THE COAUTHORS' PROBLEM REVISITED: FROM NETWORKS TO COVERS

A Master's Thesis

by
AYŞE GÜL MERMER

Department of
Economics
Bilkent University
Ankara
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The Institute of Economics and Social Sciences
of
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by

AYŞE GÜL MERMER

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray

Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Tarık Kara

Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Ali Doğanaksoy

Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Dr. Erdal Erel

Director

ABSTRACT

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MERMER Ayşe Gül

M.A., Department of Economics

Supervisor: Prof. Semih Koray

January 2010

In this thesis, we reexamine the Coauthors' Problem, introduced by Jackson and Wolinsky, 1996. We propose the Extended Coauthor Model using the cover notion, allowing for multilateral links among authors. We study the model under two utility functions, which are the extreme members of the class of utility functions induced by different synergy terms. We find the structure of the efficient and the link-wise stable covers formed under different utility functions, which depend on the synergy term under consideration. Moreover we introduce the core and core stability concepts for covers and investigate the properties possessed by core-stable covers. We find the relationship between the allocation induced by core-stable covers and the allocations in core, under player based flexible allocation rule. Finally we investigate the endogenous cover formation via a strategic form game, called the hyper-link formation game. We define Nash stability and Strong Nash stability for covers and study the characteristics of such covers.

Keywords: Coauthors' Problem, Covers, Efficieny, Stability, Core, Hyperlink Formation Game.

ÖZET

ORTAK YAZAR MODELİNİN GENELLEŞTİRİLMESİ : AĞLARDAN ÖRTÜLERE

MERMER Ayşe Gül

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Semih Koray

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Bu tez çalışmamızda, daha önce Jackson ve Wolinsky, 1996, tarafından önerilmiş olan Ortak Yazar Modelinini yeniden inceliyoruz. Örtü kavramı ile çoklu bağlara izin vererek Genelleştirilmiş Ortak Yazar Modelini sunuyoruz. Modeli sinerji ailesinin iki uç üyesi olan farklı sinerji terimlerini kullandığımız fayda fonksiyonları altında inceliyoruz. Verimli ve bağa göre kararlı olan örtülerin yapılarının sinerji terimine bağlı olarak değiştiğini buluyoruz. Daha sonra, örtüler için çekirdek ve çekirdek kararlılık kavramlarını tanımlıyor ve çekirdek kararlı yapıların özelliklerini inceliyoruz. Oyuncu merkezli esnek dağıtım kuralı altında çekirdek ve çekirdek kararlılık kavramları arasındaki ilişkiyi buluyoruz. Son olarak, içsel örtü oluşumunu hiperkenar oluşturma oyunu olarak adlandırdığımız stratejik oyun vasıtası ile inceliyoruz. Örtüler için Nash kararlılık ve Güçlü Nash kararlılık kavramlarını tanımlıyor ve bu kararlılık yapısına sahip örtülerin özelliklerini buluyoruz.

Anahtar Kelimeler: Ortak Yazar Problemi, Örtü, Verimlilik, Kararlılık, Çekirdek, Hiperkenar Oluşturma Oyunu.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Network Theory is one of the main tools which is used to model the relationships between individuals in many economic and social frameworks. These relationships play a critical role in a variety of contexts like information sharing, trade agreements, treaties among nations and political unions. For example the trade agreements among nations can be modelled using network structure by representing the countries as individuals and the trade agreements as the links between them. One well-known example modelling the research collaborations among authors is the "Co-author Model", due to Jackson and Wolinsky, (1996). In this model authors are represented by the nodes in a network, the collaboration between them is represented by the links in the network. The utility of an author is formulated using the papers he is included in and the papers that his co-authors are included in. In this framework, the individual utilities of the authors correspond to the individual productivities of the authors. Jackson and Wolinsky, (1996), examined the efficiency and pair-wise stability of the networks formed under the proposed productivity, or the utility, function.

One of the main assumptions in network theory is that relationships are formed bilaterally, which are represented as a link between two agents. However relationships between agents are not necessarily bilateral always, namely as agreements among nations such as European Union or G-8, or the joint work of people as co-authorships. In fact, the size of the coauthorships changes depending on the area under study. The average number of coauthorships in Mathematics and Economics is 2, in Physics 5, whereas in Medical Sciences the number drastically increases to 50.

In this study, we use the *cover structure*, introduced by Koray, to model these multilateral collaborations between co-authors. We propose several utility functions to model the co-authorships in The Extended Co-author Model. We examined mainly two synergy terms, two extreme members of the whole synergy family, in defining the utility function. The utility function of an author depends on the number of papers he is included in, the number of co-authors included in each paper, and the number of papers that his co-authors included in. We investigate the efficiency and the stability of the covers in this setting under different productivity functions. We ask the question that whether the efficient and the stable cover structures are the ones from whose restrictions to bilateral links one gets the efficient and the stable networks in The Co-author Model. We obtain different results from those in the Co-author Model. Jackson, (2002), showed the existence of a pairwise stable network under the Myerson allocation rule and any value function. We extend this result to covers, namely show the existence of link-wise stable covers under Myerson allocation rule and any value function. Here the role of Myerson allocation rule is crucial in the sense that the existence of link-wise stable covers is not necessarily true under different allocation rules.

We examine the core and the core-stability concepts for covers. We extended the Player Based Flexible Allocation Rule for networks, due to Jackson,(2003), to covers. The link-wise stability, which is the counterpart of pairwise stability in networks, allows only one hyper-link addition or destruction in one step . We define the core of $C \in \mathcal{C}^N$ and the value function for the covers, which allows several hyper-link additions and destructions in one step, and show that the player based flexible cover allocation rule belongs to the core. Differently from the cooperative game theory results, we show that the the allocation induced by the Myerson allocation rule, which is the counterpart of the Shapley value, is not necessarily a member of the core relative to a convex value function. We define the core-stability for covers allowing for addition or destruction of several hyper-links in one step, which is a more flexible stability notion compared to the core concept, in the sense that it allows the agents outside the coalition under consideration to form hyper-links also. We show the equivalence of efficiency of a cover with respect to any value function and the core stability of a cover with respect to any value function under the player based flexible cover allocation rule. We show the relationship between the core-stability and core under the player based flexible cover allocation rule and any convex value function. Besides taking the cover structures as given, we define the strategic link formation game for covers, namely the hyper-link formation game, and examine the covers induced by Strong Nash equilibrium of this game. We show that the cover induced by Strong Nash Equilibrium of the hyper-link formation game for any convex value function and the player based flexible cover allocation rule belongs to the core of $C \in \mathcal{C}^N$ and the convex value function.

CHAPTER 2

DEFINITIONS AND NOTATIONS

2.1 The Model

In many economic frameworks individuals form relationships from which they benefit. These relationships play a critical role in a variety of contexts like information sharing, trade agreements, treaties among nations and political unions. Network structure is a well known tool to model these relationships. One of the main assumptions in network structure is that relationships are formed bilaterally, which are represented as a link between two agents. However relationships between agents are not necessarily bilateral always, namely as agreements among nations such as European Union or G-8, joint degree programs among collages as exchange programs, joint work of people as co-authorships and so on. We use the model, the *cover structure*, introduced by Koray, to capture these relationships whose restriction to bilateral relationships is a network structure. We call these multi-agent relationships *hyper-links*, whose counterpart in network structure are links. Then, the *cover structure* can be seen as a hyper-graph which consists of the vertices, representing the agents, and the hyper-links, representing the relationships between agents. We also make the assumption that, if a group of agents form a hyper-link, then agents included in that group can not form another hyper-link among themselves. That is, if a group of nations are agreed to have a trade alliance, then sub-groups are not allowed to make an alliance among themselves. However, possible overlapping of agents in different relationships are allowed. That is a country in a trade alliance among the nations in Europe may also be in some other trade alliance in Asia. One possible scenario to motivate our model is on trade agreements. Assume that there is good to be traded among nations, which can be produced in every country identically with the same cost and can be sold at the same price in each country. A group of nations make an alliance if they want to trade this identical good within the group. Say countries X, Y, and Z agree upon this alliance. Then the good produced in country X can be traded to country Y and Z. Say country X also wants to make an alliance with the counties K and L, and K and L agree, however Y and Z does not. Then the group X, K and L make a new alliance among themselves. Being the alliance among the countries X, Y, and Z present, Y and Z do not make a new alliance as the quality, cost and price of the good is identical. Any country which is not in an alliance can only produce and sell the good in his own borders. This is interpreted as his only relation is the trivial one, namely with himself. Notice that the union of all relationships gives the set of countries. A very similar scenario can be considered for exchange program agreements among collages, with the assumption that the cost and benefit of having an exchange of students within each collage is identical.

2.2 Definitions and Notations

Let N be a finite non-empty set of agents that will be fixed throughout the paper. First we define the *hyper-link* concept :

Definition 1. Given the set of players N, a hyper-link E is an element of $2^N \setminus \emptyset$.

Now we define the *cover structure* formally as follows.

Definition 2. A subset C of 2^N is said to be a cover for N agents if

1. $\bigcup_{E\in C} E = N$ and

2. $\nexists E, E' \in C : E \subsetneq E'$.

We will denote the set of all covers for N by $\mathcal{C}^N$.

Definition 3. Given a cover $C \in \mathcal{C}^N$, a member $E \in C$ is said to be a hyper-link of order t, written $ord(E) = t$ if $\mid E \mid = t + 1$.

Note that a cover C is a collection of subsets $E \in 2^N$, such that these subsets are allowed to be overlapping but are not allowed to contain each other and the union of these subsets cover the player set N. If $E = \{i_1, i_2, \dots, i_k\} \in C$ then we say that the players $i_1, i_2, \dots, i_k$ are linked in the cover C. For the easiness of notation, we will use $(i_1 i_2 \dots i_k, \dots, j_1 j_2 \dots j_l)$ to denote the cover $\{\{i_1 i_2 \dots i_k\}, \dots, \{j_1 j_2 \dots j_l\}\}$. Note that the hyper-link notion in our model differs from the link notion in networks in the sense that players are allowed to be related to more than one player. For any $S \subset N$, C^S denotes a subset of $\mathcal{C}^N$ such that the players in $N \setminus S$ are only allowed to form the hyper-links of order 0 and the remaining players in S are allowed to form the hyper-links of any order, at most with the highest order $|S| - 1$.

Example 1. Let $N = \{1, 2, \dots, 10\}$ and $C = (123, 23, 34, 5678, 89, 10)$ The cover C has 6 hyper-links, with the orders $2, 1, 1, 3, 1, 0$ respectively.

We define adding a hyper-link to a cover C or severing a hyper-link from a cover C as follows.

Definition 4. Given any cover $C \in \mathcal{C}^N$, severing a hyper-link from C or adding a hyper-link to C is defined as follows: Given any cover $C \in \mathcal{C}^N$, and any hyper-link $E \notin C$ such that there is no $E' \in C$ with $E \subseteq E'$, $C + E$ denotes adding the hyper-link $E \notin C$ to the cover C and is defined by: $C + E = C \setminus \{E' \in C : E' \subseteq E\} \cup \{E\}$. Given any cover $C \in \mathcal{C}^N$ and any hyper-link $E \in C$, $C - E$ denotes severing the hyper-link $E \in C$ and is defined by: $C - E = C \setminus \{E\} \cup \{\{i\} : i \in E \text{ and } \nexists E' \in C \setminus \{E\} \text{ with } \in E'\}$

Definition 5. A cover $C \in \mathcal{C}^N$ is said to be connected if for any $i, j \in N$, there exists a sequence $E_1, E_2, \ldots, E_k \in C$ of hyper-links such that $i \in S_1$,$j \in S_k$ and for any $l \in 1, 2, \ldots, k-2 : E_l \cap E_{l+1} \neq \emptyset$.

A cover is connected if one can follow a path $E_1, \ldots, E_k$ for a cover with k hyper-links, in such a way that two hyper-links E_i and E_{i+1} has a nonempty intersection.

Given a cover $C \in \mathcal{C}^N$, the set $N(C) = \{i \in N : \exists E \in C \text{ with } |E| \neq 0 : i \in E\}$ denotes the agents which are included in at least one hyper-link E with an order greater then 0 in the cover C, that is all the players except the isolated ones.

Definition 6. Given a cover $C \in \mathcal{C}^N$, a subcover is defined to ve a subset of C which possesses the properties of being a cover.

If a cover is not connected, then we say it has components, which are defined as follows.

Definition 7. A nonempty sub-cover $T \subset C$ is said to be a component of the cover C if:

1. For every i and j in $N(T)$ with $i \neq j$: there exists a sequence $S_1, \ldots, S_k \in T$ of hyper-links with $i \in S_1$ and $j \in S_k$ and for all $t \in 1, \ldots, k-1 : E_t \cap E_{t+1} \neq \emptyset$, that is any two players in T are connected.

2. For every $i \in N(T)$, for every $E \in C$ with $i \in E$ one has $E \in T$.

A connected component of a cover is the maximal sub-cover which is connected. We will use $Cp(C)$ to denote the set of all connected components of the cover C.

In example, 1, the cover C is not connected, since for the players 4 and 5, there is no sequence of hyper-links $E_1, E_2, \ldots, E_k \in C$ such that $4 \in S_1, 5 \in S_k$ with $i \in 1, 2, \ldots, k-2 : E_i \cap E_{i+1} \neq \emptyset$. The cover C has 3 components which are $Cp(C) = \{(123, 23, 34), (5678, 89), (10)\}$. The set $N(C) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as 10 is isolated.

We now define a *complete cover*, which is quite different from the concept of a complete network.

Definition 8. A cover $C \in \mathcal{C}^N$ is said to be complete cover if: $C = \{(1\ 2\ \ldots\ n)\}$, that is if it is composed of only one hyper-link containing all of the players.

Complete cover consists of the hyper-link which contains all of the players. Note that, a cover consisting of the hyper-link with the highest possible order can not contain any other hyper-link by definition. In the above example, 1, the complete cover is $C = \{12345678910\}$. This definition of complete cover is different from the complete network in the sense that, any network for n agents is a subset of the complete network, which is not true for complete covers.

A value function gives the *value* generated by the players under different cover structures. Similar to the value functions of networks, a value function is different from a transferable utility game in the sense that players forming different covers in the same society can create different values. So, the value created by players depends on how the relations between players are formed. We define a value function formally as follows.

Definition 9. A function $v : \mathcal{C}^N \to \mathbb{R}$ is said to be a value function if $v(E) = 0$ whenever for every $E \in C$ one has $ord(E) = 0$. Let V denote the set of all value functions.

We now define efficiency for covers.

Definition 10. Given a value function v for $\mathcal{C}^N$, a cover $C \in \mathcal{C}^N$ is said to be efficient if $v(C) = \max_{C' \in \mathcal{C}^N} v(C')$.

We will denote $\max_{C' \in \mathcal{C}^N} v(C')$ by $\hat{v}(C^N)$. We will denote the set of all efficient covers by $\mathcal{C}^e$.

Definition 11. A value function $v \in V$ is said to be convex if for every $i \in N$, for every $S, T \in 2^{N \setminus \{i\}}$:$S \subset T \Rightarrow \hat{v}(C^{S \cup i}) - \hat{v}(C^S) \leq \hat{v}(C^{T \cup i}) - \hat{v}(C^T)$.

We will denote the set of all convex value functions by V^c. We now define super-additivity for value functions.

Definition 12. A value function $v \in V$ is said to be super additive if for all $S, T \in 2^N \setminus \{\emptyset\} : S \cap T = \emptyset \Rightarrow \hat{v}(C^{S \cup T}) \geq \hat{v}(C^S) + \hat{v}(C^T)$.

We will denote the set of all super additive value functions by V^{sa}.

An allocation rule determines how the total value of a given cover is distributed among the players. An allocation rule both depends on the value function v and how the players form their relations, namely the cover structure.

Definition 13. A function $Y : \mathcal{C}^N \times V \to \mathbb{R}^n$ with $\sum_{i=1}^n Y_i(C, v) = v(C)$ is called an allocation rule.

Since the formulations of allocation problem include the value function directly, we will write $Y_i(C)$ instead of $Y_i(C, v)$.

Definition 14. Given a cover $C \in \mathcal{C}^N$, and a value function $v \in V$, the Myerson allocation rule Y^{MV} for any player $i \in N$ is defined as follows:

$Y_i^{MV}(C) = \sum_{S \subset N \setminus \{i\}} (v(C|_{S \cup \{i\}}) - v(C|_S))(\frac{|S|!(n-|S|-1)!}{n!})$

where the restriction of a cover to a set $v(C|_S)$ is defined as $(C|_S) = \{E \cap S : E \in C \text{ with } E \cap S \neq \emptyset \text{ and } \nexists E' \in C : E \cap S \subset E' \cap S\} \cup \{\{i\} : i \in N \setminus S\}$.

When we restrict a cover C to a subset $S \subset N$, we require the players not in S to be singletons, preserve the hyper-links that are subsets of S, and for the hyper-links that are not in S but has a nonempty intersection with S, we keep the elements that are in S as a new hyper-link in case there is no other hyper-link in C containing it.

Example 2. Let $C = (1234, 345, 235)$ and $S = \{2, 3, 4\}$. Now $(C|_S) = (234, 1, 5)$. Although the intersection of the hyper-link (345) with S is nonempty, it is not included in the restricted cover as $\{3, 4, 5\} \cap \{2, 3, 4\} = \{3, 4\} \subset \{2, 3, 4\} = \{1, 2, 3, 4\} \cap \{2, 3, 4\}$ where $(1234) \in C$.

Definition 15. Given any subset $\bar{\mathcal{C}} \subseteq \mathcal{C}^N$ and $\bar{V} \subseteq V$, an allocation rule $Y : \mathcal{C}^N \times V \to \mathbb{R}^n$ is said to be individually rational (IR) at $(\bar{\mathcal{C}}, \bar{V})$ if for all $\in \bar{\mathcal{C}}$ for all $v \in \bar{V}$, $\forall i \in N$: $Y_i(C, v) \geq \hat{v}(C^{\{i\}})$.

We also define some stability notions in covers. The conditions required for stability change relative to the stability concept under consideration, but the underlying idea that players are allowed to severe or add hyper-links is the same. In all stability notions, the main idea is that players can not become better off by deviating from the present cover, that is either by adding or severing a hyper-link. Before defining the stability concepts we use, we first define some auxiliary concepts.

Definition 16. Two covers C and C' are said to be adjacent if they differ by a hyper-link, that is, either $C' = C + E$ for some $E \notin C$ or $C' = C - E$ for some $E \in C$.

Two covers are adjacent if one can be obtained from the other by simply adding or severing a hyper-link. Considering two adjacent covers, one blocks the other one either by adding a hyper-link if all players in the new link become better off while at least one them becomes strictly better off, or by severing a hyper-link if there is a player in that link which becomes better of after deletion of the hyper-link. We define this argument formally as follows.

Definition 17. Given a cover $C \in \mathcal{C}^N$, a value function $v \in V$, an allocation rule Y, and a hyper-link $E \notin C$, $C+E$ blocks C if: $\forall i \in E : Y_i(E+C) \geq Y_i(C)$

and $\exists j \in E : Y_i(E + C) > Y_i(C)$. For a hyper-link $E \in C$, $C - E$ blocks C if: $\exists i \in E : Y_i(C - E) > Y_i(C)$.

Definition 18. An improving path from a cover C to a cover C' is a sequence of covers $C_1, C_2, \ldots, C_k$ with $\forall i \in 1, \ldots, k-1 : C_i$ and C_{i+1} are adjacent covers such that C_{i+1} blocks C_i.

Definition 19. A cycle is an improving path $C_1, C_2, \ldots, C_k$ with $C_1 = C_k$.

Definition 20. Given a value function v and an allocation rule Y, a cover $C \in \mathcal{C}^N$ is said to be link-wise stable if there is no cover $C' \in \mathcal{C}^N$ with either $C' = C - E$ or $E' = C + E$ which blocks C, that is: For every $E \in C : \forall i \in E : Y_i(C) \geq Y_i(C - E)$ $\forall E \notin C : [\exists i \in E : Y_i(C + E) > Y_i(C)] \Rightarrow [\exists j \in E : Y_j(C + E) < Y_j(C)]$.

The link-wise stability definition is a counter-part of the pairwise stability definition in networks. Link-wise stability is stronger than the pairwise stability in the sense that it allows deviations by any coalition whereas pairwise stability only allows deviations by at most two players at a time. Notice that a cover $C \in \mathcal{C}^N$ is link-wise stable if there is no improving path starting from C. In order a new hyper-link to be formed, the mutual consent of all players in that hyper-link is needed, whereas hyper-link to be severed requires only unilateral consent of one player in that link. It is worth noting that, link-wise stability notion explains the situation where agents are unable to become better off by cooperation in myopic sense, that is in each step only one hyper-link can be formed or destroyed. In order to explain the situation where agents are unable to become better of by cooperation in far-sighted sense, that is the formation or destruction of more than one hyper-link are allowed in each step, we use the core-stability notion. Before defining core-stability, we define the necessary auxiliary concepts as follows.

Definition 21. A cover C' is said to beat a cover C if C and C' are adjacent and if $C' = C + E$ then $C + E$ blocks C, if $C' = C - E$ then $C - E$ blocks C.

Definition 22. Given a cover $C \in \mathcal{C}^N$, and $T \subset N$, a function $f : C \to 2^N \setminus \{\emptyset\}$ is called a T-function on C if $\forall E \in C : f(E) \subset E$ with $E \setminus f(E) \subset T$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from $C \in \mathcal{C}^N$ via T if $C' \subset \{f(E) : E \in C\} \cup 2^T$ for some T-function for C.

A cover C' being obtainable from C via some T-function where $T \subset N$ means that, the players in the coalition T are allowed to form hyper-links among themselves that are not contained in C or severe some hyper-links included in C in which at least one player of T is a member.

Definition 23. Given a value function $v \in V$ and an allocation rule Y associated with v, let $C, C' \in \mathcal{C}^N$ and $T \subset N$. We say that T can improve upon C via C' relative to v and Y if C' is obtainable from C via T and $\forall i \in T : Y_i(C') \geq Y_i(C)$ and $\exists j \in T : Y_j(C') > Y_j(C)$.

Definition 24. A cover $C \in \mathcal{C}^N$ is said to be core stable relative to v and Y if there is no $T \subset N$ such that T can improve upon C via some $C' \in \mathcal{C}^N$.

Core stability, similar to link-wise stability, allows any coalition to deviate including the grand coalition. Different from link-wise stability, these deviations are allowed to lead several hyper-links to be formed or severed. In this sense, core stability is a stronger concept. A coalition S improving upon the cover C via some cover C' can be viewed as C' blocking C by the deviations in the coalition S. Examining the "core" concept in a TU-game, our core-stability definition here differs from the core in the sense that, the players outside S are still contributing to the formation of the new cover as the hyper-links contained by those players are preserved. We define the "core" for a cover in order to prevent the players outside S to contribute to the formation of the cover.

Definition 25. Given a value function $v \in V$, and a cover $C \in \mathcal{C}^N$, an allocation $y \in \mathbb{R}^n$ for C, is said to be in the core relative to (N, v) if

1. $\sum_{i\in\{N\}} y_i \leq v(C)$ and,

2. $\forall S \subseteq N : \sum_{i\in S} y_i \geq \hat{v}(C^S)$.

This definition of core prevents the players outside S to contribute to the formation of a cover. Notice that the cover under consideration should be efficient, for otherwise taking $S = N$, $\sum_{i\in N} y_i \geq \hat{v}(C^N)$ and $\sum_{i\in\{N\}} y_i \leq v(c)$ would lead to a contradiction. In other words, for any cover which is not efficient, grand coalition can cooperate to jointly severe or add the necessary hyper-links in order to form the new cover C' which is efficient.

Note that core characterizes the allocations for efficient covers such that no coalition $S \subset N$ can deviate from the efficient cover under consideration to generate a higher value than the sum of the allocations of agent in S. Whereas core stability notion characterizes the covers that can not be improved upon via some coalition S, that is no coalition S can deviate from the cover structure under consideration to become better off where at least one agent becomes strictly better off. Also note that, the "deviations" in two concepts are different. In the core, the players outside the deviating coalition are assumed to forming hyper-edges of order 0, whereas in the core stability, the players outside the deviating coalition are assumed to preserve their hyper-link structure.

Definition 26. Given a set $\bar{V} \subseteq V$ of value functions, an allocation rule Y is said to be core-consistent relative to $\bar{V}$, if for any $v \in \bar{V}$ there exists an efficient cover $C \in \mathcal{C}^N$ such that for some $v \in \bar{V}$, for C the core relative to (N, v) is nonempty. [1]

An allocation rule Y being core-consistent relative to $\bar{V}$, for some $\bar{V} \subseteq V$, means that for some efficient cover $C \in \mathcal{C}^N$, $Y(C)$ belongs to the core for C relative to (N, v) for every $v \in \bar{V}$.

[1] the counterpart of the core-consistency in networks, defined by Jackson

CHAPTER 3

RESULTS

3.1 Link-wise Stability

We extend the Co-author Model introduced by Jackson and Wolinsky, (1996), to our model. In our extension, authors are assumed not to be restricted to having binary co-authorships. An author is allowed to work on several papers, and allowed to work with more than one co-author for each paper. There is also a synergy between co-authors depending on the time they devote for the paper, which in turn depends on how many paper each author is involved in. Each paper is symbolized by a hyper-link E, and for the author i the co-authors are symbolized by the agents in that hyper-link, namely $j \in E$. The number of papers an author is involved in, namely the hyper-links he is involved, is denoted by n_i, and so the time he devotes for paper is denoted by $\frac{1}{n_i}$. The synergy term is then captured by $\frac{1}{n_i}\sum_{j\in E}\frac{1}{n_j}$. The payoff of the individual i is represented by $u_i = \sum_{E\in C:i\in E}[\frac{1}{n_i} + \sum_{j\in E\setminus i}\frac{1}{n_j} + \frac{1}{n_i}\sum_{j\in E\setminus i}\frac{1}{n_j}]$ whenever the player is involved at least one paper-work with a co-author, $u_i = 1$ whenever the player is working on a paper by himself only. The payoff of an author depends on not only how many papers he is involved in, but also the number of co-authors he works with.

Jackson and Wolinsky, (1996), showed that in the co-author model, if the number of individuals n is even, then the efficient network structure consists of $\frac{n}{2}$ separate links. They also showed that, if a network is pairwise stable, and the number of individuals $n \geq 4$, then it is inefficient. The pairwise stable network in the co-author model consists of the maximum number of possible links, that is, for n players, the pairwise stable network is of the form: $\{12, 23, 34, 45, \ldots, (n-1)n\}$.

In our extended model, we show that for $n \geq 4$, the efficient cover structure consists of only one hyper-link with order $n-1$, that is the hyper-link containing all of the individuals, $\{1234\ldots n\}$. This can be interpreted as when authors are allowed to have only binary co-authorships then the efficient structure is formed when no paper have common authors, whereas when authors are allowed to have co-authorships of different orders then the efficient structure is formed when all of the authors are common in one paper. We also show that if a cover is link-wise stable, for $n \geq 4$, then it is efficient. That is in the co-author model pairwise stable networks lead to inefficiency, whereas in our extended model, link-wise stability leads to efficiency. We show this result as follows:

Proposition 1. *In the extended co-author model, when $n \geq 4$ the only efficient cover is the complete cover. Moreover the unique link-wise stable cover is the complete cover.*

Proof.

Claim. *For any $i \in N$ and for any cover $C \in \mathcal{C}^N \setminus \{(1\ 2\ \ldots\ n)\}$: $u_i(C) < u_i((1\ 2\ \ldots\ n))$.*

Proof will be done by induction on n.

Let $N = \{1, 2, 3, 4\}$, that is $n = 4$. Then the payoffs of the all possible covers are listed as follows:

$u(1, 2, 3, 4) = (1, 1, 1, 1)$ $u(12, 3, 4) = (3, 3, 1, 1)$ $u(12, 34) = (3, 3, 3, 3)$
$u(12, 23, 4) = (2, 4, 2, 1)$ $u(12, 13, 14) = (5, \frac{5}{3}, \frac{5}{3}, \frac{5}{3})$

$u(12,23,34) = (2, 3+\frac{1}{4}, 3+\frac{1}{4}, 2)$ $u(12,13,14,23) = (3+\frac{2}{3}, 2+\frac{1}{4}, 2+\frac{1}{4}, 1+\frac{2}{3})$

$u(12,13,14,24) = (2+\frac{7}{9}, 2+\frac{7}{9}, 2, 2)$

$u(12,13,14,23,24,34) = (2+\frac{1}{3}, 2+\frac{1}{3}, 2+\frac{1}{3}, 2+\frac{1}{3}))$

$u(12,234) = (2, 5+\frac{1}{2}, 4, 2), u(123,4) = (5,5,5,1)$

$u(123,124) = (4+\frac{1}{2}, 4+\frac{1}{2}, 3, 3)$ $u(123,234) = (2, 2+\frac{3}{4}, 2+\frac{3}{4}, 2)$

$u(123,234,341) = (3+\frac{1}{2}, 3+\frac{1}{2}, 4+\frac{1}{2}, 3+\frac{1}{2})$ $u(1234) = (7,7,7,7)$.

All the remaining covers that are not listed here are of the same shape with one of the listed one, and its value can easily be seen by the symmetric structure of the co-author model.

Here, the unique efficient cover is (1234) which is also the unique cover which is link-wise stable. To see this, in the complete cover each player gets a strictly higher payoff than the remaining covers, so all of the players can come together and form the hyper-link (1234) so that no other cover is link-wise stable, and the only cover that can be obtained from the complete cover is by severing the link, as there are no links to be added, and forming the new structure $(1,2,3,4)$, in which each player gets the payoff 1.

Induction hypothesis: Let the finite set of agents be K with $|K| = k$. For any $i \in K$ and for any cover $C \in (\mathcal{C}^K \setminus \{(1\ 2\ \ldots\ k)\})$: $u_i(C) < u_i(1\ 2\ \ldots\ k)$.

Now we show that our claim is true for $n = k+1$.

Consider the situation where there are k agents and an outsider agent comes and joins to the society, so that we have $k+1$ agents. First let us compute the contribution of $(k+1)$-th agent to others in the complete cover structure: For any $i \in K$: $u_i(\{1,\ldots,k+1\}) - u_i(\{1,\ldots,k\}) = 1 + (k) + 1(k) - 1 + (k-1) + 1(k-1) = 2$

Let C' be a cover different from the complete cover for k agents. Then there are at least two hyper-links in C'. Assume that there are more than 2 hyper-links with possibly different orders. Let $j \in N \setminus \{k+1\}$. Now let us compute the utility of j-th agent in this cover arbitrary cover C'. Then let us compute possible highest increase in his utility when $k+1$ joined.

Assume that j has m hyper-links with orders $d_1, d_2, \ldots, d_m$. Now $u_j(C') = \sum_{E\in\{E_1,\ldots,E_m\}}[\frac{1}{n_j} + (1+\frac{1}{n_j})(\sum_{i\in(E\setminus\{j\})}\frac{1}{n_i})]$. If each $i \in E \setminus \{j\}$ is included in hyper-links in which j is not included, the utility will decrease. Let us consider the possible highest value, so assume that each i is included in hyper-links that contain j. Then we get: $u_j = [1+(1+\frac{1}{m})(d_1-1)]+[1+(1+\frac{1}{m})(d_2-1)+[1+(1+\frac{1}{m})(d_m-1)]]$ where for some i,we obtain 1 when we sum $\frac{1}{n_i}$s over all hyper-edges that i is in, that is over all hyper-edges that j is in by the above assumption we made. So we have $u_j = 1+(1+\frac{1}{m})((d_1-1)+(d_2-1)+\cdots+(d_m-1)) = 1+(1+\frac{1}{m})(d_1+d_2+\cdots+d_m-m)$.The possible maximum value that the sum $(d_1+d_2+\cdots+d_m)$ can get is $k-1$, as we have k agents. Thus $u_j = k-m-1+\frac{k}{m}-\frac{1}{m}$.

Now let us compute the maximum possible utility of j when $(k+1)$-th agent joining to several hyper-links. If $k+1$-th agent joins to the hyper-links without j, then there will be no change. So assume that $k+1$-th agent joins at least one link that j is in. If he also joins hyper-links that j is not in, then due to the $\frac{1}{n_{k+1}}$ the utility of j will increase less compared to the case where he only joins to the links with j. But since we are trying to find maximum utility, we assume that he only joins to the hyper-links with j. Without loss of generality assume that he joins all the hyper-links that j is in. Denote this cover by C''. Then we have $u_j(C'') = \sum_{E\in\{E_1,\ldots,E_m\}}[\frac{1}{n_j} + (1+\frac{1}{n_j})(\sum_{i\in(E\setminus\{j\})}\frac{1}{n_i})] = 1+(1+\frac{1}{m})((d_1-1+\frac{1}{m})+(d_2-1+\frac{1}{m})+\cdots+(d_m-1+\frac{1}{m})) = 1+(1+\frac{1}{m})(d_1+d_2+\cdots+d_m-m+1)$. The possible highest value of the sum $d_1+d_2+\ldots+d_m$ is the total number of agents minus except j and $k+1$, that is $(k+1)-2 = k-1$. So $u_j = 1+(1+\frac{1}{m})(d_1+d_2+\cdots+d_m-m+1) = 1+(1+\frac{1}{m})(k-1-m+1) = k-m+\frac{k}{m}$. The maximum possible marginal contribution of $k+1$ to j is then: $[k-m+\frac{k}{m}]-[k+m-1+\frac{k}{m}-\frac{1}{m}] = 1+\frac{1}{m} < 2$ since $m \neq 1$. By induction hypothesis $u_j(C') < u_j(1\ 2\ \ldots\ k)$, so that $u_j(C'') = u_j(C')+1+\frac{1}{m} < 2+u_j(1\ 2\ \ldots\ k) = u_j(1\ 2\ \ldots\ k\ k+1)$ implying our claim for $k+1$.

Now it remains the check the above arguments for $k+1$-th agent. In the complete cover for $k+1$ agents each agent gets $2k+1$. Let C' be any cover except then the complete one, for $k+1$ agents. Assume that C' has m hyper-links with orders d_1, d_2,. . .,d_m. Assume that $k+1$-th agent joins all of the hyper-links. Then his utility will be: $u_{k+1}(C') = \sum_{E\in\{E_1,\ldots,E_m\}}[\frac{1}{n_{k+1}} + (1 + \frac{1}{n_{k+1}})(\sum_{i\in(E\setminus\{k+1\})}\frac{1}{n_i})] = 1 + (1+\frac{1}{m})\sum_{E\in\{E_1,\ldots,E_m\}}[\sum_{i\in(E\setminus\{k+1\})}(\frac{1}{n_i})]$, where the sum $\sum_{E\in\{E_1,\ldots,E_m\}}[\sum_{i\in(E\setminus\{k+1\})}\frac{1}{n_i}]$ can be at most k. As $(1+\frac{1}{m}) < 2$, we have $u_{k+1}(C') = 1 + (1+\frac{1}{m})k < 2k+1 = u_{k+1}(1\ 2\ \ldots\ (k+1))$, implying that the utility of the $(k+1)$-th agent is also strictly better in the complete cover for each $n \geq 4$. Thus we have shown that any agent has strictly better payoff when the cover structure is complete. Thus we have proved the claim. Now since each agents gets strictly better in the complete cover, it is the unique efficient. Moreover, as each agent become strictly better off by adding the the hyper-link $(1\ 2\ \ldots\ (k+1))$, the unique link-wise stable cover is the complete cover.

□

In the foregone discussion we used the synergy term $\frac{1}{n_i}[\sum_{j\in E\setminus i}\frac{1}{n_j}]$, which in fact counts for sum of the binary interactions in the sense $\frac{1}{n_i}[\sum_{j\in E\setminus i}\frac{1}{n_j}] = \sum_{j\in E\setminus i}[\frac{1}{n_i}\frac{1}{n_j}]$. Another formulation for the synergy term can be thought of the interaction of all the agents in the hyper-link under consideration, namely can be defined as $\prod_{j\in E}\frac{1}{n_j}$. The payoff of the individual i under this alternative synergy term is represented by $u_i = \sum_{E\in C: i\in E}[\frac{1}{n_i} + \sum_{j\in E\setminus i}\frac{1}{n_j} + \prod_{j\in E}\frac{1}{n_j}]$ whenever the player is involved at least one paper-work with a co-author, $u_i = 1$ whenever the player is working on a paper by himself only. In this alternative Extended Co-authorship model, we show that for $n \geq 4$, the only efficient cover structure again consists of only one hyper-link with order $n-1$, that is the hyper-link containing all of the individuals, $1, 2, ..., n$. We also show that, if a cover is link-wise stable under the above utility function for $n \geq 4$, then it is also efficient.

Proposition 2. *In the extended co-author model with the utility function*

$u_i = \sum_{E \in C: i \in E} [\frac{1}{n_i} + \sum_{j \in E \setminus i} \frac{1}{n_j} + \prod_{j \in E} \frac{1}{n_j}]$ *whenever the individual i is not isolated, $u_i = 1$ otherwise, when $n \geq 4$ the only efficient cover is the complete cover. Moreover the unique link-wise stable cover is the complete cover.*

Proof. To show the efficiency of the cover $C = (1, 2, ..., n)$, first let us compute the utility of agent i for C. Since the structure is symmetric for all agents in C, it suffices to compute u_i for an arbitrary i. Pick $i \in N$. Then $\frac{1}{n_i} = 1$, and for every $j \in N \setminus i$, $\frac{1}{n_j} = 1$. So, $u_i = \sum_{E \in C: i \in E} [\frac{1}{n_i} + \sum_{j \in E \setminus i} \frac{1}{n_j} + \prod_{j \in E} \frac{1}{n_j}] = 1 + (n-1)1 + 1 = n + 1$. Now, consider any other cover $C' \neq C$. For an individual $i \in N$, $u_i(C')$ to be maximum, C' should include hyper-links in which all individuals except i, $j \in E \setminus i$ is contained in only one hyper-link. Otherwise $\frac{1}{n_j} < 1$, where 1 is the maximum value that $\frac{1}{n_j}$ can attain, so that the value of $\sum_{j \in E \setminus i} \frac{1}{n_j}$ and $\prod_{j \in E} \frac{1}{n_j}$ becomes smaller. The other parameter that affects the value of $u_i(C')$ is the number of hyper-links. Notice that, if all the individuals other than i are contained in only one hyper-link, then the possible cover structures are: $C = (12)(13)(14)...(1n)$ (namely when the hyper-links of order 1 are formed), $C = (123)(145)...(1(n-1)n)$ (namely when the hyper-links of order 2 are formed),..., $C = (1234...\frac{n-1}{2})(1(\frac{n-1}{2} + 1)...n)$(namely when the hyper-links of order $(\frac{n-1}{2}) - 1$ is formed), or hyper-links of different order are formed $C = (12)(134...k)...(1k(k+1)...(n))$. Consider the most general case, let $C = (1...k)(1(k+1)...r)...(1(r+1)...n)$, where $2 \leq k \leq (r-2)$, $(n-2) \leq r \leq n$. Assume that the number of hyper-links in C is m and the order of the hyper-links in C are $ord(E_1)$,...,$ord(E_m)$ respectively. Now $u_i = \sum_{E \in C: i \in E} [\frac{1}{n_i} + \sum_{j \in E \setminus i} \frac{1}{n_j} + \prod_{j \in E} \frac{1}{n_j}] = [\frac{1}{m} + ord(E_1) + \frac{1}{m}] + [\frac{1}{m} + ord(E_2) + \frac{1}{m}] + ... + [\frac{1}{m} + ord(E_m) + \frac{1}{m}]$. Note that, as each $j \in E \setminus i$ is contained in only one hyper-link, $\sum_{l \in 1,...,m} (ord(E_l)) = n - 1$. In each term in the above sum, we have $2\frac{1}{m} + ord(E_l)$. Also note that we have m terms, since there are m hyper-links. Thus we have $u_i = \sum_{E \in C: i \in E} [\frac{1}{n_i} + \sum_{j \in E \setminus i} \frac{1}{n_j} + \prod_{j \in E} \frac{1}{n_j}] = [\frac{1}{m} + ord(E_1) + \frac{1}{m}] + [\frac{1}{m} + ord(E_2) + \frac{1}{m}] + ... + [\frac{1}{m} + ord(E_m) + \frac{1}{m}] = m.2.\frac{1}{m} + \sum_{l \in 1,...,m} (ord(E_l)) = 2 + (n-1) = n+1$. That is, whenever each agent $j \in E \setminus i$

is contained in only one hyper-link in $C' \neq C$, where $C = (1, 2, ..., n)$, we have $u_i(C') = n+1 = u_i(C)$. Since $j \in E \backslash i$ being included in more than one hyper-link will decrease the value of $u_i C'$, the maximum utility that individual i can achieve is $n + 1$. Considering the same argument from the view of the agent $j \neq i$, we conclude that the payoff of j in the cover C' is strictly less that $n + 1$, implying that the only efficient cover is $C = (12...n)$. The maximum payoff that any agent can achieve in a cover structure $C' \neq C$ is $n + 1$, while the payoffs for remaining players $j \neq i$ is strictly less than $n + 1$ we have, for any cover $C' \neq (12...n)$ and for any $i \in N : u_i(C') \leq n + 1 = u_i(C)$. Thus the only link-wise stable cover is C. □

In the following, we use the alternative pay-off function $u_i = \sum_{E \in C: i \in E} \frac{1}{|E|} [\frac{1}{n_i} + \sum_{j \in E \backslash i} \frac{1}{n_j} + \prod_{j \in E} \frac{1}{n_j}]$ whenever the player is involved at least one paper-work with a co-author, $u_i = 1$ whenever the player is working on a paper by himself only. In this case, the congestion effect observed in the foregone discussion disappears. For different number of players n, the structure of link-wise stable covers changes, however the efficient cover structure is the cover consisting the disjoint hyper-links of order 1 whenever the n is even, and the cover consisting the disjoint hyper-links of order 1 with an additional hyper-link of order 0 whenever n is odd. We first establish the efficiency in the following Proposition and then discuss the stability by means of examples.

Proposition 3. *: In the extended co-author model with the utility function* $u_i = \sum_{E \in C: i \in E} \frac{1}{|E|} [\frac{1}{n_i} + \sum_{j \in E \backslash i} \frac{1}{n_j} + \prod_{j \in E} \frac{1}{n_j}]$ *whenever the player is not isolated, and* $u_i = 1$ *otherwise, when* $n \geqslant 4$ *the only efficient cover is the cover consisting of disjoint hyper-links of order 1.*

Proof. : Consider a cover C. For an individual $i \in N$, $u_i(C)$ to be maximum, C should include hyper-links in which all individuals except i, $j \in E \setminus i$ is contained in only one hyper-link. Otherwise $\frac{1}{n_j} < 1$, where 1 is the maximum value that $\frac{1}{n_j}$ can attain, so that the value of $\sum_{j \in E \backslash i} \frac{1}{n_j}$ and $\prod_{j \in E} \frac{1}{n_j}$ becomes smaller. Thus for efficiency we must have disjoint hyper-links. The other parameter that affects the value of $u_i(C)$ is the number of hyper-links,and in turn the number of players included in an hyper-link. Since we have the coefficient $\frac{1}{|E|}$ in front of each term of the summation, as $|E|$ increases, the synergy term decreases, so for efficiency we must have $|E| = 2$, to obtain the maximum synergy. $\square$

Now we examine the stability in the following example:

Example 3. Consider the case where $n = 4$: We will show $C = \{123, 14, 24, 34\}$ is link-wise stable. Notice that this cover includes two disjoint groups, namely 123 and 4, and all other possible hyper-links of order 1. $u_1(123, 14, 24, 34) = \frac{1}{3}[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{8}] + \frac{1}{2}[\frac{1}{2} + \frac{1}{3} + \frac{1}{6}] = 1 + \frac{1}{24}$, and $u_1 = u_2 = u_3$ by symmetry. $u_4(123, 14, 24, 34) = \frac{1}{2}[\frac{1}{2}+\frac{1}{3}+\frac{1}{6}]+\frac{1}{2}[\frac{1}{2}+\frac{1}{3}+\frac{1}{6}]+\frac{1}{2}[\frac{1}{2}+\frac{1}{3}+\frac{1}{6}] = 1+\frac{1}{2}$. Deletion of any hyper-link of order 1 will yield the same result by symmetry, so it suffices to examine one, WLOG consider the deletion of the hyper-link (34), then the pay-offs of the third and the fourth players decrease: $u_3(123, 14, 24) = \frac{1}{3}[\frac{1}{2}+\frac{1}{2}+1+\frac{1}{4}] = \frac{2}{3}+\frac{1}{12} < 1+\frac{1}{24}$, and $u_4(123, 14, 24) = \frac{1}{2}[\frac{1}{2}+\frac{1}{2}+\frac{1}{4}] = 1+\frac{1}{8} < 1 + \frac{1}{2}$. Thus $(123, 14, 24, 34)$ blocks $(123, 14, 24)$. Now consider the deletion of the hyper-link (123): $u_1(14, 24, 34) = \frac{1}{2}[1 + \frac{1}{3} + \frac{1}{3}] = \frac{1}{2} + \frac{2}{3} < 1 + \frac{1}{24}$. As $u_1(14, 24, 34) = u_2(14, 24, 34) = u_3(14, 24, 34)$, $(123, 14, 24, 34)$ blocks $(14, 24, 34)$. Now consider the addition of a new hyper-link, (1234) and (234) respectively. $u_4(1234) = 1 + \frac{1}{4} < 1 + \frac{1}{2}$, so $(123, 14, 24, 34)$ blocks (1234). $u_4(123, 234, 14) = \frac{1}{3}[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{8}] + \frac{1}{2}[\frac{1}{2} + \frac{1}{2} + \frac{1}{4}] = 1 + \frac{1}{8} + \frac{1}{24} < 1 + \frac{1}{2}$, so $(123, 14, 24, 34)$ blocks $(123, 234, 14)$. Since we examined all possible covers C' of the form $C' = C + E$ with $E \notin C$ and $C' = C - E$ with $E \in C$, we conclude that $C = (123, 14, 24, 34)$ is link-wise stable.

Consider the case where $n = 5$: We will show $C = \{125, 35, 14, 24, 23, 45, 13, 34\}$ is link-wise stable. Notice that this cover includes two disjoint groups, namely 125 and 34, and all other possible hyper-links of order 1.

As in the above case, let us first consider hyper-link deletion, and then hyper-link formation. By symmetry it suffices to consider the deletion of the hyper-links $(34),(35),(14),(125)$ respectively. For (34) :

$u_3(125,35,14,24,23,45,13) = \frac{1}{2}[\frac{1}{3}+\frac{1}{3}+\frac{1}{9}].3 = 1+\frac{1}{6} < 1+\frac{1}{4}+\frac{1}{32} = \frac{1}{2}[\frac{1}{4}+\frac{1}{3}+\frac{1}{12}].3+\frac{1}{2}[\frac{1}{4}+\frac{1}{4}+\frac{1}{16}] = u_3(C)$. Notice that the positions of the player 3 and 4 are symmetric, so that C blocks $(125,35,14,24,23,45,13)$. For (35):

$u_3(125,14,24,23,45,13,34) = \frac{1}{2}[\frac{1}{3}+\frac{1}{3}+\frac{1}{9}].3 = 1+\frac{1}{6} < 1+\frac{1}{4}+\frac{1}{32} = u_3(C)$, and $u_5(125,14,24,23,45,13,34) = \frac{1}{3}[\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{18}]+\frac{1}{2}[\frac{1}{4}+\frac{1}{2}+\frac{1}{8}] < 1+\frac{1}{3^4} = u_3(C)$. So we conclude that C blocks $(125,14,24,23,45,13,34)$. For (14):

$u_1(125,35,24,23,45,13,34) = \frac{1}{3}[\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{18}]+\frac{1}{2}[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}] < 1+\frac{1}{3^4}$ and $u_4(125,35,24,23,45,13,34) = \frac{1}{2}[\frac{1}{3}+\frac{1}{3}+\frac{1}{9}].2+\frac{1}{2}[\frac{1}{3}+\frac{1}{4}+\frac{1}{12}] = 1+\frac{1}{9} < 1+\frac{1}{4}+\frac{1}{32}$. So we conclude that C blocks $(125,35,24,23,45,13,34)$. For (125):

$u_1(35,14,24,23,45,13,34) = \frac{1}{2}[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}].2 < 1+\frac{1}{3^4}$, and also note that the roles of the players 1, 2, 5 are symmetric. So, we conclude that C blocks $(35,14,24,23,45,13,34)$. Now let us consider possible hyper-link additions, namely (12345), (1253), (1234), (134), (135) respectively. Note that all other possible additions have the same impact by the symmetry of players, so we will not write them explicitly. For (12345):

$u_3(12345) = 1+\frac{1}{5} < 1+\frac{1}{4}+\frac{1}{32} = u_3(C)$. So, C blocks (12345). For (1253):

$u_3(1254,14,24,45,43) = 1+\frac{1}{4}[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{16}]+\frac{1}{2}[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}] < 1 < 1+\frac{1}{4}+\frac{1}{32}$. So we conclude that C blocks $(1254,14,24,45,43)$. For (1234):

$u_3(1234,125,35,45) = \frac{1}{4}[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{16}]+\frac{1}{2}[\frac{1}{3}+\frac{1}{2}+\frac{1}{6}] = 1+\frac{1}{64} < 1+\frac{1}{4}+\frac{1}{32}$. So, C blocks $(1234,125,35,45)$. For (134):

$u_1(125,134,35,24,23,45) = \frac{1}{3}[\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{18}].2 < 1 < 1+\frac{1}{3^4}$. So, C blocks $(125,134,35,24,23,45)$. For (135):

$u_3(125,135,14,24,23,45,34) = \frac{1}{3}[\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{27}].2+\frac{1}{2}[\frac{1}{3}+\frac{1}{3}+\frac{1}{9}].2 = 1+\frac{1}{9}+\frac{1}{3^4} < 1+\frac{1}{4}+\frac{1}{32}$. So, C blocks $(125,135,14,24,23,45,34)$. Since we considered all possible cases, we conclude that $C = \{125,35,14,24,23,45,13,34\}$ is link-wise stable.

Now let us extend the same structure to $n = 6$. We will show that the cover $C = (123, 45, 16, 26, 36, 46, 56, 14, 15, 24, 25, 34, 35, 45)$ is link-wise stable. Notice that this cover includes two disjoint groups, namely 123 and 45, and all other possible hyper-links of order 1. Another possible structure is $(123, 456, 14, 15, 16, 24, 25, 26, 34, 35, 36)$, which is not link-wise stable. Let us first show that the former cover is link-wise stable.

First consider the possible deletion of hyper-links, (14) and (45) respectively. For (14):

$u_1(123, 15, 16, 24, 25, 26, 34, 35, 36, 45, 46, 56) = \frac{1}{3}[\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{48}]+\frac{1}{2}[\frac{1}{3}+\frac{1}{5}+\frac{1}{15}].2$ $=0.87 < 1 + \frac{1}{3.4^3} = u_1(C)$. $u_4(123, 15, 16, 24, 25, 26, 34, 35, 36, 45, 46, 56) =$ $\frac{1}{2}[\frac{1}{4}+\frac{1}{4}+\frac{1}{16}].2+\frac{1}{2}[\frac{1}{5}+\frac{1}{4}+\frac{1}{20}].2 < 1.19 = u_4(C)$. So, C blocks $(123, 15, 16, 24, 25$ $, 26\ , 34, 35, 36, 45, 46, 56)$. For (45):

$u_4(123, 14, 15, 16, 24, 25, 26, 34, 35, 36, 56, 46) =$

$u_5(123, 14, 15, 16, 24, 25,26, 34, 35, 36, 56, 46) = \frac{1}{2}[\frac{1}{4} + \frac{1}{4} + \frac{1}{12}].3 + \frac{1}{2}[\frac{1}{5} + \frac{1}{4} + \frac{1}{20}]$ $=1.125 < 1.19 = u_4(C) = u_5(C)$.

So, C blocks $(123, 14, 15, 16, 24, 25, 26, 34, 35, 36, 56, 46)$. Now consider possible additions of hyper-links, (456),(234), (1234), (12345), (123456). For (456):

$u_4(123, 456, 14, 15, 16, 2425, 26, 34, 35, 36) = \frac{1}{3}[\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{64}] + \frac{1}{2}[\frac{1}{4} + \frac{1}{4} + \frac{1}{16}].3$ $=1.098 < 1.19$. Thus, C blocks $(123, 456, 14, 15, 16, 2425, 26, 34, 35, 36)$. For (234):

$u_4(123, 234, 14, 15, 16, 25, 26, 35, 36, 45, 46, 56) = \frac{1}{3}[\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{64}] + \frac{1}{2}[\frac{1}{2} + \frac{1}{2} + \frac{1}{16}] + \frac{1}{2}[\frac{1}{4} + \frac{1}{5} + \frac{1}{20}] = 0.79 < 1.19 = u_4(C)$.

So, C blocks

$(123, 234, 14, 15, 16, 25, 26, 35, 36, 45, 46, 56)$. For (1234):

$u_4(1234, 15, 16, 25, 26, 35, 36, 45, 46, 56) = \frac{1}{4}[\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{81}]+\frac{1}{2}[\frac{1}{3}+\frac{1}{5}+\frac{1}{15}] < 1$.

So, C blocks $(1234, 15, 16, 25, 26, 35,36, 45, 46, 56)$. For (12345):

$u_5(12345, 16, 26, 36, 46, 56) = \frac{1}{5}[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{32}]+\frac{1}{2}[\frac{1}{2}+\frac{1}{5}+\frac{1}{10}] = 0.906 < 1$.

So, C blocks $(12345, 16, 26, 36, 46, 56)$. For (123456):

$u_4(123456) = 1 + \frac{1}{6} = 1.16 < 1.19 = u_4(C)$. So, C blocks (123456). Thus, we conclude that C is link-wise stable. Now let us examine the cover

$C = (123, 456, 14, 15, 16, 23, 24, 25, 34, 35, 36)$.

$u_1(C) = \frac{1}{3}[\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{64}]+\frac{1}{2}[\frac{1}{4}+\frac{1}{4}+\frac{1}{16}].3 = 1.098 < 1.166 = u_1(123456)$. Thus, the cover (123456) blocks C. One may suspect that, whether this structure is link-wise stable for all n, the following examples shows this is not the case: For $n = 7$:

Let $C = (123, 14, 15, 16, 17, 24, 25, 26, 27,34, 35, 36, 37, 45, 46, 47, 56, 57, 67)$.

$u_1(C) = \frac{1}{3}[\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{125}] + \frac{1}{2}[\frac{1}{5} + \frac{1}{6} + \frac{1}{30}].4 = 1.002 < 1.14 = u_1(1234567)$. Thus C is not link-wise stable.

$C = (123, 456, 17, 27, 37, 47, 57, 67, 14, 15, 16, 24, 25, 26, 34, 35, 36)$, whose structure is same with $C' = (123, 456, 14, 15, 16, 23, 24, 25, 34, 35, 36)$, is link-wise stable whereas C' is not. As we have seen from this examples above, the structure of the link-wise stable cover changes for each n.

This example motivated us to study link-wise stability further to capture the relations between networks and covers. Studying on link-wise stability under the Myerson value for for different values of n, motivated us to use the "cycles" in order to prove the existence of link-wise stable covers under the Myerson-value. We use the idea that Jackson (2002)used to prove existence of pairwise stability under Myerson value in networks, to show the existence of the link-wise stable covers under Myerson value.

Theorem 1. *Let a value function $v \in V$, and an allocation rule Y be given. If there is some function $f : \mathcal{C}^N \to \mathbb{R}$ such that $[C' beats C] \Leftrightarrow [f(C') > f(C)]$ then there is no cycle.*

Proof. Assume that there exists a function $f : \mathcal{C}^N \to \mathbb{R}$ such that $[C'$ beats $C] \Leftrightarrow [f(C') > f(C)]$. Suppose that there there is a cycle. Let C be an element of this cycle, then there is a cover C' beating C. By definition a cycle is a closed improving path, that is we have the improving path $C, C', \ldots, C'', C$. Now by assumption $f(C') > f(C), f(C) > f(C'')$ and by the transitivity of the relation $>$, $f(C'') > f(C')$. Again using the transitivity, we obtain

$f(C') > f(C')$ which is a contraction. Thus under our assumption, there is no cycle. □

In the construction of the function f and in the above proof we do not use the cover structure explicitly. Rather we view the concept of a cover beating another one as a transitive relation. Interpreting the function f as the tool giving some degree (or value) to each cover, the assumption in the theorem implies that if a cover C' beats C, then the degree of C' should be larger than the degree of C. Also note that, the degree function f does not depend explicitly on v and Y. Now we show the existence of link-wise stability under the Myerson value as a corollary of this theorem. We first prove a lemma which will be used in the corollary. Also, we will use v and Y implicitly to define the degree function f in the following corollary. Before, we will prove a lemma that will be used in the proof of the corollary.

Lemma 1. *Given any value function $v \in V$ and any allocation rule Y, either there exists cycles of covers or there exists a link-wise stable cover.*

Proof.

Claim. *A cover C is link-wise stable iff there is no improving path $C_1, C_2, \ldots, C_k$ such that $\exists$ some $i \in \{1, 2, \ldots, k\} : C_i = C$ for $i \in \{1, 2, \ldots, k-1\}$.*

Assume that the cover C is link-wise stable and there is an improving path $C_1, C_2, \ldots, C_k$ such that $\exists$ some $i \in \{1, 2, \ldots, k\} : C_i = C$. But then C_{i+1} blocks C, contradicting with C being link-wise stable. Conversely assume that there is no improving path $C_1, C_2, \ldots, C_k$ such that $\exists$ some $i \in \{1, 2, \ldots, k\}$: $C_i = C$ for $i \in \{1, 2, \ldots, k-1\}$, but C is not link-wise stale. But then there exists some cover C' such that C' and C are adjacent and C' blocks C. So we have the improving path C, C' contradicting with assumption.

Let C be any cover. By our claim, either it is link-wise stable, or there is an improving path $\exists$ some $i \in \{1, 2, \ldots, k\} : C_i = C$ for $i \in \{1, 2, \ldots, k-1\}$. In the former case the statement is automatically shown. In the latter case, either the improving path stops in some cover C' or the path traces all the possible covers. If it stops in some cover C' then this cover is link-wise stable. If it traces all the possible covers and still does not stop at some cover, then this means that there is a cycle.

$\square$

Corollary 1. *Given any value function $v \in V$, and Myerson value Y^{MV}, there exists a link-wise stable cover relative to v and Y^{MV}.*

Proof. Assume that C' beats C. Then either $C' = C + E$ and $\forall i \in E$: $Y_i^{MV}(E + C) \geq Y_i^{MV}(C)$ and $\exists j \in E$: $Y_i^{MV}(E + C) > Y_i^{MV}(C)$, or $C' = C - E$ and $\exists i \in E$: $Y_i^{MV}(C - E) > Y_i^{MV}(C)$. Define $f(C) = \sum_{T \subseteq N} v(C|_T)[\frac{(|T|-1)!(n-|T|!)}{n!}$ By direct calculation we have: $Y_i^{MV}(C + E) - Y_i^{MV}(C) = f(C + E) - f(C)$. Now if $C' = C + E$ and C' beats C, then by the above equality we have $f(C + E) > f(C)$. If $C' = C - E$ and C' beats C, then replacing $C + E$ by C and C by $C - E$ in the above equality, we get $f(C - E) > f(C)$ again. Thus by the Theorem 1, there is no cycles with respect to Myerson value. By the 1, we then have a link-wise stable cover. $\square$

Thus we established the existence of link-wise stable covers with respect to the Myerson value. However, there are also some allocations with respect to which there is no link-wise stable cover. The following example captures this.

Example 4. Let $n = 3$, and the value function be given as follows: $v(1, 2, 3) = 0$, $v(12, 3) = 2$, $v(13, 2) = 3$, $v(1, 23) = 2$, $v(12, 23) = 1$, $v(12, 13) = 2$, $v(13, 23) = 4$, $v(12, 23, 31) = 3$, $v(123) = 6$ Let the value of each cover be allocated as follows: $v(1, 2, 3) = (0, 0, 0)$, $v(12, 3) = (1, 1, 0)$, $v(13, 2) = (3, 2, -2)$, $v(1, 23) = (2, 1, -1)$, $v(12, 23) = (2, 0, -1)$, $v(12, 13) = (2, -1, 1)$, $v(13, 23) = (3, 1, 0)$,$v(12, 23, 31) = (1, 0, 2)$, $v(123) = (4, 3, -1)$. Now there is no link-wise stable cover as we have the following cycle: $(1, 2, 3) \to (12, 3) \to (12, 13) \to (12, 23, 31) \to (12, 23) \to (1, 23) \to (13, 23) \to (13, 2) \to (123) \to$

$(1, 2, 3)$.

3.2 Core Stability and Core

In link-wise stability, deviations of coalitions are assumed to consist of adding or severing a hyper-link in one step. We extend this deviation assumption so that they may include addition and/or deletion of several hyper-links in one step. We define this idea in two different ways; in the first one we allow player in the deviating coalition, to form relations (hyper-links) only among themselves, and players outside the deviating coalition are not allowed to form relations (hyper-links of order 0), whereas in the second one, we allow players in the deviating coalition to severe the relations in the former cover structure, to form new relations among themselves, but the players outside the deviating coalition are assumed to preserve their relations as in the former cover structure. We investigate the core and the core stability notions respectively, and find an environment in which these two concepts coincide.

Given a value function v, for an efficient cover $C \in \mathcal{C}^N$, core relative to v may be empty. In the following example, we show that for $N = 3$ players, core for an efficient cover may be empty.

Example 5. Let $n = 3$ and the value function be given by: $v(1, 2, 3) = 0, v(12, 3) = v(13, 2) = v(23, 1) = 6, v(12, 23) = v(13, 23) = v(12, 13) = 3, v(12, 13, 23) = 2$ and $v(123) = 2$. Note that the efficient covers are : $(12, 3), (13, 2), (23, 1)$.

Claim. Given one of the efficient covers, the core relative to v is empty.

Since the structure is symmetric, without loss of generality, consider the cover $(12, 3)$. Assume that $y = (y_1, y_2, y_3)$ is an element of the core. Then we have:

$y_1 + y_2 + y_3 \leq v(12, 3) = 6$ $y_1 + y_2 \geq 6$ $y_1 + y_3 \geq 6$ $y_2 + y_3 \geq 6$

where the last three inequalities imply that $y_1 + y_2 + y_3 \geq 9$ which is a contradiction. Hence the core is empty.

Given a value function v, for an efficient cover $C \in \mathcal{C}^N$, we will prove that the core is nonempty if the value function is convex. We state this result in the following theorem.

We first define the marginal contribution vector $\varphi_\sigma(v)$ as follows.

Definition 27. Let $\sigma = (i_1, \ldots, i_n) \in S_n$ be a permutation of the player set N. Define $\theta_1 = \hat{v}(C^{i_1}) - \hat{v}(1, 2, \ldots, n)$ and for any $k \geq 2$: $\theta_2 = \hat{v}(C^{i_1,i_2}) - \hat{v}(C^{i_1})$. The vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}$ is called the marginal contribution vector and is denoted by $\varphi_\sigma(v)$.

Theorem 2. *Let $v : \mathcal{C}^N \to \mathbb{R}$ be a convex value function. Let C be an efficient cover relative to v. For any permutation $\sigma \in S_n$, the associated marginal contribution vector $\varphi_\sigma(v)$ belongs to the core for C relative to (N, v).*

Proof. Without loss of generality, assume that $\sigma = (1, \ldots, n)$ and C is an efficient cover. Now

$\sum_{i \in N} (\varphi_\sigma(v))_i = \sum_{i=1^n} (\hat{v}(C^{1,\ldots,i}) - \hat{v}(C^{1,\ldots,i-1}))$
$= (\hat{v}(C^1) - \hat{v}(1, \ldots, n)) + \ldots + (\hat{v}(C^{1,\ldots,n}) - \hat{v}(C^{1,\ldots,n-1}))$
$= \hat{v}(C^{1,\ldots,n}) - \hat{v}(1, \ldots, n) = \hat{v}(C^{1,\ldots,n}) = v(C)$ as C is efficient.

Take any coalition $S \subset N$, say $S = \{i_1, \ldots, i_s\}$ with $i_1 \leq i_2 \leq \ldots \leq i_s$. For any $t \in \{2, \ldots, s\} : \{i_1, \ldots, i_{t-1}\} \subset \{1, 2, \ldots, i_{t-1}\}$. Now by the convexity of the value function v we have: For any $t \in \{1, 2, \ldots, s\} : \hat{v}(C^{i_1,\ldots,i_t}) - \hat{v}(C^{i_1,\ldots,i_{t-1}}) \leq \hat{v}(C^{1,\ldots,i_t}) - \hat{v}(C^{1,\ldots,i_{t-1}})$, and for $t = 1 : \hat{v}(C^{i_1}) - \hat{v}(\{1, \ldots, n\}) \leq \hat{v}(C^{1,\ldots,i_1}) - \hat{v}(C^{1,\ldots,i_1-1})$. Summing these inequalities over t, $\hat{v}(C^{i_1,\ldots,i_s}) - v(1, 2, \ldots, n) \leq \sum_{i_t \in S} (\varphi_\sigma(v))_{i_t} = \sum_{i \in S} (\varphi_\sigma(v))_i$

Hence given an efficient cover, the marginal contribution vector $\varphi_\sigma(v)$ belongs to the core. □

We now define the Player Based Flexible Cover allocation rule, which is defined by Jackson (2003)in network setting, as follows.

Given a value function v and a cover C, the allocation of the $i-th$ agent is defined by: $Y_i^{PBFC}(C) = \frac{v(C)}{\hat{v}(C^N)} \sum_{S \subset N \setminus \{i\}} (\hat{v}(C^{S \cup \{i\}}) - \hat{v}(C^S)) \frac{|S|!(n-|S|-1)!}{n!}$

We now prove that, given any convex value function $v \in V^c$, and an efficient cover C, the allocation obtained by using Player Based Flexible Cover, (PBFC), allocation rule always belongs to the core. In other words, given any $v \in V^c$, and any efficient cover C, the core for C relative to (N, v) is nonempty.

Corollary 2. *Let $v \in V^c$ be a convex value function, and C be an efficient cover relative to v. Now the value obtained by Player Based Flexible Cover allocation rule belongs to the core.*

Proof. Fix an efficient cover C.

Claim. *Core relative to v is a convex set.*

Assume y^1 and y^2 belong to the core. Now, $\lambda \sum_{i \in N} y_i^1 + (1-\lambda) \sum_{i \in N} y_i^2 \leq \lambda v(C) + (1-\lambda) v(C) = v(C)$ Let $S \subset N$, $\lambda \sum_{i \in S} y_i^1 + (1-\lambda) \sum_{i \in S} y_i^2 \geq \lambda \hat{v}(C^S) + (1-\lambda) \hat{v}(C^S) = \hat{v}(C^S)$ Thus core is a convex set. Since core is a convex set, and $\varphi_\sigma(v)$ is a member of the core, the convex combinations of it also belong to the core. Noting that $\frac{v(C)}{\hat{v}(C^N)} = 1$, we have: $Y^{PBFC}(C) = \frac{v(C)}{\hat{v}(C^N)} \sum_{\sigma \in S_n} \varphi_\sigma(v) \frac{1}{n!}$ belong to the core. $\square$

Corollary 3. *The player based flexible cover allocation rule Y^{PBFC} is core consistent relative to V^c.*

Remark 1. For convex TU-games, we know that the allocation induced by the Shapley value is a member of the core. However, for covers the allocation induced by Myerson value, which is the counterpart of the allocation induced by Shapley value, is not necessarily a member of the core for an efficient cover C.

The following example points out this situation.

Example 6. Let $n = 3$, and the value function as follows: $v(1,2,3) = 0$, $v(12,3) = v(13,2) = v(1,23) = -6$, $v(12,13) = v(12,23) = v(13,23) = 0$, $v(12,23,13) = -12$, $v(123) = -3$.

Claim. v defined as above is a convex value funciton.

Recall the convexity definition: A value function $v \in V$ is said to be convex if $\forall i \in N, \forall S, T \in 2^{N \setminus \{i\}} : S \subset T \Rightarrow \hat{v}(C^{S \cup i}) - \hat{v}(C^S) \leq \hat{v}(C^{T \cup i}) - \hat{v}(C^T)$.

Now notice that, for any $R \subseteq N$ we have $\hat{v}(C^R) = 0$ by the definition of v. Thus convexity is satisfied trivially. The Myerson value allocates the values to individuals as follows:

$v(1,2,3) = (0,0,0)$, $v(12,3) = (-3,-3,0)$, $v(13,2) = (-3,0,-3)$, $v(1,23) = (0,-3,-3)$, $v(12,13) = (-2,1,1)$, $v(12,23) = (1,-2,1)$, $v(13,23) = (1,1,-2)$, $v(12,23,13) = (-4,-4,-4)$, $v(123) = (-1,-1,-1)$.

Now, the cover $(12,23)$ is efficient, however the allocation $(1,-2,1)$ is not an element of the core for $(12,23)$, since $\sum_{i \in \{2\}} y_i = y_2 = -2 < 0 = \hat{v}(C^{\{2\}})$.

Remark 2. One of the other well known property of Shapley value in TU games is individually rationality at any super additive TU game v. The allocation induced by Myerson value for covers, however fails to satisfy this property.

We point out this situation by further investigating the above example 6 in more detail.

In the example above 6, the value function is super additive trivially. The Myerson value Y^{MV} is IR at $((1,2,3), v)$ that is individually rational at only the disconnected cover. We show in the following proposition that player based flexible cover allocation rule Y^{PBFC} is individually rational at $(\mathcal{C}^e, v)$ for any value function.

Proposition 4. *The player based flexible cover allocation rule Y^{PBFC} is individually rational at $(\mathcal{C}^e, V)$.*

Proof. Take any value function $v \in V$. Let $i \in N$. Now for any $S \subset N \in \{i\}$ we have $\hat{v}(C^{S\cup\{i\}}) - \hat{v}(C^S) \geq 0$, if not $\hat{v}(C^{S\cup\{i\}}) < \hat{v}(C^S)$ which can not happen since $S \cup \{i\}$ can obtain the value at least guaranteed by S by making the same hyper-links and leaving $\{i\}$ isolated. Also note that $\hat{v}(C^{\{i\}}) = 0$, so we have $\hat{v}(C^{\{i\}}) \leq \hat{v}(C^{S\cup\{i\}}) - \hat{v}(C^S)$. Now multiplying both sides by $\frac{|S|!(n-|S|-1)!)}{n!}$ and summing both sides over $S \subseteq N \setminus \{i\}$ we get $\sum_{S\subset N\setminus\{i\}} \hat{v}(C^{\{i\}})\frac{|S|!(n-|S|-1)!)}{n!} \leq \sum_{S\subset N\setminus\{i\}}(\hat{v}(C^{S\cup\{i\}})-\hat{v}(C^S))\frac{|S|!(n-|S|-1)!)}{n!}$. Now multiplying both sides by $\frac{v(C)}{\hat{v}(C^N)}$, we get $\frac{v(C)}{\hat{v}(C^N)}\sum_{S\subset N\setminus\{i\}} \hat{v}(C^{\{i\}})\frac{|S|!(n-|S|-1)!)}{n!} \leq \frac{v(C)}{\hat{v}(C^N)}\sum_{S\subset N\setminus\{i\}}(\hat{v}(C^{S\cup\{i\}}) - \hat{v}(C^S))\frac{|S|!(n-|S|-1)!)}{n!}$. Note that $\frac{v(C)}{\hat{v}(C^N)} = 1$ whenever $C \in \mathcal{C}^e$ and note also that $\sum_{S\subset N\setminus\{i\}} \frac{|S|!(n-|S|-1)!)}{n!} = 1$. Then for any efficient cover C and any super additive value function v, we have $\hat{v}(C^{\{i\}}) \leq Y_i^{PBFC}(C, v)$. Thus the player based flexible allocation rule is individually rational at $(\mathcal{C}^e, V)$. $\square$

We will show in the next section that the Myerson value is IR at $(\bar{\mathcal{C}}, V)$ for some special class of covers $\bar{\mathcal{C}}$ to be defined in the next section.

We now investigate the relationship between the allocations induced by core-stable covers and the allocations in core, and see that under the PBFC allocation rule, in fact the two allocations coincide. That is, given any convex value function v, and an efficient cover C with respect to v, the allocation obtained by applying the PBFC allocation rule belong to the core. In the following theorem, we will show that any efficient cover C with respect to v is core stable relative to v and the PBFC allocation rule, moreover any core stable cover relative to v and the PBFC allocation rule is efficient.

Theorem 3. *Given any value function $v \in V$, a cover C is efficient relative to v iff C is core stable relative to v and player based flexible cover allocation rule Y^{PBFC}.*

Proof. Assume that C is efficient, and not core stable relative to v and Y^{PBFC}. Then there exists a coalition $T \subseteq N$ such that $\exists C' \in \mathcal{C}^N$ obtainable from C

via T and $\forall i \in T : Y_i^{PBFC}(C') \geq Y_i^{PBFC}(C)$ and $\exists j \in T : Y_j^{PBFC}(C') > Y_j^{PBFC}(C)$. That is there exists some $j \in T : \frac{v(C')}{\hat{v}(C^N)} \sum_{S \subset N\setminus\{j\}} (\hat{v}(C^{S\cup\{j\}}) - \hat{v}(C^S))\frac{|S|!(n-|S|-1)!}{n!} > \frac{v(C)}{\hat{v}(C^N)} \sum_{S \subset N\setminus\{j\}} (\hat{v}(C^{S\cup\{j\}}) - \hat{v}(C^S))\frac{|S|!(n-|S|-1)!}{n!}$. Note that the part $\frac{1}{C^N} \sum_{S \subset N\setminus\{j\}} (\hat{v}(C^{S\cup\{j\}}) - \hat{v}(C^S))\frac{|S|!(n-|S|-1)!}{n!}$ is same in both sides so we get the inequality: $v(C') > v(C)$ which contradicts with the efficiency of C. Conversely, assume that C is core stable with respect to v and Y^{PBFC} but not efficient relative to v. Take $T = N$, and any efficient cover C'. Then since $v(C') > v(C)$, we have $\forall i \in N : \frac{v(C')}{\hat{v}(C^N)} \sum_{S \subset N\setminus\{i\}} (\hat{v}(C^{S\cup\{i\}}) - \hat{v}(C^S))\frac{|S|!(n-|S|-1)!}{n!} > \frac{v(C)}{\hat{v}(C^N)} \sum_{S \subset N\setminus\{i\}} (\hat{v}(C^{S\cup\{i\}}) - \hat{v}(C^S))\frac{|S|!(n-|S|-1)!}{n!}$, equivalently $\forall i \in N : Y_i^{PBFC}(C') > Y_i^{PBFC}(C)$. Implying that N can improve upon C via C', contradicting with C being core stable. $\square$

[1]

Corollary 4. *Given any convex value function $v \in V^c$, if a cover C is core stable relative to v and Y^{PBFC}, then $Y^{PBFC}(C)$ belongs to the core for C relative to (N, v).*

3.3 Hyper-link Formation Game

We investigated some stability notions in cover theory, namely link-wise stability and core stability. Then we defined core for cover structures, and investigated its properties and pointed out the relationship between core and core stability. Besides taking a cover structure as given and then working with it, another way to investigate the structure is via strategic form games, which allows us to consider the endogenous formation of covers. We define the hyper-link formation game in which the player set is nonempty finite set of agents, strategy spaces consists of the independently announced links to be formed, and the utility function is taken as an allocation rule.

[1]The corresponding result for networks was examined by Kapan T.

We define these concepts more precisely as follows. Given a value function $v : \mathcal{C}^N \to \mathbb{R}$, and an allocation rule $Y : \mathcal{C}^N \times V \to \mathbb{R}^n$, we define the normal form game $\Gamma(v, Y) = (N,\, S_1,\, \ldots, S_n,\, u^{v,Y})$ where N denotes the nonempty finite set of players, S_i denote the strategy space of agent i, and $u^{v,Y}$ denotes the utility function. For all $i \in N$, $\xi_i = \{E \in 2^N \setminus \{\emptyset\} : i \in E\}$ is the set of all possible hyper-links that agent i can form. The strategy set of each agent is then defined by $S_i = 2^{\xi_i}$. That is each agent's strategy is a list of hyper-links that he is willing to form. Denote the strategy space by $S = \prod_{i \in N} S_i$. Given a hyper-link formation game $\Gamma(v, Y)$, and a strategy profile $s \in S$, the cover C induced by s is defined by $C(s) = \{E \in 2^N \setminus \{\emptyset\}$ $: \forall i \in E : E \in s_i\} \setminus \{E' \in 2^N \setminus \{\emptyset\} : \forall i \in E' : E' \in s_i$ with $E' \subseteq E\}$. The utility function $u^{v,Y} : S \to \mathbb{R}^n$ is defined by $u_i^{v,Y}(s) = Y_i(v, C(s))$ where $C(s)$ is the cover induced by the strategy profile s.

We will write u_i for $u_i^{v,Y}$ from this point on.

Given the strategic form game $\Gamma(v, Y)$, a strategy of an agent $i \in N$ is the list of hyper-links that agent i is willing to form. Each agent announces his list, that is hyper-links he is willing to form, and then a hyper-link is formed if it is included in the list of each member of that hyper-link. Note that a hyper-link is formed under the mutual consent of the players involved in that hyper-link. It is worth noting that, if two hyper-links E and E' satisfy the requirement of being formed and if $E \subseteq E'$, then the hyper-link E' is formed but not E.

We now define the Nash Equilibrium and Strong Nash Equilibrium for hyper-link formation game and Nash stability and Strong Nash stability for covers induced by these solution concepts as follows.

Definition 28. A strategy profile $s \in S$ is said to be a Nash Equilibrium of the game $\Gamma(v, Y)$ if for any $i \in N$ and $s_i' \in S_i : u_i(s) > u_i(s_i', s_{-i})$. A cover $C \in \mathcal{C}^N$ is called Nash stable relative to (v, Y) if there exists $s \in S$ such that s is a Nash Equilibrium of the game $\Gamma(v, Y)$ and $C = C(s)$, that is C is the cover induced by the strategy profile s.

We will denote the set of all Nash stable covers by $\mathcal{C}^{NS}$.

In the previous section we stated that the Myerson value is IR at $(\bar{\mathcal{C}}, V)$ for some special class of covers $\bar{\mathcal{C}}$ to be defined. We now provide this result.

Proposition 5. *The Myerson value Y^{MV} is individually rational at $(\mathcal{C}^{NS}, V)$.*

Proof. Let $v \in V$ be any value function, and $C \in \mathcal{C}^{NS}$ be any Nash stable cover. Considering the hyper-link formation game, there is a Nash equilibrium of the game $\Gamma(v, Y^{MV})$ from which C is induced. Suppose that there is some $i \in N$ such that $Y_i^{MV}(C, v) < \hat{v}(C^{\{i\}})$. Note that $\hat{v}(C^{\{i\}}) = 0$ since the cover that $\{i\}$ can form individually is $(1, 2, \ldots, n)$. This means that $Y_i^{MV}(C, v) < 0$, but then i can guarantee getting 0 by changing his strategy to $s_i' = \{i\}$, that is by changing his strategy in such a way that he becomes isolated. But then $Y_i^{MV}(C(s_i', s_{N\setminus\{i\}}), v) = 0 > Y_i^{MV}(C(s), v)$, contradicting with s being a Nash equilibrium of the game $\Gamma(v, Y^{MV})$. Thus $Y_i^{MV}(C, v) \geq \hat{v}(C^{\{i\}})$ for any Nash stable cover C and any value function v, implying that Myerson value is individually rational at $(\mathcal{C}^{NS}, V)$. $\square$

Remark 3. Given a value function $v : \mathcal{C}^N \to \mathbb{R}$, for a Nash stable cover $C \in \mathcal{C}^{NS}$, the set $\{Y(C)\}$, where Y is any allocation rule $Y : \mathcal{C}^N \times V \to \mathbb{R}^n$, contains the core for C relative to (N, v) whenever C is efficient.

Definition 29. A strategy profile $s \in S$ is said to be a Strong Nash Equilibrium of the game $\Gamma(v, Y)$ if there exist no coalition $T \subseteq N$ such that there exists $s_T' \in S_T = \prod_{i \in T} s_i$ with $\forall i \in T$: $u_i(s_T', s_{N\setminus T}) \geq u_i(s)$ and $\exists j \in T$: $u_i(s_T', s_{N\setminus T}) > u_i(s)$. A cover $C \in \mathcal{C}^N$ is called Strong Nash stable relative to (v, Y) if there exists $s \in S$ such that s is a Strong Nash Equilibrium of the game $\Gamma(v, Y)$ and $C = C(s)$, that is C is the cover induced by the strategy profile s.

The resemblance between strong Nash stability and the core stability is worth noting. We state the relationship between the two in the following proposition.

Proposition 6. *Given a hyper-link formation game $\Gamma(v, Y)$, if a strategy profile $s \in S$ is a Strong Nash Equilibrium of the game $\Gamma(v, Y)$, then the cover induced by the strategy profile s, namely $C(s)$ is core-stable relative to (v, Y).*

Proof. Assume that s is a strong Nash equilibrium of the game $\Gamma(v, Y)$, and let C be the cover induced by the strategy profile s, that is $C = C(s)$. Suppose that C is not core stable. Then there is some $T \subset N$ that can improve upon C via some C'. That is, there is some $T \subset N$ such that C' is obtainable from C and for every $i \in T$: $Y_i(C') \geq Y_i(C)$ and there is some $j \in T$: $Y_j(C') > Y_j(C)$. Let s' be the strategy profile inducing the cover C'. By definition of obtainability of C' from C, s' is of the form $s' = (s'_T, s_{-T})$. For if the agents outside the T also change their strategy profiles, then the hyper-links having no agents in T may also change which is not allowed. So we have: there exists some $T \subset N$: for all $i \in T$: $Y_i(C(s'_T, s_{-T})) \geq Y_i(C(s))$ and there is some agent $j \in T$: $Y_j(C(s'_T, s_{-T})) > Y_j(C(s))$, that is $u_i(s'_T, s_{-T}) \geq u_i(s)$ for all $i \in T$ and $u_j(s'_T, s_{-T}) > u_j(s)$ for some $j \in T$, which is contradiction with s being a strong Nash equilibrium of the game $\Gamma(v, Y)$. Thus $C(s)$ is core stable relative to (v, Y).

□

As a result of this proposition, we state the relationship between Strong Nash stability and the core in the following corollary.

Corollary 5. *Given a convex value function $v \in V^c$, and a hyper-link formation game $\Gamma(v, Y^{PBFC})$, if a strategy profile $s \in S$ is a Strong Nash Equilibrium of the game $\Gamma(v, Y^{PBFC})$, and if the cover induced by the strategy profile s is $C(s)$, then $Y^{PBFC}(C)$ belongs to the core relative to (N, v).*

Proof. Let s be a strong Nash equilibrium of the game $\Gamma(v, Y^{PBFC})$, and let C be the cover induced by s. By the above proposition 6, C is core stable relative to v and Y^{PBFC}. By the theorem Theorem 3, C is efficient relative to v. By the corollary 2, since v is convex and C is efficient, $Y^{PBFC}(C)$ belongs to the core for C relative to (N, v). $\square$

CHAPTER 4

CONCLUSION

In this study, we proposed the Extended Co-author Model which we defined using the *cover structure*, introduced by Koray, as an extension of the Co-author Model proposed by Jackson and Wolinsky, (1996). We showed the efficient and the stable cover structures under four different productivity, or utility, functions we proposed. The results we found differ from those found by Jackson and Wolinsky. We showed the existence of link-wise stable covers under the Myerson value. We examined the far-sighted stability notions, namely the core-stability and the core of the *cover structure* in addition to link-wise stability, which is a myopic stability notion. We showed the non-emptiness of the core similar to that of in the cooperative game theory. Different from cooperative game theory, we showed that the Myerson value which is the counterpart of the Shapley value in the cooperative game theory, is not necessarily an element of the core. We showed the equivalence of efficiency of a cover with respect to any value function and the core stability of a cover with respect to any value function under the player based flexible cover allocation rule. We showed the relationship between the core-stability and core under the player based flexible cover allocation rule and any convex value function. Lastly we defined the strategic form game, namely the hyper-link formation game for covers. We showed that the cover induced by Strong Nash Equilibrium of the hyper-link formation game for any convex value function and the player based flexible cover allocation rule belongs to the core of $C \in \mathcal{C}^N$ and the convex value function.

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