EXPLORING UNITIZING STRATEGIES IN MIDDLE SCHOOL STUDENTS: INSIGHTS FROM NUMBER LINE, FRACTION CIRCLE, AND WORD PROBLEM TASKS

by

Züleyha Terzi

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ABSTRACT

EXPLORING UNITIZING STRATEGIES IN MIDDLE SCHOOL STUDENTS: INSIGHTS FROM NUMBER LINE, FRACTION CIRCLE, AND WORD PROBLEM TASKS

The concept of a unit is fundamental to mathematics, deeply intertwined with the notion of "one" and other key mathematical constructs. This study investigates middle school students' strategies for "unitizing" — defining and using units — across three different tasks: number lines, fractional circles, and word problems. The research employs an explanatory mixed-methods approach, combining quantitative and qualitative data. First, a total of ninety-one students in grades 5, 6, and 7 completed a Fraction Test to evaluate their achievement and a Unitizing Test to assess their strategies. Tasks include identifying fractions on number lines, using fractional circles, and solving word problems. Selected students then participated in interviews to explore their reasoning processes in greater depth. Nine students across grade levels and performance levels attended the interviews. The findings revealed that students struggled with the Unitizing Test, as their solutions reflected a notably higher proportion of none or unknown strategies. For others, two main strategies had emerged, namely Effective Unitizing and Procedural Unitizing. Students' use of either strategy varied for different grade levels and task types. Finally, the study highlighted the relationship between students' fraction performance and their unitizing strategies, suggesting that students with higher fraction achievement levels were able to use more sophisticated and successful unitizing strategies, and students with lower fraction achievement levels were more likely to show none or unknown strategies. These findings altogether can provide valuable insights into improving mathematical curricula by highlighting the diverse ways students conceptualize and use units.

ÖZET

ORTAOKUL ÖĞRENCİLERİNDE BİRİMLEŞTİRME STRATEJİLERİNİN KEŞFİ: SAYI DOĞRUSU, KESİR ÇEMBERİ VE SÖZEL PROBLEM GÖREVLERİNDEN ELDE EDİLEN BULGULAR

Birim kavramı, matematikte temel olup “bir” kavramıyla ve diğer önemli matematiksel yapılarla fazlasıyla ilişkilidir. Bu çalışma, ortaokul öğrencilerinin üç farklı görevde (sayı doğrusu, kesir çemberleri ve sözel problemleri) “birimleştirme” (birimleri tanımlama ve kullanma) stratejilerini incelemektedir. Araştırma, nicel ve nitel verileri birleştiren açıklayıcı bir karma yöntem yaklaşımı kullanmaktadır. İlk olarak, 5, 6 ve 7. sınıflardan toplam 91 öğrenci, başarılarını değerlendirmek için Kesir Testi ve stratejilerini değerlendirmek için Birimleştirme Testi'ni tamamladı. Görevler arasında, sayı doğrusunda kesirleri belirleme, kesir çemberlerini kullanma ve sözel problemleri çözme yer almaktadır. Seçilen öğrenciler daha sonra, akıl yürütme süreçlerinin incelenmesi için görüşmelere katıldı. Sınıf seviyeleri ve performans seviyeleri farklı olan dokuz öğrenci görüşmelere katıldı. Bulgular, öğrencilerin Birimleştirme Testinde zorlandıklarını ortaya koydu, çözümleri çoğunlukla hiç strateji kullanmama veya bilinmeyen stratejiler kullanma eğilimi gösterdi. Diğerleri için ise iki ana strateji ortaya çıktı: Etkili Birimleştirme ve Prosedürel Birimleştirme. Öğrencilerin her iki stratejiyi kullanımı, farklı sınıf seviyeleri ve görev türlerine göre değişiklik gösterdi. Son olarak, çalışma öğrencilerin kesir performansları ile birimleştirme stratejileri arasındaki ilişkiyi vurgulayarak, kesir başarı seviyesi daha yüksek olan öğrencilerin daha sofistike ve başarılı birimleştirme stratejileri kullanabildiklerini gösterdi. Bulgular, öğrencilerin birimleri kavramsallaştırma ve kullanma yollarının çeşitliliğini vurgulayarak, matematik müfredatının iyileştirilmesine yönelik değerli içgörüler sağlayabilir.

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1. INTRODUCTION

The concept of a 'unit' is the cornerstone of mathematical thought, yet its treatment in mathematics education is often narrowly confined to its role within fractions. This study challenges that narrow view, positing that the ability to 'unitize'—to flexibly conceptualize what constitutes '1' in a given context—is a foundational thinking process that begins with a child's first efforts to count. We argue that the 'idea of 1' is not static; it is a dynamic and flexible concept that underpins topics ranging from whole number operations to proportional reasoning. This research, therefore, explores the landscape of this fundamental skill in middle school students, examining how they create, use, and coordinate units across diverse mathematical tasks.

A unit can be defined as a single quantity that can be used to form larger quantities through repetition or smaller quantities through partitioning (Lamon, 2012). The most fundamental form of this single quantity is the number "1", as we built the whole numbers on repeating the number "1" multiple times, and form fractions through partitioning the number "1". Hence, to be able to understand the concept of a "unit", we need to understand the number "1".

Historically, the number "1" is the cornerstone of mathematics. It is believed that the emergence of mathematics stems from the need to count and keep records of numbers (Burton, 2010). The first traces of counting were found as marks on bones, and we believe that the purpose of those marks was for counting since they were so systematic as to be random (Rob, 2015). The marks evolved into tokens and undisputed rulers in various civilizations, laying the groundwork for arithmetic and measurement (Rob, 2015). Hence, as we search the history of mathematics, we come across different shapes of "1". Another interesting aspect of the history of "1" is how different civilizations represented the number "1" and developed their number systems based on it. The representation of the number "1" hasn't undergone significant changes, from the marks on bones to the "1" we use today, across different geographies and cultures; it is usually a vertical (rarely a horizontal) line or a dot (Özel and Terzi, 2023). Smaller numbers also resemble each other, symbolized through the use of the "1" multiple times. However, as the numbers increase, a need for grouping and forming new symbols arises. Behind all these, marks on the bones, tokens, rulers, number groups, and other numerous examples, we can see the need for creating the unique "1", or as we call it today, unit.

A child's mathematical development mirrors the history of the unit, beginning with the concrete act of counting. They first learn to use single objects as their fundamental unit—such as their fingers or a collection of pencils (Ifrah, 2000). At this stage, defining the unit is as simple as answering the question, "What am I counting?" However, proficient mathematical thinking quickly evolves from counting by ones to counting in groups. This act of grouping, where a collection of objects like a tens block becomes a new composite unit, is a student's first critical step in flexible unitizing. It is the first time they must understand that '1' can represent more than one thing. This flexible "idea of 1" is essential across the curriculum, influencing the understanding of place value, measurement, and fractions. As Allen (2014) stated, the complexity of the unit arises from its variable meaning, which can shift depending on the context. Therefore, the ultimate educational goal is to foster students' ability to think flexibly about any given quantity by mastering this dynamic understanding of the unit (Lamon, 2012).

Lamon (2012) defined unitizing as "the process of constructing mental chunks in terms of which to think about a given quantity" (p.104). In other words, we can explain the unitizing process as mentally creating various sizes and numbers of groups about a quantity. For example, we can think of 12 cookies as one dozen, two six-pack pieces, six pairs, or even twenty-four half cookies (Lamon, 2012). The total quantity did not change at all, but the way we explained the same quantity of cookies, i.e., the units we used, has changed. Although there has been limited research on students' use of unitizing strategies, studies on the understanding of unit coordination (Acar and Sevinç, 2021; Boyce and Norton, 2016; Norton et al., 2015), composite units (Lamon, 2012; Tzur et al., 2017), unitizing discrete and continuous quantities (Lamon, 1996), and the unit concept in fractional knowledge (Cramer et al.,2019; Hunting, 1983; Tunç-Pekkan, 2015; Yanık et al., 2008;) can offer insight into the aspects of the concept to focus on. Aside from providing ideas about the areas and specific tasks that can be used to assess students' unitizing strategies, some studies highlighted the importance of using multiple representations to reveal students' understanding of units and their unitizing abilities (Behr et al., 1983; Ni, 2000; Tunç-Pekkan, 2015)

Fractions represent one of the most challenging areas of mathematics for students, largely because a precise understanding of the part-whole relationship is contingent upon correctly identifying the "whole" or "unit". As Pantziara and Philippou (2012) argue, this ability to define the "one" is a key feature of mastering fractions, and misidentifying it is a primary source of error. While graphical representations are crucial for building this understanding, they can also lead to misinterpretations if not employed carefully. For example, students might view a half-shaded rectangle as two separate objects rather than as part of a single whole (Tunç-Pekkan, 2015), or perceive 0-5 number line as a single unit because they are accustomed to 0-1 number lines (Yanık et al., 2008). However, common instructional practices can inadvertently promote the misinterpretations and the kind of over-generalizations that limit flexible thinking. When, for instance, a whole circle is always presented as "1," students are not encouraged to think critically about the referent unit in contexts where the whole might be different. Similarly, teaching rigid procedures for placing fractions on number lines, such as treating the denominator only as the 'number of pieces to divide' and the numerator as the 'number of steps to take,' hinders the development of a more flexible and robust understanding of the unit.

While many studies in the literature focus on students' understanding of fractions, the foundational concept of unitizing remains less explored. Furthermore, existing research on unitizing often narrowly frames the concept as an act of partitioning or sharing – breaking a whole into smaller parts. Nonetheless, as defined earlier, unitizing is a bi-directional process that also includes creating larger quantities by iterating smaller ones. A significant need exists in the literature to address this more comprehensive view of unitizing across multiple perspectives. The present study was designed to address this gap. By employing purposefully designed tasks that move beyond rote procedures, this research encourages students to use units flexibly and provides a comprehensive synthesis of how they unitize by examining both their strategies and the thought processes behind them.

To achieve this, the purpose of this research is threefold. First, it seeks to analyze how students demonstrate and describe their unitizing strategies across number lines, fractional circles, and word problems, identifying the methods they employ and the obstacles they encounter. Second, the study examines the consistency of these strategies across task types to determine if, for instance, success in one context transfers to another. Third, the research explores the relationship between students' overall fraction performance and their use of unitizing strategies.

Ultimately, this study addresses an essential but underexplored area of mathematics education. By focusing on the nuances of unitizing strategies, it highlights the critical role of the unit in fraction learning and its implications for broader mathematical understanding. The findings are intended to provide valuable insights for educators, enabling them to design curricula and instructional strategies that promote a more flexible and meaningful interpretation of the unit, thereby fostering deeper and more comprehensive mathematical reasoning.

Research Questions:

(i) How do middle school students demonstrate and describe their use of unitizing strategies when solving different types of tasks, including number lines, fractional circles, and word problems?
(ii) What types of strategies do students use to solve tasks involving number lines, fractional circles, and word problems, and how do these strategies differ across task types?

(iii) Do students' solution strategies in one task type (e.g., number lines) exhibit consistency when applied to the other two task types (fractional circles and word problems)?

(iv) Is there a relationship between students' overall fraction performance and their ability to use unitizing strategies effectively?

2. LITERATURE REVIEW

When learning about fractions or rational numbers, students reorganize their number knowledge and develop meaning for the new symbols and numbers they are using (Pitkethly and Hunting, 1996). Previous research suggests that competence in fractions is crucial for achievement in various mathematical contents (Bailey et al., 2012; Hansen et al., 2015; Torbeyns et al., 2015), since it can be considered as the foundation of important mathematical topics such as algebra, proportional reasoning, and problem solving. Additionally, Ünal et al. (2024) highlighted through a meta-analysis that number line performance is more strongly associated with fraction performance than with other domains. Reorganizing the number knowledge in order to get to fractions requires the mastery of part-whole relationships, which rely on the identification of the whole and part. As discussed earlier, the identification of parts and the wholes could imply unit understanding.

A considerable amount of research has been conducted on the representations of fractions. Fractions can be represented in various ways, such as linear models (e.g., number lines and fraction rods), non-linear models (e.g., fraction circles), discrete objects, and real-world contexts. These representations are presented in the literature with a connection to students' unit understanding.

Number lines are powerful tools to represent fractions as they require students to reconcile their understanding of units with the spatial relationships on the line (Bright et al., 1988; Ni, 2000). The ways different studies used number lines to reveal students' understanding of units have some differences and similarities. There are many studies, including tasks that require placing fractions on number lines, that present students' fraction understanding through the use of number lines. These tasks typically involve 0-1 number lines and various fractions to be placed on these lines (e.g., Siegler, 2011). In this review, we will focus on the number line tasks designed to assess students' understanding of the unit. Yanık et al. (2008), for example, used number line tasks that asked students to place proper or improper fractions on given number lines, like a 0-5 number line. Their findings revealed various strategies that students employed and misinterpretations they made when solving the tasks. For example, some students treated a 0-3 number line as a unit of 1, and located 3/4 via dividing the whole number line into four parts and marking the third place. They argued that students' challenges in placing fractions as measures on number lines suggest that unit understanding does not develop naturally. Cramer et al. (2019) presented a different approach to number line tasks, considering the reconstruction of units. They presented unique number line tasks and aimed to distinguish how students solve and apply their prior knowledge in solutions. It was concluded that students solve the tasks generally by finding the unit fraction first (Cramer et al., 2019).

Area models can also be considered valuable tools for introducing fractions. Circular and rectangular models are also commonly used in instructions. Tunç-Pekkan (2015) used number line tasks alongside rectangular and circular area models to investigate whether students demonstrate similar performance across the tasks. She presented six problem types for each task, with changing levels of difficulty and purposes of the problems. Number line tasks included items that required placing fractions on existing number lines and items that required using unit fraction lengths as units. Tasks that include the other area models reflect similar learning objectives. Results were presented in terms of students' success in the task. The findings were important in revealing the changing abilities across task types. Similarly, Albin and Brown (2019) used these three representations to reveal students' conceptions of fractions. The important differentiable issue was that whole areas of a circle or rectangle represented the number one in all of the tasks. Number line tasks require placing fractions on already partitioned number lines. Their findings suggested that students struggle with the tasks when the quantity is bigger than 1.

The concept of unitizing surely is not specific to fractions. Discrete objects can be used to demonstrate or describe unitizing strategies as well. Lamon (1996; 2012) demonstrated the unitizing idea behind the use of discrete objects with real-world connections in her studies. In her study in 1996, sharing and marking strategies were defined for tasks requiring unitizing discrete objects, like a dozen eggs for three people, and continuous objects, like chocolate bars and pizzas. There are examples of problems with real-life context, including prices for different quantities, other than sharing problems. Some ways of chunking can be more advantageous than others depending on the context, and we can manipulate context to encourage flexible use of unitizing (Lamon, 2012). Lamon (2012) suggests that practices of finding unit prices are not useful if the interest is to build students' reasoning abilities. Tasks stressing the use of proportional reasoning or thinking multiplicatively can be helpful on this issue. In the study by Hansen et al. (2015), the importance of non-symbolic proportional reasoning was highlighted through its association with students' conceptual understanding of fractions.

Briefly, graphical representations such as number lines and fraction circles have been widely used to study students' understanding of fractions. Research by Bright et al. (1988) and Ni (2000) presents the power of number lines for illustrating fractions, through their association and the requirement of understanding units. Fraction circles, on the other hand, help students visualize fractions as parts of a whole, offering a different but complementary perspective. Purposefully designed word problems, as suggested by Lamon (2006), further enrich this exploration by requiring students to apply their conceptual understanding of units in practical scenarios.

This study builds on these insights to examine middle school students' use of unitizing strategies in three task contexts: number lines, fraction circles, and word problems. By exploring how students define and apply units across these diverse representations, the study aims to identify patterns and challenges in their reasoning processes. Furthermore, it investigates whether students' strategies remain consistent across different task types and whether their fraction achievement correlates with their ability to effectively use units.

2.1. Theoretical Framework

This study is built on multiple theoretical frameworks to explain students' unitizing strategies across multiple representations. Development of numerical thinking and the cognitive structure of representing rational numbers can be considered as the background of the thinking aimed to be investigated in this study. Numerical Development Theory, as presented by Siegler et al. (2011), highlights the gradual and essential shift from understanding whole numbers to rational numbers. The theory suggests that fraction magnitude knowledge differs from whole number magnitude knowledge, as it requires a rather complex way of thinking, and the use of strategies plays a larger role in the estimation of fraction magnitude in number line estimation tasks. Their findings also stress the importance of mental number lines in representing whole number and fraction magnitude. The consideration of students' understanding of numbers and the number development process was crucial for the current study to determine the appropriateness of the instruments for the chosen grade levels. Moreover, Steffe's (1992) Schemes of Actions Involving Composite Units further contribute to explaining students' understanding of the concept of unit through unit-coordinating and unit segmenting schemes, and the accommodations of the number sequences they made. The schemes by Steffe (1992) do not fit completely with the purposes of the current study, but they provide insights into how we can categorize students' unitizing strategies, as they explain how students sequence numbers under three categories. This framework presents the different number sequences students displayed through whole numbers. Hence, although it provides an important background idea for the present study, there is still a need for additional ideas and supporting frameworks. The analysis framework provided by Lamon (1996) provides insights for the framework used to analyze the findings of this study. The study by Lamon (1996) focused on the economy in students' partitioning and marking strategies, which inspired the investigation of how students can use unitizing strategies in the most effective way possible with less partitioning. However, the tasks used in the study by Lamon (1996) mostly suggest the generation of smaller quantities through sharing a larger quantity; in other words, they focus on partitioning strategies in unitizing, unlike the current study. The present study considers students' partitioning strategies as a step to understand students' unitizing strategies, as the tasks in this study require the generation of a larger quantity from smaller quantities. So, again, even though the framework provided essential insights, it was not possible to use the exact framework in the current study due to the differences in the structure of the tasks.

3. METHOD

3.1. Research Design

This study employs an explanatory mixed-methods design, in which quantitative research is conducted, and results are analyzed first, followed by qualitative research that builds upon those results to provide a thorough explanation (Creswell and Creswell, 2017). The research aims to examine students' use of units across number lines, fraction circles, and word problems, with a particular focus on the diversity and consistency of their strategies. The study consists of two phases: quantitative data collection through assessments and qualitative data collection through semi-structured interviews.

Quantitative research is primarily concerned with deduction, confirmation, explanation, prediction, and statistical analysis, while the major characteristics of qualitative research are induction, discovery, and exploration with the researcher as the main instrument of the data collection (Johnson and Onwuegbuzie, 2004). For this study, quantitative data obtained through the Fraction Test and the Unitizing Test provide an overview of the students' fractional knowledge and cues to understand their methods and strategies for unitizing tasks across a broader sample. The results of the quantitative data allow for making deductions and predictions about students' thinking. Meanwhile, the qualitative data built on the quantitative findings through semi-structured interviews allows for an in-depth exploration of unique insights and challenges and adds depth and meaning to the existing results. Hence, this study integrates quantitative and qualitative approaches to explore middle school students' unitizing strategies in various mathematical contexts.

3.2. Participants

The study involved a total of 91 middle school students (47 boys and 44 girls) from 5th, 6th, and 7th grades, selected through convenience sampling due to the accessibility and availability of schools to participate. The sample was drawn from two schools—one public and one private—to ensure a diverse representation of socioeconomic and educational backgrounds. The distribution of participants by grade level and school type is presented in Table 3.1. Testing was conducted in two sessions on two separate days; therefore, not all participants were present during the administration of both tests. 80 students completed the fraction test, and 83 students completed the unitizing test. These differences were taken into account during the data analysis phase. 9 students out of the total participants were interviewed. This subset included four 5th grade, two 6th grade, and three 7th grade students, representing a range of abilities, to provide comprehensive insights into unitizing strategies across different skill levels.

Table 3.1: Number of participants according to grade levels and school types.

Grade Level	**School Type**		**Total**
	Public School	**Private School**	
5th Grade	13	19	32
6th Grade	18	11	29
7th Grade	14	16	30
Total	**45**	**46**	**91**

Prior to the main study, a pilot study was conducted to evaluate the clarity, relevance, and feasibility of the translated fraction test. A separate group of 32 students from 5th and 6th grades completed the fraction test by providing feedback. These grade levels were purposefully selected to check if there are phrases or words that younger students are not familiar with. The data collected in the pilot study were not included in the main analysis.

All necessary permissions were obtained prior to the data collection. Official approval from the Ministry of National Education was secured to administer the tests in schools, and school administrations provided permission and collaboration. Parental consent was obtained via an online consent form that provided information about the study, the participation procedure, and the participants' privacy.

For ethical considerations, data collection was confidential. Participants are assigned codes to protect their identities, and any personally identifiable information is kept separately. Participants' data is and will only be known to the researcher, and no personal information will be disclosed to third parties. Additionally, all records are securely stored, and participants are assured that their involvement is voluntary and that they have every right to withdraw at any time without consequence.

3.3. Instruments

This study utilizes two primary instruments to explore students' understanding of fractions and their unitizing strategies. A Fraction Test assesses students' overall fraction knowledge, focusing on conceptual and procedural understanding. Additionally, a Unitizing Test, developed specifically for this research, evaluates how students identify and use units across various tasks. In the qualitative phase, semi-structured interviews based on the Unitizing Test results provide deeper insights into students' reasoning processes and the challenges they face. These instruments together ensure a robust exploration of students' mathematical understanding, forming the foundation for both quantitative and qualitative analyses.

3.3.1. Fraction Test

The Fraction Test, developed by Pantziara and Philippou (2012), is used to assess students' overall fraction knowledge. This test consists of 21 items designed to evaluate students' conceptual and procedural understanding of fractions across three stages: interiorization, condensation, and reification (Sfard, 1991, as cited in Pantziara and

Philippou, 2012). These stages progressively assess deeper levels of understanding, from step-by-step procedures to abstract reasoning about fractions. This test was translated and adapted into Turkish for this research.

The Fraction Test was translated into Turkish using a streamlined approach that integrates the Machine Translation + Human Quality Check method from Kunst and Bierwiaczonek (2023) with the validation framework by Sousa and Rojjanasrirat (2010). This process eliminated the back-translation step, focusing instead on forward translation and rigorous human validation. The translation procedure followed three main steps: (1) machine translation + human quality check, (2) pilot testing and cognitive debriefing, and (3) full psychometric testing.

In the first step of the translation, the original Fraction test was translated into Turkish using the AI-based translation tool DeepL to generate an initial draft. Then, two independent bilingual experts evaluated the machine-generated translation for accuracy, cultural relevance, and clarity and readability. A multidisciplinary committee, including educators, language experts, and researchers, reviewed the revised draft to resolve ambiguities and inconsistencies, resulting in the pre-test version of the Turkish Fraction Test. Examples for the revision process for some items are presented in Table F.1. Next, the pre-test version was administered to 32 middle school students, and they were asked to evaluate the clarity, relevance, and comprehensibility of the test items. Students who found items unclear were asked to provide feedback and suggestions for improvement. Any items identified as unclear by more than 20% of the participants were decided to be revised based on their feedback. According to the pilot test results, this was not necessary for any item. Lastly, the final Turkish version was administered as part of the main study.

A total score was obtained through scoring each item on a scale ranging from 0 to 4, based on the rubric developed for this study (Table F.2). In the original version of the test, scores 0 and 1 were used for incorrect and correct answers, and the pilot test results were assessed as similar to the original version. However, the pilot test results revealed the necessity of partial scoring. To better reveal the depth of students' fraction understanding, a rubric was developed based on the patterns in students' answers from pilot testing. Internal consistency was assessed using Cronbach's alpha. The overall alpha for the translated test was .87, indicating a good reliability (George and Mallery, 2010).

3.3.2. Unitizing Test

The Unitizing Test, developed by the researchers, was designed to assess students' unitizing strategies across three task types: number lines, fractional circles, and word problems. This test includes 11 number line tasks, 8 fractional circle tasks, and 5 word problems, with overlapping items allowing comparison of strategies across contexts. Task development was guided by prior studies (e.g., Cramer et al., 2019; Lamon, 2006; Tunç-Pekkan, 2015). To ensure the test's validity, nine expert reviews are sought, and necessary revisions are made based on their feedback.

The tasks in this study are designed to evaluate students' ability to employ unitizing strategies across different contexts: number lines (NL), fraction circles (FC), and word problems (WP). Each task requires students to use and explain their unitizing approaches, providing insights into their reasoning.

For instance, in NL tasks, students must iterate a given interval (e.g., "2/3") to locate a specific point on the number line. They may achieve this by using "2/3" as their unit directly or by finding the unit fraction first and iterating it. Similarly, FC tasks require students to demonstrate analogous strategies. They might represent "2" using "2/3" as a repeated unit or determine what "one circle" represents before finding the required fraction. In WP tasks, students solve word problems using comparable strategies, such as multiplying a given quantity to make comparisons or finding the unit price of items for direct comparison.

The initial version of the test included 18 number line, 8 fraction circle, and 9 word problem tasks (Appendix C). Expert opinions were acquired for the initial version of the developed Unitizing Test, and necessary revisions were made according to their comments and suggestions. Mathematics education researchers and middle school mathematics teachers were invited to rate the scope, clarity, and appropriateness of each question for the target audience. They were asked to evaluate each question using the following categorized evaluation criteria: (a) appropriate, (b) needs improvement, (c) not appropriate. Nine experts shared their evaluations, feedback, and suggestions, which were later analyzed to determine the questions that needed revision. Questions marked as "appropriate" with less than 60% were significantly refined or removed. Experts were also invited to indicate comments and suggestions for individual questions. All comments are examined and summarized for the refinement of the questions.

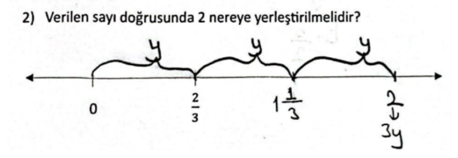
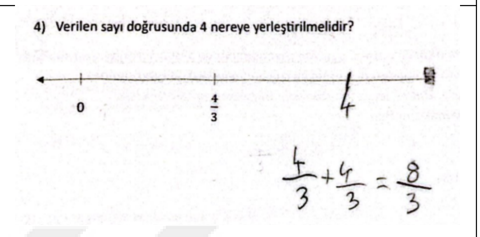
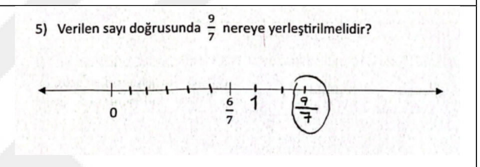
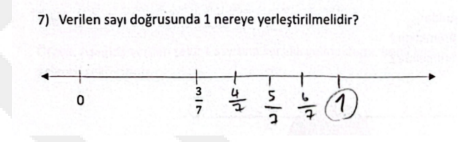
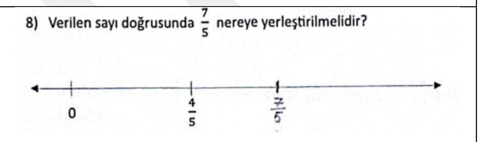
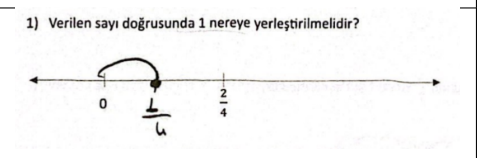
The number of prepared questions was more than needed, in case some of them would be eliminated after the expert review. There were two questions having the same learning objectives in some cases, so the more appropriate ones were selected to be included in the finalized version. For example, three number line (questions # 6, 16, and 18) and two word-problem (questions # 3 and 8) tasks were removed due to their higher computational load, and their alternatives were included. From the other question pairs with the same learning objective, the better-fitted one was kept in the finalized version of the test. Additional minor revisions were made in response to the experts' feedback. These revisions include the change of words used in the problems, the adjustment of the order of some tasks, and the refinement of the tasks. For instance, in the initial version, the instructions for the number line questions were "Where is ... on the given number line?", and it was converted into "Where should ... be placed on the given number line?" after the expert feedback. Additionally, some tasks used to demand placing, drawing, or calculating more than one number in a single task, like "...how much does 7 kg of flour cost? How much does 5 kg of flour cost?" Experts suggested that students may overlook and not respond to all of the questions asked. So instead, we created sub-questions in these types of tasks and presented them underneath each other to make them more visible. Another important revision was the addition of a more basic task at the beginning of the number line and fraction circle tasks (1st question of the number line and fraction circle tasks in the finalized version of the test). These types of tasks are somehow different than the type of tasks they used to deal with, especially fraction circle tasks, as stressed by some of the experts as well. So, a simpler question for the two types was created to familiarize students with the tasks.

The finalized version of the test was administered through the main study. The test results were evaluated through coding students' strategies. The codes were generated and revised consistently before, during, and after the data collection process. Initially, some categories were created during the test development process based on expected or possible student strategies, the mathematical properties involved, and the characteristics of the tasks. These categories, with their explanations, were presented to the experts as well to provide validation for future coding procedures. All categories were listed below each item, and experts were asked to select the categories that could be applied for the item. The categories presented in Appendix D. They were also requested to write down additional categories if necessary. By revising the categories through expert feedback, possible solution strategies and mathematical properties students need to use in some tasks were identified before data collection began.

After the initial reviews of students' solutions, a need to consider possible misconceptions and error patterns emerged. Students' answer sheets were reviewed multiple times to spot the error patterns and common misconceptions. New categories were added for those, and additional revisions were made to complete the initial coding. These initial codes were used to detect the areas that required further attention, so they were used to select the participants for semi-structured interviews.

Following the completion of the semi-structured interviews, codes were refined and finalized, and all data from the Unitizing Test were coded from scratch for quantitative analysis. Finalized codes are presented in Table 3.2. The coding structure consists of two steps, represented by two-digit numbers. The first step was to determine the unitizing strategy used. The strategy was determined as “Effective Unitizing” when students created units with less partitioning possible. Students could use the given intervals, shapes, or numbers as their unit directly in some of the questions, or they could create a new unit with minimum partitioning to be considered as using this strategy. On the other hand, the strategy was labeled as “Procedural Unitizing” if students used unit fractions, unit prices, or the number “1” to reach the solution. Finally, if the students’ solutions do not indicate the use of the two strategies, their solutions were coded as “Unknown-None”. These three main categories determine the first digit of the code. The second digit of the “Effective Unitizing” and “Procedural Unitizing” was determined according to the correctness of the solution; if the students’ solution was correct, the second digit was coded as “1”, otherwise, if there were minor errors or omissions in the solution, the second digit was coded as “2”. For the “Unknown-None” category, the number “1” in the second digit also represents the correct answer, but in this category, it means that the strategy used by students is unknown. The correct answer might be due to estimations, mental work, or luck. The code “32” was used when students did not demonstrate the use of either unitizing strategy, or reach the right answer due to an identified error pattern. Finally, if there are no meaningful solutions or no solution at all, the answer was coded as “33”. Some examples from students’ answers on number line tasks are presented in Table 3.2 to provide more insights into the codes.

Table 3.2: Coding structure for the Unitizing Test.

Category	Description	Example
Effective Unitizing	The strategy was implemented and explained/demonstrated successfully. (Code: 11)	
	Strategy displayed, but there are errors or omissions in the solution. (Code: 12)	
Procedural Unitizing	The strategy was implemented and explained/demonstrated successfully. (Code: 21)	
	Strategy displayed, but there are errors or omissions in the solution. (Code: 22)	
Unknown – None	The answer is correct, but the solution strategy is unknown (Code: 31)	
	Error patterns identified. (Code: 32)	
	No solution or random work. (Code: 33)	

3.3.3. Semi-Structured Interviews

Following the assessments, semi-structured interviews aim to explore how students demonstrate, explain, and reflect on their unitizing strategies in number line, fraction circle, and word problem tasks. The interviews follow the hierarchical method described by Tomlinson (1989) to ensure detailed exploration of students' thought processes. This approach ensures that the interviews remain both flexible and structured, enabling the researcher to explore students' reasoning processes in-depth while systematically addressing key aspects of their unitizing strategies.

The hierarchical method involves structuring the interview into layers or levels of focus. The process starts with broad, open-ended questions that allow participants to express their initial thoughts and strategies. Gradually, the questions become more specific, directing attention to particular aspects of their reasoning and solution processes. This method encourages participants to reflect on their understanding and provides the researcher with insights into the progression and structure of their thinking.

In the opening layer, General Exploration, students were asked broad questions about their initial approach to the tasks, such as "Can you explain how you approached this problem?" or "What were you thinking when you started solving this task?" This phase allowed students to describe their thought processes in their own words, providing a general overview of their strategies.

In the intermediate layer, Focused Inquiry, the interviewer gradually shifted to more targeted questions, encouraging students to elaborate on specific aspects of their solution. Examples include "How did you decide to make this division in this task?" or "What are the things you paid attention to in this task?" These questions aim to clarify the reasoning behind their chosen strategies and identify patterns or inconsistencies in their thinking.

In the in-depth layer, Deep Probing, the interviewer addressed potential misconceptions, alternative strategies, or deeper reasoning about the concept of the unit, such as "Could you explain what you think the whole represents in this problem?" or "If you were solving this task again, would you do anything differently? Why?" This phase was designed to uncover the underlying principles of their reasoning and highlight any gaps in understanding.

Finally, in the closing layer, Reflection and Synthesis, Students were encouraged to reflect on their overall approach and learning through questions like "Which type of task do you think was the easiest or hardest to solve? Why?" and "Do you think the same strategy can work for all types of problems? Why or why not?" This reflection helps students consolidate their understanding and provides valuable insights for the researcher about the generalizability of their strategies.

To analyze the semi-structured interviews, the six-phase approach to thematic analysis by Braun and Clarke (2006) was used. To gain insight into the data, transcripts of the interview were prepared and read to form initial ideas. To create initial codes, the systematically identified notable features of the data were coded. Relevant data and potential themes were explored to identify themes. Then, the themes and coded extracts were reviewed and adjusted to check how they fit together. Following this step, clear definitions and names for each theme were created. Finalized codes are presented in Appendix E. Finally, the findings of the interview data are reported in the following sections.

3.4. Procedure

The data collection procedure consisted of two sessions: assessments and interviews. In the first session, assessments, students were tested in their classrooms using a paper-and-pencil method. The two tests, Unitizing Test and Fraction Test, were administered on two separate days, and students were given 40 minutes to complete each test. The tests were administered under the supervision of the researcher. Students were encouraged to show all their work and provide detailed explanations for their solutions. An extra ten minutes was provided to students if needed.

Following the assessments, a subset of students participated in semi-structured interviews to explore their reasoning processes in greater depth. The interviews, which lasted approximately 30–45 minutes per student, were audio-recorded (with consent) for subsequent transcription and thematic analysis. The hierarchical structure ensures consistency across participants while allowing for individualized exploration of unique insights and challenges.

3.5. Data Analytic Strategies

This study employs an explanatory mixed-method design by integrating quantitative and qualitative designs. First, for the quantitative data analysis, the Fraction Test results for the pilot study were calculated to check the internal consistency of the test through KR-20. For the main study, z-scores of the Fraction Test results were calculated to determine students' fraction achievement levels compared to their peers. For the Unitizing Test, students' solution strategies for each task were coded following the developed coding structure. A graduate student, trained in the coding structure, coded 12% of the data, and the consistency of the codes was compared to provide reliability evidence, the percent agreement rate found as 84%. Coded unitizing test assessments were used to summarize students' tendency to use different unitizing strategies for different task types. Percentages of each code used in different tasks were calculated separately for each grade level. Total scores for the Unitizing Test were also calculated. The relationship between the Fraction Test and the Unitizing Test scores was investigated through the Pearson's correlation coefficient, to provide validity evidence for the developed Unitizing Test via a previously validated Fraction Test.

Second, students' strategy use on the Unitizing Test was summarized for each of the three task types (NL, FC, and WP). The analysis focused on the frequency of each strategy category, calculating the percentages for 'Effective Unitizing,' 'Procedural Unitizing,' and 'None/Unknown' separately. This allowed for a detailed comparison of which strategies were most prevalent among students at different achievement levels. To present these findings, both summary tables and heat maps were created to visualize the relationship between fraction achievement and strategy use for each grade level.

To determine the relationship between students' fraction achievement and their use of unitizing strategies, their performance was analyzed in two stages. First, students were categorized into three fraction achievement levels (high, medium, and low) for each grade, based on the percentile ranks calculated from their Fraction Test z-scores. Second, students' strategy use on the Unitizing Test was summarized for each of the three task types (NL, FC, and WP). The analysis focused on the frequency of each strategy category, calculating the percentages for Effective Unitizing, Procedural Unitizing, and None/Unknown separately. This allowed for a detailed comparison of which strategies were most prevalent among students at different achievement levels. To present these findings, both summary tables and heat maps were created to visualize the relationship between fraction achievement and strategy use for each grade level.

To analyze qualitative data collected through the semi-structured interviews, the six-phase approach to thematic analysis by Braun and Clarke (2006) was used. This method produced a comprehensive, well-structured, and reflective database by following these steps: (1) gaining insight into the data, (2) creating initial codes, (3) exploring and identifying themes, (4) examining and refining themes, (5) defining and labeling themes, and (6) compiling the report. Interviews were transcribed and reviewed multiple times to form initial codes. Identified features and potential themes were labeled using MAXQDA 2024. The themes and coded extracts were reviewed and revised through discussion to create the finalized version of the coding structure. Following this procedure, the interview data were recoded and prepared for analysis.

4. RESULTS

Descriptive analyses were conducted on the scores of both the Fraction and Unitizing tests. Scores of both tests were out of 100 total points. The Fraction Test scores ranged from 4.76 to 90.48 ($M = 39.87$, $SD = 20.48$, $N = 80$), while the Unitizing Test scores ranged from 0.00 to 95.83 ($M = 28.71$, $SD = 25.60$, $N = 83$). The boxplots in Figure 4.1 demonstrate a noticeable difference in students' performance across the two tests. Beyond the differences in mean scores, the median score for the Unitizing Test was 18.75%, which is notably lower than the 37.50% found for the Fraction Test. The distribution of the Unitizing Test scores is skewed towards the lower range, resulting in the presence of nine outliers. This distribution, along with the outliers, suggests that while most students struggled with the test, a few demonstrated strong understanding. In contrast, the distribution for the Fraction Test scores was more even, with only two outliers at the higher end. These differences indicate a generally poorer performance on the Unitizing Test, suggesting it may have been more challenging for students.

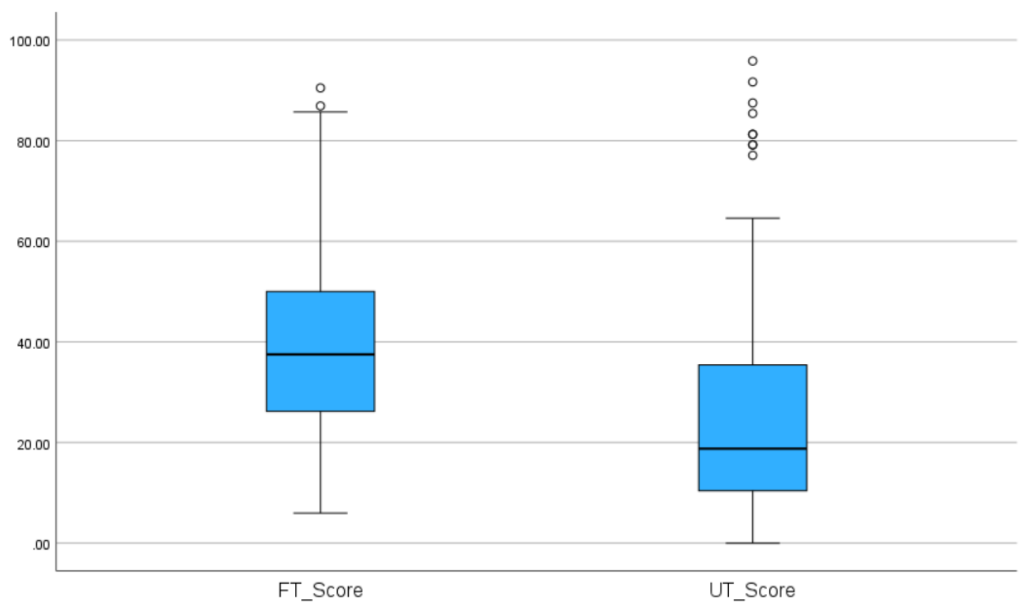


Figure 4.1: Boxplots for the Fraction Test scores and Unitizing Test scores.

Initially, a Pearson correlation analysis was conducted to examine the relationship between the Fraction and Unitizing tests. However, the assumptions of normality, linearity, and homoscedasticity were violated. Shapiro-Wilk normality test results suggested that the distributions were significantly different from a normal distribution, and the scatterplot demonstrated a quadratic relationship rather than a linear one. Considering the violations and distribution, a regression analysis was conducted to examine the relationship between students' Fraction and Unitizing test performances. It was found that students' fraction performance can predict their unitizing skills, $F(2, 69) = 94.34$, $p < .001$. This model explains 73% of the variation. The distribution of the relation presented in Figure 4.2 demonstrates the curvilinear relationship between the test scores. Parameter estimates for the model are presented in Table 4.1.

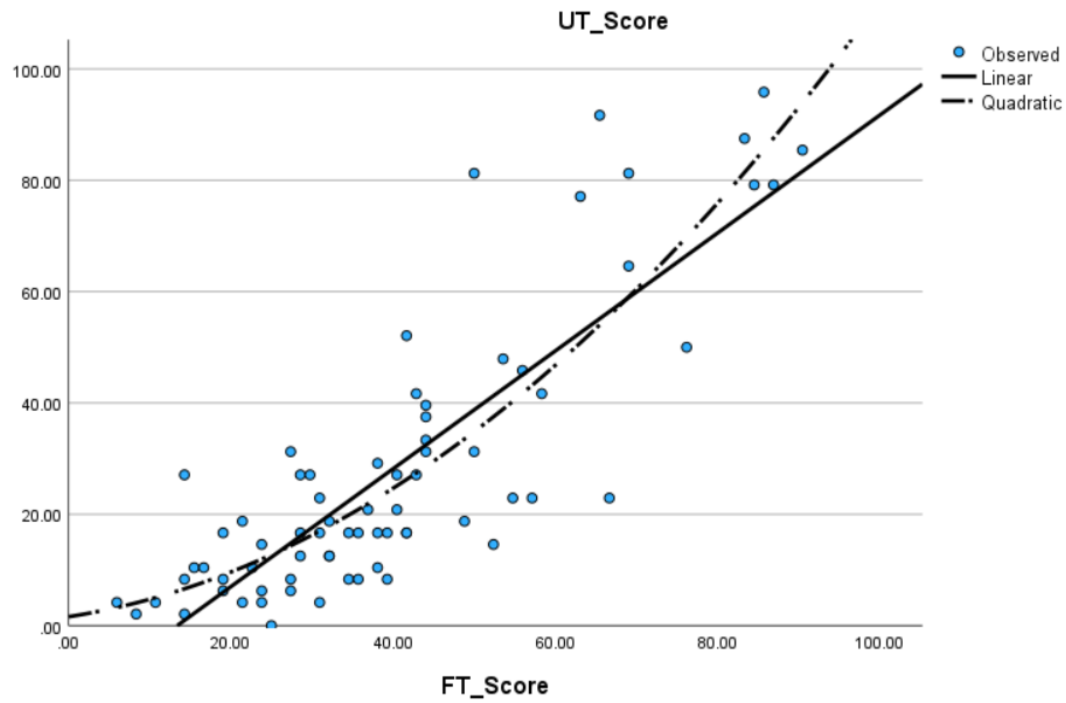


Figure 4.2: Relationship between the Fraction Test scores and Unitizing Test scores.

Based on these results, the final regression equation is:

$$\mathrm{UT} = 0.77 + 0.13\,(FT\ Score) + 0.006\,(FT\ Score)^2.$$

The analysis confirms that the quadratic term (FT Scores^2) is a significant predictor of unitizing performance (p = .008), while the linear term (FT Scores) is not (p = .478). This supports the initial observation of a curvilinear, rather than purely linear, relationship between the two test scores.

Table 4.1: Parameter estimates.

Term	**Estimate**	**Std. Error**	***t***	**Sig.**
Intercept	0.769	3.247	0.24	0.814
FT Scores	0.129	0.18	0.71	0.478
FT Scores^2	0.006	0.002	2.73	0.008*

Table 4.2 clearly demonstrates that the use of effective unitizing strategies is strongly linked to students' fraction achievement levels across all grades. A distinct pattern emerges where high-achieving students consistently favored Effective Unitizing strategies, showing a developmental trend where the use of successful effective strategies (Code 11) increased from 37% in 5th grade to 56% in 7th grade. Conversely, low-achieving students overwhelmingly produced solutions categorized as "Unknown-None". For this group, the "No solution or random work" category (Code 33) was the most frequent outcome, accounting for 56% of responses in 5th grade, 48% in 6th grade, and peaking at 67% in 7th grade. Meanwhile, mid-level students showed a mix of strategies, often relying on procedural methods with errors (Code 22) and frequently resorting to no discernible strategy (Code 33). The frequencies of students' strategy use across task types and fraction achievement levels are presented for each grade level more detailly in Table F.3-5. Overall, the data indicate that stronger foundational fraction knowledge corresponds with the ability to deploy more sophisticated and successful unitizing strategies.

Table 4.2: Percentages of students' summarized unitizing strategies according to their fraction achievement level.

	Effective Unitizing		**Procedural Unitizing**		**Unknown–None**		
	SI (11)	**EM (12)**	**SI (21)**	**EM (22)**	**CU (31)**	**ER (32)**	**N (33)**
5th Grade							
High-Level	37%	4%	11%	19%	4%	26%	0%
Mid-Level	6%	0%	9%	21%	9%	18%	36%
Low-Level	7%	0%	4%	4%	0%	30%	56%
6th Grade							
High-Level	38%	0%	33%	10%	0%	10%	10%
Mid-Level	4%	8%	8%	13%	4%	29%	33%
Low-Level	11%	0%	4%	15%	0%	22%	48%
7th Grade							
High-Level	56%	0%	22%	11%	6%	0%	6%
Mid-Level	19%	0%	29%	10%	0%	14%	29%
Low-Level	0%	0%	11%	17%	0%	6%	67%

Note. Achievement levels (High, Mid, Low) were determined by students' performance on the Fraction Test. The codes represent the type and outcome of the strategy used: 11 = Effective Unitizing, successful; 12 = Effective Unitizing, with errors; 21 = Procedural Unitizing, successful; 22 = Procedural Unitizing, with errors; 31 = Unknown strategy, correct answer; 32 = Identified error patterns; 33 = No solution or random work.

Figure 4.3 provides a visual representation of the direct relationship between students' fraction achievement and their choice of unitizing strategy across different task types. In Figure 4.3, each panel represents a grade level, with individual students shown as rows and sorted vertically by their percentile rank on the Fraction Test. Horizontal lines separate students into high, mid, and low fraction achievement groups. Columns represent Number Line (NL), Fraction Circle (FC), and Word Problem (WP) task types. The two-digit codes in each cell indicate the strategy used: the first digit represents the strategy type (1 = Effective Unitizing, 2 = Procedural Unitizing, 3 = None/Unknown), and the second digit represents the outcome (1 = Correct, 2 = Incorrect). The shading corresponds to the effectiveness of the strategy, with darker shades indicating more effective strategies (i.e., codes beginning with '1').

A clear pattern emerges across all three grade levels: students with higher fraction achievement percentiles, located at the top of each panel, consistently employed more effective unitizing strategies (codes beginning with '1', indicated by darker shading). Conversely, students in the lower achievement percentiles predominantly used

'None/Unknown' strategies (codes beginning with '3', indicated by pale or no shading), particularly the 'empty or random work' strategy. This trend is especially pronounced among 7th graders, where high-achieving students almost exclusively used effective or procedural strategies, while low-achieving students almost entirely defaulted to no discernible strategy. The heat map visually confirms that a stronger foundational knowledge of fractions is strongly associated with the ability to deploy more sophisticated and successful unitizing strategies.

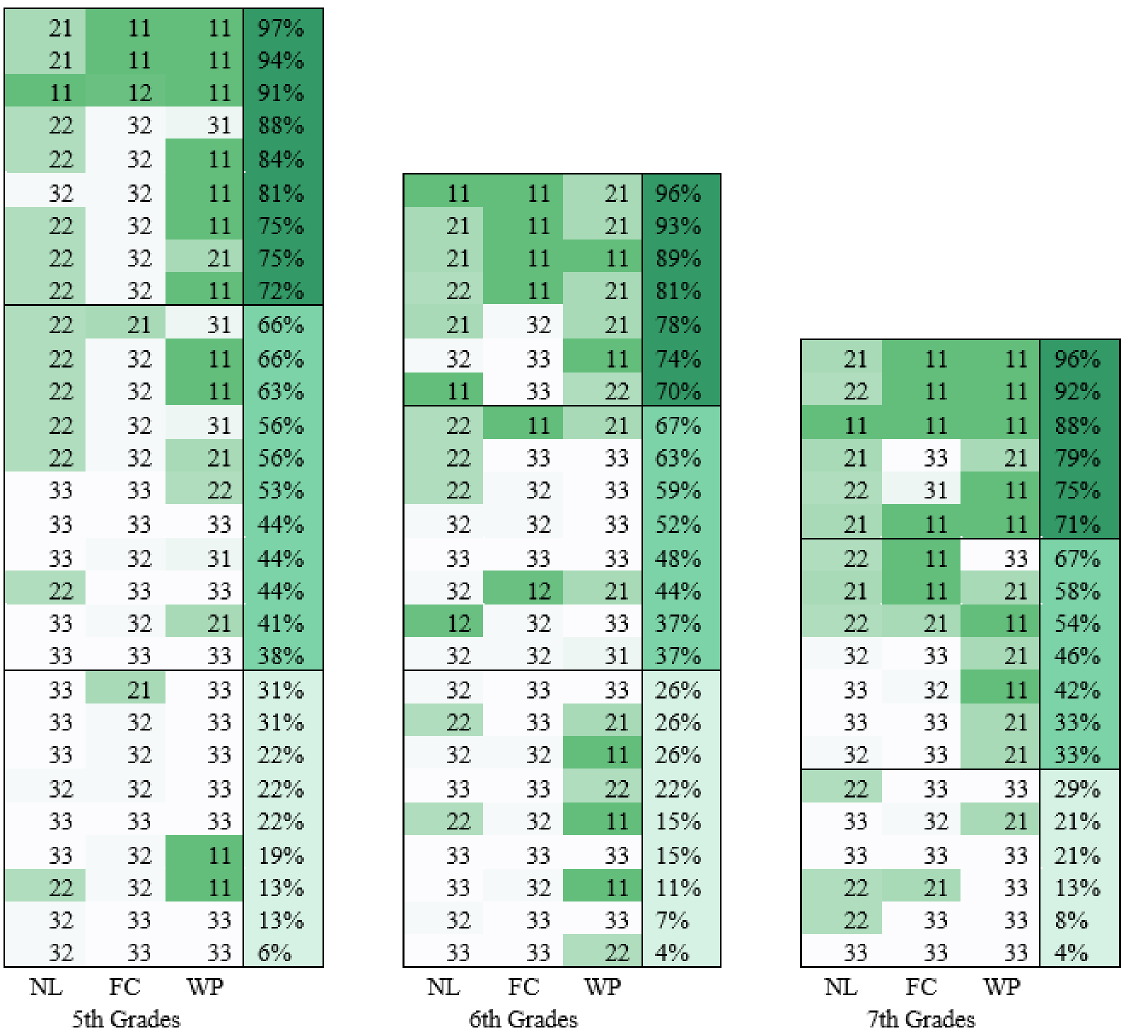


Figure 4.3: Heat map of unitizing strategies used by individual students, organized by fraction achievement percentile.

Analysis of strategy use across different task types, as presented in Table 4.3, reveals several key patterns. The most prominent finding is that more than half of the solutions for each task type and each grade level were attributed to the "none/unknown" category, indicating a general difficulty with the unitizing test. One of the important findings is that students mostly demonstrated a procedural unitizing strategy rather than the effective unitizing strategy for the number line tasks. In contrast, for Fraction Circle (FC) tasks, 6th and 7th graders were more likely to use Effective Unitizing, although this task type proved especially challenging for 5th graders, with 90% of their responses falling into the "none/unknown" category. A plausible reason for this difference is that 5th grade students tended to ignore the referent they were given in the fraction circle tasks; rather, they identified a whole circle as "one" and solved the tasks accordingly. The choice of strategy was quite close for word problem tasks for 5th and 7th grade students in favor of the effective unitizing strategy. However, 6th grade students preferred the procedural unitizing strategy more, through calculating unit prices in their solutions. Another notable result is that students of all grades struggled more in fraction circle tasks, and errors appeared more in this task type. Moreover, 5th and 6th grade students displayed any type of strategy more in the number line tasks, while it was the word problem tasks for 7th graders.

4.1. Unitizing Strategies

While the preceding data quantifies the frequency of different strategies, a deeper understanding requires a qualitative examination of what these strategies entail. To provide this clarity, this section delves into the nature of the solution strategies identified through semi-structured interviews and analysis of student work. Through constant review and revision of interview transcripts, two primary approaches emerged: Effective Unitizing, which is characterized by conceptual understanding and Procedural Unitizing, which relies on the application of learned algorithms.

Table 4.3: Percentages for total use of each strategy across task types.

Grade Level	Task Type	Effective Unitizing	Procedural Unitizing	None/ unknown
5th Grade				
	NL5	3%	29%	68%
	FC5	5%	5%	90%
	WP5	11%	10%	79%
6th Grade				
	NL6	10%	28%	62%
	FC6	14%	8%	77%
	WP6	7%	18%	75%
7th Grade				
	NL7	8%	33%	59%
	FC7	22%	12%	67%
	WP7	22%	24%	54%

4.1.1. Effective Unitizing

In the number line tasks, an Effective Unitizing strategy was identified when students used a given interval as a composite unit, iterating it to locate the target number. This approach demonstrates an understanding of the number line as a measurement model where lengths can be preserved and repeated. The two student solutions in Figure 4.4 and Figure 4.5, for example, illustrate two distinct manifestations of this strategy to locate the integer "2."

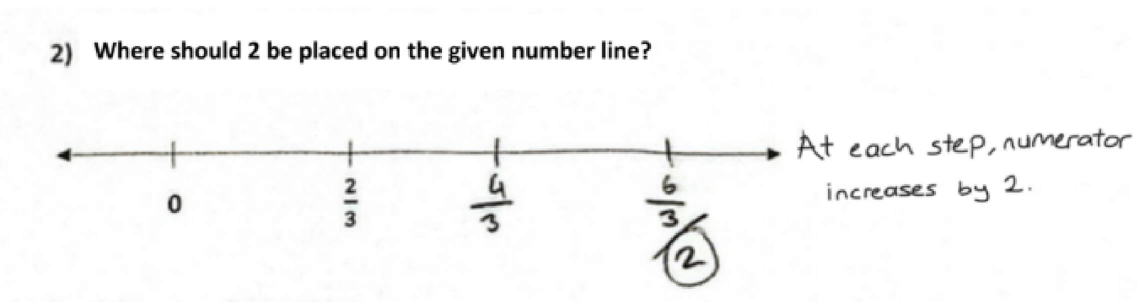


Figure 4.4: An example solution from the Unitizing Test.

In the top example, the student uses an arithmetic approach. They identify the initial interval as having a value of 2/3 and then repeatedly add this value, tracking the change in the numerator: 2/3, 4/3, 6/3. Their annotation, "At each step, numerator increases by 2," confirms this thinking, leading them to correctly identify 6/3 as equivalent to 2.

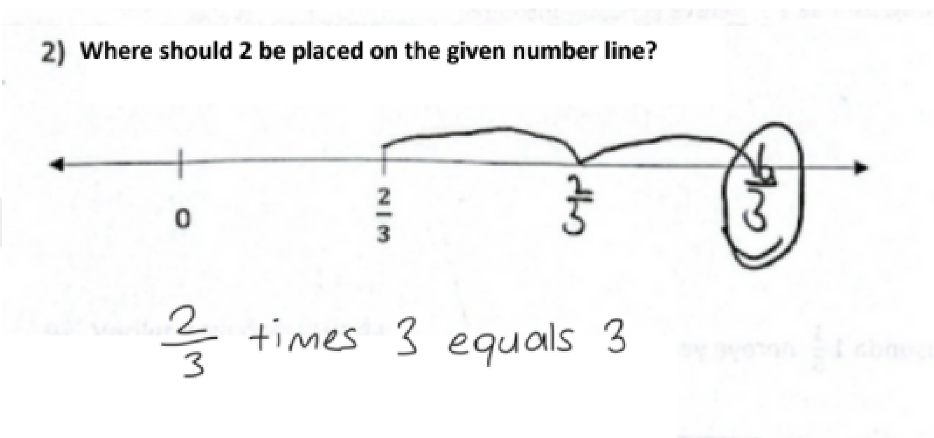


Figure 4.5: An example solution from the Unitizing Test.

In the bottom example, the student employs a more spatial strategy by visually iterating the length of the interval from 0 to 2/3 three times to land on the target location. This iterative thinking is confirmed in the interview, with the reasoning the student provided:

> "Student: By measuring it in my mind, I calculated that it would be 3 times more than that, 6 out of 3, since it is equal to 2."

This quote shows the student understood that three iterations of the 2/3 unit were needed. The student in the figure successfully executes this spatial strategy, correctly labeling the final location as 6/3. Together, these two examples powerfully illustrate that Effective Unitizing is not a single method, but a flexible conceptual understanding that students can express in different valid ways. While some students increased the denominator by multiples of a number, as in the examples above, others explained the use of the interval through objects or labels. For instance, one student articulated this "measurement" mindset clearly when describing his approach. The cognitive foundation for this advanced skill-the ability to treat intervals as tangible, measurable lengths-was evident in students' descriptions of their thinking:

> "S: When I looked at this question, at first, I wanted to calculate the distance. I may have used tools like a pencil sharpener to measure the same distance. At the same time, I found that distance in my mind by doubling it and moving it to the side. So, I applied such a solution."

While many tasks could be solved by iterating the given interval, some were designed to elicit more complex reasoning where the target number was not a direct multiple of the given unit. In these cases, students could not simply repeat the unit; they had to create, smaller unit. The most successful students demonstrated a sophisticated form of effective unitizing by applying minimum partitioning. This strategy required coordinating two units at once: the given interval and a smaller, common unit they created by partitioning, as in Figure 4.6.

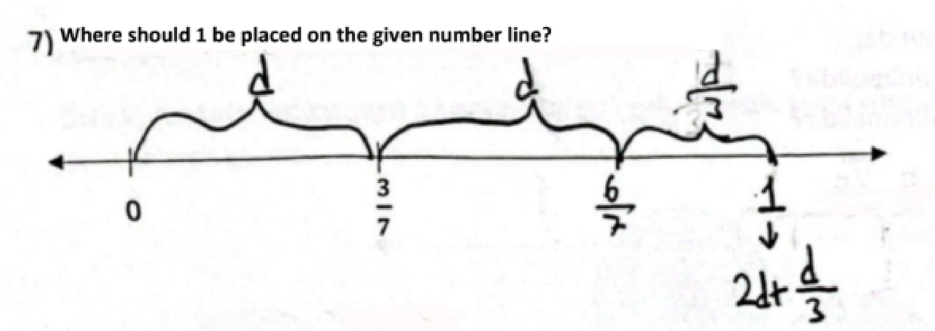


Figure 4.6: An example solution from the Unitizing Test.

A similar strategy was elaborated on in an interview with another student. Notably, this student reported this spatial approach as his alternative solution, with his primary method being the use of unit fractions. This strategic flexibility, rooted in a conceptual understanding of intervals as manipulable quantities, is precisely what enables the more complex partitioning needed for challenging tasks:

"S: Now we could have drawn one more, the previous distance of 3 in 7. Then we could have drawn one of the smaller distances that I have drawn here.
Researcher: When we draw another one from 3 divided by 7, where do we reach?
S: We have reached 6 divided by 7. To reach 7 divided by 7, we could have drawn one of the small lines I drew from 3 in 7 to 7 in 7."

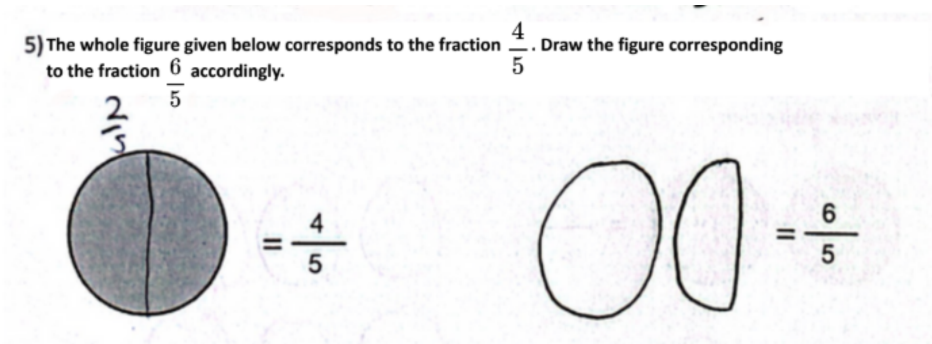


Figure 4.7: An example solution from the Unitizing Test.

For the fraction circle tasks, the linear distances in the number line tasks were replaced by circular shapes and their corresponding parts. As with the number line tasks, students could either use the given circular pieces as their unit or apply minimum partitioning to create a new unit for constructing the target fraction or integer. The following interview excerpts reveal how different students explained their reasoning for the same solution method, which is illustrated in Figure 4.7:

"S: In our solution here, I thought that half of four-fifths multiplied by three is six-fifths, but we cannot directly use 4 in 5 to find 6 in 5. So, I created a unit for myself.
R: What was the unit you created here?
S: Half of the circle, or 2 in 5. Multiplying this unit by 3 equals the whole we want to find."

"S: If 4 out of 5 is equal to one circle, it asks us about 6 out of 5. Then we will take half of the circle since 2 out of 5 is half. Then one of them is already equal to 4 out of 5. I have already drawn one circle there. I drew the half circle on the side, which is 2 out of 5."

"S: Now, if this place is equal to 4 divided by 5, I thought we could add another half to equal 6 divided by 5. So, I drew something like this."

In word problems, this strategy is characterized by the ability to create and use the given chunks, either by mentally grouping items or by using the groups provided in the problem. This requires students to identify and apply the multiplicative relationship between given quantities or prices. The solution in Figure 4.8, and the accompanying explanation illustrates a key principle of this approach: finding the direct multiplicative relationship between numbers is more efficient than performing unnecessary intermediate calculations, such as division.

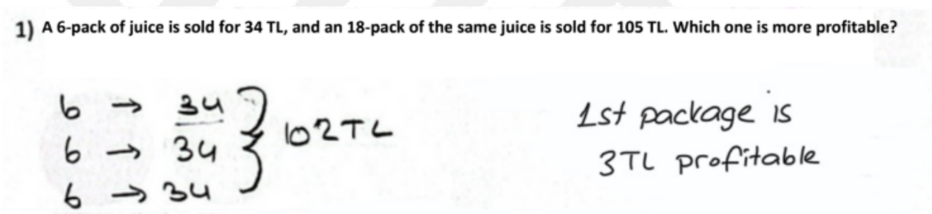


Figure 4.8: An example solution from the Unitizing Test.

"S: I've completed the sixes into 18-packs. Here... I found out how much it would cost in total. Then, a pack of 18 normally costs 105 TL. The pack of 6 is 34 TL each. Adding the 34 together, or multiplying by 3, makes 102. So, it's more profitable to buy this."

When word problems involved numbers that were not direct multiples of one another, the most successful students demonstrated sophisticated reasoning by creating a common basis for comparison. A powerful example of this is seen in the explanation for the problem in Figure 4.9. Instead of immediately performing a calculation, the student first articulated the principle of the strategy they intended to use by providing a simpler, analogous example of a multiplicated relationship:

"S: ... As 9 is one and a half times 6, 180 is one and a half times 120."

Here, the student uses a hypothetical case to explain the concept of scalar reasoning, which involves finding a single multiplicative factor that connects both quantities and prices. After explaining the general principle, the student then applied this type of relational thinking to solve the actual problem in Figure 4.9:

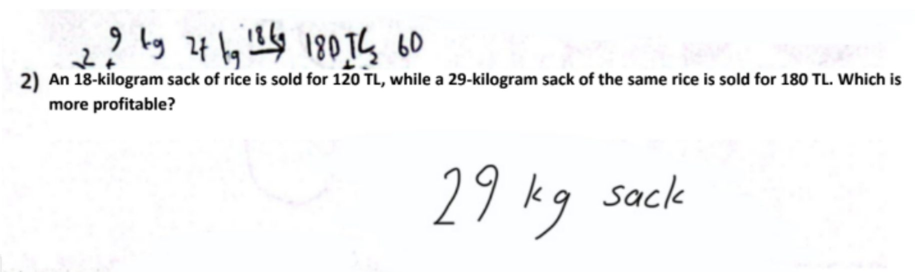


Figure 4.9: An example solution from the Unitizing Test.

A relevant quote reveals the student's elegant solution. By taking the first option (18 kg for 120 TL) and adding half of its quantity and price (9 kg for 60 TL), they mentally calculated that 27 kg of rice would cost 180 TL. This created a common price point for a direct comparison: 27 kg for 180 TL versus the second option of 29 kg for 180 TL. This strategy of establishing a common measure is a hallmark of effective and flexible unitizing in complex comparison tasks:

> "S: Since we have a portion of 18, if we find half of 18 and add the original kilogram, which bags will weigh more for 180 TL."

4.1.2. Procedural Unitizing

The strategy of Procedural Unitizing is characterized by the use of unit fractions or unit prices to reach a solution. In the number line tasks, this approach typically involved students partitioning a given interval into unit fractions. For example, a student might break down an interval of 2/3 into two separate 1/3 segments and then use these smaller units to find the target number. Evidence of this method was identified through students' written work or their explanations during interviews.

Figure 4.10 and its accompanying analysis demonstrate this use of unit fraction strategy on number line tasks.

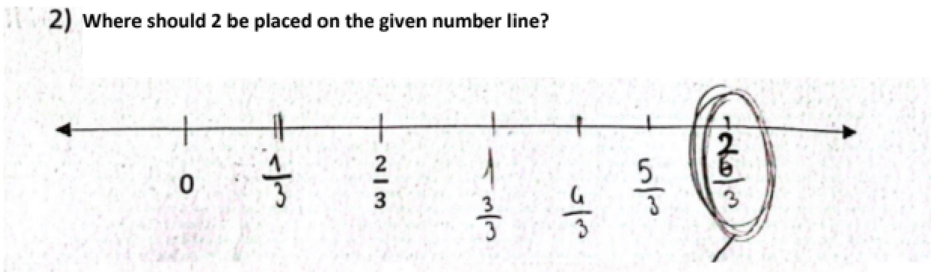


Figure 4.10: An example solution from the Unitizing Test.

> "S: First, I found the whole, that is, I found the part up to 1. Then I wrote 2 as whole. And at the end, I already divided it by 3, then 2.
> R: I see. And what did you pay attention to when finding one whole? How did you find one whole?
> S: Since there were two-thirds, I realized that it was divided into 3 parts. I divided it by 3. Then one whole."

In some cases, students' solutions reflected an incomplete use of the procedural unitizing strategy, highlighting a potential disconnect between written artifacts and cognitive processing. The student's response in Figure 4.11 serves as a clear example. On paper, the solution appears flawed; while the student correctly iterates by the unit fraction after 6/7 to place 7/7 and 9/7, the drawn intervals are not proportional, and the preceding fractions between 0 and 6/7 are missing.

However, the student's explanation during the interview revealed that their mental model of the number line was far more complete than their drawing suggested:

> "S: According to the previous ones, for example, 1 in 7, 2 in 7, 3 in 7, 4 in 7, 5 in 7, 6 in 7, 7 in 7, 8 in 7. After that, I added 7 in 9 here."

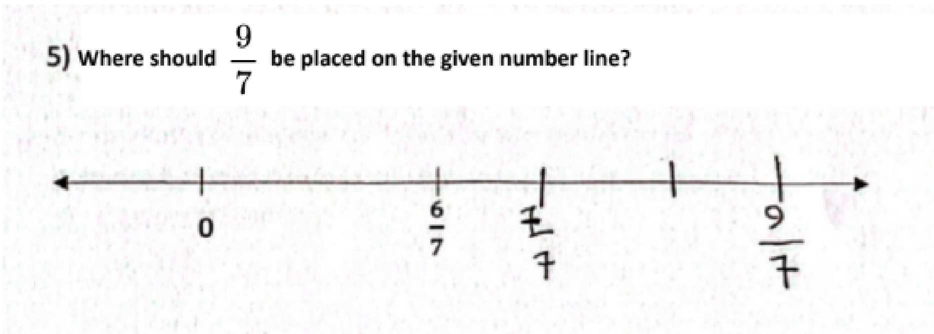


Figure 4.11: An example solution from the Unitizing Test.

This quote demonstrates that the student had mentally accounted for all the preceding unit fractions, even though they were not transcribed onto the paper. This example underscores a key methodological insight: assessing student work solely on written evidence can sometimes lead to an underestimation of their underlying procedural understanding.

For the fraction circle tasks, students were considered to use this strategy when they tried to find the representation for the unit fraction and use it as a building block to construct the target number. The two examples in Figure 4.12 and Figure 4.13 demonstrate how students apply this strategy.

In the first problem (Figure 4.12), where a circle represents 4/5, the student's solution involved partitioning the given shape to find a common unit and using that to build the target fraction of 6/5. The student's interview revealed their reliance on this partitioning procedure:

> "R: Now, when we look here, it is actually divided into four, let's say quarters. How many quarters do we see?
> S: 4.
> R: I mean, in the whole shape?
> S: A total of 6 quarters, I mean, when there is a scratch in the center, there are 6 quarters in total.
> R: Could you have made a comparison according to that?
> S: Most probably, I made a comparison by looking at the quarters."

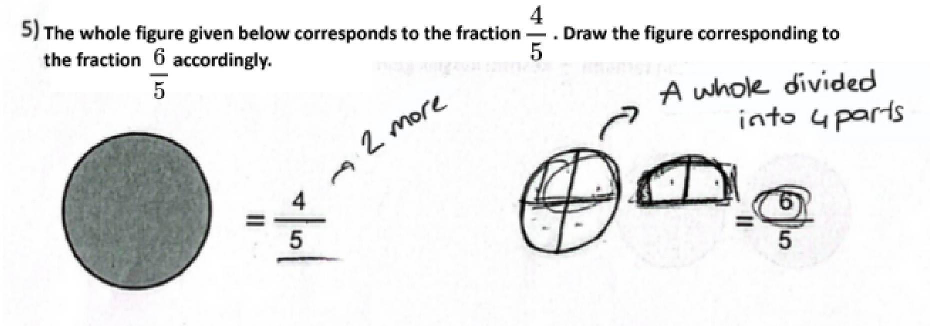


Figure 4.12: An example solution from the Unitizing Test.

In the second example (Figure 4.13), students were given two circles representing 2/3 and were asked to construct the whole number 2. A successful procedural approach involved first calculating the value of a single circle to find the unit fraction, as one student explained.

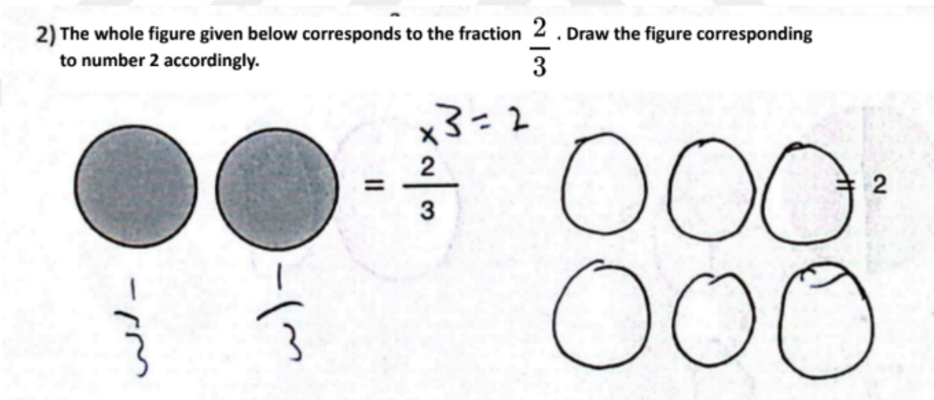


Figure 4.13: An example solution from the Unitizing Test.

"S: Here, the circles are already given as measurements. If it is two-thirds here, it means that if we divide it by two, we find the value of each circle. In other words, if we divide two-thirds by 2, we find one-third. This means that each of these circles is equal to one-third.
R: Yes.
S: Then, if it is a shape equal to the number two, there are 6 one-thirds. So, it is six-thirds.
R: Okay, good.
S: That makes 6 circles equal."

This student's clear, step-by-step process—dividing to find the unit fraction (1 circle = 1/3) and then multiplying to construct the target (2 = 6/3 = 6 circles) — is a quintessential example of procedural unitizing in action.

In word problem tasks, when students tried to make the comparisons through finding the unit prices, i.e., prices for one product, they were classified as using this strategy. Figure 4.14 demonstrates such a strategy clearly, and another student's explanation amplifies the suggestion.

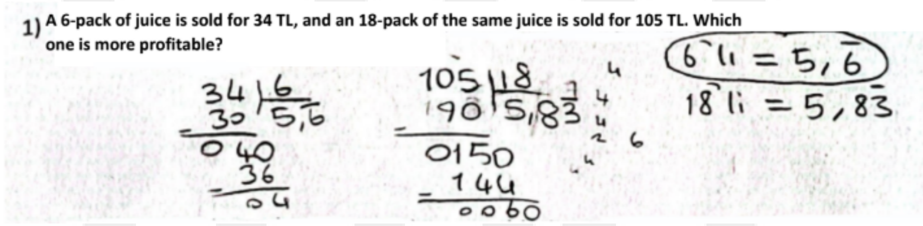


Figure 4.14: An example solution from the Unitizing Test.

"S: 18 kilograms here too. I divide 120 by 18 to get 1 kilogram. I divide 180 by 29 to get one kilogram. I compared the two."

Here, the student used division to find the price of one kilogram of rice (i.e., unit price) of an 18 kg sack of rice sold for 120 liras and a 29 kg sack of rice sold for 180 liras before making a comparison.

4.2. Unit in Interview Responses

4.2.1. Students' Definitions and Examples of the Unit

Towards the end of the interviews, students were asked to define the unit or give examples of it. Eight out of nine interviewed students provided a definition or at least one example to demonstrate their knowledge about the unit. The definitions and examples each student provided, and their elaborations on their explanations, are presented in Table 4.4. Most of the students struggled to provide a solid definition;

they rather gave examples to explain what they knew about the unit. “Unit fraction” and “unit square” examples were common among students’ examples.

Table 4.4: Students’ definitions and examples of unit, and elaborations.

Student ID	**Definition or Examples of Unit**	**Elaborations on the Definitions or Examples**
P20704	The simplest number in a fraction or a group of numbers	S: It is the simplest number in a fraction or a group of numbers. In other words, a fraction is the simplest number in fractions, that is, a unit fraction. So it is a unit fraction. That’s how we define it. R: I think you also said it in number groups. What did you mean by that? S: Actually, when I say number group, I meant it as a thing—as a denominator.
P20708	One of...	S: Unit... One of them, I think. ... S: The unit appears more often in the questions in shapes, circles, and rounds. R: How do they appear, for example? Can you give an example, or do you remember any? S: It gives one unit at the top. It asks how many units later this can be reached.
P20705	Unit square	S: Let me give an example. For example, on a square area, on a surface. A square... I mean, I don’t know how to say it... A square... a unit... I don’t know how to explain it.

Table 4.4. Students' definitions and examples of unit, and elaborations. (cont.)

Student ID	**Definition or Examples of Unit**	**Elaborations on the Definitions or Examples**
	Distance	S: The unit could be, for example, the distance.
	Unit price	S: In problems... One thing just came to my mind. For example, in problems, sometimes they say a unit price. I don't know how to explain the price of a unit right now, but it can probably be used in problems. I just remembered it now.
P20605	Unit fraction	S: A fraction whose numerator is one.
	Units of Measurement	S: For example, kilogram. Centimeter.
P20513	One part of a whole / Unit fraction	S: For example, if it is divided by 5, it is 1 in 5. So, if it is divided by 10, it is 1 in 10. Again, it is a unit fraction.
P20509	Distance	S: Now it is 6 in 5; the place between 6 in 5 and 7 in 5 is called a unit.
	Unit square	S: Any other... We use it in unit squares. I can't think of any others for now.
P20505	Unit square	S: I remember it as an area. Unit squares combine to form an area.
P10501	Unit fraction	S: Unit fraction; for example, I divided the square into five parts, and when I colored one of them, it became one unit. ... S: A unit fraction is when you divide something and take one part of it. It becomes a unit fraction.

4.2.2. Unit Use in Tasks

Although the test aimed to assess students' unitizing strategies, the concept was not mentioned directly, almost exclusively. Out of nine interviewed students, only one mentioned the unit during the interview. The following dialogue presents the part of the interview when the student used the term directly. The student indicates that he used 1/7 as the unit and reached the target, 1, through multiplying the distance by seven:

> "S: Since we are far from our target number here, I found 1 in 7 more useful. Because it is smaller and can be used in more units. We can reach 1 by multiplying the distance between 0 and one seventh by 7 times. . . .
> R: . . . Well, you said unit, what is the unit you are using here, then?
> S: It is 1 in 7."

The following excerpt presents how he created the unit 2/5 and used it to represent the asked fraction, 6/5. As a complete circle represented the fraction 4/5, he deduced that half of the fraction 4/5, which can be used to reach 6/5, should be represented by a half circle:

> "S: Our solution here is that I thought that half of four-fifths multiplied by three is six-fifths, but we cannot directly use four-fifths to find six-fifths. So I created a unit for myself.
> R: What was the unit you created here?
> S: Half of a circle, or two-fifths. Multiplying this unit by 3 equals the whole we want to find."

Since almost all student did not mention the unit during the elaboration on their solution, it was important to gather clues that may provide information on the aspects of building the unit understanding. One of these aspects is attention to equal parts. Students were given some prompts like "What did you pay attention to when solving these tasks?" and encouraged to explain whether the parts they used in their solutions were equal, especially in Number Line and Fraction Circle tasks.

> "S: I mean, I didn't pay much attention to anything. I just made the lines a little bit, you know, I tried to be careful with it. That's just why it's so much of a thing.
> R: How did you make the lines?
> S: Actually, the pieces should have been equal. They turned out a little bit unequal. I tried to pay attention to that.
> R: For example, which ones were supposed to be equal here? I mean, your drawing is not important. You drew those by considering which ones should be equal.
> S: I mean, for example, between zero and one-fourth, between one-fourth and two-fourths, they should be equal to the same parts."

Here, the student explains that the parts should be as equal as possible and provides examples to illustrate the distances that are supposed to be equal. Another student also demonstrates the importance of equal partitioning to approach the correct answer.

> "S: Yes. I mean, for prediction, they need to be equal. The more they are equal and correct, the closer we get to the correct answer."

Some students stated that they were careful about the ordering of the numbers, but nothing else, and their solutions suggested this as well. However, there was one exception: the following pieces of an interview imply that this student ignored equal partitioning at first, even though his drawings seemed rather equal. When the case was directly asked, the student explained it by chance. At the end, he stated that the parts should be equal, but he was still not sure about why:

> "S: According to the order here, 1 in 7, 2 in 7.
> R: Right. So, you placed it anywhere between zero and three sevenths, between the two. Did you understand correctly?
> S: Yes.
> R: So you could have placed it somewhere else.
> S: Yes, but I couldn't place it after three sevenths.
> ...
> R: Well, I see this, when you were placing them, I noticed that the distances between them looked a bit equal. Did you draw it with consideration, or did it just happen that way?

S: It just happened that way. I had never looked at it.
...
S: Yes. That's what I would do. I would make sure the intervals were a little bit equal.
R: Why would you make sure that the intervals were equal?
S: I don't know, to my eyes. Because it looks nice like that."

Questions like "What is one/whole in this question?" were asked specifically aiming to reveal the unit connection in students' minds. For number line tasks, students usually answered this question referring to the unit fraction of the task or the number 1. This was quite similar for fraction circle tasks as well; their answers mostly included a unit fraction or one whole circle. Finally, for word problems, they mostly referred to the unit price.

"R: So, what is one in this question, for you?
S: 1, I mean, exactly where I would put the 7 in 7."

"R: Okay, good. Then what is one in this question, for you?
S: One is to find the quarter circle."

"R: What is one in this question, for you?
S: Price per kilogram."

One other interview question asked students to reflect on their thoughts on units towards the end of the interviews, which was, "Do you think you used similar or different solution methods in different types of tasks?" The purpose of this question was to check if students can state the similarity of unit use in the tasks in any way, before we discuss the definition and examples of unit. Only two of the students, both are 7th graders, stated that they used similar methods in all tasks, and they attributed it to the unit:

"R: So, did you use the same or similar methods to solve these three types of tasks? Or did you use different methods?
S: I mean, I used similar methods.
R: What kind of similar methods?

S: By finding the price of a single. By finding where to put one whole. I mean, in the shapes and on the number line as one whole. We can say that by finding one in all of them."

Some of the students pointed out that they used similar methods in the number line and fraction circle tasks. The reason for this indication could be the use of fractions in these tasks. One of the students even used number lines when solving fraction circle tasks (Figure 4.15), and explained this use as to better understand the fractions in the tasks:

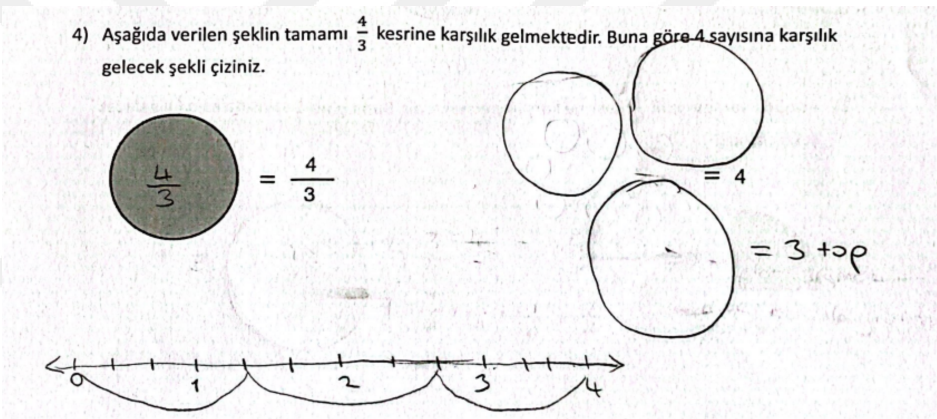


Figure 4.15: An example solution from the Unitizing Test.

"S: I used the number line on the number line and fraction circle tasks.
R: So, you used similar methods?
S: Yes.
R: Okay. Can you compare them with the word problem tasks? Did you use similar methods in those?
S: No."

Finally, students were asked if they used the unit in the tasks, at the end of the interviews, right after their definitions and examples of the unit. The answers were parallel to the previously mentioned question, in many cases. The definitions and examples they provided also influenced their answers to this question. Only one of the 5th grade students and, surprisingly, both of the 6th grade students stated that they did not use units in these tasks. Among others, two of the 5th graders indicated that they used units in number line and fraction circle tasks, and the other 5th grade student said he only used units in number line tasks. Two of the 7th graders stated that they used units in all tasks, while the other 7th grader excluded the fraction circle tasks when discussing the unit use in tasks.

4.3. Error Patterns, Misinterpretations, and Misconceptions

The unitizing test was developed to assess students' unitizing strategies and abilities, utilizing fractions, linear and non-linear models, and proportional reasoning. Although the primary purpose was to reveal students' strategies and thinking in various tasks, the emerging error patterns and misconceptions in students' solutions yielded interesting findings. The common error patterns noticed during the initial reviews were investigated further through the semi-structured interviews.

4.3.1. Placing Integers at the End of the Number Line

Through the initial reviews of the unitizing test results, it was realized that some students tended to place the asked integer at the end of the number line. Some of these students included explanations specifically indicating that they placed the number at the end of the number line in their solutions (as in Figure 4.16). The reasons for this issue were investigated further through interviews.

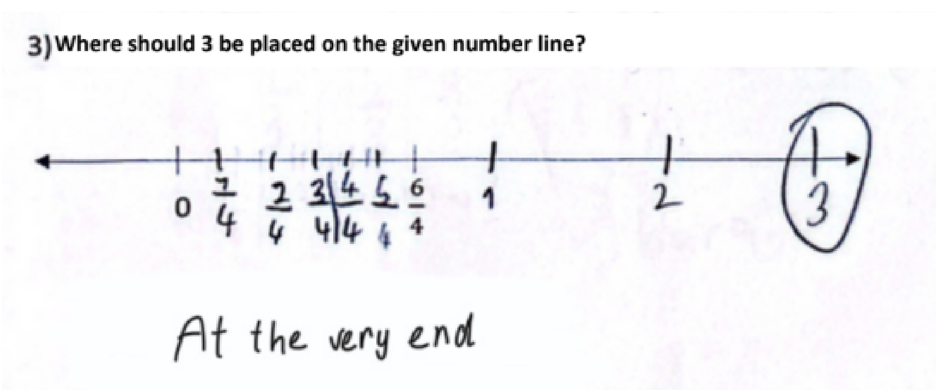


Figure 4.16: An example solution from the Unitizing Test.

Students' explanations for their solution strategy indicate that they think the biggest number mentioned should be at the very end of the number line, and if we were to add a bigger number, the previously asked number could be moved towards the center:

> "R: Now, you wrote an explanation 'at the very end' here. What was your idea when you wrote that? Can we repeat that?
> S: When I say 'at the very end', I mean at the very end. It cannot be written anywhere else.
> R: Do we always have to write the number at the end, or can we write it in different places?
> S: For example, 2, if we have 3 wholes, it can be in the middle or something like that."

Additionally, some other students explained that the integers should be at the very end to place other "numbers" in between (Figure 4.17):

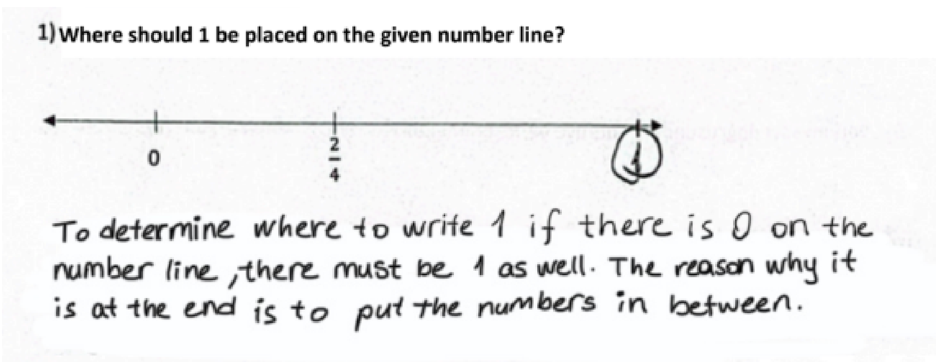


Figure 4.17: An example solution from the Unitizing Test.

The reason for this error pattern might be the tendency to fill the entire number line to use the provided space effectively. The extract from the interviews supports the argument, this student thinks that we can place one whole at the end of the number line since there is no need to place a bigger number:

> "R: Right. So, how did you place one whole here? Was it because you think that it should be at the end or did you think of something different?
> S: Well, since there is no need after one whole, I mean, there is no necessity

according to this question, I directly put one whole at the end. Because after one whole, we don't need anything right now. That's why I put it at the end."

4.3.2. Placing 0 at the Beginning of Number Line

Students tended to place zero at the very beginning of the number line. The NL10 task was purposefully designed to understand whether students place zero on the number line with attention to the given intervals or directly at the beginning. Students were asked how they placed zero during the interviews. Some of them explained that they think it should be at the beginning because they learned it like that:

"S: We can be careful to put it at the beginning.
R: Do you think we should always put the zero at the beginning?
S: That's what I have learnt."

"S: Zero is placed at the beginning. Because the beginning of the number line is zero."

On the other hand, there were some students placing zero on the number line with attention to the other numbers given on the line. However, explaining their solution was slightly challenging for them. Hypothetical changes made in the tasks were helpful to check if students were likely to move zero anywhere else in different scenarios. The following interview part demonstrates that the student placed zero at the beginning of the number line intuitively, but when the numbers were placed in different ways, which required placing zero towards the right more than the beginning, the student was able to apply the necessary changes:

"R: Then let me ask you a question about the location of zero. Zero was already given on the number line in other questions, but not in this one. Where could we place zero on a number line if no other number is given?
S: Maybe at the beginning.
R: Should we place zero at the beginning of every number line or can it be placed in different places?

S: Mostly, we learnt that zero is at the beginning, so I don't know.
R: For example, if there was a 1 in the place of 2 here. If there was a 2 somewhere over there (pointing somewhere further right). You can see my signs, right? Would the zero then be at the beginning again or somewhere else?
S: Zero would then be further to the right."

The placement of zero was also influenced by the presence of negative numbers. One student explained that zero could be at the beginning since there are no negative numbers. In the hypothetical case, where we added negative numbers, the student first placed zero in the middle, then changed its place according to the existing hypothetical numbers:

"S: ... And zero, since there will be no minus here, since there will be no negative number, I put zero at the end, sorry, at the beginning of the number line.
...
R: Right. You also said that you put zero at the beginning. You think it should always be at the beginning, as long as there are no negative numbers.
S: Exactly.
R: If there were negative numbers, where could zero be placed?
S: For example, if it continues from here, it should come between 1/2 and a negative number. I think so.
R: Okay. So, for example, if we think of another empty number line, where we will place negative and positive numbers, should we place zero at the beginning again?
S: No. If there were such a thing, if there were negative numbers here and positive numbers here, zero would be right in the middle.
R: Does it have to be in the center, or could it be in different places?
S: It depends, for example, if here is divided into 7 parts, here is divided into 6 parts. It may be closer to this side. According to the number of divisions."

4.3.3. Fraction Knowledge or Notation Errors

A common error pattern was observed in one of the 5th grade classrooms. They placed the equivalent fractions of the integers on number lines as if they were separate numbers, as in Figure 4.18. It is interesting that this was common in only one classroom. The use of examples and demonstrations in teaching might have led to this error.

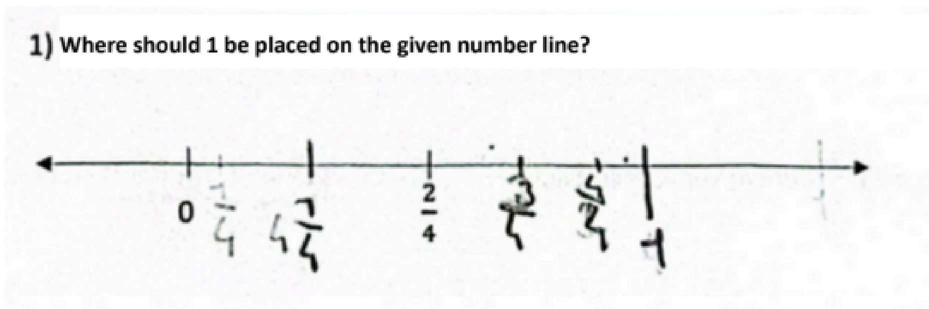


Figure 4.18: An example solution from the Unitizing Test.

Another example of one of the error patterns related to fraction notation is presented in Figure 4.19. Students manipulated the numerator of the fraction instead of the denominator to order the fractions on the number line. There are some other misinterpretations and errors in this example as well. For example, the student placed 1 on the number line when the numerators reached up to 10, which may suggest that some students connect these types of tasks with measurement tools.

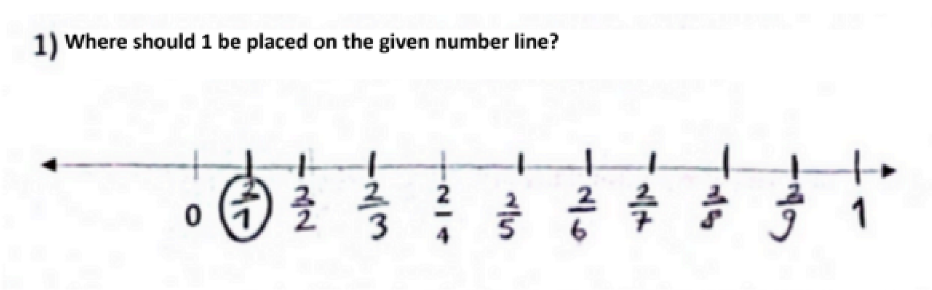


Figure 4.19: An example solution from the Unitizing Test.

It was also observed that students were confused about the fractions that they could place between two integers. This error demonstrates that some students couldn't completely grasp the meaning of fractions. The following dialogue, along with the solution presented in Figure 4.20, shows that the student was unable to explain the fractions we can place between two integers:

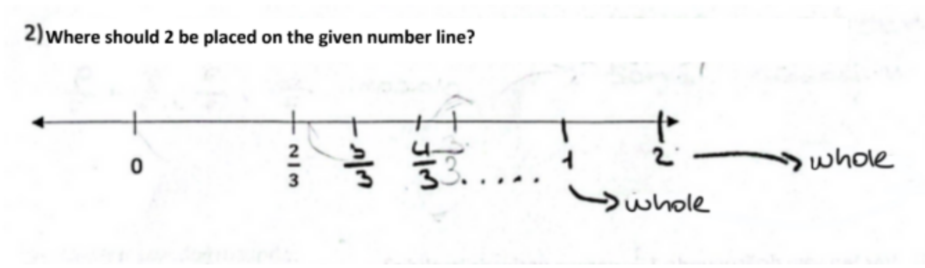


Figure 4.20: An example solution from the Unitizing Test.

"R: Well, here, you have drawn dots. What do these dots mean?
S: I did it to show that after 4 divided by 3, 5 divided by 3, as well as 6 divided by 3, continue. I put those dots to show the fractions between this one whole and this number.
R: Can we write many things until we get to one whole, or can we write a certain number of numbers there?
S: So, as I said, we can write 5 divided by 3, we can write 6 divided by 3, we can write more than one number.
. . .
R: So again, what do these dots point to here?
S: Again, I am referring to fractions between zero and one whole.
R: Okay. Again, can we write as many fractions as we want here or what?
S: What do you mean by as many as we want...
R: I mean, it goes on, for example. Is there a limit to its continuation?
S: Well, probably there is, but here's how I think about it right now. For example, we were dividing the intervals, etc. Then, we can write the numbers up to one whole. It depends on the question, as far as I know."

4.3.4. Referring to the Asked Integers as the Numerator of Fractions

Some students, especially younger ones, misinterpreted the asked number as the numerator of a fraction rather than an integer. As in the examples in Figure 4.21, when students were asked to place one on number line, some marked 1/4 as the answer, or when they were asked to place 2, they stated that the number was already given on the line, so they interpreted the asked "2" as the two in the fraction two-thirds, not "2 wholes".

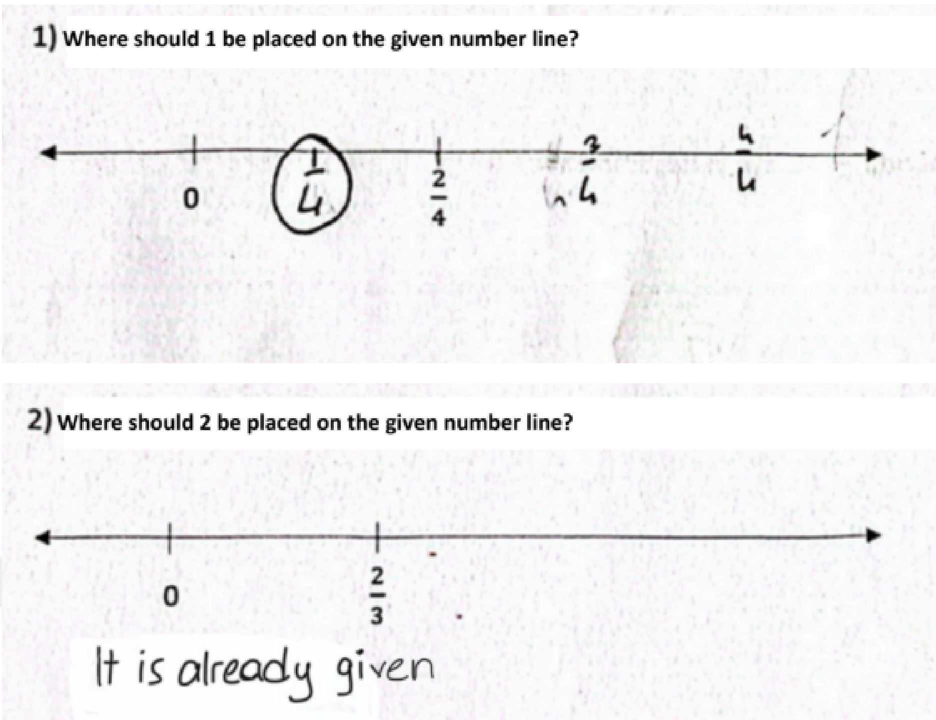


Figure 4.21: An example solution from the Unitizing Test.

Spotting these errors was rather challenging in fraction circle tasks because many students only draw circles without showing any clues for their solutions. For example, many students drew one whole circle as the answer to FC3 (as in Figure 4.22), and this answer was considered random work at first. The example in Figure X provided insights to spot the error in this task type. The student perceived two whole circles as 6, rather than 6/4, so one whole circle as 3, rather than 3/4. These misinterpretations led students to errors.

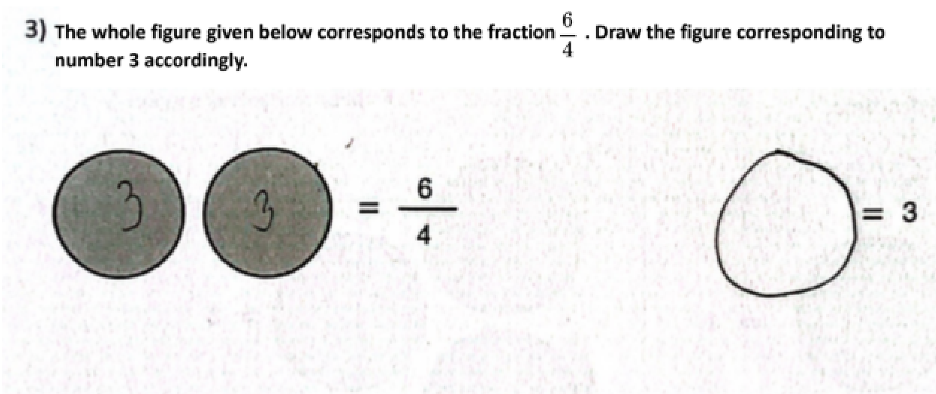


Figure 4.22: An example solution from the Unitizing Test.

"S: I mean, one is half according to 6. I mean, since the sum is 6, I divided 6 by 2 and got 3.
R: So, according to this, did you decide that the number corresponding to 3 is a circle?
S: Yes.
R: OK. So, why did you write 3 inside the circles here?
S: Thinking its half."

A similar example is presented in Figure 4.23. The student's explanation for this solution implies that she perceives 4/3 as 4 directly. So, if the fraction 4/3 corresponds to a whole circle, the number 4 should correspond to the same shape as well:

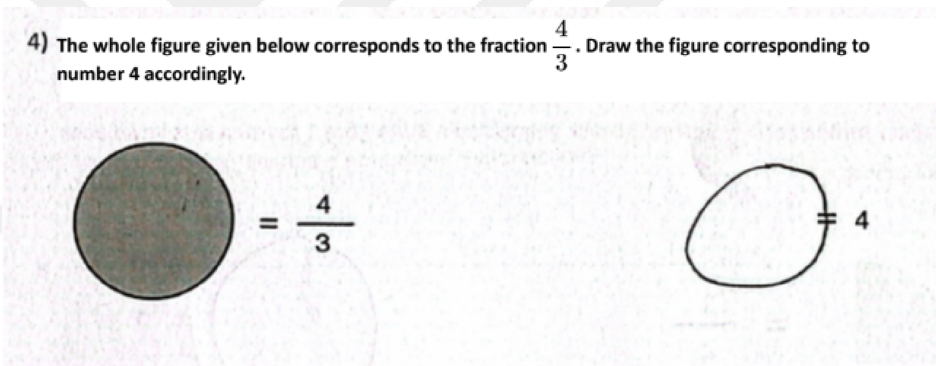


Figure 4.23: An example solution from the Unitizing Test.

"S: It is written three-fourths corresponding to the number four here. Since a circle is directly four, I drew a circle directly here."

Nevertheless, this misinterpretation did not only conclude with error. A few students specifically asked whether these "1"s or "2"s are whole or not, during the assessments and interviews. It was observed that some students were referring to the fractions by only saying the number in the numerator. The example in Figure 4.24 shows that this way of expression does not always lead to error. She only refer to the parts as 1 or 2, instead of 1/2 and 2/2, but she used them in the correct way:

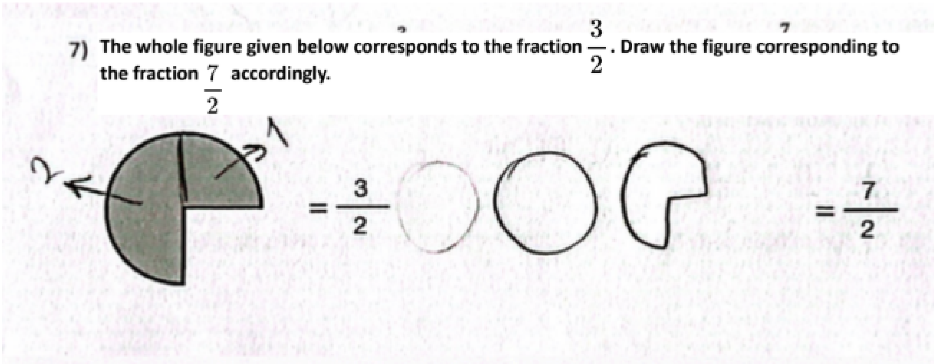


Figure 4.24: An example solution from the Unitizing Test.

> "S: 3 quarters corresponds to 3... Since 3 quarters correspond to 3, 4 quarters correspond to 4. So I put a circle that corresponds to 4, and this shape that corresponds to 3.
> R: OK, good. So what do the 2 and 1 you wrote here represent?
> S: I mean, the quarter is one, half is two."

> "S: Since 1 circle is equal to 4, half of it is equal to 2. So, 1 circle plus half, I mean 2."

4.3.5. Drawing the Shapes Without Using the Given Referents in Fraction Circles

In fraction circle tasks, many students, especially 5th graders, answer the tasks without using the given referents. Instead, they perceived a whole circle as "1" and drew the asked numbers accordingly. Figure 4.25 demonstrates this error pattern in students' responses. One of the interviewed students stated that he did not use the given shapes in his solutions:

> "R: Did you use the shapes that are given here on this side?
> S: No. I just looked at them to see what was in the question."

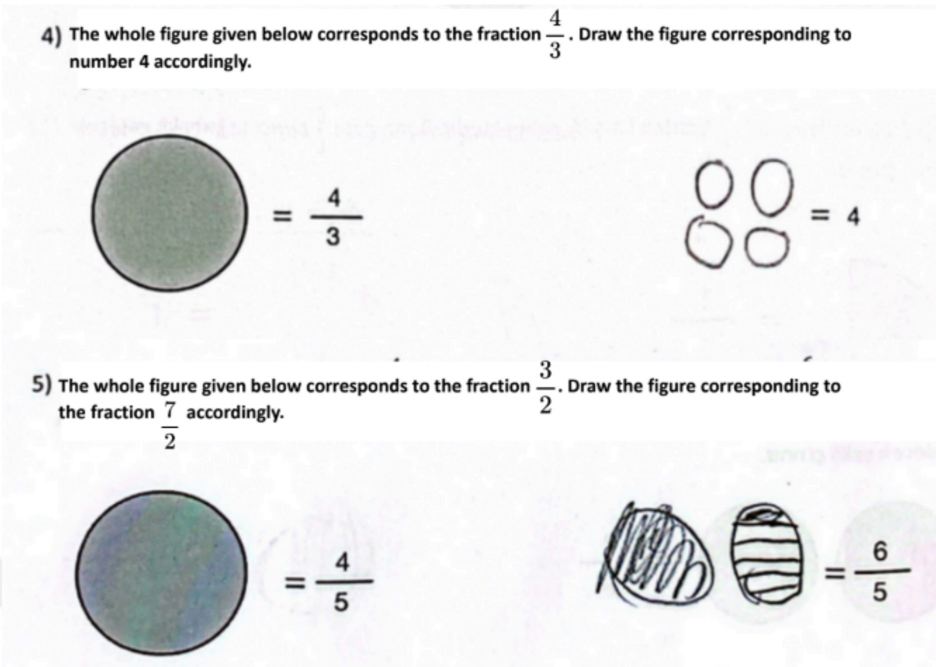


Figure 4.25: An example solution from the Unitizing Test.

4.3.6. Additive Thinking Instead of Multiplicative in Word Problems

There were students who displayed the common misconception of using additive thinking instead of multiplicative thinking. They compared the differences in some word problem tasks. Interestingly, in some cases, the same student was able to demonstrate multiplicative thinking on another word problem task, even though the tasks were quite similar. In these cases, the numbers being compared may influence students' thinking. When one number is a multiple of the other, students were able to easily multiply and make comparisons. However, in other cases, when comparing the differences is easier for them, they fall into error. The following interview excerpts belong to the same student. As he was able to answer in a multiplicative way, through dividing 18 by 3 to compare the prices for 6-packs, he was unable to think multiplicatively since the multiplicative relations for the numbers 24 for 30, and 18 for 22, were not that straightforward:

> "S: So, the profitability, for example, of a 6-pack of fruit juice is 34 liras. When I divide 6 packs of 18 by 3, I get 35 liras.
> R: So which one is more profitable
> S: I think the 6-pack is."

“S: So 30 minus 24 makes 6. 22 minus 18 makes 4.
R: Why did we subtract here?
S: To learn the differences between 22 and 18.
R: The difference between 22 and 18. Okay. Why did we want to know that?
S: To find out the profitability.
R: Okay. You said one of the differences is 6. If we subtract 24 from 30, you said 6, and if we subtract 18 from 22, you said 4. Which is more profitable, then?
S: Package of 24”

5. DISCUSSION

This study aimed to explore students' unitizing strategies across three types of tasks and to determine whether these strategies are related to their fraction achievement. Two main strategies were identified based on students' solutions in the tests and elaborations in the interviews: procedural unitizing, where students use unit fractions or the number 1 as their unit, and effective unitizing, where students demonstrate a more flexible use of units. Findings revealed that as students' fraction achievement levels increase, their ability to use unitizing strategies flexibly increases as well. Another purpose of the study was to reveal whether the strategies students used are consistent across task types. The results did not demonstrate clear patterns of such consistency. One other significant finding is the error patterns that emerged in students' solutions. Some of these error patterns reflect an incomplete understanding of fractions, overgeneralizations about number lines, and confusion about tasks.

5.1. Unitizing Strategies and Relation to Fraction Achievement

The fourth research question sought to determine if a relationship exists between overall fraction performance and their ability to use unitizing strategies effectively. The findings confirm a strong and significant positive relationship between these two constructs. This connection was evident not only in the strong correlation between students' scores on the Fraction Test and the Unitizing Test but also in the qualitative analysis, which showed that high-achieving students overwhelmingly used more effective unitizing strategies. This powerful association is logical, as the foundational skill required for fraction competency – understanding the part-whole relationship – is itself an act of unitizing. To understand a fraction, one must first define the 'whole' as a unit and then relate its parts back to it. Therefore, it follows that students with a robust conceptual knowledge of fractions would possess the cognitive flexibility needed to create, partition, and iterate various units, which is the hallmark of the effective unitizing strategies observed in this study. While this finding primarily addresses a core research question, the strong correlation with the previously validated Fraction Test also provides important evidence for the construct validity of the Unitizing Test developed for this research.

A central finding of this study is that a student's fraction achievement level is profoundly linked to the sophistication of their unitizing strategies. The data shows not just a correlation, but a qualitative difference in cognitive approach: students with higher fraction achievement demonstrated a greater capacity for effective unitizing, which the interview data suggest is a proxy for cognitive flexibility. For instance, students who employed effective strategies were often the same students who could articulate alternative solution paths during interviews, even if their initial written solution appeared procedural. This suggests that their robust fraction knowledge provided them with a more diverse mental toolkit to draw upon.

Conversely, students with low-level fraction achievement were less likely to engage in any meaningful strategic work, and their accuracy, even when a strategy was attempted, was notably lower. This reinforces the idea that the link between fraction knowledge and unitizing is not superficial. Rather, a deep understanding of fractions appears to be a prerequisite for the ability to flexibly create and coordinate units in novel situations. This finding aligns with and extends previous research linking fraction competency to broader mathematics achievement (e.g., Bailey et al., 2012; Siegler et al., 2011), specifying that one of the key mechanisms behind this relationship may be the student's underlying unitizing ability.

To identify students' strategies in the tasks and address the first and second research questions, possible solution strategies were considered and categorized prior to the data collection procedure, and experts were also requested to share their opinions on these categories. Following the data collection, students' responses were reviewed multiple times to extract the patterns in solutions, and emerging patterns were discussed. The frameworks by Lamon (1996) and Steffe (1992) contributed to our understanding and from which perspectives we could assess students' work. The framework by Lamon (1996) focuses on students' partitioning and marking strategies, and the economy in these actions, which inspired the construction of Effective Unitizing strategy, as it also focuses on the "economy" in partitioning through most possible conservation of the unit, and adds to the idea through focusing on iteration of the constructed units as well. Coding structure for the use of students' strategies was finalized following the initial analysis of the interviews. In line with the principles of naturalistic inquiry (Armstrong, 2010), the aim is to present interpretations that ensure credibility and trustworthiness through the verification and cross-checking of findings. The continuous ongoing process of data analysis and interpretation is also consistent with a naturalistic approach to research (Armstrong, 2010).

Finally, students' test responses were coded under three main categories, reflecting their strategy use, accuracy of strategy implementation, error patterns, and unknown responses. Together with the interviews, the results from the percentages of the used strategies revealed many key findings. Firstly, the strategies students presented in the number line tasks aligned with the study by Cramer et al. (2019), which guided the design of the number line tasks in the present study. Students using unit fractions to determine the place of asked numbers outnumbered the students who used units more flexibly in both studies. Only, the number of students presenting one of the two strategies was closer in the present study, which may be related to the differences in the number of participants, or the high percentages of students demonstrating "none/unknown" strategy in the present study. For fraction circle tasks, the results were consistent with those of Tunç-Pekkan (2015) in terms of accuracy in solving similar tasks. Unlike the present study, Tunç-Pekkan (2015) focused on the accuracy of the responses rather than solution strategies. Both studies revealed that these types of tasks were challenging for students, as they differ from the traditional representations presented in mathematics lessons and textbooks. Among the students presenting any strategy, effective unitizing strategy use was more common, especially among 7th graders. Their reflections indicated that they attempted to find a multiplicative relationship between the given and asked numbers, which led them to create units effectively, without dividing the pieces more than necessary. The presented interview extracts can support the claim. In word problem tasks, on the other hand, the choice of strategies was different across grade levels. 6th grade students used the procedural unitizing strategy more, while the percentages for effective or procedural unitizing use were closer for 5th and 7th graders. This tendency among 6th grade students may be attributed to the mathematical content covered in their curriculum. They were introduced to repeating decimals earlier at this grade level (MEB, 2018), so they may be more comfortable applying this kind of division to find the unit prices in the problems.

Since this study aims to reveal the unitizing strategies used by middle-school students, the comparisons focused mainly on which of the two unitizing strategies students demonstrated more frequently in their solutions. However, it is essential to note the existence of a third category, "none/unknown," which was the most prevalent across all task types and grade levels, indicating that students faced challenges in solving the tasks. More than half of the solutions were coded under this third category, meaning many of the solutions demonstrated either no clues about the strategy used, error patterns, or no meaningful work. The second assessment of the Unitizing Test results, calculated test scores, and descriptive statistics of the scores, suggested something similar. The notably lower test scores compared to the Fraction Test also highlighted the test's difficulty. Hence, the accumulation in the "none/unknown" category could be explained by the quantitative results as well. There can be multiple reasons why students had difficulties with the test. One possible reason is that many of the tasks were unfamiliar to students, especially the number line tasks and the fraction circle tasks, which were different from the traditional ones. Even some experts commented on the unfamiliarity of the tasks and pointed out that students might struggle with these tasks. Most of the 5th graders' tendency to define a whole circle as one rather than what's given can be an example of students' unfamiliarity with the type of tasks. We can also examine the outliers of the Unitizing Test statistics to see which students have strikingly higher scores, and what could be the reason for their higher performance. Among the nine outliers, one was in the fifth grade, three were in the sixth grade, five were in the seventh grade, and only two of the seventh graders were in public school; the rest of the students were in private school. This finding suggests that grade levels and school types – or socio-economic status – could have an influence on students' performance in the test. Further research is needed to see if these are the cases and what other reasons could be for the lower scores on the test.

Findings indicate that 5th and 6th graders were able to provide more valid answers in number line tasks, whereas it was word problems for 7th graders. This result can be partly explained through the mathematical content covered in the curriculum of each grade as well. 5th and 6th graders were dealing with fractions on number lines in their mathematics lessons during the academic year, and 7th graders were learning the topic of proportional reasoning (MEB, 2018; MEB, 2024), which can be quite useful to answer the word problem tasks. This could also explain the notable difference in word problem tasks across the grade levels. Another notable difference across the grade levels was observed in fraction circle tasks, where 5th graders appeared to underperform substantially. The reason for this finding was found to be the error pattern appeared frequently in 5th graders' fraction circle task solutions. The tasks were different from what they usually encountered, so they were confused about understanding the tasks as well. Students expressed their confusion about these tasks in the interviews accordingly.

5.2. Unitizing Strategies Across Tasks

This study also aimed to check if students' unitizing strategies are consistent across different tasks, and the test results suggested no clear consistency, as specified in the third research question. In other words, the unitizing strategy that a student demonstrated in one task type (e.g., number line) is not necessarily similar to the other two task types (e.g., fraction circles and word problems). The study by Tunç-Pekkan (2015) revealed a similar finding in terms of number line and fraction circle tasks. Although there is no clear consistency throughout the entire data set, some connections remain. The existence of the connections was explored through semi-structured interviews. Some students explained that they used similar methods in number line and fraction circle tasks. Even as presented in the results, one student used number lines when solving fraction circle tasks. We can explain this connection by considering the inclusion of fraction knowledge in these tasks, which differs from word problems. The student's explanation that using number lines in fraction circle tasks helps them understand fractions better can serve as evidence for such idea.

5.3. Error Patterns

Although the purpose of the study was focused more on revealing students' unitizing abilities, the emerged error patterns provide valuable insights into students' ways of thinking in any way. One error that was common in students' number line solutions was that they placed the number at the end of the number line if the asked number is a whole number. Based on the results and the students' explanations, we can say that this error can be related to an overgeneralization. Because in many examples presented in textbooks and lessons, the whole number line is likely used. And in traditional tasks where students need to place fractions, they are either given whole numbers on the line, or the line is empty and they place the whole number freely, but usually at the end, to use the given space effectively. Similarly, many of the students demonstrated the error of placing the number 0 at the very beginning of the number line. This can also be explained by what students are used to encountering. Students' tendencies on these placements are also presented in the study of Cramer et al. (2017). In a given empty number line, students place the numbers 0 and 1 at the edges before placing the asked proper fractions, and 0 and 2 at the edges when they needed to place a mixed fraction (Cramer et al., 2017). They explained this tendency as students perceive the whole number line as the "whole" or "unit", which is in line with what Yanık et al. (2008) discussed. A relatable error on the placement of 0 is highlighted by Ünal et al. (2024b); students tend to place 0 in the middle of the number line, where they also place positive and negative numbers. Their findings align with the present study, as one of the interviewed students explained that zero could be in the middle if there were negative numbers as well. Although students know zero does not have to be in the middle (Ünal et al., 2024b) or at the beginning, many of them prefer to place it as such. These findings may imply that mathematics educators should take precautions to prevent such overgeneralizations. To this end, number lines could be used in a more flexible manner in mathematics lessons, rather than trying to utilize all the space all the time and constantly reminding students of the properties of number lines.

Some errors that appeared in the solutions resulted from students' misunderstanding of fractions. One of the errors presented in the results, where students placed both one and a fraction representing one at the same time, also exists in the study of Cramer et al. (2017). They explained this as an error when students are counting the "jumps" (Cramer et al., 2017). This may also be due to students' poor understanding of the connection between fractions and whole numbers. The other two errors in fraction understanding can be explained by a failure to understand the meaning of fractions. It can be observed that they know fractions to be placed between whole numbers, but they have confusion about which fractions to place between.

The error of referring to the asked integer as the numerator of the fraction is an interesting one. For example, when they are asked to draw the shape representing 4, when a whole circle represents 4/3, some draw the exact same shape, a whole circle, explaining that they are both four. One reason might be the confusion about the tasks, but this error is observed in both the number line and the fraction circle tasks. This error may also be due to failures in students' conceptions of whole numbers and rational numbers.

In fraction circle tasks, especially fifth graders, made their solutions without considering the given referent, and used one whole circle as one. For example, if they are asked to draw the shape representing 3, they draw three circles. This can be explained by their confusion about the tasks. They are accustomed to tasks where a complete circle is always one, so they probably failed to devise any alternative solution method. They might also think that there is a problem with the solution, so they did not pay attention to what they were given. This error was not as common among 6th and 7th graders, possibly because they did not ignore the information provided in the tasks as 5th graders. This error pattern highlights the importance of using non-routine representations to deepen the understanding.

Finally, some students used an additive approach to word problem tasks instead of a multiplicative one. This was only the case when the multiplicative relation between the numbers given in the tasks was not that straightforward. Since they were able to answer at least one other word problem task with a multiplicative approach, the reason is not entirely their failure to think multiplicatively, but rather their attempt to come up with an easier solution.

5.4. Limitations

An socio-economically balanced sample was tried to be established through selecting two types of schools – public and private. However, no external data was collected on this issue, so such a claim cannot be provided with confidence. Further, the number of interviewed students was not balanced across the school types. An attempt was made to establish a balance, and students to be interviewed were carefully selected, but there were problems in establishing communication with the parents of public school students. Alternative methods were tried to contact parents, but the number remained limited. Due to the time constraints, no further attempt to provide balance could be made.

The number of participating schools can also be considered a limitation. Both schools were within the same district, which can limit the generalizability of the results to the whole population. The number of interviewed students was also limited; a greater number of interview data could provide deeper insights and different views of students' thinking.

Students were given a specific time to complete the tests in their classrooms. Even though there were a few students who could not complete the tests on time, they could perform better with a more flexible time to think deeper on tasks. Especially, the Unitizing Test is quite a comprehensive test that may require some time to understand, but the time had to be limited to some extent since they were tested in class time.

Lastly, due to the nature of the qualitative research method, that part of the study, including coding and interpretation of the interview data, may include subjectivity, despite the attempts to ensure trustworthiness through discussions with a second researcher, expert opinions, and double-checking the codes over time.

There are many important findings of the current study; however, those findings could be enriched through additional assessments and analyses. For example, the data was collected from two types of schools, but the results were not compared in terms of school types or socio-economic status. Future research could explore whether there is any difference in students' responses and scores depending on their socio-economic status. Furthermore, additional information on students' abilities, like their reasoning skills, could be assessed to enhance the implications. Future studies can investigate if there is a relation between students' reasoning abilities and their unitizing performance.

5.5. Educational Implications

This study aimed to reveal students' unitizing strategies and highlight the importance of unit understanding in mathematics education. The definitions and examples of a unit that students made revealed that students are mostly unable to explain what a unit is. This may not be a surprise since they are not clearly introduced to the concept. Some of the students were able to give examples of unit fractions and unit squares and explain those concepts more easily, probably because they were engaged in those concepts in mathematics lessons due to their places in the curriculum. Hence, we can indicate that by increasing the visibility of the unit concept in mathematics, we can support students' understanding of the concept. The study also emphasizes the importance of helping students identify and interpret the referent unit across representations. Findings suggest the need for explicit instruction using coordinated representations—such as number lines and area models—to illustrate how the unit can shift by context. Teachers can use guided questions to support reasoning about the "whole," while curriculum designers should embed unitizing tasks across grade levels. These insights can inform more effective instructional strategies and curricula that foster a flexible, conceptually grounded understanding of fractions.

REFERENCES

Acar, F., and Ş. Sevinç, 2021, "Investigation of Middle School Students' Unit Coordination Levels in Mathematics Problems Involving Multiplicative Relations", *İlköğretim Online*, Vol. 20, No. 1, pp. 90-114.

Albin, S., and B. Brown, 2019, "Developing Namibian Grade 8 Learners' Conceptions of Fractions Using Visual Models", *African Journal of Research in Mathematics, Science and Technology Education*, Vol. 23, No. 2, pp. 206-218.

Allen, G. D., 2014, "The Remarkable Number "1" ", *Science & Education*, Vol. 23, pp. 1845-1852.

Armstrong, J., 2010, "Naturalistic Inquiry", *Encyclopedia of Research Design*, pp. 881-885.

Bailey, D. H., M. K. Hoard, L. Nugent, and D. C. Geary, 2012, "Competence With Fractions Predicts Gains in Mathematics Achievement", *Journal of Experimental Child Psychology*, Vol. 113, No. 3, pp. 447-455.

Behr, M. J., R. Lesh, T. Post, and E. A. Silver, 1983, "Rational Number Concepts", *Acquisition of Mathematics Concepts and Processes*.

Boyce, S., and A. Norton, 2016, "Co-Construction of Fractions Schemes and Units Coordinating Structures", *The Journal of Mathematical Behavior*, Vol. 41, pp. 10-25.

Braun, V., and V. Clarke, 2006, "Using Thematic Analysis in Psychology", *Qualitative Research in Psychology*, Vol. 3, No. 2, pp. 77-101.

Bright, G. W., M. J. Behr, T. R. Post, and I. Wachsmuth, 1988, “Identifying Fractions on Number Lines”, *Journal for Research in Mathematics Education*, Vol. 19, No. 3, pp. 215-232.

Burton, D., 2010, *The History of Mathematics: An Introduction.*, Seventh Edition, McGraw-Hill Education, New York.

Cramer, K., S. Ahrendt, D. Monson, T. Wyberg, and C. Miller, 2017, “Making Sense of Third-Grade Students’ Misunderstandings of the Number Line”, *Investigations in Mathematics Learning*, Vol. 9, No. 1, pp. 19-37.

Cramer, K., D. Monson, S. Ahrendt, T. Wyberg, C. Pettis, and C. Fagerlund, 2019, “Reconstructing the Unit on the Number Line: Tasks to Extend Fourth Graders’ Fraction Understandings”, *Investigations in Mathematics Learning*, Vol. 11, No. 3, pp. 180-194.

Creswell, J. W., and J. D. Creswell, 2017, *Research Design: Qualitative, Quantitative, and Mixed Methods Approaches.*, Fifth Edition, SAGE Publications, California.

DeepL Translate: Dünyanın En İsabetli Çevirisi, 2017, https://www.deepl.com/tr/translator, accessed on November 15, 2024.

George, D., and P. Mallery, 2010, *SPSS for Windows Step by Step: A Simple Guide and Reference (18.0 update)*, Eleventh Edition, Allyn & Bacon / Pearson, Boston.

Hansen, N., N. C. Jordan, E. Fernandez, R. S. Siegler, L. Fuchs, R. Gersten, and D. Micklos, 2015, “General and Math-Specific Predictors of Sixth-Graders’ Knowledge of Fractions”, *Cognitive Development*, Vol. 35, pp. 34-49.

Hunting, R. P., 1983, “Alan: A Case Study of Knowledge of Units and Performance with Fractions”, *Journal for Research in Mathematics Education*, Vol. 14, No. 3, pp. 182-197.

Ifrah, G., 2000, *The Universal History of Numbers: From Prehistory to the Invention of the Computer*, Wiley, Canada.

Johnson, R. B., and A. J. Onwuegbuzie, 2004, “Mixed Methods Research: A Research Paradigm Whose Time Has Come”, *Educational Researcher*, Vol. 33, No. 7, pp. 14-26.

Kunst, J. R., and K. Bierwiaczonek, 2023, “AI in Cross-Cultural Research: The Promise and Pitfalls of Machine Translation”, *Current Opinion in Behavioral Sciences*, Vol. 49, pp. 1-11.

Lamon, S. J., 1996, “The Development of Unitizing: Its Role in Children’s Partitioning Strategies”, *Journal for Research in Mathematics Education*, Vol. 27, No. 2, pp. 170-193.

Lamon, S. J., 2006, *More: In-Depth Discussion of the Reasoning Activities in “Teaching Fractions and Ratios for Understanding”*, Second Edition, Lawrence Erlbaum Associates, New Jersey.

Lamon, S. J., 2012, *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*, Third Edition, Routledge, New York.

Milli Eğitim Bakanlığı [MEB], 2018, *Matematik Dersi Öğretim Programı (İlkokul ve Ortaokul 1–8. Sınıflar)*, T.C. Milli Eğitim Bakanlığı, Ankara.

Milli Eğitim Bakanlığı [MEB], 2024, *Ortaokul Matematik Dersi Öğretim Programı (5, 6, 7 ve 8. sınıflar).* T.C. Milli Eğitim Bakanlığı, Ankara.

Ni, Y., 2000, “How Valid Is It to Use Number Lines to Measure Children’s Conceptual Knowledge about Rational Numbers?”, *Educational Psychology*, Vol. 20, No. 2, pp. 139-152.

Norton, A., S. Boyce, N. Phillips, T. Anwyll, C. Ulrich, and J. L. Wilkins, 2015, "A Written Instrument for Assessing Students' Units Coordination Structures", *Mathematics Education*, Vol. 10, No. 2, pp. 111-136.

Özel, S., and Z. Terzi, 2023, "'1' ve Birim Nedir?", *Matematik Dünyası*, Vol. 30, No. 118, pp. 96-100.

Pantziara, M., and G. Philippou, 2012, "Levels of students' "Conception" of Fractions", *Educational Studies in Mathematics*, Vol. 79, pp. 61-83.

Pitkethly, A., and R. Hunting, 1996, "A Review of Recent Research in the Area of Initial Fraction Concepts", *Educational Studies in Mathematics*, Vol. 30, No. 1, pp. 5-38.

Rob, K., 2015, "The Story of 1 One - documentary", https://www.youtube.com/watch?v=Z5dky0oEDo8, accessed on April 29, 2023.

Sfard, A., 1991, "On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin", *Educational Studies in Mathematics*, Vol. 22, pp. 1-36.

Siegler, R. S., C. A. Thompson, and M. Schneider, 2011, "An Integrated Theory of Whole Number and Fractions Development", *Cognitive Psychology*, Vol. 62, No. 4, pp. 273-296.

Sousa, V. D., and W. Rojjanasrirat, 2010, "Translation, Adaptation and Validation of Instruments or Scales for Use in Cross-Cultural Health Care Research: A Clear and User-Friendly Guideline", *Journal of Evaluation in Clinical Practice*, Vol. 17, No. 2, pp. 268-274.

Steffe, L. P., 1992, "Schemes of Action and Operation Involving Composite Units", *Learning and Individual Differences*, Vol. 4, No. 3, pp. 259-309.

Tomlinson, P., 1989, "Having It Both Ways: Hierarchical Focusing as Research Interview Method", *British Educational Research Journal*, Vol. 15, No. 2, pp. 155-176.

Torbeyns, J., M. Schneider, Z. Xin, and R. S. Siegler, 2015, "Bridging the Gap: Fraction Understanding Is Central to Mathematics Achievement in Students from Three Different Continents", *Learning and Instruction*, Vol. 37, pp. 5-13.

Tunç-Pekkan, Z., 2015, "An Analysis of Elementary School Children's Fractional Knowledge Depicted with Circle, Rectangle, and Number Line Representations", *Educational Studies in Mathematics*, Vol. 89, pp. 419-441.

Tzur, R., H. L. Johnson, A. Norton, A. Davis, X. Wang, M. Ferrara, and B. Wei, 2017, "Conception of Number as a Composite Unit Predicts Students' Multiplicative Reasoning: Quantitative Corroboration of Steffe's Model", in *Proceedings of the 41st Annual Conference of the International Group for Psychology of Mathematics Education*, Denver, Vol. 4, pp. 289-296.

Ünal, Z. E., A. M. Ala, G. Kartal, S. Özel, and D. C. Geary, 2024b, "Ninth-Grade Students' Conceptual Understanding of the Number Line", *Journal of Numerical Cognition*, Vol. 10, pp. 1-22.

Ünal, Z. E., Z. Terzi, B. Yalvaç, and D. C. Geary, 2024, "The Relation Between Number Line Performance and Mathematics Outcomes: Two Meta-Analyses", *Developmental Science*, Vol. 27, No. 4, pp. 1-19.

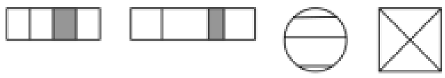
Yanik, H. B., B. Helding, and A. Flores, 2008, "Teaching the Concept of Unit in Measurement Interpretation of Rational Numbers", *İlköğretim Online*, Vol. 7, No. 3, pp. 693-705.

APPENDIX A: FRACTION TEST

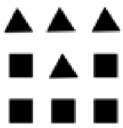
Kesir Testi

Testteki her bir soruyu, çözümlerinizi açıkça göstererek ve düşüncelerinizi açıklayarak cevaplayınız. Çözümünden emin olmadığınız sorularda aklınızdaki çözüm yöntemini veya soru hakkındaki düşüncelerinizi paylaşabilirsiniz.

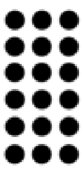
A.1) $\frac{1}{4}$'ü temsil eden şekil veya şekilleri işaretleyiniz.



A.2) Verilen şekillerin kaçta kaçının üçgen olduğunu kesir olarak yazınız.



A.3) Aşağıdaki bütünün $\frac{2}{3}$'ünü daire içine alınız.



A.4) $\frac{3}{5}$ kesrini aşağıda verilen sayı doğrusu üzerine yerleştiriniz.

A.5) Aşağıda verilen kesirlerde eşitliğin sağlanabilmesi için boşluğa gelmesi gereken pay değerini yazınız.

$$\frac{1}{3} = \frac{\quad}{12}$$

A.6) Verilen iki kesirden daha büyük olanı işaretleyiniz.

$\frac{4}{7}$ $\frac{2}{7}$

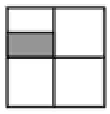
Figure A.1: The Fraction Test handed out to students.

A.7) Kesirlerin toplamını hesaplayınız.

$\frac{1}{6}+\frac{3}{6}=$

B.1) Her bir şekilde taralı alana karşılık gelen kesri yazınız.

a)



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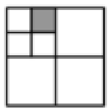
b)



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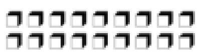
B.2)

Örnek:

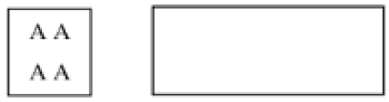
Yukarıdaki örnek size yardımcı olacaktır. Soldaki şekil bir kesri temsil etmektedir. Aynı kesri gösterecek biçimde sağdaki şekil oluşturulmuştur.

Aşağıda verilen şekiller arasından soldaki şekil bir kesri temsil etmektedir. Buna göre, aynı kesri gösterecek biçimde sağdaki şekli işaretleyiniz.

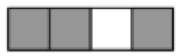
A A A A A A



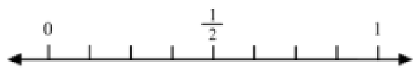
B.3) Dört adet A, bir bütünün $\frac{2}{3}$ 'ünü temsil ediyorsa, aşağıdaki kutuya bütünü çizin.



B.4) Şekil B' deki taralı alana karşılık gelen kesri sayı doğrusu üzerinde gösteriniz.



Şekil B



B.5) Aşağıdaki beş daireden hangisi dikdörtgenin temsil ettiği kesirle aynı kesri temsil etmektedir? Cevabınızı açıklayınız.

Figure A.1: The Fraction Test handed out to students. (cont.)

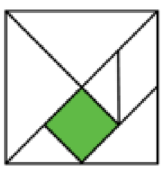
B.6) Aşağıdaki kesirleri iki farklı yöntem kullanarak karşılaştırınız ve yöntemlerinizi açıklayınız.

$$\frac{8}{18} \qquad \frac{5}{9}$$

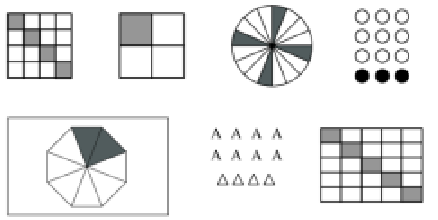
B.7) Kesirlerin toplamını hesaplayınız.

$$\frac{2}{5}+\frac{1}{6}=$$

C.1) Şekildeki taralı alana karşılık gelen kesri yazınız.



C.2) Aynı kesri ifade eden şekilleri işaretleyiniz.



C.3) Aşağıdaki harflerin 7/6'sını kutunun içine çiziniz.

AAAAAA
AAAAAA

C.4) Aşağıda verilen sayı doğrusunda boş kutu yerine gelmesi gereken sayıyı yazınız.

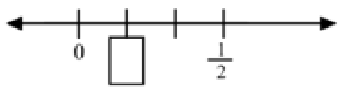


Figure A.1: The Fraction Test handed out to students. (cont.)

C.5) Üç arkadaş aynı boyut ve şekilde üç pizza sipariş etti. Gökhan pizzasının $\frac{4}{16}$ 'sını yedi. Arda pizzasının $\frac{3}{12}$ 'sini yedi. Cemal pizzasından x parça yedi. Üç arkadaş da aynı miktarda pizza yediyse, Cemal'in pizzasının toplam kaç parça olabileceğini x'i kullanarak ifade ediniz.

C.6) $\frac{1}{9}$ 'dan büyük ve $\frac{1}{8}$ 'den küçük bir kesir yazınız. Nasıl bulduğunuzu açıklayınız.

C.7)

a) Kesirlerin toplamını hesaplayınız.

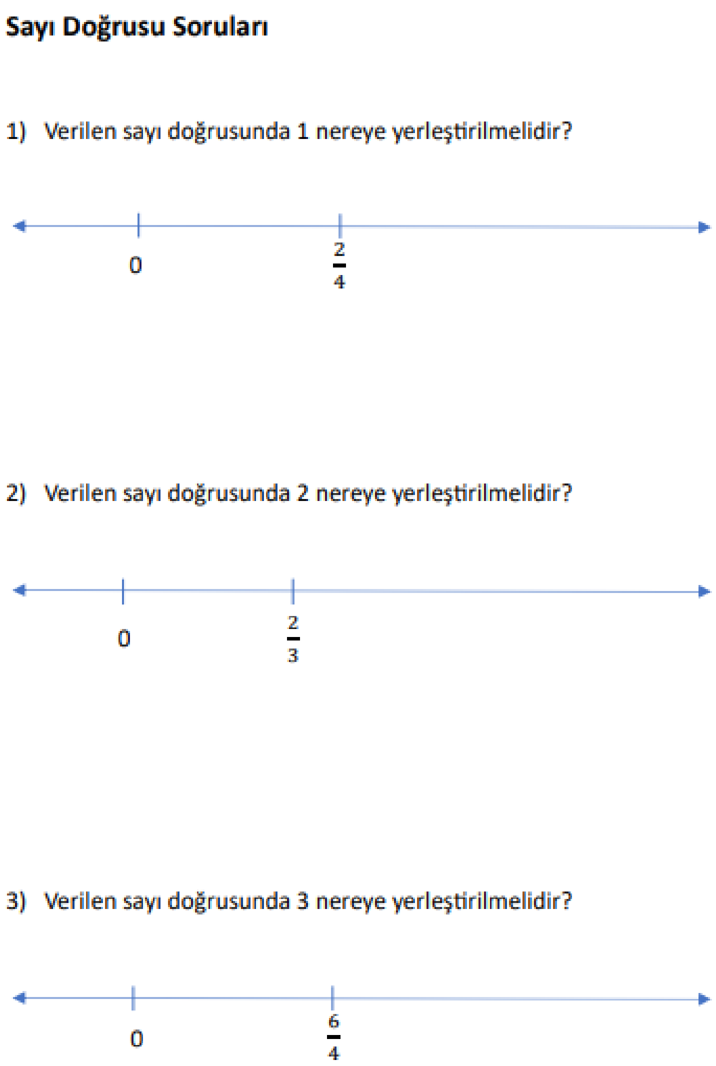
$\frac{2}{5} + \frac{1}{6} =$

b) Yukarıda verilen toplama işlemini şekil çizerek gösteriniz.

c) Çözümünde $\frac{2}{5} + \frac{1}{6}$ denklemini kullanabileceğiniz bir problem oluşturunuz.

Figure A.1: The Fraction Test handed out to students. (cont.)

APPENDIX B: UNITIZING TEST


Sayı Doğrusu Soruları

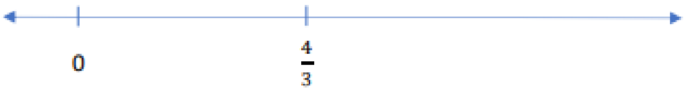
1) Verilen sayı doğrusunda 1 nereye yerleştirilmelidir?

2) Verilen sayı doğrusunda 2 nereye yerleştirilmelidir?

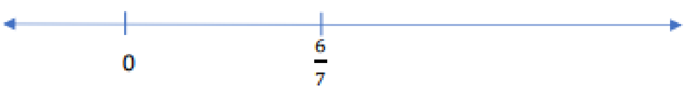
3) Verilen sayı doğrusunda 3 nereye yerleştirilmelidir?

Figure B.1: The Unitizing Test handed out to students.

4) Verilen sayı doğrusunda 4 nereye yerleştirilmelidir?



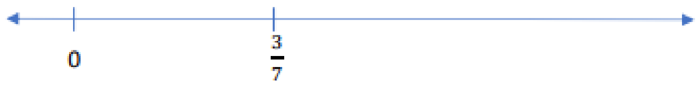
5) Verilen sayı doğrusunda $\frac{9}{7}$ nereye yerleştirilmelidir?



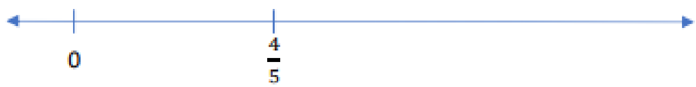
6) Verilen sayı doğrusunda $1\frac{1}{5}$ nereye yerleştirilmelidir?

Figure B.1: The Unitizing Test handed out to students. (cont.)

7) Verilen sayı doğrusunda 1 nereye yerleştirilmelidir?



8) Verilen sayı doğrusunda $\frac{7}{5}$ nereye yerleştirilmelidir?



9) Verilen sayı doğrusunda;
   a) 1 nereye yerleştirilmelidir?
   b) $\frac{1}{2}$ nereye yerleştirilmelidir?

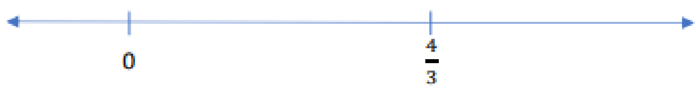
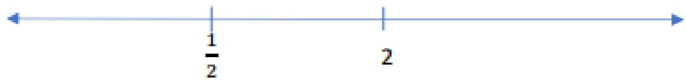


Figure B.1: The Unitizing Test handed out to students. (cont.)

10) Verilen sayı doğrusunda;

a) 1 nereye yerleştirilmelidir?

b) 0 nereye yerleştirilmelidir?



11) Verilen sayı doğrusunda $\frac{5}{3}$ nereye yerleştirilmelidir?

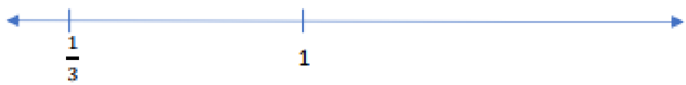
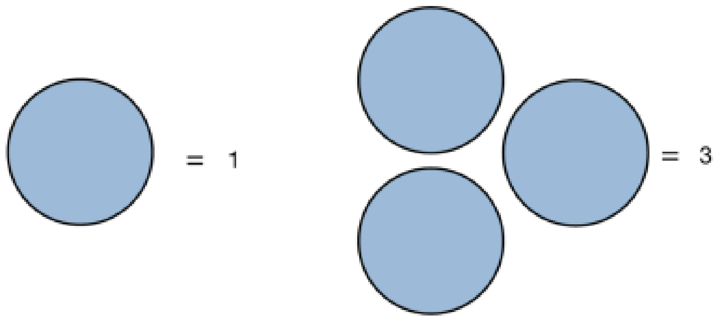


Figure B.1: The Unitizing Test handed out to students. (cont.)

Kesir Çemberi Soruları

Örnek: Aşağıda verilen şekil 1 sayısına karşılık gelmektedir. Buna göre 3 sayısına karşılık gelecek şekli çiziniz.



1) Aşağıda verilen şekil $\frac{1}{4}$ kesrine karşılık gelmektedir. Buna göre 1 sayısına karşılık gelecek şekli çiziniz.

2) Aşağıda verilen şeklin tamamı $\frac{2}{3}$ kesrine karşılık gelmektedir. Buna göre 2 sayısına karşılık gelecek şekli çiziniz.

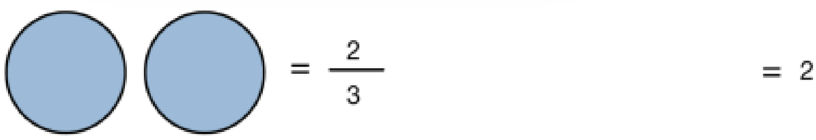
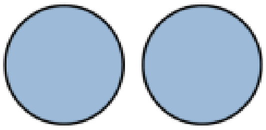


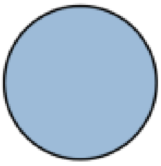
Figure B.1: The Unitizing Test handed out to students. (cont.)

3) Aşağıda verilen şeklin tamamı $\frac{6}{4}$ kesrine karşılık gelmektedir. Buna göre 3 sayısına karşılık gelecek şekli çiziniz.

 $= \frac{6}{4}$

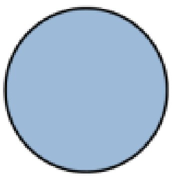
$= 3$

4) Aşağıda verilen şeklin tamamı $\frac{4}{3}$ kesrine karşılık gelmektedir. Buna göre 4 sayısına karşılık gelecek şekli çiziniz.

 $= \frac{4}{3}$

$= 4$

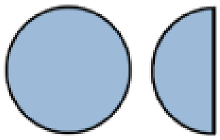
5) Aşağıda verilen şeklin tamamı $\frac{4}{5}$ kesrine karşılık gelmektedir. Buna göre $\frac{6}{5}$ kesrine karşılık gelecek şekli çiziniz.

 $= \frac{4}{5}$

$= \frac{6}{5}$

Figure B.1: The Unitizing Test handed out to students. (cont.)

6) Aşağıda verilen şeklin tamamı $\frac{3}{5}$ kesrine karşılık gelmektedir. Buna göre $1\frac{1}{5}$ kesrine karşılık gelecek şekli çiziniz.



$= \frac{3}{5}$ $= 1\frac{1}{5}$

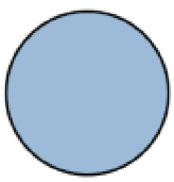
7) Aşağıda verilen şeklin tamamı $\frac{3}{2}$ kesrine karşılık gelmektedir. Buna göre $\frac{7}{2}$ kesrine karşılık gelecek şekli çiziniz.

$= \frac{3}{2}$ $= \frac{7}{2}$

8) Aşağıda verilen şeklin tamamı $\frac{4}{3}$ kesrine karşılık gelmektedir. Buna göre;

a. 1 sayısına karşılık gelecek şekli çiziniz.

b. $\frac{1}{2}$ kesrine karşılık gelecek şekli çiziniz.



$= \frac{4}{3}$ $= 1$

$= \frac{1}{2}$

Figure B.1: The Unitizing Test handed out to students. (cont.)

Problemler

1) 6'lı paket meyve suyu 34 TL'ye, aynı meyve suyunun 18'li paketi 105 TL'ye satılmaktadır. Hangisini satın almak daha kârlıdır?

2) 18 kilogramlık bir çuval pirinç 120 TL'ye, aynı pirincin 29 kilogramlık çuvalı 180 TL'ye satılmaktadır. Hangisini almak daha kârlıdır?

3) Bir markette 3 kilogram un 105 TL'ye satılmaktadır. Bu unun kilogram fiyatının miktara göre değişmediği biliniyorsa;
   a) 4 kg unun fiyatı nedir?

   b) 8 kg unun fiyatı nedir?

4) 24'lü paket sakız 30 TL'ye, aynı sakızın 18'li paketi 22 TL'ye satılmaktadır. Hangisini almak daha kârlıdır?

5) Hakan aracıyla çıktığı 130 kilometrelik yolun 10. kilometresinde benzinliğe uğrayarak benzin almış ve 70. kilometresinde mola vermiştir. Hakan, aracının benzinlikten mola verdiği yere kadar yaklaşık 5 litre benzin yaktığını hesaplamıştır. Buna göre Hakan'ın aracı mola verdiği yerden varacağı yere gidene kadar yaklaşık kaç litre benzin yakar?

Figure B.1: The Unitizing Test handed out to students. (cont.)

APPENDIX C: INITIAL VERSION OF THE UNITIZING TEST

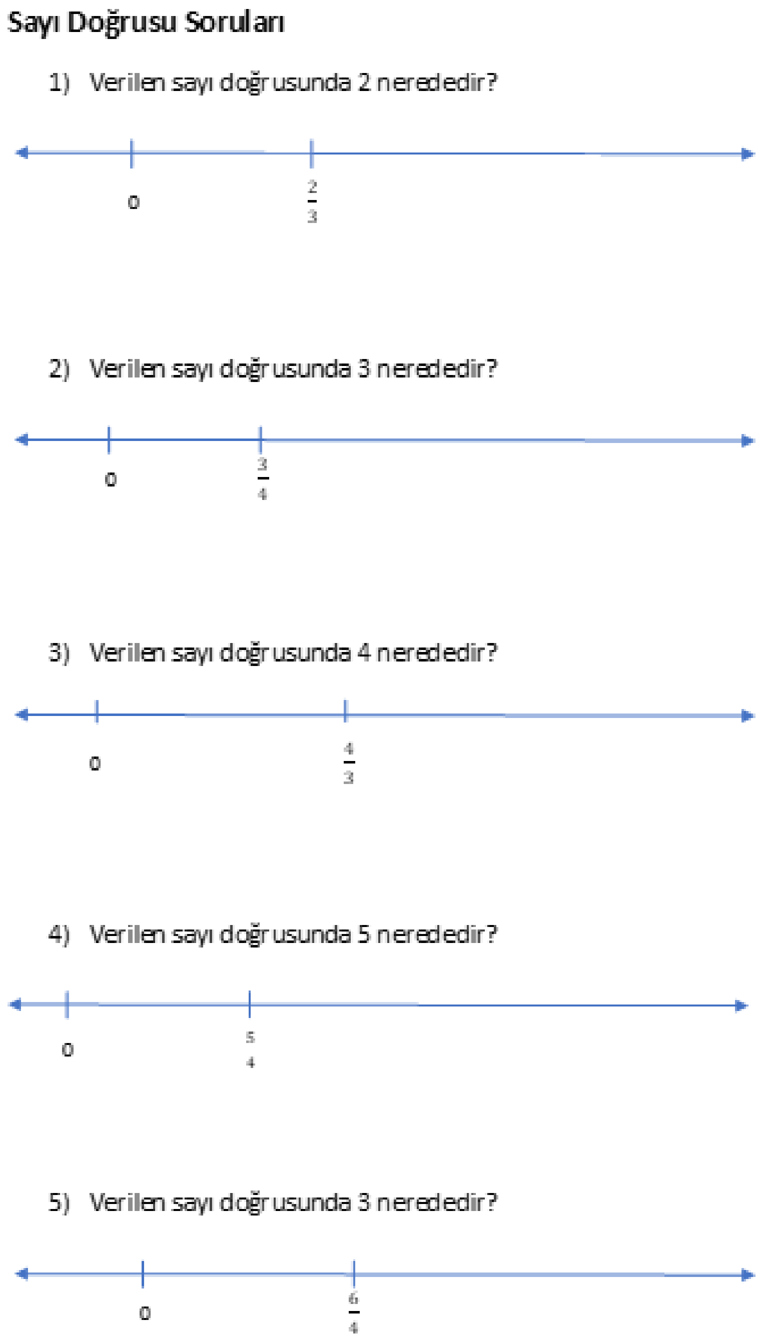
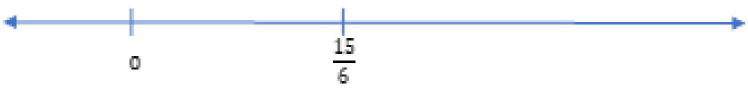
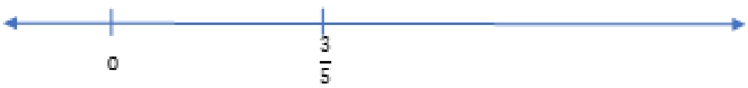


Figure C.1: Initial version of the Unitizing Test.

6) Verilen sayı doğrusunda 5 nerededir?

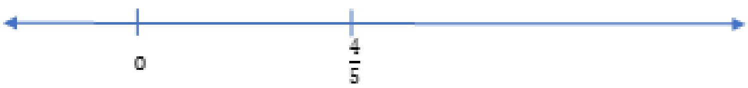


7) Verilen sayı doğrusunda 1 1/5 nerededir?

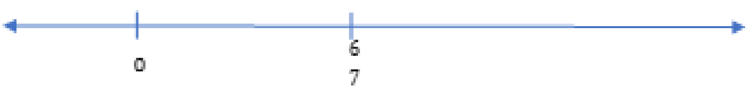


8) Verilen sayı doğrusunda 1 1/7 nerededir?

9) Verilen sayı doğrusunda 6/5 nerededir?



10) Verilen sayı doğrusunda 9/7 nerededir?



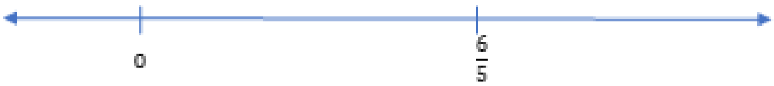
11) Verilen sayı doğrusunda 1 nerededir? 5/7 nerededir?

Figure C.1: Initial version of the Unitizing Test. (cont.)

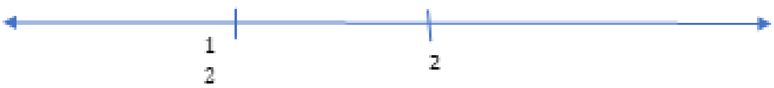
12) Verilen sayı doğrusunda 1 nerededir? 7/5 nerededir?

13) Verilen sayı doğrusunda 1 nerededir? ½ nerededir?

14) Verilen sayı doğrusunda 1 nerededir? ½ nerededir?

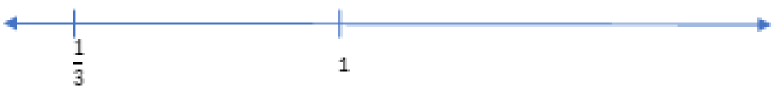


15) Verilen sayı doğrusunda 1 ve 0 nerededir? 7/2 nerededir?



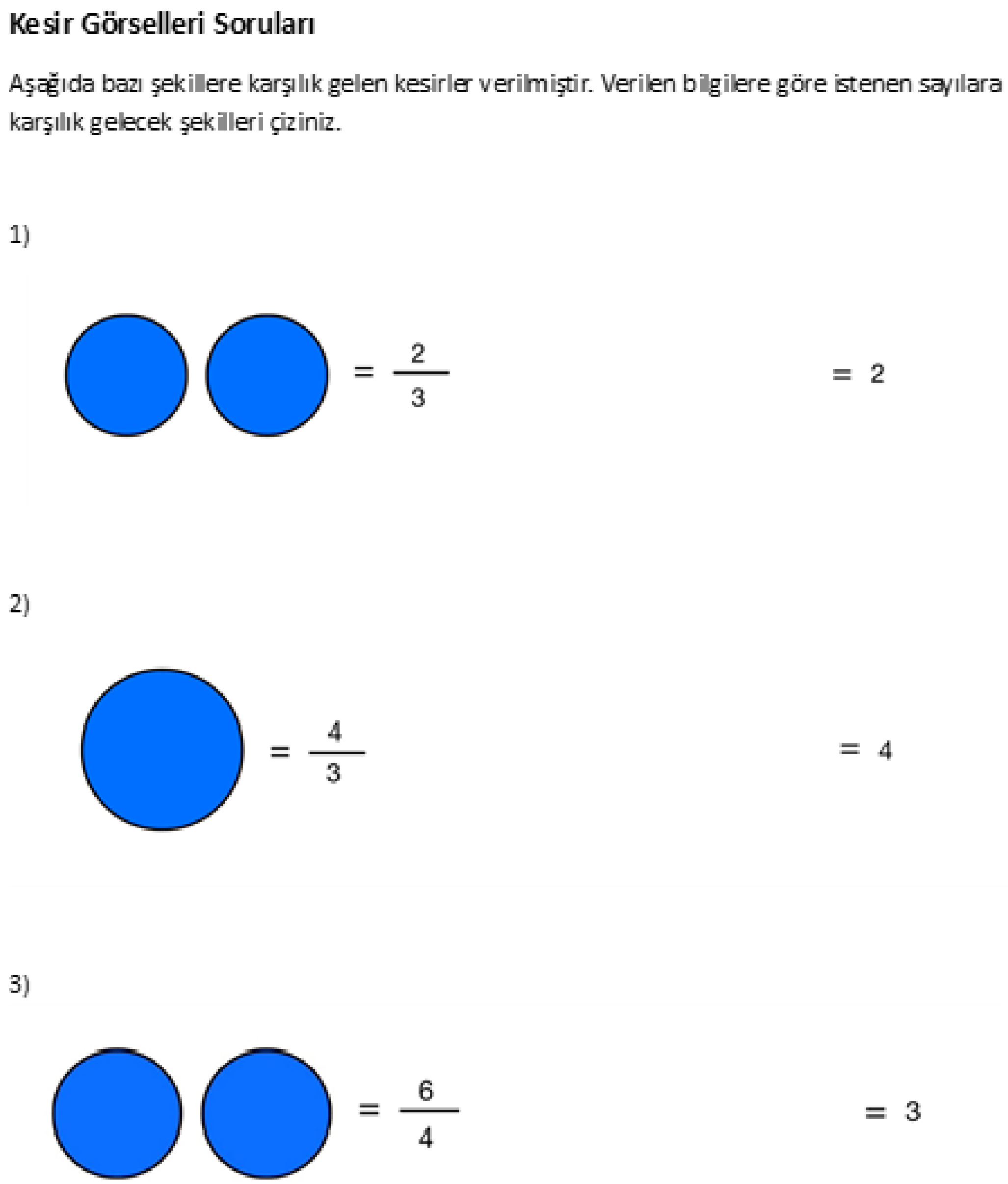
16) Verilen sayı doğrusunda 1 ve 0 nerededir? 11/2 nerededir?

17) Verilen sayı doğrusunda 5/3 nerededir?



18) Verilen sayı doğrusunda 12/7 nerededir?

Figure C.1: Initial version of the Unitizing Test. (cont.)



Kesir Görselleri Soruları

Aşağıda bazı şekillere karşılık gelen kesirler verilmiştir. Verilen bilgilere göre istenen sayılara karşılık gelecek şekilleri çiziniz.

1)

$= \frac{2}{3}$ $= 2$

2)

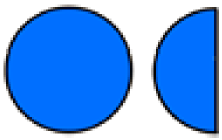
$= \frac{4}{3}$ $= 4$

3)

$= \frac{6}{4}$ $= 3$

Figure C.1: Initial version of the Unitizing Test. (cont.)

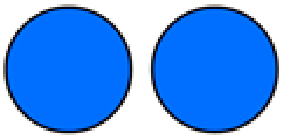
4)



$= \frac{3}{5}$

$= 1\frac{1}{5}$

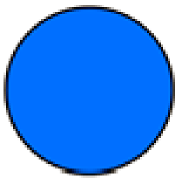
5)



$= \frac{4}{7}$

$= 1\frac{1}{7}$

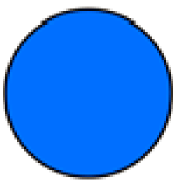
6)



$= \frac{4}{5}$

$= \frac{6}{5}$

7)



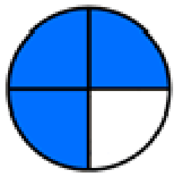
$= \frac{4}{3}$

$= \quad 1$

$= \frac{1}{2}$

Figure C.1: Initial version of the Unitizing Test. (cont.)

8)



$= \frac{3}{2}$

$= \frac{7}{2}$

Problemler

1) 6'lı paket meyve suyu 34 TL'ye, aynı meyve suyunun 18'li paketi 105 TL'ye satılmaktadır. Hangisini satın almak daha karlıdır?

2) 10'lu paket kabartma tozu 18 TL'ye, aynı kabartma tozunun 32'li paketi 54 TL'ye satılmaktadır. Hangisini satın almak daha karlıdır?

3) 10 kilogramlık bir paket şeker 428 TL'ye, aynı şekerin 15 kilogramlık paketi 632 TL'ye satılmaktadır. Hangisini satın almak daha karlıdır?

4) 18 kilogramlık bir çuval pirinç 120 TL'ye, aynı pirincin 29 kilogramlık çuvalı 180 TL'ye satılmaktadır. Hangisini almak daha karlıdır?

5) Bir markette 3 kilogram un 105 TL'ye satılmaktadır. Bu unun kilogram fiyatının miktara göre değişmediği biliniyorsa 4 kg unun fiyatı nedir? 8 kg unun fiyatı nedir?

6) Bir kahvecide 45 TL'ye 96 gram kahve alınabilmektedir. Aynı kahveden 105 TL'ye kaç gram alınabilir? 75 TL'ye kaç gram alınabilir?

7) 24'lü paket sakız 30 TL'ye, aynı sakızın 18'li paketi 22 TL'ye satılmaktadır. Hangisini almak daha karlıdır?

8) İçinde 160 adet bir koli kağıt bardak 180 TL'ye, aynı kağıt bardağın 135'li kolisi 150 TL'ye satılmaktadır. Hangisini almak daha karlıdır?

9) Hakan aracıyla çıktığı 130 kilometrelik yolun 10. kilometresinde benzinliğe uğrayarak benzin almış ve 70. kilometresinde mola vermiştir. Hakan, aracının benzinlikten mola verdiği yere kadar yaklaşık 5 litre benzin yaktığını hesaplamıştır. Buna göre Hakan'ın aracı mola verdiği yerden varacağı yere gidene kadar yaklaşık kaç litre benzin yakar?

Figure C.1: Initial version of the Unitizing Test. (cont.)

APPENDIX D: CATEGORIES PRESENTED TO EXPERTS

Which categories do you think this question covers?

a) Use the given fraction/number as the unit
b) Improper fraction is given as the unit
c) The answer is beyond the given NL
d) Given fraction/number is half of the asked
e) Needs to understand mixed fractions can be written as an improper fraction
f) Use half of the fraction/number as the unit
g) Find the unit (fraction)/form a unique unit
h) Find half of 1
i) Needs to know the number of halves between $\frac{1}{2}$ and the asked integer
j) 0 cannot be placed into the given NL

APPENDIX E: CODING STRUCTURE FOR INTERVIEWS

Error Patterns

- Placing integers at the end of the NL
- Placing 0 at the beginning of the NL
- Errors due to misconceptions in fraction knowledge or notation
- Referring to asked integers as the numerator of fractions or vice versa
- Additive thinking instead of multiplicative in WP
- Drawing the shapes without using the given referents in FC

Confusions

- Students can't explain their solutions or have doubts
- They explain their confusion or difficulties with tasks

Unitizing strategies

- Effective
- Procedural

Equal parts

Estimations

Def. and examples of unit

Unit use in tasks

Corrections (when they realize their error and change the answer)

"I don't know/remember"

APPENDIX F: SUPPLEMENTARY TABLES

Table F.1: Fraction Test translation procedure examples.

Item	Original	Machine Translation and Human Quality Check	Expert Suggestions	Finalized Version
A.2	Write in fraction what part of the shapes are triangles.	Şekillerin kaçta kaçının üçgen olduğunu kesir olarak yazınız.	1. Uygundur. 2. "Verilen 9 şeklin bir bütünü oluşturduğu" şeklinde bir açıklama eklenmesine ihtiyaç olabilir. 3. Cümle açık ve anlaşılır. *Alternatif:* Şekillerin kaçta kaçı üçgendir? Kesir olarak yazınız.	Verilen şekillerin kaçta kaçının üçgen olduğunu kesir olarak yazınız.
A.7	Calculate the sum.	Toplamı hesaplayınız.	1. Kesirlerin toplamını hesaplayınız. 2. "Toplama işlemini yapınız" veya "İşlemin sonucu kaçtır?" seçenekleri düşünülebilir. 3. "Toplamı hesaplayınız" kısa ve net; "sum" ifadesi Türkçede genellikle "verilen işlemin toplamı" şeklinde uzatılır. *Alternatifler:* Kesirlerin toplamını bulunuz/hesaplayınız. Verilen işlemi yapınız.	Kesirlerin toplamını hesaplayınız.

Table F.1. Fraction Test translation procedure examples. (cont.)

Item	Original	Machine Translation and Human Quality Check	Expert Suggestions	Finalized Version
B.1	Write in fraction the shaded part in each shape.	Her bir şekildeki taralı alanın temsil ettiği kesri yazınız.	1. Nereye yazacakları belirsiz; "Yukarıdaki boşluğa her bir şekildeki taralı alanın temsil ettiği kesri yazınız." daha iyi olur. 2. "Her bir şekilde taralı alana karşılık gelen (denk olan) kesri yazınız." da uygundur. 3. Anlaşılır.	Her bir şekilde taralı alana karşılık gelen kesri yazınız.
C.6	Write a bigger fraction than 1/9 and smaller than 1/8. Explain your thought.	1/9'dan büyük ve 1/8'den küçük bir kesir yazınız. Düşüncelerinizi açıklayınız.	1. Düşüncelerinizi ve bulma yolunuzu açıklayınız. 2. "Nasıl bulduğunuzu açıklayınız." ifadesi eklenebilir.	1/9'dan büyük ve 1/8'den küçük bir kesir yazınız. Nasıl bulduğunuzu açıklayınız.

Table F.2: Fraction Test scoring rubric.

Item	Scores for each item			
	4	**3**	**2**	**1**
A.1	Notices that the shape should be divided into four equal pieces AND 1 out of 4 pieces should be filled.		Notices that the shape should be divided into four equal pieces OR 1 out of 4 pieces should be filled.	
A.2	Successfully writes the numerator and the denominator of the fraction.			
A.3	Circles the 2/3 of the whole successfully.		Notices to mark 2 out of 3 shapes.	
A.4	Places the number in the right location.			Notices the number line is divided into 5 equal pieces
A.5	Reaches the correct answer through proportion or expansion			

Table F.2 Fraction Test scoring rubric. (cont.)

Item	**Scores for each item**			
	4	**3**	**2**	**1**
A.6	Compares the nominators correctly when denominators are equal.			
A.7	Successfully calculates the sum.			
B.1	Writes the fractions that represent the shapes successfully.		Indicates that the parts are not equal OR Divides the shapes further to make them equal.	
B.2	Marks the shape on the right successfully.	Writes down the fraction that represents the shape on the left AND Writes the equivalent fraction suited to the shape on the right.		Writes down the fraction that represents the shape on the left.
B.3	Draws the shape using the given referent correctly.			

Table F.2 Fraction Test scoring rubric. (cont.)

Item	**Scores for each item**			
	4	**3**	**2**	**1**
B.4	Successfully places the fraction on number line	Write down the fraction that represents the shape AND Writes the equivalent fractions of the numbers on number line.	Writes the equivalent fractions of the numbers on number line.	Writes down the fraction that represents the shape.
B.5	Writes the shape represented by the rectangle AND Finds its equivalent shaded circle		Writes the shape represented by the rectangle	
B.6	Uses two different methods to make a reasonable comparison.		Uses one method to make a reasonable comparison.	
B.7	Equates the denominators of fractions AND Makes the summation correctly		Equates the denominators of fractions	

Table F.2 Fraction Test scoring rubric. (cont.)

Item	**Scores for each item**			
	4	**3**	**2**	**1**
C.1	Writes the fraction that represents the shape.		Indicates that the parts are not equal OR Divides the shape further to make them equal.	Writes down the fractions that represent the shapes.
C.2	Marks all five shapes that represent the same fraction correctly.		Writes down the fractions that represent the shapes AND Marks at least two equal shapes	Writes down the fractions that represent the shapes.
C.3	Draws the shape using the given referent correctly.			
C.4	Places the number on the number line correctly.	Expands the fraction on number line to match it with number of pieces.		Notices that the given fraction and number of parts do not match.
C.5	Indicates the total number of slices as 4x.		Reaches the ratio 1/4 by realizing that fractions are equal	

Table F.2 Fraction Test scoring rubric. (cont.)

Item	Scores for each item			
	4	**3**	**2**	**1**
C.6	Reaches the correct answer through equating nominators or denominators.			
C.7	The total of 3 sub-questions			
C.7a				Equates the denominators of fractions AND Makes the summation correctly
C.7b				Visualizes the sum successfully.
C.7c			Includes expressions that require the use of addition AND Selects objects that can be represented through fractions.	Includes expressions that require the use of addition OR Selects objects that can be represented through fractions.

Table F.3: Frequencies of 5th grade students' strategy use across task types and fraction achievement levels.

5th Grade	Effective Unitizing		Procedural Unitizing		Unknown–None		
	SI	**EM**	**SI**	**EM**	**CU**	**ER**	**N**
High-Level							
NL	1	–	2	5	–	1	–
FC	2	1	–	–	–	6	–
WP	7	–	1	–	1	–	–
Mid-Level							
NL	–	–	–	6	–	–	5
FC	–	–	1	–	–	6	4
WP	2	–	2	1	3	–	3
Low-Level							
NL	–	–	–	1	–	3	5
FC	–	–	1	–	–	5	3
WP	2	–	–	–	7	–	7

Table F.4: Frequencies of 6th grade students' strategy use across task types and fraction achievement levels.

6th Grade	Effective Unitizing		Procedural Unitizing		Unknown–None		
	SI	**EM**	**SI**	**EM**	**CU**	**ER**	**N**
High-Level							
NL	2	–	3	1	–	1	–
FC	4	–	–	–	–	1	2
WP	2	–	4	1	–	–	–
Mid-Level							
NL	–	1	–	3	–	3	1
FC	1	1	–	–	–	4	2
WP	–	–	2	–	1	–	5
Low-Level							
NL	–	–	–	2	–	3	4
FC	–	–	–	–	–	3	6
WP	3	–	1	2	–	–	3

Table F.5: Frequencies of 7th grade students' strategy use across task types and fraction achievement levels.

7th Grade	**Effective Unitizing**		**Procedural Unitizing**		**Unknown–None**		
	SI	**EM**	**SI**	**EM**	**CU**	**ER**	**N**
High-Level							
NL	1	–	3	2	–	–	–
FC	4	–	–	–	1	–	1
WP	5	–	1	–	–	–	–
Mid-Level							
NL	–	–	1	2	–	2	2
FC	2	–	1	–	–	1	3
WP	2	–	4	–	–	–	1
Low-Level							
NL	–	–	–	3	–	–	3
FC	–	–	1	–	–	1	4
WP	–	–	1	–	–	–	5