{\mathcal{P}_2}) = 1$. By strong merge proofness,

$$\Phi_{12}(v|_{\mathcal{P}_3}) = 1 \leq \Phi_1(v) + \Phi_2(v) \text{ and,}$$

$$\Phi_{13}(v|_{\mathcal{P}_2}) = 1 \leq \Phi_1(v) + \Phi_3(v).$$

Combined with the fact that $\sum_{i \in N} \Phi_i(v) = v(N)$, addition of the last two inequalities gives $\Phi_1(v) = 1$. So, $\Phi_2(v) + \Phi_3(v) = 0$. By anonymity, $\Phi_{23}(v|_{\mathcal{P}_1}) = \Phi_1(v|_{\mathcal{P}_1}) = 1/2$, thus,

$$1/2 = \Phi_{23}(v|_{\mathcal{P}_1}) > \Phi_2(v) + \Phi_3(v) = 0,$$

which contradicts the assumption that Φ is strongly merge proof. $\square$

As in the case of Theorem 3.2.9, although the example in the proof of Theorem 3.2.10 is with three players, it can easily be extended to n players. Also, Theorem 3.2.10 combined with Remark 3.2.8 implies that no allocation rule is

anonymous, individually rational and strictly-strongly merge proof, and no allocation rule is anonymous, individually rational and merge proof.

Note that, individually rationality can be removed from the above theorem. The equal division allocation rule is an example that is anonymous, merge proof and strictly-strongly merge proof, thus also strongly merge proof. Yet, it can be replaced by some other axioms. Next, we give a result where it is replaced by the null player axiom.

Theorem 3.2.11. *No allocation rule is anonymous, satisfies null player axiom and strong merge proofness.*

Proof. Suppose, by contradiction, Φ be an allocation rule satisfying the hypothesis of the theorem. Consider the game v with the player set $N = \{1, 2, 3\}$, defined by: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, and $v(123) = 1$.

For each $i \in N$, let $\mathcal{P}_i := (i)(jk)$.

By the null player axiom, $\Phi_3(v|_{\mathcal{P}_3}) = \Phi_2(v|_{\mathcal{P}_2}) = 0$. So,

$$\Phi_{12}(v|_{\mathcal{P}_3}) = \Phi_{13}(v|_{\mathcal{P}_2}) = 1.$$

By strong merge proofness,

$$1 = \Phi_{12}(v|_{\mathcal{P}_3}) \leq \Phi_1(v) + \Phi_2(v) \text{ and,}$$

$$1 = \Phi_{13}(v|_{\mathcal{P}_2}) \leq \Phi_1(v) + \Phi_3(v).$$

Combined with the fact that $\sum_{i \in N} \Phi_i(v) = v(N)$, addition of the last two inequalities gives $\Phi_1(v) = 1$. So, $\Phi_2(v) + \Phi_3(v) = 0$. By anonymity, $\Phi_{23}(v|_{\mathcal{P}_1}) = \Phi_1(v|_{\mathcal{P}_1}) = 1/2$, thus,

$$1/2 = \Phi_{23}(v|_{\mathcal{P}_1}) > \Phi_2(v) + \Phi_3(v) = 0,$$

which contradicts the assumption that Φ is strongly merge proof. $\square$

We want to note that in proof of Theorems 3.2.9, 3.2.10 and 3.2.11, anonymity is used at only one place. Basically, it is used just to have $\Phi_i(v) = \Phi_j(v)$ at a

game $v \in G_{trv}^2$. Although anonymity is usually a very desirable, and a weak axiom itself, in fact anonymity in both theorems can be placed with weaker axioms. For example, since ‘equal treatment of equals’ implies ‘anonymity on G_{trv} ’ -which is enough for the proofs-, one can replace ‘anonymity’ axiom with ‘equal treatment of equals’ axiom in both Theorems 3.2.10 and 3.2.11. In fact, one can replace ‘anonymity’ even with weaker conditions, namely either with ‘equal treatment of equals on G_{trv} ’ or ‘egalitarian on G_{trv} ’.

3.2.3.1 Convex games

In this section, we restrict ourselves to the set of convex games and study two well-known allocation rules, namely the Shapley value and the Fujishige-Dutta Ray egalitarian solution.

Consider the convex game defined by the setting: $v(1) = 1$, $v(2) = v(3) = 0$, $v(12) = v(13) = 5$, $v(23) = 1$ and $v(123) = 9$. Note $Sh(v) = (28/6, 13/6, 13/6)$. Considering $\mathcal{P} = (1)(23)$, one easily sees that

$$Sh(v|_{\mathcal{P}}) = (Sh_1(v|_{\mathcal{P}}), Sh_{\underline{23}}(v|_{\mathcal{P}})) = (\frac{27}{6}, \frac{27}{6}).$$

Therefore,

$$\frac{27}{6} = Sh_{\underline{23}}(v|_{\mathcal{P}}) > Sh_2(v) + Sh_3(v) = \frac{26}{6}.$$

Thus, the Shapley value is not strongly merge proof on G_{con} . Thus, by Remark 3.2.8, the Shapley is neither strictly-strongly merge proof nor merge proof on G_{con} .¹³

In [36] it is shown that the Fujishige-Dutta Ray egalitarian solution is merge proof on the class of monotonic convex games, thus by Remark 3.2.8, it is strongly merge proof on the class of monotonic convex games, as well. However, it is not strictly-strongly merge proof on G_{con} .

To see this, consider the game $v \in G_{con}^4$ defined by:

$$v(1) = v(2) = v(3) = 0, v(4) = 5,$$

¹³In [36] it is also shown that the Shapley value is not merge proof on G_{con} .

$v(12) = 6$, $v(14) = v(24) = v(34) = 5$, $v(13) = v(23) = 0$,
 $v(123) = 6$, $v(124) = 11$, $v(134) = v(234) = 5$, and $v(1234) = 13$.

Note that, the Fujishige-Dutta Ray egalitarian solution outcome of v is $FDR(v) = (3, 3, 2, 5)$. Now, suppose that player 1 and player 2 merge.

First consider $\mathcal{P}' = (12)(3)(4)$. Then $v|_{\mathcal{P}'}$ is defined by setting,
 $v|_{\mathcal{P}'}(\underline{12}) = 6$, $v|_{\mathcal{P}'}(3) = 0$, $v|_{\mathcal{P}'}(4) = 5$,
 $v|_{\mathcal{P}'}(34) = 5$, $v|_{\mathcal{P}'}(\underline{123}) = 6$, $v|_{\mathcal{P}'}(\underline{124}) = 11$,
 $v|_{\mathcal{P}'}(\underline{1234}) = 13$.

One easily sees that

$$FDR_{\underline{12}}(v|_{\mathcal{P}'}) = FDR_1(v) + FDR_2(v) = 6. \quad (3.2.1)$$

Next consider $\mathcal{P} = (12)(34)$. Then $v|_{\mathcal{P}}$ is defined by setting,
 $v|_{\mathcal{P}}(\underline{12}) = 6$, $v|_{\mathcal{P}}(\underline{34}) = 5$, and $v|_{\mathcal{P}}(\underline{1234}) = 13$.

One also easily sees that $FDR(v|_{\mathcal{P}}) = (13/2, 13/2)$, which yields

$$\frac{13}{2} = FDR_{\underline{12}}(v|_{\mathcal{P}}) > FDR_1(v) + FDR_2(v) = 6. \quad (3.2.2)$$

Equations (3.2.1) and (3.2.2) show that $\{1, 2\}$ improves strongly via merging in v under FDR . Therefore, the Fujishige-Dutta Ray egalitarian solution is not strictly-strongly merge proof on G_{con} .

Obviously, the example above also brings a potential problem about strictly-strong merge proofness, because on the example, $\{1, 2\}$ gets strictly better off only if 3 and 4 merge. However, by merging, 3 and 4 will get worse off. Thus, 3 and 4 may not merge. Yet, there is no harm for players 1 and 2 to merge. Moreover, 1 and 2 at least guarantees a fixed payoff by merging, no matter what players 3 and 4 do. In general, for to see the need of the notion strictly-strong merge proofness, the reader may refer to the discussion in Section 3.2.2 after Definition 3.2.5. Moreover, for the formal form of the same discussion, the reader can pass to Section 3.2.5.

3.2.4 Possibility results

Natural questions arise: if there exists any allocation rule that is strictly-strongly merge proof, strongly merge proof or merge proof.¹⁴ To all, the answer is affirmative. ED is strictly-strongly merge proof obviously, by induction. Thus by Remark 3.2.8, it is strongly merge proof. Also one can easily see that it is merge proof.

Also, under the assumption that for each $n \in \mathbb{N}$, $d \in N$, one easily sees that Γ^{Dct} is strictly-strongly merge proof. Thus by Remark 3.2.8, it is strongly merge proof. Also one can easily check that it is merge proof.

Next, we consider convex combination of some allocation rules.

Theorem 3.2.12. *Let Φ and Φ' be two allocation rules both satisfying merge proofness. For each $v \in G$ and each $\alpha \in [0, 1]$, if*

$$\Phi_\alpha(v) := \alpha\Phi(v) + (1 - \alpha)\Phi'(v),$$

then Φ_α is an allocation rule that is merge proof.

Proof. If Φ and Φ' are two allocation rules both satisfying merge proofness, then given any $S \subset N$ with $|S| > 1$ and $\mathcal{P} = \bigcup_{i \in N \setminus S} \{\{i\}\} \cup \{S\}$,

$$\Phi_S(v|_{\mathcal{P}}) \leq \sum_{i \in S} \Phi_i(v) \text{ and,}$$

$$\Phi'_S(v|_{\mathcal{P}}) \leq \sum_{i \in S} \Phi'_i(v).$$

Therefore for any $\alpha \in [0, 1]$, obviously

$$\begin{aligned} (\Phi_\alpha)_S(v|_{\mathcal{P}}) &= \alpha\Phi_S(v|_{\mathcal{P}}) + (1 - \alpha)\Phi'_S(v|_{\mathcal{P}}) \\ &\leq \alpha \sum_{i \in S} \Phi_i(v) + (1 - \alpha) \sum_{i \in S} \Phi'_i(v) = \sum_{i \in S} (\Phi_\alpha)_i(v). \end{aligned}$$

Hence, for each $\alpha \in [0, 1]$, Φ_α is an allocation rule that is merge proof. $\square$

¹⁴For the latter except Fujishige-Dutta Ray egalitarian solution on G_{con} for now.

Thus, by Remark 3.2.8, each Φ_α in Theorem 3.2.12 is also strongly merge proof.

The next result gives a class of allocation rules that are strictly-strongly merge proof.

Theorem 3.2.13. *Let Φ be an allocation rule satisfying merge proofness. For each $v \in G$ and each $\alpha \in (0, 1]$, if*

$$\Phi_\alpha(v) := \alpha ED(v) + (1 - \alpha)\Phi(v),$$

then Φ_α is an allocation rule that is strictly-strongly merge proof.

Proof. For $\alpha = 1$, obviously Φ_α is strictly-strongly merge proof. For $\alpha \in (0, 1)$, we argue by contradiction. Suppose there is $\alpha' \in (0, 1)$ such that $\Phi_{\alpha'}$ is an allocation rule that is not strictly-strongly merge proof. Then, for some $v \in G^N$, there is a coalition $S \subset N$ with $|S| > 1$ such that S improves strongly via merging in v under $\Phi_{\alpha'}$.

Since Φ is an allocation rule satisfying merge proofness, for each $S' \subset N$ and each $\mathcal{P} = \bigcup_{i \in N \setminus S'} \{\{i\}\} \cup \{S'\}$,

$$\Phi_{S'}(v|\mathcal{P}) \leq \sum_{i \in S'} \Phi_i(v).$$

Also for each S' and $\mathcal{P} = \bigcup_{i \in N \setminus S'} \{\{i\}\} \cup \{S'\}$, by definition of equal division allocation rule,

$$ED_{S'}(v|\mathcal{P}) = \frac{v(N)}{|N| - |S'| + 1} \text{ and,}$$

$$\sum_{i \in S'} (ED)_i(v) = \frac{|S'|v(N)}{|N|}.$$

Note that by $S \subset N$ and $|S| > 1$,

$$|S| < |N| \iff (1 - |S|)|S| > (1 - |S|)|N| \iff \frac{|S|}{|N|} > \frac{1}{|N| - |S| + 1}.$$

Therefore,

$$ED_S(v|\mathcal{P}) = \frac{v(N)}{|N| - |S| + 1} < \sum_{i \in S} (ED)_i(v) = \frac{|S|v(N)}{|N|}.$$

Then for $\mathcal{P} = \bigcup_{i \in N \setminus S} \{\{i\}\} \cup \{S\}$,

$$\begin{aligned} (\Phi_{\alpha'})_S(v|\mathcal{P}) &= \alpha(ED)_S(v|\mathcal{P}) + (1 - \alpha)\Phi_S(v|\mathcal{P}) \\ &< \alpha \sum_{i \in S} (ED)_i(v) + (1 - \alpha) \sum_{i \in S} \Phi_i(v) = \sum_{i \in S} (\Phi_{\alpha'})_i(v), \end{aligned}$$

i.e., at $\mathcal{P} = \bigcup_{i \in N \setminus S} \{\{i\}\} \cup \{S\}$, S neither improves weakly nor strictly.

In other words, S does not improve strongly via merging in v under $\Phi_{\alpha'}$, contradicting to the assumption.

Hence, for each $\alpha \in (0, 1]$, Φ_α is strictly-strongly merge proof. $\square$

By Theorem 3.2.13, $(\Phi_\alpha)_{\alpha \in (0, 1]}$ is a class of allocation rules that are strictly-strongly merge proof.

Note that given any N with $d \in N$, dictatorial allocation rule satisfies merge proofness; therefore, as a corollary, we have the following result.

Corollary 3.2.14. *Let N with $d \in N$ and $v \in G^N$ be given. For each $\alpha \in [0, 1]$, if*

$$\Phi_\alpha(v) := \alpha ED(v) + (1 - \alpha)\Phi^{Dct}(v),$$

then Φ_α is strictly-strongly merge proof.

Note that strictly-strong merge proofness for $\alpha = 0$ follows from the fact that $\Phi_0 = \Phi^{Dct}$ itself is strictly-strongly merge proof. By Remark 3.2.8 ‘strictly-strong merge proofness’ in the above theorem and the corollary can be replaced by ‘strong merge proofness’. Moreover, following the proof, ‘strictly-strong merge proofness’ can be replaced by ‘merge proofness’.

In [36] it is shown that the Fujishige-Dutta Ray egalitarian solution is merge proof on the class of monotonic convex games. Theorem 3.2.13 combined with their result and Remark 3.2.8, we have the following.

Corollary 3.2.15. *For each v on the class of monotonic convex games, and each $\alpha \in [0, 1]$, if*

$$\Phi_\alpha(v) := \alpha ED(v) + (1 - \alpha)FDR(v),$$

then Φ_α is an allocation rule that is strictly-strongly merge proof.

Again, by Remark 3.2.8, ‘strictly-strong merge proofness’ can be replaced by ‘strong merge proofness’. Moreover, following the proof, ‘strictly-strong merge proofness’ can be replaced by ‘merge proofness’.

3.2.5 Merging as a stability notion

There is no doubt that stability notion has always been an important subject in economics’ problems or elsewhere. In this section, we study merging as a stability notion. We only discuss strictly-strong merge proofness. Analyzing strong merge proofness in a similar way is left to the reader. In other words, this section is a formal form of the discussion on Section 3.2.2 that is done in terms of rights structure.

For each player set N , each $\mathcal{P} \in \mathbb{P}(N)$ and each $v \in G^N$, we say that $\tilde{\mathcal{P}} \in \mathbb{P}(N) \setminus \{\mathcal{P}\}$ is *reachable from $\mathcal{P}$ by simultaneous coalitional movements*, denoted by $\mathcal{P} \rightarrow \tilde{\mathcal{P}}$, if $[S \in \tilde{\mathcal{P}} \Rightarrow \text{there exists a collection of coalitions, each of which belongs to } \mathcal{P} \text{ whose union is } S, \text{ i.e., } \exists T_1, T_2, \dots, T_k \subseteq N \text{ such that for each } l \in \{1, \dots, k\}, T_l \in \mathcal{P} \text{ and } \bigcup_{l=1}^k T_l = S.]$

In other words, starting with the player set $\mathcal{P}$, $\tilde{\mathcal{P}}$ is a reachable player set if some of the players in $\mathcal{P}$ form coalitions and act together as a single player. Note that $\tilde{\mathcal{P}}$ is finer than $\mathcal{P}$, i.e., it has less elements than $\mathcal{P}$.

Given $\mathcal{P} \in \mathbb{P}(N)$, let $S \subset N$ be a set such that it is a union of two or more elements of $\mathcal{P}$, $v|_{\mathcal{P}}$ be in $G^{\mathcal{P}}$, and Φ be an allocation rule. We say that *a coalition S benefits and has the right to merge at $(v|_{\mathcal{P}}, \Phi)$* if,

- (i) for each $\tilde{\mathcal{P}}$ such that $\mathcal{P} \rightarrow \tilde{\mathcal{P}}$, $\Phi_S(v|_{\tilde{\mathcal{P}}}) \geq \sum_{\{T_l: T_l \in \mathcal{P}, S = \bigcup_{l=1}^k T_k\}} \Phi_{T_l}(v|_{\mathcal{P}})$ and,
- (ii) there exists $\hat{\mathcal{P}}$ such that $\mathcal{P} \rightarrow \hat{\mathcal{P}}$, $\Phi_S(v|_{\hat{\mathcal{P}}}) > \sum_{\{T_l: T_l \in \mathcal{P}, S = \bigcup_{l=1}^k T_k\}} \Phi_{T_l}(v|_{\mathcal{P}})$.

In other words, starting with the player set $\mathcal{P}$ and the game $v|_{\mathcal{P}}$, a coalition of the player set $\mathcal{P}$ with two or more players¹⁵ benefits, and has the right to form a coalition, and thus acts together as a single player, if the coalition does not get worse off no matter how the other players in $\mathcal{P}$ form coalitions between each

¹⁵Obviously, except the grand coalition.

other and merge. In fact, when S has the right to merge, S benefits via merging and no player outside S can effect this. In other words, nobody outside of S can stop the decision of ‘merging of the players in S ’ and change its profitability for S .

Given $(v|_{\mathcal{P}}, \Phi)$, let $\mathfrak{R}$ denote the set of coalitions that benefits and has the right to merge at $(v|_{\mathcal{P}}, \Phi)$. We say that $(v|_{\mathcal{P}}, \Phi)$ is *(strictly-strongly) merge stable* if $\mathfrak{R} = \emptyset$. Thus, one can think of (strictly-strongly) merge stability as the case where there is no coalition that benefits via merging although it has the right to merge.

Given a player set N , let $\overline{\mathcal{P}} = \{\{1\}, \{2\}, \dots, \{n\}\}$. Note that each $\tilde{\mathcal{P}} \in \mathbb{P}(N) \setminus \{\overline{\mathcal{P}}\}$ is reachable from $\overline{\mathcal{P}}$ by simultaneous coalitional movements. Also, given $v \in G^N$, note that $v = v|_{\overline{\mathcal{P}}}$. Thus, we have the following proposition.

Proposition 3.2.16. *An allocation rule Φ is strictly-strongly merge proof if and only if (v, Φ) is (strictly-strongly) merge stable for each $v \in G$.*

Proof of Proposition 3.2.16 is straightforward and left to the reader. Thus, merge proofness can be thought as a stability notion. Basically, as seen above, if Φ is not strictly-strongly merge proof at a game v , i.e., if (v, Φ) is not (strictly-strongly) merge stable, then the game v will not be played when the allocation rule Φ is used for evaluation of the game. In other words, there exists some $\mathcal{P} \in \mathbb{P}(N)$ such that $v|_{\mathcal{P}}$ will surpass¹⁶ the game v .

3.3 Part II-Allocation correspondences

In this section, we extend our results in Section 3.2 to allocation correspondences. Since an allocation rule can be thought as an allocation correspondence that is non-empty and single valued, any result for allocation correspondences is true for allocation rules as well.

¹⁶Thus, $v|_{\mathcal{P}}$ will be played instead of v .

3.3.1 Preliminaries for allocation correspondences

An *allocation correspondence*¹⁷ is a map $\Gamma : G \rightarrow \bigcup_{N:n \in \mathbb{N}} \mathbb{R}^N$ that assigns a subset of payoff vectors from $F(v)$ for each $v \in G$. For example, core of a game is an allocation correspondence.

For each allocation correspondence Γ , each N , each $v \in G^N$ and each $i \in N$, the set of all possible payoff vectors that player i can receive under $\Gamma(v)$ at v is denoted by $\Gamma_i(v)$, i.e., $\Gamma_i(v) := \{x_i : x \in \Gamma(v)\}$, and for each $S \subset N$, the set of all possible total payoff vectors that S can receive under Γ at v is denoted by $\oplus_{i \in S} \Gamma_i(v)$, i.e., $\oplus_{i \in S} \Gamma_i(v) = \{\sum_{i \in S} x_i : x \in \Gamma(v)\}$.¹⁸

Next we state some axioms for allocation correspondences.

1. First is independence of the names of the players. Formally, let $\sigma : N \rightarrow N$ denote a permutation of N . Let $v \in G^N$. A new game induced by σ and v , denoted by σv , is defined by setting, for each $S \subseteq N$, $\sigma v(S) = v(\sigma(S))$. An allocation correspondence Γ is *anonymous* if for each $n \in \mathbb{N}$, each $v \in G^N$, each permutation σ , and each $i \in N$, $\Gamma_{\sigma i}(\sigma v) = \Gamma_i(v)$.
2. Second is the equal division on the games in G_{trv} . Formally, we say that Γ is *egalitarian on G_{trv}* , if for each N , each $v \in G_{trv}^N$ and $i \in N$, $\Gamma_i(v) := \frac{v(N)}{n}$.¹⁹
3. Let $v \in G^N$. A player $i \in N$ is a *null player in v* if for each $S \subseteq N$, $v(S \cup \{i\}) = v(S)$. Our next requirement is that each null player be assigned a zero payoff. Formally, for each $v \in G^N$, let $D(v)$ be the set of null players in v . An allocation correspondence Γ satisfies *the null player axiom* if for each $v \in G$ and each $i \in D(v)$, $\Gamma_i(v) = \{0\}$.
4. An allocation correspondence Γ is *individually rational* if for each $v \in G$, $\Gamma(v) \subseteq I(v)$.
5. An allocation correspondence Γ is *stable* if for each $v \in G$, $\Gamma(v) \subseteq S(v)$.

¹⁷Also called a solution concept.

¹⁸Note that ' $\oplus$ ' is not a regular summation.

¹⁹Egalitarian on G_{trv} can also be thought as equal treatments of equals on G_{trv} .

6. An allocation correspondence Γ is *efficient* if for each $v \in G$, $\Gamma(v) \subseteq E(v)$.

We work on efficient allocation correspondences, thus from now on we assume allocation correspondences to be efficient.

Note that anonymity for allocation correspondences is a bit different than anonymity for allocation rules. To explain, we briefly discuss the relation between anonymity and egalitarian on G_{trv} .²⁰ Note that if an allocation correspondence Γ is anonymous, then we only have for each N , each $v \in G_{trv}^N$ and each pair $i, j \in N$, $\Gamma_i(v) = \Gamma_j(v)$. Moreover, since we assume all allocation correspondences are efficient, for $v \in G_{trv}^N$, $i \in N$, if $x \in \Gamma_i(v)$, then there exists a collection of $(n-1)$ real numbers in $\Gamma_i(v)$, say $x_1, \dots, x_{n-1} \in \Gamma_i(v)$ such that $\sum_{k=1}^{n-1} x_k + x = v(N)$. Thus, anonymity does not imply egalitarian on G_{trv} . Also, it is obvious that egalitarian on G_{trv} does not imply anonymity. Yet, if Γ is egalitarian on G_{trv} , then Γ is anonymous on G_{trv} .

Note that, by definition, the core is an allocation correspondence that is anonymous and individually rational.

As in the case of allocation rules, we propose different definitions for merge proofness of allocation correspondences. The idea for different definitions follows from following. Assume a player is asked to choose one of the set from $A, B \subset \mathbb{R}$, where A and B are two different sets of possible payoffs that the player can get. So, if the player chooses the set A , then he will get one of the payoffs in A , and if he chooses B , then he will get one of the payoffs in B . In fact, the player mentioned here will correspond to the coalition S in the main subject, and the sets A and B will correspond to the set of possible values that the coalition S can get before merging and after merging, respectively.

Based on the above idea, we define the concept of merge proofness of an allocation correspondence nearly the same as the concept of merge proofness of allocation rules. Yet, this time, given a game if the allocation correspondence is known, then the only knowledge that is known by each of the initial players of the

²⁰Note that for allocation rules, anonymity implies egalitarian on G_{trv} . In fact, anonymity on G_{trv} is same as equality on G_{trv} . Yet, this is not the case for allocation correspondences.

game is the *set* of possible payoff vectors that each player may get. Thus, the exact value of the total payoff of each coalition is not known. Yet, the set of possible payoff vectors that the coalition can receive under the allocation correspondence at that game is known. Similarly, if one coalition merges, the only knowledge that is available for the merged coalition is again a *set* of possible payoff vectors that it may get. Thus, for merging, we compare these two sets, i.e., we take the set of possible payoff vectors of a coalition S and compare this set with the set of possible payoff vectors of the player S after merging.

Similar to merge proofness of allocation rules, when the above comparison is done only for the coalition S , i.e., by assuming the players outside of S stay same, then ‘merge proofness without externalities of allocation correspondences’ are obtained. And similar to strongly merge proofness and strictly-strongly merge proofness of allocation rules, when the above comparison is done by considering all the cases where the other players outside of S have the right to merge between themselves in any way they wish, then ‘merge proofness of allocation correspondences’ are obtained.

Moreover, different than the case for allocation rules, allocation correspondences can also give the empty set as a solution, which is taken care of at the definitions. Thus, different than the case for allocation rules, this time how to compare two non-empty sets is important, i.e., for any two sets $A, B \subset \mathbb{R}$, how to define $A \geq B$ and/or $A > B$.

We turn back to comparison of two sets. Equality between two sets is clear, i.e., $A = B \iff [a \in A \iff b \in B]$. Consider the example where $A = \{1, 3\}$ and $B = \{2\}$, obviously a comparison between A and B is hard to define. Thus, according to each of our definitions some sets will not be comparable, i.e., we may have $A \not\geq B$ and $B \not\geq A$. Thus, we define both the statements $A > B$, $A \geq B$ and their negations separately. In other words, $A \not\geq B$ does not imply $B \geq A$, and $A \not> B$ does not imply $B > A$ in the usual sense. Negation of the statement $A > B$ is equal to $A \not\geq B$ and negation of the the statement $A \geq B$ is equal to $A \not> B$. That is, $\neg(A > B) \equiv A \not\geq B$ and $\neg(A \geq B) \equiv A \not> B$.

Finally, due to our subject, we only need the above comparisons for the non-empty sets that have both minimum and maximum²¹, thus both are assumed to exist. Let $\mathbb{S} := \{A \in 2^{\mathbb{R}} : A \neq \emptyset, \inf A = \min A, \sup A = \max A\}$. For $A \geq B$, we propose two different definitions, which are simply denoted by $\geq_1$ and $\geq_2$, respectively. Roughly speaking, the two definitions below reflect two different view points: optimistic and pessimistic, respectively.

Definition 3.3.1. Given any two sets $A, B \subset \mathbb{S}$,

1. $A \geq_1 B$ if for each $a \in A$, either $a \in B$ or $a \geq \max B$. $A \not\geq_1 B$, otherwise.
2. $A \geq_2 B$ if $\min A \geq \max B$. $A \not\geq_2 B$, otherwise.

Note that $>_1$ is a partial ordering on $\mathbb{S}$, but $>_2$ is not.²² The following remark also shows the difference between the two definitions more clearly.

Remark 3.3.2.

- For each $a \in A$ and each $b \in B$, $a \geq \min A$ and $\max B \geq b$, thus we have $[A \geq_2 B \Rightarrow A \geq_1 B]$,
- Let $A = \{2, 3, 4\}$ and $B = \{1, 2, 3\}$. Note $A \geq_1 B$, but $\min A = 2 < 3 = \max B$, so $A \not\geq_2 B$, thus we have $[A \geq_1 B \not\Rightarrow A \geq_2 B]$.

For $A > B$ we suggest five different definitions driven from Definition 3.3.1.

Definition 3.3.3. Given any two sets $A, B \in \mathbb{S}$,

1. $A >_1 B$ if $A \geq_1 B$ and $A \neq B$. $A \not>_1 B$, otherwise.

²¹Efficiency of allocation correspondences guarantees the existence of the minimum and the maximum of the sets that we will be comparing. Minimum and maximum of a set can be replaced by infimum and supremum of a set, respectively. Yet, some extra care might be needed for some definitions, which makes the subject unnecessarily complicated for the main subject.

²²The relation $>_1$ is reflexive, antisymmetric and transitive, thus it is a partial ordering on $\mathbb{S}$. The relation $>_2$ is antisymmetric and transitive, yet it is not reflexive, thus is not a partial ordering on $\mathbb{S}$.

2. $A >_2 B$ if $A \geq_2 B$ and $A \neq B$. $A \not>_2 B$, otherwise.
3. $A >_3 B$ if $A \geq_1 B$ and there exists $a \in A$ such that $a > \max B$. $A \not>_3 B$, otherwise.
4. $A >_4 B$ if $A \geq_2 B$ and there exists $a \in A$ such that $a > \max B$. $A \not>_4 B$, otherwise.
5. $A >_5 B$ if $\min A > \max B$. $A \not>_5 B$, otherwise.

The following remark show the relation between the definitions.

Remark 3.3.4. Given any two sets $A, B \in \mathbb{S}$,

- $A >_5 B \Rightarrow A >_4 B \Rightarrow A >_2 B \Rightarrow A >_1 B$ and, $A >_4 B \Rightarrow A >_3 B \Rightarrow A >_1 B$,
- $A >_1 B \not\Rightarrow A >_2 B \not\Rightarrow A >_4 B \not\Rightarrow A >_5 B$ and, $A >_1 B \not\Rightarrow A >_3 B \not\Rightarrow A >_4 B$. Finally, $A >_2 B \not\Rightarrow A >_3 B$ and $A >_3 B \not\Rightarrow A >_2 B$.

For example, let $A = \{2, 3, 4\}$ and $B = \{1, 2, 3\}$, then $A >_1 B$, $A >_3 B$ but $A \not>_2 B$ and $A \not>_4 B$. Similarly, let $A_* = \{2\}$ and let $B_* = \{1, 2\}$, then $A_* >_1 B_*$, $A_* >_2 B_*$ but $A_* \not>_3 B_*$ and $A_* \not>_4 B_*$. Finally, $A_{**} = \{2, 3\} >_4 \{1, 2\} = B_*$, but $A_{**} \not>_5 B_*$.

Definition 3.3.1 and Definition 3.3.3 lead to different merge proof definitions for allocation correspondences, given in the next section.

3.3.2 Merge proofness of allocation correspondences

In this section, based on the comparison definitions (Definitions 3.3.1 and 3.3.3) we give all possible merge proof definitions for allocation correspondences. The section is organized as follows. In Section 3.3.2.1, five different definitions that correspond to Definition 3.2.4 are given. In Section 3.3.2.2, five different definitions that correspond to Definition 3.2.7-1 are given. In Section 3.3.2.3, ten different definitions that correspond to Definition 3.2.7-2 are given.

3.3.2.1 Merge proofness without externalities of allocation correspondences

As in the case of allocation rules, firstly, we only allow one coalition to merge, and the other players are not allowed to merge. Thus, as discussed for allocation rules, this is a restricted way of defining merge proofness in terms of giving rights to merge. Based on Definition 3.3.3, there are five different versions of merge proofness without externalities of an allocation correspondence, each of which is similar to the definition of merge proofness of allocation rules.

Let $v \in G^N$, $\mathcal{P} = \bigcup_{i \in N \setminus S} \{\{i\}\} \cup \{S\}$ and let Γ be an allocation correspondence such that $\Gamma(v) \neq \emptyset$. For each $s \in \{1, 2, 3, 4, 5\}$, an allocation correspondence Γ is *s-merge proof without externalities at v* if there is no $S \subset N$ with $|S| > 1$ such that $\Gamma(v|_{\mathcal{P}}) \neq \emptyset$, $\Gamma_S(v|_{\mathcal{P}}) >_s \oplus_{i \in S} \Gamma_i(v)$.

For each $s \in \{1, 2, 3, 4, 5\}$, an allocation correspondence Γ is *s-merge proof without externalities* if for each $v \in G$ where $\Gamma(v) \neq \emptyset$, Γ is *s-merge proof without externalities at v*.

It is easy to drive the relations between the five different definitions above, by Remark 3.3.4. The next figure shows all the implications.²³

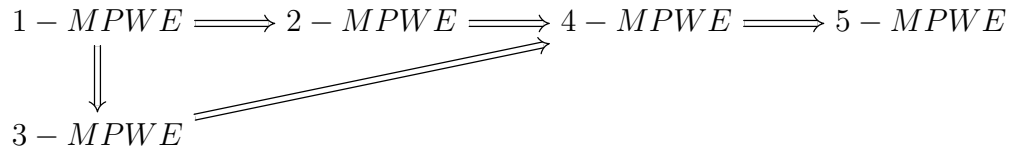


Figure 3.2: MPWE=Merge proof without externalities

3.3.2.2 s-Merge proof of allocation correspondences

Based on Definition 3.3.3, there are five different versions of merge proofness of an allocation correspondence, each of which is similar to the definition of strongly

²³The reverse of the implications in the figure are not true, and also note that $2 - MPWE \not\Rightarrow 3 - MPWE$.

merge proofness of allocation rules.

Let $v \in G^N$ and Γ be an allocation correspondence such that $\Gamma(v) \neq \emptyset$. For each $s \in \{1, 2, 3, 4, 5\}$, an allocation correspondence Γ is *s-merge proof at v* if there is no $S \subset N$ with $|S| > 1$ such that for each $\mathcal{P} \in \mathbb{P}_S$ where $\Gamma(v|_{\mathcal{P}}) \neq \emptyset$, $\Gamma(v|_{\mathcal{P}}) \neq \emptyset$, $\Gamma_S(v|_{\mathcal{P}}) >_s \oplus_{i \in S} \Gamma_i(v)$.

For each $s \in \{1, 2, 3, 4, 5\}$, an allocation correspondence Γ is *s-merge proof* if for each $v \in G$ where $\Gamma(v) \neq \emptyset$, Γ is *s-merge proof at v*.

It is easy to drive the relations between the five different definitions above, by Remark 3.3.4. The next figure shows all the implications.²⁴

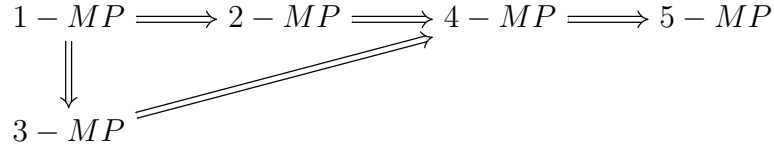


Figure 3.3: MP=Merge proofness

3.3.2.3 (m, s) -Merge proof of allocation correspondences

Based on Definitions 3.3.1 and 3.3.3, we provide ten different versions of merge proofness of an allocation correspondence, each of which is similar to the definition of strictly-strongly merge proofness of allocation rules.

Definition 3.3.5. Let $v \in G^N$, and Γ be an allocation correspondence such that $\Gamma(v) \neq \emptyset$. For each pair $(m, s) \in \{1, 2\} \times \{1, 2, 3, 4, 5\}$, Γ is *(m, s)-merge proof at v* if there is no $S \subset N$ with $|S| > 1$ satisfying the following two conditions:

- (i) for each $\mathcal{P} \in \mathbb{P}_S$ where $\Gamma(v|_{\mathcal{P}}) \neq \emptyset$, $\Gamma_S(v|_{\mathcal{P}}) \geq_m \oplus_{i \in S} \Gamma_i(v)$ and,
- (ii) there exists at least one $\mathcal{P}' \in \mathbb{P}_S$ where $\Gamma(v|_{\mathcal{P}'}) \neq \emptyset$, $\Gamma_S(v|_{\mathcal{P}'}) >_s \oplus_{i \in S} \Gamma_i(v)$.

Definition 3.3.6. For each pair $(m, s) \in \{1, 2\} \times \{1, 2, 3, 4, 5\}$, an allocation correspondence Γ is *(m, s)-merge proof* if for each $v \in G$ where $\Gamma(v) \neq \emptyset$, Γ is *(m, s)-merge proof at v*.

²⁴Inverses of the implications in the figure are not true, and also $2 - MP \not\leftrightarrow 3 - MP$.

It is easy to drive the relations between ten different merge proofness definitions by Remark 3.3.2 and Remark 3.3.4. The next figure shows all the implications.²⁵ For to read the figure: (m, s) in the below figure denotes (m, s) -merge proofness of an allocation rule, e.g. ' $(1, 1) \Rightarrow (1, 2)$ ' is read as 'If an allocation rule is $(1, 1)$ -merge proof, then it is $(1, 2)$ -merge proof.'

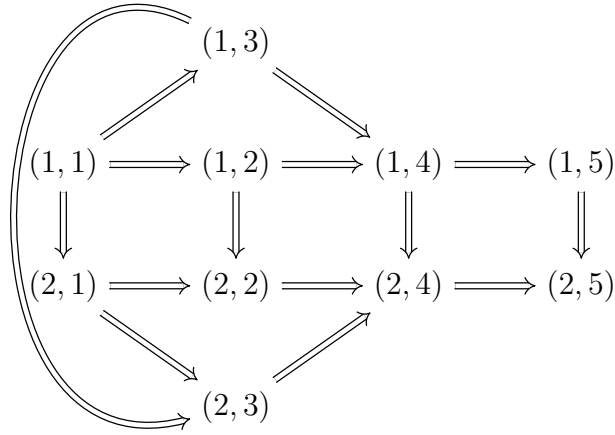


Figure 3.4: (m, s) -Merge Proofness

3.3.3 Results for allocation correspondences

In this section, first we analyze the merging behavior of the core in terms of different merge proof definitions that we have given, then we generalize these results. Indeed, all of these results are extensions of our results for allocation rules.

Theorem 3.3.7. *For each $m \in \{1, 2\}$ and each $s \in \{1, 2, 3, 4\}$, the core is not s -merge proof without externalities; the core is not s -merge proof; and the core is not (m, s) -merge proof.*

Proof. Suppose, by contradiction. Consider the game v with the player set

²⁵The reverse of the implication in the figure are not true. Also, $(1, 2) \nRightarrow (1, 3)$ and $(2, 2) \nRightarrow (2, 3)$.

$N = \{1, 2, 3\}$, defined by: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, $v(123) = 1$. Note that $C(v) = \{(1, 0, 0)\}$.

Now, let $\mathcal{P} = (1)(23)$. The game $v|_{\mathcal{P}}$ is defined by setting, $v|_{\mathcal{P}}(\underline{23}) = 0$, $v|_{\mathcal{P}}(1) = 0$, and $v|_{\mathcal{P}}(\underline{123}) = 1$.

Note that $C(v|_{\mathcal{P}}) = \{(x_1, x_{\underline{23}}) \in \mathbb{R}^2 : x_1 + x_{\underline{23}} = 1, x_1 > 0 \text{ and } x_{\underline{23}} > 0\}$.

Then for each $s \in \{1, 2, 3, 4\}$, we have

$$\{x : 0 \leq x \leq 1\} = C_{\underline{23}}(v|_{\mathcal{P}}) >_s \oplus_{i \in \{2,3\}} C_i(v) = \{0\};$$

but by assumption $C_{\underline{23}}(v|_{\mathcal{P}}) \not>_s \oplus_{i \in \{2,3\}} C_i(v)$. Hence, we get a contradiction to the assumption.²⁶ $\square$

Note that the only missing merge proof definitions at Theorem 3.3.7 are 5-merge proofness without externalities, 5-merge proofness, (1, 5)-merge proofness and (2, 5)-merge proofness. Yet, just with a minor assumption, namely equal division of the core of the trivial games, the same example in the proof of Theorem 3.3.7 gives results for those missing definitions as well. In other words, by letting,

$$C^*(v) := \begin{cases} ED(v) & \text{if } v \in G_{trv}, \\ C(v) & \text{otherwise,} \end{cases}$$

one easily sees that the allocation correspondence C^* is not 5-merge proof without externalities, not 5-merge proof, not (1, 5)-merge proof and not (2, 5)-merge proof as well.

Next, we give the generalizations of the above results, namely we extend the above results for *any* allocation correspondence.

Theorem 3.3.8. *No allocation correspondence is non-empty valued for all $v \in G$, anonymous, individually rational and 2-merge proof.*

Proof. Suppose, by contradiction, Γ be an allocation correspondence satisfying the hypothesis of the theorem. Consider the game v with the player set

²⁶Note that the example used in the proof has $|N| = 3$ players, but it can easily be modified to examples with $|N| = n$ players.

$N = \{1, 2, 3\}$, defined by: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, $v(123) = 1$.

For each $i, j, k \in N$, let $\mathcal{P}_i = (i)(j, k)$.

By individually rationality, $\Gamma_{\underline{12}}(v|_{\mathcal{P}_3}) = \Gamma_{\underline{13}}(v|_{\mathcal{P}_2}) = \{1\}$. By 2-merge proofness,

$$\Gamma_{\underline{12}}(v|_{\mathcal{P}_3}) = \{1\} \not>_2 \oplus_{i \in \{1,2\}} \Gamma_i(v) \text{ and,}$$

$$\Gamma_{\underline{13}}(v|_{\mathcal{P}_2}) = \{1\} \not>_2 \oplus_{i \in \{1,3\}} \Gamma_i(v).$$

Thus, $\oplus_{i \in \{1,2\}} \Gamma_i(v) = \{1\}$ and $\oplus_{i \in \{1,3\}} \Gamma_i(v) = \{1\}$ ²⁷. By the fact that $\oplus_{i \in N} \Gamma_i(v) = \{v(N)\}$, we obtain $\Gamma_1(v) = \{1\}$. So, $\Gamma(v) = \{(1, 0, 0)\}$. By anonymity and efficiency, $\{0\} \neq \Gamma_{\underline{23}}(v|_{\mathcal{P}_1}) = \Gamma_1(v|_{\mathcal{P}_1})$, thus,

$$\Gamma_{\underline{23}}(v|_{\mathcal{P}_1}) >_2 \oplus_{i \in \{2,3\}} \Gamma_i(v) = \{0\},$$

which contradicts the assumption that Γ is 2-merge proof. $\square$

Using the facts that 2-merge proofness without externalities implies 2-merge proofness, and (2, 2)-merge proofness implies 2-merge proofness, we have the following corollary of Theorem 3.3.8.

Corollary 3.3.9. *For each $s, m \in \{1, 2\}$, if an allocation correspondence is non-empty valued for all $v \in G$, anonymous and individually rational, then it is not s -merge proof without externalities; it is not s -merge proof; and it is not (s, m) -merge proof.*

Although a game with three players is taken in the proof of Theorem 3.3.8, one can easily find examples that has n players. One is the following example: Let $S_1, S_2 \subset N$ such that $S_1 \cap S_2 \neq \emptyset$, $S_1 \setminus S_2 \neq \emptyset$, $S_2 \setminus S_1 \neq \emptyset$, $|(S_1 \cup S_2) \setminus (S_1 \cap S_2)| > 2$, and let $v(S_1) = v(S_2) = v(N) = 1$, $v(T) = 0$ otherwise.

Also, note that the assumption of non-empty valued of an allocation correspondence in Theorem 3.3.8 can easily be removed. For example, one can replace

²⁷Note that for each $T \subseteq [0, 1], \{1\} \geq_2 T$, thus the results follow. And in fact, this is the main part why the comparison $>_2$ is used.

non-empty valued condition by non-empty on G_c . In fact, it can be removed, by just assuming non-empty valuedness of the allocation correspondence on the appropriate games.

Next, we give that the individually rationality axiom of Theorem 3.3.8 can be replaced by the null player axiom.

Theorem 3.3.10. *No allocation correspondence is non-empty valued for all $v \in G$, anonymous, satisfies null player axiom and 2-merge proofness.*

Proof. Suppose, by contradiction, Γ be an allocation correspondence satisfying the hypothesis of the theorem. Consider the game v with the player set $N = \{1, 2, 3\}$, defined by: $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, and $v(123) = 1$.

For each $i \in N$, let $\mathcal{P}_i := (i)(jk)$.

By the null player axiom, $\Gamma_3(v|_{\mathcal{P}_3}) = \Gamma_2(v|_{\mathcal{P}_2}) = \{0\}$. So,

$$\Gamma_{12}(v|_{\mathcal{P}_3}) = \Gamma_{13}(v|_{\mathcal{P}_2}) = \{1\}.$$

By 2-merge proofness,

$$\{1\} = \Gamma_{12}(v|_{\mathcal{P}_3}) \not>_2 \oplus_{i \in \{1,2\}} \Gamma_i(v) \text{ and,}$$

$$\{1\} = \Gamma_{13}(v|_{\mathcal{P}_2}) \not>_2 \oplus_{i \in \{1,3\}} \Gamma_i(v).$$

Thus, $\oplus_{i \in \{1,2\}} \Gamma_i(v) = \{1\}$ and $\oplus_{i \in \{1,3\}} \Gamma_i(v) = \{1\}$ ²⁸. By the fact that $\oplus_{i \in N} \Gamma_i(v) = \{v(N)\}$, we obtain $\Gamma_1(v) = \{1\}$. So, $\Gamma(v) = \{(1, 0, 0)\}$. By anonymity and efficiency, $\{0\} \neq \Gamma_{23}(v|_{\mathcal{P}_1}) = \Gamma_1(v|_{\mathcal{P}_1})$, thus,

$$\Gamma_{23}(v|_{\mathcal{P}_1}) >_2 \oplus_{i \in \{2,3\}} \Gamma_i(v) = \{0\},$$

which contradicts the assumption that Γ is 2-merge proof. □

²⁸Note that for each $T \subseteq [0, 1], \{1\} \geq_2 T$, thus the results follow. And in fact, this is the main part why the comparison $>_2$ is used as in the case of the previous theorem.

As in the case of Theorem 3.3.8, one can easily find examples that has n players instead of the one with three players for the proof, and the assumption of non-empty valuedness of an allocation correspondence can easily be removed from Theorem 3.3.10.

By the relations between the merge proof definitions, we have the following corollary of Theorem 3.3.10.

Corollary 3.3.11. *For each $s, m \in \{1, 2\}$, if an allocation correspondence is non-empty valued for all $v \in G$, anonymous and satisfies null player axiom, then it is not s -merge proof without externalities; it is not s -merge proof; and it is not (s, m) -merge proof.*

Finally, the result of Theorem 3.3.8 can easily be extended, one of which is given below. We mainly use the following relations: ‘egalitarian on G_{trv} ’ implies ‘anonymity on G_{trv} ’, and ‘stability’ implies ‘individually rationality’.

Proposition 3.3.12. *No allocation correspondence is non-empty valued for all $v \in G$, stable, egalitarian on G_{trv} and 5-merge proof.*

Proof of Proposition 3.3.12 is easy and thus omitted.²⁹ Again, by just assuming non-empty valuedness of the allocation correspondence on the appropriate games, the assumption of non-empty valuedness of an allocation correspondence in Proposition 3.3.12 can easily be removed.

The importance of the proposition is in fact the following corollary. Proof of the corollary follows directly from the facts that $(2, 5)$ -merge proofness implies 5-merge proofness, and 5-merge proofness without externalities implies 5-merge proofness.

Corollary 3.3.13. *For each $m \in \{1, 2\}$ and each $s \in \{1, 2, 3, 4, 5\}$, if an allocation correspondence is non-empty valued for all $v \in G$, stable, egalitarian on G_{trv} then it is not s -merge proof without externalities; it is not s -merge proof; and it is not (m, s) -merge proof.*

²⁹The same steps of the proof of Theorem 3.3.8 can be followed.

Finally, note that as allocation *correspondences* ED and Γ^{Dct} ³⁰ are both $(1, 1)$ -merge proof and 1-merge proof without externalities. Thus, both of them are merge proof in any of the senses defined in this section. In fact, as in Section 3.2.4, convex combinations of these two allocation correspondences can be considered to get some other allocation correspondences that are merge proof in the senses defined in this section. Thus, by modifying these allocation correspondences, one can find allocation correspondences that satisfy not all but some of the conditions in Theorems 3.3.8 and 3.3.10, Proposition 3.3.12, and in Corollary 3.3.9, Corollary 3.3.11 and Corollary 3.3.13.

3.4 Final remarks

We study different merge proof notions both for allocation rules and allocation correspondences. There are some questions that need addressing. What is the set of all allocation rules that are merge proof or strictly-strongly merge proof? Similarly, what is the set of all allocation rules that are strongly merge proof? Our results in this chapter give some partial answers to these questions. Yet, all are still open questions. Similarly, what are the allocation correspondences that are merge proof in different senses. Finally, modifications of the merge proof definitions may lead to similar open questions in other subjects -such as in claims problems or in networks- different than the ones in transferable utility games, where it is appropriate. Thus, as a final point, we want to note that this study is indeed a tip of an iceberg.

³⁰For Γ^{Dct} , we again assume player set N is fixed and $d \in N$.

Chapter 4

Networks: The Myerson value and the position value

4.1 Introduction

Many problems in game theory are modeled as graphs, where nodes are assumed to represent players and edges (or links) between them are assumed to represent relationships between the players. There are a wide range of valuable contributions to network games in the literature. We refer to [49] to see a good picture of the importance of networks in modeling social and economic situations.

One of the main and earliest contributions in this area is due to Myerson [6], where he adapts the cooperative game structures to network structures in which the problems are modeled as graphs. Allocation of a value among players that are connected in a network structure has been widely studied in the literature.¹ Among many others, Myerson's contribution to allocation rules for networks, named as the Myerson value [6], is to adapt the Shapley value [7]. The idea of the Shapley value can be viewed as the following. You take from the players according to their skills/ abilities/ capacities where they have full communication and form coalitions. Thus, you should give back to the players according to their

¹Some remarkable ones are [6, 50, 51, 52, 53, 54].

marginal contributions averaged over their joining orders of coalitions. So, the Shapley value is a fair payoff distribution on marginal contributions of players. In other words, it is the expectation, where expectation is taken with respect to the uniform distribution over the set of all orders of players. In network games, players do not have full communication. They communicate through the links that they form between themselves. So, the structure of the network plays an important role. Taking into account this difference between the network games and the TU games, Myerson defines an allocation rule whose formulation is similar to the Shapley value. This Shapley-value like allocation rule for network games is based on how a network structure is built up, which is known as the Myerson value. He shows that the Myerson value is characterized by the conjunction of component balancedness and equal bargaining power under the condition that the value function is component additive. Other characterizations of the Myerson value can be found in [9, 50, 52, 55, 56].

In this chapter, we study a new characterization of the Myerson value. We do not restrict ourselves to component additive value functions, thus, the value function can be any function. The key axiom for our characterization is that, given a value function, if there is an epsilon increase (or decrease) of the value function at a network, say at g , and at each network containing g , then that epsilon increase (or decrease) should be divided equally between all players in g who are not isolated. In other words, in a situation where there is an increase (or decrease) of the value function at g and at each supergraph of g ², the network g is the source of the increase (or decrease), thus, this increase (or decrease) should be divided equally *between all the players in g that has at least one link at g* . The second axiom we use for our characterization is a condition only on the value function where the value of each network is zero. It requires that if the value function is zero at any of the networks, then each player should be treated similarly, and each player gets zero payoff at each network. We call this condition as the null-game property. We show that the Myerson value is characterized by these two axioms.

In a similar manner, we study a characterization of the position value. The

²A supergraph of a graph g is a graph that contains g as a subgraph.

position value is an allocation rule that is introduced by Meessen [8] and popularized by [9]. Like the Myerson value, the position value also takes into account the structure of the network and is also a Shapley-value like allocation rule defined for network games. In that, first the links of a network are taken as if they are the players, and the value is distributed according to the Shapley value to these links. Then, the main players -who are the edges of the links- split the payoff of the links equally so that the total payoff of each player is found.

At the characterization of the position value, we again do not restrict ourselves to component additive value functions, thus, the value function can be any function. The key axiom for our characterization is that, given a value function, if there is an epsilon increase (or decrease) of the value function at a network, say at g , and at each network containing g , then that epsilon increase (or decrease) should be divided equally between the links of g . Then, the (decided) payoff of each link is distributed equally between the two edges of the link. In other words, in a situation where there is an increase (or decrease) of the value function at g and at each supergraph of g , the network g is the source of the increase (or decrease), thus, this increase (or decrease) should be divided equally *between the links of g* , and then, it is divided equally between the players who own the links. We show that the position value is characterized by this axiom and the null-game property.

Note that, our key axiom for the position value is similar to our key axiom for the Myerson value, and the difference between these two axioms captures the main difference between the Myerson value and the position value.

From plenty of characterizations of the Shapley value, Chun's characterization of the Shapley value [57] is worth mentioning in this chapter, because our characterizations of the Myerson value and the position value are similar to Chun's characterization of the Shapley value.³ Chun characterizes the Shapley value by triviality, coalitional strategic equivalence (CSE) and fair ranking (FR). In the setting of transferable utility games, triviality is the same as our null-game

³This characterization is studied independent of Chun's work, because the author was unaware of the characterization of Chun.

property, i.e., if the worth of each coalition is zero, then each player gets zero payoff. Coalition strategic equivalence requires that if there is an epsilon increase (or decrease) of the value function at a coalition, say at S , and at each coalition containing S , then that epsilon increase (or decrease) is caused by the players in S , thus, the payoff of the players that are outside of S should not be changed. So, (CSE) does not say anything how the epsilon increase (or decrease) is divided, but it says that the epsilon increase (or decrease) does not effect the players that are not in S . Fair ranking requires that given any two value functions, if these two value functions are the same except at a given coalition, say at T , then the relative payoff of the players in T should not be affected by the change in the value of the coalition T . Thus, the rankings of the players should be preserved. In his proof, as a step, Chun shows that (CSE) and (FR) together imply the following property:

“If there is an epsilon increase (or decrease) of the value function at a coalition, say at S , and at each coalition containing S , then that epsilon increase (or decrease) is divided equally between the players in S .”

Chun does not mention in his paper, but we want to note that the above property also implies (CSE) and (FR).⁴ Thus, although the above property looks like an intermediate step in Chun’s characterization, it is not, because the property given above in italic is the same as (CSE)+(FR). Now that one can see the similarities of our key axioms and Chun’s axioms.

Finally, although the two axioms that we define are enough for a characterization of the Myerson value, to compare our characterization with Shapley’s original characterization [7], we add the linearity axiom and provide an alternative characterization. That allows us to modify the axioms and study a characterization of the Shapley value. By this way, we compare the Myerson value and the Shapley value.

This chapter is organized as follows. Section 4.2 contains basic notations and definitions for networks. Our characterization of the Myerson value is given in Section 4.3. Our characterization of the position value is given in Section 4.4.

⁴It is trivial that it implies (CSE) and, by a recursive argument, one can show that it also implies (FR). See Section 4.5 for details.

Chun's characterization of the Shapley value is studied in Section 4.5. Finally, our comparison of the Shapley value and the Myerson value is given in Section 4.6.

4.2 Preliminaries for networks

Let $n \geq 2$. Let $N = \{1, \dots, n\}$ be a finite set of players who are connected in some network relationship. Throughout the chapter, we take N fixed.

Let g_K stand for the complete (undirected, loop free) graph with vertex set $N = \{1, \dots, n\}$. Note that g_K has n vertices, $\binom{n}{2} = \frac{n(n-1)}{2}$ (undirected) edges, and $2^{\frac{n(n-1)}{2}}$ subgraphs. Any subgraph of g_K will be referred as a *network* (with the player set N), and g_K will be referred as the complete network. Let g_0 stand for the network that has no edges. Let $\mathcal{G}$ denote the set of all networks with player set N . Note that, $|\mathcal{G}| = 2^{\frac{n(n-1)}{2}}$.

The vertices of a network g correspond to the players and the edges between the players correspond to bilateral relationship between the players. For any $i, j \in N$, we write $l = ij$ for the edge between the players i and j . Thus, for any $g \in \mathcal{G}$, $ij \in g$ indicates that there is an edge between the players i and j at g , and in that case, we say that there is a link between the players i and j at g .

The network obtained by adding a link l to an existing network $g \in \mathcal{G}$ is denoted by $g + l$, and similarly the network obtained by deleting a link l from an existing network $g \in \mathcal{G}$ is denoted by $g - l$.

Let $D_i(g) = \{l : \exists j \in N \text{ s.t. } l = ij \in g\}$, i.e., $D_i(g)$ denotes the set of links that player i is involved in g , and let $d_i(g)$ be the cardinality of $D_i(g)$. Let the number of links in g be denoted by $d(g)$, i.e., $d(g) = \frac{1}{2} \sum_{i \in N} d_i(g)$.

For each $g \in \mathcal{G}$, let $N(g) = \{i \in N : \exists j \text{ s.t. } ij \in g\}$, i.e., $N(g)$ is the set of players who has at least one link in g .

We write $g \subseteq \bar{g}$, if $[ij \in g \Rightarrow ij \in \bar{g}]$. Also, we write $g \subset \bar{g}$, if $[ij \in g \Rightarrow ij \in \bar{g}]$ and $[\exists ij \in \bar{g} \text{ s.t. } ij \notin g]$.

Note, $g \subseteq \bar{g} \Rightarrow N(g) \subseteq N(\bar{g})$.⁵ For $g \in \mathcal{G}$, let $E(g) = N \setminus N(g)$.

A function $v : \mathcal{G} \rightarrow \mathbb{R}$ where $v(g_0) = 0$ is called a *value function* for networks. Let $\mathcal{V}$ denote the set of all value functions for $\mathcal{G}$.

Throughout the chapter, let $v_0 \in \mathcal{V}$ denote the value function that assigns zero to each network in $\mathcal{G}$, i.e., for each $g \in \mathcal{G}$, $v_0 \in \mathcal{V}$ is defined as $v_0(g) = 0$.

For each $g \in \mathcal{G} \setminus \{g_0\}$, let v_g denote the value function that satisfies

$$v_g(\bar{g}) = \begin{cases} 1 & \text{if } g \subseteq \bar{g}, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that $\mathfrak{B} = \{v_g : g \in \mathcal{G} \setminus \{g_0\}\}$ forms a *basis* for $\mathcal{V}$.

Given $g \in \mathcal{G}$, for any $T \subseteq N$, let $g|_T \in \mathcal{G}$ denote the network such that for each pair $i, j \in N$, $ij \in g|_T$ whenever $i, j \in T$ and $ij \in g$. In other words, $g|_T = \{ij : i, j \in T \text{ and } ij \in g\}$.

For each $g \in \mathcal{G}$ and each $S \subseteq E(g)$, $g|_{N(g) \cup S} = g$ and $g|_\emptyset = g_0$. Also, $g|_T = g_0$ whenever $|T| = 1$. In general, for each $T \subseteq N$, $g|_T = g|_{T \cap N(g)}$.

Let $0 \neq \epsilon \in \mathbb{R}$. Given $g \in \mathcal{G}$ and a value function $v \in \mathcal{V}$, let $v(g)$ increase (or decrease) by ϵ units due to links between the players at g . In other words, the value of g becomes $v(g) + \epsilon$, only because of the links at g . We will assume that not only value of g increases by an epsilon units, but also each $\bar{g} \in \mathcal{G}$ with $\bar{g} \supseteq g$ increases by an epsilon unit. In such a case, a common re-distribution of the value between the players is to distribute the extra ϵ units between the players that are linked at g . To express this common behavior, given v , we first define a new value function, denoted by $v_{(g, \epsilon)}$, that depends on v , ϵ and g as below.

Definition 4.2.1. For each $g_0 \neq g \in \mathcal{G}$, each $v \in \mathcal{V}$ and each $0 \neq \epsilon \in \mathbb{R}$, we define the value function $v_{(g, \epsilon)}$ as follows:

$$v_{(g, \epsilon)}(\bar{g}) = \begin{cases} v(\bar{g}) + \epsilon & \text{if } g \subseteq \bar{g}, \\ v(\bar{g}) & \text{otherwise,} \end{cases}$$

⁵Note, $N(g) \subseteq N(\bar{g}) \not\Rightarrow g \subseteq \bar{g}$. For example, consider $N = \{1, 2, 3\}$ and $12, 13 \in \bar{g}$, but $23 \notin \bar{g}$ and $12, 13 \notin g$, $23 \in g$.

In other words, $v_{(g,\epsilon)} = v + \epsilon v_g$ where $v_g \in \mathfrak{B}$.

By taking ϵ as 1, v as v_0 at Definition 4.2.1, for each $g_0 \neq g \in \mathcal{G}$, we have $(v_0)_{(g,1)} \equiv v_g$. Hence,

$$\{(v_0)_{(g,1)} : g \in G^N \setminus \{g_0\}\} = \mathfrak{B}.$$

A rule distributing the value of a network between the players is called an allocation rule.

Definition 4.2.2. An allocation rule is a function $Y : \mathcal{G} \times \mathcal{V} \rightarrow \mathbb{R}^N$ that assigns a payoff vector $Y(g, v)$ to each $(g, v) \in \mathcal{G} \times \mathcal{V}$, such that $\sum_{i \in N} Y_i(g, v) = v(g)$. The number $Y_i(g, v)$ represents the *payoff* of player i at (g, v) .

In [52] an extension of the Myerson value to network games is provided, which we use as the definition of the Myerson value here. The Myerson value [6, 52], Y^{MV} , is an allocation rule, defined as follows:

$$Y_i^{MV}(g, v) = \sum_{S \subseteq N \setminus \{i\}} (v(g|_{S \cup \{i\}}) - v(g|_S)) \frac{|S|!(n - |S| - 1)!}{n!}.$$

As an extension of a theorem of Myerson [6], in [52] the following is shown.

Theorem [6]. An allocation rule satisfying component balancedness and equal bargaining power if and only if the allocation rule is equal to the Myerson value for each $g \in \mathcal{G}$ and each component additive $v \in \mathcal{V}$.

The *components of a network* are the distinct connected subgraphs of a network. By denoting $C(g)$ as the set of all components of g , note $g = \bigcup_{\bar{g} \in C(g)} \bar{g}$. *Component additive* requires that for each $g \in \mathcal{G}$, $v(g) = \sum_{\bar{g} \in C(g)} v(\bar{g})$.

An allocation rule Y is *component balanced* if for each $g \in \mathcal{G}$, each $\bar{g} \in C(g)$ and each component additive $v \in \mathcal{V}$, $\sum_{i \in N(\bar{g})} Y_i(g, v) = v(\bar{g})$. An allocation rule satisfies *equal bargaining power* (or *fairness*) if for each $i, j \in N$, each $g \in \mathcal{G}$ and each component additive $v \in \mathcal{V}$, $Y_i(g, v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v)$.

Note that, at the above characterization there is a restriction on the value function, namely it must be component additive.

Meessen [8] considers another allocation rule for network games, namely the position value. The position value [8], Y^{PV} , is an allocation rule, defined as follows:

$$Y_i^{PV}(g, v) = \sum_{l \in D_i(g)} \frac{1}{2} \sum_{\bar{g} \subseteq g-l} (v(\bar{g} + l) - v(\bar{g})) \frac{d(\bar{g})!(d(g) - d(\bar{g}) - 1)!}{d(g)!}.$$

A characterization of the position value that is similar to Shapley's characterization and that has no condition on the underlying network has been provided recently by [58]. Alternative characterizations of the position value can be found in [56, 59, 60].

Next, we give the key property for our characterization of the Myerson value.

Definition 4.2.3. Given $g \in \mathcal{G} \setminus \{g_0\}$, $v \in \mathcal{V}$ and $0 \neq \epsilon \in \mathbb{R}$, we say that an allocation rule Y satisfies *condition A* (*Equal division between the source players at a monotonic increment of the value function*) if the following conditions hold:

(a) for each $\bar{g} \in \mathcal{G}$ such that $\bar{g} \supseteq g$,

$$Y_i(\bar{g}, v_{(g, \epsilon)}) = \begin{cases} Y_i(\bar{g}, v) + \frac{\epsilon}{|N(g)|} & \text{if } i \in N(g), \\ Y_i(\bar{g}, v) & \text{otherwise,} \end{cases}$$

(b) for each $\bar{g} \in \mathcal{G}$ such that $\bar{g} \not\supseteq g$,

$$Y_i(\bar{g}, v_{(g, \epsilon)}) = Y_i(\bar{g}, v),$$

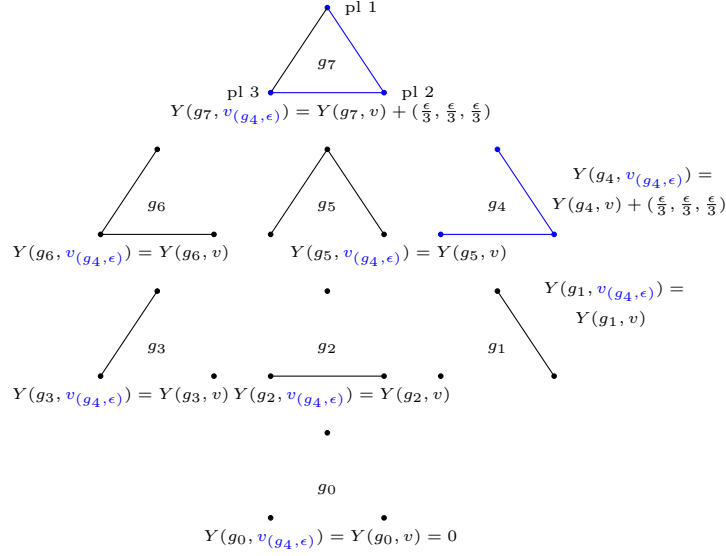
for each $i \in N$.

Let us briefly discuss condition A. According to condition A, given a value function v , when the value function changes epsilon units at a network $g \neq g_0$ and at each network containing g , then the allocation rule distributes $v(g)$ exactly the same as it was distributing before the epsilon change at each network; and at each network where there is a change, it should distribute the epsilon units equally between all the players that are in the source network g who are not isolated.⁶ In the figure below, we provide an example of an allocation rule that

⁶In other words, between the players of $N(g)$.

satisfies condition A, where an epsilon increase occurred at g_4 and at only at g_7 , since g_7 is the only network containing g_4 .

Figure 4.1: Condition A



In a similar manner, we give the key property for our characterization of the position value.

Definition 4.2.4. Given $g \in \mathcal{G} \setminus \{g_0\}$, $v \in \mathcal{V}$ and $0 \neq \epsilon \in \mathbb{R}$, we say that an allocation rule Y satisfies *condition $\hat{A}$* (Equal division between the source links at a monotonic increment of the value function) if the following conditions hold:

- (a) for each $\bar{g} \in \mathcal{G}$ such that $\bar{g} \supseteq g$,

$$Y_i(\bar{g}, v_{(g, \epsilon)}) = \begin{cases} Y_i(\bar{g}, v) + \frac{\epsilon d_i(g)}{2d(g)} & \text{if } i \in N(g), \\ Y_i(\bar{g}, v) & \text{otherwise.} \end{cases}$$

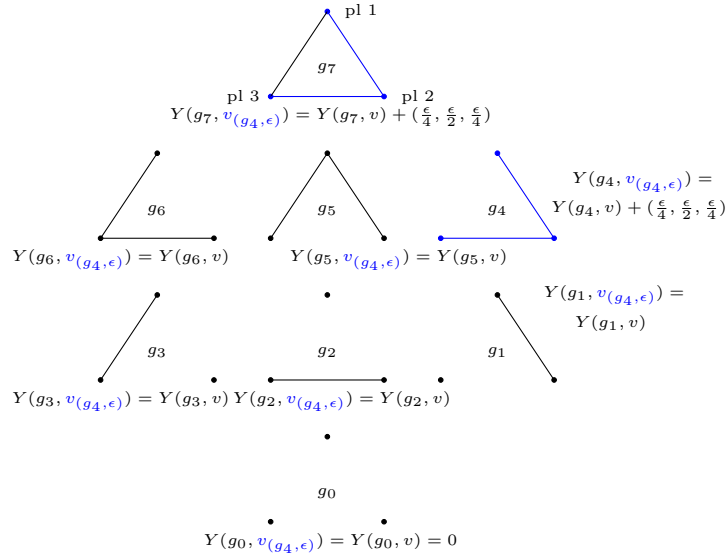
- (b) for each $\bar{g} \in \mathcal{G}$ such that $\bar{g} \not\supseteq g$,

$$Y_i(\bar{g}, v_{(g, \epsilon)}) = Y_i(\bar{g}, v),$$

for each $i \in N$.

According to condition $\hat{A}$, given a value function v , when the value function changes epsilon units at a network $g \neq g_0$ and at each network containing g , then the allocation rule distributes $v(g)$ exactly the same as it was distributing before the epsilon change at each network; and at each network where there is a change, it should distribute the epsilon units equally between the links of g , and then to the players of g . In the figure below, we provide an example of an allocation rule that satisfies condition $\hat{A}$, where an epsilon increase occurred at g_4 and at only at g_7 as in the previous figure.

Figure 4.2: Condition $\hat{A}$



Next, we give a simple property.

Definition 4.2.5. We say that an allocation rule Y satisfies *null-game property* if for each $g \in \mathcal{G}$ and each $i \in N$, $Y_i(g, v_0) = 0$.

The null-game property can be interpreted as follows. If addition of any link between any of the players does not effect the value function, then each player should be treated similarly, and each gets zero payoff at each network.

4.3 A characterization of the Myerson value

First, we show that Myerson value satisfies condition A. For that, we first show the following lemma.

Lemma 4.3.1. *For any $n, m \in \mathbb{Z}$, $n \geq 2$ and $0 \leq m \leq n - 2$,*

$$\sum_{k=0}^m \binom{n-m+k}{k} \frac{1}{n-m+k} = \binom{n}{n-m} \frac{1}{n-m}, \quad (4.3.1)$$

where $\binom{n-m+k}{k} = C(n-m+k, k) = \frac{(n-m+k)!}{k!(n-m)!}$, and similarly $\binom{n}{n-m} = C(n, n-m)$.

Proof. We prove by induction on n (and m). Note that for $n = 2$, m has to be zero, and one can easily check that Equation (4.3.1) is true for $n = 2$ and $m = 0$.

Let $l \in \mathbb{Z}$ such that $l > 2$. Let Equation (4.3.1) is true for any n , $2 \leq n \leq l$ and any m , where $0 \leq m \leq n - 2$.

Next, we prove Equation (4.3.1) for $n = l + 1$ and any $0 \leq m' \leq l - 1$. For $n = l + 1$ and $m = 0$, it is easy to check Equation (4.3.1) is true. Thus, we prove Equation (4.3.1) for $n = l + 1$ and any $1 \leq m' \leq l - 1$.

$$\sum_{k=0}^{m'} \binom{l+1-m'+k}{k} \frac{1}{l+1-m'+k} = \sum_{k=0}^{m'-1} \binom{l-(m'-1)+k}{k} \frac{1}{l-(m'-1)+k} + \binom{l+1}{m'} \frac{1}{l+1}.$$

Let $M = m' - 1$. Note $1 \leq m' \leq l - 1 \Leftrightarrow 0 \leq M \leq l - 2$. Now, by applying induction assumption on the last equation,

$$\begin{aligned} \sum_{k=0}^{m'} \binom{l+1-m'+k}{k} \frac{1}{l+1-m'+k} &= \binom{l}{l-M} \frac{1}{l-M} + \binom{l+1}{M+1} \frac{1}{l+1} \\ &= \binom{l}{l-M} \left[\frac{1}{l-M} + \frac{1}{M+1} \right] \\ &= \binom{l}{l-M} \frac{l+1}{(l-M)(M+1)} \\ &= \left[\frac{(l+1)!}{(l-M)!(M+1)!} \right] \frac{1}{l-M} \\ &= \binom{l+1}{(l+1)-m'} \frac{1}{(l+1)-m'}. \end{aligned}$$

Thus, by induction, Equation (4.3.1) is true for any $n, m \in \mathbb{Z}$, $n \geq 2$ and $0 \leq m \leq n - 2$. $\square$

Proposition 4.3.2. *The Myerson value Y^{MV} satisfies condition A.*

Proof. Consider any $g \in \mathcal{G} \setminus \{g_0\}$, $v \in \mathcal{V}$ and $0 \neq \epsilon \in \mathbb{R}$.

Case 1) Let $\bar{g} \in \mathcal{G}$ be such that $\bar{g} \supseteq g$. We show that

$$Y_i^{MV}(\bar{g}, v_{(g,\epsilon)}) = \begin{cases} Y_i^{MV}(\bar{g}, v) + \frac{\epsilon}{|N(g)|} & \text{if } i \in N(g), \\ Y_i^{MV}(\bar{g}, v) & \text{otherwise.} \end{cases} \quad (4.3.2)$$

For each $i \in N(g)$ and each $S \subseteq N \setminus \{i\}$, we want to note the following two cases:

1. if $S \supseteq N(g) \setminus \{i\}$, then $\bar{g}|_{S \cup \{i\}} \supseteq \bar{g}|_{N(g)} \supseteq g$ and $\bar{g}|_S \not\supseteq g$.⁷

Thus, $v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) = v(\bar{g}|_{S \cup \{i\}}) + \epsilon$ and $v_{(g,\epsilon)}(\bar{g}|_S) = v(\bar{g}|_S)$.

2. if $S \not\supseteq N(g) \setminus \{i\}$, then $\bar{g}|_{S \cup \{i\}} \not\supseteq g$ and $\bar{g}|_S \not\supseteq g$.⁸

Thus, $v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) = v(\bar{g}|_{S \cup \{i\}})$ and $v_{(g,\epsilon)}(\bar{g}|_S) = v(\bar{g}|_S)$.

Hence, for each $i \in N(g)$,

$$\begin{aligned} Y_i^{MV}(\bar{g}, v_{(g,\epsilon)}) &= \sum_{S \subseteq N \setminus \{i\}} (v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) - v_{(g,\epsilon)}(\bar{g}|_S)) \frac{|S|!(n - |S| - 1)!}{n!} \\ &= \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \supseteq (N(g) \setminus \{i\})}} \underbrace{(v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) - v_{(g,\epsilon)}(\bar{g}|_S))}_{(v(\bar{g}|_{S \cup \{i\}}) + \epsilon) - v(\bar{g}|_S)} \frac{|S|!(n - |S| - 1)!}{n!} \\ &\quad + \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \not\supseteq (N(g) \setminus \{i\})}} \underbrace{(v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) - v_{(g,\epsilon)}(\bar{g}|_S))}_{v(\bar{g}|_{S \cup \{i\}}) - v(\bar{g}|_S)} \frac{|S|!(n - |S| - 1)!}{n!} \\ &= \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \supseteq (N(g) \setminus \{i\})}} \epsilon \left(\frac{|S|!(n - |S| - 1)!}{n!} \right) + Y_i^{MV}(\bar{g}, v). \end{aligned}$$

By the equation above, it is enough to show that

$$\sum_{S \subseteq N \setminus \{i\}, S \supseteq (N(g) \setminus \{i\})} \left(\frac{|S|!(n - |S| - 1)!}{n!} \right) = \frac{1}{|N(g)|}. \quad (4.3.3)$$

⁷Otherwise, if $\bar{g}|_S \supseteq g$ then $S \supseteq N(g)$; contradicting to $i \notin S$ and $i \in N(g)$.

⁸Otherwise, if $\bar{g}|_{S \cup \{i\}} \supseteq g$, then $S \cup \{i\} \supseteq N(g)$; contradicting to $S \cup \{i\} \not\supseteq N(g)$. Similarly for $\bar{g}|_S \not\supseteq g$.

Let $t = |N(g)|$. Note,

$$\begin{aligned} \sum_{S \subseteq N \setminus \{i\}, S \supseteq (N(g) \setminus \{i\})} \left(\frac{|S|!(n - |S| - 1)!}{n!} \right) &= \sum_{k=0}^{n-t} \frac{(t+k-1)!(n-t-k)!}{n!} \binom{n-t}{k} \\ &= \sum_{k=0}^{n-t} \frac{(t+k-1)!(n-t)!}{n!k!} \\ &= \frac{(n-t)!t!}{n!} \sum_{k=0}^{n-t} \binom{t+k}{k} \frac{1}{t+k}. \end{aligned}$$

Now, since $2 \leq t \leq n$, by taking $m = n - t$ at Lemma 4.3.1, we have $\sum_{k=0}^{n-t} \binom{t+k}{k} \frac{1}{t+k} = \binom{n}{t} \frac{1}{t}$. Hence, we have

$$\sum_{\substack{S \subseteq N \setminus \{i\} \\ S \supseteq (N(g) \setminus \{i\})}} \left(\frac{|S|!(n - |S| - 1)!}{n!} \right) = \frac{(n-t)!t!}{n!} \sum_{k=0}^{n-t} \binom{t+k}{k} \frac{1}{t+k} = \frac{1}{t} = \frac{1}{|N(g)|}.$$

Thus, Equation (4.3.3) is satisfied. So, for $i \in N(g)$, we have $Y_i^{MV}(\bar{g}, v_{(g,\epsilon)}) = Y_i^{MV}(\bar{g}, v) + \frac{\epsilon}{|N(g)|}$.

Now, for each $i \in N \setminus N(g)$ and for each $S \subseteq N \setminus \{i\}$, we note the following two cases:

1. $S \supseteq N(g) \Rightarrow \bar{g}|_{S \cup \{i\}} \supseteq \bar{g}|_S \supseteq \bar{g}|_{N(g)} \supseteq g \Rightarrow v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) = v(\bar{g}|_{S \cup \{i\}}) + \epsilon$,
and $v_{(g,\epsilon)}(\bar{g}|_S) = v(\bar{g}|_S) + \epsilon$,
2. $S \not\supseteq N(g) \Rightarrow \bar{g}|_{S \cup \{i\}} \supseteq \bar{g}|_S \not\supseteq g \Rightarrow v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) = v(\bar{g}|_{S \cup \{i\}})$,
and $v_{(g,\epsilon)}(\bar{g}|_S) = v(\bar{g}|_S)$.

Hence, for each $i \in N \setminus N(g)$,

$$\begin{aligned} Y_i^{MV}(\bar{g}, v_{(g,\epsilon)}) &= \sum_{S \subseteq N \setminus \{i\}} (v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) - v_{(g,\epsilon)}(\bar{g}|_S)) \frac{|S|!(n - |S| - 1)!}{n!} \\ &= \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \supseteq N(g)}} \underbrace{(v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) - v_{(g,\epsilon)}(\bar{g}|_S))}_{(v(\bar{g}|_{S \cup \{i\}}) + \epsilon) - (v(\bar{g}|_S) + \epsilon)} \frac{|S|!(n - |S| - 1)!}{n!} \\ &\quad + \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \not\supseteq N(g)}} \underbrace{(v_{(g,\epsilon)}(\bar{g}|_{S \cup \{i\}}) - v_{(g,\epsilon)}(\bar{g}|_S))}_{v(\bar{g}|_{S \cup \{i\}}) - v(\bar{g}|_S)} \frac{|S|!(n - |S| - 1)!}{n!}. \end{aligned}$$

$$\begin{aligned}
Y_i^{MV}(\bar{g}, v_{(g, \epsilon)}) &= \sum_{S \subseteq N \setminus \{i\}} (v(\bar{g}|_{S \cup \{i\}}) - v(\bar{g}|_S)) \frac{|S|!(n - |S| - 1)!}{n!} \\
&= Y_i^{MV}(\bar{g}, v).
\end{aligned}$$

Hence, Equation (4.3.2) is satisfied, i.e., if $\bar{g} \supseteq g$, then

$$Y_i^{MV}(\bar{g}, v_{(g, \epsilon)}) = \begin{cases} Y_i^{MV}(\bar{g}, v) + \frac{\epsilon}{|N(g)|} & \text{if } i \in N(g), \\ Y_i^{MV}(\bar{g}, v) & \text{otherwise.} \end{cases}$$

Case 2) Let $\bar{g} \in \mathcal{G}$ be such that $\bar{g} \not\supseteq g$. Then, for each $S \subseteq N$, $\bar{g}|_S \not\supseteq g|_S$.⁹ Thus, for each $S \subseteq N$, $v_{(g, \epsilon)}(\bar{g}|_S) = v(\bar{g}|_S)$. Hence, by definition of the Myerson value, for each $i \in N$,

$$Y_i^{MV}(\bar{g}, v_{(g, \epsilon)}) = Y_i^{MV}(\bar{g}, v).$$

Hence, Y^{MV} satisfies condition A. □

Here, we want to refer some of the criticisms of Jackson [51] to the Myerson value. Jackson criticizes that the Myerson value does not use the whole information of the value function. In other words, he criticizes that the values of all the networks are not taken into account while calculating the Myerson value. The above result contributes to explanations of his criticisms. Cause, as seen above, if an allocation rule satisfies condition A, then it uses the information of the value function up to some level. Moreover, if the considered source network is a star - i.e., if the only links of the considered source network are between a central player and all other players-, then according to Myerson value, the change is distributed equally between all the players in that star. Thus, being a central player is not important in that case.

Theorem 4.3.3. *There is a unique allocation rule that satisfies condition A and the null-game property, namely Y^{MV} .*

Proof. It is easy to check that Y^{MV} satisfies the null-game property. By Proposition 4.3.2, Y^{MV} satisfies condition A.

⁹Otherwise, if $\bar{g}|_S \supseteq g|_S$, then $\bar{g} \supseteq \bar{g}|_S \supseteq g$; contradicting to $\bar{g} \not\supseteq g$.

Conversely, assume Y is an allocation rule that satisfies condition A and the null-game property. Consider any $g \in \mathcal{G}$, any $v \in \mathcal{V}$ and any labeling $g_1, g_2, \dots, g_{2^{n(n-1)/2}-1}$ of the elements in $\mathcal{G} \setminus \{g_0\}$. Since $\mathfrak{B}$ is a basis for $\mathcal{V}$, $v = \sum_{i=1}^{2^{n(n-1)/2}-1} c_i(v) v_{g_i}$ where the coefficients $c_i(v)$ are unique.

Define the sequence $v^0, v^1, \dots, v^{2^{n(n-1)/2}-1}$ of $2^{\frac{n(n-1)}{2}}$ value functions recursively as follows:

$$v^0 = v_0, v^i = v^{i-1} + c_i(v) v_{g_i} = (v^{i-1})_{(g_i, c_i(v))} \text{ for each } i \in \{1, \dots, 2^{n(n-1)/2}\}.$$

By the null-game property, $Y(g, v^0)$ is unique. Then, for each $i \in \{1, \dots, 2^{n(n-1)/2}\}$, condition A uniquely determines $Y(g, v^i)$ from $Y(g, v^{i-1})$ recursively. Note also that $v^{2^{n(n-1)/2}} = v$, so $Y(g, v)$ is uniquely determined.

Thus, for each $(g, v) \in \mathcal{G} \times \mathcal{V}$, $Y(g, v)$ is uniquely determined. Since Y^{MV} satisfies condition A and null-game property, $Y \equiv Y^{MV}$. $\square$

Condition A and the null-game property are independent of each other.

Remark 4.3.4. 1. There exists an allocation rule that satisfies condition A, but it does not satisfy the null-game property. Thus, condition A does not imply the null-game property.

Without loss in generality, let $N = \{1, 2, 3\}$. Define Y as follows:

- $v = v_0$: For each $\bar{g} \in \mathcal{G}$, $Y(\bar{g}, v_0) = (1, -1, 0)$.
- $v = (v_0)_{(g, \epsilon)}$: For each $g_0 \neq g \in \mathcal{G}$, each $\bar{g} \supseteq g$ and each $0 \neq \epsilon \in \mathbb{R}$,

$$Y_i(\bar{g}, (v_0)_{(g, \epsilon)}) = \begin{cases} Y_i(\bar{g}, v_0) + \frac{\epsilon}{|N(g)|} & \text{if } i \in N(g), \\ Y_i(\bar{g}, v_0) & \text{otherwise .} \end{cases}$$

For each $g_0 \neq g \in \mathcal{G}$, each $\bar{g} \not\supseteq g$ and each $0 \neq \epsilon \in \mathbb{R}$,

$$Y(\bar{g}, (v_0)_{(g, \epsilon)}) = (1, -1, 0).$$

- For all other cases, let $Y = Y^{MV}$.

It is easy to check that Y satisfies condition A, but it does not satisfy the null-game property.

2. There exists an allocation rule that satisfies the null-game property, but it does not satisfy condition A. Thus, null-game property does not imply condition A.

Without loss in generality, let $N = \{1, 2, 3\}$. For each $v \in \mathcal{V}$ and each $g \in \mathcal{G}$, define Y as follows:

$$Y_i(g, v) = \begin{cases} v(g) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that Y satisfies the null-game property. Next, we show it does not satisfy condition A. First define the value function $\tilde{v}$ as follows: for each $g \in \mathcal{G} \setminus \{g_0\}$, $\tilde{v}(g) = 2$.

Consider the complete graph g_K , $\tilde{v}$ and $\epsilon = 3$. Note that by definition, $Y(g_K, \tilde{v}_{(g_K, \epsilon)}) = (5, 0, 0)$.

Assume Y satisfies condition A, then for each $i \in N$, $Y_i(g_K, \tilde{v}_{(g_K, \epsilon)}) = Y_i(g_K, \tilde{v}) + 1$, which contradicts to the fact that $Y(g_K, \tilde{v}_{(g_K, \epsilon)}) = (5, 0, 0)$. In other words, $Y(g_K, \tilde{v}_{(g_K, \epsilon)}) = (2, 0, 0) + (1, 1, 1) = (3, 1, 1) \neq (5, 0, 0)$. Hence, Y does not satisfy condition A.

4.4 A characterization of the position value

First, we show that the position value satisfies condition $\hat{A}$.

Proposition 4.4.1. *The position value Y^{PV} satisfies condition $\hat{A}$.*

Proof. Consider $g \in \mathcal{G} \setminus \{g_0\}$, $v \in \mathcal{V}$ and $0 \neq \epsilon \in \mathbb{R}$.

Case 1) Let $\bar{g} \in \mathcal{G}$ be such that $\bar{g} \supseteq g$. We show that

$$Y_i^{PV}(\bar{g}, v_{(g, \epsilon)}) = \begin{cases} Y_i^{PV}(\bar{g}, v) + \frac{\epsilon d_i(g)}{2d(g)} & \text{if } i \in N(g), \\ Y_i^{PV}(\bar{g}, v) & \text{otherwise.} \end{cases} \quad (4.4.1)$$

For each $i \in N(g)$,

$$\begin{aligned}
Y_i^{PV}(\bar{g}, v_{(g,\epsilon)}) &= \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\tilde{g} \subseteq \bar{g}-l} (v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g})) \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \subseteq \tilde{g} \subseteq \bar{g}+l}} \underbrace{(v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g}))}_{v(\tilde{g}+l)+\epsilon-(v(\tilde{g})+\epsilon)} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&\quad + \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}, g \not\subseteq \bar{g}+l}} \underbrace{(v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g}))}_{v(\tilde{g}+l)-v(\tilde{g})} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&\quad + \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}, g \subseteq \bar{g}+l}} \underbrace{(v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g}))}_{v(\tilde{g}+l)+\epsilon-v(\tilde{g})} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\tilde{g} \subseteq \bar{g}-l} (v(\tilde{g}+l) - v(\tilde{g})) \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&\quad + \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}, g \subseteq \bar{g}+l}} \epsilon \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!}.
\end{aligned}$$

From the last equality, we have

$$Y_i^{PV}(\bar{g}, v_{(g,\epsilon)}) = Y_i^{PV}(\bar{g}, v) + \sum_{l \in D_i(\bar{g})} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}, g \subseteq \bar{g}+l}} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!}.$$

By the equation above, it is enough to show that

$$\sum_{l \in D_i(\bar{g})} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}, g \subseteq \bar{g}+l}} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} = \frac{\epsilon d_i(g)}{2d(g)}. \quad (4.4.2)$$

Let $d(\bar{g}) = \bar{n}$. Consider the case where $t = d(g) = 1$. Then, $|N(g)| = 2$, and

for each $i \in N(g)$, $d_i(g) = 1$. Thus, $\frac{\epsilon d_i(g)}{2d(g)} = \frac{\epsilon}{2}$. Note,

$$\begin{aligned}
& \sum_{l \in D_i(\bar{g})} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g} - l \\ g \not\subseteq \tilde{g}, g \subseteq \bar{g} + l}} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g} - l \\ D(\bar{g}) \supseteq D(g) - \{l\} = \emptyset}} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&= \frac{\epsilon}{2} \sum_{k=0}^{\bar{n}-1} \frac{k!(\bar{n} - k - 1)!}{\bar{n}!} \binom{\bar{n} - 1}{k} \\
&= \frac{\epsilon}{2} \sum_{k=0}^{\bar{n}-1} \frac{k!(\bar{n} - k - 1)!}{\bar{n}!} \frac{(\bar{n} - 1)!}{k!(\bar{n} - k - 1)!} \\
&= \frac{\epsilon}{2} \sum_{k=0}^{\bar{n}-1} \frac{1}{\bar{n}} = \frac{\epsilon}{2\bar{n}} \sum_{k=0}^{\bar{n}-1} 1 = \frac{\epsilon\bar{n}}{2\bar{n}} = \frac{\epsilon}{2}.
\end{aligned}$$

Now, consider the case where $t = d(g) \geq 2$. Then,

$$\begin{aligned}
& \sum_{l \in D_i(\bar{g})} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g} - l \\ g \not\subseteq \tilde{g}, g \subseteq \bar{g} + l}} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g} - l \\ D(\bar{g}) \supseteq D(g) - \{l\}}} \frac{d(\tilde{g})!(d(\bar{g}) - d(\tilde{g}) - 1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \sum_{k=0}^{\bar{n}-t} \frac{(t+k-1)!(\bar{n}-t-k)!}{\bar{n}!} \binom{\bar{n}-t}{k} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \sum_{k=0}^{\bar{n}-t} \frac{(t+k-1)!(\bar{n}-t)!}{\bar{n}!k!} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \frac{(\bar{n}-t)!t!}{\bar{n}!} \sum_{k=0}^{\bar{n}-t} \binom{t+k}{k} \frac{1}{t+k}.
\end{aligned}$$

Now, since $2 \leq t \leq \bar{n}$, by taking $m = \bar{n} - t$ at Lemma 4.3.1, we have

$\sum_{k=0}^{\bar{n}-t} \binom{t+k}{k} \frac{1}{t+k} = \binom{\bar{n}}{t} \frac{1}{t}$. Hence, we have

$$\begin{aligned}
& \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}, g \subseteq \tilde{g}+l}} \frac{d(\tilde{g})!(d(\bar{g})-d(\tilde{g})-1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \frac{(\bar{n}-t)!t!}{\bar{n}!} \sum_{k=0}^{\bar{n}-t} \binom{t+k}{k} \frac{1}{t+k} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \frac{(\bar{n}-t)!t!}{\bar{n}!} \binom{\bar{n}}{t} \frac{1}{t} \\
&= \sum_{l \in D_i(\bar{g}) \cap D(g)} \frac{\epsilon}{2} \frac{1}{t} = \sum_{l \in D_i(g)} \frac{\epsilon}{2} \frac{1}{d(g)} = \frac{\epsilon d_i(g)}{2d(g)},
\end{aligned}$$

where the second last equality follows from the fact that $l \in D_i(\bar{g}) \cap D(g)$ implies that $l \in D_i(\bar{g}) \cap D_i(g) = D_i(g)$, cause $i \in N(g)$ and $\bar{g} \supseteq g$.

Thus, Equation (4.4.2) is satisfied. So, for each $i \in N(g)$, we have

$$Y_i^{PV}(\bar{g}, v_{(g,\epsilon)}) = Y_i^{MV}(\bar{g}, v) + \frac{\epsilon d_i(g)}{2d(g)}.$$

Now, for each $i \in N \setminus N(g)$ and each $l \in D_i(\bar{g})$, we first show that $l \notin D(g)$.

Assume $l \in D(g)$. Then, $l \in D_i(\bar{g}) \cap D(g)$ implies that $l \in D_i(g)$. Thus, $i \in N(g)$, contradicting to the fact that $i \notin N(g)$.

Now, since $l \notin D(g)$, for each $i \in N \setminus N(g)$, each $l \in D_i(\bar{g})$ and each $\tilde{g} \subseteq \bar{g} - l$, $g \subseteq \tilde{g} \Leftrightarrow g \subseteq \tilde{g} + l$ and $g \not\subseteq \tilde{g} \Leftrightarrow g \not\subseteq \tilde{g} + l$.

Hence, for each $i \in N \setminus N(g)$,

$$\begin{aligned}
Y_i^{PV}(\bar{g}, v_{(g,\epsilon)}) &= \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \subseteq \tilde{g}}} (v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g})) \frac{d(\tilde{g})!(d(\bar{g})-d(\tilde{g})-1)!}{d(\bar{g})!} \\
&= \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \subseteq \tilde{g}}} \underbrace{(v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g}))}_{(v(\tilde{g}+l)+\epsilon)-(v(\tilde{g})+\epsilon)} \frac{d(\tilde{g})!(d(\bar{g})-d(\tilde{g})-1)!}{d(\bar{g})!} \\
&\quad + \sum_{l \in D_i(\bar{g})} \frac{1}{2} \sum_{\substack{\tilde{g} \subseteq \bar{g}-l \\ g \not\subseteq \tilde{g}}} \underbrace{(v_{(g,\epsilon)}(\tilde{g}+l) - v_{(g,\epsilon)}(\tilde{g}))}_{v(\tilde{g}+l)-v(\tilde{g})} \frac{d(\tilde{g})!(d(\bar{g})-d(\tilde{g})-1)!}{d(\bar{g})!} \\
&= Y_i^{PV}(\bar{g}, v).
\end{aligned}$$

Hence, Equation (4.4.1) is satisfied, i.e., if $\bar{g} \supseteq g$, then

$$Y_i^{PV}(\bar{g}, v_{(g, \epsilon)}) = \begin{cases} Y_i^{PV}(\bar{g}, v) + \frac{\epsilon d_i(g)}{2d(g)} & \text{if } i \in N(g), \\ Y_i^{PV}(\bar{g}, v) & \text{otherwise.} \end{cases}$$

Case 2) Let $\bar{g} \in \mathcal{G}$ be such that $\bar{g} \not\supseteq g$. Then, for each $i \in N$, each $l \in D_i(\bar{g})$ and each $\tilde{g} \subseteq \bar{g} - l$, $\tilde{g} \not\supseteq g$ and $\tilde{g} + l \not\supseteq g$.¹⁰ Thus, for each $i \in N$, each $l \in D_i(\bar{g})$ and each $\tilde{g} \subseteq \bar{g} - l$, $v_{(g, \epsilon)}(\tilde{g} + l) = v(\tilde{g} + l)$ and $v_{(g, \epsilon)}(\tilde{g}) = v(\tilde{g})$. Hence, by definition of the position value, for each $i \in N$,

$$Y_i^{PV}(\bar{g}, v_{(g, \epsilon)}) = Y_i^{PV}(\bar{g}, v).$$

Hence, Y^{PV} satisfies condition $\hat{A}$. □

Remember that, if the considered source network is a star, then according to the Myerson value, epsilon change is distributed equally between all the players in that star, thus being a central player is not important. Yet, since position value satisfies condition $\hat{A}$, being a central player is important. Assume the star network has $n - 1$ links that is formed by player 1 as the central player. Then, according to the position value, payoff of player 1 is $\frac{\epsilon}{2}$ while the payoff of each other player is $\frac{\epsilon}{2(n-1)}$.

Theorem 4.4.2. *There is a unique allocation rule that satisfies condition $\hat{A}$ and the null-game property, namely Y^{PV} .*

Proof. It is easy to check that Y^{PV} satisfies the null-game property. By Proposition 4.4.1, Y^{PV} satisfies condition $\hat{A}$.

Conversely, assume Y is an allocation rule that satisfies condition $\hat{A}$ and the null-game property. Consider any $g \in \mathcal{G}$, any $v \in \mathcal{V}$ and any labeling $g_1, g_2, \dots, g_{2^{n(n-1)/2}-1}$ of the elements in $\mathcal{G} \setminus \{g_0\}$. Since $\mathfrak{B}$ is a basis for $\mathcal{V}$, $v = \sum_{i=1}^{2^{n(n-1)/2}-1} c_i(v) v_{g_i}$ where the coefficients $c_i(v)$ are unique.

¹⁰Otherwise, if $\tilde{g} + l \supseteq g$, then $\bar{g} \supseteq \tilde{g} + l \supseteq g$; contradicting to $\bar{g} \not\supseteq g$.

Define the sequence $v^0, v^1, \dots, v^{2^{n(n-1)/2}-1}$ of $2^{\frac{n(n-1)}{2}}$ value functions recursively as follows:

$$v^0 = v_0, v^i = v^{i-1} + c_i(v)v_{g_i} = (v^{i-1})_{(g_i, c_i(v))} \text{ for each } i \in \{1, \dots, 2^{n(n-1)/2}\}.$$

By the null-game property, $Y(g, v^0)$ is unique. Then, for each $i \in \{1, \dots, 2^{n(n-1)/2}\}$, condition $\hat{A}$ uniquely determines $Y(g, v^i)$ from $Y(g, v^{i-1})$ recursively. Note also that $v^{2^{n(n-1)/2}} = v$, so $Y(g, v)$ is uniquely determined.

Thus, for each $(g, v) \in \mathcal{G} \times \mathcal{V}$, $Y(g, v)$ is uniquely determined. Since Y^{PV} satisfies condition $\hat{A}$ and the null-game property, $Y \equiv Y^{PV}$. $\square$

Condition $\hat{A}$ and the null-game property are independent of each other.

Remark 4.4.3. 1. There exists an allocation rule that satisfies condition $\hat{A}$, but it does not satisfy the null-game property. Thus, condition $\hat{A}$ does not imply the null-game property.

Without loss in generality, let $N = \{1, 2, 3\}$. Define Y as follows:

- $v = v_0$: For each $\bar{g} \in \mathcal{G}$, $Y(\bar{g}, v_0) = (1, -1, 0)$.
- $v = (v_0)_{(g, \epsilon)}$: For each $g_0 \neq g \in \mathcal{G}$, each $\bar{g} \supseteq g$ and each $0 \neq \epsilon \in \mathbb{R}$,

$$Y_i(\bar{g}, (v_0)_{(g, \epsilon)}) = \begin{cases} Y_i(\bar{g}, v_0) + \frac{\epsilon d_i(g)}{2d(g)} & \text{if } i \in N(g), \\ Y_i(\bar{g}, v_0) & \text{otherwise.} \end{cases}$$

For each $g_0 \neq g \in \mathcal{G}$, each $\bar{g} \not\supseteq g$ and each $0 \neq \epsilon \in \mathbb{R}$,

$$Y(\bar{g}, (v_0)_{(g, \epsilon)}) = (1, -1, 0).$$

- For all other cases, let $Y = Y^{PV}$.

It is easy to check that Y satisfies condition $\hat{A}$, but it does not satisfy the null-game property.

2. There exists an allocation rule that satisfies the null-game property, but it does not satisfy condition $\hat{A}$. Thus, null-game property does not imply condition $\hat{A}$.

Consider the allocation rule given in part 2. of Remark 4.3.4. Specifically, either condition A or condition $\hat{A}$ is applied for the complete graph g_K , the result obtained for $Y(g_K, v_{(g_K, \epsilon)})$ is exactly the same for any $v \in \mathcal{V}$ and any $\epsilon \in \mathbb{R}$. Thus, it satisfies the null-game property, but it does not satisfy condition $\hat{A}$.

4.5 Chun's characterization of the Shapley value

In this section we study Chun's characterization of the Shapley value and compare our key axioms with the axioms that he uses. For that, we first provide some basic definitions that we need.

For this section, we fix the player set as N . Let w_0 stand for the null-game, i.e., $w_0(S) = 0$, for each $S \subseteq N$. Remember that G^N denote the set of all TU games with player set N .

For each $\emptyset \neq S \subseteq N$, let w_S denote the TU game that satisfies:

$$w_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that $\overline{\mathfrak{B}} = \{w_S : \emptyset \neq S \subseteq N\}$ forms a *basis* for G^N .

For each $w \in G^N$, each $\emptyset \neq T \subseteq N$ and each $0 \neq \epsilon \in \mathbb{R}$, define a new TU game $w_{(T, \epsilon)}$ as follows:

$$w_{(T, \epsilon)}(S) = \begin{cases} w(S) + \epsilon & \text{if } T \subseteq S, \\ w(S) & \text{if } T \not\subseteq S. \end{cases}$$

In other words, $w_{(T, \epsilon)} = w + \epsilon w_T$ where $w_T \in \overline{\mathfrak{B}}$.

Note, by taking w as w_0 , and ϵ as 1, for each $\emptyset \neq T \subseteq N$, we have $(w_0)_{(T, 1)} \equiv w_T$. Hence, $\{(w_0)_{(T, 1)} : \emptyset \neq T \subseteq N\} = \overline{\mathfrak{B}}$.

Remember that a rule distributing the value of the grand coalition between the players is called an allocation rule for transferable utility games, and the *Shapley value* $Sh(w) = (Sh_1(w), \dots, Sh_n(w))$ is an allocation rule defined by:

$$Sh_i(w) := \sum_{S \subset N \setminus \{i\}} (w(S \cup i) - w(S)) \frac{|S|!(n - 1 - |S|)!}{n!},$$

for each $w \in G^N$ and each $i \in N$.

We now give the axioms that Chun uses for his characterization.

- An allocation rule Φ satisfies *triviality* (T) if for each $i \in N$, $\Phi_i(w_0) = 0$.
- An allocation rule Φ satisfies *coalitional strategic equivalence* (CSE) if for each $\emptyset \neq T \subseteq N$, each $\epsilon \in \mathbb{R}$, each $w \in G^N$ and each $i \in N \setminus T$, $\Phi_i(w_{(T, \epsilon)}) = \Phi_i(w)$.
- An allocation rule Φ satisfies *fair ranking* (FR) if for each $\emptyset \neq T \subseteq N$ and each pair of TU games $v, w \in G^N$ where $v(S) = w(S)$ for each $S \neq T$, $\Phi_i(v) > \Phi_j(v)$ implies $\Phi_i(w) > \Phi_j(w)$ for each $i, j \in T$.

Theorem [57]. *The Shapley value is the unique allocation rule that satisfies triviality (T), coalitional strategic equivalence (CSE) and fair ranking (FR).*

Definition 4.5.1. Given a TU game $w \in G^N$, $\emptyset \neq T \subseteq N$ and $0 \neq \epsilon \in \mathbb{R}$, we say that an allocation rule Φ satisfies *condition A'* if

$$\Phi_i(w_{(T, \epsilon)}) := \begin{cases} \Phi_i(w) + \frac{\epsilon}{|T|} & \text{if } i \in T, \\ \Phi_i(w) & \text{if } i \in N \setminus T, \end{cases}$$

In his paper, Chun shows that if an allocation rule satisfies (CSE) and (FR), then it satisfies condition A' . He does not mention that condition A' also implies (CSE) and (FR), but indeed it does. So, Chun's characterization can be rewritten as the follows:

“The Shapley value is the unique allocation rule that satisfies triviality (T) and condition A' .”

Different than Chun's proof technique, Chun's theorem can also be proven with the proof technique that we used for characterizations of the Myerson value and the position value.

Note that condition A' is similar to our key axioms for our characterizations, namely similar to condition A and condition $\hat{A}$. The common idea in all three conditions is that, if there is a change of the value function caused by a certain set of players, then this set of players *is* the source of the change, so, the change should be divided between *these players*. At the Shapley value and the Myerson value, this is divided equally between the players that are in the source coalition. So, from this behavior of the two, the Shapley value and the Myerson value are similar. Yet, also due to this similarity of the two, one can criticize that the Myerson value is not taking into account the structure of the network enough. Then, condition $\hat{A}$ steps in, where the change is divided equally between the links that are formed by the players in the source coalition, and this gives us the characterization of the position value. So, the similarities and differences of these three axioms explain the similarities and differences of the Shapley value, Myerson value and the position value.

Lastly, we give a sketch of the proof that condition A' implies (CSE) and (FR).

Remark 4.5.2. If an allocation rule satisfies condition A' , then it satisfies coalitional strategic equivalence (CSE) and fair ranking (FR).

Let Φ be an allocation rule that satisfies condition A' . By definition of condition A' , Φ satisfies (CSE) trivially.

Let $\emptyset \neq T \subseteq N$ and let $v(S) = w(S)$ for each $S \neq T$. To show that Φ satisfies (FR), we need to show that $\Phi_i(v) > \Phi_j(v) \Leftrightarrow \Phi_i(w) > \Phi_j(w)$ for each $i, j \in T$. Without loss in generality, let $v(T) = w(T) + c$, for some $c \in \mathbb{R}$.

Case 1: Let $|T| = |N|$, i.e., $T = N$. Then, by condition A' , for each $i \in N$, $\Phi_i(v) = \Phi_i(w) + \frac{c}{|N|}$, thus the result follows.

Case 2: Let $|T| = |N| - 1$, i.e., let $N \setminus \{k\}$ for some $k \in N$. Let us define an intermediate game $\bar{v}$ as follows:

$\bar{v}(N) = v(N) + c$, and $\bar{v}(S) = v(S)$ if $S \neq N$.

Since $\bar{v} = v_{(N,c)}$, by condition A' , for each $i \in N$, we have,

$$\Phi_i(\bar{v}) = \Phi_i(v) + \frac{c}{|N|}. \quad (4.5.1)$$

Observe that

$$\bar{v}(S) = \begin{cases} w(S) + c & \text{if } S = N \text{ or } S = T, \\ w(S) & \text{otherwise.} \end{cases}$$

In other words, $\bar{v} = w_{(T,c)}$. Thus, by condition A' , we have,

$$\Phi_i(\bar{v}) = \begin{cases} \Phi_i(w) + \frac{c}{|N|-1} & \text{if } i \in T, \\ \Phi_i(w) & \text{if } i \notin T, \text{ i.e., if } i = k. \end{cases} \quad (4.5.2)$$

Then, by equations 4.5.1 and 4.5.2, for each $l \in T$, we have $\Phi_l(v) + \frac{c}{|N|} = \Phi_l(w) + \frac{c}{|N|-1}$. Thus, for each $i, j \in T$,

$$\Phi_i(v) - \Phi_j(v) = \Phi_i(w) - \Phi_j(w).$$

Therefore, if $T = N \setminus \{k\}$ for some $k \in N$, then Φ satisfies (FR). In other words, $\Phi_i(v) > \Phi_j(v) \Leftrightarrow \Phi_i(w) > \Phi_j(w)$ for each $i, j \in T$.

Case s : Let $|T| = |N| - s$ for some $s \in \{2, 3, \dots, n-1\}$. In other words, let $T = N \setminus \{k_1, k_2, \dots, k_s\}$. Then, similar to the intermediate game defined in Case 2, by defining intermediate games for each $S \supseteq T$ and applying condition A' recursively, one can show that for each $i, j \in T$, $\Phi_i(v) - \Phi_j(v) = \Phi_i(w) - \Phi_j(w)$. So, if $T = N \setminus \{k_1, k_2, \dots, k_s\}$ for some $s \in \{2, 3, \dots, n-1\}$, then Φ satisfies (FR).

4.6 A comparison of the Shapley value and the Myerson value

Condition A and the null-game property are enough for a characterization of the Myerson value as we have proven in Section 4.3, but to compare with Shapley's original characterization, we add the linearity axiom to these two axioms and provide an alternative characterization in 4.6.1. That also allows us to modify

these axioms, and study a characterization of the Shapley value for cooperative games -which is the same as Chun's characterization except the addition of the linearity axiom, which is given in 4.6.2. The method used at one part of both characterizations is pretty similar to Shapley's own method [7]. Moreover, the characterizations of the Myerson value and the Shapley value are similar. Similarity of the characterizations is not that surprising, yet the similarities contribute how the information changes when one passes from transferable utility games to networks or vice versa, and allow us to see relations between different axioms.

4.6.1 The Myerson value revisited

First we give some axioms.

Definition 4.6.1. We say that an allocation rule Y is *linear* if, for each $g \in \mathcal{G}$, each pair of value functions $v, w \in \mathcal{V}$ and each $a, b \in \mathbb{R}$, $Y(g, av + bw) = aY(g, v) + bY(g, w)$ whenever $(av + bw)(g) = av(g) + bw(g)$.

Definition 4.6.2. We say that an allocation rule Y satisfies *condition C* (*Equal treatment to the linked players in basis value functions*) if for each $\bar{g} \in \mathcal{G}$ and each $v_g \in \mathfrak{B} = \{v_g : g \in \mathcal{G} \setminus \{g_0\}\}$,

$$Y_i(\bar{g}, v_g) := \begin{cases} \frac{1}{|N(g)|} & \text{if } \bar{g} \supseteq g \text{ and } i \in N(g), \\ 0 & \text{otherwise.} \end{cases}$$

Note that if an allocation rule Y satisfies condition C, then for each $\bar{g} \in \mathcal{G}$ and each $v_g \in \mathfrak{B}$, $Y(\bar{g}, v_g)$ is uniquely determined.

We first give the relations between condition A, the null-game property and condition C.

Proposition 4.6.3. *If an allocation rule Y satisfies condition A and the null-game property, then Y satisfies condition C.*

Proof. Let Y be an allocation rule that satisfies condition A and the null-game property. Let $v_g \in \mathfrak{B}$. Let g be the network that corresponds to v_g . Then,

$|N(g)| \neq 0$. We know that $v_g \equiv (v_0)_{(g,1)}$. Now, by considering ϵ as 1, v as v_0 , we apply condition A to Y .

Case 1) If $\bar{g} \supseteq g$ ¹¹, then by condition A,

$$Y_i(\bar{g}, (v_0)_{(g,1)}) = \begin{cases} Y_i(\bar{g}, v_0) + \frac{1}{|N(g)|} & \text{if } i \in N(g), \\ Y_i(\bar{g}, v_0) & \text{if } i \in N \setminus N(g). \end{cases}$$

By the null-game property, for each $i \in N$, $Y_i(\bar{g}, v_0) = 0$. Thus,

$$Y_i(\bar{g}, v_g) = Y_i(\bar{g}, (v_0)_{(g,1)}) = \begin{cases} 0 + \frac{1}{|N(g)|} & \text{if } i \in N(g), \\ 0 & \text{if } i \in N \setminus N(g). \end{cases}$$

Case 2) If $\bar{g} \not\supseteq g$ ¹², then for each $i \in N$, $Y_i(\bar{g}, v_g) = Y_i(\bar{g}, (v_0)_{(g,1)}) = Y_i(\bar{g}, v_0)$ by condition A, and that is equal to zero by the null-game property.

Thus, by cases 1) and 2), for each $\bar{g} \in \mathcal{G}$ and each $v_g \in \mathfrak{B}$,

$$Y_i(\bar{g}, v_g) := \begin{cases} \frac{1}{|N(g)|} & \text{if } \bar{g} \supseteq g \text{ and } i \in N(g), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, Y satisfies condition C. □

Theorem 4.6.4. *There is a unique allocation rule that satisfies condition A, the null-game property and linearity, namely Y^{MV} .*

Proof. By Proposition 4.3.2 we already know that Y^{MV} satisfies condition A. Also, it is already known that Y^{MV} satisfies null-game property and linearity.

Conversely, assume Y is an allocation rule that satisfies condition A, null-game property and linearity. We know that $\mathfrak{B}$ forms a basis for $\mathcal{V}$. Thus, for each $(\tilde{g}, \tilde{v}) \in \mathcal{G} \times \mathcal{V}$, $Y(\tilde{g}, \tilde{v}) = Y(\tilde{g}, \sum_{v_g \in \mathfrak{B}} \lambda_g v_g)$, where $(\lambda_g)_{g \in \mathcal{G}} \in \mathbb{R}^N$. By linearity of Y , $Y(\tilde{g}, \tilde{v}) = \sum_{v_g \in \mathfrak{B}} \lambda_g Y(\tilde{g}, v_g)$.

By Proposition 4.6.3, Y satisfies condition C. Now, by condition C, for each $\tilde{g}$ and $v_g \in \mathfrak{B}$, $Y(\tilde{g}, v_g)$ is uniquely determined. Thus, for each $(\tilde{g}, \tilde{v}) \in G^N \times V^N$, $Y(\tilde{g}, \tilde{v})$ is uniquely determined. Hence, $Y \equiv Y^{MV}$. □

¹¹One can easily check, we apply case (a) of condition A in this case.

¹²One can easily check, we apply case (b) of condition A in this case.

The next remark shows that the converse of Proposition 4.6.3 is not true in general.

Remark 4.6.5. There exists an allocation rule that satisfies condition C, but does not satisfy condition A. Thus, there exists an allocation rule that satisfies condition C, but does not satisfy condition A and the null-game property.

Without loss in generality, let $N = \{1, 2, 3\}$. Define $\tilde{v}$ as follows: for each $g \in \mathcal{G} \setminus \{g_0\}$, $\tilde{v}(g) = 5$ and $\tilde{v}(g_0) = 0$. Note that $\tilde{v} \notin \mathfrak{B} = \{v_g : g \in \mathcal{G}\}$.

Let Y be an allocation rule defined as follows:

- If $v \in \mathfrak{B}$, then for each $\bar{g} \in \mathcal{G}$ let

$$Y_i(\bar{g}, v_g) = \begin{cases} \frac{1}{|N(g)|} & \text{if } \bar{g} \supseteq g \text{ and } i \in N(g), \\ 0 & \text{otherwise.} \end{cases}$$

- If $v = \tilde{v}$, then for each $\bar{g} \in \mathcal{G}$ let

$$Y_i(\bar{g}, v) = \begin{cases} 5 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- If $v \in \mathcal{V} \setminus (\mathfrak{B} \cup \{\tilde{v}\})$, then for each $\bar{g} \in \mathcal{G}$ let $Y(\bar{g}, v) = Y^{MV}(\bar{g}, v)$.

Obviously Y satisfies condition C. Next, we show Y does not satisfy condition A. Consider g_K ¹³, $\tilde{v}$ and $\epsilon = 3$. Note that $\tilde{v}_{(g_K, \epsilon)} \in \mathcal{V} \setminus (\mathfrak{B} \cup \{\tilde{v}\})$, thus by definition $Y(g_K, \tilde{v}_{(g_K, \epsilon)}) = Y^{MV}(g_K, \tilde{v}_{(g_K, \epsilon)}) = (8/3, 8/3, 8/3)$.

If Y satisfies condition A, then for each $i \in N$, $Y_i(g_K, \tilde{v}_{(g_K, \epsilon)}) = Y_i(g_K, (\tilde{v})) + 1$, i.e., $Y(g_K, \tilde{v}_{(g_K, \epsilon)}) = (5, 0, 0) + (1, 1, 1) = (6, 1, 1) \neq (8/3, 8/3, 8/3)$, a contradiction. Thus, Y does not satisfy condition A.

By Theorem 4.3.3, Theorem 4.6.4 and Proposition 4.6.3, we have the following relations.

An allocation rule satisfies Condition A and null-game property $\Leftrightarrow$ It is the Myerson value $\Leftrightarrow$ An allocation rule satisfies Condition A, null-game property and linearity $\Leftrightarrow$ An allocation rule satisfies Condition C and linearity.

¹³Remember that g_K stands for the complete graph.

4.6.2 The Shapley value revisited

We first provide Shapley's characterization of the Shapley value.

Theorem [7].

There is a unique allocation rule which is anonymous¹⁴, additive¹⁵ and satisfies the null player axiom¹⁶, namely the Shapley value.

Shapley's proof method is as follows: Shapley shows that an allocation rule Γ that satisfies anonymity and the null player axiom uniquely determines the allocation rule at the basis elements of the set of all transferable utility games. Then, additivity together with this fact uniquely determines the allocation rule for each transferable utility game. Showing that Shapley value satisfies anonymity, additivity and the null player axiom, he completes the proof.

It is well-known that network games are related to transferable utility games and the calculation method used in Myerson value is inspired from the Shapley value. Using similar axioms that we used for the characterization of the Myerson value in Section 4.6.1, we visit a characterization of the Shapley value. The characterization theorem can be proven easily by Shapley's proof method that is given above.

First we give two definitions, for the other definitions that are used in this section we refer to Section 4.5.

Definition 4.6.6. We say that an allocation rule Γ is *linear* if, for each pair of TU games v, w and each $a, b \in \mathbb{R}$, $\Phi(av + bw) = a\Phi(v) + b\Phi(w)$ whenever $(av + bw)(S) = av(S) + bw(S)$ for each $S \subseteq N$.

Definition 4.6.7. We say that an allocation rule Φ satisfies *condition C'* (*Equal treatment of the players in basis value functions*) if for each $v \in \overline{\mathfrak{B}}$,

$$\Phi_i(v) := \begin{cases} \frac{1}{|T|} & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

¹⁴For anonymity of an allocation rule, we refer to page 64 of Section 3.2.1.

¹⁵For additivity of an allocation rule, we refer to page 28 of Section 2.3.1.

¹⁶For the null player axiom, we refer to page 64 of Section 3.2.1.

We first give the relations between the two conditions, namely condition A' and triviality that are given in Section 4.5, and condition C' . As one can easily guess, the results here are modifications of the results given for networks.

Proposition 4.6.8. *If an allocation rule Φ satisfies condition A' and triviality, then Φ satisfies condition C' .*

Proof. Let Φ be an allocation rule that satisfies condition A' and triviality. We know that $\overline{\mathfrak{B}} = \{(w_0)_{(T,1)} : \emptyset \neq T \subseteq N\}$.¹⁷ By considering v as w_0 , ϵ as 1, for each $\emptyset \neq T \subseteq N$, we apply condition A' , then

$$\Phi_i((w_0)_{(T,1)}) := \begin{cases} \Phi_i(w_0) + \frac{1}{|T|} & \text{if } i \in T, \\ \Phi_i(w_0) & \text{if } i \in N \setminus T. \end{cases}$$

By triviality, for each $i \in N$ $\Phi_i(w_0) = 0$, thus for each $\emptyset \neq T \subseteq N$,

$$\Phi_i((w_0)_{(T,1)}) := \begin{cases} \frac{\epsilon}{|T|} & \text{if } i \in T, \\ 0 & \text{if } i \in N \setminus T \end{cases}$$

In other words Φ satisfies condition C' . □

Next, we show that Shapley value satisfies condition A' . Note that, as explained in Section 4.5, this is also shown by Chun in an indirect way. The proof technique we use here is different than Chun's, and it is similar to the technique we used for the proof of Proposition 4.3.2.

Proposition 4.6.9. *Shapley value satisfies condition A' .*

Proof. Let $v \in G^N$, $\emptyset \neq T \subseteq N$ and $0 \neq \epsilon \in \mathbb{R}$ be given. We show that

$$Sh_i(v_{(T,\epsilon)}) = \begin{cases} Sh_i(v) + \frac{\epsilon}{|T|} & \text{if } i \in T, \\ Sh_i(v) & \text{if } i \in N \setminus T. \end{cases}$$

For each $i \in T$ note that:

$$S \supseteq T \setminus \{i\} \Rightarrow v_{(T,\epsilon)}(S \cup \{i\}) = v(S \cup \{i\}) + \epsilon, \quad v_{(T,\epsilon)}(S) = v(S) \text{ and,}$$

¹⁷See Section 4.5 for details.

$$S \not\supseteq T \setminus \{i\} \Rightarrow v_{(T,\epsilon)}(S \cup \{i\}) = v(S \cup \{i\}), \quad v_{(T,\epsilon)}(S) = v(S).$$

Thus, for each $i \in T$,

$$\begin{aligned} Sh_i(v_{(T,\epsilon)}) &= \sum_{S \subseteq N \setminus \{i\}} (v_{(T,\epsilon)}(S \cup \{i\}) - v_{(T,\epsilon)}(S)) \frac{|S|!(n - |S| - 1)!}{n!} \\ &= \sum_{S \subseteq N \setminus \{i\}, S \supseteq T \setminus \{i\}} \dots + \sum_{S \subseteq N \setminus \{i\}, S \not\supseteq T \setminus \{i\}} \dots \\ &= \sum_{S \subseteq N \setminus \{i\}, S \supseteq T \setminus \{i\}} \left(\frac{|S|!(n - |S| - 1)!}{n!} \right) \epsilon + Sh_i(v). \end{aligned}$$

Note,

$$\sum_{S \subseteq N \setminus \{i\}, S \supseteq T \setminus \{i\}} \left(\frac{|S|!(n - |S| - 1)!}{n!} \right) = \sum_{k=0}^{n-|T|} \frac{(|T| + k - 1)!(n - |T| - k)!}{n!} \binom{n - |T|}{k} = \frac{1}{|T|},$$

where the last equality follows from Lemma 4.3.1. Thus, for each $i \in T$, we have $Sh_i(v_{(T,\epsilon)}) = Sh_i(v) + \frac{\epsilon}{|T|}$.

Also, for each $i \in N \setminus T$,

$$\begin{aligned} Sh_i(v_{(T,\epsilon)}) &= \sum_{S \subseteq N \setminus \{i\}} (v_{(T,\epsilon)}(S \cup \{i\}) - v_{(T,\epsilon)}(S)) \frac{|S|!(n - |S| - 1)!}{n!} \\ &= \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)) \frac{|S|!(n - |S| - 1)!}{n!} = Sh_i(v). \end{aligned}$$

Hence, the Shapley value satisfies condition A' . □

Now, by Proposition 4.6.8 and Proposition 4.6.9, we have the following fact.

Corollary 4.6.10. *Shapley satisfies condition C' .*

Now, using the above results, we have the following simple characterizations of the Shapley value.

Theorem 4.6.11. *There is a unique allocation rule that satisfies condition A' , triviality and linearity, namely the Shapley value.*

The proof of Theorem 4.6.11 is easy and thus we omit it. We want to note that uniqueness of the allocation rule satisfying condition C' and linearity is in fact due to Shapley's original characterization.

The next remark shows that the converse of Proposition 4.6.8 is not true in general.

Remark 4.6.12. There exists an allocation rule that satisfies condition C' , but does not satisfy condition A' . Thus, there exists an allocation rule that satisfies condition C' , but does not satisfy condition A' and triviality.

Without loss in generality, let $N = \{1, 2, 3\}$. Define $\tilde{v}$ as follows: for each $\emptyset \neq S \subseteq N$, $\tilde{v}(S) = 5$. Note that $\tilde{v} \notin \overline{\mathfrak{B}}$.

Let Φ be an allocation rule defined as follows:

- If $v = (w_0)_{(T,1)} \in \overline{\mathfrak{B}}$, then let

$$\Phi_i(v) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

- If $v = \tilde{v}$, then for each $i \in N$, then let

$$\Phi_i(v) = \begin{cases} 5 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Otherwise, let $\Phi(v) = Sh(v)$.¹⁸

Obviously Φ satisfies condition C' . Next, we show Φ does not satisfy condition A' . Consider the TU game $(\tilde{v})_{(N,3)}$.

Note that for each $S \subset N$, $(\tilde{v})_{(N,3)}(S) = 5$, and $(\tilde{v})_{(N,3)}(N) = 8$, thus $(\tilde{v})_{(N,3)} \notin (\overline{\mathfrak{B}} \cup \{\tilde{v}\})$. Then, by definition $\Phi(\tilde{v}_{(N,3)}) = Sh(\tilde{v}_{(N,3)}) = (8/3, 8/3, 8/3)$.

If Φ satisfies condition A' , then for each $i \in N$, $\Phi_i((\tilde{v})_{(N,3)}) = \Phi_i(\tilde{v}) + 3/|N|$, i.e., $\Phi_i((\tilde{v})_{(N,3)}) = (6, 1, 1) \neq (8/3, 8/3, 8/3)$, a contradiction. Thus, Φ does not satisfy condition A' .

¹⁸In other words, for each TU game v that is not in $\overline{\mathfrak{B}} \cup \{\tilde{v}\}$, let Φ allocate the same as the Shapley value.

By the theorem of Chun, Theorem 4.6.11 and Proposition 4.6.8 we have the following relations.

An allocation rule satisfies Condition A' and triviality $\Leftrightarrow$ It is the Shapley value $\Leftrightarrow$ An allocation rule satisfies Condition A' , triviality and linearity $\Leftrightarrow$ An allocation rule satisfies Condition C' and linearity.

Chapter 5

A characterization of the Borda rule on the domain of weak preferences

This chapter is a joint work with Mehmet Karakaya [61].

5.1 Introduction

One scoring rule that has received a great deal of attention in the literature is attributed to Borda [10],¹ which is a scoring rule, and constitutes our main focus of interest in this chapter.

Young [73] shows that a social choice rule is a scoring rule if and only if it satisfies anonymity, neutrality, reinforcement and continuity, where agents have strict preference relations.²

¹See [62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72].

²Characterization of scoring rules as social welfare functions are obtained in [71] and [74], and a characterization of general scoring rules is obtained in [75]. An alternative parametrization of scoring rules that is based on the ability to ameliorate majority tyranny is given by [76].

Young [77] provides an axiomatic characterization of the Borda rule. He shows that when agents' have strict preferences over alternatives, the Borda rule is characterized by neutrality, reinforcement, faithfulness and Young's cancellation property. Neutrality is the independence of the name of the alternatives. In other words, the names of the alternatives do not affect the selected alternatives. A social choice rule satisfies the reinforcement³ if there exists common selected alternatives for any two disjoint voter sets, and these common choices are considered as the exact selected alternatives for the combined society. Faithfulness is satisfied by a social choice rule if there is only one agent in the society and the social choice rule chooses her top-ranked alternative. An social choice rule satisfies Young's cancellation property if, for every pair of alternatives, the number of agents who strictly prefer the first alternative to the second one is equal to the number of agents who strictly prefer the second alternative to the first one implies the selection of all alternatives. Another proof for Young's characterization of the Borda rule, again assuming that voters have strict preferences over alternatives, is provided in [78].

Debord [64] characterizes the Borda rule as a social choice rule by neutrality, reinforcement, faithfulness, and Young's cancellation property when agents' preferences are crisp binary relations belonging to a family that contains all linear orders. Characterization in [64] is considered as a generalization of Young's [77] result. Debord [64] also characterizes the Borda rule as a social welfare function with the same axioms as in [77] when agents have weak preferences.⁴ Marchant [80] characterizes the Borda rule by neutrality, reinforcement, faithfulness, and Young's cancellation property as a social welfare function when profiles of preferences are valued (fuzzy) relations belonging to a set which contains weak preferences and is stable by permutation and transposition.⁵ Nitzan and Rubinstein [83] provide another characterization of the Borda rule as a social welfare function in terms of neutrality, monotonicity,⁶ consistency and Young's cancellation

³This condition is called separability in [71], and consistency in [73].

⁴See also [79] for a characterization of Borda's k-choice function.

⁵We refer to [81, 82] for characterizations of the Borda rule which takes into account the cardinal properties and differences of Borda scores.

⁶Monotonicity means that if the position of one alternative is improved in preferences of a voter then the position of the alternative determined by the social welfare function is not

property, where agents' preferences need not be transitive.⁷

Rather than forcing agents to have strict preferences over alternatives allowing agents to have indifferences in their preferences yields more freedom in expressing preferences. Moreover, studies in the literature show that when agents are allowed to have weak preferences the probabilities of some paradoxes that an SCR might face decrease.

Gahrlein and Valognes [86] study the impact of agents' indifferences over the alternatives in their preferences on the existence of a Condorcet winner and the Condorcet efficiency⁸ of scoring rules for the case of three alternatives under the assumption that every agent chooses her preference independently from the set of weak preferences. They show that the increase in proportion of agents who have indifference in their preferences increases the probability of existence of the Condorcet winner, and also increases the Condorcet efficiency of all scoring rules. The same results are obtained when the number of agents who have strict preferences over alternatives are reduced. They also show that the Borda rule is the scoring rule with maximum Condorcet efficiency. Furthermore, they show that when agents are forced to report strict preferences where an agent breaks the indifferences in her preferences randomly reduces the Condorcet efficiency of all scoring rules.⁹

We study a characterization of the Borda rule on the domain of weak preferences, where the Borda rule is defined for each finite set of voters having preferences over a fixed set of alternatives. One new property that we introduce is the degree equality. A social choice rule satisfies degree equality if and only if, for any two profiles of two finite sets of voters, equality between the sums of the degrees of every alternative under these two profiles implies that the same alternatives get

worsened provided that other voters' preferences are the same.

⁷See also [84] for the characterization of the Borda rule when agents have partially ordered preferences. We refer to [85] for a survey of different characterizations of scoring rules in different contexts.

⁸The Condorcet winner is the alternative that receives a majority of votes in all pairwise comparisons [87]. The Condorcet efficiency of an SCR is the conditional probability that the Condorcet winner is chosen by the SCR, given that it exists.

⁹For other studies regarding some voting paradoxes and the Condorcet efficiency of SCRs when agents have weak preferences, see [88, 89, 90, 91, 92].

chosen at both of them. We show that degree equality and Young’s cancellation property are independent of each other in general. We also show that the Borda rule is the unique scoring rule which satisfies the degree equality, which indicates that degree equality and Young’s cancellation property are equivalent when we restrict ourselves to scoring rules. We show that the Borda rule is characterized by the conjunction of reinforcement, faithfulness and degree equality on the domain of weak preferences.

Young [77] notes that his characterization can be extended to the domain of weak preferences. Since Young’s characterization of the Borda rule is by faithfulness, reinforcement, neutrality, and Young’s cancellation property, and ours is by faithfulness, reinforcement, and degree equality, one natural question is whether Young’s cancellation property combined with neutrality implies degree equality. The answer is no. So, Young’s cancellation property combined with neutrality does not imply degree equality.

In addition, we introduce a new cancellation property and show that it characterizes the Borda rule among all scoring rules. The new cancellation property is weaker than degree equality, and it is independent of Young’s cancellation property in general.

The contributions of this chapter is threefold. First, we introduce a new axiom which we call degree equality and show that it characterizes the Borda rule among all scoring rules. Second, we provide a characterization of the Borda rule by a set axioms (faithfulness, reinforcement, and degree equality) on the domain of weak preferences. Third, we introduce a new cancellation property which is weaker than degree equality, and show that the Borda rule is the unique scoring rule that satisfies the cancellation property.

The rest of this chapter is organized as follows. Basic notions and preliminaries are given in Section 5.2. Our characterization of the Borda rule is given in Section 5.3.1. Finally, a new cancellation property is given in Section 5.4.

5.2 Preliminaries

Let A be a non-empty finite set of alternatives with $\#A = m \geq 3$.¹⁰ The set of alternatives A is fixed throughout this chapter. The (universal) set of voters is denoted by positive integers $\mathbb{N}$, the set of all non-empty finite subsets of $\mathbb{N}$ is denoted by $\mathcal{N}$, and a finite set of voters is $N = \{1, 2, \dots, n\} \in \mathcal{N}$.

Each voter $i \in \mathbb{N}$ has a complete, reflexive and transitive preference relation R_i over A . Let $W(A)$ denote the set of all preference relations over A . An n -tuple, $R^N = (R_1, \dots, R_n) \in W(A)^N$ denote a preference profile for a finite set of voters N , where $\#N = n$. For a pair $N, H \in \mathcal{N}$ with $N \cap H = \emptyset$, and $R^N \in W(A)^N$, $R^H \in W(A)^H$, we use the notation $(R^N + R^H) \in W(A)^{N \cup H}$ when we consider the union of voter sets N and H with their profiles R^N and R^H . We denote $k \in \mathbb{N}$ copies of $R^N \in W(A)^N$ by $R^{kN} \in W(A)^{kN}$, where each copy is taken on a different voter set.

For any $i \in N$, let P_i denote the strict preference relation associated with R_i and I_i denote the indifference relation associated with R_i . Let $L(A)$ denote the set of all strict preference relations over A . An n -tuple, $P^N = (P_1, \dots, P_n) \in L(A)^N$ denote a strict preference profile for a finite set of voters N .

Given any $x \in A$ and any $R \in W(A)$, let

- $U(x, R) = \{y \in A \mid yRx\}$ denote the upper contour set of x at R ,
- $SU(x, R) = \{y \in A \mid yPx\}$ denote the strict upper contour set of x at R ,
- $SL(x, R) = \{y \in A \mid xPy\}$ denote the strict lower contour set of x at R .

For any $R \in W(A)$, let $top(R) = \{x \in A \mid xRy \text{ for each } y \in A\}$ denote the best (top-ranked) alternatives at R .

Definition 5.2.1. A *social choice rule* (SCR) is a map $F : \bigcup_{N \in \mathcal{N}} W(A)^N \rightarrow 2^A \setminus \{\emptyset\}$, i.e., for each preference profile $R^N \in W(A)^N$ of a finite set of voters N , an SCR F assigns a nonempty subset $F(R^N)$ of A .

¹⁰For a non-empty finite set H , we let $\#H$ denote the cardinality of H .

For each $x \in A$ and each $R \in W(A)$, the *degree of x at R* , denoted by $d(x, R)$ is defined as follows:¹¹

$$d(x, R) = \frac{\#SU(x, R) + \#U(x, R) + 1}{2}.$$

Note that for each $x \in A$ and each $R \in W(A)$, the degree of x at R is either a positive integer or a multiple of $1/2$.

Let $s = (s_1, s_2, \dots, s_m) \in \mathbb{R}^m$ denote a score vector, where $s_1 \geq s_2 \geq \dots \geq s_m$ and $s_1 > s_m$. We now define how we determine the score of an alternative by using its degree at a preference relation R . Given $s = (s_1, s_2, \dots, s_m)$, $x \in A$ and $R \in W(A)$, we determine the score of x at R , $s(x, R) \in \mathbb{R}$, as follows:

$$s(x, R) = \begin{cases} s_{d(x, R)} & \text{if } d(x, R) \in \mathbb{Z}_+, \\ (s_{\lfloor d(x, R) \rfloor} + s_{\lfloor d(x, R) \rfloor + 1})/2 & \text{otherwise,} \end{cases}$$

where for any $\delta \in \mathbb{R}$, $\lfloor \delta \rfloor$ denote the maximal integer which is smaller than or equal to δ , and $\mathbb{Z}_+$ denote positive integers. Given any $N \in \mathcal{N}$ and any profile $R^N \in W(A)^N$, the total score of $x \in A$ at R^N , denoted by $S(x, R^N)$, is defined by $S(x, R^N) = \sum_{i \in N} s(x, R_i)$.

A *scoring rule* F^s associated with a score vector s is an SCR that selects the alternatives with the maximal total score, i.e., for any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$,

$$F^s(R^N) = \{x \in A \mid \text{for each } y \in A, S(x, R^N) \geq S(y, R^N)\}.$$

Plurality rule is a scoring rule defined by the scoring vector $(1, 0, \dots, 0)$. *Inverse plurality rule* is a scoring rule defined by the scoring vector $(1, 1, \dots, 1, 0)$. *Borda rule* is a scoring rule defined by the scoring vector $s = (s_1, s_2, \dots, s_m)$ such that $s_1 - s_2 = s_2 - s_3 = \dots = s_{m-1} - s_m$, i.e., for each $1 \leq k \leq m - 2$, $s_k - s_{k+1} = s_{k+1} - s_{k+2}$. Note that any positive affine transformation of a Borda score vector is also a Borda score vector, i.e., if s is a Borda score vector, then for any $a \in \mathbb{R}_+$ and any $b = (b, \dots, b) \in \mathbb{R}^m$, $s' = as + b$ is also a Borda score vector.

We will now define some axioms.

¹¹We are grateful to our friend Serhat Doğan for suggesting this definition.

- We say that an SCR F satisfies *neutrality* (N) if for each $N \in \mathcal{N}$, each $R^N \in W(A)^N$ and every permutation $\tau : A \rightarrow A$, we have $F((R_N)^\tau) = \tau(F(R_N))$, where for each $i \in N$ and $x, y \in A$, $\tau(x)R_i^\tau \tau(y) \Leftrightarrow xR_iy$.
- We say that an SCR F satisfies *anonymity* (A) if for each $N \in \mathcal{N}$, each $R^N \in W(A)^N$ and every permutation $\sigma : N \rightarrow N$, we have $F(R^{\sigma(N)}) = F(R^N)$, where $F(R^{\sigma(N)}) = F((R_{\sigma(i)})_{i \in N})$.
- We say that an anonymous SCR F satisfies *continuity* (CO) if for each pair $N, H \in \mathcal{N}$, each $R^N \in W(A)^N$ with $\#F(R^N) = 1$ and each $R^H \in W(A)^H$, there exists an integer $\acute{k}$ ($\acute{k}$ is sufficiently large) such that $F(R^{kN} + R^H) = F(R^N)$ for all $k \geq \acute{k}$, where R^{kN} denote the k copies of R^N .
- We say that an SCR F satisfies *reinforcement* (RE) if for each pair $N, H \in \mathcal{N}$ with $N \cap H = \emptyset$, each $R^N \in W(A)^N$ and each $R^H \in W(A)^H$, $F(R^N) \cap F(R^H) \neq \emptyset$ implies $F(R^N + R^H) = F(R^N) \cap F(R^H)$, where $(R^N + R^H) \in W(A)^{N \cup H}$.
- We say that an SCR F satisfies *Young's cancellation* (Y-Ca) property if for each $N \in \mathcal{N}$, each $R^N \in W(A)^N$ such that for all pairs x and y , $\#\{i \in N \mid xP_iy\} = \#\{i \in N \mid yP_ix\}$, then we have $F(R^N) = A$.
- We say that an SCR F satisfies *faithfulness* (F) if whenever $N = \{i\}$, F chooses agent i 's most preferred alternative(s), i.e., for each $R_i = R \in W(A)$, $F(R) = \text{top}(R_i)$.

We now give Young's characterizations of scoring rules and the Borda rule.

- **Theorem [73].** *An SCR $F : \bigcup_{N \in \mathcal{N}} L(A)^N \rightarrow 2^A \setminus \{\emptyset\}$ is a scoring rule if and only if it satisfies anonymity (A), neutrality (N), reinforcement (RE) and continuity (CO), where agents have strict preference relations.*
- **Theorem [77, 78].** *An SCR $F : \bigcup_{N \in \mathcal{N}} L(A)^N \rightarrow 2^A \setminus \{\emptyset\}$ is the Borda*

rule if and only if it satisfies neutrality (N), reinforcement (RE), faithfulness (F) and Young's cancelation (Y-Ca) property, where agents have strict preference relations.

For each $N \in \mathcal{N}$, each $R^N \in W(A)^N$ and each $x \in A$, the total degree of x at R^N , $D(x, R^N)$, is defined as $D(x, R^N) = \sum_{R_i \in R^N} d(x, R_i)$.

We now define our degree equality axiom.

Definition 5.2.2. We say that an SCR F satisfies *degree equality* (DE) if, for each $N, \dot{N} \in \mathcal{N}$ with $\#N = \#\dot{N}$ and each pair $R^N \in W(A)^N$ and $\dot{R}^{\dot{N}} \in W(A)^{\dot{N}}$, for each $x \in A$, $D(x, R^N) = D(x, \dot{R}^{\dot{N}})$ implies $F(R^N) = F(\dot{R}^{\dot{N}})$.

An SCR satisfies degree equality if, for any two profiles of two finite sets of voters with equal cardinality, equality between the total degree of every alternative under these two profiles implies that the same alternatives get chosen by the SCR at these two profiles.

We note that if an SCR satisfies degree equality then it satisfies anonymity, since the total degree of an alternative for a given preference profile does not change if the names of voters are permuted.

In [77] and [78], the Borda score vector is taken as $s^B = ((m-1), (m-3), \dots, -(m-3), -(m-1))$. Young [77] notes that when agents have weak preferences over alternatives, the Borda rule is characterized with the same axioms where the Borda score of $x \in A$ at $R \in W(A)$ is $s^B(x, R) = \#SL(x, R) - \#SU(x, R)$. Hence, all axioms that are used in Young's characterization of the Borda rule are well defined when agents have weak preferences over alternatives. So, we will look at the relation between Young's cancellation property (Y-Ca) and our axiom degree equality (DE).

We now briefly discuss the differences between Young's cancelation property (Y-Ca) and degree equality (DE). Young's cancelation property (Y-Ca) looks at only one preference profile R^N and says that the set of all alternatives should be selected under R^N if majority comparisons of each pair of alternatives tie.

However, degree equality (DE) looks at two profiles R^N and $\acute{R}^{\acute{N}}$ of two societies N and $\acute{N}$, and says the same alternatives are chosen at these two profiles if the total degree of each alternative is same under these profiles, but degree equality (DE) does not say anything about which alternatives should be selected at these two profiles.

The next proposition shows degree equality (DE) and Young's cancelation property (Y-Ca) are independent of each other.

Proposition 5.2.3. *Degree equality (DE) and Young's cancelation property (Y-Ca) are independent of each other. Moreover, an SCR satisfying Young's cancellation property (Y-Ca) and neutrality (N) may not satisfy degree equality (DE).*

Proof. The constant social choice rule $\overline{F}$ satisfies degree equality (DE), but it violates Young's cancelation property (Y-Ca). As usual, the constant social choice rule $\overline{F}$ is defined as follows: For any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$, $\overline{F}(R^N) = \{a\}$, where $a \in A$. We also note that $\overline{F}$ violates neutrality.

We now provide an SCR which satisfies Young's cancelation property (Y-Ca) but violates degree equality (DE).

Let $A = \{a, b, c\}$. The set of all weak preference orderings over A is given below.¹²

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}
a, b, c	a	a	b	b	c	c	a	b	c	a, b	a, c	b, c
	b	c	a	c	a	b	b, c	a, c	a, b	c	b	a
	c	b	c	a	b	a						

We define $\widetilde{F}$ as follows:

¹²The alternative(s) on the upper row are strictly preferred to the alternatives on the lower rows, and alternatives on the same row which are next to each other and separated by comma are indifferent among themselves. For instance, at R_8 , a is strictly preferred to b and c , and b is indifferent to c .

- for $\#N = 1$, $\tilde{F}(R_k) = \text{top}(R_k)$ for all $k \in \{1, \dots, 13\}$,
- for $\#N = 2$, $\tilde{F}(R_{11} + R_{11}) = \{c\}$, $\tilde{F}(R_{12} + R_{12}) = \{b\}$, $\tilde{F}(R_{13} + R_{13}) = \{a\}$,
and for any other profile $(R_j + R_k)$ we have $\tilde{F}(R_j + R_k) = A$,
- for $\#N \geq 3$, $\tilde{F}(R^N) = A$ for all $R^N \in W(A)^N$.

It is clear that $\tilde{F}$ satisfies Young's cancelation property (Y-Ca). Note that for each $x \in A$ we have $D(x, R_2 + R_4) = D(x, R_{11} + R_{11})$, however $\tilde{F}(R_2 + R_4) \neq \tilde{F}(R_{11} + R_{11})$. Hence, $\tilde{F}$ violates degree equality (DE).

Next, we show that the SCR $\tilde{F}$ defined above satisfies neutrality (N).

For $\#N = 1$ and for $\#N \geq 3$, $\tilde{F}$ trivially satisfies neutrality (N). We only need to check that when $\#N = 2$, $\tilde{F}$ satisfies neutrality (N) at the profiles $(R_{11} + R_{11})$, $(R_{12} + R_{12})$, and $(R_{13} + R_{13})$ since for any other profiles $\tilde{F}$ chooses the set of all alternatives.

The set of all permutations over A is given below:

τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
$\tau_1(a) = a$	$\tau_2(a) = a$	$\tau_3(a) = b$	$\tau_4(a) = b$	$\tau_5(a) = c$	$\tau_6(a) = c$
$\tau_1(b) = b$	$\tau_2(b) = c$	$\tau_3(b) = a$	$\tau_4(b) = c$	$\tau_5(b) = a$	$\tau_6(b) = b$
$\tau_1(c) = c$	$\tau_2(c) = b$	$\tau_3(c) = c$	$\tau_4(c) = a$	$\tau_5(c) = b$	$\tau_6(c) = a$

We first show that neutrality (N) is satisfied for the profile $(R_{11} + R_{11})$, i.e., for all $k \in \{1, \dots, 6\}$, $\tilde{F}((R_{11} + R_{11})^{\tau_k}) = \tau_k(\tilde{F}(R_{11} + R_{11})) = \tau_k(c)$.

Since $R_{11}^{\tau_1} = R_{11}$ and $R_{11}^{\tau_3} = R_{11}$, we have $\tilde{F}(R_{11}^{\tau_1} + R_{11}^{\tau_1}) = \tilde{F}(R_{11}^{\tau_3} + R_{11}^{\tau_3}) = \tilde{F}(R_{11} + R_{11}) = c = \tau_1(c) = \tau_3(c)$.

Since $R_{11}^{\tau_2} = R_{12}$ and $R_{11}^{\tau_5} = R_{12}$, we have $\tilde{F}(R_{11}^{\tau_2} + R_{11}^{\tau_2}) = \tilde{F}(R_{11}^{\tau_5} + R_{11}^{\tau_5}) = \tilde{F}(R_{12} + R_{12}) = b = \tau_2(c) = \tau_5(c)$,

Since $R_{11}^{\tau_4} = R_{13}$ and $R_{11}^{\tau_6} = R_{13}$, we have $\tilde{F}(R_{11}^{\tau_4} + R_{11}^{\tau_4}) = \tilde{F}(R_{11}^{\tau_6} + R_{11}^{\tau_6}) = \tilde{F}(R_{13} + R_{13}) = a = \tau_4(c) = \tau_6(c)$.

We now show that neutrality (N) is satisfied for the profile $(R_{12} + R_{12})$, i.e., for all $k \in \{1, \dots, 6\}$, $\tilde{F}((R_{12} + R_{12})^{\tau_k}) = \tau_k(\tilde{F}(R_{12} + R_{12})) = \tau_k(b)$.

Since $R_{12}^{\tau_1} = R_{12}$ and $R_{12}^{\tau_6} = R_{12}$, we have $\tilde{F}(R_{12}^{\tau_1} + R_{12}^{\tau_1}) = \tilde{F}(R_{12}^{\tau_6} + R_{12}^{\tau_6}) = \tilde{F}(R_{12} + R_{12}) = b = \tau_1(b) = \tau_6(b)$,

Since $R_{12}^{\tau_2} = R_{11}$ and $R_{12}^{\tau_4} = R_{11}$, we have $\tilde{F}(R_{12}^{\tau_2} + R_{12}^{\tau_2}) = \tilde{F}(R_{12}^{\tau_4} + R_{12}^{\tau_4}) = \tilde{F}(R_{11} + R_{11}) = c = \tau_2(b) = \tau_4(b)$,

Since $R_{12}^{\tau_3} = R_{13}$ and $R_{12}^{\tau_5} = R_{13}$, we have $\tilde{F}(R_{12}^{\tau_3} + R_{12}^{\tau_3}) = \tilde{F}(R_{12}^{\tau_5} + R_{12}^{\tau_5}) = \tilde{F}(R_{13} + R_{13}) = a = \tau_3(b) = \tau_5(b)$.

We finally show that neutrality (N) is satisfied for the profile $(R_{13} + R_{13})$, i.e., for all $k \in \{1, \dots, 6\}$, $\tilde{F}((R_{13} + R_{13})^{\tau_k}) = \tau_k(\tilde{F}(R_{13} + R_{13})) = \tau_k(a)$.

Since $R_{13}^{\tau_1} = R_{13}$ and $R_{13}^{\tau_2} = R_{13}$, we have $\tilde{F}(R_{13}^{\tau_1} + R_{13}^{\tau_1}) = \tilde{F}(R_{13}^{\tau_2} + R_{13}^{\tau_2}) = \tilde{F}(R_{13} + R_{13}) = a = \tau_1(a) = \tau_2(a)$,

Since $R_{13}^{\tau_5} = R_{11}$ and $R_{13}^{\tau_6} = R_{11}$, we have $\tilde{F}(R_{13}^{\tau_5} + R_{13}^{\tau_5}) = \tilde{F}(R_{13}^{\tau_6} + R_{13}^{\tau_6}) = \tilde{F}(R_{11} + R_{11}) = c = \tau_5(a) = \tau_6(a)$.

Since $R_{13}^{\tau_3} = R_{12}$ and $R_{13}^{\tau_4} = R_{12}$, we have $\tilde{F}(R_{13}^{\tau_3} + R_{13}^{\tau_3}) = \tilde{F}(R_{13}^{\tau_4} + R_{13}^{\tau_4}) = \tilde{F}(R_{12} + R_{12}) = b = \tau_3(a) = \tau_4(a)$.

Hence, $\tilde{F}$ satisfies neutrality (N). $\square$

Throughout the chapter, we take the Borda score vector $s^B = ((m-1), (m-3), \dots, -(m-3), -(m-1))$ as in [77] and [78]. Then, $s^B(x, R) = \#SL(x, R) - \#SU(x, R)$ denotes the Borda score of $x \in A$ at $R \in W(A)$, and $S^B(x, R^N) = \sum_{i \in N} s^B(x, R_i)$ denotes the total Borda score of $x \in A$ at $R^N \in W(A)^N$ for $N \in \mathcal{N}$.

We now examine the relation between the total Borda score and the total degree of an alternative at a preference profile.

Lemma 5.2.4. *For any $N \in \mathcal{N}$ with $\#N = n$, any $R^N \in W(A)^N$, and any $x \in A$, $S^B(x, R^N) = n(m+1) - 2D(x, R^N)$.*

Proof. We first show that for any $x \in A$ and any $R \in W(A)$, $s^B(x, R) = m + 1 - 2d(x, R)$.

Since for any $x \in A$ and any $R \in W(A)$, $\#SL(x, R) = m - \#U(x, R)$, we have $s^B(x, R) = m - \#U(x, R) - \#SU(x, R) = m - (\#U(x, R) + \#SU(x, R))$.

Since $d(x, R) = \frac{\#SU(x, R) + \#U(x, R) + 1}{2}$, we have $\#SU(x, R) + \#U(x, R) = 2d(x, R) - 1$. We now substitute $2d(x, R) - 1$ for $\#U(x, R) + \#SU(x, R)$ in $s^B(x, R)$, i.e.,

$$\begin{aligned} s^B(x, R) &= m - (\#U(x, R) + \#SU(x, R)) \\ &= m - (2d(x, R) - 1) \\ &= m + 1 - 2d(x, R). \end{aligned}$$

Hence, for any $x \in A$ and any $R \in W(A)$, we have $s^B(x, R) = m + 1 - 2d(x, R)$.

Thus, since for any $N \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $x \in A$, $S^B(x, R^N) = \sum_{i \in N} s^B(x, R_i)$ and $D(x, R^N) = \sum_{i \in N} d(x, R_i)$, we have for any $N \in \mathcal{N}$ with $\#N = n$, any $R^N \in W(A)^N$, and any $x \in A$, $S^B(x, R^N) = n(m + 1) - 2D(x, R^N)$. $\square$

By Lemma 5.2.4 we have the following corollary.

Corollary 5.2.5. (i) For any $N \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $x, y \in A$,

$$S^B(x, R^N) \begin{cases} > \\ = \\ < \end{cases} S^B(y, R^N) \Leftrightarrow D(x, R^N) \begin{cases} < \\ = \\ > \end{cases} D(y, R^N).$$

(ii) For any $N \in \mathcal{N}$ with $\#N = n$, any $R^N \in W(A)^N$, and any $x \in A$, we have,

$$S^B(x, R^N) \begin{cases} > \\ = \\ < \end{cases} 0 \Leftrightarrow D(x, R^N) \begin{cases} < \\ = \\ > \end{cases} \frac{n(m + 1)}{2}.$$

(iii) For any $N, \tilde{N} \in \mathcal{N}$ with $\#N = \#\tilde{N}$, any $R^N \in W(A)^N$, and any $\tilde{R}^{\tilde{N}} \in W(A)^{\tilde{N}}$,

$$\forall x \in A, S^B(x, R^N) = S^B(x, \tilde{R}^{\tilde{N}}) \Leftrightarrow \forall x \in A, D(x, R^N) = D(x, \tilde{R}^{\tilde{N}}).$$

(iv) An SCR F satisfies degree equality (DE) if and only if there exists a Borda score vector such that for each pair $N, \tilde{N} \in \mathcal{N}$ with $\#N = \#\tilde{N}$, each pair $R^N \in W(A)^N$, $\tilde{R}^{\tilde{N}} \in W(A)^{\tilde{N}}$, and for each $x \in A$, $S(x, R^N) = S(x, \tilde{R}^{\tilde{N}})$ implies $F(R^N) = F(\tilde{R}^{\tilde{N}})$.

Proof. (i) Lemma 5.2.4 directly implies that for any $N \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $x, y \in A$, $[S^B(x, R^N) = S^B(y, R^N) \Leftrightarrow D(x, R^N) = D(y, R^N)]$. Inequality relations follow from the fact that for any $N \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $x \in A$, $S^B(x, R^N)$ and $D(x, R^N)$ are inversely related (by Lemma 5.2.4).

(ii) It is easy to check by Lemma 5.2.4.¹³

(iii) It follows from Lemma 5.2.4.

(iv) It follows from part (iii), where the Borda score vector is $s^B = ((m-1), (m-3), \dots, -(m-3), -(m-1))$. $\square$

For two profiles of two societies with distinct cardinalities, it is impossible that the total degree of each alternative is same under the two profiles. However, it is possible that the total Borda score of each alternative is same under the two profiles of two societies with distinct cardinalities. For instance, let $A = \{a, b, c\}$, $N = \{1, 2\}$, $\tilde{N} = \{3, 4, 5\}$, and the preferences $R^N = (R_1, R_2)$, $\tilde{R}^{\tilde{N}} = (\tilde{R}_3, \tilde{R}_4, \tilde{R}_5)$ are as follows:

R_1	R_2	$\tilde{R}_3$	$\tilde{R}_4$	$\tilde{R}_5$
a, b	b, c	b	a	b
c	a	a, c	c	c
			b	a

For each $x \in A$, $S^B(x, R^N) = S^B(x, \tilde{R}^{\tilde{N}})$. However, for each $x \in A$, $D(x, R^N) \neq D(x, \tilde{R}^{\tilde{N}})$. Therefore, Corollary 5.2.5-(iii) and Corollary 5.2.5-(iv) are not correct for societies with distinct cardinalities.

Corollary 5.2.5-(iv) can be considered as an alternative definition for degree equality (DE) that clarifies the relation between degree equality and the Borda score.

¹³We also note that for any $R \in W(A)$ and any $x \in A$, if m is even, then $s^B(x, R) > 0$ if and only if $d(x, R) \leq \frac{m}{2}$, $s^B(x, R) = 0$ if and only if $d(x, R) = [\frac{m}{2} + (\frac{m}{2} + 1)]/2 = \frac{m+1}{2}$, and $s^B(x, R) < 0$ if and only if $d(x, R) \geq \frac{m}{2} + 1$; if m is odd, then $s^B(x, R) > 0$ if and only if $d(x, R) < \frac{m+1}{2}$, $s^B(x, R) = 0$ if and only if $d(x, R) = \frac{m+1}{2}$, and $s^B(x, R) < 0$ if and only if $d(x, R) > \frac{m+1}{2}$.

We now show that the Borda rule satisfies degree equality (DE).

Proposition 5.2.6. *The Borda rule satisfies the degree equality (DE) axiom.*

Proof. Let F denote the Borda rule, i.e., for any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$, $F(R^N) = \{x \in A \mid \text{for all } y \in A, S^B(x, R^N) \geq S^B(y, R^N)\}$. Let $N, \tilde{N} \in \mathcal{N}$ with $\#N = \#\tilde{N}$, $R^N \in W(A)^N$, and $\tilde{R}^{\tilde{N}} \in W(A)^{\tilde{N}}$ be such that for each $x \in A$, $D(x, R^N) = D(x, \tilde{R}^{\tilde{N}})$. Then, by Corollary 5.2.5-(iii), for each $x \in A$, $S^B(x, R^N) = S^B(x, \tilde{R}^{\tilde{N}})$. Therefore, by Corollary 5.2.5-(iv), $F(R^N) = F(\tilde{R}^{\tilde{N}})$. $\square$

5.3 Results

As it is shown by Proposition 5.2.3 that degree equality (DE) and Young's cancellation property (Y-Ca) are independent of each other in general. However, these two properties are equivalent when we restrict ourselves to scoring rules. We know that the Borda rule is the only scoring rule satisfying Young's cancellation property (Y-Ca). This is also correct for degree equality (DE). The next result shows that degree equality (DE) characterizes the Borda rule among all scoring rules.

Proposition 5.3.1. *A scoring rule satisfies degree equality (DE) if and only if it is the Borda rule.*

Proof. We first show that the proposition holds on the domain of strict preferences. By Proposition 5.2.6, the Borda rule satisfies degree equality (DE). For the other part of the proof, let F be a scoring rule that satisfies degree equality (DE). Suppose that F is not the Borda rule.

Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$. We consider following strict profiles

$P^N = \{P_1, P_2, P_3\}$ and $\tilde{P}^N = \{\tilde{P}_1, \tilde{P}_2, \tilde{P}_3\}$:

P_1	P_2	P_3	$\tilde{P}_1$	$\tilde{P}_2$	$\tilde{P}_3$
a	a	b	a	b	b
b	b	c	c	a	a
c	c	a	b	c	c

For each $x \in A$, $D(x, P^N) = D(x, \tilde{P}^N)$. So, $F(P^N) = F(\tilde{P}^N)$ by degree equality (DE). Let $s = (s_1, s_2, s_3)$, where $s_1 \geq s_2 \geq s_3$ and $s_1 > s_3$. Since it is supposed that F is not the Borda rule, we have, $s_1 - s_2 \neq s_2 - s_3$. Let $t_{12} = s_1 - s_2$ and $t_{23} = s_2 - s_3$. So, $s_1 = s_3 + t_{12} + t_{23}$ and $s_2 = s_3 + t_{23}$.

We calculate the total score of each alternative at P^N and $\tilde{P}^N$:

$$S(a, P^N) = 2s_1 + s_3 = 3s_3 + 2t_{12} + 2t_{23},$$

$$S(b, P^N) = 2s_2 + s_1 = 3s_3 + t_{12} + 3t_{23},$$

$$S(c, P^N) = 2s_3 + s_2 = 3s_3 + t_{23},$$

$$S(a, \tilde{P}^N) = s_1 + 2s_2 = 3s_3 + t_{12} + 3t_{23},$$

$$S(b, \tilde{P}^N) = s_3 + 2s_1 = 3s_3 + 2t_{12} + 2t_{23},$$

$$S(c, \tilde{P}^N) = s_2 + 2s_3 = 3s_3 + t_{23}.$$

Since F is a scoring rule, we have $t_{12} \neq 0$ or $t_{23} \neq 0$. This implies that $S(a, P^N) > S(c, P^N)$ and $S(b, P^N) > S(c, P^N)$. Thus, $c \notin F(P^N)$. Note that $S(a, P^N) = S(b, \tilde{P}^N)$ and $S(b, P^N) = S(a, \tilde{P}^N)$, that means, $a \in F(P^N)$ if and only if $b \in F(\tilde{P}^N)$, and $b \in F(P^N)$ if and only if $a \in F(\tilde{P}^N)$. This fact, together with $c \notin F(P^N)$, yields that $F(P^N) = F(\tilde{P}^N) = \{a, b\}$. Therefore, $S(a, P^N) = S(b, P^N)$ which in turn yields that $t_{12} = t_{23}$ which is the desired contradiction. Hence, F is the Borda rule. $\square$

5.3.1 Characterization and its proof

We now provide our main result and its proof.

Theorem 5.3.2. *An SCR $F : \bigcup_{N \in \mathcal{N}} W(A)^N \rightarrow 2^A \setminus \{\emptyset\}$ satisfies faithfulness*

(F), reinforcement (RE), and degree equality (DE) if and only if it is the Borda rule.

It is clear that the Borda rule satisfies faithfulness (F) and reinforcement (RE). We showed by Proposition 5.2.6 that the Borda rule satisfies degree equality (DE).

For the other part of the proof, let F be an SCR satisfying the given axioms. We will show that F is the Borda rule. First, we will show that such an SCR completely depends on the total Borda scores. That is, for any two profiles of two finite sets of voters (not necessarily with equal cardinality), equality between the total Borda score of every alternative under these two profiles implies that the same alternatives get chosen at these two profiles. These two voter sets may have equal cardinality or distinct cardinalities. When the two voter sets have equal cardinality, dependence on the total Borda scores is satisfied by Corollary 5.2.5-(iv) since F satisfies degree equality (DE). However, when the two voter sets have distinct cardinalities, equality between the total Borda score of every alternative under the two profiles may occur although it is impossible that the total degree of every alternative is same under the two profiles. Second, we will show that the SCR depends on the total Borda scores in the right way, i.e., it chooses the alternatives with the maximal total Borda score.

As mentioned earlier, we take the Borda score vector $s^B = ((m-1), (m-3), \dots, -(m-3), -(m-1))$, i.e., $\sum_{k=1}^m s_k^B = 0$ and $s_k^B - s_{k+1}^B = 2$ for all $1 \leq k \leq m-1$.

Notice that when the Borda score vector is s^B , then for any $R \in W(A)$, we have $\sum_{x \in A} s(x, R) = 0$. So, for any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$, we have $\sum_{x \in A} S^B(x, R^N) = 0$.

Before proving that our SCR completely depends on the total Borda scores, we will have two lemmata. For any $x \in A$, let Ψ_x denote the set of all permutations $\tau : A \rightarrow A$ such that $\tau(x) = x$, i.e., x is kept fixed. While proving our lemmata, we will use the *amplification* approach of Hansson and Sahlquist [78], where given a preference profile $R^N \in W(A)^N$ for $N \in \mathcal{N}$, $\sum_{\tau \in \Psi_x} (R^N)^\tau = \mathcal{R}_x$ is called the amplification of R^N at $x \in A$. Since alternative x is fixed and there

are $(m-1)!$ permutations in Ψ_x , $D(x, \mathcal{R}_x) = (m-1)!D(x, R^N)$, and for any $y, z \in (A \setminus \{x\})$ we have $D(y, \mathcal{R}_x) = D(z, \mathcal{R}_x)$, i.e., differences in total degrees among other alternatives disappear under $\mathcal{R}_x$.

For any finite set of voters N and any $R^N \in W(A)^N$, let $\widehat{R}^N \in W(A)^N$ denote the preference profile obtained from R^N by reversing each voter's preferences.

Lemma 5.3.3. *Let F be an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE). For any $N \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $x \in A$, we have*

- (i) $F(R^N + \widehat{R}^N) = A$ and
- (ii) $F(\sum_{\tau \in \Psi_x} (R^N + \widehat{R}^N)^\tau) = A$.

Proof. Let F be an SCR that satisfies faithfulness (F), reinforcement (RE), and degree equality (DE). Let $N \in \mathcal{N}$ with $\#N = n$, $R^N \in W(A)^N$, and $x \in A$.

(i) For any $R \in W(A)$ and any $x \in A$, $d(x, R) + d(x, \widehat{R}) = m+1$. Thus, for any $x \in A$, $D(x, R^N) + D(x, \widehat{R}^N) = (m+1)n$. We consider the profile $\overline{R} \in W(A)$ under which all alternatives are indifferent among themselves:

$$\begin{array}{|c|} \hline \overline{R} \\ \hline A \\ \hline \end{array}$$

For each $x \in A$, $d(x, \overline{R}) = \frac{m+1}{2}$. We consider $2n$ copies of $\overline{R}$, denoted by $\overline{R}^{2n}$. For each $x \in A$, $D(x, \overline{R}^{2n}) = (m+1)n$. So, for all $x \in A$, $D(x, R^N + \widehat{R}^N) = D(x, \overline{R}^{2n}) = (m+1)n$. Then, by degree equality (DE), $F(R^N + \widehat{R}^N) = F(\overline{R}^{2n})$. Since F satisfies faithfulness (F) and reinforcement (RE), we have $F(\overline{R}^{2n}) = A$. Hence, $F(R^N + \widehat{R}^N) = A$.

(ii) We know that for any $x \in A$, $D(x, R^N + \widehat{R}^N) = (m+1)n$. We consider the amplification of $(R^N + \widehat{R}^N)$ at any $x \in A$, i.e., for any $x \in A$, we consider the profile $\sum_{\tau \in \Psi_x} (R^N + \widehat{R}^N)^\tau = \sum_{\tau \in \Psi_x} (R^N)^\tau + \sum_{\tau \in \Psi_x} (\widehat{R}^N)^\tau = \mathcal{R}_x + \widehat{\mathcal{R}}_x$.

We find the total degree of every alternative at $\mathcal{R}_x + \widehat{\mathcal{R}}_x$. For alternative x , we have $D(x, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = (m-1)!D(x, R^N + \widehat{R}^N) = (m-1)!(m+1)n$, and for

any $y \in (A \setminus \{x\})$, we have

$$\begin{aligned}
D(y, \mathcal{R}_x + \widehat{\mathcal{R}}_x) &= \left[\sum_{a \in A} D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) - D(x, \mathcal{R}_x + \widehat{\mathcal{R}}_x) \right] / (m-1) \\
&= \left[(m-1)!m(m+1)n - (m-1)!(m+1)n \right] / (m-1) \\
&= (m-2)!m(m+1)n - (m-2)!(m+1)n \\
&= (m-2)!(m+1)n(m-1) \\
&= (m-1)!(m+1)n.
\end{aligned}$$

Hence, for every $a \in A$, $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = (m-1)!(m+1)n$.

We again consider the profile $\bar{R} \in W(A)$ under which the degree of each $a \in A$ is $(m+1)/2$. We consider $2n(m-1)!$ copies of $\bar{R}$, denoted by $\bar{R}^{2n(m-1)!}$.¹⁴ For every $a \in A$, we have $D(a, \bar{R}^{2n(m-1)!}) = 2n(m-1)!\left(\frac{m+1}{2}\right) = (m-1)!(m+1)n$.

Hence, for every $a \in A$, $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = D(a, \bar{R}^{2n(m-1)!})$. Then, by degree equality (DE), $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = F(\bar{R}^{2n(m-1)!})$. Faithfulness (F) and reinforcement (RE) imply that $F(\bar{R}^{2n(m-1)!}) = A$. So, $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = A$. $\square$

Lemma 5.3.4. *Let F be an SCR that satisfies faithfulness (F), reinforcement (RE), and degree equality (DE). For any $N \in \mathcal{N}$ with $\#N = n$ and any $R^N \in W(A)^N$, if for each $x \in A$, $D(x, R^N) = \frac{(m+1)n}{2}$, then $F(R^N) = A$.*

Proof. Let F be an SCR that satisfies faithfulness (F), reinforcement (RE), and degree equality (DE). Let $N \in \mathcal{N}$ with $\#N = n$ and $R^N \in W(A)^N$ be such that for each $x \in A$, $D(x, R^N) = \frac{(m+1)n}{2}$. We will show that $F(R^N) = A$. We consider $2(m-1)!$ copies of R^N , denoted by $R^{2N(m-1)!}$. We have $F(R^N) = F(R^{2N(m-1)!})$ by reinforcement (RE), and for each $a \in A$, $D(a, R^{2N(m-1)!}) = 2(m-1)!\left(\frac{(m+1)n}{2}\right) = (m-1)!(m+1)n$.

We now consider the profile $\sum_{\tau \in \Psi_x} (R^N + \widehat{R}^N)^\tau = \sum_{\tau \in \Psi_x} (R^N)^\tau + \sum_{\tau \in \Psi_x} (\widehat{R}^N)^\tau = \mathcal{R}_x + \widehat{\mathcal{R}}_x$. We know from the proof of Lemma 5.3.3-(ii) that for each $a \in A$, $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = (m-1)!(m+1)n$.

¹⁴Since all alternatives are indifferent among themselves at the profile $\bar{R} \in W(A)$, $2n(m-1)!$ copies of $\bar{R}$ is equal to the amplification of $\bar{R}^{2n}$ at any $x \in A$, i.e., $\bar{R}^{2n(m-1)!} = \sum_{\tau \in \Psi_x} (\bar{R}^{2n})^\tau$.

Hence, for each $a \in A$, $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = D(a, R^{2N(m-1)!})$. Then, by degree equality (DE), we have $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = F(R^{2N(m-1)!})$. By Lemma 5.3.3-(ii), we also have $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = A$. So, $F(R^{2N(m-1)!}) = A = F(R^N)$. $\square$

We now show that our SCR completely depends on the total Borda scores.

Proposition 5.3.5. *Let F be an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE). For any $N, H \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $R^H \in W(A)^H$, if for each $x \in A$, $S^B(x, R^N) = S^B(x, R^H)$, then $F(R^N) = F(R^H)$.*

Proof. Let $N, H \in \mathcal{N}$ with $\#N = n$ and $\#H = h$, $R^N \in W(A)^N$, and $R^H \in W(A)^H$ be such that for each $x \in A$, $S^B(x, R^N) = S^B(x, R^H)$. We will show that $F(R^N) = F(R^H)$.

When $n = h$, then the result follows from Corollary 5.2.5-(iv) by degree equality (DE) (without the need of faithfulness (F) and reinforcement (RE)). So, we assume that $n \neq h$.

We know that for any $x \in A$, $S^B(x, R^N + \widehat{R}^N) = S^B(x, R^N) + S^B(x, \widehat{R}^N) = 0$.

Since for each $x \in A$, $S^B(x, R^N) = S^B(x, R^H)$, we have for each $x \in A$, $S^B(x, R^H) + S^B(x, \widehat{R}^N) = 0$. So, for each $x \in A$, $S^B(x, R^H + \widehat{R}^N) = 0$. Then, by Corollary 5.2.5-(ii), for each $x \in A$ we have $D(x, R^H + \widehat{R}^N) = [(n+h)(m+1)]/2$, where there are $n+h$ voters at the profile $R^H + \widehat{R}^N$. This fact, together with Lemma 5.3.4, implies that $F(\widehat{R}^N + R^H) = A$. We now have that

$$\begin{aligned}
F(R^N) &= F(R^N) \cap A \\
&= F(R^N) \cap \underbrace{F(\widehat{R}^N + R^H)}_A \\
&= F(R^N + \widehat{R}^N + R^H) \\
&= \underbrace{F(R^N + \widehat{R}^N)}_A \cap F(R^H) \\
&= A \cap F(R^H) \\
&= F(R^H),
\end{aligned}$$

where third and forth equalities follow from reinforcement (RE) and $F(R^N + \widehat{R}^N) = A$ by Lemma 5.3.3-(i). Hence, $F(R^N) = F(R^H)$. $\square$

Hence, an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE) completely depends on the total Borda scores. It is left to prove that an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE) depends on the total Borda scores in the right way.

By Corollary 5.2.5-(i), for any $N \in \mathcal{N}$, any $R^N \in W(A)^N$, and any $x, y \in A$, $S^B(x, R^N) > S^B(y, R^N)$ if and only if $D(x, R^N) < D(y, R^N)$, $S^B(x, R^N) = S^B(y, R^N)$ if and only if $D(x, R^N) = D(y, R^N)$, and also $S^B(x, R^N) < S^B(y, R^N)$ if and only if $D(x, R^N) > D(y, R^N)$. This result, together with Proposition 5.3.5, implies that F chooses alternatives with the maximal total Borda scores if and only if F chooses alternatives with the minimal total degrees. So, for any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$,

$$\begin{aligned} F(R^N) &= \{x \in A \mid \text{for each } y \in A, D(x, R^N) \leq D(y, R^N)\} \\ &= \{x \in A \mid \text{for each } y \in A, S^B(x, R^N) \geq S^B(y, R^N)\}. \end{aligned}$$

Hence, we will show that if an SCR F satisfies faithfulness (F), reinforcement (RE), and degree equality (DE), then for any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$, $F(R^N) = \{x \in A \mid \text{for each } y \in A, D(x, R^N) \leq D(y, R^N)\}$ in order to show that F depends on the total Borda scores in the right way.

Proposition 5.3.6. *Let F be an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE). Then, for any $N \in \mathcal{N}$ and any $R^N \in W(A)^N$,*

$$F(R^N) = \{x \in A \mid \text{for each } y \in A, D(x, R^N) \leq D(y, R^N)\}.$$

Proof. Let F be an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE). Let $N \in \mathcal{N}$ with $\#N = n$ and $R^N \in W(A)^N$. We partition the set of alternatives A as follows. Let $B_1, \dots, B_K$ be a partition of A ¹⁵ such that

¹⁵That is, for each $k \in \{1, \dots, K\}$, $B_k \neq \emptyset$; for each $k, l \in \{1, \dots, K\}$ with $k \neq l$, $B_k \cap B_l = \emptyset$; and $\cup_{k=1}^K B_k = A$.

- for any $k \in \{1, \dots, K\}$ and any $x, y \in B_k$, $D(x, R^N) = D(y, R^N)$, and
- for any $k, l \in \{1, \dots, K\}$, any $x \in B_k$, and any $y \in B_l$, $k < l$ implies $D(x, R^N) < D(y, R^N)$.

So, $B_1 = \{x \in A \mid \text{for each } y \in A, D(x, R^N) \leq D(y, R^N)\}$, and we will show that $F(R^N) = B_1$.

For any $k \in \{1, \dots, K\}$ and any $x \in B_k$, let $\#B_k = b_k$ and $D(x, R^{2N}) = t_k$, where R^{2N} denote two copies of R^N . Note that $t_1 b_1 + \dots + t_K b_K = \sum_{k=1}^K t_k b_k = \sum_{a \in A} D(a, R^{2N}) = \frac{m(m+1)(2n)}{2} = m(m+1)n$, and $b_1 + \dots + b_K = \sum_{k=1}^K b_k = m = \#A$.

We now consider the profile $\bar{R} \in W(A)$ under which all alternatives are indifferent among themselves, i.e.,

$\bar{R}$
A

For each $x \in A$, $d(x, \bar{R}) = \frac{m+1}{2}$. We consider $2mn$ copies of $\bar{R}$, denoted by $\bar{R}^{2mn}$. So, for each $x \in A$, $D(x, \bar{R}^{2mn}) = mn(m+1)$. Let R^{2mN} denote m copies of R^{2N} . We now consider the profile $R^{2mN} + \bar{R}^{2mn}$. Note that for any $k \in \{1, \dots, K\}$ and any $x \in B_k$, $D(x, R^{2mN} + \bar{R}^{2mn}) = mt_k + mn(m+1)$.

We now provide a profile, say R^* , such that total numbers of voters under profile R^* is equal to that of under profile $(R^{2mN} + \bar{R}^{2mn})$, and such that for each $x \in A$, $D(x, R^*) = D(x, R^{2mN} + \bar{R}^{2mn})$ and $F(R^*) = B_1$.

For any $k \in \{1, \dots, K\}$, let $R_k \in W(A)$ is a profile such that $\text{top}(R_k) = \bigcup_{k=1}^k B_k$ and for any $x, y \notin \text{top}(R_k)$, x and y are indifferent under R_k , i.e.,

R_k
$B_1, \dots, B_k$
$B_{k+1}, \dots, B_K$

For any $k \in \{1, \dots, K-1\}$ and any $R_k \in W(A)$, $x \in \text{top}(R_k)$ implies $d(x, R_k) = \frac{b_1 + \dots + b_k + 1}{2}$, and for any $y \notin \text{top}(R_k)$ we have $d(y, R_k) = \frac{b_1 + \dots + b_k + m + 1}{2} = d(x, R_k) + \frac{m}{2}$.

For every $x \in A$ we have $d(x, R_K) = \frac{b_1 + \dots + b_K + 1}{2} = \frac{m+1}{2}$.

For any $k \in \{1, \dots, K-1\}$, any $x \in B_k$, and any $y \in B_{k+1}$, let $\theta_k = D(y, R^{2N}) - D(x, R^{2N}) = t_{k+1} - t_k$, and let $\theta_K = 2mn + t_1 - t_K$.¹⁶

For any $k \in \{1, \dots, K\}$, let $R_k^{2\theta_k}$ denote $2\theta_k \in \mathbb{Z}_+$ copies of R_k . We now consider the profile $R^* = \sum_{k=1}^K R_k^{2\theta_k}$, i.e., R^* is as follows:¹⁷

$2\theta_1$ copies				$2\theta_k$ copies				$2\theta_K$ copies
R_1	+	...	+	R_k	+	...	+	R_K
B_1				$B_1, \dots, B_k$				$B_1, \dots, B_K$
$B_2, \dots, B_K$				$B_{k+1}, \dots, B_K$				

We first note that total numbers of voters under profiles R^* and $(R^{2mN} + \bar{R}^{2mn})$ are equal. That is, $2\theta_1 + \dots + 2\theta_{K-1} + 2\theta_K = 2[(t_2 - t_1) + (t_3 - t_2) + \dots + (t_K - t_{K-1})] + 2[2mn + t_1 - t_K] = 2[t_K - t_1] + 4mn + 2[t_1 - t_K] = 4mn$.

We now show that for each $k \in \{1, \dots, K\}$ and each $x \in B_k$, $D(x, R^*) = D(x, R^{2mN} + \bar{R}^{2mn}) = mt_k + mn(m+1)$.

¹⁶Note that for each $k \in \{1, \dots, K-1\}$, $\theta_k \in \mathbb{Z}_+$. We now show that $\theta_K \in \mathbb{Z}_+$. Since $2mn$, t_1 , and t_K are integer, θ_K is an integer. It remains to show that $\theta_K > 0$. Note that the maximal value of t_K is equal to $2mn$ at which every voter considers the same alternative as the worst alternative under R^N , and the minimal value of t_1 is equal to $2n$ at which every voter considers the same alternative as the best alternative under R^N . So, minimal value of θ_K is equal to $2mn + 2n - 2mn = 2n > 0$. Hence, $\theta_K \in \mathbb{Z}_+$.

¹⁷Profiles which are similar to R^* can be found in [79, 80, 81].

Let $x \in B_1$, now

$$\begin{aligned}
D(x, R^*) &= \sum_{k=1}^K D(x, R_k^{2\theta_k}) \\
&= 2\theta_1 D(x, R_1) + 2\theta_2 D(x, R_2) + \dots + 2\theta_k D(x, R_k) + 2\theta_{k+1} D(x, R_{k+1}) \\
&\quad + \dots + 2\theta_{K-2} D(x, R_{K-2}) + 2\theta_{K-1} D(x, R_{K-1}) + 2\theta_K D(x, R_K) \\
&= (t_2 - t_1)(b_1 + 1) + (t_3 - t_2)(b_1 + b_2 + 1) + \dots \\
&\quad + (t_{k+1} - t_k)(b_1 + \dots + b_k + 1) \\
&\quad + (t_{k+2} - t_{k+1})(b_1 + \dots + b_{k+1} + 1) + \dots \\
&\quad + (t_{K-1} - t_{K-2})(b_1 + \dots + b_{K-2} + 1) \\
&\quad + (t_K - t_{K-1})(b_1 + \dots + b_{K-1} + 1) \\
&\quad + \theta_K(b_1 + \dots + b_K + 1) \\
&= b_1[(t_2 - t_1) + (t_3 - t_2) + \dots + (t_{k+1} - t_k) + (t_{k+2} - t_{k+1}) + \dots \\
&\quad + (t_{K-1} - t_{K-2}) + (t_K - t_{K-1}) + \theta_K] \\
&\quad + b_2[(t_3 - t_2) + \dots + (t_{k+1} - t_k) + (t_{k+2} - t_{k+1}) + \dots \\
&\quad + (t_{K-1} - t_{K-2}) + (t_K - t_{K-1}) + \theta_K] + \dots \\
&\quad + b_k[(t_{k+1} - t_k) + (t_{k+2} - t_{k+1}) + \dots \\
&\quad + (t_{K-1} - t_{K-2}) + (t_K - t_{K-1}) + \theta_K] + \dots \\
&\quad + b_{K-2}[(t_{K-1} - t_{K-2}) + (t_K - t_{K-1}) + \theta_K] \\
&\quad + b_{K-1}[(t_K - t_{K-1}) + \theta_K] + b_K \theta_K \\
&\quad + [(t_2 - t_1) + (t_3 - t_2) + \dots + (t_{K-1} - t_{K-2}) + (t_K - t_{K-1}) + \theta_K] \\
&= b_1[t_K - t_1 + \theta_K] + b_2[t_K - t_2 + \theta_K] + \dots + b_k[t_K - t_k + \theta_K] + \dots \\
&\quad + b_{K-2}[t_K - t_{K-2} + \theta_K] + b_{K-1}[t_K - t_{K-1} + \theta_K] \\
&\quad + b_K \theta_K + [(t_K - t_1) + \theta_K].
\end{aligned}$$

$$\begin{aligned}
D(x, R^*) &= t_K \underbrace{[b_1 + b_2 + \dots + b_k + \dots + b_{K-2} + b_{K-1}]}_{m-b_K} \\
&\quad - \underbrace{[b_1 t_1 + b_2 t_2 + \dots + b_k t_k + \dots + b_{K-2} t_{K-2} + b_{K-1} t_{K-1}]}_{m(m+1)n-t_K b_K} \\
&\quad + \theta_K \underbrace{[b_1 + b_2 + \dots + b_k + \dots + b_{K-1} + b_K]}_m + (t_K - t_1) + \theta_K \\
&= m t_K - t_K b_K - (m(m+1)n) + t_K b_K + m \theta_K + (t_K - t_1) + \theta_K.
\end{aligned}$$

We now continue using $\theta_K = 2mn + t_1 - t_K$ on the last equation.

$$\begin{aligned}
D(x, R^*) &= mt_K - (m(m+1)n) + m\theta_K + (t_K - t_1) + \theta_K \\
&= mt_K - (m(m+1)n) + m[2mn + t_1 - t_K] \\
&\quad + (t_K - t_1) + [2mn + t_1 - t_K] \\
&= mt_1 + (mn)[2 + 2m - (m+1)] \\
&= mt_1 + mn(m+1) \\
&= D(x, R^{2mN} + \bar{R}^{2mn}).
\end{aligned}$$

Now, let $x \in B_1$ and $y \in B_2$,

$$\begin{aligned}
D(y, R^*) &= D(x, R^*) - 2\theta_1 D(x, R_1) + 2\theta_1 D(y, R_1) \\
&= D(x, R^*) - 2\theta_1 \left[\frac{b_1 + 1}{2} \right] + 2\theta_1 \left[\frac{(b_1 + 1) + m}{2} \right] \\
&= D(x, R^*) + m\theta_1 \\
&= mt_1 + mn(m+1) + m(t_2 - t_1) \\
&= mt_2 + mn(m+1) \\
&= D(y, R^{2mN} + \bar{R}^{2mn}).
\end{aligned}$$

In a similar way, by recursion, for any $k \in \{2, \dots, K-1\}$, let $x \in B_k$ and $y \in B_{k+1}$, now

$$\begin{aligned}
D(y, R^*) &= D(x, R^*) - 2\theta_k D(x, R_k) + 2\theta_k D(y, R_k) \\
&= D(x, R^*) - 2\theta_k \left[\frac{b_1 + \dots + b_k + 1}{2} \right] + 2\theta_k \left[\frac{(b_1 + \dots + b_k + 1) + m}{2} \right] \\
&= D(x, R^*) + m\theta_k.
\end{aligned}$$

$$\begin{aligned}
D(y, R^*) &= mt_k + mn(m+1) + m(t_{k+1} - t_k) \\
&= mt_{k+1} + mn(m+1) \\
&= D(y, R^{2mN} + \bar{R}^{2mn}).
\end{aligned}$$

We have shown that for each $x \in A$, $D(x, R^{2mN} + \bar{R}^{2mn}) = D(x, R^*)$. So, by degree equality (DE), $F(x, R^{2mN} + \bar{R}^{2mn}) = F(R^*)$. For any $k \in \{1, \dots, K\}$,

by faithfulness (F) and reinforcement (RE), we have $F(R_k^{2\theta_k}) = \bigcup_{k=1}^K B_k$. Since $\bigcap_{k=1}^K F(R_k^{2\theta_k}) \neq \emptyset$, then by reinforcement (RE), we have $F(R^*) = F(\sum_{k=1}^K R_k^{2\theta_k}) = \bigcap_{k=1}^K F(R_k^{2\theta_k}) = B_1$. So, $F(R^{2mN} + \bar{R}^{2mn}) = B_1$. By faithfulness (F) and reinforcement (RE), $F(\bar{R}^{2mn}) = A$. Thus, $F(R^{2mN} + \bar{R}^{2mn}) = F(R^{2mN}) \cap A = F(R^{2mN}) = B_1$. By reinforcement (RE), we have $F(R^{2mN}) = F(R^N)$, therefore $F(R^N) = B_1$, which completes the proof. $\square$

By Propositions 5.3.5 and 5.3.6, an SCR satisfying faithfulness (F), reinforcement (RE), and degree equality (DE) is the Borda rule, completing the proof of Theorem 5.3.2.

For the independence of the axioms that are used in Theorem 5.3.2, we have following examples.

Remark 5.3.7. (1) Faithfulness (F)

For each $N \in \mathcal{N}$ and each $R^N \in W(A)^N$, we define $\hat{F}(R^N) = A$, i.e., $\hat{F}$ always chooses the set of all alternatives. It is clear that $\hat{F}$ satisfies reinforcement (RE) and degree equality (DE), but it violates faithfulness (F).

(2) Reinforcement (RE)

We define $\tilde{F}$ as follows:

For $\#N = 1$, $\tilde{F}(R) = \text{top}(R)$ for all $R \in W(A)$,

for $\#N \geq 2$, $\tilde{F}(R^N) = A$ for all $R^N \in W(A)^N$.

It is obvious that $\tilde{F}$ satisfies faithfulness (F) and degree equality (DE). However, $\tilde{F}$ violates reinforcement (RE).

(3) Degree equality (DE)

The plurality rule is a scoring rule defined by the scoring vector $(1, 0, \dots, 0)$. The plurality rule satisfies faithfulness (F) and reinforcement (RE). However, it does not satisfy degree equality (DE) by Proposition 5.3.1.

5.4 A cancelation property

In this section, we introduce a new cancelation property and show that the Borda rule is the unique scoring rule which satisfies this property.

For any positive integer h , $1 \leq h \leq m$, let $r_h(R_i)$ denote the h^{th} level best alternatives at R_i , that is, $r_1(R_i) = \{x \in A \mid \text{for each } y \in A, d(x, R_i) \leq d(y, R_i)\}$, and recursively for each $h > 1$, $r_h(R_i)$ is defined as follows, $r_h(R_i) = \{x \in A \mid \text{for each } y \in (A \setminus \cup_{h=1}^{h-1} r_h(R_i)), d(x, R_i) \leq d(y, R_i)\}$.¹⁸

Let $R^N \in W(A)^N$ be a profile such that there exist $i, j \in N$ and $\alpha, \beta \in \{1, \dots, m-1\}$ such that $r_\alpha(R_i) = r_{\beta+1}(R_j)$ and $r_{\alpha+1}(R_i) = r_\beta(R_j)$, i.e.,

$R^N :$

	R_1	$\dots$	R_i	$\dots$	R_j	$\dots$	R_n
r_1	$r_1(R_1)$	$\dots$	$r_1(R_i)$	$\dots$	$r_1(R_j)$	$\dots$	$r_1(R_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r_α	$r_\alpha(R_1)$	$\dots$	$r_\alpha(R_i) = r_{\beta+1}(R_j)$	$\dots$	$r_\alpha(R_j)$	$\dots$	$r_\alpha(R_n)$
$r_{\alpha+1}$	$r_{\alpha+1}(R_1)$	$\dots$	$r_{\alpha+1}(R_i) = r_\beta(R_j)$	$\dots$	$r_{\alpha+1}(R_j)$	$\dots$	$r_{\alpha+1}(R_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r_β	$r_\beta(R_1)$	$\dots$	$r_\beta(R_i)$	$\dots$	$r_\beta(R_j) = r_{\alpha+1}(R_i)$	$\dots$	$r_\beta(R_n)$
$r_{\beta+1}$	$r_{\beta+1}(R_1)$	$\dots$	$r_{\beta+1}(R_i)$	$\dots$	$r_{\beta+1}(R_j) = r_\alpha(R_i)$	$\dots$	$r_{\beta+1}(R_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now, we derive $\dot{R}^N$ from R^N as follows:

- for all voters $l \in (N \setminus \{i, j\})$, $\dot{R}_l = R_l$,

¹⁸For each R_i , any $1 \leq h \leq m$ and each alternative $x \in r_h(R_i)$, an alternative way to determine the score of x at R_i is as follows:

$$s(x, R_i) = \frac{s_{\#SU(x, R_i)+1} + \dots + s_{\#SU(x, R_i)+\#r_h(R_i)}}{\#r_h(R_i)} = \frac{\sum_{k=\#SU(x, R_i)+1}^{k=\#SU(x, R_i)+\#r_h(R_i)} s_k}{\#r_h(R_i)}.$$

- for voter i , $r_\alpha(\dot{R}_i) = r_\alpha(R_i) \cup r_{\alpha+1}(R_i)$,
for all $h < \alpha$, $r_h(\dot{R}_i) = r_h(R_i)$,
for all $h > \alpha + 1$, $r_h(\dot{R}_i) = r_{h+1}(R_i)$,
- for voter j , $r_\beta(\dot{R}_j) = r_\beta(R_j) \cup r_{\beta+1}(R_j)$,
for all $h < \beta$, $r_h(\dot{R}_j) = r_h(R_j)$,
for all $h > \beta + 1$, $r_h(\dot{R}_j) = r_{h+1}(R_j)$,

i.e., $\dot{R}^N$:

	$\forall l \in (N \setminus \{i, j\}) : \dot{R}_l = R_l$	$\dot{R}_i$	$\dot{R}_j$
r_1	$r_1(\dot{R}_l) = r_1(R_l)$	$r_1(\dot{R}_i) = r_1(R_i)$	$r_1(\dot{R}_j) = r_1(R_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{\alpha-1}$	$r_{\alpha-1}(\dot{R}_l) = r_{\alpha-1}(R_l)$	$r_{\alpha-1}(\dot{R}_i) = r_{\alpha-1}(R_i)$	$r_{\alpha-1}(\dot{R}_j) = r_{\alpha-1}(R_j)$
r_α	$r_\alpha(\dot{R}_l) = r_\alpha(R_l)$	$r_\alpha(\dot{R}_i) = r_\alpha(R_i) \cup r_{\alpha+1}(R_i)$	$r_\alpha(\dot{R}_j) = r_\alpha(R_j)$
$r_{\alpha+1}$	$r_{\alpha+1}(\dot{R}_l) = r_{\alpha+1}(R_l)$	$r_{\alpha+1}(\dot{R}_i) = r_{\alpha+2}(R_i)$	$r_{\alpha+1}(\dot{R}_j) = r_{\alpha+1}(R_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{\beta-1}$	$r_{\beta-1}(\dot{R}_l) = r_{\beta-1}(R_l)$	$r_{\beta-1}(\dot{R}_i) = r_{\beta-1}(R_i)$	$r_{\beta-1}(\dot{R}_j) = r_{\beta-1}(R_j)$
r_β	$r_\beta(\dot{R}_l) = r_\beta(R_l)$	$r_\beta(\dot{R}_i) = r_{\beta+1}(R_i)$	$r_\beta(\dot{R}_j) = r_\beta(R_j) \cup r_{\beta+1}(R_j)$
$r_{\beta+1}$	$r_{\beta+1}(\dot{R}_l) = r_{\beta+1}(R_l)$	$r_{\beta+1}(\dot{R}_i) = r_{\beta+2}(R_i)$	$r_{\beta+1}(\dot{R}_j) = r_{\beta+2}(R_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Given any profile $R^N \in W(A)^N$, let $\dot{\mathcal{R}}(R^N)$ denote the set of all profiles which are derived from R^N for any $i, j \in N$ and any $\alpha, \beta \in \{1, \dots, m-1\}$ as defined above.

Definition 5.4.1. We say that an SCR F satisfies the *cancellation property* (CA) if for each $N \in \mathcal{N}$, each $R^N \in W(A)^N$ and each $\dot{R}^N \in \dot{\mathcal{R}}(R^N)$, we have $F(R^N) = F(\dot{R}^N)$.

We now show that the cancellation property (CA) characterizes the Borda rule among all scoring rules.

Proposition 5.4.2. *A scoring rule satisfies the cancellation property (CA) if and only if it is the Borda rule.*

Proof. It is clear that the Borda rule satisfies the cancelation property (CA). For the other part of the proof, let F be a scoring rule which satisfies the cancelation property (CA). Suppose that F is not the Borda rule.

Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$. We consider following profile R^N :

R_1	R_2	R_3
a	c	a, b
b	b	c
c	a	

We now consider $\acute{R}^N \in \mathcal{R}(R^N)$ for voters 1 and 2, and $\alpha = 1, \beta = 2$, i.e., $\acute{R}^N$ is as follows:

$\acute{R}_1$	$\acute{R}_2$	$\acute{R}_3 = R_3$
a, b	c	a, b
c	a, b	c

Since F is a scoring rule satisfying cancelation property (CA), we have $F(R^N) = F(\acute{R}^N)$. Let $s = (s_1, s_2, s_3)$, where $s_1 \geq s_2 \geq s_3$ and $s_1 > s_3$. Since we supposed that F is not the Borda rule, we have $s_1 - s_2 \neq s_2 - s_3$. Let $t_{12} = s_1 - s_2$ and $t_{23} = s_2 - s_3$. So, $s_1 = s_3 + t_{12} + t_{23}$ and $s_2 = s_3 + t_{23}$.

We calculate the total score of each alternative at R^N and $\acute{R}^N$:

$$\begin{aligned}
S(a, R^N) &= s_1 + s_3 + \frac{s_1 + s_2}{2} = 3s_3 + \frac{3}{2}t_{12} + 2t_{23}, \\
S(b, R^N) &= 2s_2 + \frac{s_1 + s_2}{2} = 3s_3 + \frac{1}{2}t_{12} + 3t_{23}, \\
S(c, R^N) &= s_1 + 2s_3 = 3s_3 + t_{12} + t_{23}, \\
S(a, \acute{R}^N) &= 2\left(\frac{s_1 + s_2}{2}\right) + \frac{s_2 + s_3}{2} = 3s_3 + t_{12} + \frac{5}{2}t_{23}, \\
S(b, \acute{R}^N) &= 2\left(\frac{s_1 + s_2}{2}\right) + \frac{s_1 + s_3}{2} = 3s_3 + t_{12} + \frac{5}{2}t_{23}, \\
S(c, \acute{R}^N) &= 2s_3 + s_1 = 3s_3 + t_{12} + t_{23}.
\end{aligned}$$

Note that $S(a, \acute{R}^N) = S(b, \acute{R}^N)$. We will now show that $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$. Since F is a scoring rule, we do not have $t_{12} = t_{23} = 0$. If $t_{12} > 0$ and $t_{23} > 0$, then we have $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$. If $t_{12} = 0$ and $t_{23} > 0$, then again we have $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$. If

$t_{12} > 0$ and $t_{23} = 0$, then we have $S(a, \acute{R}^N) = S(b, \acute{R}^N) = S(c, \acute{R}^N)$ which implies that $F(\acute{R}^N) = \{a, b, c\}$. Hence, we have $F(R^N) = \{a, b, c\}$. Therefore, $S(a, R^N) = S(b, R^N) = S(c, R^N)$ implying that $t_{12} = t_{23} = 0$, a contradiction. Hence, $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$.

The fact that $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$, together with F being a scoring rule, implies that $F(\acute{R}^N) = \{a, b\}$. So, $F(R^N) = \{a, b\}$. Hence, we have $S(a, R^N) = S(b, R^N)$ which implies that $t_{12} = t_{23}$, a contradiction. Hence, F is the Borda rule. $\square$

Propositions 5.3.1 and 5.4.2 imply that when we restrict ourselves to scoring rules degree equality (DE) is equivalent to cancelation property (CA). However, in general, degree equality (DE) is stronger than cancelation property (CA).

Proposition 5.4.3. (i) *If an SCR satisfies degree equality (DE) then it also satisfies the cancelation property (CA).*

(ii) *There exists an SCR which satisfies the cancelation property (CA) but violates degree equality (DE).*

Proof. (i) Let F be an SCR which satisfies degree equality (DE). We will show that F satisfies cancelation property (CA).

Let $R^N \in W(A)^N$ be such that there exist $i, j \in N$ and $\alpha, \beta \in \{1, \dots, m-1\}$ such that $r_\alpha(R_i) = r_{\beta+1}(R_j)$ and $r_{\alpha+1}(R_i) = r_\beta(R_j)$. Let $\acute{R}^N$ (derived from R^N) be as follows:

- for all voters $l \in (N \setminus \{i, j\})$, $\acute{R}_l = R_l$,
- for voter i , $r_\alpha(\acute{R}_i) = r_\alpha(R_i) \cup r_{\alpha+1}(R_i)$,
for all $h < \alpha$, $r_h(\acute{R}_i) = r_h(R_i)$,
for all $h > \alpha + 1$, $r_h(\acute{R}_i) = r_{h+1}(R_i)$,
- for voter j , $r_\beta(\acute{R}_j) = r_\beta(R_j) \cup r_{\beta+1}(R_j)$,
for all $h < \beta$, $r_h(\acute{R}_j) = r_h(R_j)$,
for all $h > \beta + 1$, $r_h(\acute{R}_j) = r_{h+1}(R_j)$.

In order to show that F satisfies cancelation property (CA), we need to show that $F(R^N) = F(\acute{R}^N)$. It is enough to show that for each $a \in A$ we have $D(a, R^N) = D(a, \acute{R}^N)$, since F satisfies degree equality (DE).

Let $x \in r_\alpha(R_i) = r_{\beta+1}(R_j)$ and $y \in r_{\alpha+1}(R_i) = r_\beta(R_j)$.

Since $\acute{R}_l = R_l$ for all $l \in (N \setminus \{i, j\})$, for each $a \in A$ we have $D(a, R^{N \setminus \{i, j\}}) = D(a, \acute{R}^{N \setminus \{i, j\}})$. So, we will show followings:

- 1- $d(x, R_i) + d(x, R_j) = d(x, \acute{R}_i) + d(x, \acute{R}_j)$,
- 2- $d(y, R_i) + d(y, R_j) = d(y, \acute{R}_i) + d(y, \acute{R}_j)$, and
- 3- for each $z \in [A \setminus (r_\alpha(R_i) \cup r_{\alpha+1}(R_i))]$, $d(z, R_i) + d(z, R_j) = d(z, \acute{R}_i) + d(z, \acute{R}_j)$.

For voter i , we have $SU(x, R_i) = SU(x, \acute{R}_i)$, $U(x, \acute{R}_i) = U(x, R_i) \cup r_{\alpha+1}(R_i)$ and $U(x, R_i) \cap r_{\alpha+1}(R_i) = \emptyset$. For voter j , we have $U(x, R_j) = U(x, \acute{R}_j)$, $SU(x, R_j) = SU(x, \acute{R}_j) \cup r_\beta(R_j)$ and $SU(x, \acute{R}_j) \cap r_\beta(R_j) = \emptyset$. Now,

$$\begin{aligned}
d(x, R_i) + d(x, R_j) &= \left(\underbrace{\#SU(x, R_i)}_{\#SU(x, \acute{R}_i)} + \underbrace{\#U(x, R_i)}_{\#U(x, \acute{R}_i) - \#r_{\alpha+1}(R_i)} + 1 \right) / 2 \\
&\quad + \left(\underbrace{\#SU(x, R_j)}_{\#SU(x, \acute{R}_j) + \#r_\beta(R_j)} + \underbrace{\#U(x, R_j)}_{\#U(x, \acute{R}_j)} + 1 \right) / 2 \\
&= [\#SU(x, \acute{R}_i) + \#U(x, \acute{R}_i) + 1] / 2 \\
&\quad + [\#SU(x, \acute{R}_j) + \#U(x, \acute{R}_j) + 1] / 2 \\
&= d(x, \acute{R}_i) + d(x, \acute{R}_j).
\end{aligned}$$

For voter i , we have $U(y, R_i) = U(y, \acute{R}_i)$, $SU(y, R_i) = SU(y, \acute{R}_i) \cup r_\alpha(R_i)$ and $SU(y, \acute{R}_i) \cap r_\alpha(R_i) = \emptyset$. For voter j , we have $SU(y, R_j) = SU(y, \acute{R}_j)$,

$U(y, \acute{R}_j) = U(y, R_j) \cup r_{\beta+1}(R_j)$ and $U(y, R_j) \cap r_{\beta+1}(R_j) = \emptyset$. So,

$$\begin{aligned}
d(y, R_i) + d(y, R_j) &= (\underbrace{\#SU(y, R_i)}_{\#SU(y, \acute{R}_i) + \#r_\alpha(R_i)} + \underbrace{\#U(y, R_i) + 1}_{\#U(y, \acute{R}_i)}) / 2 \\
&\quad + (\underbrace{\#SU(y, R_j)}_{\#SU(y, \acute{R}_j)} + \underbrace{\#U(y, R_j)}_{\#U(y, \acute{R}_j) - \#r_{\beta+1}(R_j)} + 1) / 2 \\
&= [\#SU(y, \acute{R}_i) + \#U(y, \acute{R}_i) + 1] / 2 \\
&\quad + [\#SU(y, \acute{R}_j) + \#U(y, \acute{R}_j) + 1] / 2 \\
&= d(y, \acute{R}_i) + d(y, \acute{R}_j).
\end{aligned}$$

Notice that for each $z \in [A \setminus (r_\alpha(R_i) \cup r_{\alpha+1}(R_i))]$, we have $U(z, R_i) = U(z, \acute{R}_i)$, $SU(z, R_i) = SU(z, \acute{R}_i)$, $U(z, R_j) = U(z, \acute{R}_j)$ and $SU(z, R_j) = SU(z, \acute{R}_j)$. So, for each $z \in [A \setminus (r_\alpha(R_i) \cup r_{\alpha+1}(R_i))]$ we have $d(z, R_i) + d(z, R_j) = d(z, \acute{R}_i) + d(z, \acute{R}_j)$.

So, for each $a \in A$ we have $D(a, R^N) = D(a, \acute{R}^N)$. Then, we have $F(R^N) = F(\acute{R}^N)$ by degree equality (DE). Hence, F satisfies cancelation property (CA).

(ii) We now provide an SCR which satisfies cancelation property (CA) but violates degree equality (DE).

Let $A = \{a, b, c\}$. The set of all weak preference orderings over A is given below.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}
a, b, c	a	a	b	b	c	c	a	b	c	a, b	a, c	b, c
	b	c	a	c	a	b	b, c	a, c	a, b	c	b	a
	c	b	c	a	b	a						

We define $\acute{F}$ as follows:

- for $\#N = 1$, for each $k \in \{1, \dots, 13\}$, $\acute{F}(R_k) = \text{top}(R_k)$,
- for $\#N = 2$, $\acute{F}(R_2 + R_5) = \{c\}$, $\acute{F}(R_1 + R_4) = \{b\}$, and for any other profile $(R_j + R_k)$ we define $\acute{F}(R_j + R_k) = A$,

- for $\#N \geq 3$, for all $R^N \in W(A)^N$, $\hat{F}(R^N) = A$.

It is clear that $\hat{F}$ satisfies cancelation property (CA). For each $x \in A$ we have $D(x, R_2 + R_5) = D(x, R_1 + R_4)$, however $\hat{F}(R_2 + R_5) \neq \hat{F}(R_1 + R_4)$. Hence, $\hat{F}$ violates degree equality (DE). $\square$

Finally, we study the relation between cancelation property (CA) and Young's cancelation property (Y-Ca).

Proposition 5.4.4. *Cancelation (CA) and Young's cancelation (Y-Ca) properties are independent of each other.*

Proof. The constant social choice rule $\bar{F}$ satisfies cancelation (CA), but it violates Young's cancelation property (Y-Ca).

We now provide an SCR which satisfies Young's cancelation property (Y-Ca) but it violates cancelation (CA).

Let $A = \{a, b, c\}$. The set of all weak preference orderings over A is given below.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}
a, b, c	a	a	b	b	c	c	a	b	c	a, b	a, c	b, c
	b	c	a	c	a	b	b, c	a, c	a, b	c	b	a
	c	b	c	a	b	a						

We define $\hat{F}$ as follows:

- for $\#N = 1$, $\hat{F}(R_k) = \text{top}(R_k)$ for all $k \in \{1, \dots, 13\}$,
- for $\#N = 2$, $\hat{F}(R_2 + R_3) = \{a\}$, $\hat{F}(R_8 + R_8) = A$, and for any other profile $(R_j + R_k)$ we have $\hat{F}(R_j + R_k) = A$,
- for $\#N \geq 3$, $\hat{F}(R^N) = A$ for all $R^N \in W(A)^N$.

It is clear that $\widehat{F}$ satisfies Young's cancelation property (Y-Ca). Note that $(R_8 + R_8) \in \mathcal{R}(R_2 + R_3)$, i.e., $(R_8 + R_8)$ can be derived from $(R_2 + R_3)$ for $\alpha = 2$ and $\beta = 3$. However, $\widehat{F}(R_2 + R_3) \neq \widehat{F}(R_8 + R_8)$. Hence, $\widehat{F}$ violates cancelation (CA). $\square$

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