

SIGNALING AND INFORMATION GAMES WITH SUBJECTIVE COSTS OR PRIORS AND PRIVACY CONSTRAINTS

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COSTS OR PRIORS AND PRIVACY CONSTRAINTS

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ABSTRACT

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We investigate signaling game problems where an encoder and a decoder with misaligned objectives communicate. We consider a variety of setups involving cost criterion mismatch, prior mismatch and a particular application to privacy problems. We also consider both Nash and Stackelberg solution concepts. First, we extend the classical results on the scalar cheap talk problem which is introduced by Crawford and Sobel. In prior work, it is shown that the encoder must employ a quantization policy under any Nash equilibrium for arbitrary source distributions. We specifically consider sources with a log-concave density and investigate properties of equilibria. For sources with two-sided unbounded support, we prove that, for any finite number of bins, there exists a unique equilibrium. If the source has semi-unbounded support, then there may exist a finite upper bound on the number of bins in equilibrium depending on certain explicit conditions. Moreover, we show that an equilibrium with more bins is more informative by showing that the expected costs of the encoder and the decoder in equilibrium decrease as the number of bins increases. Furthermore, for strictly log-concave sources with two-sided unbounded support, we prove that if the encoder and decoder iteratively compute their best responses starting from a given number of bins, then the resulting policies converge to the unique equilibrium with the corresponding number of bins.

Second, we model a privacy problem as a signaling game between an encoder and a decoder. Given a pair of correlated observations modeled as jointly Gaussian random vectors, the encoder aims to hide one of them and convey the other one to the decoder. In contrast, the aim of the decoder is to accurately estimate both of the random vectors. For the resulting signaling game problem, we show that a payoff dominant Nash equilibrium among all admissible policies is attained by a set of explicitly characterized linear policies. We also show that a payoff

dominant Nash equilibrium coincides with a Stackelberg equilibrium. Moreover, we formulate the information bottleneck problem within our Stackelberg framework under the mean squared error criterion where the information bottleneck setup has a further restriction that only one of the parameters is observed at the encoder. We show that the Gaussian information bottleneck problem admits a linear solution which is explicitly characterized.

Third, we investigate communications through a Gaussian noise channel between an encoder and a decoder with prior mismatch. Although they consider the same cost function, the induced expected costs as a map of their policies are misaligned due to their prior mismatch. We analyze the resulting signaling game problem under Stackelberg equilibria. We first investigate robustness of equilibria and show that the Stackelberg equilibrium cost of the encoder is upper semi continuous, under the Wasserstein metric, as the encoder's prior approaches the decoder's prior, and it is also lower semi continuous with Gaussian priors. In addition, we show that the optimality of affine policies for Gaussian signaling no longer holds under prior mismatch. Furthermore, we provide conditions under which there exist informative equilibria under an affine policy restriction.

Fourth, we extend Crawford and Sobel's formulation to a multidimensional source setting. We first provide a set of geometry conditions that decoder actions at a Nash equilibrium has to satisfy considering any multidimensional source. Then, we consider independent and identically distributed sources and characterize necessary and sufficient conditions under which an informative linear equilibrium exists. We observe that these conditions involve the bias vector that leads to misaligned costs. Depending on certain conditions on the bias vector, the existence of linear equilibria may require sources with a Gaussian or a symmetric density. Moreover, we provide a rate-distortion theoretic formulation of the cheap talk problem and obtain achievable rate and distortion pairs for the Gaussian case.

Finally, in a communication theoretic setup, we consider modulation classification and symbol decoding problems jointly and propose optimal strategies under various settings. The aim is to decode a sequence of received signals with an unknown modulation scheme. First, the prior probabilities of the candidate modulation schemes are assumed to be known and a formulation is proposed under the Bayesian framework. Second, we address the case when the prior probabilities of the candidate modulation schemes are unknown, and provide a method under the minimax framework. Numerical simulations show that the proposed

techniques improve the performance under the employed criteria compared to the conventional techniques in a variety of system configurations.



Keywords: Game theory, signaling games, cheap talk, multidimensional cheap talk, information theory, inconsistent priors, privacy, information bottleneck, quantization, modulation classification.

ÖZET

ÖZNEL MALİYETLER VEYA OLASILIK MODELLERİ VE GİZLİLİK KISITLARI KAPSAMINDA İŞARETLEME VE BİLGİ OYUNLARI

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Farklı hedeflere sahip kodlayıcının ve kod çözücünün haberleştiği işaretleme oyunlarını araştırmaktayız. Maliyet kriterlerinin veya olasılık modellerinin uyumsuz olduğu kurgular ve gizlilik problemlerine uygulaması olan bir kurgu olmak üzere çeşitli kurguları incelemekteyiz. Önerilen kurguları Nash ve Stackelberg çözüm kavramlarıyla araştırmaktayız. İlk olarak, Crawford ve Sobel tarafından ortaya konulan tek boyutlu ucuz konuşma problemi üzerine klasik sonuçları genişletmekteyiz. Önceki bir çalışmada, herhangi bir kaynak dağılımı için Nash dengesi durumundaki kodlayıcının sayılabilir seviyeli nicemleme stratejisi uygulaması gerektiği gösterilmektedir. Özel olarak logaritması içbükey dağılıma sahip kaynakları ele almakta ve Nash dengesinin özelliklerini araştırmaktayız. Sonsuz dayanaklı dağılıma sahip kaynaklar ve verilen herhangi bir sonlu seviye sayısı için o seviye sayısına sahip eşsiz bir nicemleme stratejisinin Nash dengesi oluşturduğunu ispatlamaktayız. Eğer kaynak yarı sonsuz dayanaklı dağılıma sahip ise belli şartlara bağlı olarak denge durumundaki nicemleme stratejilerinin seviye sayısı için sonlu bir üst sınır olabilmektedir. Ayrıca, denge durumunda kullanılan nicemleme stratejisinin seviye sayısı arttıkça hem kodlayıcının hem de kod çözücünün elde ettiği ortalama maliyetin azaldığını ve dolayısıyla daha bilgilendirici bir dengenin elde edildiğini göstermekteyiz. Ek olarak, sonsuz dayanaklı ve logaritması tekdüze içbükey dağılıma sahip kaynaklar için kodlayıcı ve kod çözücü belli bir seviye sayılı nicemleme stratejisinden başlayarak yinelemeli bir şekilde en iyi tepki fonksiyonlarını hesapladıklarında stratejilerinin o seviye sayısına sahip eşsiz denge noktasına yakınsadığını ispatlamaktayız.

İkinci olarak, bir gizlilik problemini kodlayıcı ve kod çözücü arasında işaretleme oyunu olarak modellemekteyiz. Kodlayıcı Gauss dağılımına sahip birbirleriyle ilişkili iki vektör parametresinden bir tanesini gizlemeyi ve diğerini kod çözücüye

iletmeyi hedeflemektedir. Diğer yandan, kod çözücü her iki parametreyi de kestirmeyi hedeflemektedir. Elde edilen işaretleme oyunu için türettiğimiz doğrusal kodlayıcı ve kod çözücü stratejilerinin ortalama maliyet açısından dominant bir Nash dengesi oluşturduğunu göstermekteyiz. Elde ettiğimiz Nash dengesinin aynı zamanda Stackelberg dengesi olduğunu göstermekteyiz. Ayrıca, karesel hata metriği altındaki bilgi darboğazı problemini Stackelberg dengesi kullanarak oyun kuramı kapsamında araştırmaktayız. Gizlilik probleminden farklı olarak bilgi darboğazı probleminde kodlayıcı parametrelerden sadece bir tanesini gözlemlemektedir. Gauss dağılımına sahip kaynaklar için bilgi darboğazı probleminin elde ettiğimiz doğrusal bir çözümünün olduğunu ispatlamaktayız.

Üçüncü olarak, kodlayıcının ve kod çözücünün Gauss dağılımına sahip gürültülü bir kanal üzerinden haberleşmesi durumunu önsel bilgi tutarsızlığı altında araştırmaktayız. Kodlayıcı ve kod çözücü aynı maliyet fonksiyonunu kullanmalarına rağmen önsel bilgi tutarsızlığı sebebiyle farklı ortalama maliyetlere sahip olmaktadır. Elde edilen işaretleme oyunu problemini Stackelberg dengesi altında araştırmaktayız. Kodlayıcının önsel bilgisinin kod çözücünün önsel bilgisine Wasserstein metriği altında yakınsadığı durumda kodlayıcının Stackelberg dengesi altındaki ortalama maliyetinin üst yarı sürekli olduğunu ve eğer önsel bilgiler Gauss dağılımına sahip ise ayrıca alt yarı sürekli olduğunu göstermekteyiz. Ayrıca, tutarsız önsel bilgilerin Gauss dağılımına sahip olduğu durumda doğrusal kodlayıcı stratejilerinin optimal olmadığını göstermekteyiz. Ek olarak, doğrusal kodlayıcı koşulu altındaki denge stratejilerinin bilgilendirici olup olmadığına dair koşullar elde etmekteyiz.

Dördüncü olarak, Crawford ve Sobel'in formülasyonunu çok boyutlu parametre kurgusuna genişletmekteyiz. İlk olarak, herhangi bir çok boyutlu kaynak dağılımı için Nash dengesi altındaki kod çözücü stratejilerinin sağlaması gereken bir dizi geometrik koşul elde etmekteyiz. Daha sonra, bağımsız ve eş dağılıma sahip çok boyutlu kaynakları ele almakta ve kodlayıcının doğrusal olduğu bilgilendirici bir dengenin var olabilmesi için gerekli ve yeterli koşulları elde etmekteyiz. Bu koşulların kodlayıcının ve kod çözücünün farklı hedeflere sahip olmasına neden olan yanlılık vektörü ile ilişkili olduğunu gözlemlemekteyiz. Yanlılık vektörüne bağlı olarak, doğrusal bir dengenin varlığı kaynağın Gauss dağılımına ya da simetrik bir dağılıma sahip olmasını gerektirebilmektedir. Ayrıca ucuz konuşma problemini hız-bozunum kuramı çerçevesinde incelemekte ve kaynak Gauss dağılımına sahip olduğunda başarılabılır hız ve bozunum değerlerini elde etmekteyiz.

Son olarak, bir haberleşme kuramı kurgusunda, modülasyon sınıflandırma ve sembol kod çözümü problemlerini birlikte ele almakta ve çeşitli senaryolar için optimal teknikler önermekteyiz. Modülasyon tipi bilinmeyen bir sinyalden sembollerin kestirimini amaçlamaktayız. İlk olarak, olası modülasyon tiplerinin önsel olasılık dağılımlarının bilindiğini varsayıp, Bayes kriteri altında bir formülasyon önermekteyiz. İkinci olarak, olası modülasyon tiplerinin önsel olasılık dağılımlarının bilinmediği durum için minimax kriteri altında bir formülasyon önermekteyiz. Sayısal örnekler önerilen tekniklerin geleneksel tekniklere göre ele aldığımız metrikler altında daha iyi performans sergilediğini göstermektedir.

Anahtar sözcükler: Oyun kuramı, işaretleme oyunları, ucuz konuşma, çok boyutlu ucuz konuşma, bilgi kuramı, önsel bilgi tutarsızlığı, gizlilik, bilgi darboğazı, nicemleme, modülasyon sınıflandırma.

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Chapter 1

Introduction

In a networked system such as in communication and control theory, there are decision makers that exchange information with each other in order to achieve some design goal. Traditional design of such systems makes the assumption that the goals of each decision maker in the system are aligned. The common goal assumption enables a system designer not to worry about any additional information provided to a decision maker since this additional information cannot degrade the system performance. In particular, if a decision maker ignores this additional information, the system performance is kept the same while the use of additional information may improve the system performance depending on its value.¹ Namely, in systems involving decision makers with aligned goals, it is well-known that more information or communication is always at least as good for each decision maker in the system. On the other hand, the common goal assumption may no longer hold in various settings due to the presence of strategic decision makers (see e.g., [3–10]). Since the decision makers have different goals, the principle of ignoring additional information does not apply in such settings. In particular, certain information may be beneficial for a decision maker whereas this knowledge by that decision maker may hurt some other decision maker.

¹This motivates the study of comparing information structures to see which one is more valuable for the system. Towards that goal, one can employ a partial order on information structures, which is studied extensively in the literature. These studies are initiated by Blackwell in his seminal works [1, 2].

In fact, in certain settings involving decision makers with misaligned goals, it may even be the case that disclosing certain information publicly hurts all the decision makers [11]. Hence, a decision maker has to take the strategic nature of other decision makers into account while designing what information to reveal to another decision maker. This requires a game theoretic analysis of the problem where one considers a suitable equilibrium concept.

At this point, it is important to make a distinction between nonstrategic and strategic settings. If all the decision makers in the system share common goals, then the system is categorized as *team theoretic*. In a team theoretic setting, the decision makers form a team to achieve these common goals. On the other hand, if there exists a decision maker in the system whose aim is different from another decision maker, then the setting becomes *game theoretic*. In a game theoretic setting, the decision makers play against each other in order to achieve their own goals. A game theoretic setup may also arise if the decision makers have the same cost criterion but have subjective priors.

In this dissertation, we consider communications between an encoder and a decoder, viewed as two decision makers, where the decision makers are strategic due to several factors leading to different formulations. In the literature, problems investigating strategic communication between decision makers are referred to as *signaling game* problems. In a signaling game setup with an encoder and a decoder, a better informed encoder communicates with a decoder where their objective functions are not aligned. Since the objective functions of the encoder and decoder are different, the encoder designs what information to reveal strategically. A signaling game problem is referred to as *cheap talk* if the encoder can convey messages without any cost. We also consider a setup with subjective priors where even though the decision makers employ the same cost criterion, their expected costs given their policies are misaligned.

Crawford and Sobel in their seminal work [12] introduce a signaling game problem, which also corresponds to a cheap talk problem. They analyze communication scenarios between an encoder and a decoder which do not share a

common objective function due to a bias term appearing in the objective function of the encoder. They establish that under certain technical conditions, the encoder is required to apply a quantization policy in any equilibrium. In particular, due to the misalignment in the objective functions of the encoder and decoder, the encoder hides information by reporting the quantization bin that its observation lies in, rather than revealing its observation completely.² Hence, the encoder hides information due to the misaligned cost structure and not because of a cost induced by message transmission as it is costless for the encoder to transmit information. This is in significant contrast to classical communication theoretic settings involving decision makers with aligned goals since the encoder does not have any incentive to hide information in such settings.

A generic signaling game setup involving an encoder and a decoder can be described as follows: An encoder makes an observation from a source M . The encoder maps its observation to a message through an encoding policy denoted by $\gamma^e(\cdot)$. The decoder observes the encoded message directly or its noise corrupted version. Given its observation, the decoder takes an action U via a decoding policy $\gamma^d(\cdot)$, which determines the costs for both of the players. In many applications such as in communications and networked control, the induced expected costs of the encoder and the decoder given their policies may be misaligned due to the strategic or selfish nature of the players. While the misalignment in the expected costs may occur directly from the difference in the employed cost functions, subjective probabilistic beliefs of the players may also lead to misaligned expected costs even though the decision makers employ the same cost function. Hence, the encoder may aim to minimize

$$J^e(\gamma^e, \gamma^d) = \mathbb{E}_e [c^e(M, U)], \quad (1.1)$$

whereas the decoder may aim to minimize

$$J^d(\gamma^e, \gamma^d) = \mathbb{E}_d [c^d(M, U)], \quad (1.2)$$

where $c^e(m, u)$ and $c^d(m, u)$ denote the cost functions of the encoder and the decoder, respectively, when m is observed at the encoder and the decoder takes

²An equilibrium where the encoder employs a quantization policy is also referred to as a partition equilibrium in the economics literature.

the action u given its observation, and $\mathbb{E}_e[\cdot]$ and $\mathbb{E}_d[\cdot]$ denote that the expectation is taken with respect to the subjective probability measures seen by the encoder and the decoder, respectively.

A signaling game problem can be investigated using the *Nash equilibrium* concept or the *Stackelberg equilibrium* concept depending on the commitment assumption of the encoder to its policies. At a Nash equilibrium [13], the players choose their strategies simultaneously. These chosen strategies are referred to as a Nash equilibrium if no player gains by unilaterally deviating from its strategy. In other words, neither the encoder nor the decoder have any incentive to unilaterally change their strategies when they operate at a Nash equilibrium. A pair of encoding and decoding policies $\gamma^{*,e}$ and $\gamma^{*,d}$ is said to be a Nash equilibrium [13] if

$$J^e(\gamma^{*,e}, \gamma^{*,d}) \leq J^e(\gamma^e, \gamma^{*,d}) \text{ for all } \gamma^e \in \Gamma^e, \quad (1.3)$$

$$J^d(\gamma^{*,e}, \gamma^{*,d}) \leq J^d(\gamma^{*,e}, \gamma^d) \text{ for all } \gamma^d \in \Gamma^d, \quad (1.4)$$

where Γ^e and Γ^d denote all possible encoding and decoding strategies, respectively.

On the other hand, the Stackelberg equilibrium involves a sequential game play in the sense that first the encoder and then the decoder act via announcing their policies [13]. The encoder chooses and announces its strategy and then the decoder acts upon learning the strategy of the encoder. Once the encoder announces its strategy, it cannot change it, i.e., the encoder commits to employ this announced strategy. The decoder employs an optimal response to the announced strategy of the encoder, which is known by the decoder before taking an action. A pair of encoding and decoding policies $\gamma^{*,e}$ and $\gamma^{*,d}$ is said to be a Stackelberg equilibrium [13] if

$$J^e(\gamma^{*,e}, \gamma^{*,d}(\gamma^{*,e})) \leq J^e(\gamma^e, \gamma^{*,d}(\gamma^e)) \text{ for all } \gamma^e \in \Gamma^e, \quad (1.5)$$

where $\gamma^{*,d}(\gamma^e)$ satisfies

$$J^d(\gamma^e, \gamma^{*,d}(\gamma^e)) \leq J^d(\gamma^e, \gamma^d(\gamma^e)) \text{ for all } \gamma^d \in \Gamma^d, \quad (1.6)$$

where for the policy of the decoder we use the notation $\gamma^d(\gamma^e)$ to indicate that the decoder decides on its policy after observing the encoder's policy. This setup

is commonly referred to as the *Bayesian persuasion* problem in the economics literature [14].

As mentioned earlier, in the Nash setup, the encoder and decoder announce their policies at the same time. This means that the encoder does not commit to an encoding policy a priori. This type of interaction is appropriate for instance when the encoder and decoder engage in a one-shot signaling game and the encoder does not care about the long term interactions. More specifically, in a single one-shot game, the encoder does not put emphasis on its long term credibility, and thus, the encoder can backtrack its announced strategy for its own benefit or the encoder does not make an a priori announcement at all. On the other hand, in a Stackelberg game model, the encoder is committed to an encoding policy a priori.

In a signaling game problem, it is of interest to investigate existence of an equilibrium in which the encoder conveys information related to its observation. Depending on how much the objectives of the encoder and the decoder are misaligned, we may have an equilibrium where the encoder reveals its observation completely or partially, or an equilibrium where the encoder does not reveal any information at all. An equilibrium where the encoder reveals some information related to the source is referred to as an *informative equilibrium*. On the other hand, if the encoded random variable does not depend on the source, this type of equilibrium is referred to as a *noninformative (babbling) equilibrium*. In this case, there is no communication between the encoder and the decoder due to the strategic nature of the players.

In Chapter 2, we investigate a quadratic scalar cheap talk problem where the priors are identical and the misalignment in the objectives is due to a bias term appearing in the objective function of the encoder. More specifically, we investigate Crawford and Sobel's formulation [12] under quadratic cost criterion for scalar valued sources that admit a log-concave density. Due to [12] and [15], we know that a Nash equilibrium must involve quantized encoding policies with convex bins (i.e., intervals) considering arbitrary scalar valued sources. This chapter investigates various properties of the Nash equilibria such as the number

of bins in equilibrium, uniqueness of an equilibrium for a given number of bins, the behavior of expected costs in equilibrium with respect to number of bins and convergence to equilibria under best response dynamics.

In Chapter 3, we consider a signaling game problem where the costs of the encoder and decoder are misaligned due to privacy concerns of the encoder with respect to a parameter. More specifically, the encoder observes two types of correlated observations: private and nonprivate parameters. The encoder wishes to convey the nonprivate parameter to the decoder and wants to protect the private parameter whereas the decoder wishes to extract both of the parameters. We model the private and nonprivate parameters as jointly Gaussian random vectors. We analyze the interaction between the encoder and decoder by viewing it as a signaling game and derive Nash and Stackelberg equilibrium solutions. In addition, we analyze the information bottleneck problem under mean squared error cost criterion by viewing it as an instance of our signaling game formulation under the Stackelberg equilibrium concept. In the classical information bottleneck problem, one wishes to represent a parameter efficiently based on a correlated observation by formulating an optimization problem involving the mutual information between the considered parameters.

In Chapter 4, we investigate a signaling game problem where the costs are identical but the priors of an encoder and decoder are subjective. More specifically, the encoder wishes to convey a scalar source variable to the decoder through a Gaussian noise channel where the encoder and decoder holds subjective beliefs regarding the prior distribution of the source. Since the expected costs given their policies are misaligned, the problem becomes a signaling game. We investigate this problem under the Stackelberg equilibrium concept. In particular, we investigate continuity and robustness of the equilibria around the team setup by analyzing the expected cost of the encoder in equilibrium around the point of identical priors. We present more specific result for the case with Gaussian subjective priors regarding nonlinearity property of the equilibrium solutions as well as conditions for informativeness of equilibria under an affine policy restriction.

In Chapter 5, we investigate a multidimensional cheap talk problem where the

priors are identical and the misalignment in the objectives is due to a bias vector appearing in the objective function of the encoder. More specifically, we consider a multidimensional extension of Crawford and Sobel's formulation [12]. Unlike the scalar setup, there may exist a Nash equilibrium with a linear encoder in the multidimensional setup. We characterize necessary and sufficient conditions under which an equilibrium with a linear encoder exists considering independent and identically distributed sources. We also investigate necessary conditions for equilibria in terms of induced decoder actions.

In Chapter 6, we investigate a communication theoretic problem where the encoder is fixed and our aim is to design the decoder. In particular, we consider the problem of identifying an unknown modulation format of a received signal and then extract transmitted symbols. This corresponds to a team theoretic problem since there is a single decision maker. The problem of identifying modulation format of a received signal is termed as modulation classification problem. Modulation classification problems attract significant attention in the literature as it enables information extraction in a noncooperative environment for which the decoder does not have modulation format information.

1.1 A General Literature Review

In their seminal work [12] in the economics literature, Crawford and Sobel introduce a signaling game problem where a better informed encoder communicates a scalar source variable to a decoder. Here, the decoder observes the message directly and the cost function of the encoder does not depend on the transmitted message. The encoder and decoder employ general cost functions satisfying certain assumptions where the encoder's cost function includes a deterministic bias term. They show that under certain technical conditions, at a Nash equilibrium, the encoder must employ quantization policies. This means that the encoder partitions the observation space into intervals and reveals the interval that its observation lies in. This is a striking result since the presence of a bias term leads to information hiding in the form of applying a quantization policy and

the encoder cannot use a linear policy where the source is revealed completely. This holds even though a message transmission is costless. For instance, for the quadratic criteria with a uniform source, there may only exist a noninformative equilibrium when the bias term is larger than a specified threshold where the encoder reveals no information related to the source [12]. Crawford and Sobel's work on strategic communication [12] has inspired many studies that investigate signaling game problems in the economics literature (see [16] for an extensive survey). In contrast to the Nash setup, it is also possible to consider a Stackelberg game setup. An important line of work in this context in the economics literature is the Bayesian persuasion problem introduced by Kamenica and Gentzkow where signaling scenarios are investigated under the Stackelberg equilibrium concept [14]. The reader is referred to [17] for an extensive literature review on the Bayesian persuasion problem. In the following, we first provide a literature review for more general signaling game problems in the engineering literature and then we discuss more closely related works in each individual chapter.

Signaling game problems find applications in various engineering contexts including communication and control theory literature [15, 18–34]. For instance, the work in [18] considers a problem under the Stackelberg equilibrium concept for transmitting a scalar Gaussian source from an encoder to a decoder. They consider a quadratic cost structure where the encoder's cost includes a bias term which is modeled as jointly Gaussian with the source variable. For this scalar Gaussian source setting, they establish optimality property of characterized linear policies for a noiseless channel setting and a Gaussian noise channel setting. The work in [15] investigates communication scenarios for transmitting a scalar or a multidimensional Gaussian source through a Gaussian noise channel under a quadratic cost structure using either Nash or Stackelberg equilibria concepts where the encoder's objective contains a deterministic bias term. An interesting observation from [15] is the existence of a linear Nash equilibrium in the case of a Gaussian noise channel, which is in contrast to the quantized nature of the equilibrium in Crawford and Sobel considering a cheap talk setup. In addition, [15] extends the result of Crawford and Sobel [12] which considers sources with a

bounded support by showing the quantized nature of Nash equilibria for arbitrary random variables in the special case of a quadratic cost criterion. The work in [20] investigate Nash and Stackelberg equilibria in a dynamic setting under a quadratic cost structure with a deterministic bias at the encoder. In [23], the authors consider a Stackelberg game setup under general quadratic cost criteria with multidimensional Gaussian sources in a finite horizon dynamic environment and investigate linearity property of the equilibrium solutions.

There exists also signaling game setups in the literature that consider more general cost structures and sources [25–31], rather than investigating settings with a Gaussian source under quadratic cost criteria. A main theme in these studies is to investigate information theoretic limits of communication in the presence of strategic decision makers. In particular, the works in [25–27] analyze joint source-channel coding problem in a Bayesian persuasion setting where communication scenarios between an encoder and a decoder with misaligned utility functions is considered. Here, the problem corresponds to a Stackelberg setup where the encoder is committed a priori. In this context, a more recent work in [28] investigates a Bayesian persuasion setting where an encoder communicates with two decoders instead of one. On the other hand, [29, 30] consider the case when the decoder has a commitment assumption. Here, the authors introduce and investigate the notion of information extraction capacity which quantifies how much information that can be extracted in such strategic setting when the decoder is committed to a policy a priori. An interesting observation from these works is that the decoder may choose to commit to a policy that deliberately makes erroneous decisions for some of its observations so that there is an incentive for the encoder not to lie all the time. This type of commitment assumption by the decoder has applications on designing optimal questionnaires by which information is collected from strategic agents [31].

1.2 Contributions

1.2.1 Chapter 2

In this chapter, we investigate various properties of the Nash equilibrium for the quadratic cheap talk problem with log-concave sources. In particular, we prove that for any $N \in \mathbb{N}$, there exists a unique equilibrium with N bins if the source distribution is log-concave and supported on the real line. A similar result also holds for log-concave sources with semi-unbounded support except that there exists an upper bound on the number of bins in equilibrium depending on the sign of the bias term. In addition, we prove that the equilibrium costs of the encoder and the decoder improve as the number of bins increases, i.e., a more informative equilibrium is obtained. Finally, we investigate convergence behavior under best response dynamics where the encoder and decoder iteratively compute their best responses. In particular, we show that for sources with a strictly log-concave density and two-sided unbounded support, best response iterations starting from an initial set of bin edges with N bins always converge to the unique equilibrium with N bins. The results of Chapter 2 are under review [35].

1.2.2 Chapter 3

In this chapter, we consider a quadratic signaling game problem with Gaussian sources where privacy concerns of the encoder with respect to a private parameter lead to misaligned costs for the encoder and decoder. The encoder wants to hide the private parameter while at the same time wishes to convey a correlated nonprivate parameter. In contrast, the decoder wishes to estimate both of the parameters. For this signaling game problem, we show that a payoff dominant Nash equilibrium is attained by a set of characterized linear policies. This means that there does not exist any other equilibrium at which the expected costs of both players are simultaneously better than the expected costs at the payoff dominant equilibria. We also show that the characterized linear payoff dominant Nash

equilibria coincide with the Stackelberg equilibria. Note that these (Stackelberg) equilibria are obtained without an *a priori* affine policy restriction for the players. Moreover, we introduce a minimum mean squared error Gaussian information bottleneck problem as an instance of our privacy game formulation under the Stackelberg equilibrium setup. We show that a set of explicitly characterized linear policies gives the information bottleneck solution. Finally, we extend our results on privacy-signaling game formulation for scalar sources to the additive Gaussian noise channel setting as well as a discrete noiseless channel setting. The results of Chapter 3 was presented in part at [36] and are under review [37].

1.2.3 Chapter 4

In this chapter, we analyze a quadratic signaling game problem where the encoder and decoder communicate through a Gaussian noise channel. Here, the costs are identical but the encoder and decoder hold subjective probabilistic beliefs regarding the source distribution. Since the induced expected cost as a map of their policies are misaligned, the problem needs to be studied within a game theoretic framework. We show that the Stackelberg equilibrium cost of the encoder is upper semi continuous, under the Wasserstein metric, as the encoder's prior approaches the decoder's prior, and it is also lower semi continuous with Gaussian priors. This means that the Stackelberg equilibria under Gaussian subjective priors are robust with respect to perturbations around the team setup. We also show that the optimality of affine policies for Gaussian signaling no longer holds under prior mismatch, and thus, team-theoretic optimality of linear/affine policies is not robust to perturbations. In addition, we derive conditions under which there exist informative Stackelberg equilibria under an affine policy restriction. The results of Chapter 4 are under review [38].

1.2.4 Chapter 5

In this chapter, we investigate a signaling game problem where an encoder observes a multidimensional source and conveys a message to a decoder, and the quadratic objectives of the encoder and decoder are misaligned due to a bias vector. We first provide a set of geometry conditions that a Nash equilibrium has to satisfy considering any multidimensional source. Then, we consider multidimensional sources with independent and identically distributed (i.i.d.) components and completely characterize conditions under which a Nash equilibrium with a linear encoder exists. In particular, we show that if the components of the bias vector are not equal in magnitude, then there exists an informative linear equilibrium if and only if the source distribution is Gaussian. On the other hand, for an informative linear equilibrium to exist in the case of equal bias components, it is required that the source density is symmetric around its mean. Moreover, in the case of Gaussian sources, our results have a rate-distortion theoretic implication that achievable rate and decoder distortion pairs asymptotically approach their team theoretic counterpart as we increase the number of observed source components at the encoder. The results of Chapter 5 will be presented in part at [39].

1.2.5 Chapter 6

In this chapter, modulation classification and symbol decoding problems are jointly considered and optimal strategies are proposed under various settings. In the considered framework, a decoder aims at decoding a sequence of received signals whose modulation format is unknown. Hence, the decoder faces with the tasks of classifying the modulation format of the received signal among a number of candidates and decoding the symbols. To that aim, two different formulations are proposed. In the first formulation, the prior probabilities of the modulation schemes are assumed to be known and a formulation is proposed under the Bayesian framework. This formulation takes a constrained approach in which the objective function is related to symbol decoding performance whereas the

constraint is related to modulation classification performance. The second formulation on the other hand addresses the case in which the prior probabilities of the modulation schemes are unknown, and provides a method under the minimax framework. In this case, a constrained approach is employed as well; however, the introduced performance metrics differ from those in the first formulation due to the absence of the prior probabilities of the modulation schemes. The results of Chapter 6 have appeared in [40].



Chapter 2

Quadratic Signaling Games with Log-Concave Sources

2.1 Motivation

In their seminal paper [12], Crawford and Sobel introduce a signaling game problem. Here, a better informed encoder wants to convey a scalar source to a decoder where they have misaligned objectives due to a bias term at the encoder. This formulation has attracted significant attention in the economics literature as it can be used to model many strategic communication scenarios involving individuals (see [16] for an extensive survey). Recently, since strategic decision makers also arise in engineering applications, there have been a growing interest in analyzing signaling game problems in the engineering literature [15, 18–34]. As an application of Crawford and Sobel’s formulation, [41] analyzes a smart grid problem by modeling the interaction between a consumer and an electricity aggregator as a signaling game. The consumer wishes to get the amount of power he/she needs by making a request to the aggregator, and the aggregator wants to meet this power demand to some extent while at the same time takes other power distribution costs into account. This leads to a signaling game as the consumer and aggregator has misaligned cost functions.

In this chapter, our aim is to investigate Crawford and Sobel's signaling game formulation under quadratic cost criteria. We particularly focus on source distributions that admit a log-concave density, which is termed as log-concave sources. This class of distributions arises in many engineering applications since distributions such as Gaussian, exponential, Laplace and Rayleigh are log-concave. For log-concave sources, we extend and generalize some of the results concerning the cheap talk problem analyzed in [12] in three main directions as we summarize below.

2.1.1 Number of Bins

In [15, Theorem 3.2], it is shown that for the quadratic cheap talk problem, a Nash equilibrium must involve quantized encoding policies with convex bins (i.e., intervals) for arbitrary scalar sources. This is a generalization of the result in [12] that establishes the quantized nature of the Nash equilibria for sources that admit a density supported on $[0, 1]$. Hence, if we have a source with a bounded support, it directly follows that the number of bins in equilibrium is upper bounded by a finite quantity. For instance, [12] analyzes the source with a uniform distribution on $[0, 1]$ and quadratic cost structure as an example in more detail and derives an upper bound on the number of bins in equilibrium as a function of the bias. Then, it is of interest to investigate the number of bins in equilibrium for more general sources. In particular, a problem is to see whether there exists an upper bound on the number of bins in equilibrium for sources with either one-sided or two-sided unbounded support on the real line. It is also of interest to see if the upper bound on the number of bins is equal to one, which means that there can only exist a noninformative equilibrium. Another interesting problem is to find out whether an equilibrium with N bins is unique or not for a given $N \geq 1$.

2.1.2 Equilibrium Costs

If there exist multiple equilibria for the considered game, then a new question arises: among these multiple equilibria, which ones induce smaller expected costs simultaneously for both of the players? Results in [12] show that under certain assumptions an equilibrium with more bins induces smaller expected costs for both the encoder and the decoder for a source with a density supported on $[0, 1]$. Accordingly, for more general sources, the problem is to see how the expected costs of the players in equilibrium behave with respect to number of bins. In the case that both expected costs are monotonic with respect to number of bins in equilibrium, this result corresponds to the payoff dominant equilibrium selection property commonly employed in the game theory literature [42]. This also allows for a well-posed coding problem as one would like to design a coding scheme that is payoff dominant for both the encoder and the decoder.

2.1.3 Convergence to Equilibria

The interaction between the encoder and decoder can be viewed as a quantization game where the encoder decides on quantization bins and the decoder decides on reconstruction values. One might wonder how an equilibrium point is reached in this quantization game. For instance, the players may act sequentially by computing their best responses, which corresponds to a repeated play of the considered quantization game. Then, a problem of interest is to see whether these best response iterations converge to an equilibrium. Another related issue is that if they start from an arbitrary quantization policy with N bins, is it the case that these best response iterations converge to an equilibrium with N bins?

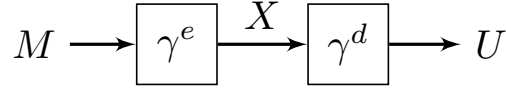


Figure 2.1: Communication setting for the cheap talk setup.

2.2 Preliminaries

Suppose that there is an encoder and a decoder. The encoder observes the value of an $\mathbb{M}$ -valued random variable M . Given its observation, the encoder conveys a message X , which is an $\mathbb{X}$ -valued random variable, to the decoder. The decoder directly observes the transmitted message X and takes an $\mathbb{M}$ -valued action U upon receiving X . Figure 2.1 illustrates the considered communication scenario. In this communication scenario, the encoder and decoder are assumed to employ deterministic policies, i.e., $x = \gamma^e(m)$ and $u = \gamma^d(x) = \gamma^d(\gamma^e(m))$ where $\gamma^e(\cdot)$ and $\gamma^d(\cdot)$ denote the encoding and decoding policies, respectively. Let $c^e(m, u)$ and $c^d(m, u)$ denote the cost functions of the encoder and the decoder, respectively, when the decoder takes the action u for the corresponding source observation m at the encoder. Then, a given pair of encoding and decoding policies induces expected costs of $J^e(\gamma^e, \gamma^d) = \mathbb{E}[c^e(M, U)]$ for the encoder and $J^d(\gamma^e, \gamma^d) = \mathbb{E}[c^d(M, U)]$ for the decoder. Let $\mathbb{M} = \mathbb{X} = \mathbb{R}$, $c^e(m, u) = (m - u - b)^2$ and $c^d(m, u) = (m - u)^2$, where $b \in \mathbb{R}$ denotes a deterministic bias term. In other words, we consider scalar real valued random variables and quadratic cost functions where the encoder's cost includes a bias term. We assume that all aspects of the game except for the realization of the source M is common knowledge as in [12]. We analyze the Nash equilibria for the described communication setup where the encoder and decoder announce their policies at the same time.¹ A pair of policies $\gamma^{*,e}$ and $\gamma^{*,d}$ is said to be a Nash equilibrium [13] if

$$\begin{aligned} J^e(\gamma^{*,e}, \gamma^{*,d}) &\leq J^e(\gamma^e, \gamma^{*,d}) \text{ for all } \gamma^e \in \Gamma^e, \\ J^d(\gamma^{*,e}, \gamma^{*,d}) &\leq J^d(\gamma^{*,e}, \gamma^d) \text{ for all } \gamma^d \in \Gamma^d, \end{aligned} \tag{2.1}$$

¹We note that if one considers the Stackelberg equilibrium concept, then the equilibrium becomes fully informative, as indicated in [15, Theorem 3.3].

where Γ^e and Γ^d are the sets of all deterministic (and Borel measurable) functions from $\mathbb{M}$ to $\mathbb{X}$ and from $\mathbb{X}$ to $\mathbb{M}$, respectively. The definition in (2.1) states that each player employs an optimal policy given the policy of the other player.

The quantized nature of Nash equilibria for the cheap talk problem [12] motivates the following definition of a scalar quantizer.

Definition 2.2.1. An N -cell scalar quantizer, q , is a (Borel) measurable mapping from $\mathbb{M} = \mathbb{R}$ to the set $\{1, 2, \dots, N\}$ characterized by a measurable partition $\{B_1, B_2, \dots, B_N\}$ such that $B_i = \{x \mid q(x) = i\}$ for $i = 1, 2, \dots, N$ and that bin probabilities are strictly positive. The B_i are called the bins (or cells) of q .

The results in [12] and [15] show that the encoder must employ a quantization policy at a Nash equilibrium where the quantization bins are convex. This means that an encoding policy at a Nash equilibrium must be a policy as in Definition 2.2.1 with the cells B_i being intervals

Now consider an equilibrium with N bins. At this equilibrium, let k th bin be the interval $[m_{k-1}, m_k)$ for $k = 1, \dots, N$ where $m_0 < m_1 < \dots < m_N$ denote bin edges. Note that for an equilibrium with N bins, we have $m_0 = m_L$ and $m_N = m_U$ for sources with bounded support with m_L and m_U denoting lower or upper boundaries, respectively. In the case of sources with semi-unbounded support, we have $m_0 = m_L$ and $m_N = +\infty$, or $m_0 = -\infty$ and $m_N = m_U$ whereas in the case of sources with two-sided unbounded support, we have $m_0 = -\infty$ and $m_N = +\infty$. Here, the encoder reports the interval its observation lies in. This can be represented by a quantization policy with $\gamma^e(m) = k$ when $[m_{k-1}, m_k)$ for $k = 1, \dots, N$. In this representation, the decoder knows that receiving k means the encoder observed $m \in [m_{k-1}, m_k)$. At an equilibrium, the best response of the decoder to a given encoding policy is obtained by the centroid conditions as follows:

$$u_k = \mathbb{E}[M \mid m_{k-1} \leq M < m_k] \quad (2.2)$$

for $k = 1, \dots, N$. Namely, the decoder computes the centroids of each bin to obtain its best response. The best response of the encoder to a given decoding

policy is obtained by the nearest neighbor condition² as follows:

$$u_{k+1} - m_k = (m_k - u_k) - 2b \Leftrightarrow m_k = \frac{u_k + u_{k+1}}{2} + b \quad (2.3)$$

for $k = 1, \dots, N - 1$. In order for a set of bin edges $m_0, \dots, m_N$ and centroids $u_1, \dots, u_N$ to yield a Nash equilibrium, these bin edges and centroids must satisfy (2.2) and (2.3) simultaneously. Under this setting, the encoder determines the positions of bin edges and the decoder determines the centroids. Alternatively, the problem can be considered as a quantization game in which the boundaries are determined by the encoder and the reconstruction values are determined by the decoder.

Based on the description of a Nash equilibrium above, we formally describe the problems considered in this chapter as follows:

2.2.1 Number of Bins

For a given finite N , we wish to see whether there exists an equilibrium with N bins. Namely, it is desired to show the existence of bin edges $m_0, m_1, \dots, m_N$ and centroids $u_1, u_2, \dots, u_N$ which satisfy (2.2) and (2.3) simultaneously. Another problem is to see whether an equilibrium with N bins is unique for a given N , i.e., whether there are unique values for bin edges $m_0, m_1, \dots, m_N$ and centroids $u_1, u_2, \dots, u_N$ which satisfy (2.2) and (2.3) simultaneously.

2.2.2 Equilibrium Costs

An equilibrium is more informative (than another one) if it results in smaller expected costs for both the encoder and the decoder. Let $J^{e,N}$ and $J^{d,N}$ denote the encoder cost and the decoder cost, respectively, at an equilibrium with N bins. Then, the aim is to see if $J^{d,N} > J^{d,N+1}$ holds or not for all finite N .³

²Although this condition can be viewed as a nearest neighbor condition with a bias term, we simply use the term nearest neighbor condition as in the quantization theory literature.

³It suffices to consider the expected cost of one of the players since Remark 2.6.7 shows that for a given equilibrium with N bins, we have that $J^{e,N} = J^{d,N} + b^2$.

2.2.3 Convergence to Equilibria

Consider an initial arbitrary set of ordered bin edges $m_0, m_1, \dots, m_N$ that does not necessarily lead to an equilibrium where m_0 and m_N correspond to boundaries of the support for the source distribution. Suppose that the encoder and decoder iteratively compute their best responses by taking the centroid conditions and the nearest neighbor conditions (i.e., (2.2) and (2.3)) into account by which bin edges $m_1, m_2, \dots, m_{N-1}$ and centroids $u_1, u_2, \dots, u_N$ are updated iteratively. Then, the aim is to see whether these bin edges and centroids converge to an equilibrium with N bins as a result of best response iterations. This corresponds to modified Lloyd's Method I [43] where the modification specific to our setting is due to the bias term in (2.3).

2.3 Literature Review

In addition to seminal work in [12], there have been many contributions to general cheap talk problems from a variety of aspects in the economics literature [44–54]. For instance, the work in [49] considers a cheap talk setup where with some probability the decoder does not observe the message transmitted by the encoder and instead observes a message coming from a distribution which is statistically independent of the encoder's transmission and the source. This setup reduces to Crawford and Sobel's formulation [12] when the probability of error is zero. They show the quantized nature of the equilibria with convex bins (i.e., intervals) in such a noisy communication setting. The work in [53] considers a cheap talk setup with two senders communicating with a receiver and an interesting observation is that more information revelation by one sender incentivizes more information revelation by the other sender. The reader is referred to [16] for a detailed survey of signaling game problems in the economics literature.

A related work to our results in this chapter is presented in [22]. The authors investigate properties such as the number of bins and expected costs in equilibrium for Gaussian and exponential sources. They prove that there always exist

equilibria with infinitely many bins in the case of Gaussian sources. On the other hand, for the exponential source case, there exists an upper bound on the number of bins depending on the sign of the bias. These distributions are in fact special cases of the distributions considered in this chapter. In that respect, our results on the number of bins and equilibrium costs can be viewed as a generalization of these results for more general distributions.

Since for the scalar cheap talk setup, an equilibrium must involve a quantization policy at the encoder, the results of this chapter are related to the quantization theory. We refer the reader to [55] for an extensive survey regarding a variety of results in the quantization theory literature. A problem of interest in this context is to obtain an optimal quantization policy with respect to a cost criterion. Lloyd in his seminal work [43] introduces two heuristic methods called Method I and Method II to obtain quantization policies where the former is also referred to as Lloyd-Max algorithm. These are iterative methods in which quantization bins and reconstruction values are updated iteratively until convergence. In [56–59], the authors investigate such problems as whether there exists a unique locally optimal quantizer and whether Lloyd’s Method I converges to a globally or locally optimal quantizer. In particular, [58] proves that there exists a unique locally optimal quantizer, which is then globally optimal, and that Lloyd’s Method I converges to this globally optimal quantizer where [58] makes the assumption that the error weighting function is convex, strictly increasing and continuously differentiable and that the source admits a log-concave density. Another related work in [60] considers entropy-constrained quantizer design problem and provides conditions under which there exist finite or infinite number of bins for an optimal quantizer. In addition, in [41], an application of Crawford and Sobel’s formulation to a smart grid setting is presented. The authors model the interaction between a consumer and an electricity aggregator as a signaling game problem and Lloyd’s Method II is employed to obtain signaling schemes between them. The work in [61] studies evolution of language in social networks by modeling the problem as a quantization game where information spreads over a network of strategic decision makers. The authors show convergence to equilibria via a distributed version of the Lloyd-Max algorithm for the considered setup.

There may exist multiple equilibria in game theoretic problems and this is also the case in the cheap talk problem investigated in this chapter. Then, it is of interest to compare performance of multiple equilibria and employ a technique to select an equilibrium point among multiple equilibria [42, 62–64]. For instance, if we have a payoff dominant Nash equilibrium, it means that there does not exist any other equilibrium where both decision makers simultaneously have better expected costs than the expected costs at the payoff dominant Nash equilibrium [42]. Our results have implications regarding the payoff dominant equilibrium selection criterion for the considered signaling game problem by showing monotonic behavior of the expected costs of the players with respect to number of bins under an assumption on the source distribution. As an alternative to the payoff dominant equilibrium selection criterion, the authors in [63] introduce the notion of no incentive to separate (NITS) condition for selection among multiple equilibria for general cheap talk problems considering a source with a density supported on $[0, 1]$ as in [12] and several applications are discussed. This equilibrium selection criterion has also relation with a monotonicity condition, which they call (M), assumed in [12]. Namely, [63] shows that if this condition is satisfied, then only the unique equilibrium with maximum number of decoder actions satisfies NITS. In fact, our results reveal that for the setup considered in this chapter, if the source density is supported on a bounded interval, then the condition (M) is satisfied.

2.4 Contributions

The main aim of this chapter is to investigate various properties of the Nash equilibria for the quadratic cheap talk problem where we consider sources with a log-concave density. The main contributions of this chapter can be summarized as follows:

- (i) For log-concave sources with two-sided unbounded support, we prove that for any $N \in \mathbb{N}$, there exists a unique equilibrium with N bins regardless of the value of b , i.e., there is no upper bound on the number of bins in equilibrium (Theorem 2.6.1).

- (ii) For log-concave sources with semi-unbounded support, we show that when the support extends to $+\infty$ and $b < 0$, or the support extends to $-\infty$ and $b > 0$, there exists an upper bound on the number of quantization bins in equilibrium and that for any N less than or equal to this upper bound there exists a unique equilibrium with N bins (Theorem 2.6.2). Otherwise, for any $N \in \mathbb{N}$, there exists a unique equilibrium with N bins.
- (iii) We prove that an equilibrium with more bins is more informative for log-concave sources, i.e., the equilibrium costs of the encoder and the decoder reduce as the number of bins increases (Theorem 2.6.5).
- (iv) We show that for sources with a strictly log-concave density and two-sided unbounded support, if the encoder and the decoder iteratively compute their best responses starting from an initial set of bin edges with N bins, then it is guaranteed that these bin edges converge to the unique equilibrium with N bins regardless of the value of b (Theorem 2.6.6).

2.5 Some Relevant Prior Results

In this section, we review and discuss closely related prior results in the literature. In this chapter, we touch upon all of these listed results and extend them. To begin with, in their seminal paper [12], Crawford and Sobel prove the quantized nature of Nash equilibria under certain technical assumptions for sources with a density supported on $[0, 1]$. It should be emphasized that the result of Crawford and Sobel holds for more general cost functions which satisfy certain conditions. These conditions hold for the quadratic criteria analyzed in this chapter. In the next theorem, $u^e(m)$ and $u^d(m)$ are defined as $u^e(m) \triangleq \arg \min_u c^e(m, u)$ and $u^d(m) \triangleq \arg \min_u c^d(m, u)$ where $c^e(m, u)$ and $c^d(m, u)$ denote the cost function of the encoder and decoder, respectively, and the former term depends also on the bias term b . As noted earlier, here we only consider the Nash (simultaneous) setup, and not the Stackelberg (Bayesian persuasion) setup.

Theorem 2.5.1 (Crawford and Sobel [12, Theorem 1]). *Let M be a real-valued*

random variable which admits a density supported on $[0, 1]$. Suppose that $u^e(m) \neq u^d(m)$ for all m . Then, there exists at least one equilibrium with a quantization policy at the encoder where there are N quantization bins with $1 \leq N \leq N_{\max}(b)$ and $N_{\max}(b) \in \mathbb{N}$ denoting the upper bound on the number of bins. Furthermore, an encoding policy at an equilibrium is equivalent to a quantized encoder policy in terms of the performance at the equilibrium.

It is also possible to consider arbitrary scalar valued random variables and to show the quantized nature of Nash equilibria in this case, as well. Namely, the following result applies to arbitrary scalar valued random variables, not necessarily sources on $[0, 1]$ that admit densities. Here, the costs functions are assumed to be quadratic, which is also the case in this chapter.

Theorem 2.5.2 (Saritaş et al. [15, Theorem 3.2 and Theorem 3.4]). *Let M be a real-valued random variable with an arbitrary probability measure. Let the strategy set of the encoder consists of the set of all measurable (deterministic) functions from $\mathbb{M}$ to $\mathbb{X}$. Then, an equilibrium encoder policy has to be quantized almost surely, that is, it is equivalent to a quantized policy for the encoder in the sense that the performance of any equilibrium encoder policy is equivalent to the performance of a quantized encoder policy. Furthermore, the quantization bins are convex. In addition, the quantized nature of equilibria does not necessarily hold when the source is vector-valued.*

A set of related results involves multi-stage (dynamic) cheap talk game setup considered in [20]. In particular, [20, Theorem 3] establishes the quantized nature of the last stage in a repeated cheap talk game and [20, Theorem 5] makes further assumptions on the single-stage setup to prove that encoding policies at all stages must be quantization policies with a finite number of bins at each stage. In this latter result, it is assumed that there exists an upper bound on the number of bins at the equilibria in addition to an upper bound on the number of equilibria for a given number of bins. It is noted that this chapter draws conclusions regarding these assumptions for log-concave sources.

Moreover, our results in this paper are closely related to some foundational

results by Kieffer [58] where optimal quantization in the classical sense (i.e., in a team theoretic setup) is considered. This problem can be viewed as a communication problem where an encoder decides on quantization bins and a decoder determines reconstruction values where the encoder and the decoder wishes to minimize a common cost function. In this setting, [58] shows that if the source is log-concave there exists a unique locally optimal quantization policy for a given number of bins under a set of assumptions on the common cost function. The uniqueness result makes the corresponding quantizer globally optimal for a given number of bins.

Theorem 2.5.3 (Kieffer [58]). *If the source is log-concave and the common cost function is convex, strictly increasing and continuously differentiable, then there exists a unique locally optimal quantizer and Lloyd-Max iterations (i.e., Lloyd's Method I) converge to this quantizer.*

Note that the special case of quadratic cost criteria with $b \neq 0$ is analyzed in this chapter. For the case of $b = 0$ leading to a team theoretic setup, Theorem 2.5.3 quoted above proves the uniqueness of the optimal quantizer for a given number of bins and in addition to convergence to this optimal quantizer via Lloyd-Max iterations. In that respect, this chapter generalizes these uniqueness and convergence results to the game theoretic setup where $b \neq 0$. We will use Kieffer's analysis in some crucial steps while proving convergence to equilibria.

2.6 Nash Equilibria

In this section, we present our results on equilibrium behavior for the scalar quadratic cheap talk problem with sources that admit a log-concave density. Before presenting our results, we make the following definitions.

Definition 2.6.1. A probability density function f on $\mathbb{R}$ is said to be (strictly) log-concave if $\log f$ is a (strictly) concave function on the support of f where the support is an interval.

We note that distributions such as Gaussian, exponential, Laplace, Rayleigh and uniform, which are commonly encountered in information and communication theoretic applications, are log-concave (see [22] or [35] for more specific results for the cases where the source is exponential and Gaussian). The reader is referred to [65] to see a list of common distributions which are log-concave or log-convex.

Definition 2.6.2. For a probability density function f , we say that:

- (i) f has a two-sided unbounded support if its support is $\mathbb{R}$.
- (ii) f has a semi-unbounded support if its support is either an interval in the form $(-\infty, m_U]$ or an interval in the form $[m_L, \infty)$ for some $m_U, m_L \in \mathbb{R}$.
- (iii) f has a bounded support if $f(x) = 0$ for all $x \notin [m_L, m_U]$ for some $-\infty < m_L < m_U < \infty$.

2.6.1 Existence and Uniqueness of Equilibria

In the following theorem, we present existence and uniqueness results for sources with a log-concave density and two-sided unbounded support. Note that an equilibrium $N = 1$ bins is equivalent to the case where the encoder and decoder do not communicate, as remarked below.

Remark 2.6.1. In the case of a single bin, since the encoder does not partition its observation space, the received message at the sender is useless and the decoder simply uses the prior distribution of the source while taking an action. This implies that an equilibrium with $N = 1$ bins is a noninformative (babbling) equilibrium [12].

Theorem 2.6.1. *Consider a source that has a log-concave density with two-sided unbounded support. Then, for any $N \geq 1$, there exists a unique equilibrium with N bins.*

Proof. See Section 2.8.1. □

Remark 2.6.2. In the proof of Theorem 2.6.1, we employ the property that the mean of a source with a log-concave density is finite. This finite mean property follows from the fact that the tails of a log-concave density are at most exponential [66, Lemma 1]. In the following, we restate this result for the special case of densities defined on $\mathbb{R}$.

Lemma 2.6.1 ([66, Lemma 1]). *For a log-concave density f defined on $\mathbb{R}$, there exists $a > 0$ and $b \in \mathbb{R}$ such that $f(x) \leq \exp(-a|x| + b)$ for all $x \in \mathbb{R}$.*

Remark 2.6.3. It is noted that in the proof of Theorem 2.6.1 the source distribution is assumed to be strictly log-concave. In fact, the result holds for distributions which are only log-concave. In that case, for the result to hold, it is required that the density is not log affine everywhere. This is indeed true for a log-concave distribution with two-sided unbounded support.

Theorem 2.6.1 reveals that there does not exist an upper bound on the number of bins for an equilibrium considering log-concave sources with two-sided unbounded support. Namely, there exist countably infinite number of distinct equilibria for log-concave sources with two-sided unbounded support.

The uniqueness result of Theorem 2.6.1 is related to the classical results for optimal quantization in [57] and [58] where the aim is to find quantization bins and reconstruction values that minimize a common cost function (i.e., team theoretic setup). As stated in Theorem 2.5.3 of this chapter, [58] particularly focuses on optimal quantization of log-concave sources. In that respect, Theorem 2.6.1 generalizes the uniqueness result of [58] to the signaling games setup under misaligned quadratic cost structure with a bias term appearing in the cost function of the encoder.

The existence and uniqueness results of Theorem 2.6.1 can be generalized to sources with semi-unbounded support as well as sources with bounded support, and the following is a generalization to sources with semi-unbounded support. For sources with semi-unbounded support, there may be an upper bound on the number of bins depending on the sign of b . Throughout this chapter, we use the following notation for the smallest upper bound (whenever it exists) on the

number of bins in equilibrium for a given problem setup.

Definition 2.6.3. For a given source density and a certain bias b , the quantity $N_{\max}(b)$ denotes the maximum possible number of bins in equilibrium.

Theorem 2.6.2. (i) Consider a source that has a log-concave density with a support on $[m_L, \infty)$ for some $m_L \in \mathbb{R}$. Then, for $b > 0$ and for any $N \geq 1$, there exists a unique equilibrium with N bins.

(ii) Consider a source that has a log-concave density with a support on $[m_L, \infty)$ for some $m_L \in \mathbb{R}$. Then, for a given $b < 0$, there exists a finite $N_{\max}(b)$ such that for any N satisfying $1 \leq N \leq N_{\max}(b)$, there exists a unique equilibrium with N bins, and for any $N_{\max}(b) < N$, there does not exist any equilibrium with N bins.

Proof. See Section 2.8.2. □

Remark 2.6.4. The result of Theorem 2.6.2 can also be extended to the case of support on $(-\infty, m_U]$ for $m_U \in \mathbb{R}$. In this case, if $b < 0$, then for any $N \geq 1$, there exists a unique equilibrium with N bins. If $b > 0$, there exists a finite upper bound on the number of bins in equilibrium.

Remark 2.6.5. In the case of semi-unbounded support, the result of Theorem 2.6.2 holds even for densities which are log affine everywhere on its support.

The following theorem specifies a necessary and sufficient condition for the existence of an informative equilibrium, which is proven in a similar manner to Theorem 2.6.2.

Theorem 2.6.3. For a source that has a log-concave density with a support on $[m_L, \infty)$ for some $m_L \in \mathbb{R}$ and has a mean $\mathbb{E}[M] = \mu$, the only equilibrium is noninformative if and only if $2b \leq -(\mu - m_L)$.

Proof. See Section 2.8.3. □

Remark 2.6.6. The result of Theorem 2.6.3 can also be extended to the case of support on $(-\infty, m_U]$ for $m_U \in \mathbb{R}$. In this case, the only equilibrium is noninformative if and only if $2b \geq (m_U - \mu)$.

We note that for sources with a density supported on $[0, 1]$ Crawford and Sobel prove that for a given b there exists an upper bound on the number of bins expressed as $N_{\max}(b)$ and that for any $1 \leq N \leq N_{\max}(b)$ there exists an equilibrium with N bins in [12, Theorem 1] (see Theorem 2.5.1 of this chapter). This existence result is valid even for a source without a log-concave density. In addition, an additional monotonicity assumption (M) with regard to the behavior of bin edges is made in [12], which ensures the uniqueness of an equilibrium for a given number of bins [12, Lemma 3]. In fact, when sources supported on a bounded interval are considered in Lemma 2.8.2 of this chapter, it can be shown that the assumption (M) is satisfied for sources with a log-concave density. Therefore, the following uniqueness result follows from Lemma 2.8.2 of this chapter and Lemma 3 of [12].

Theorem 2.6.4. *For sources with a log-concave density and a support on $[m_L, m_U]$ with $m_L < m_U$ and $m_L, m_U \in \mathbb{R}$, an equilibrium with N bins is unique for $1 \leq N \leq N_{\max}(b)$.*

2.6.2 Equilibrium Costs

So far, the number of bins at the equilibria has been investigated for sources with two-sided unbounded support, semi-unbounded support and bounded support. At this point, it is interesting to examine the behavior of the encoder's and decoder's expected costs in equilibrium with respect to number of bins. In the following theorem, we show that an equilibrium with N bins payoff dominates an equilibrium with K bins if $K < N$ [42]. In other words, an equilibrium with N bins leads to smaller expected costs for both of the players than an equilibrium with K bins if $K < N$. Before presenting this result, we first make the following remark regarding the relation between the expected costs of the encoder and decoder in equilibrium.

Remark 2.6.7. Note that at a Nash equilibrium, the relation between the encoder cost and the decoder cost can be expressed as follows:

$$\begin{aligned}
J^e(\gamma^{*,e}, \gamma^{*,d}) &= \sum_{i=1}^N \Pr(m_{i-1} \leq M < m_i) \\
&\quad \times \mathbb{E} \left[(M - \mathbb{E}[M | m_{i-1} \leq M < m_i] - b)^2 | m_{i-1} \leq M < m_i \right] \\
&= \sum_{i=1}^N \Pr(m_{i-1} \leq M < m_i) \\
&\quad \times \left(b^2 + \mathbb{E} \left[(M - \mathbb{E}[M | m_{i-1} \leq M < m_i])^2 | m_{i-1} \leq M < m_i \right] \right) \\
&= J^d(\gamma^{*,e}, \gamma^{*,d}) + b^2.
\end{aligned}$$

Thus, the difference between the expected cost of the encoder and the decoder in equilibrium is always b^2 regardless of the number of bins under the quadratic cost assumption. This implies that if an equilibrium with more bins induces smaller expected cost for one of the players, then it also induces smaller expected cost for the other player.

Theorem 2.6.5. *For sources with a log-concave density, an equilibrium with more bins induces smaller expected costs for both of the players. In other words, denoting expected costs of the encoder and the decoder by $J^{e,N}$ and $J^{d,N}$ at the unique equilibrium with N bins, we have $J^{e,N} < J^{e,K}$ and $J^{d,N} < J^{d,K}$ if $K < N$.*

Theorem 2.6.5 shows that an equilibrium with more bins leads to better expected costs in equilibrium for both of the players. When $b = 0$ (i.e., team theoretic setup), the problem reduces to the classical optimal quantization problem under quadratic cost structure and the result of Theorem 2.6.5 is in fact valid also when $b = 0$. In other words, the classical optimal quantization setup with the quadratic cost criterion is a special case of Theorem 2.6.5.

We note that for sources with a support on $[0, 1]$ Crawford and Sobel establish that an equilibrium with more bins induces smaller expected costs in [12, Theorem 3] and [12, Theorem 5] from the perspective of encoder and decoder, respectively, under a monotonicity assumption, i.e., the assumption (M), considering a general cost structure. In [12, Theorem 2], a sufficient condition for (M) to hold is

presented. Nevertheless, for the quadratic cost criteria considered in this chapter, if the density is assumed to be differentiable, then this sufficient condition only holds for sources with a uniform distribution. On the other hand, Theorem 2.6.5 of this chapter considers sources with a log-concave density, and proves such an informativeness result for sources with two-sided unbounded support, and this result also generalizes to sources with semi-unbounded support as well as sources with bounded support.

Remark 2.6.8. In Theorem 2.6.5, we compare expected costs of the players in equilibrium and prove that these costs are monotonically decreasing with respect to number of bins. This directly implies that the decoder always prefers an equilibrium with more bins. In that respect, our result is a generalization of [12, Theorem 3] for log-concave sources. On the other hand, our result implies that the encoder *ex ante* (before observing a realization of the source) prefers an equilibrium with more bins. This is a generalization of [12, Theorem 5] for log-concave sources. In that respect, our results reveal that if we compare (ex ante) expected costs of the encoder and decoder in equilibrium among all possible Nash equilibria, the one with the maximum possible number of bins gives the smallest expected cost, i.e., the most informative equilibrium.

Remark 2.6.9. Without a log-concave source assumption, such a monotonic behavior of the expected costs in equilibrium with respect to number of bins may not hold in general. For instance, we provide a simple example where there exists an equilibrium with $N = 3$ bins for which the expected costs in equilibrium is worse than an equilibrium with $N = 2$ bins. Although we provide an example with a discrete distribution for simplicity, one can also construct an example with a continuous density.

Example 2.6.1. Let the bias term be $b = 0.9$. Suppose that the distribution of the source is given by the following discrete distribution:

$$p_M(m) = \begin{cases} 0.8 & \text{if } m = 0 \\ 0.05 & \text{if } m \in \{2, 4, 6, 8\} \\ 0 & \text{otherwise} \end{cases} . \quad (2.4)$$

Consider an encoding policy with $N = 2$ bins where the quantization bins are given by $[0, m_1)$ and $[m_1, 8]$ with $m_1 = 3.9588$. The corresponding centroids are

given by $u_1 = \mathbb{E}[M|M < m_1] = 0.1176$ and $u_2 = \mathbb{E}[M|m_1 \leq M] = 6$. Notice that the nearest neighbor condition $m_1 = (u_1 + u_2)/2 + b$ is satisfied with these centroids. This leads to an expected cost value of $J^d = 0.5882$ for the decoder at the characterized equilibrium. Now consider an encoding policy with $N = 3$ bins where the quantization bins are given by $[0, m_1)$, $[m_1, m_2)$ and $[m_2, 8]$ with $m_1 = 4.0667$ and $m_2 = 7.9$. The corresponding centroids are given by $u_1 = \mathbb{E}[M|M < m_1] = 0.3333$, $u_2 = \mathbb{E}[M|m_1 \leq M < m_2] = 6$ and $u_3 = \mathbb{E}[M|m_2 \leq M] = 8$. With these centroids, the nearest neighbor conditions are satisfied, i.e., $m_1 = (u_1 + u_2)/2 + b$ and $m_2 = (u_2 + u_3)/2 + b$. This leads to an expected cost value of $J^d = 0.9$ for the decoder at the characterized equilibrium. Therefore, the characterized equilibrium with two bins payoff dominates the characterized equilibrium with three bins, which is in contrast to the result of Theorem 2.6.5 involving log-concave sources.

2.6.3 Convergence to Equilibria

It is interesting to investigate if the best response iterations converge to the unique equilibrium. Before presenting our result, we note that in the classical optimal quantization setup with aligned cost structure (i.e., team theoretic setup) and a log-concave density, the uniqueness of a fixed point and convergence to the unique fixed point via fixed point iterations are proven in [58] (see Theorem 2.5.3 of this chapter). In the following theorem, we present a signaling games counterpart of such a convergence result where the proof is based on the result of [58]. In particular, when modified Lloyd-Max iterations (i.e., by using (2.2) and (2.3)) are performed starting from an initial set of bin edges, the unique equilibrium (for the given number of bins) is reached for strictly log-concave sources with two-sided unbounded support. These iterations in the classical team theoretic setting is referred to as Lloyd's Method I [43] where bin edges and centroids are updated in a parallel fashion as described below.⁴

⁴Note that parallel updates in Lloyd's Method I indeed models the scenario where the encoder and the decoder update their policies iteratively in our game theoretic setting. Lloyd's Method II [43] is another heuristic technique to obtain a quantization policy in communication theoretic settings where bin edges and centroids are updated serially by taking the nearest

A modified Lloyd-Max iteration can be defined as follows: Let $m_1 < \dots < m_{N-1}$ be an initial set of bin edges and denote $\mathbf{m} \triangleq [m_1, \dots, m_{N-1}]^T$. Given these bin edges, the decoder first determines its best response by computing the centroids via

$$u_k = \mathbb{E}[M | m_{k-1} \leq M < m_k] \quad (2.5)$$

for $k = 1, \dots, N$, where $m_0 = m_L$ and $m_N = m_U$ remain the same during these iterations. We denote this operation by $\text{BR}_d(\mathbf{m}) = \mathbf{u}$ where $\mathbf{u} \triangleq [u_1, \dots, u_N]^T$. Then, the encoder computes the nearest neighbors for the resulting centroids $\mathbf{u}$ via

$$m_k = \frac{u_k + u_{k+1}}{2} + b \quad (2.6)$$

for $k = 1, \dots, N-1$. We denote this operation by $\text{BR}_e(\mathbf{u}) = \mathbf{m}$. With these best response characterizations, a modified Lloyd-Max iteration is defined as

$$T(\mathbf{m}) \triangleq \text{BR}_e(\text{BR}_d(\mathbf{m})). \quad (2.7)$$

The operation in (2.7) is applied iteratively to update an initial set of bin edges. In the following theorem, we show that by repeating these iterations, it is guaranteed that a set of bin edges that leads to an equilibrium with N bins is obtained under a strict log-concave source assumption.

Theorem 2.6.6. *For a strictly log-concave source with two-sided unbounded support, best response iterations (i.e., modified Lloyd-Max iterations) through (2.7) starting with N bins always converge to the unique equilibrium with N bins.*

Proof. See Section 2.8.5. □

Remark 2.6.10. We note that [58] proves convergence in a team theoretic setup even for log-concave sources which are not necessarily strictly log-concave. To prove this result, [58] uses the fact that the common cost is strictly decreasing as long as the corresponding set of bin edges is not a fixed point. Since there is no common cost in our signaling games setup, such an approach is not feasible.

neighbor and centroid conditions into account, i.e., given a value for the first centroid, compute the first bin edge and then compute the second centroid, and so on.

In the case of sources with semi-unbounded support, convergence to the unique equilibrium is guaranteed depending on the sign of b .

Theorem 2.6.7. (i) *Consider a source that has a strictly log-concave density with a support on $[m_L, \infty)$ for some $m_L \in \mathbb{R}$. If $b > 0$, then best response iterations through (2.7) starting with N bins always converge to the unique equilibrium with N bins.*

(ii) *Consider a source that has a strictly log-concave density with a support on $(-\infty, m_U]$ for some $m_U \in \mathbb{R}$. If $b < 0$, then best response iterations through (2.7) starting with N bins always converge to the unique equilibrium with N bins.*

Proof. The result essentially follows from Theorem 2.6.6 after observing that under the given assumptions, the bin edges are always greater than the lower boundary or larger than the upper boundary during best response iterations. $\square$

For sources with semi-unbounded support, if the sign of b does not satisfy the conditions of Theorem 2.6.7, convergence is not guaranteed, see the following remark. This is in contrast to the case of no bias [58].

Remark 2.6.11. For sources with semi-unbounded support, even if there exists an equilibrium with N bins for a given bias b , an initial set of bin edges forming a partition with N bins may not converge to the unique equilibrium with N bins. For instance, for a source with a support on $[m_L, \infty)$, if the initial values for the first two bin edges (i.e., m_1 and m_2 with $m_L < m_1 < m_2$) are close to m_L and the bias is negative (i.e., $b < 0$), then after the first iteration, the resulting first bin edge may be less than the value of m_L . As a result, the sender reduces the number of bin edges during the best response iterations. This implies that even though iterations start with N bins, they may converge to an equilibrium with K bins with $K < N$. In this case, it is guaranteed that best response iterations converge to an informative equilibrium with at least two bins (assuming it exists).

A similar result also holds for sources with bounded support.

Remark 2.6.12. For sources with bounded support, best response iterations starting with N bins may not converge to the unique equilibrium with N bins regardless of the value or the sign of b . Nevertheless, best response iterations always converge to an informative equilibrium with at least two bins (assuming it exists).

2.7 Conclusion

In this chapter, the Nash equilibria of cheap talk have been investigated for general log-concave sources. It has been shown that for any finite N , there exist a unique equilibrium with N bins for sources with two-sided unbounded support. Similarly, it has been proven that for sources with semi-unbounded support, when the support extends to $+\infty$ and $b > 0$, or the support extends to $-\infty$ and $b < 0$, for any finite N , there exist a unique equilibrium with N bins. On the other hand, in the converse case for sources with semi-unbounded support, it has been proven that there exists a finite upper bound on the number of bins in equilibrium. Furthermore, it has been shown that, as the number of bins increases, the expected costs of the encoder and the decoder in equilibrium improve, i.e., a more informative equilibrium is obtained. Moreover, we have proven that best response iterations converge to the unique equilibrium for sources with strictly log-concave density and two-sided unbounded support, generalizing the classical convergence results in optimal quantization via Lloyd's Method I.

2.8 Proofs

2.8.1 Proof of Theorem 2.6.1

We first discuss the existence of an equilibrium with a single bin. A quantizer with a single bin means that the encoder does not reveal any information related to the source. The optimal receiver action in this case becomes the mean of the source.

This pair of encoding and decoding policies leads to a Nash equilibrium. In this case, since the encoder does not convey information related to the source, the resulting equilibrium is a babbling equilibrium. Moreover, such an equilibrium always exists regardless of whether the source distribution is log-concave or not.

In order to prove the existence of an equilibrium with N bins for any $N \geq 2$, we take the following approach. We first show the existence of an equilibrium with $N = 2$ bins and then show that one can obtain an equilibrium with $N + 1$ bins starting from an equilibrium with N bins. These reveal that there exists an equilibrium with any finite number of bins. In addition, while obtaining the existence result, we also prove the uniqueness of an equilibrium for a given number of bins. In the following proof, we assume that the source is strictly log-concave. Nevertheless, our result also holds for a source with a log-concave density. In this case, one of the inequalities in Lemma 2.8.2 must be strict for the result to hold and this is indeed the case for a density with two-sided unbounded support.

To begin with, we restate the centroid condition and nearest neighbor condition that need to be satisfied at an equilibrium. An ordered set of bin edges $m_1 < \dots < m_{N-1}$ leads to an equilibrium with N bins if the following conditions are satisfied:

$$\int_{-\infty}^{m_1} (u_1 - m) f(m) dm = 0, \quad (2.8)$$

$$\int_{m_i}^{m_{i+1}} (u_{i+1} - m) f(m) dm = 0, \quad \text{for } i = 1, \dots, N-2, \quad (2.9)$$

$$\int_{m_{N-1}}^{\infty} (u_N - m) f(m) dm = 0, \quad (2.10)$$

$$u_{i+1} = 2m_i - u_i - 2b, \quad \text{for } i = 1, \dots, N-1 \quad (2.11)$$

where $u_1, \dots, u_N$ denote receiver actions. The aim is to show the existence of bin edges $m_1, \dots, m_{N-1}$ and receiver actions $u_1, \dots, u_N$ that satisfy (2.8)-(2.11) for any $N \geq 2$ and also to prove that these bin edges and receiver actions are indeed unique. To that aim, we need some auxiliary results which are presented in the following lemmas. The first lemma derives a set of inequalities which always holds for a log-concave density.

Lemma 2.8.1. *For a random variable M with a strictly log-concave density f ,*

the following inequalities are always satisfied:

$$\Pr(M < u) > (u - \mathbb{E}[M|M < u])f(u) \quad \text{for any } u \in \mathbb{R} \quad (2.12)$$

$$\Pr(M > u) > (\mathbb{E}[M|u < M] - u)f(u) \quad \text{for any } u \in \mathbb{R} \quad (2.13)$$

$$\begin{aligned} \Pr(v < M < u) &> (u - \mathbb{E}[M|v < M < u])f(u) \\ &+ (\mathbb{E}[M|v < M < u] - v)f(v) \end{aligned} \quad (2.14)$$

for any $u, v \in \mathbb{R}$ with $v < u$.

Before presenting the proof, we note that a variant of the inequalities (2.12) and (2.13) can be found in the literature considering log-concave densities with bounded support [65]. On the other hand, these inequalities need to be derived for the general case of densities with two-sided unbounded support for our purpose. Moreover, it is required to prove the additional property in (2.14), which is crucial in obtaining the monotonic behavior of the bin edges $m_2, \dots, m_N$ with respect to a change in m_1 in Lemma 2.8.2.

Proof. We first prove the inequality in (2.14) under an additional differentiable density assumption and then show that the result holds for any log-concave density. The proof for the case with a differentiable density relies on the fact that for a strictly log-concave density f , we have that $\frac{f'(x)}{f(x)}$ is strictly monotonically decreasing in x , which directly follows from the definition of being log-concave.

It is seen that the inequality in (2.14) is equivalent to

$$\begin{aligned} g(u) &\triangleq (F(u) - F(v))^2 + (u - v) \left(F(v)f(u) - F(u)f(v) \right) \\ &- f(u) \int_v^u F(m)dm + f(v) \int_v^u F(m)dm > 0. \end{aligned} \quad (2.15)$$

For fixed v , the derivative of $g(u)$ with respect to u is given by

$$\begin{aligned} g'(u) &= f(u)(F(u) - F(v)) - f'(u) \int_v^u F(m)dm \\ &+ (u - v) \left(F(v)f'(u) - f(u)f'(v) \right) \\ &= \int_v^u h(m)dm, \end{aligned} \quad (2.16)$$

where

$$h(m) \triangleq f(u)f(m) - f'(u)F(m) + F(v)f'(u) - f(u)f(v).$$

Next, we show that the integrand $h(m)$ is always positive for $m > v$:

$$\begin{aligned} h(m) &= f(u)(f(m) - f(v)) - f'(u) \int_v^m f(t)dt \\ &= f(u)(f(m) - f(v)) - f(u) \frac{f'(u)}{f(u)} \int_v^m f(t)dt \\ &> f(u)(f(m) - f(v)) - f(u) \int_v^m \frac{f'(t)}{f(t)} f(t)dt \\ &= f(u)(f(m) - f(v)) - f(u)(f(m) - f(v)) = 0, \end{aligned} \tag{2.17}$$

where the inequality follows from the fact that f is a strictly log-concave density. This implies that $g'(u) > 0$ when $u > v$. Since $g(u) = 0$ when $u = v$ and $g'(u) > 0$ when $u > v$, it follows that $g(u) > 0$ for all $u > v$. This shows that (2.14) is satisfied for all $u > v$.

Now, we show that the result holds for any strictly log-concave density. Since f is log-concave, it follows that $\log f$ is left and right differentiable. This implies that f is left and right differentiable, as well. Denote the left and right derivative of f by f'_- and f'_+ , respectively. Let $\log f(x) \triangleq \ell(x)$ for notational convenience. Since f is strictly log-concave, we have that $\ell'_-(x) = \frac{f'_-(x)}{f(x)}$ and $\ell'_+(x) = \frac{f'_+(x)}{f(x)}$ are strictly monotonically decreasing in x . We also know that g defined in (2.15) has left and right derivatives since f has left and right derivatives. Then, by taking the left derivative of g with respect to u for fixed v , we get

$$g'_-(u) = \int_v^u \left(f(u)f(m) - f'_-(u)F(m) + F(v)f'_-(u) - f(u)f(v) \right) dm.$$

Then, a similar analysis as in (2.17) yields $g'_-(u) > 0$ when $u > v$, where we use the fact that $\frac{f'_-(x)}{f(x)}$ is strictly monotonically decreasing and that $\int_v^m f'_-(t)dt = f(m) - f(v)$ since $\log f$, and therefore $e^{\log f}$, is an absolutely continuous function in the interior of the support of f (see e.g. [67, p. 34-35] for the absolute continuity of convex/concave functions). Also, we analogously obtain $g'_+(u) > 0$ when $u > v$. We also know that $g(u) = 0$ if $u = v$. Hence, we get $g(u) > 0$ for all $u > v$, as desired.

The inequality in (2.12) is equivalent to

$$\begin{aligned} F(u)^2 - F(u)f(u)u + f(u) \int_{-\infty}^u mf(m)dm &> 0 \\ \Leftrightarrow F(u)^2 &> f(u) \int_{-\infty}^u F(m)dm \end{aligned} \quad (2.18)$$

where $F(\cdot)$ denotes the cumulative distribution function of M , the second expression is obtained by applying integration by parts and using the fact that $F(m)m \rightarrow 0$ as $m \rightarrow -\infty$. We know that f has left and right derivatives. Note also that $\int_{-\infty}^x f'_-(t)dt = f(x)$ since $\log f$, and therefore $e^{\log f}$, is an absolutely continuous function in the interior of the support of f (see e.g. [67, p. 34-35] for the absolute continuity of convex/concave functions). In addition, since $\frac{f'_-(x)}{f(x)}$ is strictly monotonically decreasing, we get

$$\begin{aligned} f'_-(x)F(x) - f(x)^2 &= f(x) \frac{f'_-(x)}{f(x)} \int_{-\infty}^x f(m)dm - f(x)^2 \\ &< f(x) \int_{-\infty}^x f'_-(m)dm - f(x)^2 = 0, \end{aligned}$$

where the last equality is due to $\int_{-\infty}^x f'_-(t)dt = f(x)$. Also, a similar analysis yields $f'_+(x)F(x) - f(x)^2 < 0$. These imply that $\frac{f(x)}{F(x)}$ is strictly monotonically decreasing. Then, by using the monotonic behavior of $\frac{f(x)}{F(x)}$, we get

$$\frac{f(u)}{F(u)} \int_{-\infty}^u F(m)dm < \int_{-\infty}^u \frac{f(m)}{F(m)} F(m)dm = F(u), \quad (2.19)$$

which shows that (2.18) and equivalently (2.12) always hold. A similar approach can be taken to prove (2.13). $\square$

Next, we use the inequalities derived in Lemma 2.8.1 to establish a monotonicity property of centroids and subsequent bin edges with respect to a shift in the first bin edge. Before presenting the lemma, for notational convenience, we make the following definitions which specify a collection consisting of a set of bin edges either in an ascending or descending order such that the nearest neighbor conditions with the corresponding centroids are satisfied at all bin edges except at the last bin edge.

- Definition 2.8.1.** (i) Let $m_1 < m_2 < \dots < m_N$, $u_1 = \mathbb{E}[M|M < m_1]$ and $u_i = \mathbb{E}[M|m_{i-1} \leq M < m_i]$ for $i = 2, \dots, N$. Define $\mathcal{M}_A^N$ as the collection of all $\{m_i\}_{i=1}^N$ for which the nearest neighbor conditions at the boundaries $m_1, \dots, m_{N-1}$ are satisfied with the corresponding centroids $u_1, \dots, u_N$, i.e., $2m_i = u_i + u_{i+1} + 2b$ for $i = 1, \dots, N-1$.
- (ii) Let $m_1 > m_2 > \dots > m_N$, $u_1 = \mathbb{E}[M|m_1 \leq M]$ and $u_i = \mathbb{E}[M|m_i \leq M < m_{i-1}]$ for $i = 2, \dots, N$. Define $\mathcal{M}_D^N$ as the collection of all $\{m_i\}_{i=1}^N$ for which the nearest neighbor conditions at the boundaries $m_1, \dots, m_{N-1}$ are satisfied with the corresponding centroids $u_1, \dots, u_N$, i.e., $2m_i = u_i + u_{i+1} + 2b$ for $i = 1, \dots, N-1$.

Notice that when a set of bin edges belongs to $\mathcal{M}_A^N$ or $\mathcal{M}_D^N$, it does not necessarily mean that these set of bin edges lead to an equilibrium. For these set of bin edges to form an equilibrium, the nearest neighbor condition at the last bin edge should also be satisfied. In our proof, we show that by varying the value of the first bin edge, it is possible to satisfy the last nearest neighbor condition. In fact, in the proof, it will be apparent that there exists a unique value for the first bin edge that leads to an equilibrium with N bins and these unique values are monotone in N . This monotonicity property of the bin edges is also used to prove monotonically decreasing behavior of the costs with respect to the number of bins in Theorem 2.6.5.

Lemma 2.8.2. *Suppose that there exists a set of bin edges $\{m_i\}_{i=1}^N$ which belongs to $\mathcal{M}_A^N$ (resp. $\mathcal{M}_D^N$) with the corresponding set of centroids $\{u_i\}_{i=1}^N$. Let $\bar{u}_{N+1} = \mathbb{E}[M|m_N \leq M]$ (resp. $\bar{u}_{N+1} = \mathbb{E}[M|M < m_N]$) denote the centroid of the last bin and let $\tilde{u}_{N+1} = 2m_N - u_N - 2b$ denote the receiver action that satisfies the nearest neighbor condition at the bin edge m_N . Then, the following hold:*

$$\frac{dm_i}{du_i} > 1, \quad \text{for } i = 1, \dots, N, \quad (2.20)$$

$$\frac{du_{i+1}}{dm_i} > 1, \quad \text{for } i = 1, \dots, N-1, \quad (2.21)$$

$$\frac{d\tilde{u}_{N+1}}{dm_N} > 1, \quad (2.22)$$

$$\frac{dm_N}{d\bar{u}_{N+1}} > 1. \quad (2.23)$$

Proof. We prove the result for the case of $\{m_i\}_{i=1}^N \in \mathcal{M}_A^N$ and the case of $\{m_i\}_{i=1}^N \in \mathcal{M}_D^N$ can be proven in a similar manner. The centroid condition requires that

$$\int_{-\infty}^{m_1} (u_1 - m)f(m)dm = 0, \quad (2.24)$$

$$\int_{m_i}^{m_{i+1}} (u_{i+1} - m)f(m)dm = 0, \quad \text{for } i = 1, \dots, N-1, \quad (2.25)$$

$$\int_{m_N}^{\infty} (\bar{u}_{N+1} - m)f(m)dm = 0. \quad (2.26)$$

Since it is assumed that the nearest neighbor conditions at $m_1, \dots, m_{N-1}$ are satisfied, we have

$$u_{i+1} = 2m_i - u_i - 2b, \quad \text{for } i = 1, 2, \dots, N-1. \quad (2.27)$$

By differentiating (2.24) with respect to u_1 , we get

$$F(m_1) + u_1 f(m_1) \frac{dm_1}{du_1} - \frac{d}{du_1} \int_{-\infty}^{m_1} m f(m) dm = 0, \quad (2.28)$$

where the integral is finite since $\mathbb{E}[M]$ is finite. From (2.28), it follows that

$$\frac{dm_1}{du_1} = \frac{\int_{-\infty}^{m_1} f(m) dm}{(m_1 - u_1) f(m_1)} > 1, \quad (2.29)$$

where $\frac{dm_1}{du_1}$ is finite and the inequality follows from (2.12) in Lemma 2.8.1. Then, $\frac{dm_1}{du_1} > 1$ and (2.27) imply that $\frac{du_2}{dm_1} > 1$. Next, we show that if $\frac{du_{i+1}}{dm_i} > 1$, then $\frac{dm_{i+1}}{du_{i+1}} > 1$ by using (2.25). Differentiating (2.25) with respect to u_{i+1} leads to

$$\int_{m_i}^{m_{i+1}} f(m) dm = (u_{i+1} - m_i) f(m_i) \frac{dm_i}{du_{i+1}} + (m_{i+1} - u_{i+1}) f(m_{i+1}) \frac{dm_{i+1}}{du_{i+1}}. \quad (2.30)$$

From Lemma 2.8.1, we know that

$$\int_{m_i}^{m_{i+1}} f(m) dm > (u_{i+1} - m_i) f(m_i) + (m_{i+1} - u_{i+1}) f(m_{i+1}). \quad (2.31)$$

By combining (2.30) and (2.31), we get

$$\begin{aligned}
& (u_{i+1} - m_i)f(m_i)\frac{dm_i}{du_{i+1}} + (m_{i+1} - u_{i+1})f(m_{i+1})\frac{dm_{i+1}}{du_{i+1}} \\
& > (u_{i+1} - m_i)f(m_i) + (m_{i+1} - u_{i+1})f(m_{i+1}) \\
& \Leftrightarrow \left(\frac{dm_i}{du_{i+1}} - 1\right)(u_{i+1} - m_i)f(m_i) \\
& > \left(1 - \frac{dm_{i+1}}{du_{i+1}}\right)(m_{i+1} - u_{i+1})f(m_{i+1}) \tag{2.32}
\end{aligned}$$

From (2.32), it is seen that $\frac{du_{i+1}}{dm_i} > 1$ leads to $\frac{dm_{i+1}}{du_{i+1}} > 1$ since $(u_{i+1} - m_i)f(m_i) > 0$ and $(m_{i+1} - u_{i+1})f(m_{i+1}) > 0$. Next, $\frac{dm_{i+1}}{du_{i+1}} > 1$ and (2.27) imply that $\frac{du_{i+2}}{dm_{i+1}} > 1$. By iteratively using (2.27), (2.30) and (2.31) in this manner, one can show (2.20) and (2.21). Moreover, $\frac{dm_N}{du_N} > 1$ and $\tilde{u}_{N+1} = 2m_N - u_N - 2b$ lead to (2.22). Finally, differentiating (2.26) with respect to $\bar{u}_{N+1}$, we get

$$\frac{dm_N}{d\bar{u}_{N+1}} = \frac{\int_{m_N}^{\infty} f(m)dm}{(\bar{u}_{N+1} - m_N)f(m_N)} > 1, \tag{2.33}$$

where $\frac{dm_N}{d\bar{u}_{N+1}}$ is finite and the inequality follows from Lemma 2.8.1. $\square$

Now, equipped with Lemma 2.8.2, we are ready to prove the theorem. We first prove the existence of a unique equilibrium with two bins. In this case, for a bin edge m , the centroids of the resulting bins are denoted by $u_1(m) = \mathbb{E}[M|M < m]$ and $u_2(m) = \mathbb{E}[M|m \leq M]$. The point where the sender is indifferent between these receiver actions is given by

$$\tilde{m}(m) = \frac{u_1(m) + u_2(m)}{2} + b. \tag{2.34}$$

For an equilibrium, it is required that $m = \tilde{m}(m)$. Lemma 2.8.2 states that $\frac{dm}{du_1} > 1$ and $\frac{dm}{du_2} > 1$ hold due to the fact that the source distribution follows a log-concave density and these imply that $\frac{dm}{d\tilde{m}} > 1$ due to (2.34). This last property is crucial in obtaining the uniqueness of the equilibrium. In the following, we construct an interval such that $\tilde{m}(\cdot)$ maps this interval into itself and then use the Brouwer's fixed point theorem [68] to show the existence of equilibrium. Towards that goal, let m^α be a bin edge for which $m^\alpha \neq \tilde{m}(m^\alpha)$. Without loss of generality, suppose that $m^\alpha < \tilde{m}(m^\alpha)$. Let $m^\beta \triangleq u_2(m^\alpha) - m^\alpha + \mu + 2b$. Note that $u_1(m^\alpha)$

and $u_2(m^\alpha)$ are finite for a finite m^α as the source has a finite mean. Observe that $m^\alpha < m^\beta$ holds since

$$\begin{aligned} m^\alpha &< 2\tilde{m}(m^\alpha) - m^\alpha \\ &= u_2(m^\alpha) + u_1(m^\alpha) + 2b - m^\alpha \\ &< u_2(m^\alpha) + \mu + 2b - m^\alpha = m^\beta, \end{aligned}$$

where the first inequality is by assumption and the second inequality is due to $u_1(m) = \mathbb{E}[M|M < m] < \mathbb{E}[M|M < \infty] = \mu$. We also have $\tilde{m}(m^\beta) < m^\beta$ as shown in the following:

$$\begin{aligned} 2(\tilde{m}(m^\beta) - m^\beta) &= u_2(m^\beta) - m^\beta + u_1(m^\beta) - m^\beta + 2b \\ &< u_2(m^\alpha) - m^\alpha + u_1(m^\beta) - m^\beta + 2b \\ &< u_2(m^\alpha) - m^\alpha + \mu + 2b - m^\beta = 0, \end{aligned}$$

where the first inequality follows from the facts that $m^\alpha < m^\beta$ and $u_2(m) - m$ is monotonically decreasing in m due to (2.23) in Lemma 2.8.2, and the second inequality is due to $u_1(m) = \mathbb{E}[M|M < m] < \mathbb{E}[M|M < \infty] = \mu$. By using $\tilde{m}(m^\beta) < m^\beta$ together with the fact that $\tilde{m}(m)$ is monotone in m , we have $\tilde{m}(m) \in (m^\alpha, m^\beta)$ for any $m \in [m^\alpha, m^\beta]$. Therefore, by Brouwer's fixed point theorem [68], there exists $m \in (m^\alpha, m^\beta)$ such that $m = \tilde{m}(m)$ holds since $\tilde{m}(m)$ is continuous and maps the interval $[m^\alpha, m^\beta]$ into itself. The uniqueness, on the other hand, follows from $\frac{dm}{d\tilde{m}} > 1$, which makes $(\tilde{m}(m) - m)$ monotonically decreasing in m . In the case of $\tilde{m}(m^\alpha) < m^\alpha$, a similar procedure can be applied. This shows the existence of a unique equilibrium with two bins.

Next, the aim is to prove that it is possible to obtain an equilibrium with $N + 1$ bins starting from an equilibrium with N bins. Depending on whether b is positive or negative, we need to take slightly different approaches. In the following, we first focus on the case where b is negative. Let

$$K_A(x) = \max \left\{ k \mid \text{there exists } x = m_1 < \dots < m_{k-1} \text{ with } \{m_i\}_{i=1}^{k-1} \in \mathcal{M}_A^{k-1} \right\}. \quad (2.35)$$

Denote the bin edges that attain the maximum in (2.35) for a given x by $m_1^x, \dots, m_{K_A(x)-1}^x$ and denote the centroids with these bin edges by $u_1^x \triangleq$

$\mathbb{E}[M|M < m_1^x]$ and $u_i^x \triangleq \mathbb{E}[M|m_{i-1}^x \leq M < m_i^x]$ for $i = 2, \dots, K_A(x) - 1$. In the following, an equilibrium with $N + 1$ bins is obtained by continuously decreasing $m_1^x = x$ starting from its value for the equilibrium with N bins. Let $m_i^A(N)$ for $i = 1, \dots, N - 1$ denote bin edges for an equilibrium with N bins where these bin edges are sorted in an ascending order, i.e., $m_1^A(N) < \dots < m_{N-1}^A(N)$.

To begin with, we show that when $m_1^x < m_1^A(N)$, there exists $m_1^x < \dots < m_N^x$ such that the nearest neighbor conditions at the boundaries $m_1^x, \dots, m_{N-1}^x$ hold with the corresponding centroids, i.e., $x < m_1^A(N)$ implies that $K_A(x) \geq N + 1$. Towards that goal, we first show that for all $m_1^x < m_1^A(N)$ there exist $m_1^x < \dots < m_{N-1}^x$ for which the nearest neighbor conditions at the boundaries $m_1^x, \dots, m_{N-2}^x$ are satisfied. These nearest neighbor conditions are in fact satisfied when $m_1^x = m_1^A(N)$ and our aim is to show the existence of boundaries which still satisfy these nearest neighbor conditions when m_1^x is decreased. These nearest neighbor conditions require that

$$u_{i+1}^x - m_i^x = m_i^x - u_i^x - 2b, \quad \text{for } i = 1, \dots, N - 2. \quad (2.36)$$

It is always guaranteed that $u_1^x < m_1^x$ since $u_1^x = \mathbb{E}[M|M < m_1^x]$. From $u_1^x < m_1^x$, $b < 0$ and (2.36), it follows that $u_1^x < m_2^x$. Then, it is always possible to find $m_2^x < m_2^A(N)$ such that the centroid of the bin $[m_1^x, m_2^x]$ is given by u_2^x , where $m_2^x < m_2^A(N)$ is due to the fact that m_2^x is monotonically increasing in m_1^x . This implies that while m_1^x is decreased $m_1^x < m_2^x$ is guaranteed. Next, via a similar argument, $u_i^x < m_i^x$, $b < 0$ and (2.36) imply that there exist $m_i^x < m_{i+1}^x < m_{i+1}^A(N)$ such that the nearest neighbor condition at the boundary m_i^x holds for $i = 2, \dots, N - 2$. Now, we show the existence of the last bin edge m_N^x for which the nearest neighbor condition at m_{N-1}^x holds. Let $\bar{u}_N^x = \mathbb{E}[M|m_{N-1}^x \leq M]$ and $\tilde{u}_N^x = 2m_{N-1}^x - u_{N-1}^x - 2b$ where the former term corresponds to the centroid of the last bin and the latter term is the receiver action that makes the nearest neighbor condition at m_{N-1}^x hold. We know that $\bar{u}_N^x = \tilde{u}_N^x$ when $m_1^x = m_1^A(N)$ as this bin edge leads to an equilibrium with N bins. Since $0 < \frac{d\bar{u}_N^x}{dx} < \frac{d\tilde{u}_N^x}{dx}$ by Lemma 2.8.2, it follows that $\tilde{u}_N^x < \bar{u}_N^x$ for any $m_1^x < m_1^A(N)$. Moreover, from $u_{N-1}^x < m_{N-1}^x$, $\tilde{u}_N^x = 2m_{N-1}^x - u_{N-1}^x - 2b$ and $b < 0$, it is guaranteed that $m_{N-1}^x < \tilde{u}_N^x$. Therefore, for any $m_1^x < m_1^A(N)$, there exists m_N^x such that $\tilde{u}_N^x = \mathbb{E}[M|m_{N-1}^x \leq M < m_N^x]$. These establish the existence of boundaries $m_1^x, \dots, m_N^x$ for which the nearest

neighbor conditions at $m_1^x, \dots, m_{N-1}^x$ are satisfied when $m_1^x < m_1^A(N)$, i.e., $x < m_1^A(N)$ leads to $K_A(x) \geq N + 1$.

In the previous analysis, the existence of $m_1^x < \dots < m_N^x$ with $\{m_i^x\}_{i=1}^N \in \mathcal{M}_A^N$ is established when $m_1^x < m_1^A(N)$. Now, we prove that by continuously decreasing m_1^x starting from $m_1^A(N)$, it is possible to reach at a point where this set of bin edges $\{m_i^x\}_{i=1}^N$ forms an equilibrium, i.e., the nearest neighbor conditions at all boundaries are satisfied. Since $m_1^x = m_1^A(N)$ leads to an equilibrium with N bins and all of the subsequent bin edges move in a monotone manner with respect to m_1^x by Lemma 2.8.2, it follows that $m_N^x \rightarrow \infty$ as m_1^x approaches $m_1^A(N)$ from the left. This observation ensures that it is possible to find a left neighborhood of $m_1^A(N)$ such that a value of m_1^x in this neighborhood leads to $\bar{u}_{N+1}^x < \tilde{u}_{N+1}^x$ where $\bar{u}_{N+1}^x = \mathbb{E}[M|m_N^x \leq M]$ and $\tilde{u}_{N+1}^x = 2m_N^x - u_N^x - 2b$. In order to see this, we write

$$\begin{aligned} \tilde{u}_{N+1}^x - \bar{u}_{N+1}^x &= 2m_N^x - u_N^x - 2b - \bar{u}_{N+1}^x \\ &= m_N^x - u_N^x - (\bar{u}_{N+1}^x - m_N^x) - 2b. \end{aligned} \quad (2.37)$$

Since $(\mathbb{E}[M|x \leq M] - x)$ is a decreasing function of x by Lemma 2.8.2, the term $(\bar{u}_{N+1}^x - m_N^x)$ decreases as m_1^x increases. Moreover, since $u_N^x \rightarrow \mathbb{E}[M|m_{N-1}^A(N) < M]$ as m_1^x approaches $m_1^A(N)$ from the left and u_N^x is monotonically increasing in m_1^x , u_N^x is upper bounded by $\mathbb{E}[M|m_{N-1}^A(N) < M]$. Hence, the last three terms in (2.37) can be upper bounded by a finite value. Thus, by choosing a value of m_1^x sufficiently close to $m_1^A(N)$, one can make m_N^x sufficiently large, which leads to $\tilde{u}_{N+1}^x - \bar{u}_{N+1}^x > 0$. As a result, it holds that $\tilde{u}_{N+1}^x > \bar{u}_{N+1}^x$ for m_1^x at a left neighborhood of $m_1^A(N)$. On the other hand, a sufficiently small value of m_1^x yields $\tilde{u}_{N+1}^x < \bar{u}_{N+1}^x$. To see this, observe that since $\frac{d\tilde{u}_{N+1}^x}{dm_1^x} > 1$ by Lemma 2.8.2, $\tilde{u}_{N+1}^x$ can be made sufficiently small by taking m_1^x sufficiently small. However, $\bar{u}_{N+1}^x$ is lower bounded by the mean. Thus, a sufficiently small value of m_1^x makes $\tilde{u}_{N+1}^x < \bar{u}_{N+1}^x$. In addition, we know that $(\tilde{u}_{N+1}^x - \bar{u}_{N+1}^x)$ is continuous and monotonically increasing in m_1^x . Therefore, there exists a particular value of m_1^x for which $\tilde{u}_{N+1}^x = \bar{u}_{N+1}^x$ holds, and hence, this particular value leads to an equilibrium with $N + 1$ bins.

To sum up, it is shown that there always exists an equilibrium with two bins

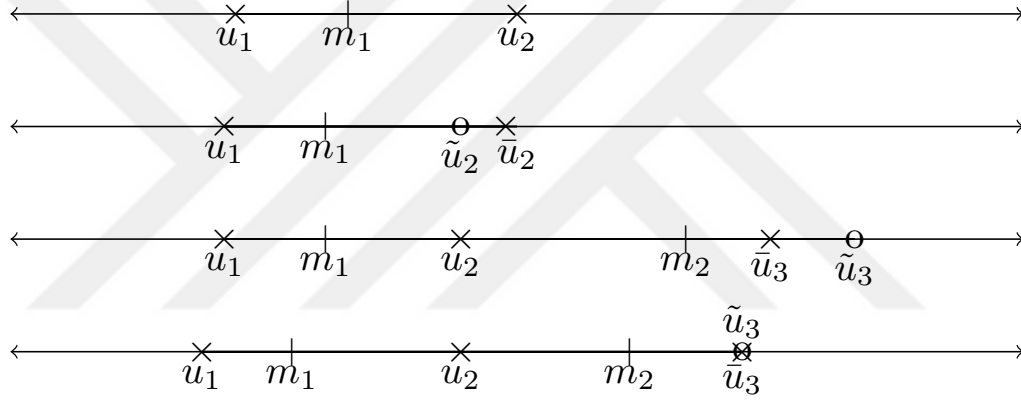


Figure 2.2: Illustration of the proof technique while showing the existence of equilibria with more bins when $b < 0$. The first plot shows an equilibrium with two bins. In order to obtain an equilibrium with three bins, we decrease the value of bin edge m_1 . In the second plot, u_1 is the centroid of the bin $(-\infty, m_1)$ and $\bar{u}_2$ is the centroid of the bin $[m_1, \infty)$, and $\tilde{u}_2$ is where the nearest neighbor condition with u_1 is satisfied at m_1 . Since $\tilde{u}_2 < \bar{u}_2$, we need to introduce a second bin edge m_2 so that $\tilde{u}_2$ is the centroid of the bin $[m_1, m_2)$ and this is shown in the third plot. With this bin edge, we obtain a new centroid $\bar{u}_3$ of the bin $[m_2, \infty)$. In order for this centroid to coincide with $\tilde{u}_3$, we decrease m_1 further. This leads to the final plot where an equilibrium with three bins is obtained.

and that if there exists an equilibrium with N bins, it is possible to construct an equilibrium with $N + 1$ bins by continuously decreasing the left-most bin edge. Fig. 2.2 illustrates this technique to obtain equilibria with more bins. These results imply that there exists an equilibrium with N bins for any $N \geq 2$. Moreover, the proof for the existence also reveals that an equilibrium with N bins is unique. To see this, observe that $K_A(x)$ is a piecewise constant and monotonically non-increasing function with jumps by one at discontinuities where these discontinuities lead to equilibrium scenarios, i.e., when $x \in [m_1^A(N+1), m_1^A(N))$, we have that $K_A(x) = N + 1$ for $N \in \{2, 3, \dots\}$. Hence, this monotonic behavior of $K_A(x)$ ensures the uniqueness of an equilibrium for a given number of bins.

Finally, we briefly describe the generalization of the existence and uniqueness results to the case of $b > 0$. We take bin edges in a descending order, i.e., $m_1 > \dots > m_{N-1}$, and continuously increase m_1 starting from $m_1^D(N)$ to obtain an equilibrium with $N + 1$ bins where $m_1^D(N) > \dots > m_{N-1}^D(N)$ denote bin edges for an equilibrium with N bins. Let

$$K_D(x) = \max \left\{ k \mid \text{there exists } x = m_1 > \dots > m_{k-1} \text{ with } \{m_i\}_{i=1}^{k-1} \in \mathcal{M}_D^{k-1} \right\}. \quad (2.38)$$

Denote the bin edges that attain the maximum in (2.38) for a given x by $m_1^x, \dots, m_{K_D(x)-1}^x$ and denote the centroids with these bin edges by $u_1^x \triangleq \mathbb{E}[M | m_1^x \leq M]$ and $u_i^x \triangleq \mathbb{E}[M | m_i^x \leq M < m_{i-1}^x]$ for $i = 2, \dots, K_D(x) - 1$. Via a similar reasoning, it can be shown that when $m_1^x > m_1^D(N)$ there exists $m_1^x > \dots > m_N^x$ such that the nearest neighbor conditions at $m_1^x, \dots, m_{N-1}^x$ are satisfied. Moreover, there exists a right neighborhood of $m_1^D(N)$ such that a value of m_1^x in this neighborhood leads to $\tilde{u}_{N+1}^x < \bar{u}_{N+1}^x$ where $\bar{u}_{N+1}^x = \mathbb{E}[M | M < m_N^x]$ and $\tilde{u}_{N+1}^x = 2m_N^x - u_N^x - 2b$. On other hand, a sufficiently large value of m_1^x results in $\tilde{u}_{N+1}^x > \bar{u}_{N+1}^x$. Since $(\tilde{u}_{N+1}^x - \bar{u}_{N+1}^x)$ is continuous and monotonically increasing in m_1^x , there exists a particular value of m_1^x leading to an equilibrium with $N + 1$ bins by making $\tilde{u}_{N+1}^x = \bar{u}_{N+1}^x$. These imply that $K_D(x)$ is a piecewise constant monotonically non-decreasing function with jumps by one at discontinuities which correspond to equilibrium scenarios, i.e., we have that $K_D(x) = N + 1$ when $x \in (m_1^D(N), m_1^D(N+1)]$ for $N \in \{2, 3, \dots\}$. Therefore, for any given $N \geq 2$, there exists a unique equilibrium with N bins.

2.8.2 Proof of Theorem 2.6.2

For the cases when there is no upper bound on the number of bins, the existence and uniqueness results can be obtained in a similar manner to Theorem 2.6.1. Namely, for a density with a support on $[m_L, \infty)$ and $b > 0$, the approach is to increase the value of the right-most bin edge to construct an equilibrium with $N + 1$ bins starting from an equilibrium with N bins.

We now prove that there exists an upper bound on the number of bins in the case of support on $[m_L, \infty)$ and $b < 0$. Note that if there does not exist an informative equilibrium, then the upper bound on the number of bins is equal to one. Suppose that the bias term is such that there exists at least one informative equilibrium. Let m_1 be the right-most bin edge and $u_1 = \mathbb{E}[M|m_1 \leq M]$. For the nearest neighbor condition at m_1 to be satisfied, there must be a receiver action $\tilde{u}_2 = 2m_1 - u_1 - 2b$. We know that $(\mathbb{E}[M|m_1 \leq M] - m_1)$ monotonically decreases as m_1 increases due to Lemma 2.8.2 and approaches zero as $m_1 \rightarrow \infty$. Therefore, existence of an informative equilibrium and monotonic behavior of $(\mathbb{E}[M|m_1 \leq M] - m_1)$ guarantee that there exists a particular value of m_1 , say $\bar{m}_1$, which makes $u_1 - \bar{m}_1 = \mathbb{E}[M|\bar{m}_1 \leq M] - \bar{m}_1 = -2b$. Since $(\mathbb{E}[M|m_1 \leq M] - m_1)$ monotonically decreases as m_1 increases, we have $(m_1 - \mathbb{E}[M|m_1 \leq M] - 2b) > 0$ for all $m_1 > \bar{m}_1$. This implies that for all $m_1 > \bar{m}_1$, we have $(m_1 - \mathbb{E}[M|m_1 \leq M] - 2b) = \tilde{u}_2 - m_1 > 0$. Therefore, there cannot be an equilibrium with a right-most bin edge m_1 satisfying $m_1 > \bar{m}_1$ since it is required to find $m_2 < m_1$ such that the centroid of the bin $[m_2, m_1)$ is given by $\tilde{u}_2 > m_1$, which is not possible. Thus, there exists an upper bound on the value of the right-most bin edge in equilibrium, which reveals that the number of bins at an equilibrium is upper bounded. Via a construction similar to Theorem 2.6.1, the existence and uniqueness of an equilibrium with number of bins $N \in \{1, \dots, N_{\max}\}$ can be proven.

2.8.3 Proof of Theorem 2.6.3

Suppose first that $2b \leq -(\mu - m_L)$. Recall that an equilibrium with two bins requires that $\tilde{m}(m) = m$ holds for a bin edge m where

$$\tilde{m}(m) = \frac{\mathbb{E}[M|m \leq M] + \mathbb{E}[M|M < m]}{2} + b. \quad (2.39)$$

Note that for an equilibrium with two bins we need $m > m_L$ as otherwise there is no information conveyed related to the source. If we take $m_L = m$, we get

$$\tilde{m}(m_L) = \frac{\mu + m_L}{2} + b \leq m_L, \quad (2.40)$$

where the inequality is by assumption. Moreover, we know that $(m - \tilde{m}(m))$ is strictly monotonically increasing in m due to Lemma 2.8.2. Therefore, for any bin edge $m > m_L$, we have $m > \tilde{m}(m)$. Therefore, there does not exist a bin edge that leads to an equilibrium with two bins. On the other hand, when $2b > -(\mu - m_L)$, it is possible to construct an interval for which $\tilde{m}(\cdot)$ maps this interval into itself in a similar manner to Theorem 2.6.1 and then apply Brouwer's fixed point theorem [68].

2.8.4 Proof of Theorem 2.6.5

We first focus on the case of $b < 0$ and then generalize the result to the case of $b > 0$. Theorem 2.6.1 obtains an equilibrium with $N + 1$ bins starting from an equilibrium with N bins by continuously decreasing the value of the left-most bin edge. In the following analysis, we instead start from an equilibrium with $N + 1$ bins and continuously decrease the value of the right-most bin edge to reach an equilibrium with N bins. Then, we show that the objective function for the decoder is monotonically decreasing in the value of the right-most bin edge. Note that the encoder and decoder costs only differ by a constant as pointed out in Remark 2.6.7. Thus, it suffices to prove the monotonic behavior for the cost of the decoder.

Recall that $m_1^D(N) > \dots > m_{N-1}^D(N)$ denote the bin edges arranged in a descending order for the unique equilibrium with N bins. We note that

$m_1^D(N) < m_1^D(N+1)$. In order to show this, we assume otherwise and reach at a contradiction. When $m_1^x = m_1^D(N)$, we have that $\tilde{u}_N^x = \bar{u}_N^x$ where $\bar{u}_N^x = \mathbb{E}[M|M < m_{N-1}^x]$ and $\tilde{u}_N^x = 2m_{N-1}^x - u_{N-1}^x - 2b$. Since Lemma 2.8.2 states that $\frac{dm_{i+1}^x}{dm_i^x} > 1$ and $\frac{du_{i+1}^x}{dm_i^x} > 1$ hold for $i = 1, \dots, N-2$, it is guaranteed that $m_{N-1}^x < \dots < m_1^x$ holds as m_1^x is decreased. Moreover, by Lemma 2.8.2, we have $0 < \frac{d\bar{u}_N^x}{dm_1^x} < \frac{d\tilde{u}_N^x}{dm_1^x}$. Thus, $m_1^x < m_1^D(N)$ leads to $\tilde{u}_N^x < \bar{u}_N^x$. These observations imply that $x < m_1^D(N)$ leads to $K_D(x) = N$. However, this contradicts with $K_D(x) = N+1$ when $x = m_1^D(N+1)$. Therefore, it always holds that $m_1^D(N) < m_1^D(N+1)$. Now, we take $m_1^x = m_1^D(N+1)$ and decrease until $m_1^x = m_1^D(N)$. Since all the subsequent bin edges decrease monotonically as m_1^x is decreased, it follows that $m_i^x \rightarrow m_i(N)$ for $i = 2, \dots, N-1$ and $m_N^x \rightarrow -\infty$ as m_1^x approaches $m_1^D(N)$ from the right. Namely, in order to obtain an equilibrium with N bins, one can decrease m_1^x starting from its value for the unique equilibrium with $N+1$, i.e., $m_1^D(N+1)$, bins until its value for the unique equilibrium with N bins, i.e., $m_1^D(N)$.

Next, the aim is to show that when $m_1^x \in (m_1^D(N), m_1^D(N+1))$ the objective function of decoder is monotonically decreasing in m_1^x where the sender employs the bin edges $m_N^x < \dots < m_1^x$ with $m_1^x = x$ such that the nearest neighbor conditions at the boundaries $m_1^x, \dots, m_{N-1}^x$ hold with the centroids $u_1^x, \dots, u_{N+1}^x$ computed based on these bin edges. With these bin edges, the objective function of the receiver can be written as

$$\begin{aligned} J^d(x) &= \int_{m_1^x}^{\infty} (u_1^x - m)^2 f(m) dm \\ &\quad + \sum_{i=1}^{N-1} \int_{m_{i+1}^x}^{m_i^x} (u_{i+1}^x - m)^2 f(m) dm \\ &\quad + \int_{-\infty}^{m_N^x} (u_{N+1}^x - m)^2 f(m) dm, \end{aligned}$$

where u_1^x , u_{N+1}^x and u_i^x denote the centroid of the bins $[m_1^x, \infty)$, $(-\infty, m_N^x)$ and $[m_i^x, m_{i-1}^x)$ for $i = 2, \dots, N$, respectively. If we take the derivative with respect to x , we get

$$\frac{dJ^d(x)}{dx} = \sum_{i=1}^N \frac{dm_i^x}{dx} f(m_i^x) \left((u_{i+1}^x - m_i^x)^2 - (u_i^x - m_i^x)^2 \right).$$

Since the nearest neighbor conditions at $m_1^x, \dots, m_{N-1}^x$ are satisfied, the following holds:

$$(u_{i+1}^x - m_i^x)^2 - (u_i^x - m_i^x)^2 = (u_i^x - u_{i+1}^x)2b < 0$$

for $i = 1, \dots, N-1$, where the inequality follows from $b < 0$. Moreover, if $m_1^x = m_1^A(N+1)$, then

$$(u_{N+1}^x - m_N^x)^2 - (u_N^x - m_N^x)^2 = (u_N^x - m_{N+1}^x)2b < 0.$$

As m_1^x decreases, the term $(u_{N+1}^x - m_N^x)^2 = (m_N^x - \mathbb{E}[M|M < m_N^x])^2$ decreases by Lemma 2.8.2. On the other hand, $(u_N^x - m_N^x)^2$ increases as m_1^x decreases because $\frac{dm_N^x}{du_N^x} > 1$ by Lemma 2.8.2. These ensure that when m_1^x is decreased, we still have

$$(u_{N+1}^x - m_N^x)^2 - (u_N^x - m_N^x)^2 < 0. \quad (2.41)$$

As a result, since $f(m_i^x)$ and $\frac{dm_i^x}{dx}$ are all positive and $((u_{i+1}^x - m_i^x)^2 - (u_i^x - m_i^x)^2)$ are all negative, it follows that $\frac{dJ^d(x)}{dx} < 0$. This shows that the cost monotonically increases as $x = m_1^x$ decreases. Since we obtain the equilibrium with N bins by decreasing m_1^x , the equilibrium with $N+1$ bins induces better cost than the equilibrium with N bins.

We now generalize the result to the case of $b > 0$ where we take bin edges in an ascending order, i.e., $m_1^x < \dots < m_N^x$. We know that $m_1^A(N+1) < m_1^A(N)$ in this case, which can be proven in a similar manner to the case of $b < 0$. Due to the monotonicity of the subsequent bin edges with respect to the first bin edge, we know that $m_i^x \rightarrow m_i^A(N)$ for $i = 2, \dots, N-1$ and $m_N^x \rightarrow \infty$ as m_1^x approaches $m_1^A(N)$ from the left. Therefore, by increasing m_1^x from $m_1^A(N+1)$ to $m_1^A(N)$, one can obtain the unique equilibrium with N bins starting from the unique equilibrium with $N+1$ bins.

If we write the objective function of the decoder as a function of $m_1^x = x$, we

get

$$\begin{aligned}
J^d(x) &= \int_{-\infty}^{m_1^x} (u_1^x - m)^2 f(m) dm \\
&\quad + \sum_{i=1}^{N-1} \int_{m_i^x}^{m_{i+1}^x} (u_{i+1}^x - m)^2 f(m) dm \\
&\quad + \int_{m_N^x}^{\infty} (u_{N+1}^x - m)^2 f(m) dm.
\end{aligned}$$

Taking the derivative with respect to x yields

$$\frac{dJ^d(x)}{dx} = \sum_{i=1}^N \frac{dm_i^x}{dx} f(m_i^x) \left((u_i^x - m_i^x)^2 - (u_{i+1}^x - m_i^x)^2 \right).$$

Since the nearest neighbor conditions at $m_1^x, \dots, m_{N-1}^x$ are satisfied, we have that

$$(u_i^x - m_i^x)^2 - (u_{i+1}^x - m_i^x)^2 = (u_{i+1}^x - u_i^x) 2b > 0 \quad (2.42)$$

for $i = 1, \dots, N-1$. Moreover, when $m_1^x = m_1^A(N+1)$, we have

$$(u_N^x - m_N^x)^2 - (u_{N+1}^x - m_N^x)^2 = (u_{N+1}^x - u_N^x) 2b > 0.$$

As m_1^x increases, the term $(u_{N+1}^x - m_N^x)^2 = (\mathbb{E}[M|m_N^x < M] - m_N^x)^2$ decreases by Lemma 2.8.2. On the other hand, $(u_N^x - m_N^x)^2$ increases as m_1^x increases since $\frac{dm_N^x}{du_N^x} > 1$ by Lemma 2.8.2. These ensure that when m_1^x is decreased, we still have

$$(u_{N+1}^x - m_N^x)^2 - (u_N^x - m_N^x)^2 > 0. \quad (2.43)$$

As a result, we get $\frac{dJ^d(x)}{dx} > 0$ since $f(m_i^x)$, $\frac{dm_i^x}{dx}$ and $((u_i^x - m_i^x)^2 - (u_{i+1}^x - m_i^x)^2)$ are all positive. This shows that the cost monotonically increases as m_1^x increases. Since we obtain the equilibrium with N bins by increasing $x = m_1^x$, the equilibrium with $N+1$ bins induces better cost than the equilibrium with N bins.

2.8.5 Proof of Theorem 2.6.6

Given an initial set of bin edges $\mathbf{m} = [m_1, \dots, m_{N-1}]^T$ with $m_1 < \dots < m_{N-1}$, a modified Lloyd-Max iteration defined in (2.7) is expressed as

$$\begin{aligned} T_1(\mathbf{m}) &= \frac{\mathbb{E}[M|M < m_1] + \mathbb{E}[M|m_1 \leq M < m_2]}{2} + b \\ T_2(\mathbf{m}) &= \frac{\mathbb{E}[M|m_1 \leq M < m_2] + \mathbb{E}[M|m_2 \leq M < m_3]}{2} + b \\ &\vdots \\ T_{N-1}(\mathbf{m}) &= \frac{\mathbb{E}[M|m_{N-2} \leq M < m_{N-1}] + \mathbb{E}[M|m_{N-1} \leq M]}{2} + b. \end{aligned}$$

Notice that as a result of an iteration it is guaranteed that the resulting bin edges maintain the order, i.e., $T_1(\mathbf{m}) < \dots < T_{N-1}(\mathbf{m})$. Thus, after each iteration, the corresponding edges form a valid partition. Let $\|\mathbf{x}\| = \max_i |x_i|$. Let $\mathbf{m}$ and $\tilde{\mathbf{m}}$ be two set of bin edges, which do not necessarily lead to an equilibrium, with $m_1 < \dots < m_{N-1}$ and $\tilde{m}_1 < \dots < \tilde{m}_{N-1}$. Observe that

$$\begin{aligned} \|T(\mathbf{m}) - T(\tilde{\mathbf{m}})\| &= \|(T(\mathbf{m}) - b) - (T(\tilde{\mathbf{m}}) - b)\| \\ &< \|\mathbf{m} - \tilde{\mathbf{m}}\| \end{aligned} \tag{2.44}$$

where the inequality follows from [58, Lemma 6] since $(T(\mathbf{m}) - b)$ may be viewed as a fixed point iteration considering the team theoretic setup with no bias. Notice that the inequality in (2.44) is strict as the source distribution is assumed to be strictly log-concave. We also know that there exists a unique fixed point of $T(\cdot)$ from Theorem 2.6.1. Let $\mathbf{m}^*$ denote the unique fixed point. By utilizing (2.44), we get

$$\|T(\mathbf{m}) - \mathbf{m}^*\| = \|T(\mathbf{m}) - T(\mathbf{m}^*)\| < \|\mathbf{m} - \mathbf{m}^*\|,$$

which implies that $\|T^n(\mathbf{m}) - \mathbf{m}^*\|$ is a non-negative and monotonically decreasing sequence. Hence, the limit $\lim_{n \rightarrow \infty} \|T^n(\mathbf{m}) - \mathbf{m}^*\|$ exists. In the following, we show that this limit is zero. To prove this, suppose otherwise, that is, $\lim_{n \rightarrow \infty} \|T^n(\mathbf{m}) - \mathbf{m}^*\| = c$ for some $c > 0$. Since $\{T^n(\mathbf{m})\}$ is bounded, there exists a convergent subsequence, i.e., $T^{n_k}(\mathbf{m}) \rightarrow \tilde{\mathbf{m}}$ as $n_k \rightarrow \infty$ where $\tilde{\mathbf{m}}$ denotes

the limit point. Now, observe that

$$\begin{aligned}
c &= \lim_{n \rightarrow \infty} \|T^n(\mathbf{m}) - \mathbf{m}^*\| \\
&= \lim_{n_k \rightarrow \infty} \|T^{n_k}(\mathbf{m}) - \mathbf{m}^*\| \\
&= \left\| \lim_{n_k \rightarrow \infty} T^{n_k}(\mathbf{m}) - \mathbf{m}^* \right\| \\
&= \|\tilde{\mathbf{m}} - \mathbf{m}^*\|,
\end{aligned}$$

where the third equality is due to the continuity of the norm. By using (2.44), we get $\|T(\tilde{\mathbf{m}}) - \mathbf{m}^*\| < \|\tilde{\mathbf{m}} - \mathbf{m}^*\| = c$. This contradicts with the assumption that the sequence $\|T^n(\mathbf{m}) - \mathbf{m}^*\|$ converges to $c > 0$ as a monotonically decreasing sequence. Thus, $\|T^n(\mathbf{m}) - \mathbf{m}^*\|$ converges to zero as n goes to infinity, which implies that the fixed point iterations converge to the unique fixed point.

Chapter 3

Information Privacy as a Signaling Game and the Information Bottleneck Problem

3.1 Motivation

Various applications such as social networks, networked control, smart grid and crowd sensing benefit from data collected from decision makers. In these applications, users share information with a service provider which wishes to improve the quality of service by utilizing information gathered from the users. The users as well are interested in enhanced service quality as they benefit from the service while at the same time they wish to retain a certain level of privacy. The privacy objective arises from the fact that the information they wish to convey to the service provider may be correlated with certain private information they want to protect. For instance, in smart grid applications, power usage information shared by the users with the service provider may disclose some information related to users such as their habits and behaviors [69, 70]. For that reason, privacy is a major challenge in smart grid applications and this is a current research topic in numerous studies (see [70–73] and references therein). In addition to smart

grid applications, the problem of preserving privacy while maintaining reasonable system performance appears many different contexts, see e.g., [74–94].

In this chapter, we consider the following communication scenario between a sender and a receiver motivated by the aforementioned applications. There is a pair of messages at the sender and the perspective of the sender is such that one message needs to be protected and the other message needs to be conveyed. As opposed to the sender, the receiver desires to accurately estimate both messages with the aim of acquiring as much information as possible. Under this setting, we investigate the interactions between the sender and the receiver whose objectives are different from each other due to the privacy concerns of the transmitter.

In addition, we provide an estimation theoretic perspective to the information bottleneck problem [95], which has gained a significant attention in the recent literature [96–104]. In the era of big data, it is of paramount importance to represent data efficiently. The information bottleneck problem investigates efficient representation of data using tools from information theory. Here, the aim is to compress an observed parameter while preserving information related to another correlated parameter which is not observed. The principle of efficiently representing useful information has applications in machine learning whose performance improve with more useful data. Therefore, various machine learning applications of the information bottleneck technique such as clustering and deep learning are studied in the literature [98, 102, 105, 106].

3.2 Literature Review

In this chapter, we introduce a privacy-signaling game problem under mean squared error cost criterion. In the literature, several studies consider the strategic information transmission problem in which the encoder takes privacy of certain information into account by employing a suitable privacy measure, under either the Nash or Stackelberg criteria [107–110]. In these studies, a common theme

is to model private and nonprivate parameters as jointly Gaussian random variables. In [107], a communication scenario between an encoder and a decoder is investigated using the Stackelberg equilibrium concept in which an additional side information is assumed to be available at the decoder. The estimation errors are measured using quadratic costs and a family of Stackelberg equilibria is characterized under an *a priori* affine policy assumption. In contrast, here, we do not restrict the policies to be affine *a priori* and we consider a setting with no side information. We investigate Nash equilibria as well and show that a payoff dominant equilibrium is attained by linear policies. We also show that these linear policies at the payoff dominant Nash equilibria lead to a Stackelberg equilibrium even when the encoding policy is not restricted to be linear. The work in [108] also investigates a Stackelberg game where the utility measure for the nonprivate parameter is quadratic and the privacy measure is entropy based. Both noiseless and noisy communication scenarios are considered and essentially unique linear encoding and decoding policies that form a Stackelberg equilibrium are characterized. In [109], a Nash game is studied where the privacy measure is based on mutual information and the utility measure for the nonprivate parameter is quadratic. In [109], apart from the previously described Gaussian scenario, another scenario in which private and nonprivate data are treated as discrete random variables is considered.

In the economics literature, an important related work [111] considers a multi-dimensional signaling scenario under a Stackelberg game setup where the sender employs a general quadratic cost structure. An upper bound on the performance of the encoder is obtained via formulating a semidefinite program. For jointly Gaussian sources, a linear policy that achieves this upper bound is characterized which shows the optimality of linear policies for such a Stackelberg game. We use this characterization in some of our results in this chapter rather prominently.

In the literature, parameter privacy is investigated in a multitude of studies as it is desired to protect sensitive information from third parties in wide range of applications. Our work on privacy-signaling game can also be viewed as a setup

where the encoder makes a trade off between privacy for protecting sensitive information and utility for conveying intended information. The trade off between utility and privacy appears also in various other contexts [76, 77, 79–94, 97, 112, 113]. One line of related work that attracted significant attention in the current literature is the differential privacy problem where the aim is to protect private information on publicly available databases [76, 77]. The notion of differential privacy ensures that private information provided by an individual to a database is not compromised by a third party, e.g., data analyst, who retrieves information from this database. In this context, an interesting result from [91] is the application of the Laplacian or Gaussian perturbations to guarantee differential privacy. For a comprehensive treatment of differential privacy on such problems as filtering and estimation, please see [114]. Another line of work is the privacy funnel problem [112] where it is desired to convey as much information as possible related to an observed random variable while trying to leak as low information as possible related to an unobserved private parameter. It should be noted that in the privacy funnel problem, only the nonprivate parameter is observed at the sender whereas in our framework, we assume that both the nonprivate and private parameters are observed at the sender. Another related work is [80] where the trade off between utility and privacy is investigated through formulating constrained optimization problems that consider settings with a discrete parameter and a continuous parameter. The continuous parameter case focuses on Gaussian perturbations applied to the nonprivate parameter to protect private information.

A further related problem for our privacy-signaling game formulation is the information bottleneck problem [95, 96, 98–102, 106, 115] which also has connections with the privacy funnel problem [97]. In the information bottleneck technique [95], the aim is to compress an observed random variable while trying to preserve information related to another correlated random variable which is not observed. It is important to note that the information bottleneck technique is closely related to an earlier seminal work [116] which considers a similar constrained optimization problem by employing conditional entropy to assess the performance. The information bottleneck problem specializing to Gaussian sources

is investigated in [117] where the parameters of interest are jointly Gaussian random vectors. The compression objective in the information bottleneck problem can also be viewed as a privacy objective as in our framework in the sense that the corresponding information is desired to be removed from the revealed message. In the information bottleneck problem, the costs involve mutual information and only one of the parameters is received at the sender whereas in our framework the costs include mean squared error terms and both of the parameters are observed at the sender. In order to provide an estimation theoretic perspective on the information bottleneck problem, we formulate a similar problem where we use mean squared error terms for the costs as in our original setting and we show that there are operational and consequential differences when the encoder is allowed to use both of the hidden variables.

3.3 Contributions

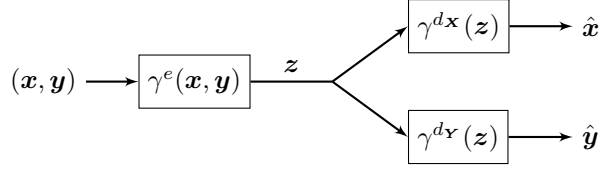
The main aim of this chapter is to provide both Nash and Stackelberg equilibria analyses for the considered privacy-signaling game problem. In game theory, since the solution concept involves an equilibrium (Nash, Stackelberg, and refinements), one cannot talk about an optimal equilibrium in general. Nonetheless, as a main contribution of our work, we establish and compute an equilibrium, which is desirable among all, for both of the players. In addition, we introduce an estimation theoretic formulation of information bottleneck problem for Gaussian sources where we use mean squared error instead of mutual information as our metric. The main contributions of this chapter can be summarized as follows:

- In the literature, a Nash equilibrium analysis of the privacy game problem, in which both the privacy and the utility (for the nonprivate parameter) are measured via the mean squared error cost, has not been available. In this chapter, we consider this problem for the first time in the literature to our knowledge. More importantly, we show that a payoff dominant Nash equilibrium is attained by linear policies in Theorem 3.5.1. These equilibria are

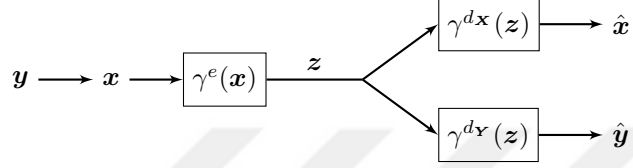
the most desirable equilibria for both of the players among any set of policies. We show that the characterized linear payoff dominant Nash equilibria coincide with the Stackelberg equilibria in Theorem 3.6.1. It should be emphasized that these (Stackelberg) equilibria are obtained without an *a priori* affine policy restriction for the players. In other words, if we consider the optimization problem that the encoder needs to solve while obtaining the Stackelberg equilibria, these linear policies are the optimal solution among any sets of policies.

- We introduce a minimum mean squared error (MMSE) Gaussian information bottleneck problem, which is a modification of the classical information bottleneck problem that has been considered under mutual information criteria. By viewing this as an instance of the privacy-signaling game under the Stackelberg formulation, we show that the solution to the MMSE Gaussian information bottleneck problem is attained by a set of explicitly characterized linear policies in Theorem 3.7.1. Namely, even when the policies are allowed to be nonlinear, a set of linear policies arises as the optimal solution.
- We extend our results for scalar sources to the additive Gaussian noise channel setting. Under this setting, it is shown that a payoff dominant Nash equilibrium is attained by linear policies in Theorem 3.8.1. This theorem also establishes that the characterized linear Nash equilibrium is unique among the affine class. We also show that the payoff dominant Nash equilibrium coincides with the Stackelberg equilibrium in Theorem 3.8.2. In addition, the characterized linear Stackelberg equilibrium is unique among any set of policies. We also establish the existence of nonlinear Nash and Stackelberg equilibria considering a discrete channel setting in which the encoding function is restricted to take discrete values in Theorem 3.8.3 and Theorem 3.8.4, respectively.

The remainder of this chapter is organized as follows: Section 3.4 describes the problem. Section 3.5 and Section 3.6 provide, respectively, the Nash and Stackelberg equilibria analyses for the considered privacy-signaling game. Section 3.7



(a) Privacy-signaling game.



(b) Information bottleneck.

Figure 3.1: Information flow under the quadratic privacy-signaling game and the information bottleneck problems.

investigates the information bottleneck problem as an instance of our proposed framework. Section 3.8.1 and Section 3.8.2 extend the results for scalar sources to the Gaussian noise channel and discrete channel, respectively. Section 3.9 provides numerical examples, and Section 3.10 concludes the chapter with some final remarks.

3.4 Problem Formulation

Consider an information transmission scenario in which a sender encodes a pair of correlated random vectors $\mathbf{X}$ and $\mathbf{Y}$ into $\mathbf{Z}$ using an encoding function denoted by $\mathbf{z} = \gamma^e(\mathbf{x}, \mathbf{y})$ and a receiver wants to decode both of the random variables based on its observation $\mathbf{Z} = \mathbf{z}$. We denote the size of the random vectors $\mathbf{X}$ and $\mathbf{Y}$ by $n_{\mathbf{X}}$ and $n_{\mathbf{Y}}$, respectively. In this communication scenario, the sender wishes to convey information contained in $\mathbf{Y}$ whereas it views $\mathbf{X}$ as a private parameter that needs to be hidden from the receiver. The aim of the receiver is to accurately estimate both of the parameters given its observation $\mathbf{Z} = \mathbf{z}$. Let the decoders for estimating $\mathbf{X}$ and $\mathbf{Y}$ at the receiver be denoted by $\gamma^{d\mathbf{x}}(\mathbf{z})$ and $\gamma^{d\mathbf{y}}(\mathbf{z})$, respectively. Figure 3.1 (a) illustrates the considered information transmission scenario.

We model the parameters $\mathbf{X}$ and $\mathbf{Y}$ as jointly Gaussian random vectors. Let $\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$ be a zero mean Gaussian random vector with a positive definite covariance

matrix $\Sigma \triangleq \begin{bmatrix} \Sigma_{\mathbf{X}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{Y}} \end{bmatrix}$. The parameters $\mathbf{X}$ and $\mathbf{Y}$ are not independent of each other, i.e., $\Sigma_{\mathbf{XY}} \neq \mathbf{0}$. It is assumed that the joint distribution of $\mathbf{X}$ and $\mathbf{Y}$ is common knowledge, i.e., both players know $\Sigma_{\mathbf{X}}$, $\Sigma_{\mathbf{Y}}$ and $\Sigma_{\mathbf{XY}}$. Since $\mathbf{X}$ and $\mathbf{Y}$ are correlated, transmitting $\mathbf{Y}$ directly discloses information related to the private parameter $\mathbf{X}$. In other words, the objectives of hiding $\mathbf{X}$ and conveying $\mathbf{Y}$ are conflicting. These conflicting objectives at the transmitter are modeled via the following objective function:

$$J^e(\gamma^e, \gamma^{d\mathbf{X}}, \gamma^{d\mathbf{Y}}) = \mathbb{E}[\|\mathbf{Y} - \gamma^{d\mathbf{Y}}(\mathbf{Z})\|^2] - \delta \mathbb{E}[\|\mathbf{X} - \gamma^{d\mathbf{X}}(\mathbf{Z})\|^2] \quad (3.1)$$

which is to be minimized, where δ is a positive design parameter that determines the level of desired privacy in terms of hiding $\mathbf{X}$. On the other hand, the receiver aims to extract both of the parameters. Thus, the receiver wishes to minimize the following objective function:

$$J^d(\gamma^e, \gamma^{d\mathbf{X}}, \gamma^{d\mathbf{Y}}) = \mathbb{E}[\|\mathbf{Y} - \gamma^{d\mathbf{Y}}(\mathbf{Z})\|^2] + \mathbb{E}[\|\mathbf{X} - \gamma^{d\mathbf{X}}(\mathbf{Z})\|^2] \quad (3.2)$$

In (3.2), the mean squared errors corresponding to parameters $\mathbf{X}$ and $\mathbf{Y}$ are incorporated into the objective function with equal weights since taking different weights does not alter the problem. In this work, we investigate the Nash and the Stackelberg equilibria for the described strategic information transmission scenario in which the objectives of the sender and the receiver are as defined above.

At a Nash equilibrium, each player chooses their strategies at the same time and no player gains by unilaterally deviating from its strategy. Suppose that the set of possible strategies for the encoder is denoted by Γ^e , i.e., $\gamma^e \in \Gamma^e$, and those for the decoders of each parameter are denoted by $\Gamma^{d\mathbf{X}}$ and $\Gamma^{d\mathbf{Y}}$, i.e., $\gamma^{d\mathbf{X}} \in \Gamma^{d\mathbf{X}}$ and $\gamma^{d\mathbf{Y}} \in \Gamma^{d\mathbf{Y}}$. A set of policies $\gamma^{*,e}$, $\gamma^{*,d\mathbf{X}}$ and $\gamma^{*,d\mathbf{Y}}$ forms a Nash

equilibrium [13] if

$$J^e(\gamma^{*,e}, \gamma^{*,d\mathbf{X}}, \gamma^{*,d\mathbf{Y}}) \leq J^e(\gamma^e, \gamma^{*,d\mathbf{X}}, \gamma^{*,d\mathbf{Y}}) \quad (3.3)$$

for all $\gamma^e \in \Gamma^e$ and

$$J^d(\gamma^{*,e}, \gamma^{*,d\mathbf{X}}, \gamma^{*,d\mathbf{Y}}) \leq J^d(\gamma^{*,e}, \gamma^{d\mathbf{X}}, \gamma^{d\mathbf{Y}}) \quad (3.4)$$

for all $\gamma^{d\mathbf{X}} \in \Gamma^{d\mathbf{X}}$ and $\gamma^{d\mathbf{Y}} \in \Gamma^{d\mathbf{Y}}$.

On the other hand, at a Stackelberg equilibrium, the sender first decides on a policy and announces this policy, and the receiver acts given the announcement of the sender. Under this setting, the sender is committed to employ this announced policy. A set of policies $\gamma^{e,*}$, $\gamma^{*,d\mathbf{X}}$ and $\gamma^{*,d\mathbf{Y}}$ forms a Stackelberg equilibrium [13] if

$$J^e(\gamma^{*,e}, \gamma^{*,d\mathbf{X}}(\gamma^{*,e}), \gamma^{*,d\mathbf{Y}}(\gamma^{*,e})) \leq J^e(\gamma^e, \gamma^{*,d\mathbf{X}}(\gamma^e), \gamma^{*,d\mathbf{Y}}(\gamma^e)) \quad (3.5)$$

for all $\gamma^e \in \Gamma^e$, where $\gamma^{*,d\mathbf{X}}(\gamma^e)$ and $\gamma^{*,d\mathbf{Y}}(\gamma^e)$ are such that

$$J^d(\gamma^e, \gamma^{*,d\mathbf{X}}(\gamma^e), \gamma^{*,d\mathbf{Y}}(\gamma^e)) \leq J^d(\gamma^e, \gamma^{d\mathbf{X}}(\gamma^e), \gamma^{d\mathbf{Y}}(\gamma^e)) \quad (3.6)$$

for all $\gamma^{d\mathbf{X}} \in \Gamma^{d\mathbf{X}}$ and $\gamma^{d\mathbf{Y}} \in \Gamma^{d\mathbf{Y}}$.

We also consider an instance of the problem above as the *MMSE Gaussian Information Bottleneck Problem*. The difference with the setup above is that the sender only has access to $\mathbf{X}$, which is the message it intends to hide while revealing as much information on $\mathbf{Y}$ as possible. This is depicted in Fig. 3.1b. Details are provided in Section 3.7.

3.5 Nash Equilibria

In this section, we characterize linear Nash equilibria of the considered privacy-signaling game. More importantly, we show that special cases of these equilibria lead to payoff dominant Nash equilibria. These payoff dominant Nash equilibria are the most desirable equilibria for both of the players (among all

coding/decoding policies, including those that are nonlinear) in a sense that is made explicit in the following definition.

Definition 3.5.1. A Nash equilibrium that is not Pareto dominated¹ by any other Nash equilibrium of the game is said to be a payoff dominant Nash equilibrium [42] .

In order to characterize linear Nash equilibria, we propose an equivalent formulation by applying an invertible linear transformation of variables from Tamura [111]. We note that [111] considers a general multidimensional signaling setup under quadratic costs and characterizes a set of linear policies that forms a Stackelberg equilibrium for jointly Gaussian sources. We use this characterization for our special case of privacy-signaling game scenario to formulate an equivalent problem and this approach facilitates our Nash equilibrium analysis.

Theorem 3.5.1. (i) *There exist informative linear Nash equilibria with an encoding policy*

$$\mathbf{z} = \begin{bmatrix} \alpha_1 \mathbf{q}_1 & \cdots & \alpha_{n_Y} \mathbf{q}_{n_Y} \end{bmatrix}^T \Sigma^{-1/2} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \quad (3.7)$$

for scalars $\{\alpha_i\}_{i=1}^{n_Y}$ with at least one of these scalars being nonzero² where $\{\mathbf{q}_i\}_{i=1}^n$ are normalized eigenvectors of $W = \Sigma^{1/2} \text{diag}(-\delta I, I) \Sigma^{1/2}$ with $n \triangleq n_X + n_Y$ and these eigenvectors are arranged in such a way that the corresponding eigenvalues $\{\lambda_i\}_{i=1}^n$ satisfy $\lambda_i > 0$ for $i = 1, \dots, n_Y$ and $\lambda_i < 0$ for $i = n_Y + 1, \dots, n$. The corresponding decoding policy is given by

$$\begin{bmatrix} \gamma^{d_X}(\mathbf{z}) \\ \gamma^{d_Y}(\mathbf{z}) \end{bmatrix} = \Sigma^{1/2} \begin{bmatrix} \beta_1 \mathbf{q}_1 & \cdots & \beta_{n_Y} \mathbf{q}_{n_Y} \end{bmatrix} \mathbf{z} \quad (3.8)$$

where $\beta_i = 1/\alpha_i$ if $\alpha_i \neq 0$ and $\beta_i = 0$ otherwise for $i = 1, \dots, n_Y$. These Nash equilibria exist for any set of scalars $\{\alpha_i\}_{i=1}^{n_Y}$. When the indices with

¹A set of policies $\gamma^e(\cdot, \cdot)$, $\gamma^{d_X}(\cdot)$ and $\gamma^{d_Y}(\cdot)$ Pareto dominates another set of policies $\tilde{\gamma}^e(\cdot, \cdot)$, $\tilde{\gamma}^{d_X}(\cdot)$ and $\tilde{\gamma}^{d_Y}(\cdot)$ if $J^e(\gamma^e, \gamma^{d_X}, \gamma^{d_Y}) \leq J^e(\tilde{\gamma}^e, \tilde{\gamma}^{d_X}, \tilde{\gamma}^{d_Y})$, $J^d(\gamma^e, \gamma^{d_X}, \gamma^{d_Y}) \leq J^d(\tilde{\gamma}^e, \tilde{\gamma}^{d_X}, \tilde{\gamma}^{d_Y})$ and at least one of these inequalities is strict.

²The case when $\alpha_i = 0$ for all $i = 1, \dots, n_Y$ leads to a noninformative Nash equilibrium.

$\alpha_i \neq 0$ are the same for two sets of scalars, they lead to the same performance values, i.e., the resulting equilibria with these sets of scalars are informationally equivalent.

(ii) These informative equilibria are payoff dominant Nash equilibria if $\alpha_i \neq 0$ for all $i = 1, \dots, n_Y$.

(iii) In addition to informative equilibria, there always exist noninformative Nash equilibria with the transmitted message being independent of the parameters, e.g., $\gamma^e(\mathbf{x}, \mathbf{y}) = C$ for some constant C , and with the decoding policies $\gamma^{d\mathbf{x}}(\mathbf{z}) = \mathbf{0}$ and $\gamma^{d\mathbf{y}}(\mathbf{z}) = \mathbf{0}$.

Proof. We apply a linear transformation of variables that gives an orthonormal coordinate system and then show that in this transformed coordinate system the sender wishes to convey some of the coordinates and to hide the remaining coordinates. The advantage of this transformation is that as these coordinates are orthogonal to each other, conveying one coordinate does not give information related to other coordinates.

Since $W = \Sigma^{1/2} \text{diag}(-\delta I, I) \Sigma^{1/2}$ is symmetric, we can decompose it as $W = Q\Lambda Q^T$ for orthonormal Q and diagonal Λ . We denote the columns of Q by $\{\mathbf{q}_i\}_{i=1}^n$ and the diagonal elements of Λ by $\{\lambda_i\}_{i=1}^n$. By Sylvester's law of inertia, W and $\text{diag}(-\delta I, I)$ have the same number of positive and negative eigenvalues [118, p. 282]. Therefore, W has n_Y positive and n_X negative eigenvalues. We sort these eigenvalues $\{\lambda_i\}_{i=1}^n$ in such a way that the first n_Y of them are positive and the remaining ones are negative. For notational convenience, we define

$$\mathbf{S} \triangleq \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \gamma^{d\mathbf{S}}(\mathbf{z}) \triangleq \begin{bmatrix} \gamma^{d\mathbf{x}}(\mathbf{z}) \\ \gamma^{d\mathbf{y}}(\mathbf{z}) \end{bmatrix}. \quad (3.9)$$

Now, we make the following transformation of variables:

$$\mathbf{T} \triangleq Q^T \Sigma^{-1/2} \mathbf{S}, \quad (3.10)$$

where $\Sigma^{-1/2}$ is well-defined since Σ is assumed to be positive definite. We note

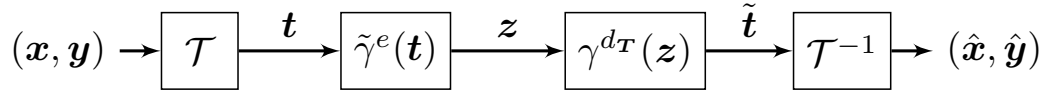


Figure 3.2: Equivalent formulation where $\mathcal{T}$ denotes the linear transformation specified in (3.10).

that

$$\begin{aligned}
\mathbb{E}[\mathbf{T}\mathbf{T}^T] &= \mathbb{E}[Q^T \Sigma^{-1/2} \mathbf{S} \mathbf{S}^T \Sigma^{-1/2} Q] \\
&= Q^T \Sigma^{-1/2} \mathbb{E}[\mathbf{S} \mathbf{S}^T] \Sigma^{-1/2} Q \\
&= Q^T \Sigma^{-1/2} \Sigma \Sigma^{-1/2} Q = I.
\end{aligned}$$

Thus, each components of $\mathbf{T}$ are independent and identically distributed zero-mean Gaussian random variables each with a unit variance.

Next, we propose an equivalent problem under this linear transformation of variables. The encoder consists of two consecutive mappings, one of which is fixed as above and the other one can arbitrarily be chosen by the sender. In other words, there is a linear mapping from $(\mathbf{x}, \mathbf{y})$ to $\mathbf{t}$ fixed as (3.10) and then an encoding function $\mathbf{z} = \tilde{\gamma}^e(\mathbf{t})$. At the receiver side, we also consider a similar setting. The observation at the receiver is mapped into $\tilde{\mathbf{t}}$ via $\gamma^{d_T}(\mathbf{z})$, which can arbitrarily be selected by the receiver. Then, these auxiliary variables are mapped into estimates of $\mathbf{x}$ and $\mathbf{y}$ as follows:

$$\gamma^{d_S}(\mathbf{z}) = \Sigma^{1/2} Q \gamma^{d_T}(\mathbf{z}). \quad (3.11)$$

Fig. 3.2 provides an illustration for the equivalent formulation. The aim is to characterize $\tilde{\gamma}^e(\mathbf{t})$ as well as $\gamma^{d_T}(\mathbf{z})$ at a Nash equilibrium. Since the proposed transformation is invertible, the problem in the transform domain is equivalent to the original problem.

Next, we express the objective function of each player in terms of the parameters in the transformed coordinate system. The objective function of the sender

can be written as

$$\begin{aligned}
J^e(\tilde{\gamma}^e, \gamma^{d\mathbf{T}}) &= \mathbb{E}[(\mathbf{S} - \gamma^{d\mathbf{S}}(\mathbf{Z}))^T \text{diag}(-\delta I, I)(\mathbf{S} - \gamma^{d\mathbf{S}}(\mathbf{Z}))] \\
&= \mathbb{E}[(\mathbf{T} - \gamma^{d\mathbf{T}}(\mathbf{Z}))^T Q^T \Sigma^{1/2} \text{diag}(-\delta I, I) \Sigma^{1/2} Q(\mathbf{T} - \gamma^{d\mathbf{T}}(\mathbf{Z}))] \\
&= \mathbb{E}[(\mathbf{T} - \gamma^{d\mathbf{T}}(\mathbf{Z}))^T \Lambda (\mathbf{T} - \gamma^{d\mathbf{T}}(\mathbf{Z}))] \\
&= \sum_{i=1}^n \lambda_i \mathbb{E}[(T_i - \gamma^{dT_i}(\mathbf{Z}))^2],
\end{aligned} \tag{3.12}$$

where the first equation is obtained from (3.1) and (3.9), the second equation is based on (3.10) and (3.11), and $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of W which satisfy $\lambda_i > 0$ for $i = 1, \dots, n_{\mathbf{Y}}$ and $\lambda_i < 0$ for $i = n_{\mathbf{Y}} + 1, \dots, n$. Similarly, if we express the objective function of the receiver in terms of the parameters in the introduced coordinate system, we get

$$\begin{aligned}
J^d(\tilde{\gamma}^e, \gamma^{d\mathbf{T}}) &= \mathbb{E}[(\mathbf{S} - \gamma^{d\mathbf{S}}(\mathbf{Z}))^T (\mathbf{S} - \gamma^{d\mathbf{S}}(\mathbf{Z}))] \\
&= \mathbb{E}[(\mathbf{T} - \gamma^{d\mathbf{T}}(\mathbf{Z}))^T K (\mathbf{T} - \gamma^{d\mathbf{T}}(\mathbf{Z}))]
\end{aligned} \tag{3.13}$$

where $K \triangleq Q^T \Sigma Q$ is a positive definite matrix since Σ is positive definite. In the proposed equivalent problem, the optimal $\gamma^{d\mathbf{T}}(\mathbf{z})$ for a given encoding policy $\tilde{\gamma}^e(\mathbf{x}, \mathbf{y})$ is the minimum mean squared error estimator of the corresponding parameter, as stated below.

Lemma 3.5.1. *For a fixed encoding function $\tilde{\gamma}^e(\mathbf{x}, \mathbf{y})$, the optimal $\gamma^{d\mathbf{T}}(\mathbf{z})$ that minimizes (3.13) is given by $\mathbb{E}[\mathbf{T}|\mathbf{Z} = \mathbf{z}]$.*

Proof. The result is standard but for completeness we present a short proof. Suppose that $\gamma^{d\mathbf{T}}(\mathbf{z}) = \mathbb{E}[\mathbf{T}|\mathbf{Z} = \mathbf{z}] + g(\mathbf{z})$. Inserting this expressions into the objective function of the receiver in (3.13), we get

$$\begin{aligned}
J^d(g) &= \mathbb{E}[(\mathbf{T} - \mathbb{E}[\mathbf{T}|\mathbf{Z}] - g(\mathbf{Z}))^T K (\mathbf{T} - \mathbb{E}[\mathbf{T}|\mathbf{Z}] - g(\mathbf{Z}))] \\
&= \mathbb{E}[(\mathbf{T} - \mathbb{E}[\mathbf{T}|\mathbf{Z}])^T K (\mathbf{T} - \mathbb{E}[\mathbf{T}|\mathbf{Z}])] + \mathbb{E}[g(\mathbf{Z})^T K g(\mathbf{Z})] \\
&\geq \mathbb{E}[(\mathbf{T} - \mathbb{E}[\mathbf{T}|\mathbf{Z}])^T K (\mathbf{T} - \mathbb{E}[\mathbf{T}|\mathbf{Z}])]
\end{aligned}$$

where the inequality follows from K being positive definite. This proves the optimality of the minimum mean squared error estimator in the transformed coordinate system for a given encoding policy. $\square$

In the equivalent formulation, the objective function of the sender is expressed as a weighted sum of mean squared error terms corresponding to independent random variables where there are both positive and negative weights. We partition the transformed random vector as

$$\mathbf{T} \triangleq \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \quad (3.14)$$

where $\mathbf{U} \in \mathbb{R}^{n_Y}$ and $\mathbf{V} \in \mathbb{R}^{n_X}$ such that the positive and negative coefficients in (3.12) correspond to $\mathbf{U}$ and $\mathbf{V}$, respectively. The receiver still employs the minimum mean squared error estimators for the corresponding parameters for a given encoding policy. Next, we use the equivalent formulation to characterize the Nash equilibria. Towards that goal, we first present the following lemma.

Lemma 3.5.2. *At a Nash equilibrium, the sender does not reveal any information related to the linear combinations (of the private and nonprivate parameters) $\mathbf{V}$ specified in (3.14).*

Proof. Consider a set of policies where the sender employs an encoding policy $\tilde{\gamma}^e(\mathbf{t}) = f(\mathbf{t})$ which conveys information related to T_j for some j with $\lambda_j < 0$. In response to this encoding policy, it is optimal for the receiver to employ the minimum mean squared error estimators of each parameter, as shown in Lemma 3.5.1. Denote these estimators for estimating T_i by $g_i(\mathbf{z})$ for $i = 1, \dots, n$. Since we assume that the encoding policy $f(\mathbf{t})$ conveys information related to T_j , the mean squared error for estimating T_j with the corresponding optimal estimator $g_j(\mathbf{z})$ is lower than $\sigma_{T_j}^2$, i.e., $\mathbb{E}[(T_j - g_j(\mathbf{Z}))^2] < \sigma_{T_j}^2$. In response to the decoding policies of $\{g_i(\cdot)\}_{i=1}^n$, the sender can switch to the following policy to improve its objective value. Instead of sending $\mathbf{z} = f(\mathbf{t})$, the sender can transmit $\mathbf{z} = f(\bar{\mathbf{t}})$ while keeping the encoding function $f(\cdot)$ the same where $\bar{\mathbf{t}} \triangleq [t_1, \dots, t_{j-1}, w, t_{j+1}, \dots, t_n]^T$ and w is a realization of a random variable that follows the same distribution as T_j and is independent of $\mathbf{T}$. In that case, the performance for estimating T_i for $i \neq j$ remains the same since receiving $f(\bar{\mathbf{t}})$ or $f(\mathbf{t})$ are equivalent for the decoding policy $g_i(\mathbf{z})$. However, the performance for

estimating T_j degrades as shown in the following:

$$\begin{aligned}\mathbb{E}[(T_j - g_j(\mathbf{Z}))^2] &= \mathbb{E}[T_j^2] - 2\mathbb{E}[T_j g_j(\mathbf{Z})] + \mathbb{E}[g_j(\mathbf{Z})^2] \\ &= \mathbb{E}[T_j^2] - 2\mathbb{E}[T_j]\mathbb{E}[g_j(\mathbf{Z})] + \mathbb{E}[g_j(\mathbf{Z})^2] \\ &= \mathbb{E}[T_j^2] + \mathbb{E}[g_j(\mathbf{Z})^2] \geq \sigma_{T_j}^2,\end{aligned}$$

where the second equality is due to the fact that T_j and $\mathbf{Z}$ are independent in case $f(\bar{\mathbf{t}})$ is transmitted. Since the parameter T_j is chosen such that $\lambda_j < 0$ in (3.12), the sender gains by employing $\mathbf{z} = f(\bar{\mathbf{t}})$ instead of $\mathbf{z} = f(\mathbf{t})$. As a result, any encoding policy which yields $\mathbb{E}[(T_j - \gamma^{d_{T_j}}(\mathbf{Z}))^2] < \sigma_{T_j}^2$ for an index j with $\lambda_j < 0$ cannot be a Nash equilibrium since in that case the sender can change its strategy to improve its objective value.

In game theory, when a unilateral change by a decision maker occurs, the perturbed policies may cease to be an equilibrium. However, a subtle aspect of our proof is that, the revised sender policy does not alter the policy of the decoder, therefore the perturbation is still an equilibrium. $\square$

From Lemma 3.5.2, it follows that the sender is restricted to send information related to $\mathbf{U}$, which is specified in (3.14), at a Nash equilibrium. This implies that the objective function of the sender at a Nash equilibrium reduces to

$$J^e(\tilde{\gamma}^e, \gamma^{d_{\mathbf{T}}}) = \sum_{i=1}^{n_{\mathbf{Y}}} \lambda_i \mathbb{E}[(T_i - \gamma^{d_{T_i}}(\mathbf{Z}))^2] \quad (3.15)$$

where $\{1, \dots, n_{\mathbf{Y}}\}$ are the only indices with $\lambda_i > 0$ in $\{\lambda_i\}_{i=1}^n$. Since the receiver wishes to extract all the random variables in this transformed coordinate system including the ones with these indices, the receiver also shares the objective of minimizing these mean squared error terms. As a result, conveying all or a subset of the random variables in $\mathbf{U}$ yields a Nash equilibrium and this gives the linear policies stated in (3.7). In addition, conveying all of these random variables yields the minimum attainable performance at a Nash equilibrium for both of the players. Thus, if $\alpha_i \neq 0$ for all $i = 1, \dots, n_{\mathbf{Y}}$ in (3.7), the corresponding linear Nash equilibria are payoff dominant Nash equilibria. Namely, there does not exist any other Nash equilibrium which Pareto dominates these characterized equilibria. $\square$

Theorem 3.5.1 characterizes linear Nash equilibria in which there is communication between the transmitter and the receiver. Hence, the considered game always admits informative linear Nash equilibria regardless of the system parameters. More importantly, special cases of these linear equilibria leads to payoff dominant Nash equilibria, which are the most desirable equilibria for both of the players among all coding/decoding policies.

Next, we specialize to the case of scalar sources with the aim to provide an explicit characterization of the payoff dominant Nash equilibria in this case. In particular, X and Y are assumed to be zero-mean jointly Gaussian with variances σ_X^2 and σ_Y^2 , respectively and a nonzero correlation ρ , i.e., $\Sigma = \begin{bmatrix} \sigma_X^2 & \rho \\ \rho & \sigma_Y^2 \end{bmatrix}$. We present the following as a corollary of Theorem 3.5.1.

Corollary 3.5.1. *For scalar sources X and Y , there exist informative linear Nash equilibria with an encoding policy $\gamma^e(x, y) = Ax + By$ which satisfies*

$$\frac{B}{A} = -\frac{\delta\sigma_x^2 + \sigma_y^2}{2\delta\rho} - \frac{\sqrt{(\delta\sigma_x^2 + \sigma_y^2)^2 - 4\delta\rho^2}}{2\delta\rho} \quad (3.16)$$

and decoding policies

$$\gamma^{dx}(z) = \left(\frac{A\sigma_X^2 + B\rho}{A^2\sigma_X^2 + B^2\sigma_Y^2 + 2AB\rho} \right) z, \quad (3.17)$$

$$\gamma^{dy}(z) = \left(\frac{A\rho + B\sigma_Y^2}{A^2\sigma_X^2 + B^2\sigma_Y^2 + 2AB\rho} \right) z. \quad (3.18)$$

These equilibria are payoff dominant Nash equilibria. In addition, these equilibria are unique among linear policies.

Proof. In order to characterize the Nash equilibria, we need to find the eigenvalues and eigenvectors of $W = \Sigma^{1/2} \text{diag}(-\delta, 1) \Sigma^{1/2}$. We note that W has the same eigenvalues as $D \triangleq \Sigma \text{diag}(-\delta, 1)$ and these can be computed as

$$\lambda_1 = \frac{-\delta\sigma_X^2 + \sigma_Y^2}{2} + \frac{\sqrt{(\delta\sigma_X^2 + \sigma_Y^2)^2 - 4\delta\rho^2}}{2}, \quad (3.19)$$

$$\lambda_2 = \frac{-\delta\sigma_X^2 + \sigma_Y^2}{2} - \frac{\sqrt{(\delta\sigma_X^2 + \sigma_Y^2)^2 - 4\delta\rho^2}}{2}. \quad (3.20)$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus, the sender is restricted to transmit $u = \mathbf{q}_1^T \Sigma^{-1/2} \begin{bmatrix} x \\ y \end{bmatrix}$ where $\mathbf{q}_1$ is the normalized eigenvector of W associated with the eigenvalue λ_1 computed in (3.19). It is seen that $\mathbf{q}_1^T \Sigma^{-1/2}$ is a left eigenvector of D associated with its eigenvalue λ_1 . By expressing this left eigenvector, an encoding policy which satisfies (3.16) is obtained. Then, the conditional expectation formula for jointly Gaussian distributions can be employed to obtain (3.17) and (3.18) [119, p. 155]. As a result, these characterized policies lead to payoff dominant Nash equilibria. Moreover, the only possible linear equilibria are attained by transmitting a scaled version of u . $\square$

Remark 3.5.1. Note that when $\delta \rightarrow 0^+$, the encoding policy satisfies $A/B \rightarrow 0$ as can be deduced from (3.16). Therefore, $\delta \rightarrow 0^+$ implies that the encoder transmits directly Y at a Nash equilibrium. When $\delta = 0$, the sender also transmits Y directly at a Nash equilibrium since it does not have any privacy concern in this case. Hence, the Nash equilibrium specified in Corollary 3.5.1 as $\delta \rightarrow 0^+$ coincides with the Nash equilibrium when $\delta = 0$.

Remark 3.5.2. It is seen that the ratio of B and A converges to $-\sigma_x^2/\rho$ as $\delta \rightarrow \infty$, which can be verified from (3.16). This shows that in the high privacy regime, the encoder makes the revealed information Z and the private parameter X essentially uncorrelated.

3.6 Stackelberg Equilibria

In this section, we characterize the Stackelberg equilibria of the considered quadratic privacy-signaling game. Our main result is that the payoff dominant Nash equilibria characterized in the previous section are also Stackelberg equilibria. It is important to note that the set of possible encoding strategies, i.e., Γ^e , is not restricted to be affine. If the sender performs an optimization of its objective function by anticipating the best response of the receiver, these linear policies become the optimal solution among any set of policies.

We note that the Stackelberg equilibria can be obtained from the analysis presented by Tamura [111, Theorem 2] which characterizes linear policies that form a Stackelberg equilibrium for a general quadratic multidimensional signaling setup for jointly Gaussian sources with some generalizations (in the cost setup considered) but also some restrictions, such as the *a priori* restriction of the decoder to an affine class in the conditional expectation (that limits the applicability for the noisy channel setup to be considered later in the chapter). Accordingly, we consider an alternative approach where we use the equivalent formulation employed in Section 3.5.

- Theorem 3.6.1.** *(i) The encoding policies in (3.7) with $\alpha_i \neq 0$ for all $i = 1, \dots, n_{\mathbf{Y}}$ and the decoding policies in (3.8) form a Stackelberg equilibrium, i.e., a payoff dominant Nash equilibrium and a Stackelberg equilibrium coincide.*
- (ii) In contrast to Nash setup for which there exist both informative and noninformative equilibria, a Stackelberg equilibrium is always informative, where the sender uses the private and nonprivate parameters in constructing its message.*
- (iii) When the nonprivate parameter is not scalar valued, i.e., $n_{\mathbf{Y}} > 1$, there exist informative Nash equilibria which do not coincide with the Stackelberg equilibria. These Nash equilibria are attained by an encoding policy in (3.7) and a decoding policy in (3.8) where the scalars $\{\alpha_i\}_{i=1}^{n_{\mathbf{Y}}}$ can take any value with at least one zero term and one nonzero term.*

Proof. We apply a transformation of variables as in (3.10). In this transformed coordinate system, the objective functions of the sender and the receiver are expressed as in (3.12) and (3.13), respectively. If we partition the random vector $\mathbf{T}$ in this coordinate system as in (3.14) according to the sign of coefficients in (3.12), we can show that the sender can only convey information related to $\mathbf{U}$. The following lemma presents this result in a similar manner to Lemma 3.5.2 with the exception that in this case the sender announces its policy first.

Lemma 3.6.1. *At a Stackelberg equilibrium, the sender does not reveal any information related to the linear combinations (of the private and nonprivate parameters) $\mathbf{V}$ specified in (3.14).*

Proof. We show that any encoding policy which yields $\mathbb{E}[(T_j - \gamma^{d_{T_j}}(\mathbf{Z}))^2] < \sigma_{T_j}^2$ for an index j with $\lambda_j < 0$ cannot be a Stackelberg equilibrium via a similar analysis to that employed in Lemma 3.5.2. Towards that goal, we compare the performance of two scenarios from the perspective of the sender. Recall that in a Stackelberg equilibrium the sender chooses a policy and announces this policy to the receiver and the receiver acts with the knowledge of sender's policy. Denote the encoding policy by $\hat{\gamma}^e(\mathbf{t}) = f(\mathbf{t})$ in the first scenario. The receiver takes an optimal response to this announced encoding policy. Assume that $\mathbb{E}[(T_j - \gamma^{d_{T_j}}(\mathbf{Z}))^2] < \sigma_{T_j}^2$ with the corresponding set of policies. In the second scenario, suppose that the encoder chooses the same policy as before with the exception that the sender replaces the realization $T_j = t_j$ by an independent noise following the same distribution as T_j . Namely, the sender uses $f(\bar{\mathbf{t}})$ where $\bar{\mathbf{t}} = [t_1, \dots, t_{j-1}, w, t_{j+1}, \dots, t_n]^T$ and w is a realization of a random variable that follows the same distribution as T_j and is independent of $\mathbf{T}$. As the sender announces its strategy, the optimal response of the receiver for the parameter T_j becomes $\gamma^{d_{T_j}}(\mathbf{z}) = \mathbb{E}[T_j | \mathbf{Z} = \mathbf{z}] = \mathbb{E}[T_j] = 0$ due to the independence of T_j and $\mathbf{Z}$ in this scenario. Therefore, we get $\mathbb{E}[(T_j - \gamma^{d_{T_j}}(\mathbf{Z}))^2] = \sigma_{T_j}^2$ in this case. Notice that the mean squared error performance in estimating T_i for $i \neq j$ is the same for both scenarios. As a result, the second scenario yields better performance for the sender. Thus, transmitting information related to T_j with $\lambda_j < 0$ in (3.12) is not desirable for the sender. $\square$

As a result of Lemma 3.6.1, the sender is restricted to transmit $\mathbf{U}$. Since the linear encoding policies in (3.7) with $\alpha_i \neq 0$ for all $i = 1, \dots, n_{\mathbf{Y}}$ reveals $\mathbf{U}$ completely, these encoding policies yield the minimum attainable performance for the sender among all encoding policies. Therefore, the policies in the statement of the theorem lead to a Stackelberg equilibrium.

We note that the parameter $\mathbf{U}$, which is desired to be conveyed at a Stackelberg

equilibrium, has a size of $n_{\mathbf{Y}} \geq 1$. Therefore, there always exists informative Stackelberg equilibrium where the sender reveals $\mathbf{U}$ completely. On the other hand, it is always possible to construct a noninformative Nash equilibrium as noted in Theorem 3.5.1.

When $n_{\mathbf{Y}} > 1$, the parameter $\mathbf{U}$ is not scalar valued. Thus, there exist informative Nash equilibria with an encoding policy in (3.7) where the scalars $\{\alpha_i\}_{i=1}^{n_{\mathbf{Y}}}$ take any value with at least one zero term and one nonzero term. In these informative Nash equilibria, the sender conveys only a subset of the random variables in $\mathbf{U}$. Since at these informative Nash equilibria, the performance of the sender is strictly worse than that at the payoff dominant Nash equilibria, these policies do not lead to a Stackelberg equilibrium. $\square$

Remark 3.6.1. In fact, employing any invertible function $h(\mathbf{u})$ at the sender also yields a Stackelberg equilibrium. Since the receiver knows the commitment of the sender, it can simply employ $h^{-1}(\mathbf{u})$ to perfectly recover $\mathbf{u}$.

3.7 The MMSE Gaussian Information Bottleneck Problem

We now visit and re-formulate the information bottleneck problem [95], as depicted in Figure 3.1 (b). We will interpret this problem as an instance of our formulation under the Stackelberg equilibrium concept in the following sense: in contrast to the privacy game setup, only the parameter $\mathbf{X}$ is observed at the sender in the information bottleneck setup. In particular, we provide an estimation theoretic perspective on the information bottleneck problem by using quadratic distortion criteria as in our privacy game setup.

The information bottleneck problem considers a similar objective to that employed in this chapter where the performance metrics involve mutual information rather than the mean squared error. In the information bottleneck problem, the

aim is to compress an observed random variable while trying to preserve information related to a correlated random variable. These conflicting objectives are analyzed by formulating an optimization problem involving the mutual information between the considered random variables. Although the problem can be cast as a constrained optimization problem of maximizing the released (useful) information related to the unobserved parameter under a compression constraint, the Lagrangian dual of this constrained problem is commonly considered (see Remark 3.7.2 for a constrained version in our setting). The aim is to find the optimal solution to the following optimization problem:

$$\min_{\mathbf{Z}=\gamma^e(\mathbf{X})} I(\mathbf{X}; \mathbf{Z}) - \beta I(\mathbf{Y}; \mathbf{Z}), \quad (3.21)$$

where $\beta \geq 0$ is a tradeoff parameter. Here, the goal is to compress an observation $\mathbf{X}$ while at the same time to maximize the released information related to $\mathbf{Y}$.

The information bottleneck problem considering jointly Gaussian multidimensional sources is studied in [117] where the structure of the optimal solution, which is jointly Gaussian with $\mathbf{X}$ [120], is identified. The objective function in (3.21) resembles the objective function considered in this chapter in the sense that in both problems the parameter $\mathbf{Y}$ is desired to be conveyed while information related to the parameter $\mathbf{X}$ is desired to be removed from the displayed message.

The information bottleneck problem can in fact be viewed as a Stackelberg game between a sender and a receiver. In this game, the sender wants to compress the observed random variable and to convey the unobserved random variable. The use of mutual information as a performance metric effectively means that the receiver uses all the available information related to both of the parameters, i.e., it always employs its best response as in the Stackelberg equilibrium. Thus, the receiver is concerned with extracting information related to both of the parameters, which is also the case in our framework.

In the following, we consider a setting which is similar to the information bottleneck technique by employing mean squared error terms as our metric. As in the information bottleneck framework, the sender observes only the parameter $\mathbf{X}$, rather than observing both of the parameters. Namely, the encoder has access

to only partial information and is of the form $\mathbf{z} = \gamma^e(\mathbf{x})$. The objective functions of the sender and receiver are as defined in (3.1) and (3.2), respectively. Since the receiver is concerned with estimating both of the parameters, it employs the minimum mean squared error estimators of each parameter. Since the equilibrium concept is the Stackelberg equilibrium, the objective function of the sender can be written as

$$J^e(\gamma^e) = -\delta \mathbb{E}[\|\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}]\|^2] + \mathbb{E}[\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{Z}]\|^2]. \quad (3.22)$$

In the following theorem, we characterize the Stackelberg equilibria solutions for the considered estimation theoretic information bottleneck problem.

Theorem 3.7.1. *(i) When $\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}} - \delta\Sigma_{\mathbf{X}}^2$ is not negative definite, the MMSE Gaussian information bottleneck solution is attained by an encoding policy*

$$\mathbf{z} = \begin{bmatrix} \alpha_1 \mathbf{q}_1 & \dots & \alpha_k \mathbf{q}_k \end{bmatrix}^T \Sigma_{\mathbf{X}}^{-1/2} \mathbf{x} \quad (3.23)$$

for nonzero scalars $\{\alpha_i\}_{i=1}^k$ where k denotes the number of nonnegative eigenvalues of $W = \Sigma_{\mathbf{X}}^{-1/2}\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}}\Sigma_{\mathbf{X}}^{-1/2} - \delta\Sigma_{\mathbf{X}}$ and $\{\mathbf{q}_i\}_{i=1}^k$ are the normalized eigenvectors of W associated with its nonnegative eigenvalues. The corresponding decoding policy is given by

$$\begin{bmatrix} \gamma^{d\mathbf{x}}(\mathbf{z}) \\ \gamma^{d\mathbf{y}}(\mathbf{z}) \end{bmatrix} = \text{diag}(\Sigma_{\mathbf{X}}^{1/2}, \Sigma_{\mathbf{Y}\mathbf{X}}\Sigma_{\mathbf{X}}^{-1/2}) \times \begin{bmatrix} \beta_1 \mathbf{q}_1 & \dots & \beta_k \mathbf{q}_k \end{bmatrix} \mathbf{z} \quad (3.24)$$

where $\beta_i = 1/\alpha_i$ for $i = 1, \dots, k$.

(ii) In the particular case when $\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}} - \delta\Sigma_{\mathbf{X}}^2$ is positive definite, the MMSE Gaussian information bottleneck solution is attained by a fully informative encoding policy, where the sender reveals the parameter $\mathbf{X}$ completely.

(iii) When $\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}} - \delta\Sigma_{\mathbf{X}}^2$ is negative definite, the MMSE Gaussian information bottleneck solution is noninformative, i.e., the sender does not reveal any information related to its observation.

Before we present the proof, we contrast our estimation theoretic solution of information bottleneck problem in Theorem 3.7.1 with the information theoretic solution in [117]. Towards that goal, we restate [117, Theorem 3.1] which gives the solution to the optimization problem in (3.21).

Theorem 3.7.2 ([117, Theorem 3.1]). *For the Gaussian information bottleneck problem under (3.21), the optimal solution is given by*

$$\mathbf{z} = A(\beta)\mathbf{x} + \mathbf{n} \quad (3.25)$$

where $\mathbf{n}$ is a realization of a zero-mean Gaussian random vector with identity covariance matrix and

$$A(\beta) = \begin{cases} [\mathbf{0} \ \cdots \ \mathbf{0}]^T & \text{if } 0 \leq \beta < \beta_1^c \\ [\alpha_1 \mathbf{p}_1 \ \mathbf{0} \ \cdots \ \mathbf{0}]^T & \text{if } \beta_1^c \leq \beta < \beta_2^c \\ [\alpha_1 \mathbf{p}_1 \ \alpha_2 \mathbf{p}_2 \ \mathbf{0} \ \cdots \ \mathbf{0}]^T & \text{if } \beta_2^c \leq \beta < \beta_3^c \\ \vdots & \end{cases}$$

where $\{\mathbf{p}_i^T\}_{i=1}^{n_{\mathbf{X}}}$ are left eigenvectors of

$$\Sigma_{\mathbf{X}|\mathbf{Y}} \Sigma_{\mathbf{X}}^{-1} \triangleq (\Sigma_{\mathbf{X}} - \Sigma_{\mathbf{X}\mathbf{Y}} \Sigma_{\mathbf{Y}}^{-1} \Sigma_{\mathbf{Y}\mathbf{X}}) \Sigma_{\mathbf{X}}^{-1}$$

with the corresponding eigenvalues $\{\lambda_i\}_{i=1}^{n_{\mathbf{X}}}$, which are sorted in ascending order, $\beta_i^c = 1/(1 - \lambda_i)$ and $\alpha_i = \sqrt{\frac{\beta(1-\lambda_i)-1}{\lambda_i(\mathbf{p}_i^T \Sigma_{\mathbf{X}} \mathbf{p}_i)}}$ for $i = 1, \dots, n_{\mathbf{X}}$.

Remark 3.7.1. We note that for the information bottleneck problem under (3.21), the optimal solution is jointly Gaussian with $\mathbf{X}$ which is also the case for our estimation theoretic solution in Theorem 3.7.1. However, in the information theoretic formulation, the solution involves applying an independent Gaussian perturbation to a linear function of $\mathbf{X}$ as given in (3.25) whereas in our estimation theoretic solution the encoder reveals a linear function of $\mathbf{X}$ without applying any perturbation. We note that if one considers the original unconstrained version of the information bottleneck problem, then randomization will be needed to ensure that the constraint condition holds with an equality in some constraint regime. See Remark 3.7.2 for further details.

Next, we present the proof of Theorem 3.7.1.

Proof. We have that $(\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}])$ is orthogonal to $\mathbf{X}$ since

$$\mathbb{E}[\mathbf{Y}\mathbf{X}] = \mathbb{E}[\mathbb{E}[\mathbf{Y}\mathbf{X}|\mathbf{X}]] = \mathbb{E}[\mathbb{E}[\mathbf{Y}|\mathbf{X}]\mathbf{X}].$$

Since the sender observes only the parameter $\mathbf{X}$ and determines its message based on $\mathbf{X}$, it follows that $\mathbf{Y} \rightarrow \mathbf{X} \rightarrow \mathbf{Z}$ is a Markov chain in that order. This Markov property implies that $(\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}])$ is orthogonal to $\mathbf{Z}$ as well. To see this, observe that

$$\mathbb{E}[\mathbf{Z}\mathbb{E}[\mathbf{Y}|\mathbf{X}]] = \mathbb{E}[\mathbf{Z}\mathbb{E}[\mathbf{Y}|\mathbf{X}, \mathbf{Z}]] = \mathbb{E}[\mathbb{E}[\mathbf{Z}\mathbf{Y}|\mathbf{X}, \mathbf{Z}]] = \mathbb{E}[\mathbf{Z}\mathbf{Y}]$$

where the first equality is due to the Markov property and the last equality is due to iterated expectations. By using these orthogonality properties, we can express the second term in (3.22) as

$$\begin{aligned} \mathbb{E}[\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{Z}]\|^2] &= \mathbb{E}[\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}] + \mathbb{E}[\mathbf{Y}|\mathbf{X}] - \mathbb{E}[\mathbf{Y}|\mathbf{Z}]\|^2] \\ &= \mathbb{E}[\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}]\|^2] + \mathbb{E}[\|\mathbb{E}[\mathbf{Y}|\mathbf{X}] - \mathbb{E}[\mathbf{Y}|\mathbf{Z}]\|^2] \\ &= \mathbb{E}[\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}]\|^2] + \mathbb{E}[\|\mathbb{E}[\mathbf{Y}|\mathbf{X}] - \mathbb{E}[\mathbb{E}[\mathbf{Y}|\mathbf{X}]\mathbf{Z}]\|^2] \end{aligned}$$

where the second equality follows from the fact that $(\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}])$ is orthogonal to $\mathbf{X}$ and $\mathbf{Z}$ and the third equality is due to iterated expectations. Observing that

$$\mathbb{E}[\mathbf{Y}|\mathbf{X} = \mathbf{x}] = \Sigma_{\mathbf{Y}\mathbf{X}}\Sigma_{\mathbf{X}}^{-1}\mathbf{x},$$

the objective function can be written as

$$\begin{aligned} J^e(\gamma^e) &= \mathbb{E}[\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}|\mathbf{X}]\|^2] - \delta \mathbb{E}[\|\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}]\|^2] \\ &\quad + \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])^T (\Sigma_{\mathbf{X}}^{-1} \Sigma_{\mathbf{X}\mathbf{Y}} \Sigma_{\mathbf{Y}\mathbf{X}} \Sigma_{\mathbf{X}}^{-1}) (\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])], \end{aligned}$$

where the first term is independent of the encoder. Thus, we obtain an optimization problem of the form

$$\min_{\mathbf{Z}=\gamma^e(\mathbf{X})} \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])^T M (\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])], \quad (3.26)$$

where

$$M \triangleq (\Sigma_{\mathbf{X}}^{-1} \Sigma_{\mathbf{X}\mathbf{Y}} \Sigma_{\mathbf{Y}\mathbf{X}} \Sigma_{\mathbf{X}}^{-1} - \delta I).$$

The optimization problem in (3.26) can be viewed as a quadratic multidimensional signaling game problem considered earlier in the chapter and the solution can be

obtained by using the analysis in Section 3.6. In particular, we can rewrite the problem in (3.26) as a Stackelberg game between a sender and a receiver with objective functions

$$J^e(\gamma^e, \gamma^{d\mathbf{x}}) = \mathbb{E}[(\mathbf{X} - \gamma^{d\mathbf{x}}(\mathbf{Z}))^T M (\mathbf{X} - \gamma^{d\mathbf{x}}(\mathbf{Z}))], \quad (3.27)$$

$$J^d(\gamma^e, \gamma^{d\mathbf{x}}) = \mathbb{E}[(\mathbf{X} - \gamma^{d\mathbf{x}}(\mathbf{Z}))^T (\mathbf{X} - \gamma^{d\mathbf{x}}(\mathbf{Z}))]. \quad (3.28)$$

Notice that for a given encoding policy, the best response of the receiver under (3.28) is given by the minimum mean squared estimator of $\mathbf{X}$ given $\mathbf{Z}$, which is consistent with (3.26).

Next, we apply a transformation of variables to express the objective function of the sender in a simplified form. Towards that goal, let $W = \Sigma_{\mathbf{X}}^{-1/2} \Sigma_{\mathbf{X}\mathbf{Y}} \Sigma_{\mathbf{Y}\mathbf{X}} \Sigma_{\mathbf{X}}^{-1/2} - \delta \Sigma_{\mathbf{X}} = Q \Lambda Q^T$ for orthonormal Q and diagonal Λ . Now, consider the following invertible transformation of variables

$$\mathbf{T} \triangleq Q^T \Sigma_{\mathbf{X}}^{-1/2} \mathbf{X}. \quad (3.29)$$

Under this transformation of variables, we introduce an equivalent problem in a similar manner to Section 3.5. The observation $\mathbf{X}$ is mapped into $\mathbf{T}$ via a fixed transformation (3.29). The encoder chooses an arbitrary policy $\tilde{\gamma}^e(\cdot)$ which maps the transformed random vector $\mathbf{T}$ into the message $\mathbf{Z}$. The receiver applies an arbitrary policy $\gamma^{d\mathbf{T}}(\cdot)$ to its observation $\mathbf{Z}$ and then the estimate of $\mathbf{X}$ is obtained via a fixed relation

$$\gamma^{d\mathbf{x}}(\mathbf{z}) = \Sigma_{\mathbf{X}}^{1/2} Q \gamma^{d\mathbf{T}}(\mathbf{z}). \quad (3.30)$$

In this transformed coordinate system, for a given encoding policy the receiver still employs the minimum mean squared estimator of the parameter $\mathbf{T}$, which can be established via a similar analysis to that in Lemma 3.5.1. The objective function of the sender in this transformed coordinate system can be written

$$J^e(\tilde{\gamma}^e, \gamma^{d\mathbf{T}}) = \sum_{i=1}^{n_{\mathbf{X}}} \lambda_i \mathbb{E}[(T_i - \gamma^{d\mathbf{T}}_i(\mathbf{Z}))^2], \quad (3.31)$$

where $\{\lambda_i\}_{i=1}^{n_{\mathbf{X}}}$ are eigenvalues of W .

If all of these eigenvalues $\{\lambda_i\}_{i=1}^{n_{\mathbf{x}}}$ are positive, which is equivalent to $\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}} - \delta\Sigma_{\mathbf{X}}^2$ being positive definite, then the minimum can be attained by revealing $\mathbf{T}$, which corresponds to the fully informative scenario. In case all of these eigenvalues are negative, i.e., $\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}} - \delta\Sigma_{\mathbf{X}}^2$ is negative definite, then revealing information related to any component of $\mathbf{T}$ is not desirable for the sender, and thus, this scenario leads to a noninformative Stackelberg equilibrium. In the remaining case, i.e., $\Sigma_{\mathbf{X}\mathbf{Y}}\Sigma_{\mathbf{Y}\mathbf{X}} - \delta\Sigma_{\mathbf{X}}^2$ is neither positive definite nor negative definite, we partition the transformed vector according to the sign of the coefficients $\{\lambda_i\}_{i=1}^{n_{\mathbf{x}}}$ in (3.31) as follows:

$$\mathbf{T} \triangleq \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \quad (3.32)$$

where $\mathbf{U} \in \mathbb{R}^k$ and $\mathbf{V} \in \mathbb{R}^{n_{\mathbf{x}}-k}$ correspond to nonnegative and negative coefficients in (3.31), respectively, with k denoting the number of nonnegative eigenvalues of W . Next, we can apply Lemma 3.6.1 for this particular Stackelberg game setup to establish that the sender cannot convey information related to $\mathbf{V}$ and is restricted to send information related to $\mathbf{U}$. As the encoding policy in (3.23) reveals $\mathbf{U}$ completely, this encoding policy achieves the minimum attainable cost for the sender among any set of policies. Thus, the pair of policies (3.23) and (3.24) yield a Stackelberg equilibrium, which gives the solution to the MMSE Gaussian information bottleneck problem. $\square$

In the special case of scalar sources, Theorem 3.7.1 simplifies. In particular, depending on the value of δ , the equilibrium is either fully informative or noninformative and we summarize this result in the following corollary.

Corollary 3.7.1. *The Stackelberg equilibrium of the information bottleneck problem for scalar Gaussian sources is given by one of the following cases:*

- (i) *If $(\rho^2/\sigma_X^4) > \delta$, then the sender completely reveals X .*
- (ii) *If $(\rho^2/\sigma_X^4) < \delta$, then the sender does not reveal information related to X .*

(iii) If $(\rho^2/\sigma_X^4) = \delta$, then both informative and noninformative scenarios lead to a Stackelberg equilibrium.

Remark 3.7.2. [Constrained MMSE Information Bottleneck Problem.]

It is also possible to apply the ideas used in the proof of Theorem 3.7.1 to a constrained version of the MMSE Gaussian information bottleneck problem where the aim is to minimize the mean squared error for estimating $\mathbf{Y}$ under constraint that the mean squared error for estimating $\mathbf{X}$ is above a certain threshold α . In this case, the problem is defined with

$$\begin{aligned} \min_{\mathbf{Z}=\gamma^e(\mathbf{X})} \quad & \text{Tr}\left(\Upsilon \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])^T]\right) \\ \text{subject to} \quad & \text{Tr}\left(\mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])^T]\right) \geq \alpha \end{aligned} \quad (3.33)$$

where $\Upsilon \triangleq \Sigma_{\mathbf{X}\mathbf{Y}}^{-1} \Sigma_{\mathbf{X}\mathbf{Y}} \Sigma_{\mathbf{Y}\mathbf{X}} \Sigma_{\mathbf{X}\mathbf{X}}^{-1}$, which is always positive semidefinite. Since any positive semidefinite

$$\Phi \triangleq \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])(\mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Z}])^T]$$

is attainable via a linear encoding policy with a Gaussian perturbation, the problem reduces to

$$\begin{aligned} \min_{\Phi \succeq 0} \quad & \text{Tr}(\Upsilon \Phi) \\ \text{subject to} \quad & \text{Tr}(\Phi) \geq \alpha. \end{aligned} \quad (3.34)$$

Let the minimum eigenvalue of Υ be denoted by $\lambda_{\min}$. Observe that

$$\begin{aligned} \text{Tr}(\Upsilon \Phi) &= \text{Tr}(\Upsilon \Phi - \lambda_{\min} \Phi + \lambda_{\min} \Phi) \\ &= \text{Tr}((\Upsilon - \lambda_{\min} I) \Phi) + \lambda_{\min} \text{Tr}(\Phi) \\ &\geq \alpha \lambda_{\min} \end{aligned}$$

where the inequality uses the constraint along with the observation that $(\Upsilon - \lambda_{\min} I)$ and Φ are positive semidefinite. As a result, by using a linear encoding policy possibly with a Gaussian perturbation, one can attain $\text{Tr}(\Phi) = \alpha$ where the solution satisfies the orthogonality condition under the trace inner-product defining a Hilbert space on square matrices:

$$\text{Tr}((\Upsilon - \lambda_{\min} I) \Phi) = 0.$$

Since such an encoding policy achieves the characterized lower bound, it becomes the optimal solution to the constrained MMSE Gaussian information bottleneck problem. That the constrained problem with inequality is equivalent to a problem with an equality constraint applies more broadly to information bottleneck problems, see e.g. [116].

Remark 3.7.3. We emphasize that the solution presented in Theorem 3.7.1 is obtained without making an *a priori* affine policy restriction. These policies are the optimal solution among any set of policies for the optimization problem constructed at the sender by anticipating the best response of the receiver.

Remark 3.7.4. In the information bottleneck problem, the sender uses partial information since only parameter $\mathbf{X}$ is available at the sender whereas in our privacy-signaling game formulation the sender has access to both of the parameters. Due to this further restriction that only partial information is available at the sender, our information bottleneck analysis provides a lower bound on the performance of our original Stackelberg game setting.

Remark 3.7.5. It should be emphasized that the information bottleneck problem involving mutual information corresponds to the Stackelberg equilibrium concept since employing mutual information effectively means that the receiver uses all the available information, i.e., it employs its best response. On the other hand, the Nash problem would require an explicit dependence of the functions (considered in the optimization) on the receiver policy.

3.8 A Channel between the Sender and the Receiver

In this section, we generalize our results on the considered privacy-signaling game problem to scenarios when there is a channel between the sender and the receiver. In fact, the proposed equivalent formulation employed in the proof of Theorem 3.5.1 is also applicable when there is a channel between the sender and the receiver. Namely, Lemma 3.5.2 and Lemma 3.6.1 generalize to scenarios with a channel between the players represented by a conditional distribution $p(\mathbf{r}|\mathbf{z})$

where $\mathbf{R} = \mathbf{r}$ denotes the observation of the receiver. These generalizations imply that the sender cannot transmit information related to $\mathbf{V}$ and is restricted to send information related to $\mathbf{U}$ at a Nash equilibrium or at a Stackelberg equilibrium, where $\mathbf{U}$ and $\mathbf{V}$ are partitions of the transformed coordinate system specified in (3.14). Thus, for a given channel, the aim is to find an encoder/decoder pair that is optimal in conveying a sequence of independent zero-mean Gaussian distributed sources over that particular channel in a mean squared error sense and such an optimal encoding/decoding policy pair leads to a payoff dominant Nash equilibrium as well as a Stackelberg equilibrium.

In the following, we focus on the particular case of scalar sources and investigate the Nash and the Stackelberg equilibria for two important channel settings.

3.8.1 Gaussian Noise Channel between the Sender and the Receiver

In this subsection, we consider the same problem as before for scalar sources except that there is an additive Gaussian noise (e.g., measurement noise) between the sender and the receiver. More specifically, the sender encodes X and Y into Z which is subject to additive noise W and the receiver uses the observation $R = Z + W$ while decoding both of the random variables. The additive noise term is independent of X and Y and it is modeled as zero-mean Gaussian with variance σ_W^2 . In addition, we assume that there is an average power constraint at the sender, i.e., $\mathbb{E}[Z^2] \leq P$.

3.8.1.1 Nash Equilibria

Theorem 3.8.1. (i) *There exist informative linear Nash equilibria with an encoding policy $\gamma^e(x, y) = Ax + By$ that satisfies*

$$\frac{B}{A} = -\frac{(\delta\sigma_X^2 + \sigma_Y^2) + \sqrt{(\delta\sigma_X^2 + \sigma_Y^2)^2 - 4\delta\rho^2}}{2\delta\rho}, \quad (3.35)$$

$$A^2\sigma_X^2 + B^2\sigma_Y^2 + 2AB\rho = P \quad (3.36)$$

and with decoding policies

$$\gamma^{dx}(r) = \left(\frac{A\sigma_X^2 + B\rho}{P + \sigma_W^2} \right) r, \quad (3.37)$$

$$\gamma^{dy}(r) = \left(\frac{A\rho + B\sigma_Y^2}{P + \sigma_W^2} \right) r. \quad (3.38)$$

(ii) *These informative equilibria are payoff dominant Nash equilibria and they are the only possible payoff dominant Nash equilibria. Moreover, these equilibria are unique among the affine class of policies.*

Proof. Lemma 3.5.2 implies that the sender is restricted to convey U which corresponds to information conveyed by the transmitter at the equilibria specified in Corollary 3.5.1. Then, it is easy to verify that sending U after scaling up to the maximum available power level yields a Nash equilibrium. The decoding policies at this equilibrium are given by the minimum mean squared error estimators corresponding to each parameter. Since the observation R is jointly Gaussian with X and Y , the conditional expectation formula for Gaussian distributions can be employed to obtain (3.37) and (3.38) [119, p. 155].

The proof for the payoff dominance property of the equilibria uses the observation that the performance of both players is determined by $\mathbb{E}[(U - \gamma^{dv}(R))^2]$ at a Nash equilibrium. Since the source is scalar and the Gaussian noise is additive, we can employ the well-known result that the problem of transmitting a scalar Gaussian source over a scalar Gaussian channel under an average power constraint admits a unique solution with linear encoding scaled to satisfy the power constraint with equality (see e.g. [121, p. 376]). Hence, the result immediately follows.

As it can be shown that an encoding policy $\tilde{\gamma}^e(u) = Au + C$ with $\mathbb{E}[(AU)^2] < P$ cannot be a Nash equilibrium, it follows that the only possible informative affine Nash equilibria are attained by an encoding policy that satisfies (3.35) and (3.36). $\square$

3.8.1.2 Stackelberg Equilibria

Theorem 3.8.2. *The Stackelberg equilibria coincide with the payoff dominant Nash equilibria characterized in Theorem 3.8.1. These equilibria are unique among any set of policies.*

Proof. Lemma 3.6.1 implies that the encoder cannot convey V and it can only use U in constructing its message. As the objectives of each player then becomes the minimization of the mean squared error for estimating U , the optimal strategy of the sender is to employ an encoding policy which is linear in U with an average power equal to P . Moreover, this encoding strategy is unique due to the fact that it is the unique solution to the problem of transmitting a scalar Gaussian source over a scalar Gaussian channel under an average power constraint [121]. $\square$

It is important to emphasize that the encoder is not restricted to be affine. Since the problem reduces to transmitting a scalar Gaussian source over a scalar Gaussian channel under an average power constraint, we obtain these linear policies as the optimal unique solution to this reduced problem.

3.8.2 Discrete Noiseless Channel between the Sender and the Receiver

In this subsection, we consider scalar sources and investigate the discrete channel setting where the sender is restricted to transmit a discrete value, i.e., $Z \in \{0, \dots, M-1\}$ for some $M \geq 2$. We assume that the channel is noiseless, i.e., $R = Z$.

While investigating the discrete channel setting, we again employ the equivalent formulation which facilitates the analysis. Lemma 3.5.2 and Lemma 3.6.1 imply that both players share the common objective of minimizing $\mathbb{E}[(U - \gamma^{d_U}(R))^2]$ under both of the equilibrium concepts. Since the sender is restricted to transmit discrete values, it is required to quantize U at the sender. Since this would

correspond to classical quantization, the existence of an optimal quantizer follows from the classical results in the literature, e.g., [55]. Namely, there exist quantization bins and reconstruction points which minimize the corresponding mean squared error. Thus, by assigning each bin to a discrete value of Z and then using the corresponding optimal reconstruction points at the receiver yield a Nash equilibrium. We summarize this result in the following theorem.

Theorem 3.8.3. *Consider the quantization of U into M bins where each bin is assigned to a discrete value of Z at the encoder and the corresponding reconstruction points at the receiver such that $\mathbb{E}[(U - \gamma^{dv}(R))^2]$ is minimized. This pair of encoding and decoding policies, which always exists, forms an informative Nash equilibrium. In addition, this equilibrium is a payoff dominant Nash equilibrium.*

It is worth pointing out that for any number of bins lower than M , there exists a Nash equilibrium. In other words, even if Z can take M discrete values, a quantization policy using lower than M bins at the sender and the corresponding reconstruction points at the receiver is also a Nash equilibrium. In addition, the case of a single bin is also a Nash equilibrium where no information related to U is conveyed to the receiver.

It is noted that using a large number of bins yields a lower $\mathbb{E}[(U - \gamma^{dv}(R))^2]$. Since the mean squared error for estimating U is desired to be minimized for both players, using a large number of bins results in improved objectives for both players. This monotonicity property with respect to the number of bins implies that at the Stackelberg equilibrium there must be M bins.

Theorem 3.8.4. *The pair of policies in Theorem 3.8.3 leads to a Stackelberg equilibrium.*

3.9 Numerical Examples

In this section, the proposed privacy-signaling game problem is investigated through numerical examples, where the following scenario is considered: The

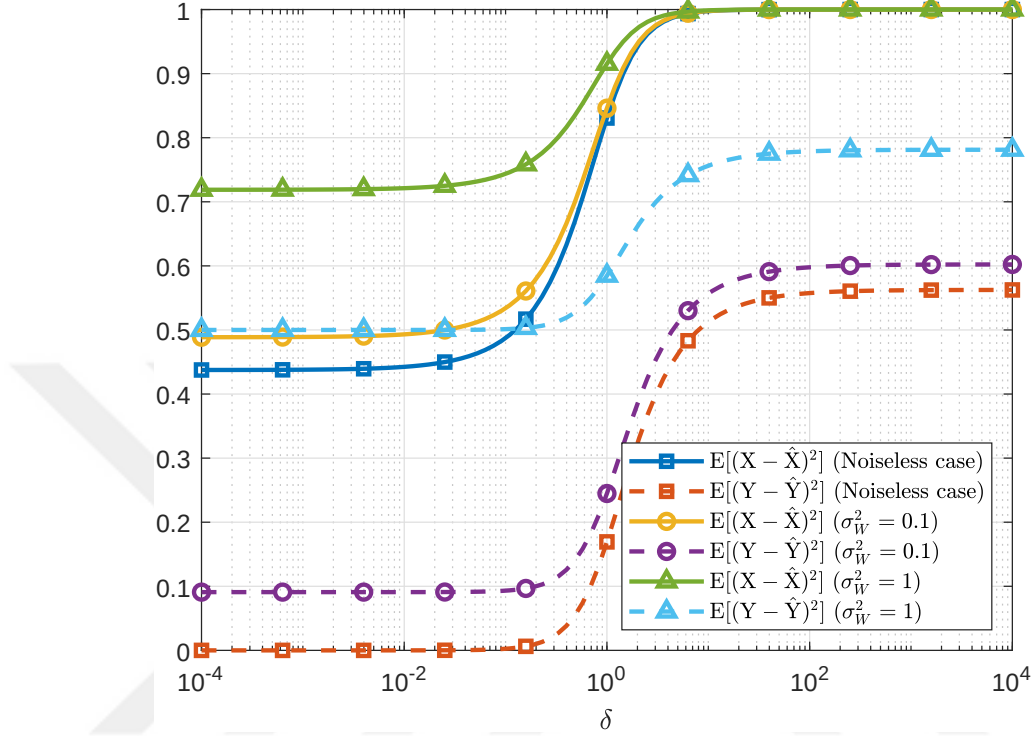


Figure 3.3: Mean squared errors for private and nonprivate parameters at the equilibrium with respect to privacy ratio for different noise levels, where $\rho = 0.75$.

variances of the private and nonprivate parameters are set as $\sigma_X^2 = \sigma_Y^2 = 1$, and the average power level constraint at the sender for the noisy communication scenario is taken as $P = 1$. In the following, we illustrate the performances at the characterized equilibria. Since the informative Nash equilibria coincide with the Stackelberg equilibria, we do not make a distinction between them.

Fig. 3.3 plots the estimation errors for the private and nonprivate parameters with respect to the privacy ratio. The estimation error for the private parameter increases with the privacy ratio since the transmitter removes information related to the private parameter due to enhanced privacy concerns. This removal also distorts the information conveyed related to the nonprivate parameter and hence the corresponding estimation error also increases.

Fig. 3.4 plots the estimation errors for the private and nonprivate parameters with respect to the correlation between them. In the low privacy scenario, the estimation error for Y does not change significantly with respect to the correlation

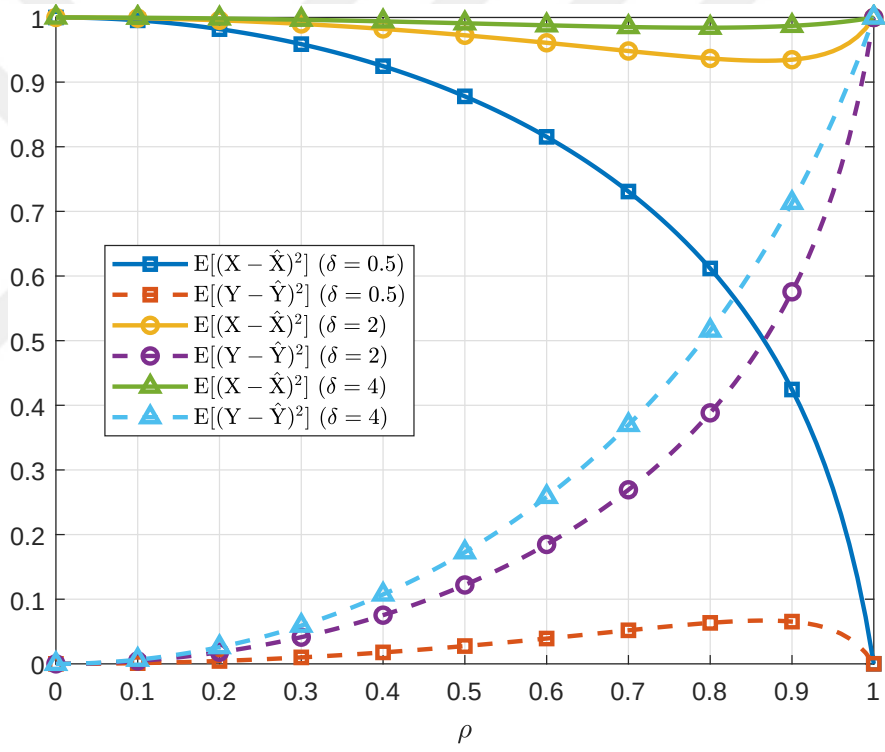


Figure 3.4: Mean squared errors for private and nonprivate parameters at the equilibrium in the absence of noise with respect to correlation between X and Y for different values of privacy ratio.

Table 3.1: The ratio of coefficients at the encoder for the derived informative equilibria when $\sigma_X^2 = 1$ and $\sigma_Y^2 = 1$.

Scenario	B/A
$\rho = 0.3$ and $\delta = 0.1$	-36.39
$\rho = 0.3$ and $\delta = 1$	-6.51
$\rho = 0.3$ and $\delta = 10$	-3.63
$\rho = 0.7$ and $\delta = 0.1$	-15.04
$\rho = 0.7$ and $\delta = 1$	-2.44
$\rho = 0.7$ and $\delta = 10$	-1.50

since most of the information contained in Y is conveyed to the receiver regardless of the correlation. As a result, more information is leaked related to the private parameter as the correlation is increased. In contrast, in the high privacy scenario, regardless of the correlation, most of the information related to X is removed from the transmitted message. Thus, the estimation error for the nonprivate parameter increases with the correlation whereas no significant changes in the estimation error for the private parameter are observed.

Finally, Table 3.1 illustrates the tradeoff between *utility* in terms of conveying Y and *privacy* in terms of hiding X by providing the structure of the encoder at the equilibrium for various values of the privacy ratio and correlation. It can be inferred that if the privacy ratio is increased while the correlation is kept the same, the information leakage related to the private parameter reduces, as expected.

3.10 Conclusion

A communication setting between a sender with privacy concerns and a receiver has been investigated in a game theoretic framework. The private and nonprivate parameters have been modeled as jointly Gaussian random vectors. It has

been proven that a payoff dominant Nash equilibrium is attained by linear policies. It has been shown that these linear policies at the payoff dominant Nash equilibria lead to Stackelberg equilibria as well. These results have been further generalized to the Gaussian noisy channel setting as well as a discrete noiseless channel setting for the special case of scalar sources. We have also provided an estimation theoretic perspective on the information bottleneck problem under the Stackelberg equilibrium concept. We have shown that the Stackelberg equilibria are attained by a set of characterized linear policies.

Chapter 4

Quadratic Signaling Games under Prior Mismatch

4.1 Motivation

The team theoretic formulation in systems theory (e.g., as studied by Witsenhausen [122]) requires that all decision makers have the same probabilistic system model, even though they may have different local information. While this also has been the norm for nearly all information theoretic applications, in some applications, an encoder and a decoder may have subjective probabilistic models, especially when an encoder may realize that the model as seen by a remote decoder is inaccurate. Even though the decision makers employ the same cost function, induced expected costs, given encoding and decoding functions, are different from the perspective of the encoder and decoder due to their subjective probabilistic beliefs, which turns the team problem into a game theoretic one. For cooperative setups, the encoder needs to account for this inconsistency, which leads to a leader-follower (Stackelberg) game formulation. In some further applications, the encoder and the decoder may be engaged in a signaling game where their models may not be available to each other a priori or they may belong to separate organizations, leading to a Nash game theoretic setup. Accordingly, we

study communications between an encoder and a decoder, viewed as two decision makers, which have subjective beliefs on the probabilistic model of the source distribution. Depending on the cooperation or commitment nature of the encoder to its policies, one can analyze Stackelberg and Nash equilibria for signaling games with subjective priors. In this chapter, we focus on the Stackelberg equilibrium setup.¹

In scenarios modeled by the Stackelberg equilibrium concept, there is a hierarchy in the decision making procedure [13]. In particular, the encoder first makes a decision and announces its decision and then the decoder acts after observing the encoder's decision. In this setting, the encoder commits to employ this announced strategy, and the decoder trusts the encoder and employs its best response. In this scenario, the encoder knows its own prior as well as the prior of the decoder whereas the decoder only knows its own prior. This happens especially when the encoder realizes that the decoder employs an inaccurate prior. In this setting, the encoder decides on what information to reveal to the decoder in order to optimize its objective function. This scenario can be viewed as a cooperative communication scenario since we know that the encoder's announcement is observed by the decoder and the decoder acts by using this information. We note that the classical communication setup with no strategic decision makers corresponds to the Stackelberg equilibrium concept since the decision makers trust each other in such a setting. Different from the classical communication setup, we incorporate prior mismatch into the problem. This type of cooperative communication setup with mismatched priors can be used to model scenarios where it is not feasible for the encoder to share its prior probability distribution with the decoder, the encoder conveys only the message, and the encoding function used for generating this message (which is designed off-line).

¹For results concerning the Nash equilibrium setup, the reader is referred to [123] and [38] where the latter includes also the results presented in this chapter.

4.2 Literature Review

In this chapter, we investigate a signaling game problem for conveying a source from an encoder to a decoder where the encoder and decoder have subjective probabilistic beliefs on the source distribution. A related work in this context [124] investigates equilibrium behavior under either Nash or Stackelberg equilibria in a two-person decision making scenario with quadratic cost where the decision makers have subjective probabilistic beliefs. An interesting observation from [124] is that in the special case of Gaussian priors, the decision makers employ linear policies under Nash equilibria whereas the policies under Stackelberg equilibria are in general nonlinear. The authors in [19] investigate Nash and Stackelberg equilibria for a binary hypothesis testing problem where the players may have subjective costs and/or subjective views on prior probabilities of the hypotheses. In addition, motivated by scenarios where individuals hold different prior beliefs in applications in the economics literature, the work [125] analyzes the Bayesian persuasion problem with general cost functions where the encoder and decoder have subjective probabilistic beliefs. Moreover, the subjectivity may appear when there are prospect theoretic agents in the system where the decision makers may have different views on the probabilistic models due to their subjective biases [32, 126, 127]. In prospect theory, the subjective views of the agents on the prior probabilities are modeled via a weight function which for instance reflects a common misconception of overestimating (underestimating) the probability of less (more) likely events. For instance, in [32], communication scenarios through an additive noisy channel with an encoder and a decoder which have different weight functions is analyzed under the Stackelberg equilibrium concept where there is an affine policy restriction at the encoder. It is shown that for Gaussian source and noise case policies at the equilibrium are not affected by subjective biases whereas for exponential source and noise case policies at the equilibrium depend on subjective biases. Another related work [128] investigates optimal stochastic signaling in a binary communication setup with aligned cost structure for an encoder and a decoder and provides sufficient conditions under which stochastic signaling improves the performance or not.

In this chapter, we also have results regarding robustness/continuity properties of the costs in equilibrium for the considered signaling game problem. In the literature, one wishes to analyze such analytical properties in different contexts [129–131]. For instance, [131] studies analytical properties such as continuity of mean squared error for estimation under additive noise in a classical team theoretic setup where there is no prior mismatch. In addition, [129] investigates robustness and continuity with respect to prior probability measures for partially observed stochastic control problems where the priors can be incomplete or incorrect. Here, [129] shows that indeed under total variation, strong continuity results with a rate of continuity/convergence hold, but in our work, the presence of an encoder adds further challenges.

In addition, our results in this chapter are related to works that investigate performance loss arising from model mismatches [132–139]. In these settings, it is aimed to analyze the effects of a mismatch between the true prior distribution and the prior distribution employed in the system. In contrast to these works, we model the problem as a game between players which hold different beliefs on the source distribution. In a game theoretic setup, a player may be aware of this prior mismatch and may use this knowledge for its own benefit.

4.3 Contributions

Main contributions of this chapter can be summarized as follows:

- (i) We prove that the Stackelberg equilibrium cost of the encoder is upper semi continuous (under the Wasserstein metric) as the subjective prior of the encoder approaches the subjective prior of the decoder considering any subjective prior distributions for the players (Theorem 4.5.1). For the special case of Gaussian subjective priors, it is proven that the cost around the team setup is lower semi continuous, as well. Therefore, for the Gaussian subjective priors case, the equilibrium is robust with respect to perturbations around the team setup (under the Wasserstein metric or the weak

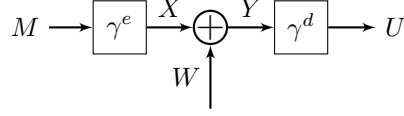


Figure 4.1: Signaling game setup.

convergence topology). In addition, we also provide a duality result which states that as the subjective prior of the decoder approaches the subjective prior of the encoder, the Stackelberg equilibrium cost of the encoder is lower semi continuous (under the Wasserstein metric or the weak convergence topology).

- (ii) We show that the Stackelberg equilibrium solution for Gaussian signaling with subjective priors is in general nonlinear by providing specific examples where nonlinear encoding policies outperform the best affine encoding policy (Theorem 4.5.2). Thus, team-theoretic optimality of linear/affine policies is not robust to prior perturbations.
- (iii) For the signaling game problem with Gaussian subjective priors (under a soft power constraint or a hard power constraint), we show that the Stackelberg equilibrium solution under affine policy restriction is either informative or noninformative depending on conditions stated explicitly (Theorem 4.5.3 and Theorem 4.5.4). Moreover, these game theoretic solutions do not coincide with the corresponding team theoretic solutions in general. In particular, when there is a hard power constraint, game theoretic solution may be noninformative whereas team theoretic solution is always informative regardless of the parameters. In addition, we show that the Stackelberg equilibria under affine policy restriction are robust with respect to perturbations around the team setup.

4.4 Problem Formulation

A *signaling game* problem, which is depicted in Fig. 4.1, can be formulated as follows: Suppose that there is an encoder and a decoder. The encoder wishes

to transmit a random variable M taking values in $\mathbb{M}$. The encoder encodes M into an $\mathbb{X}$ -valued random variable X via an encoding policy denoted by $\gamma^e \in \Gamma^e$ where Γ^e denotes the set of possible encoding strategies. We take Γ^e to be the set of all stochastic kernels from $\mathbb{M}$ to $\mathbb{X}$.² There exists an additive zero-mean Gaussian noise channel between the encoder and decoder where the additive noise is independent of the source. In this setup, the encoder designs its policy by incorporating either a soft or a hard power constraint into its objective function, as explained below. The decoder observes a noise corrupted version of the message denoted by $Y = X + W$ which is a $\mathbb{Y}$ -valued random variable. The decoder generates its optimal decision U , which is also an $\mathbb{M}$ -valued random variable, given its observation Y via a decoding policy $\gamma^d \in \Gamma^d$. Here, Γ^d denotes the set of possible decoding strategies which is the set of all stochastic kernels from $\mathbb{Y}$ to $\mathbb{M}$.

As motivated before, the encoder and the decoder may have different views on the prior probability distribution of the source. For example, the encoder may aim to minimize $J^e(\gamma^e, \gamma^d) = \mathbb{E}_e[c^e(m, u)]$ whereas the decoder may aim to minimize $J^d(\gamma^e, \gamma^d) = \mathbb{E}_d[c^d(m, u)]$ where $c^e(m, u)$ and $c^d(m, u)$ denote the cost functions of the encoder and the decoder, respectively, when m is observed at the encoder and the action u is taken at the decoder, and $\mathbb{E}_e[\cdot]$ and $\mathbb{E}_d[\cdot]$ denote that the expectation is taken over the probability measure from the perspective of the encoder and the decoder, respectively. Note that each decision maker computes their expected cost with respect to its own subjective prior since it believes that its subjective prior is the true prior distribution. Each player designs its policy by minimizing its expected cost computed with respect to its subjective prior distribution. Thus, although the actual true distribution can be a different distribution than these subjective prior distributions, this true distribution does not affect policies employed by the decision makers. Nevertheless, in the following remark, we provide a motivation for a case that the encoder's prior is the true prior distribution.

Remark 4.4.1. Note that the encoder designs its message by observing the

² P is a stochastic kernel from $\mathbb{M}$ to $\mathbb{X}$ if $P(\cdot|m)$ is a probability measure on $\mathcal{B}(\mathbb{X})$ for every $m \in \mathbb{M}$, and $P(A|\cdot)$ is a Borel measurable function of m for every $A \in \mathcal{B}(\mathbb{X})$. Note that if $\mathbb{M}$ and $\mathbb{X}$ are finite, then P corresponds to a transition matrix.

source random variable. Thus, in some applications, it may be possible for the encoder to correct its prior before the information transmission stage by observing a large number of samples. Therefore, in these applications, it is reasonable to assume that the encoder's view corresponds to the true distribution of the source. On the other hand, the decoder may have an incorrect belief since its observations are limited to what the encoder reveals.

In this chapter, we consider a quadratic cost structure where either a soft power constraint or a hard power constraint is employed at the encoder. In the case of soft power constraint, the encoder employs the objective function

$$J^e(\gamma^e, \gamma^d) = \mathbb{E}_e [c^e(m, x, u)], \quad (4.1)$$

and the decoder employs the objective function

$$J^d(\gamma^e, \gamma^d) = \mathbb{E}_d [c^d(m, u)], \quad (4.2)$$

where $c^e(m, x, u) = (m - u)^2 + \lambda x^2$ and $c^d(m, x) = (m - u)^2$ and λ represents the appended soft power constraint to encoder's objective. Appending a soft power constraint in this manner is encountered in stochastic control problems, see e.g., [140, 141]. Notice that when $\lambda = 0$, the case without any power constraint is recovered. Note also that it is possible append this additional λx^2 term to the cost function of the decoder and this does not make a difference in the analysis. In other words, misalignment between the encoder and decoder essentially arises from subjective probabilistic beliefs of the players and not from the considered costs. For the case in which the encoder has a hard power constraint instead of a soft power constraint, the goal of the encoder is to minimize

$$\begin{aligned} J^e(\gamma^e, \gamma^d) &= \mathbb{E}_e [c^e(m, u)] \\ \text{s.t. } \mathbb{E}_e [(\gamma^e(m))^2] &\leq \bar{P}, \end{aligned} \quad (4.3)$$

whereas the decoder aims to minimize

$$J^d(\gamma^e, \gamma^d) = \mathbb{E}_d [c^d(m, u)], \quad (4.4)$$

where $c^e(m, u) = (m - u)^2$ and $c^d(m, u) = (m - u)^2$. In communication theoretic settings, a hard power constraint for an encoder is commonly imposed,

and many results in information theory with regard to communication through a Gaussian channel involves a hard power constrained encoder, see e.g., [142, Ch. 9] and [121, Ch. 11].

Our aim is to analyze the previously described communication scenarios using the Stackelberg equilibrium concept. Under the Stackelberg equilibrium concept, the game is played in a sequential manner where first the encoder chooses and announces its policy and then the decoder determines its policy given the announcement of the encoder. In this scenario, the encoder commits to employ its announced strategy and the encoder cannot change its strategy once the decoder learns the strategy of the encoder. A pair of encoding and decoding policies $\gamma^{*,e}$ and $\gamma^{*,d}$ forms a Stackelberg equilibrium [13] if

$$J^e(\gamma^{*,e}, \gamma^{*,d}(\gamma^{*,e})) \leq J^e(\gamma^e, \gamma^{*,d}(\gamma^e)) \quad \forall \gamma^e \in \Gamma^e, \quad (4.5)$$

where $\gamma^{*,d}(\gamma^e)$ satisfies

$$J^d(\gamma^e, \gamma^{*,d}(\gamma^e)) \leq J^d(\gamma^e, \gamma^d(\gamma^e)) \quad \forall \gamma^d \in \Gamma^d. \quad (4.6)$$

Remark 4.4.2. Since the encoder performs an optimization of its objective by anticipating the best response of the decoder, it is required that the encoder knows the subjective prior distribution of the decoder in addition to its own subjective prior in the case of Stackelberg equilibria. On the other hand, for a Nash equilibrium, this knowledge is not required as the players announce their policies at the same time [38].

An important quantity in a Stackelberg setup is the induced cost for the encoder as the encoder performs an optimization of its objective given that the decoder best responds. In that respect, we investigate the continuity properties of the encoder's cost with respect to perturbations of the priors around the team setup. We say that a Stackelberg equilibrium is *robust* with respect to perturbations around the team setup if the encoder's expected cost continuously changes as the prior of the encoder (decoder) approaches the prior of the decoder (encoder). We also have results where one can only guarantee upper semi continuity or lower semi continuity of the encoder's cost.

In the case that the cost is upper semi continuous, there cannot be a drastic degradation in encoder’s performance with prior mismatch around the point of identical priors. In that respect, upper semi continuity ensures that the worst case performance under prior mismatch behaves continuously around the point of identical priors. On the other hand, in the case that the cost is only lower semi continuous, a small prior perturbation in principle may lead to a large performance degradation of the game theoretic cost in comparison with the team theoretic cost.

4.5 Stackelberg Equilibria

In this section, we analyze the Stackelberg equilibria when there is either a soft power constraint or a hard power constraint at the encoder. Before presenting our results, we make the following remark.

Remark 4.5.1. We note that if both players share the same probabilistic belief on the source distribution, then the problem reduces to the classical team theoretic setup with a power constrained encoder. Although obtaining optimal coding/decoding policies for general source distributions is in general difficult, for the special case that the source is scalar Gaussian, the optimal solution involves linear policies at the encoder and the decoder [143] (see e.g., [121, Theorem 11.2.2]). This optimality result for linear policies is obtained by using channel capacity and rate distortion bounds.

4.5.1 Continuity and Robustness to Perturbations around the Team Setup

In this subsection, we investigate continuity and robustness of the Stackelberg equilibria around the team setup for general subjective source priors with more specific results for the Gaussian priors case.

In our work, subjective prior of a decision maker deviates from that of the other decision maker, i.e., a deviation from the team theoretic setup. In this case, we analyze if the cost function of the encoder behaves continuously around the point of identical priors. Note that the encoder's cost determines whether the equilibrium is robust or not since the encoder performs an optimization of its objective in the Stackelberg setup. To be more precise, a robust Stackelberg equilibrium means that $J^{*,e}(\phi_e, \phi_d) \rightarrow J^{*,e}(\phi_d, \phi_d)$ as $\phi_e \rightarrow \phi_d$, where $J^{*,e}(\phi_e, \phi_d)$ denotes the Stackelberg equilibrium cost of the encoder when the subjective priors of the encoder and the decoder are $\phi_e(\cdot)$ and $\phi_d(\cdot)$, respectively. Such a continuity property is equivalent to $J^{*,e}(\phi_e, \phi_d) \rightarrow J^{*,e}(\phi_e, \phi_e)$ as $\phi_d \rightarrow \phi_e$. We also have semi continuity results where depending on whether $\phi_e \rightarrow \phi_d$ or $\phi_d \rightarrow \phi_e$ is considered, upper or lower semi continuity can be guaranteed.

In order to perform a continuity analysis with respect to subjective priors, we apply a perturbation to these subjective priors around the point of identical priors. This perturbation is quantified via a convergence notion for probability measures. We also emphasize that these continuity results hold only at the point of identical priors. In order to investigate continuity properties of the encoder's cost, we need to define a probability space and a convergence notion for probability measures in this space. Towards that goal, let $\mathbb{X} = \mathbb{R}$ and let $\mathcal{P}(\mathbb{X})$ denote the family of all probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ where $\mathcal{B}(\mathbb{X})$ denotes the Borel σ -algebra on $\mathbb{X}$. Let $\{\mu_n, n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}(\mathbb{X})$. A sequence $\{\mu_n\}$ is said to converge to $\mu \in \mathcal{P}(\mathbb{X})$ as n tends to infinity *weakly* if the following convergence relation holds as n tends to infinity:

$$\int_{\mathbb{R}} c(x) \mu_n(dx) \rightarrow \int_{\mathbb{R}} c(x) \mu(dx)$$

for every continuous and bounded $c : \mathbb{X} \rightarrow \mathbb{R}$. The Prokhorov metric can be used to metrize this space. As a more practical metric, the Wasserstein metric can also be used (for compact $\mathbb{X}$).

Definition 4.5.1 (Wasserstein metric). The *Wasserstein metric* of order $p \geq 1$ for two distributions $\mu, \nu \in \mathcal{P}(\mathbb{X})$ with finite p th order moments is defined as

$$W_p(\mu, \nu) = \inf_{\eta \in \mathcal{H}(\mu, \nu)} \left(\int_{\mathbb{X} \times \mathbb{X}} \eta(dx, dy) \|x - y\|^p \right)^{\frac{1}{p}},$$

where $\mathcal{H}(\mu, \nu)$ denotes the set of probability measures on $\mathbb{X} \times \mathbb{X}$ with first marginal μ and second marginal ν and $\|\cdot\|$ is a norm (such as the Euclidean norm).

As noted, for compact $\mathbb{X}$, the Wasserstein distance of order p metrizes the weak topology on the set of probability measures on $\mathbb{X}$ (see [144, Theorem 6.9]). For non-compact $\mathbb{X}$, weak convergence combined with convergence of moments (that is of $\int \|x\|^q \mu_n(dx) \rightarrow \int \|x\|^q \mu(dx)$ for all orders $q \leq p$) is equivalent to convergence in W_p (see [144, Definition 6.8] and [144, Theorem 6.9]).

In the following theorem, we investigate continuity and robustness properties of the encoder's cost around the point of identical priors.

Theorem 4.5.1. *Suppose that the prior of the source is $\phi_e(\cdot)$ and $\phi_d(\cdot)$ from the perspective of the encoder and the decoder, respectively, where these prior distributions are arbitrary. Suppose further that variance of the source under $\phi_d(\cdot)$ is finite. Then, the following are true when there is either a soft power constraint or a hard power constraint at the encoder.*

- (i) *The Stackelberg equilibrium cost of the encoder is upper semi continuous (under the Wasserstein metric) as the subjective prior of the encoder approaches the subjective prior of the decoder, i.e., $\phi_e(\cdot) \rightarrow \phi_d(\cdot)$.*
- (ii) *If the subjective prior distributions $\phi_e(\cdot)$ and $\phi_d(\cdot)$ are Gaussian, the Stackelberg equilibrium cost of the encoder is lower semi continuous (under the Wasserstein metric or the weak convergence topology) as the subjective prior of the encoder approaches the subjective prior of the decoder, i.e., $\phi_e(\cdot) \rightarrow \phi_d(\cdot)$.*
- (iii) *If the subjective prior distributions $\phi_e(\cdot)$ and $\phi_d(\cdot)$ are Gaussian, the Stackelberg equilibria are robust (under the Wasserstein metric or the weak convergence topology) with respect to perturbations around the team setup .*
- (iv) *The Stackelberg equilibrium cost of the encoder is lower semi continuous (under the Wasserstein metric or the weak convergence topology) as the subjective prior of the decoder approaches the subjective prior of the encoder, i.e., $\phi_d(\cdot) \rightarrow \phi_e(\cdot)$.*

Proof. See Section 4.7.1. □

Theorem 4.5.1 reveals an interesting duality property of encoder's cost in the sense that if $\phi_e \rightarrow \phi_d$, then upper semi continuity holds whereas if $\phi_d \rightarrow \phi_e$, then lower semi continuity holds. On the other hand, for mismatched Gaussian priors, both upper semi continuity and lower semi continuity hold, which proves the robustness of the Stackelberg equilibria around the point of identical priors.

Remark 4.5.2. For a communication setup, it is reasonable to assume that the source to be conveyed has a finite average power, which holds when the variance of the source is finite. Note that if the variance is unbounded, infinite Shannon capacity would be implied, which would reduce the problem to a cheap talk problem, as the noise can be suppressed with arbitrarily large encoder gain. Therefore, the assumption of having a finite variance from the perspective of the decoder in Theorem 4.5.1 is not unnatural, and this assumption is commonly imposed in practice.

Remark 4.5.3. We note that the property of upper semi continuity also holds when the source is multidimensional. On the other hand, when the source is multidimensional Gaussian from the perspective of both players, then the analysis regarding lower semi continuity in (ii) of Theorem 4.5.1 does not apply since the optimal encoding policy may not be linear for the team theoretic setup with a multidimensional Gaussian source. Nonetheless, if there is an affine policy restriction for the encoder, a similar analysis to that in (4.18) can be carried out to obtain lower semi continuity result for a setup with multidimensional Gaussian priors.

Remark 4.5.4. We note that in a related work [131] in this context, continuity property of minimum mean squared error is investigated in a team theoretic setup. In particular, it is shown that minimum mean squared error is continuous for the case with a linear encoder and an additive channel where the noise density is continuous and bounded [131, Theorem 4] and the Gaussian density satisfies these properties. In other words, there is essentially no encoding in [131] other than a scaling factor and the analysis takes only the decoder into account. As opposed to [131], our work analyzes the scenario with prior mismatch and there is an encoder which may also be nonlinear.

4.5.2 Affine Policies may no longer be Optimal with Gaussian Subjective Priors

Here, the subjective probabilistic beliefs of the encoder and the decoder are taken as Gaussian. In particular, the Gaussian source has different mean and variance from the perspectives of the encoder and the decoder, i.e., the source is $M \sim \phi_e(m) = \mathcal{N}(\mu_e, \sigma_e^2)$ and $M \sim \phi_d(m) = \mathcal{N}(\mu_d, \sigma_d^2)$ from encoder's and decoder's perspective, respectively. In addition, the additive noise, which is independent of the source, between the encoder and the decoder is modeled by a zero-mean Gaussian random variable, i.e., the noise is $W \sim \mathcal{N}(0, \sigma_W^2)$.

For setups with a Gaussian source and an additive Gaussian noise channel, affine class of policies is an important class of policies due to its desirable optimality property for the classical team theoretic communication setup with identical costs and priors (see e.g., [121, Theorem 11.2.2]). On the other hand, for the Stackelberg setup, the optimality of affine policies for Gaussian signaling no longer holds due to the presence of subjective probabilistic beliefs of the players, and thus team-theoretic optimality of linear/affine policies are not robust to perturbations. In the following theorem, we prove this result by providing examples where nonlinear encoding policies yield better performance for the encoder than the best affine policies. In particular, we consider a quantization policy at the encoder motivated by the seminal work of Witsenhausen [140] where he provides a striking example in control theory that such a nonlinear control policy outperforms the best affine policy for the considered control system.

Theorem 4.5.2. *Consider the quadratic signaling games problem with Gaussian subjective priors where there is no affine policy restriction at the encoder. Then, for a soft power constrained or a hard power constrained encoder, it is not necessarily true that an affine encoding policy always gives the Stackelberg equilibrium solution.*

Proof. It suffices to provide examples where a nonlinear policy yields a better cost than the best affine policy. First, consider the soft power constrained setup. We provide an example where a quantization policy leads to a cost (for the

encoder) which is smaller than the optimal cost under affine policy restriction. Let $\mu_e = \mu_d = 0$, $\sigma_e^2 = 6.25$, $\sigma_d^2 = 0.25$, $\sigma_W^2 = 0.25$ and $\lambda = 1.5$. The aim of the encoder is to solve the optimization problem (4.1) after plugging in the best response of the decoder which is given by $u = \mathbb{E}_d[m|\gamma^e(m) + w]$. Under an affine policy restriction at the encoder, the best response of the decoder also becomes affine. If we numerically solve the optimization problem at the encoder under affine policy restriction, the optimal encoding policy becomes $\gamma^{*,e}(m) = 0.30m$ and the corresponding cost of the encoder is given by $J_{\text{affine}}^{*,e} = 6.12$. Now, consider an encoding policy in the form of a quantization policy specified by $\gamma^e(m) = \sqrt{P} \text{sgn}(m)$ with $P = 0.5$. The best response of the decoder to such a quantization policy at the encoder can be computed as $\gamma^d(y) = \sqrt{2/\pi} \sigma_d \tanh(\sqrt{P}y/\sigma_W)$. With this best response of the decoder, we compute the expected cost of the encoder via numerical integration and obtain $J_{\text{quantization}}^e \approx 5.90$. This shows that such a quantization policy outperforms the best affine policy.

Now, consider the hard power constrained case. Let $\mu_e = \mu_d = 0$, $\sigma_e^2 = 1$, $\sigma_d^2 = 3$, $\sigma_W^2 = 0.4$ and $\bar{P} = 0.1$. In this case, if we numerically solve the optimization problem (4.3) after inserting the best response of the decoder under affine policy restriction, we obtain a noninformative equilibrium where the transmitted message is independent of the source, e.g., $\gamma^{*,e}(m) = 0$. In this case, the cost of the encoder becomes $J_{\text{affine}}^{*,e} = \sigma_e^2 = 1$. Next, consider an encoding policy in the form of a quantization policy specified by $\gamma^e(m) = \sqrt{\bar{P}} \text{sgn}(m)$. In a similar manner to the soft power constrained case, by computing the expected cost of the encoder numerically, we obtain $J_{\text{quantization}}^e \approx 0.94$. This reveals that such a quantization policy outperforms the best affine policy. $\square$

Theorem 4.5.2 shows that when the players have different perception on the prior probability of the message, then optimality of linear/affine policies (which holds under identical priors) may no longer hold. Nonetheless, it is observed through numerical simulations that such a quantization policy outperforms the best affine policy when the players have very different perceptions on the prior probability of the source.

4.5.3 Gaussian Signaling under Affine Policy Restriction

In this subsection, for the Stackelberg equilibrium solution, we impose an affine policy restriction at the encoder. In addition to its simplicity for the Gaussian case, one other motivation for employing affine policies is as follows:

Remark 4.5.5. Note that in the classical team theoretic setup with identical priors, the optimal solution is attained by affine policies. We also know that the Stackelberg equilibrium concept is used to model scenarios where the decoder trusts the encoder and employs its best response. In such a setting, if the encoder employs a nonlinear policy, then the decoder will be able to learn that its prior is incorrect (as otherwise the encoder must employ an affine policy). As a result, the decoder realizes that the setup is a game setup with mismatched priors (or the decoder may even think that the encoder employs a different cost function). Therefore, the encoder may prefer employing an affine policy rather than a nonlinear policy in order not to damage its credibility. In the case of affine encoding policies, since the decoder may not have the knowledge of power constraint at the encoder, the decoder cannot extract the subjective prior distribution of the encoder using the announced encoding policy. Thus, from an affine policy announcement, the decoder may not realize that its prior is inconsistent in general.

In the following theorem, we analyze the Stackelberg equilibrium when there is an affine policy restriction and there exists a soft power constraint at the encoder.

Theorem 4.5.3. *In the quadratic signaling games with Gaussian subjective priors under a soft power constraint, the affine Stackelberg equilibrium is informative if*

$$\lambda\sigma_e^2\sigma_W^2 < \sigma_d^2(2\sigma_e^2 + 2(\mu_e - \mu_d)^2 - \sigma_d^2). \quad (4.7)$$

When (4.7) does not hold, there exists an informative affine Stackelberg equilibrium if the following conditions simultaneously hold:

$$3(\sigma_e^2 + (\mu_d - \mu_e)^2) < 2\sigma_d^2, \quad (4.8)$$

$$4\lambda\sigma_e^2\sigma_W^2(\sigma_d^2 - \sigma_e^2 - (\mu_e - \mu_d)^2) \leq \sigma_d^2(\sigma_e^2 + (\mu_e - \mu_d)^2)^2. \quad (4.9)$$

Otherwise, the affine Stackelberg equilibrium is noninformative.

Proof. See Section 4.7.2. □

Remark 4.5.6. It is noted that the nature of the affine Stackelberg equilibrium can be noninformative or informative depending on the system parameters. This is because the conditions in (4.7)-(4.9) may or may not hold. For instance, the following are examples of informative and noninformative scenarios:

1. Let $\sigma_e^2 = 1$, $\sigma_d^2 = 4$, $\sigma_W^2 = 0.25$, $\lambda = 2$ and $\mu_e = \mu_d$. In this case, (4.8) holds whereas (4.7) and (4.9) do not hold, which lead to a noninformative equilibrium.
2. Let $\sigma_e^2 = 1$, $\sigma_d^2 = 4$, $\sigma_W^2 = 0.25$, $\lambda = 1$ and $\mu_e = \mu_d$. As (4.8) and (4.9) are satisfied in this case, the affine Stackelberg equilibrium is informative.

Remark 4.5.7. The conditions in Theorem 4.5.3 for the informativeness of the affine Stackelberg equilibrium depend on the subjective priors of both players. In particular, Theorem 4.5.3 shows that the Stackelberg equilibrium solution in general does not coincide with a team theoretic solution when the common prior is the subjective prior of either of the players.

Remark 4.5.8. When the priors are consistent and the source is zero-mean, i.e., $\mu_e = \mu_d = 0$ and $\sigma_e^2 = \sigma_d^2 = \sigma_M^2$, then (4.7) turns into the condition that $\lambda\sigma_W^2 < \sigma_M^2$ whereas (4.8) is always violated. Therefore, we recover the result of [15, Theorem 4.5] (with $b = 0$) whose proof indicates that the affine equilibrium is informative if $\lambda < \sigma_M^2/\sigma_W^2$ and noninformative otherwise.

Next, we analyze the Stackelberg equilibrium under the affine policy restriction when there is a hard power constraint at the encoder. Due to prior mismatch, the affine equilibrium solution may be noninformative in contrast with the informative nature of team theoretic solution. The following theorem presents a necessary and sufficient condition for the informativeness of the affine Stackelberg equilibria.

Theorem 4.5.4. *In the quadratic signaling games with Gaussian subjective priors under a hard power constraint, there exists a unique informative affine Stackelberg*

equilibrium if³

$$\frac{\sigma_e^2}{\sigma_e^2 + (\mu_e - \mu_d)^2} - \frac{2\sigma_e^2}{\sigma_d^2} \leq \frac{\bar{P}}{\sigma_W^2}. \quad (4.10)$$

Otherwise, the affine equilibrium is always noninformative.

Proof. See Section 4.7.3. □

Remark 4.5.9. When $\mu_e = \mu_d = \mu_M$ and $\sigma_e = \sigma_d = \sigma_M$, i.e., the team theoretic setup, the left hand side of (4.10) becomes negative leading to an informative scenario, as expected.

Remark 4.5.10. Unlike the team theoretic solution which is always informative, the affine Stackelberg equilibrium solution is either informative or noninformative depending on the system parameters.

Now, we derive the costs at the affine Stackelberg equilibrium in the hard power constrained case to illustrate the effects of subjectivity.

Theorem 4.5.5. *In the quadratic signaling games with Gaussian subjective priors under a hard power constraint, if (4.10) holds, then the cost of the encoder and decoder at the informative affine Stackelberg equilibrium are*

$$J_s^{*,e} = \frac{\bar{P}\sigma_d^4\sigma_e^2\sigma_W^2 + \sigma_W^4\sigma_e^6 + \sigma_W^4\sigma_e^4(\mu_e - \mu_d)^2}{(\bar{P}\sigma_d^2 + \sigma_e^2\sigma_W^2)^2}, \quad (4.11)$$

$$J_s^{*,d} = \frac{\sigma_e^2\sigma_d^2\sigma_W^2}{\bar{P}\sigma_d^2 + \sigma_e^2\sigma_W^2}. \quad (4.12)$$

On the other hand, if (4.10) does not hold, the affine Stackelberg equilibrium is noninformative with costs $J_s^{*,e} = \sigma_e^2 + (\mu_e - \mu_d)^2$ and $J_s^{*,d} = \sigma_d^2$.

Proof. See Section 4.7.4. □

³In the case of equality in (4.10), it is possible to have a noninformative equilibrium as well and the informative and noninformative equilibria induce the same cost for the encoder in this case.

It is noted that if both players share a common prior with $\sigma_e = \sigma_d = \sigma_M$ and $\mu_e = \mu_d = \mu_M$ leading to a team theoretic setup, then (4.11) and (4.12) reduces to

$$J_t^{*,e} = J_t^{*,d} = \frac{\sigma_M^2 \sigma_W^2}{P + \sigma_W^2}. \quad (4.13)$$

Remark 4.5.11. Note that one can analyze the Stackelberg equilibrium solutions under affine policy restriction to investigate continuity and robustness properties. On the other hand, we can conclude the continuity result directly from Theorem 4.5.1. In particular, the analysis in the proof of Theorem 4.5.1 can be carried out when the optimization problem imposes an additional affine policy restriction. This implies that for the Gaussian prior case, the Stackelberg equilibrium cost of the encoder under affine policy restriction is robust with respect to perturbations around the team setup under either a soft or a hard power constraint.

4.6 Conclusion

We have investigated Stackelberg equilibria for quadratic signaling games under subjective/inconsistent priors. We have established that the Stackelberg equilibrium cost is upper or lower semi continuous around the point of identical priors and in the particular case of Gaussian priors it is continuous, which shows the robustness of the Stackelberg equilibria around the team setup. Moreover, we have shown that for Gaussian signaling, the Stackelberg equilibrium solution is in general nonlinear, which is in contrast to the linearity property of team optimal solution. This shows that team theoretic optimality of affine/linear policies is not robust to perturbations. Finally, we have investigated the Stackelberg equilibrium solution under affine policy restriction. In particular, we have shown that depending on certain condition stated explicitly, the affine Stackelberg equilibrium solution can be informative or noninformative.

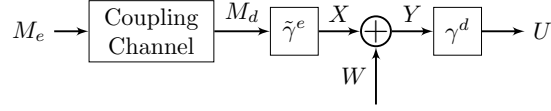


Figure 4.2: Suboptimal scheme used in proving upper semi continuity result where the coupling channel $f(\cdot)$ is such that marginal distribution of its output is $\phi_d(\cdot)$.

4.7 Proofs

4.7.1 Proof of Theorem 4.5.1

In the following, we focus on the case with a soft power constraint at the encoder and the case with a hard power constraint can be shown in a similar manner.

- (i) Let M_e and M_d denote random variables distributed according to the priors $\phi_e(\cdot)$ and $\phi_d(\cdot)$, respectively. The Stackelberg equilibrium cost is given by

$$\min_{\gamma^e(\cdot)} \mathbb{E}[(m_e - u)^2] + \lambda \mathbb{E}[\gamma^e(m_e)^2] \quad (4.14)$$

where there is an implicit constraint that the decoder employs its best response with respect to its own prior. In particular, the decoder's action is a deterministic function of its observation given by $u = \mathbb{E}_d[m|\gamma^e(m) + w]$. Now consider a suboptimal scheme in which the encoder maps its observation into an auxiliary variable via $f(\cdot)$ which is then mapped into the transmitted message through $\tilde{\gamma}^e(\cdot)$ under constraint that the marginal distribution of this auxiliary random variable is fixed to $\phi_d(\cdot)$. This scheme is depicted in Fig. 4.2. By introducing such an auxiliary random variable which has a fixed marginal distribution, the proposed scheme is in general a suboptimal scheme. Here, the encoder is given by $\gamma^e(m_e) = \tilde{\gamma}^e(f(m_e))$

where $f(m_e) \triangleq m_d$ has a fixed marginal distribution $\phi_d(\cdot)$. Observe that

$$\begin{aligned}
& \min_{\gamma^e(\cdot)} \mathbb{E}[(m_e - u)^2] + \lambda \mathbb{E}[\gamma^e(m_e)^2] \\
& \leq \min_{\tilde{\gamma}^e(\cdot), f(\cdot)} \mathbb{E}[(m_e - u)^2] + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \\
& = \min_{\tilde{\gamma}^e(\cdot), f(\cdot)} \mathbb{E}[(m_e - m_d + m_d - u)^2] + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \\
& = \min_{\tilde{\gamma}^e(\cdot), f(\cdot)} \mathbb{E}[(m_e - m_d)^2] + \mathbb{E}[(m_d - u)^2] \\
& \quad + 2\mathbb{E}[(m_e - m_d)(m_d - u)] + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \\
& \leq \min_{\tilde{\gamma}^e(\cdot), f(\cdot)} \mathbb{E}[(m_e - m_d)^2] + \mathbb{E}[(m_d - u)^2] \\
& \quad + 2\sqrt{\mathbb{E}[(m_e - m_d)^2]}\sqrt{\mathbb{E}[(m_d - u)^2]} + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \tag{4.15}
\end{aligned}$$

where the first inequality is due to suboptimality of the scheme with $f(\cdot)$ and $\tilde{\gamma}^e(\cdot)$, and the second inequality follows from Cauchy-Schwarz inequality. The final optimization problem in (4.15) can be first solved with respect to $f(\cdot)$ which leads to

$$\begin{aligned}
& \min_{\tilde{\gamma}^e(\cdot), f(\cdot)} \mathbb{E}[(m_e - m_d)^2] + \mathbb{E}[(m_d - u)^2] + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \\
& \quad + 2\sqrt{\mathbb{E}[(m_e - m_d)^2]}\sqrt{\mathbb{E}[(m_d - u)^2]} \\
& = \min_{\tilde{\gamma}^e(\cdot)} W_2(\phi_e, \phi_d)^2 + \mathbb{E}[(m_d - u)^2] + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \\
& \quad + 2W_2(\phi_e, \phi_d)\sqrt{\mathbb{E}[(m_d - u)^2]} \tag{4.16}
\end{aligned}$$

where $W_2(\phi_e, \phi_d)$ denotes quadratic Wasserstein distance between the distributions $\phi_e(\cdot)$ and $\phi_d(\cdot)$. Note that the optimal value for the solution to (4.16) is finite when $W_2(\phi_e, \phi_d)$ is finite. To see this, observe that if $\tilde{\gamma}^e(m_d) = 0$ for all m_d , then the optimal decoder action becomes $u = \mu_d$ where μ_d denotes the mean of the source message from decoder's perspective. Then, the objective function in (4.16) takes the value of $(W_2(\phi_e, \phi_d) + \mathbb{E}[(m_d - \mu_d)^2]^{1/2})^2$, which is finite as the variance of the source is finite from the perspective of decoder. Thus, an optimal solution to (4.16) must yield a finite objective value. From this observation, it follows that as $\phi_e \rightarrow \phi_d$, the optimal solution of (4.16) leads to a finite value for the term

$\mathbb{E}[(m_d - u)^2]$ since $W_2(\phi_e, \phi_d) \rightarrow 0$ in this case. As a result, we get

$$\begin{aligned}
& \limsup_{\phi_e \rightarrow \phi_d} \min_{\gamma^e(\cdot)} \mathbb{E}[(m_e - u)^2] + \lambda \mathbb{E}[\gamma^e(m_e)^2] \\
& \leq \limsup_{\phi_e \rightarrow \phi_d} \min_{\tilde{\gamma}^e(\cdot)} W_2(\phi_e, \phi_d)^2 + \mathbb{E}[(m_d - u)^2] \\
& \quad + 2 W_2(\phi_e, \phi_d) \sqrt{\mathbb{E}[(m_d - u)^2]} + \lambda \mathbb{E}[\tilde{\gamma}^e(m_d)^2] \\
& = \mathbb{E}[(m_d - \mathbb{E}[m_d | \gamma^{*,e}(m_d) + w])^2] + \lambda \mathbb{E}[\gamma^{*,e}(m_d)^2]
\end{aligned}$$

where the inequality is due to (4.15) and (4.16), $\gamma^{*,e}(\cdot)$ denotes the optimal encoding policy under team theoretic setup with common prior distribution $\phi_d(\cdot)$ and the equality follows from the facts that $\mathbb{E}[(m_d - u)^2]$ is finite and $W_2(\phi_e, \phi_d) \rightarrow 0$ as $\phi_e \rightarrow \phi_d$. Since the last term corresponds to the cost for team theoretic setup with prior $\phi_d(\cdot)$, this analysis shows that the Stackelberg equilibrium cost is upper semi continuous around the team setup.

- (ii) We note that if the decoder employs the encoder's prior rather than its own prior, the objective function of the encoder is improved. Therefore, the following inequality holds:

$$\begin{aligned}
& \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_d[m | \gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2] \\
& \geq \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_e[m | \gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2]
\end{aligned} \tag{4.17}$$

For the latter optimization problem, it is well-known that the optimal encoding and decoding policies are affine. By using this observation, we get

$$\begin{aligned}
& \liminf_{\phi_e \rightarrow \phi_d} \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_d[m | \gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2] \\
& \geq \liminf_{\phi_e \rightarrow \phi_d} \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_e[m | \gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2] \\
& = \liminf_{\phi_e \rightarrow \phi_d} \mathbb{E}_e[(m - \beta_e^*(\alpha_e^*(m - \mu_e) + w) - \mu_e)^2] + \lambda \mathbb{E}_e[(\alpha_e^*(m - \mu_e))^2] \\
& = \mathbb{E}_d[(m - \beta_d^*(\alpha_d^*(m - \mu_d) + w) - \mu_d)^2] + \lambda \mathbb{E}_d[(\alpha_d^*(m - \mu_d))^2]
\end{aligned} \tag{4.18}$$

where μ_e and μ_d denote the mean of the source from the perspective of the encoder and the decoder, respectively, and α_e^* and β_e^* (respectively α_d^* and β_d^*) are the optimal coefficients at the linear encoder and decoder for

the problem of transmitting a zero-mean Gaussian source $(M_e - \mu_e)$ (respectively $(M_d - \mu_d)$) over an independent additive Gaussian channel under quadratic criterion with a soft power constraint at the encoder. This establishes the lower semi continuity of the Stackelberg equilibrium cost around the point of identical priors.

(iii) The result follows from (i) and (ii).

(iv) The lower semi continuity result in this case can be established easily. In particular, we note that the inequality in (4.17) holds for general subjective priors. In other words, if the decoder employs encoder's prior rather than its own prior while computing its estimate, the cost of the encoder improves. From this inequality, we get

$$\begin{aligned} & \liminf_{\phi_d \rightarrow \phi_e} \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_d[m|\gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2] \\ & \geq \liminf_{\phi_d \rightarrow \phi_e} \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_e[m|\gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2] \\ & = \min_{\gamma^e(\cdot)} \mathbb{E}_e[(m - \mathbb{E}_e[m|\gamma^e(m) + w])^2] + \lambda \mathbb{E}_e[x^2], \end{aligned}$$

where the equality follows from the fact that the optimization problem does not depend on $\phi_d(\cdot)$. This proves the lower semi continuity of encoder's cost.

4.7.2 Proof of Theorem 4.5.3

If the encoder is affine in the form $x = \gamma^e(m) = Am + C$, we know that the optimal decoder is affine with $u^* = \gamma^{*,d}(y) = \mathbb{E}_d[m|y] = \frac{A\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2}y + \frac{\sigma_W^2\mu_d - AC\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2}$, where $y = Am + C + w$. Then, by inserting the best response of the decoder into the objective function of the encoder and after some manipulation, the goal of the encoder becomes

$$\begin{aligned} & \min_{x=\gamma^e(m)=Am+C} \mathbb{E}_e[(m - u)^2 + \lambda x^2] \\ & = \min_{A, C} \frac{\sigma_W^4((\mu_e - \mu_d)^2 + \sigma_e^2 - \sigma_d^2)}{(A^2\sigma_d^2 + \sigma_W^2)^2} \\ & \quad + \frac{\sigma_d^2\sigma_W^2}{A^2\sigma_d^2 + \sigma_W^2} + \lambda A^2\sigma_e^2 + \lambda(A\mu_e + C)^2. \end{aligned} \tag{4.19}$$

Here, the optimal encoder cost is achieved when $C^* = -A^*\mu_e$, and A^* can be found by solving

$$A^* = \arg \min_A \frac{\sigma_W^4(\sigma_e^2 + (\mu_e - \mu_d)^2 - \sigma_d^2)}{(A^2\sigma_d^2 + \sigma_W^2)^2} + \frac{\sigma_d^2\sigma_W^2}{A^2\sigma_d^2 + \sigma_W^2} + \lambda A^2\sigma_e^2 \triangleq f(A^2). \quad (4.20)$$

In order to investigate the optimization problem in (4.20), we need the first and second derivatives of $f(A^2)$ with respect to A^2 . Taking the first derivative of $f(A^2)$ with respect to A^2 , we get

$$\frac{df(A^2)}{d(A^2)} = -\frac{2\sigma_d^2\sigma_W^4(\sigma_e^2 + (\mu_e - \mu_d)^2 - \sigma_d^2)}{(A^2\sigma_d^2 + \sigma_W^2)^3} - \frac{\sigma_d^4\sigma_W^2}{(A^2\sigma_d^2 + \sigma_W^2)^2} + \lambda\sigma_e^2.$$

Differentiating further leads to

$$\frac{d^2f(A^2)}{d(A^2)^2} = \frac{\sigma_d^4\sigma_W^4(6(\mu_e - \mu_d)^2 + 6\sigma_e^2 - 4\sigma_d^2) + 2A^2\sigma_d^8\sigma_W^2}{(A^2\sigma_d^2 + \sigma_W^2)^4}.$$

It is seen that if (4.8) holds, the objective function is first concave and then convex when A^2 is increased starting from zero. If (4.8) does not hold, the objective function is always convex. Therefore, a solution to (4.20) always exists.

Now, we wish to see if the optimal solution of (4.20) leads to $A \neq 0$ or not. If the first derivative at $A^2 = 0$ is negative, it means that there exists $A \neq 0$ which induces a lower cost than the noninformative scenario for the encoder. Therefore, the affine Stackelberg equilibrium is informative under the condition that $\frac{df(A^2)}{d(A^2)} < 0$ when $A = 0$, and this condition can be expressed as in (4.7).

If (4.7) does not hold, then the objective function is non-decreasing at $A^2 = 0$. Notice that if (4.8) does not hold, then the objective function becomes strictly convex when $A^2 > 0$. Since a strictly convex function with a non-negative first derivative at $A^2 = 0$ is an increasing function for $A^2 > 0$, the optimal solution becomes noninformative with $A^* = 0$ in this case.

In the remainder of the proof, we consider the remaining scenario when (4.8) holds and (4.7) does not hold. In this case, we wish to see if there exists a solution to $f(x) = f(0)$ with a positive x . We note that in this remaining case, $f(x)$ is non-decreasing at $x = 0$ and strictly concave when $x < 0$. Hence, when there

exists a solution to $f(x) = f(0)$, this solution must be attained with a positive x . Now, we are looking for a solution to the following equation for an informative equilibrium:

$$\frac{\sigma_W^4(\sigma_e^2 + (\mu_e - \mu_d)^2 - \sigma_d^2)}{(A^2\sigma_d^2 + \sigma_W^2)^2} + \frac{\sigma_d^2\sigma_W^2}{A^2\sigma_d^2 + \sigma_W^2} + \lambda A^2\sigma_e^2 = \sigma_e^2 + (\mu_e - \mu_d)^2. \quad (4.21)$$

After some manipulation, it can be shown that (4.21) is equivalent to

$$\begin{aligned} & A^4\lambda\sigma_d^4\sigma_e^2 + A^2\left(2\lambda\sigma_e^2\sigma_d^2\sigma_W^2 - \sigma_d^4(\sigma_e^2 + (\mu_e - \mu_d)^2)\right) \\ & + \sigma_d^4\sigma_W^2 + \lambda\sigma_e^2\sigma_W^4 - 2\sigma_d^2\sigma_W^2(\sigma_e^2 + (\mu_e - \mu_d)^2) = 0, \end{aligned}$$

which is a second order polynomial in A^2 whose discriminant is expressed as $\Delta \triangleq \sigma_d^8(\sigma_e^2 + (\mu_e - \mu_d)^2)^2 + 4\lambda\sigma_e^2\sigma_d^2\sigma_W^2(\sigma_e^2 + (\mu_e - \mu_d)^2 - \sigma_d^2)$. Hence, when $\Delta \geq 0$ or equivalently (4.9) holds, there exists a solution to (4.21). This implies that there exists an informative affine Stackelberg equilibrium in this case.

4.7.3 Proof of Theorem 4.5.4

If the encoder is affine with $x = \gamma^e(m) = Am + C$, then the optimal decoder is $u^* = \gamma^{*,d}(y) = \mathbb{E}_d[m|y] = \frac{A\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2}y + \frac{\sigma_W^2\mu_d - AC\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2}$, where $y = Am + C + w$. Then, after inserting the best response of the decoder, the objective function of the encoder becomes

$$\begin{aligned} \mathbb{E}_e[(m - u)^2] &= \mathbb{E}_e[(m - AKm - KC - Kw - L)^2] \\ &= \mathbb{E}_e\left[\left(\frac{\sigma_W^2}{A^2\sigma_d^2 + \sigma_W^2}(m - \mu_d)\right)^2\right] + K^2\sigma_W^2 \\ &= \frac{A^2\sigma_d^4\sigma_W^2 + \sigma_W^4(\sigma_e^2 + (\mu_e - \mu_d)^2)}{(A^2\sigma_d^2 + \sigma_W^2)^2} \triangleq f(A^2) \end{aligned} \quad (4.22)$$

Thus, the goal of the encoder is to solve the following optimization problem:

$$J^{*,e} = \min_{A, C} f(A^2) \quad \text{s.t.} \quad (A\mu_e + C)^2 + A^2\sigma_e^2 \leq \bar{P}.$$

It is seen that the cost to be minimized is independent of C . Therefore, if we set $C = -A\mu_e$, we obtain all feasible values of A . Hence, the optimization problem

can be solved under the constraint that $A^2\sigma_e^2 \leq \bar{P}$. Taking the derivative of the objective function with respect to A^2 , we get

$$\frac{df(A^2)}{dA^2} = \frac{\sigma_d^2\sigma_W^2(A^2\sigma_d^2 + \sigma_W^2)}{(A^2\sigma_d^2 + \sigma_W^2)^4} \times \underbrace{\left(\sigma_W^2(\sigma_d^2 - 2\sigma_e^2 - 2(\mu_e - \mu_d)^2) - A^2\sigma_d^4\right)}_{\triangleq \mathfrak{P}}.$$

Here, if $\sigma_d^2 - 2\sigma_e^2 - 2(\mu_e - \mu_d)^2 \leq 0$, then $f(A^2)$ is a decreasing function of A^2 as $\mathfrak{P} < 0$ in this case. Therefore, the minimization can be accomplished by choosing A^2 as large as possible satisfying the constraint, which makes the optimal encoding policy $(A^*)^2 = \frac{\bar{P}}{\sigma_e^2}$ and $C^* = -A^*\mu_e$.

On the other hand, if $\sigma_d^2 - 2\sigma_e^2 - 2(\mu_e - \mu_d)^2 > 0$, then $f(A^2)$ is first an increasing and then a decreasing function of A^2 . As the goal is to minimize the cost, the optimal A^2 value is obtained by choosing either the largest or the smallest A^2 depending on the value of $\bar{P}$. The objective function takes the value of $\sigma_e^2 + (\mu_e - \mu_d)^2$ when $A^2 = 0$. It is necessary to check if the solution of $f(A^2) = \sigma_e^2 + (\mu_e - \mu_d)^2$ for nonzero A^2 is feasible. In particular, $f(\tilde{A}^2) = (\sigma_e^2 + (\mu_e - \mu_d)^2)$ yields $\tilde{A}^2 = \frac{\sigma_W^2(\sigma_d^2 - 2\sigma_e^2 - 2(\mu_e - \mu_d)^2)}{\sigma_d^2(\sigma_e^2 + (\mu_e - \mu_d)^2)}$ and such a value is feasible when $\tilde{A}^2\sigma_e^2 \leq \bar{P}$. As a result, if (4.10) holds, then there exists an informative affine equilibrium with $A^* = \pm\sqrt{\frac{\bar{P}}{\sigma_e^2}}$. On the other hand, if (4.10) does not hold, then the largest possible A^2 gives an objective value larger than the objective value at $A = 0$, which leads to a noninformative equilibrium. Hence, for a noninformative equilibrium, we need $\frac{\sigma_W^2\sigma_e^2(\sigma_d^2 - 2\sigma_e^2 - 2(\mu_e - \mu_d)^2)}{\sigma_d^2(\sigma_e^2 + (\mu_e - \mu_d)^2)} > \bar{P}$ and $\sigma_d^2 - 2\sigma_e^2 - 2(\mu_e - \mu_d)^2 > 0$ to hold simultaneously and otherwise the affine equilibrium is informative. Since the former condition implies the latter condition, the former condition becomes the only condition which leads to a noninformative equilibrium.

4.7.4 Proof of Theorem 4.5.5

As shown in Theorem 4.5.4, an encoding policy $\gamma^{*,e}(m) = Am + C$ with $A = \sqrt{\bar{P}}/\sigma_e$ and $C = -A\mu_e$ leads to an informative affine equilibrium. In response to this encoding policy, the decoder takes its optimal action as $u^* = \gamma^{*,d}(y) = Ky + L = AKm + KC + L + Kw$ with $K = \frac{\sqrt{\bar{P}}\sigma_d^2\sigma_e}{\bar{P}\sigma_d^2 + \sigma_e^2\sigma_W^2}$ and $L = \mu_d - K\sqrt{\bar{P}}(\mu_d - \mu_e)/\sigma_e$.

By inserting the optimal value of A into (4.22), the cost of the encoder becomes as in (4.11). Now, we derive the cost of the decoder at the informative equilibrium. By using best response characterization of the decoder, it can be shown that $KC + L = (1 - AK)\mu_d$. Then, the cost of the decoder becomes

$$\begin{aligned}
\mathbb{E}_d[(m - u)^2] &= \mathbb{E}_d[(m - K(Am + C + w) - L)^2] \\
&= \mathbb{E}_d[((1 - AK)m - (KC + L))^2] + K^2\sigma_W^2 \\
&= (1 - AK)^2\mathbb{E}_d[(m - \mu_d)^2] + K^2\sigma_W^2 \\
&= \frac{\sigma_W^4\sigma_e^4}{(\bar{P}\sigma_d^2 + \sigma_W^2\sigma_e^2)^2}\sigma_d^2 + \frac{\bar{P}\sigma_d^4\sigma_e^2\sigma_W^2}{(\bar{P}\sigma_d^2 + \sigma_W^2\sigma_e^2)^2} \\
&= \frac{\sigma_e^2\sigma_d^2\sigma_W^2}{\bar{P}\sigma_d^2 + \sigma_e^2\sigma_W^2}.
\end{aligned}$$

When (4.10) does not hold, an encoding policy $\gamma^e(m) = C$ with $C \leq \sqrt{\bar{P}}$ leads to an affine Stackelberg equilibrium. In this case, the best response of the decoder becomes $\gamma^d(y) = \mathbb{E}_d[m|y] = \mathbb{E}_d[m] = \mu_d$. Then, the costs are given by

$$\begin{aligned}
J_s^{*,e} &= \mathbb{E}_e[(m - u)^2] = \mathbb{E}_e[(m - \mu_d)^2] = \sigma_e^2 + (\mu_e - \mu_d)^2, \\
J_s^{*,d} &= \mathbb{E}_d[(m - u)^2] = \mathbb{E}_d[(m - \mu_d)^2] = \sigma_d^2.
\end{aligned}$$

Chapter 5

Quadratic Signaling Games with General Sources in Higher Dimensions

5.1 Motivation

In a team theoretic setup involving decision makers with a common goal, the decision makers do not wish to hide information to improve the performance since revealing more information cannot lead to a degradation of system performance. On the other hand, there exist setups where the decision makers have misaligned objectives. As mentioned earlier, a foundational contribution is made by Crawford and Sobel by introducing a signaling game problem where an encoder and a decoder with misaligned objectives communicate [12]. Interestingly, although there is no restriction imposed by the system, the encoder must employ quantization policies at a Nash equilibrium (see Theorem 2.5.1 and Theorem 2.5.2 of this dissertation). In Chapter 2, we studied Crawford and Sobel's cheap talk setup with scalar sources; but many applications involve vector sources and an information theoretic formulation requires a multidimensional setting.

In this chapter, our aim is to investigate a multi-dimensional extension of Crawford and Sobel's formulation [12] where the encoder jointly encodes its multi-dimensional observation to a message observed by the decoder. For the multi-dimensional source setting, an important problem is to see whether there exists a linear Nash equilibrium. In [15, Theorem 3.4], it is shown that an equilibrium for quadratic multidimensional cheap talk problem may be linear or quantized in certain cases by providing examples. This means that in contrast to the one-dimensional scenario, having more than one dimension allows the encoder to employ a linear encoder under certain conditions. In this chapter, we completely characterize these conditions under which linear equilibria exist considering independent and identically distributed (i.i.d.) sources. In addition, we investigate geometric properties of decoder actions in equilibrium where the source distribution is arbitrary and the source components can be dependent or independent. Note that in the case of scalar quadratic cheap talk, a necessary condition for equilibria is that the distance between any two decoder action is lower bounded by a constant that depends on the bias term as shown in [15, Theorem 3.2]. Our results on geometric properties can be viewed as a generalization of this result. In contrast to scalar cheap talk setup where equilibria require a distance condition for decoder actions, in the case of multidimensional sources, necessary conditions for equilibria involve expressions containing the bias vector and the decoder actions expressed as vectors.

An important implication of our results is related to information theoretic limits of the cheap talk problem. This problem can be formulated as the maximum rate of communication between an encoder and decoder in a multidimensional signaling setup where the rate is defined as the number of distinct messages per source symbol. This is an important problem in communication theory where the objectives are aligned (i.e., team theoretic setup) but there may be an upper bound on the achievable communication rate due to system requirements such as a power constraint at the encoder and having a noisy channel. This maximum rate of communication possible is Shannon's channel capacity and forms the foundations for many results in information theory. In particular, one wishes to design a communication system achieving this channel capacity in order to convey as

much information as possible. On the other hand, an interesting question arises in a game theoretic setup: Does there exist an upper bound on achievable rate of communication with an encoder and decoder where the encoder and decoder have misaligned objectives? In certain cases, our analysis gives a conclusive answer to this question for the multidimensional cheap talk setup. In particular, if there exists an equilibrium with a linear encoder, then there is no upper bound on the maximum rate of communication. Note that we consider the Nash setup for such an information theoretic problem. We refer the reader to [26] for a Stackelberg setup.

5.2 Literature Review

In this chapter, we formulate a multidimensional cheap talk problem where an encoder observes a multidimensional source and jointly encodes components of the source into a message which is directly observed by a decoder. Since we considered scalar cheap talk problem in Chapter 2, the results as well as literature review presented in that chapter are relevant. The reader is referred to Chapter 2 for details. In addition, cheap talk problems specifically with multidimensional sources have also been considered in the economics literature [45–48, 54]. For instance, [54] investigates a two-dimensional scenario where an encoder communicates with a decoder. Different from our work, in [54], the encoder’s preferences over different decoder actions are primarily determined by preferences in a certain dimension. In particular, if the encoder prefers one decoder action over the other in this dimension, then the decoder action in the second dimension does not matter. In this case, [54] shows the existence of an upper bound on the number of decoder actions. In addition, the work in [45] considers a multidimensional cheap talk problem when there are two senders referred to as experts and a receiver referred to as a policy-maker. These decision makers including the policy-maker have inconsistent ideal points regarding an outcome of the game and these ideal point vectors can be viewed as bias vectors as in our framework. They show that if these ideal point vectors of the experts in the multidimensional space are linearly independent, then there exist fully revealing equilibria. In contrast, we focus

on a scenario with a single encoder which jointly encodes its multi-dimensional observation and employs a single quadratic cost function.

In addition, we provide a rate-distortion theoretic perspective for the cheap talk problem by investigating achievable rate and distortion pairs for the considered game. In that respect, our results have relation with rate-distortion theory in the information theory literature. In rate-distortion theory, the aim is to obtain achievable rate and distortion pairs so that one can better understand the trade off between them. The reader is referred to [142, Chapter 10] for a treatment of rate-distortion theory.

5.3 Contributions

The main aim of this chapter is to analyze a quadratic multidimensional cheap talk problem. In contrast to Chapter 2 where the encoder's observation is scalar valued, in this case the encoder communicates a multidimensional observation to the decoder. Main contributions of this chapter can be summarized as follows:

- (i) We derive a necessary condition that decoder actions in any equilibrium has to satisfy. In particular, we show that two decoder actions in equilibrium cannot get arbitrarily close if the bias vector and the difference of these decoder actions as vectors are not orthogonal (Lemma 5.5.1).
- (ii) We derive necessary conditions that an equilibrium with a continuum of decoder actions needs to satisfy in the case of two-dimensional observations. In particular, we show that a continuum of decoder actions in equilibrium can only exist in a certain direction in the two-dimensional space, which depends on the bias vector (Lemma 5.5.3). In addition, if there exists a continuum of decoder actions, then this continuum of decoder actions must be supported for all possible values in this direction, i.e., there cannot be any discontinuity (Lemma 5.5.4).
- (iii) We completely characterize necessary and sufficient conditions under which

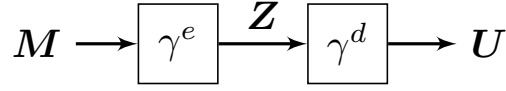


Figure 5.1: Communication setting for the multidimensional cheap talk setup.

linear Nash equilibria exist considering independent and identically distributed two-dimensional observations. In particular, we prove that if the bias vector has nonzero components which are not equal in magnitude, then there exist linear Nash equilibria if and only if the source distribution is Gaussian. In the remaining cases, the existence of linear equilibria depends on whether the distribution of the source is symmetric and whether the signs of bias components are the same (Theorem 5.5.1).

- (iv) We provide a rate-distortion theoretic formulation of cheap talk problem. We obtain achievable rate and distortion values for the particular case of independent and identically distributed Gaussian sources (Theorem 5.6.1).

5.4 Preliminaries

In this chapter, we consider the following multidimensional cheap talk problem where an encoder and a decoder communicates. The encoder knows the value of an n -dimensional random vector $\mathbf{M} = [M_1, \dots, M_n]^T$ where $M_1, \dots, M_n$ are $\mathbb{M}$ -valued i.i.d. random variables.¹ The encoder conveys a message $\mathbf{Z} = [Z_1, \dots, Z_n]^T$ via an encoding policy $\gamma^e(\cdot)$, i.e., $\mathbf{Z} = \gamma^e(\mathbf{M})$, where Z_i is a $\mathbb{Z}_i$ -valued random variable for $i = 1, \dots, n$. The decoder directly observes $\mathbf{Z}$ and takes an action $\mathbf{U} = [U_1, \dots, U_n]^T$ where U_i 's are $\mathbb{M}$ -valued random variables. The aim of the encoder is to minimize $J^e(\gamma^e, \gamma^d) = \mathbb{E}[c^e(\mathbf{M}, \mathbf{U})]$ where

$$c^e(\mathbf{m}, \mathbf{u}) = \sum_{i=1}^n (m_i - u_i - b_i)^2 = \|\mathbf{m} - \mathbf{u} - \mathbf{b}\|^2. \quad (5.1)$$

¹While we present general results for n -dimensional cheap talk problem for any $n \geq 2$, some of our results focus on the case of two-dimensional cheap talk where $n = 2$. Nevertheless, it is also possible to generalize these results to cheap talk with an arbitrary number of i.i.d. sources by using the results for the two-dimensional scenario.

In (5.1), $\mathbf{b}$ denotes a deterministic bias vector which is common knowledge between the players and quantifies the degree of misalignment between the objective functions of the encoder and decoder. On the other hand, the decoder wishes to estimate the source random vector as accurately as possible, and thus, its objective function does not include a bias vector. In particular, the aim of the decoder is to minimize $J^d(\gamma^e, \gamma^d) = \mathbb{E}[c^d(\mathbf{M}, \mathbf{U})]$ where

$$c^d(\mathbf{m}, \mathbf{u}) = \sum_{i=1}^n (m_i - u_i)^2 = \|\mathbf{m} - \mathbf{u}\|^2. \quad (5.2)$$

The communication scenario is depicted in Figure 5.1.

Our aim is to characterize the Nash equilibrium where the decision makers announce their policies at the same time. At a Nash equilibrium, none of the players wishes to unilaterally deviate from their current strategies as their cost cannot get better by doing so. In particular, a set of policies $\gamma^{*,e}$ and $\gamma^{*,d}$ forms a Nash equilibrium if

$$\begin{aligned} J^e(\gamma^{*,e}, \gamma^{*,d}) &\leq J^e(\gamma^e, \gamma^{*,d}) \text{ for all } \gamma^e \in \Gamma^e, \\ J^d(\gamma^{*,e}, \gamma^{*,d}) &\leq J^d(\gamma^{*,e}, \gamma^d) \text{ for all } \gamma^d \in \Gamma^d, \end{aligned} \quad (5.3)$$

where Γ^e and Γ^d are the sets of all deterministic (and Borel measurable) functions from $\mathbb{M}^n$ to $(\mathbb{Z}_1, \dots, \mathbb{Z}_n)$ and from $(\mathbb{Z}_1, \dots, \mathbb{Z}_n)$ to $\mathbb{M}^n$, respectively.

We make the following assumption for the source distribution.

Assumption 5.4.1. Considering each component M_i of the source random vector $\mathbf{M}$, every non-empty open set on its support has a positive measure.

The following is an implication of this assumption. Consider a convex set C with a non-empty interior. Then, its centroid $\mathbb{E}[\mathbf{M} | \mathbf{M} \in C]$ must be in the interior of set C . We will use this implication later in this chapter.

5.5 Nash Equilibria

5.5.1 Necessary Conditions for Equilibria

We first present some preliminary results which are useful for providing conditions that ensure the existence of an equilibrium with a linear encoding policy. To begin with, due to the definition of $J^d(\gamma^e, \gamma^d)$, we know that for a given encoding policy $\mathbf{z} = \gamma^e(\mathbf{m})$, the decoder takes the action $\mathbf{u} = \mathbb{E}[\mathbf{M} | \gamma^e(\mathbf{M}) = \mathbf{z}]$. All such possible values of $\mathbf{u}$ at the equilibria are called the set of decoder actions. While investigating the existence of a linear equilibrium, we make a frequent use of the following geometric definition regarding the set of decoder actions in equilibrium.

Definition 5.5.1. We say that a non-empty set of decoder actions forms a *continuum* if it is a closed and connected set (i.e., it cannot be expressed as the union of two or more disjoint closed sets.)

The following lemma derives a geometry condition that any two decoder actions at a Nash equilibrium need to satisfy. In addition to its usefulness while deriving conditions for the existence of linear equilibria, this result is important on its own as it provides a necessary condition for a Nash equilibrium in terms of the induced decoder actions. It is noted that the following result holds regardless of the source distribution and applies to both i.i.d. and non-i.i.d. sources.

Lemma 5.5.1. Consider n -dimensional cheap talk problem with the source random vector $\mathbf{M} = [M_1, \dots, M_n]^T$ where each element of $\mathbf{M}$ can have different distributions and can be dependent or independent. Let $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ be two bins and let $\mathbf{u}^\alpha = \mathbb{E}[\mathbf{M} | \mathbf{M} \in \mathcal{B}^\alpha]$ and $\mathbf{u}^\beta = \mathbb{E}[\mathbf{M} | \mathbf{M} \in \mathcal{B}^\beta]$ denote their centroids which are the decoder actions taken when the encoder reveals $\mathbf{M} \in \mathcal{B}^\alpha$ and $\mathbf{M} \in \mathcal{B}^\beta$, respectively.

- (i) These decoder actions must satisfy the following necessary condition at a Nash equilibrium:

$$2 |(\mathbf{u}^\beta - \mathbf{u}^\alpha)^T \mathbf{b}| \leq \|\mathbf{u}^\beta - \mathbf{u}^\alpha\|^2. \quad (5.4)$$

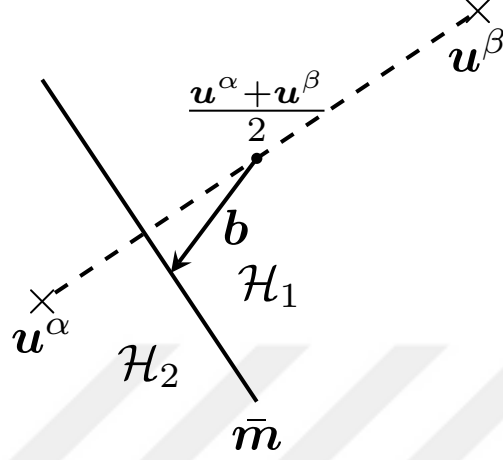


Figure 5.2: Illustration of half spaces induced by decoder actions $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ in Lemma 5.5.1 considering a two-dimensional scenario. Here, the crosses represent the decoder actions and the arrow represents the bias vector. These decoder actions and the bias vector lead to a line of $\bar{\mathbf{m}}$ values for which the encoder is indifferent between reporting these observations as $\mathbf{u}^\alpha$ or $\mathbf{u}^\beta$. On the other hand, the half spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ cannot intersect with $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$, respectively.

- (ii) *At a Nash equilibrium, the encoder decomposes the complete observation space into two regions via a hyperplane orthogonal to $(\mathbf{u}^\alpha - \mathbf{u}^\beta)$ and intersecting the line connecting $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$, and $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ are subsets of these respective regions. In particular, $\mathcal{B}^\alpha$ must be a subset of $\mathbf{m}$ values satisfying $h(\mathbf{m}) \geq 0$ whereas $\mathcal{B}^\beta$ must be a subset of $\mathbf{m}$ values satisfying $h(\mathbf{m}) \leq 0$ where*

$$h(\mathbf{m}) \triangleq \left(\mathbf{m} - \left(\frac{\mathbf{u}^\beta + \mathbf{u}^\alpha}{2} + \mathbf{b} \right) \right)^T (\mathbf{u}^\beta - \mathbf{u}^\alpha), \quad (5.5)$$

and $h(\mathbf{m}) = 0$ defines the hyperplane on which the encoder is indifferent between either decoder actions, i.e., these $\mathbf{m}$ values can belong to both $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$.

- (iii) *At an equilibrium where the encoder uses quantization policies, the quantization bins are always convex.*

Proof. Given two bins $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ and their centroids $\mathbf{u}^\alpha = \mathbb{E}[\mathbf{M} | \mathbf{M} \in \mathcal{B}^\alpha]$ and $\mathbf{u}^\beta = \mathbb{E}[\mathbf{M} | \mathbf{M} \in \mathcal{B}^\beta]$, the aim is to find the values of $\mathbf{m}$ for which the encoder is indifferent between the decoder actions $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$, i.e., for these observations at

the encoder, taking the actions $\mathbf{u}^\alpha$ or $\mathbf{u}^\beta$ by the decoder leads to the same cost for the encoder. These set of indifference points determines possible regions for $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$. In particular, the encoder is indifferent between the decoder actions $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ for values $\bar{\mathbf{m}}$ which satisfy the following:

$$\begin{aligned} c^e(\bar{\mathbf{m}}, \mathbf{u}^\alpha) &= c^e(\bar{\mathbf{m}}, \mathbf{u}^\beta) \\ \Leftrightarrow \|\bar{\mathbf{m}} - \mathbf{u}^\alpha - \mathbf{b}\|^2 &= \|\bar{\mathbf{m}} - \mathbf{u}^\beta - \mathbf{b}\|^2 \\ \Leftrightarrow \left(\bar{\mathbf{m}} - \left(\frac{\mathbf{u}^\beta + \mathbf{u}^\alpha}{2} + \mathbf{b} \right) \right)^T &(\mathbf{u}^\beta - \mathbf{u}^\alpha) = 0. \end{aligned} \quad (5.6)$$

From this final expression, it is seen that these $\bar{\mathbf{m}}$ values for which the encoder is indifferent between the decoder actions $\mathbf{u}^\beta$ and $\mathbf{u}^\alpha$ defines a hyperplane orthogonal to $(\mathbf{u}^\beta - \mathbf{u}^\alpha)$. Note also that for an observation $\mathbf{m} = \bar{\mathbf{m}} + \Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha)$ with $\Delta > 0$ where $\bar{\mathbf{m}}$ satisfies (5.6), the encoder wishes to report this observation as $\mathbf{u}^\beta$ rather than as $\mathbf{u}^\alpha$ since the following holds:

$$\begin{aligned} c^e(\bar{\mathbf{m}} + \Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha), \mathbf{u}^\beta) &- c^e(\bar{\mathbf{m}} + \Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha), \mathbf{u}^\alpha) \\ &= \|\mathbf{m} - \mathbf{u}^\beta - \mathbf{b}\|^2 - \|\mathbf{m} - \mathbf{u}^\alpha - \mathbf{b}\|^2 \\ &= (\mathbf{u}^\beta + \mathbf{u}^\alpha + 2\mathbf{b} - 2\mathbf{m})^T (\mathbf{u}^\beta - \mathbf{u}^\alpha) \\ &= (\mathbf{u}^\beta + \mathbf{u}^\alpha + 2\mathbf{b} - 2\bar{\mathbf{m}} - 2\Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha))^T (\mathbf{u}^\beta - \mathbf{u}^\alpha) \\ &= (-2\Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha))^T (\mathbf{u}^\beta - \mathbf{u}^\alpha) \\ &= -2\Delta\|\mathbf{u}^\beta - \mathbf{u}^\alpha\|^2 < 0. \end{aligned} \quad (5.7)$$

This implies that $\mathcal{B}^\alpha$ and $\mathcal{H}_1$ are disjoint sets where

$$\mathcal{H}_1 \triangleq \{\mathbf{m} \mid \mathbf{m} = \bar{\mathbf{m}} + \Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha) \text{ where } \bar{\mathbf{m}} \text{ satisfies (5.6) and } \Delta > 0\}. \quad (5.8)$$

Similarly, for any observation $\mathbf{m} = \bar{\mathbf{m}} + \Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha)$ with $\Delta < 0$, the encoder wishes to report this observation as $\mathbf{u}^\alpha$ rather than as $\mathbf{u}^\beta$. It follows that $\mathcal{B}^\beta$ and $\mathcal{H}_2$ are disjoint sets where

$$\mathcal{H}_2 \triangleq \{\mathbf{m} \mid \mathbf{m} = \bar{\mathbf{m}} + \Delta(\mathbf{u}^\beta - \mathbf{u}^\alpha) \text{ where } \bar{\mathbf{m}} \text{ satisfies (5.6) and } \Delta < 0\}. \quad (5.9)$$

Hence, the plane specified by (5.6) decomposes the complete observation space into two regions $\mathcal{H}_1$ and $\mathcal{H}_2$ (i.e., half spaces) where the encoder strictly prefers reporting its observation as $\mathbf{u}^\alpha$ or $\mathbf{u}^\beta$ in respective regions, and on this plane the

encoder is indifferent between reporting its observation as either of the decoder actions. Fig. 5.2 illustrates these regions for an example setup.

Furthermore, the plane specified by (5.6) intersects the affine set $\lambda \mathbf{u}^\beta + (1 - \lambda) \mathbf{u}^\alpha$ with $\lambda \in \mathbb{R}$ at a single point due to the fact that $(\mathbf{u}^\beta - \mathbf{u}^\alpha)$ and the hyperplane specified by (5.6) are orthogonal. In order to find the value of λ that gives this intersection point, we solve (5.6) and $\bar{\mathbf{m}} = \bar{\lambda} \mathbf{u}^\beta + (1 - \bar{\lambda}) \mathbf{u}^\alpha$ together and obtain an expression for $\bar{\lambda}$ in the following:

$$\begin{aligned}
& (2\bar{\mathbf{m}} - \mathbf{u}^\beta - \mathbf{u}^\alpha - 2\mathbf{b})^T (\mathbf{u}^\beta - \mathbf{u}^\alpha) = 0 \\
& \Leftrightarrow (2\bar{\lambda} \mathbf{u}^\beta + (2 - 2\bar{\lambda}) \mathbf{u}^\alpha - \mathbf{u}^\beta - \mathbf{u}^\alpha - 2\mathbf{b})^T (\mathbf{u}^\beta - \mathbf{u}^\alpha) = 0 \\
& \Leftrightarrow ((1 - 2\bar{\lambda})(\mathbf{u}^\beta - \mathbf{u}^\alpha) + 2\mathbf{b})^T (\mathbf{u}^\beta - \mathbf{u}^\alpha) = 0 \\
& \Leftrightarrow (2\bar{\lambda} - 1) \|\mathbf{u}^\beta - \mathbf{u}^\alpha\|^2 = 2(\mathbf{u}^\beta - \mathbf{u}^\alpha)^T \mathbf{b} \\
& \Leftrightarrow (2\bar{\lambda} - 1) = \frac{2(\mathbf{u}^\beta - \mathbf{u}^\alpha)^T \mathbf{b}}{\|\mathbf{u}^\beta - \mathbf{u}^\alpha\|^2}. \tag{5.10}
\end{aligned}$$

We know that centroid conditions require $\mathbf{u}^\alpha = \mathbb{E}[\mathbf{M} | \mathbf{M} \in \mathcal{B}^\alpha]$ and $\mathbf{u}^\beta = \mathbb{E}[\mathbf{M} | \mathbf{M} \in \mathcal{B}^\beta]$. Since $\mathcal{B}^\alpha$ and $\mathcal{H}_1$ are disjoint sets, and $\mathcal{B}^\beta$ and $\mathcal{H}_2$ are disjoint sets, we need $0 \leq \bar{\lambda} \leq 1$ which is equivalent to $|2\bar{\lambda} - 1| \leq 1$. By combining this inequality with (5.10), it follows that (5.4) holds.

The convexity of the quantization bins follows from the fact that the decision regions for two decoder actions must be constructed by computing and intersecting half spaces. The intersection of these half spaces yields convex bins. $\square$

Remark 5.5.1. In the special case of one-dimensional cheap talk as investigated in [15, Theorem 3.2], the condition in (5.4) reduces to $|u^\alpha - u^\beta| \geq 2|b|$. In fact, the only way this inequality holds with equality is when one of the decoder actions is at the boundary of the support. In this case, the bin for the corresponding decoder action, say u^β , becomes a singleton consisting only of u^β , i.e., $\mathcal{B}^\beta = \{u^\beta\}$. In this case, the scenario without the bin $\mathcal{B}^\beta = \{u^\beta\}$ is still an equilibrium which is equivalent to the equilibrium with $\mathcal{B}^\beta = \{u^\beta\}$. Therefore, for one-dimensional cheap talk, a necessary condition for an equilibrium can essentially be stated as $|u^\alpha - u^\beta| > 2|b|$ by disregarding the decoder action at the boundary of the support.

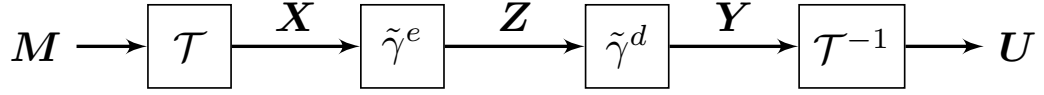


Figure 5.3: Equivalent formulation where $\mathcal{T}$ denotes the linear transformation specified by $X_1 = b_1 M_2 - b_2 M_1$ and $X_2 = b_1 M_1 + b_2 M_2$, and $\mathcal{T}^{-1}$ denotes its inverse.

Remark 5.5.2. In the case of one-dimensional cheap talk, the condition (5.4) directly implies that the equilibrium must involve quantization policies as concluded in [15, Theorem 3.2]. In contrast, such a direct conclusion does not hold for the multidimensional cheap talk problem. Nevertheless, this condition is useful in deriving conditions for the existence of linear equilibria when the source is multidimensional.

Lemma 5.5.1 presents a geometry condition that any two decoder actions at the equilibrium must satisfy. It is important to emphasize that this condition applies to any joint distribution for the multidimensional observations. In particular, the result of Lemma 5.5.1 holds even for joint distributions which are not independent and identically distributed since the derivation of the condition in Lemma 5.5.1 uses only the cost function of the encoder and does not use the joint distribution of the sources.

Now, we focus on equilibria with continuously connected set of decoder actions. We provide a necessary condition that an equilibrium with decoder actions supported on a continuum has to satisfy. An equilibrium with a continuum of decoder actions means that the encoding policy is not a quantization policy.

In the remainder of this section, we specialize to the case of two-dimensional cheap talk problem where $n = 2$.

Lemma 5.5.1 implies that for decoder actions $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ satisfying $(\mathbf{u}^\beta - \mathbf{u}^\alpha)^T \mathbf{b} = 0$, it is possible to make their distance $\|\mathbf{u}^\alpha - \mathbf{u}^\beta\|$ arbitrarily small. On the other hand, for decoder actions $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ with $(\mathbf{u}^\beta - \mathbf{u}^\alpha)^T \mathbf{b} \neq 0$, since the distance $\|\mathbf{u}^\alpha - \mathbf{u}^\beta\|$ is lower bounded by a positive value, these decoder actions $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ cannot get arbitrarily close. This motivates an equivalent formulation

by introducing the following transformation of variables. In particular, we define

$$\mathbf{X} = \mathcal{T}\mathbf{M}, \quad (5.11)$$

$$\mathbf{U} = \mathcal{T}^{-1}\mathbf{Y}, \quad (5.12)$$

where

$$\mathcal{T} = \begin{bmatrix} -b_2 & b_1 \\ b_1 & b_2 \end{bmatrix}, \quad \mathcal{T}^{-1} = \frac{1}{b_1^2 + b_2^2} \begin{bmatrix} -b_2 & b_1 \\ b_1 & b_2 \end{bmatrix}, \quad (5.13)$$

and $\mathbf{X} \triangleq [X_1, X_2]^T$ and $\mathbf{Y} \triangleq [Y_1, Y_2]^T$ respectively denote the observation at the encoder and the decoder action in the transformed coordinate system. The proposed equivalent formulation is depicted in Fig. 5.3 where the linear transformation $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ is fixed, and the encoder and decoder simply design $\tilde{\gamma}^e(\cdot)$ and $\tilde{\gamma}^d(\cdot)$. In this coordinate system, the cost function of the encoder becomes

$$\begin{aligned} c^e(\mathbf{m}, \mathbf{u}) &= (\mathbf{m} - \mathbf{u} - \mathbf{b})^T (\mathbf{m} - \mathbf{u} - \mathbf{b}) \\ &= (\mathcal{T}^{-1}\mathcal{T}(\mathbf{m} - \mathbf{u} - \mathbf{b}))^T (\mathcal{T}^{-1}\mathcal{T}(\mathbf{m} - \mathbf{u} - \mathbf{b})) \\ &= (\mathcal{T}(\mathbf{m} - \mathbf{u} - \mathbf{b}))^T (\mathcal{T}^{-1})^T (\mathcal{T}^{-1})(\mathcal{T}(\mathbf{m} - \mathbf{u} - \mathbf{b})). \end{aligned}$$

Since $(\mathcal{T}^{-1})^T (\mathcal{T}^{-1}) = (1/(b_1^2 + b_2^2))I$ with I denoting identity matrix, $\mathcal{T}\mathbf{b} = [0, b_1^2 + b_2^2]^T$, $\mathcal{T}\mathbf{m} = \mathbf{x}$ and $\mathcal{T}\mathbf{u} = \mathbf{y}$, we get

$$c^e(\mathbf{m}, \mathbf{u}) = \frac{(x_1 - y_1)^2}{\tilde{b}} + \frac{(x_2 - y_2 - \tilde{b})^2}{\tilde{b}} \triangleq \frac{c_t^e(\mathbf{x}, \mathbf{y})}{\tilde{b}}, \quad (5.14)$$

where

$$\tilde{b} \triangleq b_1^2 + b_2^2, \quad (5.15)$$

and the common factor of $(1/\tilde{b})$ is canceled in the definition of $c_t^e(\mathbf{x}, \mathbf{y})$ for notational convenience. This implies that there is no bias for the first coordinate whereas there is a bias of $\tilde{b}$ for the second coordinate. In a similar manner, the cost function of the decoder in this coordinate system can be expressed as

$$\begin{aligned} c^d(\mathbf{m}, \mathbf{u}) &= (m_1 - u_1)^2 + (m_2 - u_2)^2 \\ &= \frac{(x_1 - y_1)^2}{\tilde{b}} + \frac{(x_2 - y_2)^2}{\tilde{b}} \triangleq \frac{c_t^d(\mathbf{x}, \mathbf{y})}{\tilde{b}}. \end{aligned} \quad (5.16)$$

With this transformation of variables, the aim of the encoder and decoder is to minimize $J^e(\tilde{\gamma}^e, \tilde{\gamma}^d) = \mathbb{E}[c_t^e(\mathbf{X}, \mathbf{Y})]$ and $J^d(\tilde{\gamma}^e, \tilde{\gamma}^d) = \mathbb{E}[c_t^d(\mathbf{X}, \mathbf{Y})]$, respectively. The expression in (5.16) shows that the cost of the decoder still involves a sum of squared error terms in the transformed coordinate system. This observation implies that for a given encoding policy, the best response of the decoder is still given by the conditional mean expressions in the transformed coordinate system. We summarize this result in the following lemma.

Lemma 5.5.2. *For a fixed encoding function $\tilde{\gamma}^e(\mathbf{x})$, the optimal $\tilde{\gamma}^d(\cdot)$ that minimizes $J^d(\tilde{\gamma}^e, \tilde{\gamma}^d)$ is given by $\mathbb{E}[\mathbf{X}|\mathbf{Z} = \mathbf{z}]$.*

By using the cost function of the encoder and decoder in this transformed coordinate system, it can be shown that the condition Lemma 5.5.1 translates to the condition that for any decoder actions $\mathbf{y}^\alpha = [y_1^\alpha, y_2^\alpha]^T$ and $\mathbf{y}^\beta = [y_1^\beta, y_2^\beta]^T$ in this coordinate system, we have that

$$0 \leq (y_1^\alpha - y_1^\beta)^2 + (y_2^\alpha - y_2^\beta)^2 - 2\tilde{b}|y_2^\alpha - y_2^\beta| \triangleq g(\mathbf{y}^\alpha, \mathbf{y}^\beta). \quad (5.17)$$

Equipped with this transformation of variables and the cost functions in this transformed coordinate system, we are now ready to present our results on necessary conditions which need to be satisfied for any equilibrium with a continuum of decoder actions.

Lemma 5.5.3. *At an equilibrium, any continuum of decoder actions must have a constant y_2 coordinate, i.e., a continuum of decoder actions can only be supported on $y_2 = \kappa$ for some κ in the support of X_2 .*

Proof. See Appendix 5.8.1. □

Lemma 5.5.3 implies that a continuum of actions is allowed only in a specific direction that depends on the bias terms in the original coordinate system. There exists another constraint for such continuum of decoder actions as the next lemma shows.

Lemma 5.5.4. *Let there be a continuum of decoder actions with a constant second coordinate, i.e., $y_2 = \kappa$ where κ is in the support of X_2 . Then, it must be that there exist decoder actions for all values of $y_1 \in [x_1^L(\kappa), x_1^U(\kappa)]$ and $y_2 = \kappa$ where $x_1^L(\kappa)$ and $x_1^U(\kappa)$ denote lower and upper boundaries of the support of X_1 when $X_2 = \kappa$.*

Proof. See Appendix 5.8.2. □

Lemma 5.5.3 states that a continuum of decoder actions must have a constant y_2 coordinate and Lemma 5.5.4 states that these continuum of decoder actions must be supported for all values of y_1 in the support of X_1 given that $X_2 = y_2$. This means that a continuum of decoder actions cannot have a discontinuity. This type of continuum of actions can be attained by revealing the value of X_1 completely. In certain scenarios depending on the distribution and the bias vector, revealing X_1 can be an equilibrium, as investigated in the next section.

5.5.2 Necessary and Sufficient Conditions for Existence of Linear Equilibria

In this subsection, we present our main results on the existence of linear equilibria. In order to establish our main results, we employ Lemma 5.5.3 and Lemma 5.5.4 together with an interesting result from the literature known as Kagan-Linnik-Rao Theorem [145, Theorem 5.3.1]. This theorem characterizes conditions under which $\mathbb{E}[M_1 - \alpha M_2 | M_1 + \beta M_2]$ is a constant for i.i.d. M_1 and M_2 where α and β are non-zero constants.

Theorem 5.5.1. *Consider multidimensional cheap talk problem with sources M_1 and M_2 which are i.i.d. with the corresponding bias terms b_1 and b_2 .*

- (i) *For $b_1 = 0$ and/or $b_2 = 0$, there always exists an informative equilibrium with a linear encoder where the encoder completely reveals the source(s) corresponding to zero bias term(s).*

- (ii) For $|b_1| \neq |b_2|$, there exists an informative equilibrium with a linear encoder if and only if the source distribution is Gaussian.
- (iii) For $b_1 = b_2$, there exists an informative equilibrium with a linear encoder if and only if the source distribution is symmetric around its mean, i.e., denoting the density of M_1 by $f(\cdot)$, we have that $f(\mu + x) = f(\mu - x)$ for all x where $\mathbb{E}[M_1] = \mu$.
- (iv) For $b_1 = -b_2$, there always exists an informative equilibrium with a linear encoder regardless of the source distribution.

Proof. In the case of zero bias term(s), it is easy to verify that revealing the corresponding source(s) leads to a Nash equilibrium. This is a consequence of the fact that M_1 and M_2 are independent, and thus, the problem decouples into two one-dimensional cheap talk problems where one or both of the problems involve an encoder with a zero bias. In this case, an equilibrium may be attained without joint encoding of the sources due to the decoupled nature of the problem. In the following, we treat the remaining cases where $b_1 \neq 0$ and $b_2 \neq 0$.

As mentioned earlier, main ingredients in the proof are Lemma 5.5.3 and Lemma 5.5.4 of this chapter and Kagan-Linnik-Rao Theorem. From Lemma 5.5.3 and Lemma 5.5.4, we know that a continuum of actions must have a constant y_2 coordinate and must be supported for all values of y_1 in the support of X_1 given that $X_2 = y_2$. To conclude the results, we use conditions when $\mathbb{E}[X_2|X_1 = x_1] = \mathbb{E}[b_1 M_1 + b_2 M_2 | b_1 M_2 - b_2 M_1 = x_1]$ is a constant from [145, Theorem 5.3.1].

Now, suppose without loss of generality that $\mathbb{E}[M_1] = 0$. This is without loss of generality since one can introduce an equivalent formulation for removing the mean in a similar manner to the one illustrated in Fig. 5.3. If $|b_1| \neq |b_2|$, [145, Theorem 5.3.1] states that it is required to have a Gaussian source to make $\mathbb{E}[X_2|X_1 = x_1]$ a constant. This means that the condition for the existence of a continuum of actions in Lemma 5.5.3 and Lemma 5.5.4 holds for the Gaussian distribution. If the encoder reveals X_1 completely, we obtain a linear informative equilibrium with a single continuum of decoder actions. To see this, observe

that if the source distribution is Gaussian, X_1 and X_2 are independent random variables. Thus, we obtain decoupled one-dimensional cheap talk problems where there is a non-zero bias for one of them. Then, revealing the random variable with a zero bias yields an informative Nash equilibrium where the encoder is linear. Now, suppose that the source distribution is not Gaussian and $|b_1| \neq |b_2|$. We know that $\mathbb{E}[X_2|X_1 = x_1]$ is not a constant in this case. Hence, if the encoder reveals $X_1 = b_1M_2 - b_2M_1$ completely without giving additional information, then we obtain a single continuum of decoder actions whose second coordinate is not constant. Since this contradicts with Lemma 5.5.3, there cannot be a Nash equilibrium where the encoder conveys $X_1 = b_1M_2 - b_2M_1$ only. Moreover, if there are multiple continuum of decoder actions with constant y_2 coordinates, then we have that $\mathbb{E}[X_2|X_1 = x_1]$ is a constant. Thus, if there exist more than one continuum of decoder actions with constant y_2 coordinates, then it must be that the source distribution is Gaussian. This proves that if the source distribution is not Gaussian, then there cannot exist a linear informative equilibrium.

In the case of $b_1 = b_2$, [145, Theorem 5.3.1] states that $\mathbb{E}[X_2|X_1 = x_1]$ is constant if and only if the source distribution is symmetric. It follows that if the source distribution is not symmetric, there cannot exist a linear informative Nash equilibrium. Now, suppose that the source distribution is symmetric. If the encoder reveals X_1 completely, then the best response of the decoder yields a single continuum of decoder actions. In particular, there only exists decoder actions for all values of $y_1 \in [x_1^L(0), x_1^U(0)]$ and $y_2 = 0$ where $x_1^L(0)$ and $x_1^U(0)$ respectively denote lower and upper boundaries of the support for X_1 given that $X_2 = 0$. Note that the bias corresponding to X_1 is zero for the encoder in this coordinate system. Therefore, the best response of the encoder to this single continuum of decoder actions is given by revealing X_1 completely as otherwise the performance of the encoder degrades. Hence, it follows that revealing X_1 completely leads to a Nash equilibrium. Moreover, in the case of $b_1 = -b_2$, [145, Theorem 5.3.1] implies that $\mathbb{E}[X_2|X_1 = x_1]$ is constant regardless of the source distribution. Therefore, there always exists a linear informative Nash equilibrium for any source distribution when $b_1 = -b_2$. $\square$

Remark 5.5.3. Theorem 5.5.1 shows that depending on certain conditions there

exists an equilibrium with a linear encoder even for large values of $|b_1|$ and $|b_2|$. On the other hand, in the case of one-dimensional cheap talk, when the source distribution has bounded support, there exists an upper bound on the number of bins in equilibrium, and this upper bound may even be equal to one (i.e., a noninformative equilibrium) when the bias term is large. Hence, even though an equilibrium in the case of one-dimensional scenario can be noninformative, in the case of two-dimensional scenario with the same bias as in the one-dimensional scenario in both dimensions, it is possible to obtain an informative equilibrium.

Remark 5.5.4. In the case of a Gaussian source, the problem decouples into two one-dimensional cheap talk problems. In particular, $X_1 = b_2M_1 - b_1M_2$ and $X_2 = b_1M_1 + b_2M_2$ become independent random variables when M_1 and M_2 are i.i.d. Gaussian. In fact, due to Darmois-Skitovich Theorem [145, Theorem 3.1.1], X_1 and X_2 are independent only when M_1 and M_2 are Gaussian. As a result, the problem reduces to obtaining equilibria for decoupled two one-dimensional cheap talk problems where an encoder wishes to convey X_1 with a zero bias and another encoder wishes to convey X_2 with a bias of $\tilde{b}$. From Theorem 2.6.1, we know that in the case of one-dimensional cheap talk with a Gaussian source, for any $N \geq 1$, there exists a unique equilibrium with N bins. Thus, for two-dimensional cheap talk problem with a Gaussian source, there exists an equilibrium where the encoder reveals X_1 completely and applies a quantization policy to X_2 with an arbitrary number of bins.

We can also consider n -dimensional i.i.d. Gaussian sources. In this case, one can apply an orthogonal transformation of variables in a similar manner to the two-dimensional case where random variables in each dimension are independent. Under this transformation of variables, there remains a bias term only for a single random variable. Due to Theorem 2.6.1 and independence of the random variables in the transformed coordinate system, it follows that there exists an equilibrium where the encoder applies a quantization policy to this remaining random variable with any number of bins. Thus, we obtain the following result.

Theorem 5.5.2. *If the source distribution is n -dimensional i.i.d. Gaussian, then there exists an equilibrium with a linear encoding policy where the encoder reveals all or a subset of $(n - 1)$ dimensions completely in a transformed coordinate*

system, and applies a signaling game policy for the remaining dimension with any number of bins.

Proof. By applying a suitable linear transformation of variables, one can obtain an equivalent problem. In this equivalent problem, the aim is to convey a sequence of independent Gaussian sources and the bias is zero for all sources except for one of them. Due to independence of these sources, the problem decouples. By using the result of Theorem 2.6.1, it follows that there exists an equilibrium where the encoder uses a quantization policy with any number of bins for the source corresponding to a non-zero bias. On the other hand, since the bias is zero for all other sources, revealing these sources leads to an equilibrium. $\square$

5.6 Large Dimensions and a Rate-Distortion Theoretic Formulation of Cheap Talk

We have analyzed the multidimensional cheap talk problem where the bias vector at the encoder can be arbitrary. In the special case that the components of the bias vector are the same leads to an important problem from an information theoretic perspective. In this case, the problem is to convey an i.i.d. source with a certain bias and the bias is the same for each source component. In other words, the encoder observes independent copies from a random source and wishes to introduce the same bias for each independent copy. In such a problem, one may wish to obtain information theoretic limits of the communication. In a sense, this problem is a game theoretic counterpart of rate-distortion theory that is studied in a classical communication theoretic setup. Our findings reveal that if the distribution is Gaussian, then there always exists a linear equilibrium where the sender completely reveals $(n - 1)$ dimensions in a transformed coordinate system. For the remaining dimension, the encoder has to employ a quantization policy with an arbitrary number of bins including the case with one bin. This result holds because the problem can be transformed into decoupled problems consisting of a team theoretic problem for conveying a $(n - 1)$ -dimensional i.i.d.

source without any bias and a one-dimensional cheap talk problem with a certain bias in the remaining dimension. If we increase the number of observed sources at the encoder, the effect of employed policy for this remaining dimension becomes negligible. This implies that the problem of finding achievable rate and distortion pairs is asymptotically equivalent to obtaining achievable rate and distortion pairs for a team theoretic setup in a transformed coordinate system.

The problem of interest is in fact can be more generally expressed in a rate-distortion theoretic formulation. The aim is to find the achievable rate and distortion region. In particular, we have the following problem:

Problem 5.6.1. Consider the n -dimensional cheap talk problem with i.i.d. sources and $b = b_1 = \dots = b_n$. We say that a tuple of rate and distortion pairs (R, D_e, D_d) is *achievable* at a Nash equilibrium if there exists a sequence of encoders and decoders that leads to a Nash equilibrium with the following properties:

- (i) The encoder is given by $\gamma_n^e : \mathbb{M}^n \rightarrow \mathcal{Z}(n)$ with $|\mathcal{Z}(n)| \leq 2^{Rn}$ where $|\mathcal{Z}(n)|$ denotes the cardinality of $\mathcal{Z}(n)$.
- (ii) The decoder is given by $\gamma_n^d : \mathcal{Z}(n) \rightarrow \mathbb{M}^n$ such that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} [\sum_{i=1}^n (M_i - U_i - b)^2]}{n} \leq D_e, \quad (5.18)$$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} [\sum_{i=1}^n (M_i - U_i)^2]}{n} \leq D_d. \quad (5.19)$$

Then, the problem is to determine if a given tuple (R, D_e, D_d) is *achievable* at a Nash equilibrium.

If the bias term is zero in this problem, then we obtain a team theoretic problem since the corresponding distortion values are identical at the encoder and decoder. We denote the corresponding rate and distortion values by R_T and D_T , respectively, where the subscript refers to the fact that the setup is team theoretic.

While we leave the study of Problem 1 for general sources for future work, the Gaussian case is completely solvable. Our result in the previous section shows that if the source distribution is Gaussian, one can apply a suitable transformation of variables to obtain an equivalent problem for which the encoder has bias only for a single random variable. The following theorem uses this idea to relate achievable rate and distortion values of the original problem to that of a team theoretic problem. Before presenting the theorem, we note that at a Nash equilibrium we have $\mathbb{E}[\sum_{i=1}^n (M_i - U_i - b)^2] = \mathbb{E}[\sum_{i=1}^n (M_i - U_i)^2] + b^2 n$. Thus, we have the same rate region for any D_e value satisfying $D_e \geq D_d + b^2$.

Theorem 5.6.1. *Consider multidimensional cheap talk problem with i.i.d. Gaussian sources where the bias term b is the same at each dimension. Suppose that a rate and a distortion pair (R_T, D_T) is achievable for the team theoretic problem with a zero bias. Then, for the game theoretic problem with a non-zero bias, the following rate and distortion values are achievable:*

$$R = R_T, \quad (5.20)$$

$$D_e \geq D_T + b^2, \quad (5.21)$$

$$D_d \geq D_T. \quad (5.22)$$

Proof. Suppose without loss of generality that $\mathbb{E}[M_1] = 0$. We first apply a linear transformation of variables. In a similar manner to the two-dimensional case, we obtain an equivalent problem in a transformed coordinate system. The proposed linear transformation is such that the cost function of the encoder becomes

$$\begin{aligned} c^e(\mathbf{m}, \mathbf{u}) &= \sum_{k=1}^n (m_k - u_k - b)^2 \\ &= \sum_{k=1}^{n-1} (x_k - y_k)^2 + (x_n - y_n - \sqrt{n}b)^2 \triangleq c_t^e(\mathbf{x}, \mathbf{y}), \end{aligned}$$

where

$$x_k = \frac{\sum_{i=1}^k m_i - km_{k+1}}{\sqrt{k(k+1)}}, \quad \text{for } k = 1, \dots, n-1, \quad (5.23)$$

$$y_k = \frac{\sum_{i=1}^k u_i - ku_{k+1}}{\sqrt{k(k+1)}}, \quad \text{for } k = 1, \dots, n-1, \quad (5.24)$$

$$x_n = \frac{\sum_{i=1}^n m_i}{\sqrt{n}}, \quad (5.25)$$

$$y_n = \frac{\sum_{i=1}^n u_i}{\sqrt{n}}. \quad (5.26)$$

On the other hand, the cost function of the encoder in this transformed coordinate system is given by

$$c^d(\mathbf{m}, \mathbf{u}) = \sum_{k=1}^n (m_k - u_k)^2 = \sum_{k=1}^n (x_k - y_k)^2 \triangleq c_t^d(\mathbf{x}, \mathbf{y}).$$

The random variables $X_1, \dots, X_n$ defined by (5.23) and (5.25) are i.i.d. and follow the same distribution as $M_1, \dots, M_n$. Then, one can obtain an equivalent problem in a similar manner to one illustrated in Fig. 5.3 where the linear transformation denoted by $\mathcal{T}$ from $\mathbf{M}$ to $\mathbf{X}$ is specified by (5.23) and (5.25). In particular, we fix the linear transformation $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$, and the encoder and decoder design $\tilde{\gamma}^e(\cdot)$ and $\tilde{\gamma}^d(\cdot)$, respectively. An important feature special to the Gaussian case is that $X_1, \dots, X_n$ are independent as a result of the employed linear transformation of variables. Thus, the problem decouples to n one-dimensional cheap talk problems where only for one of the problems there is a non-zero bias. If the encoder does not reveal information related to X_n , we obtain a team theoretic problem for the remaining $(n-1)$ dimensions. In this case, the rate of the original problem is identical to that of a team theoretic problem with $(n-1)$ i.i.d. sources. On the other hand, since the term that contributes to the objective of the sender is $\mathbb{E}[(X_n - Y_n - \sqrt{nb})^2] = nb^2$ in the case that no information is conveyed related to X_n , we obtain an additional b^2 term in (5.21). In particular, in this case, we get

$$\frac{\mathbb{E}[\sum_{i=1}^n (M_i - U_i - b)^2]}{n} = \frac{\mathbb{E}[\sum_{i=1}^{n-1} (X_i - Y_i)^2]}{n} + b^2.$$

By taking the limit of both sides, we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{\mathbb{E}[\sum_{i=1}^n (M_i - U_i - b)^2]}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\mathbb{E}[\sum_{i=1}^{n-1} (X_i - Y_i)^2]}{n} + b^2 \\
&= \lim_{n \rightarrow \infty} \frac{\mathbb{E}[\sum_{i=1}^{n-1} (X_i - Y_i)^2]}{n-1} + b^2,
\end{aligned}$$

where the last equality is due to the fact that $n \rightarrow \infty$. This leads to a reduced team theoretic problem where the encoder wishes to convey an i.i.d. source $X_1, \dots, X_{n-1}$ with a zero bias. Therefore, if the pair (R_T, D_T) is achievable in a team theoretic setup, $R = R_T$ and $D_e \geq D_T + b^2$ are achievable for the original game theoretic setup. For the distortion bound of the decoder, we can simply use the relation $\mathbb{E}[\sum_{i=1}^n (M_i - U_i - b)^2] = \mathbb{E}[\sum_{i=1}^n (M_i - U_i)^2] + b^2 n$. $\square$

In rate-distortion theory, an important concept is the rate-distortion function. In a classical communication theoretic setup, this is defined as the infimum of rates R such that (R, D) is achievable. A similar definition of rate-distortion function in a game theoretic setup yields

$$R(D_e, D_d) \triangleq \inf\{R \mid (R, D_e, D_d) \text{ is achievable}\}. \quad (5.27)$$

By using the result of Theorem 5.6.1, we can upper bound the rate-distortion function for the Gaussian case, as stated in the following.

Theorem 5.6.2. *Consider multidimensional cheap talk problem with i.i.d. Gaussian sources where the bias term b is the same at each dimension. The rate-distortion function for such a setup is upper bounded by the following:*

$$R(D_e, D_d) \leq \frac{1}{2} \log_2 \frac{\sigma^2}{\min\{D_d, D_e - b^2\}} \quad (5.28)$$

if $0 \leq \min\{D_d, D_e - b^2\} \leq \sigma^2$. On the other hand, we have $R(D_e, D_d) = 0$ if $\min\{D_d, D_e - b^2\} > \sigma^2$.

Proof. From Theorem 5.6.1, we know that if we take $D_T = \min\{D_e - b^2, D_d\}$ for the team theoretic setup, then the corresponding rate R_T is achievable for the

game theoretic setup. Then, the following holds:

$$\begin{aligned}
& \inf\{R \mid (R, D_e, D_d) \text{ is achievable}\} \\
& \leq \inf\{R \mid (R, \min\{D_e - b^2, D_d\}) \text{ is achievable}\} \\
& = \begin{cases} \frac{1}{2} \log_2 \frac{\sigma^2}{\min\{D_d, D_e - b^2\}} & \text{if } 0 \leq \min\{D_d, D_e - b^2\} \leq \sigma^2 \\ 0 & \text{if } \min\{D_d, D_e - b^2\} > \sigma^2 \end{cases},
\end{aligned}$$

where the inequality uses the reduction to team theoretic setup and the equality uses a well-known result in the literature for the Gaussian setup, see e.g., [142, Theorem 10.3.2]. $\square$

Remark 5.6.1. A related result in this context is presented in Theorem 2.6.5 considering scalar quadratic cheap talk problem. Here, it is shown that for one-dimensional cheap talk having more bins in the quantized encoding policy leads to reduced distortion values if the source distribution is log-concave. In other words, a large rate leads to better expected costs for both of the players when the source distribution is log-concave.

Remark 5.6.2. For multi-dimensional i.i.d. Gaussian sources, we know that there exists an equilibrium where the encoder reveals $(n - 1)$ dimensions and applies a quantization policy for the remaining dimension X_n with an arbitrary number of bins. Since the Gaussian distribution is log-concave, from Theorem 2.6.5, it follows that the expected costs of both players reduce when the number of bins for the quantization policy applied to X_n is increased. In addition, it is also possible to have an equilibrium with infinitely many bins applied to X_n due to [35, Theorem 13]. This implies that an equilibrium where $X_1, \dots, X_{n-1}$ are revealed and a quantization policy with infinitely many bins applied to X_n corresponds to a payoff dominant Nash equilibrium [42].

Remark 5.6.3. Note that in the case of symmetric sources if we do not impose a distortion constraint, then the maximum attainable rate among all possible Nash equilibria cannot be upper bounded by a finite value. This is due to the fact that a linear equilibrium exists in this case. On the other hand, if this symmetry condition is broken, one can analyze information theoretic bounds on the rate of communication.

5.7 Conclusion

We have analyzed a quadratic multidimensional cheap talk problem building on our analysis in Chapter 2, which involved scalar sources. First, we have derived general necessary conditions for a Nash equilibrium considering any joint source distribution. In particular, we have shown that decoder actions at a Nash equilibrium need to satisfy a geometry condition that essentially prevents any two decoder actions being arbitrarily close to each other depending on their difference as vectors and the bias vector. Then, we have derived conditions under which a linear equilibrium exists considering i.i.d. sources. In particular, we have proven that if the bias vector has components which are not equal in magnitude, then there exists a linear equilibrium if and only if the source distribution is Gaussian. On the other hand, if the bias vector has components equal to each other, then there exists a linear equilibrium if and only if the source distribution is symmetric. In contrast, if the components are equal in magnitude with different signs, then there always exists a linear equilibrium regardless of the source distribution. This complete characterization of conditions that ensure existence of linear equilibria has implications on the information theoretic limits of the cheap talk problem. If there exists a linear equilibrium, then there is no upper bound on the maximum rate of communication arising from the biased nature of the encoder. Motivated by this observation, we have proposed a rate-distortion theoretic formulation of cheap talk problem and we have analyzed i.i.d. Gaussian case.

5.8 Proofs

5.8.1 Proof of Lemma 5.5.3

This result is a consequence of Lemma 5.5.1. Let there be a continuum of decoder actions which does not have a constant y_2 coordinate. Suppose that this continuum of decoder actions can be expressed as decoder actions of the form $\mathbf{y} = [y_1, h(y_1)]^T$ for some continuous function $h(\cdot)$ where the continuity of $h(\cdot)$ is

by the assumption of having a continuum of decoder actions.² Here, we represent the second coordinate of the continuum as a function of the first coordinate. In the following, we prove that such a continuum of decoder actions leads to a contradiction when $h(\cdot)$ is not a constant function.

Suppose, by contradiction, that $h(\cdot)$ is not a constant function. It follows that we can find two decoder actions $\mathbf{y}^\alpha = [y_1^\alpha, y_2^\alpha]^T$ and $\mathbf{y}^\beta = [y_1^\beta, y_2^\beta]^T$ on the continuum with the property that $y_2^\alpha \neq y_2^\beta$ and that $y_1^\alpha < y_1^\beta$. Without loss of generality, take $y_2^\alpha < y_2^\beta$. In the following, we first prove the result by making the assumption that $h(\cdot)$ is differentiable. By this assumption, we are able to invoke the standard mean value theorem of calculus to conclude the result. However, the result holds also for a non-differentiable $h(\cdot)$. We first prove the result for a differentiable $h(\cdot)$ since this proof is more intuitive. Then, we prove the result in the general case where $h(\cdot)$ is non-differentiable.

Let $h(\cdot)$ be differentiable. By the mean value theorem of calculus, there exists $y_1^\gamma \in [y_1^\alpha, y_1^\beta]$ such that $h'(y_1^\gamma) = (y_2^\beta - y_2^\alpha)/(y_1^\beta - y_1^\alpha) > 0$. In particular, we can find a decoder action $\mathbf{y}^\gamma$ on the continuum with $y_1^\gamma \in [y_1^\alpha, y_1^\beta]$ and $y_2^\gamma = h(y_1^\gamma)$ such that the derivative of $h(\cdot)$ at y_1^γ is strictly positive. Next, we take another decoder action $\mathbf{y}^\eta = [y_1^\eta, h(y_1^\eta)]^T$ on the continuum and vary its first coordinate to reach a contradiction to (5.17). In particular, if we express the condition imposed by (5.17) for the decoder action $\mathbf{y}^\gamma$ and a decoder action on the continuum denoted by $\mathbf{y}^\eta$, we get

$$g(\mathbf{y}^\eta, \mathbf{y}^\gamma) = (y_1^\eta - y_1^\gamma)^2 + (h(y_1^\eta) - h(y_1^\gamma))^2 - 2\tilde{b}|h(y_1^\eta) - h(y_1^\gamma)| \geq 0.$$

When $y_1^\eta > y_1^\gamma$, and $(y_1^\eta - y_1^\gamma)$ is sufficiently small, we have $h(y_1^\eta) > h(y_1^\gamma)$ due to positive derivative at y_1^γ . For fixed y_1^γ , if we take the derivative of $g(\mathbf{y}^\eta, \mathbf{y}^\gamma)$ with respect to y_1^η , we get

$$\frac{dg(\mathbf{y}^\eta, \mathbf{y}^\gamma)}{dy_1^\eta} = 2(y_1^\eta - y_1^\gamma) + 2(h(y_1^\eta) - h(y_1^\gamma))h'(y_1^\eta) - 2\tilde{b}h'(y_1^\eta).$$

²Note that a continuum of decoder actions with a constant y_1 coordinate is not allowed, which follows from (5.17). In particular, for a given $\mathbf{y}^\alpha = [y_1^\alpha, y_2^\alpha]^T$, there cannot be a decoder action $\mathbf{y}^\beta = [y_1^\alpha, y_2^\alpha + \Delta]^T$ with $-2\tilde{b} < \Delta < 2\tilde{b}$. Therefore, if we take a continuum such that for a given y_1 value, there are multiple decoder actions with different y_2 coordinates, then it is possible to partition this continuum so that it consists of representations of the form $\mathbf{y} = [y_1, h_i(y_1)]^T$ for some continuous functions $h_i(\cdot)$ with $i \in \{1, \dots, k\}$.

Since $h'(y_1^\gamma) > 0$, we have that $\left. \frac{dg(\mathbf{y}^\eta, \mathbf{y}^\gamma)}{dy_1^\eta} \right|_{y_1^\eta = y_1^\gamma} < 0$. This is a contradiction to $g(\mathbf{y}^\eta, \mathbf{y}^\gamma) \geq 0$ since $g(\mathbf{y}^\gamma, \mathbf{y}^\gamma) = 0$ and $\left. \frac{dg(\mathbf{y}^\eta, \mathbf{y}^\gamma)}{dy_1^\eta} \right|_{y_1^\eta = y_1^\gamma} < 0$. Therefore, it must be that $h(\cdot)$ is a constant function, which means that a continuum of action must have a constant y_2 coordinate.

Now, consider the general case where $h(\cdot)$ is not differentiable. By the mean value theorem in [146, Corollary 1], there exists $y_1^* \in [y_1^\alpha, y_1^\beta]$ such that either $s^* \in \partial h(y_1^*)$ or $s^* \in -\partial(-h(y_1^*))$ holds where $s^* \triangleq (y_2^\beta - y_2^\alpha)/(y_1^\beta - y_1^\alpha) > 0$ and $\partial h(y_1^*)$ denotes the set of subgradients of $h(\cdot)$ at y_1^* . We take two decoder actions and employ the condition in (5.17) to reach at a contradiction. Let $\mathbf{y}^\gamma$ and $\mathbf{y}^\eta$ denote these two decoder actions which are expressed as $y_1^\gamma = y_1$, $y_2^\gamma = h(y_1^\gamma)$, $y_1^\eta = y_1 + td'$ and $y_2^\eta = h(y_1^\eta)$. If we express the condition imposed by (5.17) for the decoder actions $\mathbf{y}^\gamma$ and $\mathbf{y}^\eta$, we get

$$\begin{aligned} g(\mathbf{y}^\eta, \mathbf{y}^\gamma) &= (y_1^\eta - y_1^\gamma)^2 + (h(y_1^\eta) - h(y_1^\gamma))^2 - 2\tilde{b}|h(y_1^\eta) - h(y_1^\gamma)| \\ &= (td')^2 + (h(y_1 + td') - h(y_1))^2 - 2\tilde{b}|h(y_1 + td') - h(y_1)| \geq 0. \end{aligned} \quad (5.29)$$

If t and d' are positive and sufficiently small, the inequality in (5.29) can be expressed as

$$|h(y_1 + td') - h(y_1)| + \sqrt{\tilde{b}^2 - (td')^2} - \tilde{b} \leq 0. \quad (5.30)$$

Since t is positive, it follows from (5.30) that

$$\frac{|h(y_1 + td') - h(y_1)|}{t} + \frac{\sqrt{\tilde{b}^2 - (td')^2} - \sqrt{\tilde{b}^2}}{t} \leq 0. \quad (5.31)$$

Thus, (5.31) implies that

$$\limsup_{\substack{y_1 \rightarrow y_1^* \\ t \downarrow 0}} \inf_{d' \rightarrow d} \frac{|h(y_1 + td') - h(y_1)|}{t} + \frac{\sqrt{\tilde{b}^2 - (td')^2} - \sqrt{\tilde{b}^2}}{t} \leq 0, \quad (5.32)$$

where we take $d > 0$ small so that the assumption of having a small d' holds. Since the limit of the second term in (5.32) is zero, we get

$$\limsup_{\substack{y_1 \rightarrow y_1^* \\ t \downarrow 0}} \inf_{d' \rightarrow d} \frac{|h(y_1 + td') - h(y_1)|}{t} = 0. \quad (5.33)$$

However, this is a contradiction to the fact that either $s^* \in \partial h(y_1^*)$ or $s^* \in -\partial(-h(y_1^*))$ holds.

5.8.2 Proof of Lemma 5.5.4

In order to prove this result, we assume that there exists a continuum of decoder actions whose support does not extend to the boundaries of the support and then reach at a contradiction. Let $\mathbf{y}^\alpha = [y_1^\alpha, y_2^\alpha]^T$ and $\mathbf{y}^\beta = [y_1^\beta, y_2^\beta]^T$ be decoder actions on the continuum with $y_1^\alpha < y_1^\beta$ and $y_2^\alpha = y_2^\beta$ such that there exist decoder actions for all values of y_1 satisfying $y_1^\alpha \leq y_1 \leq y_1^\beta$ and $y_2 = y_2^\alpha$ where $x_1^L(y_2^\alpha) < y_1^\alpha$ and $y_1^\beta < x_1^U(y_2^\beta)$. As mentioned earlier, $x_1^L(x_2)$ and $x_1^U(x_2)$ respectively denote the lower and upper boundaries of the support for X_1 given that $X_2 = x_2$. In addition, suppose that there exists $\delta > 0$ such that there does not exist any decoder action with $y_1^\beta < y_1 < y_1^\beta + \delta$ and $y_2 = y_2^\beta$. Similarly, suppose that there exists $\tilde{\delta} > 0$ such that there does not exist any decoder action with $y_1^\alpha - \tilde{\delta} < y_1 < y_1^\alpha$ and $y_2 = y_2^\alpha$. In the following, we focus on the decoder action $\mathbf{y}^\beta$ and show that the assumption of $y_1^\beta < x_1^U(y_2^\beta)$ leads to a contradiction. A similar approach can be taken for the decoder action $\mathbf{y}^\alpha$ to prove that the assumption of $x_1^L(y_2^\alpha) < y_1^\alpha$ leads to a contradiction.

Let the bin at the encoder corresponding to the decoder action $\mathbf{y}^\beta$ be denoted by $\mathcal{B}^\beta$, i.e., $\mathbf{y}^\beta = \mathbb{E}[\mathbf{X} | \mathbf{X} \in \mathcal{B}^\beta]$. We will obtain a contradiction that $\mathcal{B}^\beta$ contains observations with $x_1 > y_1^\beta$ whereas it does not contain any observation with $x_1 < y_1^\beta$. This is a contradiction to $\mathbf{y}^\beta = \mathbb{E}[\mathbf{X} | \mathbf{X} \in \mathcal{B}^\beta]$ since we assume that every non-empty open set has a positive probability measure.

We first show that $\mathcal{B}^\beta$ does not contain any observations with $x_1 < y_1^\beta$. Towards that goal, we prove that there always exists a decoder action that induces a better cost than $\mathbf{y}^\beta$ for any observation with $x_1 < y_1^\beta$. Now, take an observation $\mathbf{x}$ with $x_1 < y_1^\beta$. Since the continuum of decoder actions is supported on $y_1 \in [y_1^\alpha, y_1^\beta]$ and $y_2 = y_2^\beta$, we can find a decoder action $\mathbf{y}^\nu$ on the continuum with coordinates $y_1^\nu = \max\{x_1, y_1^\alpha\}$ and $y_2^\nu = y_2^\beta$. For this decoder action, when $x_1 < y_1^\beta$, we have that $c_t^e(\mathbf{x}, \mathbf{y}^\nu) < c_t^e(\mathbf{x}, \mathbf{y}^\beta)$. This implies that any observation with $x_1 < y_1^\beta$ cannot

be an element of $\mathcal{B}^\beta$.

Next, we prove that $\mathcal{B}^\beta$ must contain observations with $x_1 > y_1^\beta$ under the assumed configuration, which however contradicts with $\mathbf{y}^\beta = \mathbb{E}[\mathbf{X} | \mathbf{X} \in \mathcal{B}^\beta]$. While proving that $\mathcal{B}^\beta$ must contain observations with $x_1 > y_1^\beta$, we use the result from Lemma 5.5.1 which imposes the condition in (5.17) in the transformed coordinate system considering any two decoder actions at an equilibrium. In particular, for any decoder action $\mathbf{y}^\eta$, we need $g(\mathbf{y}^\beta, \mathbf{y}^\eta) \geq 0$ where $g(\cdot, \cdot)$ is as defined in (5.17). Note that other decoder actions imposes additional constraints on the region where there can exist decoder actions at an equilibrium. Nonetheless, for our purpose, it suffices to use the condition $g(\mathbf{y}^\beta, \mathbf{y}^\eta) \geq 0$. Our aim is to show that for all $\mathbf{y}^\eta$ satisfying $g(\mathbf{y}^\beta, \mathbf{y}^\eta) \geq 0$, it is possible to find a region of observations with $x_1 > y_1^\beta$ where $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\eta)$. This implies that the intersection of these observation regions must be the bin for $\mathbf{y}^\beta$, i.e., $\mathcal{B}^\beta$. To conclude the result, we will show that this intersection is not empty. Note that $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\eta)$ is equivalent to

$$(y_1^\eta - y_1^\beta)(y_1^\eta + y_1^\beta - 2x_1) + (y_2^\eta - y_2^\beta)(y_2^\eta + y_2^\beta + 2\tilde{b} - 2x_2) > 0. \quad (5.34)$$

We will use this equivalent expression in the remainder of the proof.

Note that when $y_2^\eta = y_2^\beta$, due to the discontinuity assumption at $\mathbf{y}^\beta$, for any observation with $y_1^\beta < x_1 < y_1^\beta + \delta/2$, the inequality $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\eta)$ holds. Therefore, it is sufficient to consider decoder actions with $y_2^\eta \neq y_2^\beta$. We can express a decoder action $\mathbf{y}^\eta$ satisfying $y_2^\eta \neq y_2^\beta$ and $g(\mathbf{y}^\beta, \mathbf{y}^\eta) \geq 0$ as $\mathbf{y}^\eta = \mathbf{y}^\beta + \lambda(\mathbf{y}^\gamma - \mathbf{y}^\beta)$ for some $\lambda \geq 1$ where $\mathbf{y}^\gamma$ satisfies $g(\mathbf{y}^\beta, \mathbf{y}^\gamma) = 0$. In particular, we have

$$\begin{aligned} & (y_1^\eta - y_1^\beta)^2 + (y_2^\eta - y_2^\beta)^2 - 2\tilde{b}|y_2^\eta - y_2^\beta| \\ &= \lambda^2(y_1^\gamma - y_1^\beta)^2 + \lambda^2(y_2^\gamma - y_2^\beta)^2 - 2|\lambda||y_2^\gamma - y_2^\beta|\tilde{b} \\ &\stackrel{(a)}{=} \lambda^2(2\tilde{b}|y_2^\gamma - y_2^\beta|) - 2|\lambda||y_2^\gamma - y_2^\beta|\tilde{b} \\ &= (2\tilde{b}|y_2^\gamma - y_2^\beta|)(\lambda^2 - |\lambda|) \end{aligned}$$

where (a) follows from $g(\mathbf{y}^\beta, \mathbf{y}^\gamma) = 0$ and the final expression implies that we can write $\mathbf{y}^\eta = \mathbf{y}^\beta + \lambda(\mathbf{y}^\gamma - \mathbf{y}^\beta)$ with $\lambda \geq 1$ so that $g(\mathbf{y}^\beta, \mathbf{y}^\eta) \geq 0$ holds. It is seen that when $\lambda > 1$, for a given observation $\mathbf{x}$ satisfying $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$, the

following holds:

$$\begin{aligned}
& (y_1^\eta - y_1^\gamma)(y_1^\eta + y_1^\gamma - 2x_1) \\
& + (y_2^\eta - y_2^\gamma)(y_2^\eta + y_2^\gamma + 2\tilde{b} - 2x_2) \\
& = (\lambda - 1)(y_1^\gamma - y_1^\beta)((\lambda + 1)y_1^\gamma + (1 - \lambda)y_1^\beta - 2x_1) \\
& + (\lambda - 1)(y_2^\gamma - y_2^\beta)((\lambda + 1)y_2^\gamma + (1 - \lambda)y_2^\beta + 2\tilde{b} - 2x_2) \\
& > (\lambda - 1)(y_1^\gamma - y_1^\beta)((\lambda + 1)y_1^\gamma + (1 - \lambda)y_1^\beta - y_1^\gamma - y_1^\beta) \\
& + (\lambda - 1)(y_2^\gamma - y_2^\beta)((\lambda + 1)y_2^\gamma + (1 - \lambda)y_2^\beta - y_2^\gamma - y_2^\beta) \\
& = \lambda(\lambda - 1)((y_1^\gamma - y_1^\beta)^2 + (y_2^\gamma - y_2^\beta)^2) > 0,
\end{aligned}$$

where the first inequality is due to $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ and $\lambda > 1$ and the last inequality follows from $\mathbf{y}^\beta \neq \mathbf{y}^\gamma$ and $\lambda > 1$. This shows that $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ implies $c_t^e(\mathbf{x}, \mathbf{y}^\gamma) < c_t^e(\mathbf{x}, \mathbf{y}^\eta)$. Hence, it is sufficient to consider decoder actions $\mathbf{y}^\gamma$ with $g(\mathbf{y}^\beta, \mathbf{y}^\gamma) = 0$.

Next, we show that for any decoder action $\mathbf{y}^\gamma$ satisfying $g(\mathbf{y}^\beta, \mathbf{y}^\gamma) = 0$, there exists a nonempty region of observations with $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ and $x_1 > y_1^\beta$. Towards that goal, we consider different cases and treat each of these cases separately. In the following, we take $y_2^\beta < x_2 < y_2^\beta + 2\tilde{b}$ and specify a nontrivial interval for x_1 so that the resulting observation satisfies the desired property.

- (i) Let $\mathbf{y}^\gamma$ satisfy $y_2^\beta < y_2^\gamma < y_2^\beta + 2\tilde{b}$ and $y_1^\gamma = y_1^\beta + \sqrt{(y_2^\gamma - y_2^\beta)(y_2^\beta - y_2^\gamma + 2\tilde{b})}$.

Let

$$y_1^\beta < x_1 < y_1^\beta + \frac{(y_2^\gamma - y_2^\beta)(y_2^\beta - x_2 + 2\tilde{b})}{(y_1^\gamma - y_1^\beta)}. \quad (5.35)$$

For these observations, we have that

$$\begin{aligned}
& (y_1^\gamma - y_1^\beta)(y_1^\gamma + y_1^\beta - 2x_1) \\
& + (y_2^\gamma - y_2^\beta)(y_2^\gamma + y_2^\beta + 2\tilde{b} - 2x_2) \\
& = (y_1^\gamma - y_1^\beta)^2 + (y_2^\gamma - y_2^\beta)^2 - 2(y_1^\gamma - y_1^\beta)(x_1 - y_1^\beta) \\
& \quad + 2(y_2^\gamma - y_2^\beta)(\tilde{b} - x_2 + y_2^\beta) \\
& \stackrel{(a)}{=} 2\tilde{b}(y_2^\gamma - y_2^\beta) - 2(y_1^\gamma - y_1^\beta)(x_1 - y_1^\beta) \\
& \quad + 2(y_2^\gamma - y_2^\beta)(\tilde{b} - x_2 + y_2^\beta) \\
& = 2(y_2^\gamma - y_2^\beta)(2\tilde{b} - x_2 + y_2^\beta) \\
& \quad - 2(y_1^\gamma - y_1^\beta)(x_1 - y_1^\beta) > 0
\end{aligned}$$

where $\stackrel{(a)}{=}$ follows from the definition of y_1^γ in this case and the inequality is due to (5.35). This shows that (5.34) and equivalently $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ hold for the observations specified by (5.35) and $y_2^\beta < x_2 < y_2^\beta + 2\tilde{b}$. Notice also that if (5.35) holds for some y_2^γ , then it automatically holds for any $\tilde{y}_2^\gamma$ satisfying $y_2^\gamma < \tilde{y}_2^\gamma < y_2^\beta + 2\tilde{b}$. Therefore, if there are multiple decoder actions satisfying the assumptions of this case, it is sufficient to consider the one with the minimum y_2 coordinate.

(ii) Let $\mathbf{y}^\gamma$ be such that $y_2^\beta < y_2^\gamma \leq y_2^\beta + 2\tilde{b}$ and $y_1^\gamma = y_1^\beta - \sqrt{(y_2^\gamma - y_2^\beta)(y_2^\beta - y_2^\gamma + 2\tilde{b})}$ hold. Let

$$y_1^\beta < x_1. \tag{5.36}$$

A similar analysis to that in the previous case yields

$$\begin{aligned}
& (y_1^\gamma - y_1^\beta)(y_1^\gamma + y_1^\beta - 2x_1) \\
& + (y_2^\gamma - y_2^\beta)(y_2^\gamma + y_2^\beta + 2\tilde{b} - 2x_2) \\
& = 2(y_2^\gamma - y_2^\beta)(2\tilde{b} - x_2 + y_2^\beta) \\
& \quad + 2(y_1^\beta - y_1^\gamma)(x_1 - y_1^\beta) > 0,
\end{aligned}$$

where the inequality follows from the facts that $y_2^\beta < y_2^\gamma$, $x_2 < y_2^\beta + 2\tilde{b}$, $y_1^\beta \leq y_1^\gamma$ and $y_1^\beta < x_1$. This proves that (5.34) and equivalently $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ hold for the observations specified by (5.36) and $y_2^\beta < x_2 < y_2^\beta + 2\tilde{b}$.

(iii) Let $\mathbf{y}^\gamma$ be a decoder action satisfying $y_2^\beta - 2\tilde{b} < y_2^\gamma < y_2^\beta$ and $y_1^\gamma = y_1^\beta + \sqrt{(y_2^\gamma - y_2^\beta)(y_2^\beta - y_2^\gamma - 2\tilde{b})}$. Let

$$y_1^\beta < x_1 < y_1^\beta + \frac{(y_2^\beta - y_2^\gamma)(x_2 - y_2^\beta)}{(y_1^\gamma - y_1^\beta)}. \quad (5.37)$$

For these observations, observe that

$$\begin{aligned} & (y_1^\gamma - y_1^\beta)(y_1^\gamma + y_1^\beta - 2x_1) \\ & + (y_2^\gamma - y_2^\beta)(y_2^\gamma + y_2^\beta + 2\tilde{b} - 2x_2) \\ & = (y_1^\gamma - y_1^\beta)^2 + (y_2^\gamma - y_2^\beta)^2 - 2(y_1^\gamma - y_1^\beta)(x_1 - y_1^\beta) \\ & \quad + 2(y_2^\gamma - y_2^\beta)(\tilde{b} - x_2 + y_2^\beta) \\ & \stackrel{(a)}{=} -2\tilde{b}(y_2^\gamma - y_2^\beta) - 2(y_1^\gamma - y_1^\beta)(x_1 - y_1^\beta) \\ & \quad + 2(y_2^\gamma - y_2^\beta)(\tilde{b} - x_2 + y_2^\beta) \\ & = 2(y_2^\gamma - y_2^\beta)(y_2^\beta - x_2) \\ & \quad - 2(y_1^\gamma - y_1^\beta)(x_1 - y_1^\beta) > 0, \end{aligned}$$

where $\stackrel{(a)}{=}$ follows from the definition of y_1^γ in this case and the inequality is due to (5.37). This shows that (5.34) and equivalently $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ hold for the observations specified by (5.37) and $y_2^\beta < x_2 < y_2^\beta + 2\tilde{b}$. In addition, notice that if (5.37) holds for some y_2^γ , then it automatically holds for any $\tilde{y}_2^\gamma$ satisfying $y_2^\gamma - 2\tilde{b} < \tilde{y}_2^\gamma < y_2^\gamma$. Therefore, if there are multiple decoder actions satisfying the assumptions in this case, it is sufficient to consider the one with the maximum y_2 coordinate.

(iv) Let $\mathbf{y}^\gamma$ be such that $y_2^\beta - 2\tilde{b} \leq y_2^\gamma < y_2^\beta$ and $y_1^\gamma = y_1^\beta - \sqrt{(y_2^\gamma - y_2^\beta)(y_2^\beta - y_2^\gamma - 2\tilde{b})}$. Let

$$y_1^\beta < x_1. \quad (5.38)$$

A similar analysis to that in the previous case yields

$$\begin{aligned} & (y_1^\gamma - y_1^\beta)(y_1^\gamma + y_1^\beta - 2x_1) \\ & + (y_2^\gamma - y_2^\beta)(y_2^\gamma + y_2^\beta + 2\tilde{b} - 2x_2) \\ & = 2(y_2^\gamma - y_2^\beta)(y_2^\beta - x_2) \\ & \quad + 2(y_1^\beta - y_1^\gamma)(x_1 - y_1^\beta) > 0, \end{aligned}$$

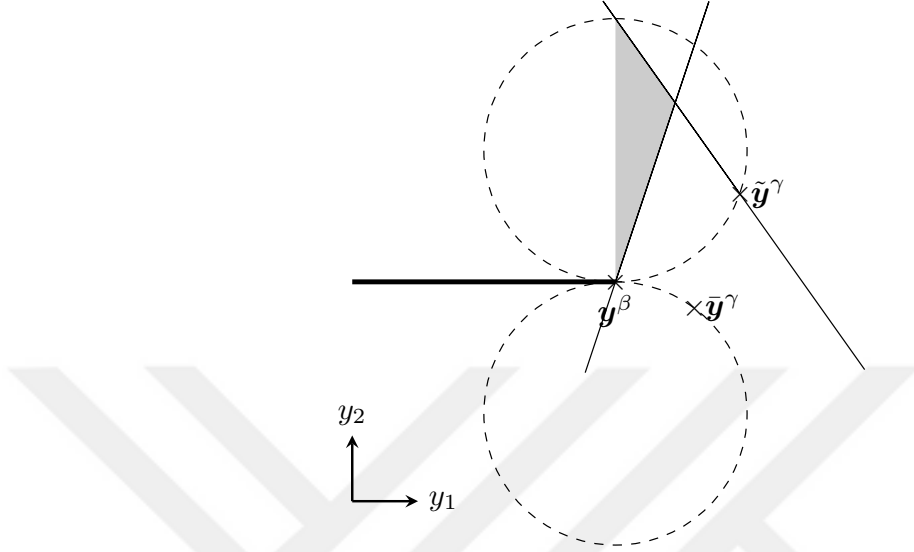


Figure 5.4: Illustration of the proof technique employed in Lemma 5.5.4. Here, the horizontal solid line represents a continuum of decoder actions and the decoder action on this continuum with the largest y_1 coordinate is denoted by $\mathbf{y}^\beta$. Inside the dashed circles, there cannot be a decoder action due to the condition imposed by Lemma 5.5.1 that for any $\mathbf{y}$ we have $g(\mathbf{y}^\beta, \mathbf{y}) \geq 0$ where $g(\cdot, \cdot)$ is defined in (5.17). If we place decoder actions $\bar{\mathbf{y}}^\gamma$ and $\tilde{\mathbf{y}}^\gamma$ on the dashed circles, then the shaded area must be the bin for $\mathbf{y}^\beta$, i.e., $\mathcal{B}^\beta$. However, the centroid of this shaded area cannot be $\mathbf{y}^\beta$, which is a contradiction.

where the inequality follows from the facts that $y_2^\gamma < y_2^\beta$, $y_2^\beta < x_2$, $y_1^\gamma \leq y_1^\beta$ and $y_1^\beta < x_1$. This proves that (5.34) and equivalently $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\gamma)$ hold for the observations specified by (5.38) and $y_2^\beta < x_2 < y_2^\beta + 2\tilde{b}$.

As a result, for any decoder action $\mathbf{y}^\eta$ satisfying $g(\mathbf{y}^\beta, \mathbf{y}^\eta) \geq 0$ and for a given x_2 with $y_2^\beta < x_2 < y_2^\beta + 2\tilde{b}$, there exists a nontrivial interval for x_1 with $x_1 > y_1^\beta$ such that $c_t^e(\mathbf{x}, \mathbf{y}^\beta) < c_t^e(\mathbf{x}, \mathbf{y}^\eta)$ holds. This implies that there exists a nonempty region of observations with $x_1 > y_1^\beta$ that must belong to $\mathcal{B}^\beta$. Fig. 5.4 illustrates this region of observations for an example scenario. Note that as long as $y_1^\beta < x_1^U(y_2^\beta)$, a nonempty subset of this observation region is in the support of the joint distribution. As a result, we obtain a contradiction to $\mathbf{y}^\beta = \mathbb{E}[\mathbf{X} | \mathbf{X} \in \mathcal{B}^\beta]$ since $\mathcal{B}^\beta$ does not contain any observation with $x_1 < y_1^\beta$ and $\mathcal{B}^\beta$ contains a nonempty region of observations with $x_1 > y_1^\beta$.

Chapter 6

Optimal Joint Modulation Classification and Symbol Decoding

6.1 Motivation

Recent years have witnessed a surge of interest in modulation classification motivated by a variety of contemporary applications in which it is paramount to recognize the modulation format of a received signal prior to decoding the transmitted message [147–164]. In the classical signal reception scenario, it is assumed that the characteristics of the transmitted signal are known and the channel parameters can be estimated reliably at the receiver. Under this setting with perfect information regarding the modulation format employed by the transmitter, traditional digital receivers first perform several tasks such as downconversion, timing synchronization, matched filtering, sampling and channel equalization, and then demodulate the signal samples to recover the transmitted symbol sequence. If the modulation format employed by the transmitter is also unknown, the receiver

needs to be equipped with a modulation classification mechanism before the symbol decoding stage [147]. For instance, in signal monitoring, there is no cooperation between the transmitting and the intercepting nodes to obtain information regarding the modulation format. Hence, in that case, it is imperative to perform modulation classification in order to convey the messages emitted by the target communications system. As another example, in cognitive and software defined radio based applications, intelligent receivers equipped with modulation classification algorithms can be utilized to reduce the communication overhead resulting from the exchange of information regarding the modulation format adapted to the varying channel conditions [147, 150, 162].¹

It should be noted that the primary goal of the aforementioned techniques is to improve the modulation classification accuracy. It is sensible to assume that better modulation classification performance yields better symbol decoding performance, the latter being the ultimate task for any receiver equipped with a modulation classifier. However, since these approaches have not been optimized for the subsequent symbol decoding stage, it is also reasonable to question their optimality. In this work, it is shown that by jointly designing the modulation classification and the symbol decoding stages, significant performance improvements for the symbol detection performance over the conventional strategy that treats two problems separately can be achieved.

6.2 Literature Review

In this chapter, we propose joint frameworks for modulation classification and symbol decoding in a communication theoretic setting. Modulation classification techniques can be grouped into feature-based and likelihood-based techniques [147, 149]. Feature-based techniques utilize features such as cumulants extracted from the received signal to identify the modulation format. Although

¹In adaptive modulation, the transmitter employs a low order modulation scheme under low signal-to-noise ratio (SNR) in order to ensure reliable transmission of the data whereas a high order modulation scheme is employed to increase the data rate in the case of high SNR [165].

feature-based techniques can be preferred due to their simplicity, these techniques are developed in an ad-hoc manner which do not guarantee optimality. On the other hand, likelihood-based techniques are developed using a statistical characterization of the received signal which facilitates the development of optimal (or near optimal in case the analysis uses some approximations, simplifying assumptions, etc.) techniques. In this study, in order to come up with optimal techniques for the solution of the considered problem, a statistical characterization of the received signal is utilized, and hence, related literature focusing on likelihood-based techniques are summarized in the following. In the context of likelihood-based modulation classification techniques, three prominent approaches are adopted based on how the unknown channel parameters and the transmitted symbol sequence are handled in the statistical model: the average likelihood ratio test (ALRT) [147, 148, 151, 154, 156, 161, 164], the generalized likelihood ratio test (GLRT) [119, 147, 148, 151, 159, 163] and the hybrid likelihood ratio test (HLRT) [147, 148, 151, 155, 156, 159, 160, 164, 166]. In the case of ALRT, all unknowns are treated as random variables with known prior distributions so that the likelihood expressions can be marginalized over the space of channel parameters and modulation symbols. Then, the resulting average likelihood scores are compared to decide on the modulation format. On the other hand, GLRT treats all unknowns as deterministic and substitutes their maximum likelihood (ML) estimates in the corresponding likelihood expressions to obtain a likelihood ratio based decision metric. HLRT takes a compromise between ALRT and GLRT by treating some parameters as random quantities with known priors while considering the rest as deterministic.

Our perspective on the the problem has been motivated by the success of joint design frameworks such as joint detection and estimation [167–172] and joint detection and decoding [173–175]. The work in [168] considers a binary hypothesis testing problem in which only one of the hypothesis is composite, i.e., it contains an unknown random parameter whose prior distribution is assumed to be known. Under this setting, the aim is to jointly determine the correct hypothesis and to provide an estimate of the unknown parameter when a decision is declared in favor of the composite hypothesis. It is shown that the optimal

decision rule depends on both the likelihood ratio and the posterior estimation cost for the unknown parameter while the optimal parameter estimate is given by the classical Bayes estimator that minimizes the posterior estimation cost. In [169], the optimal estimators and decision rule are determined in order to minimize the maximum of the conditional estimation costs under constraints on the false alarm and miss detection probabilities. In this case, unknown random parameters are assumed under both hypotheses. In [170], the authors consider the average (Bayesian) estimation cost as the objective function. Different from the previous work, unknown parameters common to both hypotheses are assumed in addition to parameters unique to each hypothesis. The optimization problems considered in [168–170] are formulated in a similar manner in the sense that the estimation performance metric appears in the objective as a function of both the estimators and the decision rule whereas the constraints are related to the detection performance metric, being solely dependent on the decision rule (i.e., uncoupled with the estimator), which facilitates ease of analysis. In [173], the authors consider intermittent communication over a noisy channel. It is assumed that the receiver acquires either pure noise or a codeword and the aim is to determine the presence of a codeword and to decode if declared present. The optimal decision rule that minimizes the probability of not correctly deciding on the true codeword transmitted subject to constraints on the false alarm and miss detection probabilities of rejecting pure noise hypothesis is derived. In [174], this idea is extended to channel detection in coded communications.

6.3 Contributions

The main contributions of this chapter can be summarized as follows:

- For the first time in the literature, modulation classification and symbol decoding problems are considered in a joint framework. Both the Bayesian and the minimax formulations are provided depending on the availability of

the prior probabilities for the employed modulation schemes. In the presence of prior information, the optimal decision rule that minimizes the overall average probability of symbol error (at the subsequent stage following modulation classification) subject to a constraint on the average probability of erroneous modulation classification is obtained in closed form. In the absence of such information, the optimal decision rule that minimizes the maximum symbol error probability given that the modulation is correctly identified is derived under individual constraints on the conditional correct classification probability of each modulation scheme.

- Since the proposed modulation classification rules are optimal with respect to each criterion, no other classifier can achieve a lower average (or maximum) symbol error probability while delivering equal or smaller average (or individual) modulation classification error(s).
- The proposed framework provides an optimal tradeoff between the probabilities of symbol error and modulation classification error. It is general in the sense that it covers the conventional method of performing LRT based modulation classification followed by ML decoding of the transmitted symbols as a special case.

The rest of this chapter is organized as follows. Section 6.4 presents the system model for the problem of modulation classification and symbol decoding as well as the conventional technique that relies on separate treatment. Section 6.5.1 formulates the problem of joint modulation classification and symbol decoding for the case of known prior probabilities of the modulation scheme and derives the optimal modulation classification and symbol decoding rules. Next, Section 6.5.2 considers a minimax formulation of the joint problem in the absence of modulation scheme priors. Section 6.5.3 presents some remarks on the proposed approach. Section 6.6 discusses the computational complexity of the proposed techniques as well as that of the conventional techniques. Section 6.7 describes an extension of the proposed framework obtained by slightly altering the metric used to measure the symbol decoding performance. Section 6.8 provides numerical examples to illustrate the performance of the proposed solutions and their comparison with

the conventional approach. Section 6.9 concludes the chapter with some remarks.

6.4 Problem Formulation

6.4.1 System Model

Consider a coherent and synchronous signal reception scenario under which the discrete-time baseband signal model is given by [176–179]

$$x_k = s_k + \eta_k, \quad k = 1, \dots, K, \quad (6.1)$$

where s_k and η_k are the transmitted constellation symbol and the additive noise term at time k , respectively, and K denotes the number of observations. Suppose there are L candidate digital amplitude-phase modulation schemes and let the set of constellation points corresponding to the ℓ th modulation scheme be denoted as $\mathcal{S}^{(\ell)} \triangleq \{\mu_1^{(\ell)}, \dots, \mu_{M^{(\ell)}}^{(\ell)}\}$ for $\ell = 1, \dots, L$, where $\mu_m^{(\ell)}$ represents a certain amplitude and/or phase modulated complex signal corresponding to the m th symbol of the ℓ th modulation scheme and $M^{(\ell)}$ denotes the modulation order of the ℓ th modulation scheme. It is assumed that during the reception of K signal samples, the modulation scheme remains the same, i.e., $(s_k)_{k=1}^K \in \mathcal{S}^{(\ell)}$ for some fixed ℓ . Suppose that under each modulation scheme the symbols are equally likely and the symbols transmitted at distinct time instants are independent. Furthermore, let $(\eta_k)_{k=1}^K$ be independent and identically distributed (i.i.d.) according to a known probability density function (PDF) denoted by $f_\eta^{(\ell)}(\cdot)$ when the ℓ th modulation scheme is employed. By allowing different PDFs for the noise term under each modulation scheme, the effect of different channel conditions such as that of SNR variations in an adaptive modulation system are incorporated into the model given in (6.1). The conditional PDF of the observed signal sample x_k given that the m th symbol of the ℓ th modulation scheme is sent at time instant k is expressed as $f^{(\ell)}(x_k | \mu_m^{(\ell)}) \triangleq f_\eta^{(\ell)}(x_k - \mu_m^{(\ell)})$.²

²Although an additive noise channel is assumed in this chapter, the proposed framework generalizes in a straightforward manner to other memoryless channel models thanks to the general form of the statistical characterization $f^{(\ell)}(x_k | \mu_m^{(\ell)})$.

Given the observed samples $\mathbf{x} \triangleq [x_1, \dots, x_K]^T$, the aim is to identify the modulation scheme $\mathcal{S}$ and to decode the transmitted symbol sequence $\mathbf{s} \triangleq [s_1, \dots, s_K]^T$. We consider two types of decision functions: $\boldsymbol{\delta} \triangleq \{\delta^{(\ell)}(\cdot)\}_{\ell=1}^L$ and $\boldsymbol{\phi} \triangleq \{\phi^{(\ell)}(\cdot)\}_{\ell=1}^L$. $\delta^{(\ell)}(\mathbf{x})$ denotes the probability of choosing the ℓ th modulation scheme based on the observation $\mathbf{x}$ such that $\sum_{\ell=1}^L \delta^{(\ell)}(\mathbf{x}) = 1$ and $\delta^{(\ell)}(\mathbf{x}) \geq 0$ for every ℓ and $\mathbf{x}$. On the other hand, $\phi^{(\ell)}(x)$ represents the decision rule for decoding transmitted symbols under the ℓ th modulation scheme based on the observed sample x . More specifically, $\phi^{(\ell)}(x_k) = m$ for some $m \in \{1, \dots, M^{(\ell)}\}$ means that the transmitted symbol s_k is decoded as $\mu_m^{(\ell)}$. Whenever a decision is declared in favor of the ℓ th modulation scheme, all samples are decoded using $\phi^{(\ell)}(\cdot)$.

Before proceeding with the proposed framework for joint modulation classification and symbol decoding, we summarize the conventional approach that treats two problems separately along with their solutions.

6.4.2 Conventional Method

6.4.2.1 Modulation Classification

Identifying the modulation scheme based on the samples given in (6.1) can be accomplished in two different ways depending on the availability of the prior probabilities for the modulation schemes under consideration. For example, in an adaptive modulation system that employs different constellations for low and high SNR regimes, the proportion of time that the channel spends in each SNR regime can be employed in the assessment of prior probabilities. If modulation type priors are known, the Bayes classifier is optimal [119]. Since the received samples are i.i.d., the likelihood function (LF) of the ℓ th modulation scheme is written as

$$f^{(\ell)}(\mathbf{x}) = \prod_{k=1}^K f^{(\ell)}(x_k), \quad (6.2)$$

where

$$f^{(\ell)}(x_k) \triangleq \sum_{m=1}^{M^{(\ell)}} \frac{1}{M^{(\ell)}} f^{(\ell)}(x_k | \mu_m^{(\ell)}). \quad (6.3)$$

Let $\mathbf{c} \triangleq [c_1, \dots, c_L]$, where c_ℓ denotes the prior probability of the ℓ th modulation scheme. Then, the probability of classification error is minimized by the maximum *a posteriori* (MAP) decision rule [119]:

$$\delta_B^{(j)}(\mathbf{x}) = \begin{cases} 1, & \text{if } j = \arg \max_{\ell \in \{1, \dots, L\}} c_\ell f^{(\ell)}(\mathbf{x}) \\ 0, & \text{otherwise} \end{cases}. \quad (6.4)$$

The Bayes classifier (i.e., the MAP classifier) in (6.4) reduces to the ML classifier when equal priors for modulation schemes are assumed. On the other hand, if the prior probabilities of the modulation schemes are unknown, Neyman Pearson (NP) framework can be employed for classification between two candidate modulation schemes.³ In this case, two types of classification errors are given by $P_1(\delta^{(1)}, \delta^{(2)}) \triangleq \int \delta^{(2)}(\mathbf{x}) f^{(1)}(\mathbf{x}) d\mathbf{x}$ and $P_2(\delta^{(1)}, \delta^{(2)}) \triangleq \int \delta^{(1)}(\mathbf{x}) f^{(2)}(\mathbf{x}) d\mathbf{x}$, where $\delta^{(1)}(\mathbf{x}) \geq 0$, $\delta^{(2)}(\mathbf{x}) \geq 0$ and $\delta^{(1)}(\mathbf{x}) + \delta^{(2)}(\mathbf{x}) = 1$ are satisfied for all $\mathbf{x}$. In the NP framework, the decision rule that minimizes one type of error subject to a constraint on the other is desired. Due to asymmetry between the objective and the constraint, we denote by $\{\delta_{\text{NP},i}^{(\ell)}(\cdot)\}_{\ell=1}^2$ the NP type modulation classifier that places the constraint on the i th type of error, i.e., $P_i(\delta^{(1)}, \delta^{(2)}) \leq \beta_i$, and minimizes the other type of error. Based on the NP lemma [119], the corresponding modulation classifier is given by the following LRT:

$$\delta_{\text{NP},i}^{(2)}(\mathbf{x}) = \begin{cases} 1, & \text{if } \frac{f^{(2)}(\mathbf{x})}{f^{(1)}(\mathbf{x})} > \tau_i \\ 0, & \text{otherwise} \end{cases}, \quad (6.5)$$

with $\delta_{\text{NP},i}^{(1)}(\mathbf{x}) = 1 - \delta_{\text{NP},i}^{(2)}(\mathbf{x})$ for $i = 1, 2$ where τ_i is chosen to satisfy $P_i(\delta_{\text{NP},i}^{(1)}, \delta_{\text{NP},i}^{(2)}) = \beta_i$. It should be noted that if the likelihood ratio in (6.5) contains any point mass, a randomization may be required in order to satisfy the constraint with equality.

³In the case of more than two modulation schemes, a generalization of NP lemma to more than one side condition can be utilized [180, Sec. 3.6].

Another alternative for modulation classification in the absence of modulation scheme prior probabilities is to employ the minimax hypothesis testing framework in which the maximum of individual probabilities of classification error, i.e., $\max_{\ell \in \{1, \dots, L\}} 1 - \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x}$, is minimized [119]. Under this framework, the modulation classification rule is a Bayes rule that corresponds to a least favorable prior.

6.4.2.2 Symbol Decoding

Once a decision is declared in favor of a modulation scheme, the MAP decision rule, which selects the symbol with the maximum *a posteriori* probability, can be employed to minimize the probability of symbol error for the corresponding constellation [165]. The *a posteriori* probability that the m th symbol of the ℓ th modulation scheme is transmitted given the observed sample x_k is computed as

$$p^{(\ell)}(\mu_m^{(\ell)} | x_k) \triangleq \frac{1}{M^{(\ell)}} \frac{f^{(\ell)}(x_k | \mu_m^{(\ell)})}{f^{(\ell)}(x_k)} \quad (6.6)$$

and the MAP symbol decoding rule for the ℓ th modulation scheme is expressed as

$$\phi_{\text{MAP}}^{(\ell)}(x_k) = \arg \max_{m \in \{1, \dots, M^{(\ell)}\}} p^{(\ell)}(\mu_m^{(\ell)} | x_k). \quad (6.7)$$

To sum up, in the conventional approach, the modulation scheme is determined first based on either (6.4) or (6.5) (or, the Bayes rule for least favorable prior when the minimax framework is adopted) depending on the availability of the prior probabilities of the modulation schemes and then symbols are decoded according to (6.7) for the selected modulation.

6.5 Optimal Joint Modulation Classification and Symbol Decoding

6.5.1 Known Priors for the Modulation Schemes

In this part, a joint formulation of the modulation classification and the symbol decoding problems is proposed for the case when prior probabilities of the modulation schemes are known or accurately estimated. A constrained optimization framework is considered in which the objective function reflects our desire to accurately recover as much symbols as possible while the constraint function controls the probability of classification error for the modulation type. In particular, consider the following cost function related to symbol error rate:

$$C(\phi^{(\ell)}(\mathbf{x}), \mathbf{m}) \triangleq \frac{1}{K} \sum_{k=1}^K \mathbb{1}_{\{\phi^{(\ell)}(x_k) \neq m_k\}} \quad (6.8)$$

where $\mathbf{m} \triangleq [m_1, \dots, m_K]^T$ contains the indices of the actual transmitted symbols from the ℓ th modulation scheme with $m_k \in \{1, \dots, M^{(\ell)}\}$ and $\mathbb{1}_{\{\cdot\}}$ is the indicator function that returns 1 if the condition in its argument is satisfied and 0 otherwise. Our aim is to minimize the average cost of symbol error when the employed modulation scheme is correctly identified while controlling the probability of modulation classification error (and consequently, the erroneous symbol decoding due to an incorrect modulation decision). To that aim, the average cost of symbol error in the presence of a correct modulation classification decision can be obtained by taking the expectation of (6.8) over the joint PDF of transmitted symbols and received samples when the employed modulation scheme is correctly identified and averaging over all candidate modulation schemes as follows:

$$\begin{aligned} \mathcal{J}_a(\delta, \phi) &\triangleq \sum_{\ell=1}^L c_\ell \mathbb{E}_\ell[\delta^{(\ell)}(\mathbf{x}) C(\phi^{(\ell)}(\mathbf{x}), \mathbf{m})] \\ &= \sum_{\ell=1}^L c_\ell \int \delta^{(\ell)}(\mathbf{x}) \sum_{m_1=1}^{M^{(\ell)}} \cdots \sum_{m_K=1}^{M^{(\ell)}} \frac{1}{K} \\ &\quad \sum_{k=1}^K \mathbb{1}_{\{\phi^{(\ell)}(x_k) \neq m_k\}} \left(\prod_{k=1}^K \frac{f^{(\ell)}(x_k | \mu_{m_k}^{(\ell)})}{M^{(\ell)}} \right) d\mathbf{x}, \end{aligned} \quad (6.9)$$

where $\mathbb{E}_\ell[\cdot]$ denotes expectation with respect to (w.r.t.) the conditional joint PDF of transmitted symbols and received samples given that the ℓ th modulation scheme is employed and the subscript a in $\mathcal{J}_a(\cdot)$ indicates averaging over the candidate modulation schemes. The aim is to minimize the objective function proposed in (6.9) with respect to $\{\delta^{(\ell)}(\cdot)\}_{\ell=1}^L$ and $\{\phi^{(\ell)}(\cdot)\}_{\ell=1}^L$. Notice that the cost function given in (6.9) takes into account the symbol decoding error when the modulation is correctly recognized. On the other hand, if the modulation is not correctly identified, the symbol decoding stage uses the wrong set of constellation symbols while recovering the transmitted symbol sequence, i.e., although $(s_k)_{k=1}^K \in \mathcal{S}^{(\ell)}$ in (6.1), $\phi^{(\ell')}(\cdot)$ is utilized to decode symbols for some $\ell' \neq \ell$. As a result, symbol decoding performance in the case of incorrect modulation classification becomes significantly poor.⁴ In that respect, there is also a need to control the probability of modulation classification error as it causes erroneous symbol decoding. For fixed modulation classification rule $\boldsymbol{\delta}$, the average probability of classification error is given by

$$\mathcal{H}_a(\boldsymbol{\delta}) = 1 - \sum_{\ell=1}^L c_\ell \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x}, \quad (6.10)$$

which is seen to be independent of the symbol decoders $\boldsymbol{\phi}$. Subsequently, a constraint on the average probability of modulation classification error is introduced, which, together with (6.9), yields the following optimization problem:

$$\begin{aligned} \min_{\boldsymbol{\delta}, \boldsymbol{\phi}} \quad & \mathcal{J}_a(\boldsymbol{\delta}, \boldsymbol{\phi}) \\ \text{subject to} \quad & \mathcal{H}_a(\boldsymbol{\delta}) \leq \alpha, \end{aligned} \quad (\text{P1})$$

where α is the constraint on the average probability of modulation classification error. Let α_B denote the average probability of modulation classification error corresponding to the Bayes rule given in (6.4). Since no detector can yield lower probability of modulation classification error than that of the Bayes detector, it is required that $\alpha \geq \alpha_B$ is satisfied for feasibility of the optimization problem. More generally, α can be regarded as a design parameter that governs the tradeoff between modulation classification performance and symbol decoding performance.

⁴A modulation classification error does not always mean that all the symbols are decoded erroneously (e.g., when some constellation points from different modulation schemes coincide). However, since the mapping of bits onto symbols will be different for distinct constellations, a modulation classification error, in general, significantly degrades the decoding performance.

In order to obtain the solution of the optimization problem given in (P1), we first present the following lemma that expresses optimality of the MAP rule based symbol decoding.

Lemma 6.5.1. *For any given priors $\mathbf{c}$ and feasible modulation classification rule specified by $\boldsymbol{\delta}$, the optimal symbol decoders that minimize $\mathcal{J}_a(\boldsymbol{\delta}, \boldsymbol{\phi})$ are obtained as the MAP decision rules $\boldsymbol{\phi}_{\text{MAP}} \triangleq \{\phi_{\text{MAP}}^{(\ell)}(\cdot)\}_{\ell=1}^L$ given in (6.7).*

Proof. Since the modulation classification rule $\boldsymbol{\delta} = \{\delta^\ell(\cdot)\}_{\ell=1}^L$ is fixed and the constraint function does not depend on $\boldsymbol{\phi} = \{\phi^{(\ell)}(\cdot)\}_{\ell=1}^L$, (P1) reduces to an unconstrained problem in this case. Moreover, the problem decouples across ℓ since we minimize the summation of L terms each of which depend only on $\phi^{(\ell)}(\cdot)$. The optimization problem related to the symbol decoder under the ℓ th modulation scheme is expressed as

$$\min_{\phi^{(\ell)}} \int \delta^{(\ell)}(\mathbf{x}) \sum_{m_1=1}^{M^{(\ell)}} \cdots \sum_{m_K=1}^{M^{(\ell)}} \left(\sum_{k=1}^K \mathbb{1}_{\{\phi^{(\ell)}(x_k) \neq m_k\}} \right) \left(\prod_{k=1}^K \frac{1}{M^{(\ell)}} f^{(\ell)}(x_k \mid \mu_{m_k}^{(\ell)}) \right) d\mathbf{x}. \quad (6.11)$$

By noting that $\mathbb{1}_{\{\phi^{(\ell)}(x_k) \neq m_k\}} = 1 - \mathbb{1}_{\{\phi^{(\ell)}(x_k) = m_k\}}$ and using (6.2), (6.3), and (6.6), the optimization problem in (6.11) can be written as

$$\max_{\phi^{(\ell)}} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) \sum_{m_1=1}^{M^{(\ell)}} \cdots \sum_{m_K=1}^{M^{(\ell)}} \left(\sum_{k=1}^K \mathbb{1}_{\{\phi^{(\ell)}(x_k) = m_k\}} \right) \left(\prod_{k=1}^K p^{(\ell)}(\mu_{m_k}^{(\ell)} \mid x_k) \right) d\mathbf{x}, \quad (6.12)$$

which after some manipulations reduces to

$$\max_{\phi^{(\ell)}} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) \sum_{k=1}^K \sum_{m_k=1}^{M^{(\ell)}} \mathbb{1}_{\{\phi^{(\ell)}(x_k) = m_k\}} p^{(\ell)}(\mu_{m_k}^{(\ell)} \mid x_k) d\mathbf{x}. \quad (6.13)$$

From (6.13), it is seen that the objective function is maximized by selecting the symbol with the maximum *a posteriori* probability $p^{(\ell)}(\mu_{m_k}^{(\ell)} \mid x_k)$, which corresponds to the MAP decoder given in (6.7). $\square$

Lemma 6.5.1 states that the MAP symbol decoder is optimal irrespective of the employed modulation classification rule. From a practical perspective, it means that there is no need to change the existing receiver demodulator architecture. On the other hand, as it will be shown next, the proposed joint formulation leads to a modulation classification rule that is different from the conventional Bayes classifier given in (6.4).

Substituting the optimal MAP symbol decoders into the objective function of (P1), we get

$$\mathcal{J}_a(\boldsymbol{\delta}, \boldsymbol{\phi}_{\text{MAP}}) = \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) C_p^{(\ell)}(\mathbf{x}) d\mathbf{x}. \quad (6.14)$$

In (6.14), $C_p^{(\ell)}(\mathbf{x})$ is the posterior cost associated with the optimal MAP based symbol decoder for the ℓ th constellation, which is given by

$$C_p^{(\ell)}(\mathbf{x}) = 1 - \frac{1}{K} \sum_{k=1}^K \max_{m \in \{1, \dots, M^{(\ell)}\}} p^{(\ell)}(\mu_m^{(\ell)} | x_k). \quad (6.15)$$

Consequently, the optimization problem in (P1) is obtained as

$$\begin{aligned} \min_{\boldsymbol{\delta}} \quad & \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) C_p^{(\ell)}(\mathbf{x}) d\mathbf{x} \\ \text{subject to} \quad & 1 - \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x} \leq \alpha. \end{aligned} \quad (6.16)$$

Next, the following theorem provides the optimal modulation classification rule that solves (6.16), which together with Lemma 6.5.1, characterize the solution to the original optimization problem (P1).

Theorem 6.5.1. *For $\alpha \geq \alpha_B$, and assuming that $\{f^{(\ell)}(\mathbf{x})(C_p^{(\ell)}(\mathbf{x}) - \lambda)\}_{\ell=1}^L$ do not contain any nonzero probability mass for $\lambda \geq 0$, the optimal modulation classification rule that solves (6.16) is given by*

Case 1:

$$\begin{aligned} \delta^{(j)}(\mathbf{x}) &= 1 \\ \text{if } j &= \arg \min_{\ell \in \{1, \dots, L\}} c_{\ell} f^{(\ell)}(\mathbf{x}) C_p^{(\ell)}(\mathbf{x}) \end{aligned} \quad (6.17)$$

and $\delta^{(j)}(\mathbf{x}) = 0$ otherwise, for $j = 1, \dots, L$, if the constraint on the classification error probability is satisfied, i.e., $\mathcal{H}_a(\boldsymbol{\delta}) \leq \alpha$; if not go to Case 2.

Case 2:

$$\begin{aligned} \delta^{(j)}(\mathbf{x}) &= 1 \\ \text{if } j &= \arg \min_{\ell \in \{1, \dots, L\}} c_\ell f^{(\ell)}(\mathbf{x}) (C_p^{(\ell)}(\mathbf{x}) - \lambda) \end{aligned} \quad (6.18)$$

and $\delta^{(j)}(\mathbf{x}) = 0$ otherwise, for $j = 1, \dots, L$, where $\lambda > 0$ is selected such that the probability of classification error is equal to α , i.e., $\mathcal{H}_a(\boldsymbol{\delta}) = \alpha$.

Proof. The optimization problem given in (6.16) is linear and hence convex in $\{\delta^{(\ell)}(\cdot)\}_{\ell=1}^L$. Moreover, with the assumption of $\alpha \geq \alpha_B$, the problem is feasible and Slater's condition is satisfied. Therefore, Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality. We begin with writing the Lagrangian as follows:

$$\mathcal{L}(\boldsymbol{\delta}, \lambda) = \lambda(1 - \alpha) + \int \sum_{\ell=1}^L \delta^{(\ell)}(\mathbf{x}) c_\ell f^{(\ell)}(\mathbf{x}) (C_p^{(\ell)}(\mathbf{x}) - \lambda) d\mathbf{x} \quad (6.19)$$

Using the fact that $\delta^{(\ell)}(\mathbf{x}) \geq 0$ for all ℓ and $\sum_{\ell} \delta^{(\ell)}(\mathbf{x}) = 1$, it is seen that (6.19) is minimized when modulation classifier selects the modulation that minimizes $c_\ell f^{(\ell)}(\mathbf{x}) (C_p^{(\ell)}(\mathbf{x}) - \lambda)$ for each given $\mathbf{x}$. Based on the remaining KKT conditions, λ can be determined. Namely, dual feasibility, complementary slackness and primal feasibility conditions require that $\lambda \geq 0$, $\lambda(1 - \alpha - \sum_{\ell=1}^L c_\ell \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x}) = 0$ and $1 - \sum_{\ell=1}^L c_\ell \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x} \leq \alpha$, respectively. First, let us substitute $\lambda = 0$. If the primal feasibility condition is satisfied, then the corresponding classifier given in (6.17) is the optimal solution to (6.16). On the other hand, if the classifier given in (6.17) violates the constraint, we go to Case 2, where $\lambda > 0$ is assumed. In this case, modulation classification constraint must be satisfied with equality due to complementary slackness. As $\lambda \rightarrow \infty$, it is seen from (6.18) that the optimal decision rule converges to the conventional MAP decision rule given in (6.4), which yields the lowest modulation classification error, i.e., $\mathcal{H}_a(\boldsymbol{\delta}_B) = \alpha_B$, among all detectors. Noting that the probability of

modulation classification error corresponding to the detector given in (6.18) is a continuous function of λ , by the intermediate value theorem, there exists $\lambda > 0$ such that $\mathcal{H}_a(\delta(\lambda)) = \alpha$. Hence, a solution in the form of (6.18) that satisfies all KKT conditions is guaranteed. $\square$

It is seen from (6.17) and (6.18) that the optimal modulation classification rule takes into account *a posteriori* probabilities of the transmitted symbols in addition to the likelihood of the observed samples and the prior probabilities of the modulation schemes. As a result, symbol decoding performance is taken into account in the proposed modulation classification mechanism. For a given feasible constraint on the modulation classification error, no other mechanism can achieve better average symbol decoding performance when the modulation type is correctly identified. In that respect, the proposed framework provides the flexibility of tradeoff between performances of modulation classification and symbol decoding. Furthermore, the conventional method, which treats two problems separately, is obtained as a special case of the proposed method by selecting $\alpha = \alpha_B$. By relaxing the constraint on the modulation classification performance, i.e., taking $\alpha > \alpha_B$, it is possible to improve the average symbol decoding performance in the case of correct modulation classification. To sum up, the solution to the proposed joint optimization problem stated in (P1) is given by the modulation classifier specified in (6.17) and (6.18), and the standard MAP rule based symbol decoder corresponding to the classified modulation scheme which is given in (6.7).

6.5.2 Unknown Priors for the Modulation Schemes

The approach proposed in the previous part requires that the prior probabilities of the modulation schemes are known or accurately estimated. In order to come up with a robust joint scheme that can correctly identify the modulation scheme in the absence of this information and decode the received samples, we consider the maximum conditional symbol decoding error among candidate modulation schemes given that each is identified correctly as our performance metric. More

explicitly, the objective function is expressed as

$$\begin{aligned}\mathcal{J}_m(\boldsymbol{\delta}, \boldsymbol{\phi}) &\triangleq \max_{\ell \in \{1, \dots, L\}} \mathbb{E}_\ell[C(\phi^{(\ell)}(\mathbf{x}), \mathbf{m}) \mid \mathbf{D} = \mathcal{S}^{(\ell)}] \\ &= \max_{\ell \in \{1, \dots, L\}} \frac{\mathbb{E}_\ell[C(\phi^{(\ell)}(\mathbf{x}), \mathbf{m})\delta^{(\ell)}(\mathbf{x})]}{\mathbb{E}_\ell[\delta^{(\ell)}(\mathbf{x})]},\end{aligned}\quad (6.20)$$

where the subscript m in $\mathcal{J}_m$ denotes that maximum is taken over the modulation schemes, $\{\mathbf{D} = \mathcal{S}^{(\ell)}\}$ denotes the event that a decision is declared in favor of the ℓ th modulation scheme, and the subscript ℓ denotes that the ℓ th modulation scheme is employed. As for the constraints, since an average probability of classification error cannot be defined in this case, an individual constraint on each conditional probability of classification error is employed. This results in the following optimization problem:

$$\begin{aligned}\min_{\boldsymbol{\delta}, \boldsymbol{\phi}} \quad & \mathcal{J}_m(\boldsymbol{\delta}, \boldsymbol{\phi}) \\ \text{subject to} \quad & 1 - \int \delta^{(\ell)}(\mathbf{x})f^{(\ell)}(\mathbf{x})d\mathbf{x} \leq \beta_\ell, \quad \text{for } \ell = 1, \dots, L.\end{aligned}\quad (\text{P2})$$

The feasibility of the problem in (P2) can be verified via the NP lemma or its generalization to more than one side condition [180, Sec. 3.6]. As an example, consider a binary modulation classification scenario. Let $\{\delta_{\text{NP},1}^{(\ell)}(\cdot)\}_{\ell=1}^2$ denote the optimal NP decision rule under type-1 error constraint β_1 , i.e., $\int \delta_{\text{NP},1}^{(2)}(\mathbf{x})f^{(1)}(\mathbf{x})d\mathbf{x} = \beta_1$. If its type-2 error satisfies $\int \delta_{\text{NP},1}^{(1)}(\mathbf{x})f^{(2)}(\mathbf{x})d\mathbf{x} \leq \beta_2$, then the problem in (P2) is feasible with the constraints β_1 and β_2 . By relaxing the constraints compared to the NP classifier, it is desired to improve the worst case symbol decoding performance.

We follow a similar procedure to that in Section 6.5.1 and first derive the optimal symbol decoder for fixed modulation classifier, which is presented in the following lemma.

Lemma 6.5.2. *For any feasible modulation classification rule specified by $\boldsymbol{\delta}$, the optimal symbol decoders that minimize $\mathcal{J}_m(\boldsymbol{\delta}, \boldsymbol{\phi})$ are obtained as the MAP decision rules $\{\phi_{\text{MAP}}^{(\ell)}(\cdot)\}_{\ell=1}^L$ given in (6.7).*

Proof. Since the constraint functions in (P2) do not depend on the symbol decoding rules $\{\phi^{(\ell)}(\cdot)\}_{\ell=1}^L$, the optimal symbol decoders can be derived by restricting

our attention to the objective function $\mathcal{J}_m(\boldsymbol{\delta}, \boldsymbol{\phi})$ in (P2). Then, the problem of interest becomes

$$\min_{\boldsymbol{\phi}} \mathcal{J}_m(\boldsymbol{\delta}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi}} \max_{\ell \in \{1, \dots, L\}} \frac{\mathbb{E}_\ell[\mathcal{C}(\phi^{(\ell)}(\mathbf{x}), \mathbf{m})\delta^{(\ell)}(\mathbf{x})]}{\mathbb{E}_\ell[\delta^{(\ell)}(\mathbf{x})]}. \quad (6.21)$$

In the proof of Lemma 1, it is shown that $\mathbb{E}_\ell[\mathcal{C}(\phi^{(\ell)}(\mathbf{x}), \mathbf{m})\delta^{(\ell)}(\mathbf{x})]$ is minimized when MAP rule based symbol decoders are employed under each modulation scheme. Furthermore, for fixed modulation classification rule, the correct classification probability of the ℓ th modulation scheme, i.e., $\mathbb{E}_\ell[\delta^{(\ell)}(\mathbf{x}) = 1]$, is constant. Consequently, for any set of symbol decoding rules $\{\phi^{(\ell)}(\cdot)\}_{\ell=1}^L$ and fixed modulation classification rule $\{\delta^{(\ell)}(\cdot)\}_{\ell=1}^L$, the following holds for all $\ell \in \{1, \dots, L\}$:

$$\frac{\mathbb{E}_\ell[\mathcal{C}(\phi^{(\ell)}(\mathbf{x}), \mathbf{m})\delta^{(\ell)}(\mathbf{x})]}{\mathbb{E}_\ell[\delta^{(\ell)}(\mathbf{x})]} \geq \frac{\mathbb{E}_\ell[\mathcal{C}(\phi_{\text{MAP}}^{(\ell)}(\mathbf{x}), \mathbf{m})\delta^{(\ell)}(\mathbf{x})]}{\mathbb{E}_\ell[\delta^{(\ell)}(\mathbf{x})]}, \quad (6.22)$$

where $\phi_{\text{MAP}}^{(\ell)}(\cdot)$ is as given in (6.7). Taking the maximum over all $\ell \in \{1, \dots, L\}$ on both sides, it is seen that the solution to (6.21) is given by the MAP rule based symbol decoders. $\square$

After inserting the optimal MAP decoders into the objective function, the optimization problem (P2) is written as

$$\begin{aligned} \min_{\boldsymbol{\delta}} \max_{\ell \in \{1, \dots, L\}} & \left\{ \frac{\int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) \mathcal{C}_p^{(\ell)}(\mathbf{x}) d\mathbf{x}}{\int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x}} \right\} \\ \text{subject to } & 1 - \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x} \leq \beta_\ell, \\ & \text{for } \ell = 1, \dots, L \end{aligned} \quad (6.23)$$

where the posterior symbol decoding cost $\mathcal{C}_p^{(\ell)}(\mathbf{x})$ is as given in (6.15). An optimization problem in the form of (6.23) is considered by Jajamovich et al. in the context of joint detection and estimation [169]. Based on the result in [169, Theorem 7], the globally optimal solution to (6.23) is given by

$$\begin{aligned} \delta^{(j)}(\mathbf{x}) &= 1 \\ \text{if } j &= \arg \min_{\ell \in \{1, \dots, L\}} f^{(\ell)}(\mathbf{x}) (a_\ell^1(\mathcal{C}_p^{(\ell)}(\mathbf{x}) - \gamma) - a_\ell^2) \end{aligned} \quad (6.24)$$

and $\delta^{(j)}(\mathbf{x}) = 0$ otherwise, for $j = 1, \dots, L$. In (6.24), $(a_\ell^1)_{\ell=1}^L$ and $(a_\ell^2)_{\ell=1}^L$ are nonnegative multipliers selected to satisfy $\sum_{\ell=1}^L a_\ell^1 + \sum_{\ell=1}^L a_\ell^2 = 1$ as well as the constraints on modulation classification error probabilities and γ denotes the optimal value of the solution to (6.23). The nonnegative multipliers $\mathbf{a} = [(a_\ell^1)_{\ell=1}^L, (a_\ell^2)_{\ell=1}^L]$ can be found along with the optimal value γ by performing a numerical search as described in [169, Section III.B]. To put it briefly, this can be accomplished by first discretizing the standard $(2L - 1)$ -simplex and then, for each point $\mathbf{a}$ in the discretized grid, computing the decision rules in (6.24) for the current value of γ , checking whether the resulting decision rules satisfy

$$\begin{aligned} \frac{\int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) C_p^{(\ell)}(\mathbf{x}) d\mathbf{x}}{\int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x}} &\leq \gamma \\ 1 - \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x} &\leq \beta_\ell \end{aligned} \quad (6.25)$$

for all $\ell = 1, \dots, L$, decreasing (increasing) the value of γ via a line or bisection search if all conditions are (not) satisfied, and repeating these steps in an iterative manner until γ can no further decreased. All in all, the modulation classification rule specified by (6.24) together with the MAP decoders in (6.7) characterize the solution to (P2). In Algorithm 1, a summary of the proposed techniques in this part and the previous part is provided.

It is noted that the optimal modulation classifier specified in (6.24) depends on *a posteriori* probabilities of the transmitted symbols through $\{C_p^{(\ell)}(\cdot)\}_{\ell=1}^L$ as well as the likelihood of the observed samples (i.e., $\{f^{(\ell)}(\cdot)\}_{\ell=1}^L$) similar to the previous part (see (6.17) and (6.18)). On the other hand, as opposed to the previous part where there is only one Lagrange multiplier term to be determined (based on the KKT conditions) in the modulation classification rule, in this case, multiple coefficients appear in the classification rule. Moreover, it should be remarked that the proposed modulation classifier is jointly optimal in the sense that no other modulation classifier can deliver a lower maximum symbol error probability in the case of correct modulation classification while satisfying given constraints on individual modulation classification error probabilities. Furthermore, if the individual modulation classification error probabilities corresponding to the NP classifier (or its generalization when $L > 2$) are used as constraints in (P2), the conventional method of treating modulation classification separately from the

subsequent symbol decoding is optimal. By relaxing the constraints, the symbol decoding performance quantified by the worst case metric specified in (6.20) can be improved.

Algorithm 1 Joint Modulation Classification and Symbol Decoding

```

procedure JOINTMODCLASSSYMDEC( $\mathbf{x}$ )
  for each candidate modulation  $\ell \in \{1, 2, \dots, L\}$  do
    for  $k = 1, 2, \dots, K$  do
      Compute  $f^{(\ell)}(x_k | \mu_m^{(\ell)}) = f_\eta^{(\ell)}(x_k - \mu_m^{(\ell)})$ 
         $\forall m \in \{1, 2, \dots, M^{(\ell)}\}$ 
      Compute  $p^{(\ell)}(\mu_m^{(\ell)} | x_k)$  using (6.6)
         $\forall m \in \{1, 2, \dots, M^{(\ell)}\}$ 
      Decode the symbol using (6.7), i.e.,
         $\hat{m}_k^{(\ell)} = \arg \max_{m \in \{1, \dots, M^{(\ell)}\}} p^{(\ell)}(\mu_m^{(\ell)} | x_k)$ 
    end for
    Compute  $f^{(\ell)}(\mathbf{x})$  using (6.2) and (6.3)
    Compute  $C_p^{(\ell)}(\mathbf{x})$  by inserting previously computed
       $p^{(\ell)}(\mu_{\hat{m}_k^{(\ell)}}^{(\ell)} | x_k)$  values into (6.15)
    end for
    Select the modulation format based on (6.17) and (6.18) or
      based on (6.24), in the known or unknown modulation
      scheme priors cases, respectively
    Output the symbols  $\{\hat{m}_k^{(\hat{\ell})}\}_{k=1}^K$  where  $\hat{\ell}$  denote the index
      of selected modulation format
end procedure

```

6.5.3 Discussion

In this part, we present some remarks about the proposed framework. To this end, let an adaptive modulation system be configured such that the employed modulation format can be changed after every transmitted K symbols. Suppose that a sequence of length $N \times K$ symbols with $N > 1$ is observed according to this adaptive modulation scenario where the received signal sequence between indices $1 + (n - 1)K$ and nK for $n \in \{1, \dots, N\}$ follows the signal model given in (6.1). For convenience, each of these signal sequences of length K is termed as a single block in the following. Suppose also that the channel parameters and

the noise levels are such that if the conventional technique is utilized, the average probability of symbol error in the case of correct classification is not acceptable for the application in hand. In order to reduce the symbol error to an acceptable level, the proposed approach in (P1) compromises modulation classification performance. By relaxing the constraint on the modulation classification error w.r.t. that of the MAP based classifier, the resulting modulation classifier is allowed to decide in favor of a modulation scheme that delivers a lower individual symbol error rate (even though it may not be the most probable modulation scheme for the observed block) optimally in the sense that the symbol error rate averaged over all candidate constellations is minimized subject to the relaxed constraint on the modulation classification error. (Recall that since all of the decoded symbols in that block will be erroneous if the modulation scheme is not identified correctly and the constellation points of candidate modulation schemes do not overlap, it is necessary to constrain modulation classification errors.) Another way to think of this is as follows: Suppose that a block of signal samples that falls around a small neighborhood of the decision boundary of the MAP based classifier is observed at the receiver. As far as the MAP based modulation classifier is concerned, this block can be assigned to any modulation scheme on either side of the decision boundary since that would not alter the performance metric of the MAP based classifier noticeably. On the other hand, one of these modulation schemes is likely to yield a lower symbol error probability than the other in the case of a correct decision on the employed modulation scheme. In that respect, the overall symbol error performance in the case of correct classification can be reduced by assigning the received blocks close to the decision boundary to the modulation scheme with smaller symbol error score while controlling the erroneous symbol decoding due to an incorrect classification decision with the constraint on the tolerable modulation classification error.

6.6 Computational Complexity

In this section, we examine the computational complexity of joint modulation classification and symbol decoding. In order to implement the proposed methods, it is required to find λ or the pair $\mathbf{a}$ and γ when the prior probabilities of the modulation formats are known or unknown, respectively. In practice, these parameters can be determined in an offline manner for possible system configurations (e.g., different SNR and K values) and the corresponding receiver can directly use these values. Therefore, in the following, we assume that these parameters have been computed and focus on the computational complexity of the joint modulation classification and symbol decoding scheme. Then, the complexity of computing these parameters is explained.

In order to identify the employed modulation scheme, both of the proposed techniques need to compute the likelihood scores (i.e., $\{f^{(\ell)}(\cdot)\}_{\ell=1}^L$) and the posterior symbol decoding costs (i.e., $\{C_p^{(\ell)}(\cdot)\}_{\ell=1}^L$) based on the observed samples for each candidate modulation format. Since $C_p^{(\ell)}(\cdot)$ depends on the *a posteriori* probabilities of the symbols selected by the MAP decoding rule corresponding to the ℓ th modulation scheme, the received signal sequence is first decoded using the MAP decoding rule of the ℓ th modulation scheme and then *a posteriori* probability values achieved by the decoded symbols are inserted into (6.15). With this approach, modulation classification and symbol decoding stages are combined. Thus, the overall computational complexity of the proposed techniques depends on the number of operations required to obtain $\{f^{(\ell)}(\cdot)\}_{\ell=1}^L$ and $\{C_p^{(\ell)}(\cdot)\}_{\ell=1}^L$. They are given in Table 6.1, where function evaluations represent the computation of $f_\eta^{(\ell)}(\cdot)$, and M_s and $M_{\max}$ are defined as $M_s \triangleq \sum_{\ell=1}^L M^{(\ell)}$ and $M_{\max} \triangleq \max_{\ell \in \{1, \dots, L\}} M^{(\ell)}$, respectively. In the table, the particular values corresponding to the numerical examples in Section 6.8 are also presented.

On the other hand, the conventional techniques decide on the modulation format based only on the likelihood of the observed samples and then decode the symbols according to the MAP symbol detector of the selected modulation

Table 6.1: Number of operations required for conventional and proposed techniques (Final expressions correspond to the numerical examples in Section 6.8.)

	Conventional	Proposed
# of function evaluations	$K M_s = 112K$	$K M_s = 112K$
# of additions or subtractions	$K(2M_s - L)$ $\approx 2K M_s = 224K$	$2K M_s = 224K$
# of multiplications or divisions	$2K(M_{\max} + L) - L$ $\approx 2K M_{\max} = 128K$	$2K(M_s + L)$ $\approx 2K M_s = 224K$
# of maxima or minima	$K(M_{\max} - 1)$ $\approx K M_{\max} = 64K$	$K(M_s - L)$ $\approx K M_s = 112K$

format. Therefore, the main difference between the conventional and the proposed techniques in terms of complexity arise from the fact that the conventional techniques need to perform symbol decoding only under the selected modulation scheme whereas the proposed techniques perform symbol decoding under each candidate to obtain $\{C_p^{(\ell)}(\cdot)\}_{\ell=1}^L$. Yet, since the MAP symbol decoders under each modulation format needs to be implemented in practice, the complexity of the hardware for the proposed techniques can be reduced by sharing the resources for modulation classification and symbol decoding stages.

From Table 6.1, it is seen that the number of samples (K) and the sum of modulation orders (M_s) determine the complexity of the proposed techniques. On the other hand, as the conventional techniques decode symbols only under the selected modulation format, the total number of multiplication and maximum operations depend on $M_{\max}$ instead of M_s considering the worst case scenario when the highest order modulation format is selected.

Next, we examine the computational complexity of the procedure for setting the parameters that appear in the modulation classification rules, i.e., λ , $\mathbf{a}$ and γ . For (P1), Monte Carlo simulations can be performed to determine λ . Towards this goal, we consider the discretized interval $[0, \lambda_{\max}]$ where $\lambda_{\max}$ is such

that $\mathcal{H}_a(\boldsymbol{\delta}(\lambda_{\max})) \approx \alpha_B$. For a given λ in this discretized interval, we compute the decision rule for each Monte Carlo trial and then empirically obtain the average probability of modulation classification error using the results of these trials. Next, we repeat this procedure for each λ in this discretized interval until $|\mathcal{H}_a(\boldsymbol{\delta}(\lambda)) - \alpha| \leq \epsilon$ for some small ϵ . Treating each iteration as $\mathcal{O}(1)$, the computational complexity of this complete procedure is given by $\mathcal{O}(N_T N_G^\lambda)$ where N_T and N_G^λ denote the number of Monte Carlo trials and the size of the discretized grid, respectively. For (P2), the optimal values of $\mathbf{a}$ and γ are determined based on the algorithm in [169]. In this algorithm, a single iteration of the bisection search requires $\mathcal{O}(N_T N_G^{\mathbf{a}})$ computations where $N_G^{\mathbf{a}}$ denotes the size of the discretized grid for $\mathbf{a}$ and a few iterations is in practice sufficient for convergence. It should be noted that the complexity of this procedure can be significantly higher than the previous case since the discretization for $\mathbf{a}$ is on a standard $(2L - 1)$ -simplex whereas the discretization for λ is simply over an interval on the real line.

6.7 Extension

In this section, an extension to the proposed framework is considered by utilizing a different cost function to evaluate the symbol decoding performance. In Section 6.5.1 and Section 6.5.2, the symbol decoding performance of the proposed techniques is measured by the cost function defined in (6.8), which appears in the objective functions of (P1) and (P2). The cost function in (6.8) essentially counts the number of symbol errors made by the employed detector out of K symbol transmissions that constitute the received signal sequence in (6.1). In other words, the resulting performance metric penalizes each erroneous symbol decoding separately in a given block of length K . However, in certain applications, rather than individual symbol errors, it may be important to decode the whole block of symbols (e.g., codewords, frames, etc.) correctly. In order to meet the requirements of such applications, the following cost function is introduced:

$$C_b(\boldsymbol{\phi}^{(\ell)}(\mathbf{x}), \mathbf{m}) = \mathbb{1}_{\{\boldsymbol{\phi}^{(\ell)}(\mathbf{x}) \neq \mathbf{m}\}} \quad (6.26)$$

which yields 1 if the transmitted and decoded messages do not agree and 0 otherwise, where $\boldsymbol{\phi}^{(\ell)}(\mathbf{x}) \triangleq [\phi^{(\ell)}(x_1), \dots, \phi^{(\ell)}(x_K)]^T$.

In order to characterize the solution to (P1) and (P2) when the objective function is computed using (6.26) instead of (6.8), we begin by deriving the optimal symbol decoding rule for a fixed modulation classifier. To that end, the optimality of the MAP decoding is shown first in the following. Furthermore, it is sufficient to only consider (P1) here since the optimality in the case of (P2) is clear following a similar reasoning as in the proof of Lemma 6.5.2.

For a fixed modulation classification rule $\boldsymbol{\delta}$, the optimization problem in (P1) with the new cost function is written as

$$\min_{\boldsymbol{\phi}^{(\ell)}} \int \delta^{(\ell)}(\mathbf{x}) \sum_{m_1=1}^{M^{(\ell)}} \cdots \sum_{m_K=1}^{M^{(\ell)}} \mathbb{1}_{\{\boldsymbol{\phi}^{(\ell)}(\mathbf{x}) \neq \mathbf{m}\}} \left(\prod_{k=1}^K \frac{1}{M^{(\ell)}} f^{(\ell)}(x_k \mid \mu_{m_k}^{(\ell)}) \right) d\mathbf{x}. \quad (6.27)$$

After some manipulation, we get

$$\begin{aligned} \max_{\boldsymbol{\phi}^{(\ell)}} \int & \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) \\ & \left(\sum_{m_1=1}^{M^{(\ell)}} \mathbb{1}_{\{\boldsymbol{\phi}^{(\ell)}(x_1)=m_1\}} p^{(\ell)}(\mu_{m_1}^{(\ell)} \mid x_1) \right) \\ & \vdots \\ & \left(\sum_{m_K=1}^{M^{(\ell)}} \mathbb{1}_{\{\boldsymbol{\phi}^{(\ell)}(x_K)=m_K\}} p^{(\ell)}(\mu_{m_K}^{(\ell)} \mid x_K) \right) d\mathbf{x}. \end{aligned} \quad (6.28)$$

It is now clear from (6.28) that the optimal symbol decoder is given by the MAP rule, i.e., $\phi_{\text{MAP}}^{(\ell)}(\cdot)$.

Having identified the symbol decoder, we proceed with the derivation of the modulation classification rule. By inserting MAP based symbol decoders into the objective function (6.27), the problem reduces to

$$\begin{aligned} \min_{\boldsymbol{\delta}} \quad & \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) C_{\text{p,b}}^{(\ell)}(\mathbf{x}) d\mathbf{x} \\ \text{subject to} \quad & 1 - \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x} \leq \alpha \end{aligned} \quad (6.29)$$

where $C_{p,b}^{(\ell)}(\mathbf{x})$ denotes the posterior cost associated with the optimal MAP based symbol decoder for the ℓ th constellation computed using the cost function in (6.26) and it is given by

$$C_{p,b}^{(\ell)}(\mathbf{x}) = 1 - \prod_{k=1}^K \max_{m \in \{1, \dots, M^{(\ell)}\}} p^{(\ell)}(\mu_m^{(\ell)} | x_k). \quad (6.30)$$

Notice that the problem in (6.29) is the same as that in (6.16) after replacing $C_p^{(\ell)}(\mathbf{x})$ with $C_{p,b}^{(\ell)}(\mathbf{x})$. Therefore, the solution to (6.29) is given by the modulation classification rule specified in (6.17) and (6.18) after substituting $C_{p,b}^{(\ell)}(\mathbf{x})$ in place of $C_p^{(\ell)}(\mathbf{x})$. Furthermore, a similar approach can be employed in the case of (P2) leading to the modulation classification rule specified by (6.24), where $C_p^{(\ell)}(\mathbf{x})$ is again replaced with $C_{p,b}^{(\ell)}(\mathbf{x})$.

Moreover, we can analyze an interesting case in which the optimization problem does not involve any constraints. Recall that in (P1), the objective function measures the symbol error probability in the case of correct modulation classification while the constraint ensures a given level of modulation classification accuracy. Instead of such a constrained problem, an unconstrained problem can also be formulated by combining these two performance measures into a single objective function with equal weights. More specifically, consider the following optimization problem

$$\min_{\boldsymbol{\delta}, \boldsymbol{\phi}} \mathcal{J}_{a,b}(\boldsymbol{\delta}, \boldsymbol{\phi}) + \mathcal{H}_a(\boldsymbol{\delta}) \quad (6.31)$$

where

$$\mathcal{J}_{a,b}(\boldsymbol{\delta}, \boldsymbol{\phi}) \triangleq \sum_{\ell=1}^L c_\ell \mathbb{E}_\ell [C_b(\boldsymbol{\phi}^{(\ell)}(\mathbf{x}), \mathbf{m}) \delta^{(\ell)}(\mathbf{x})] \quad (6.32)$$

and $\mathcal{H}_a(\boldsymbol{\delta})$ is as given in (6.10). Under the assumption that the modulation classification errors automatically lead to symbol errors (i.e., when constellation points of candidate modulation schemes do not overlap), the solution of the optimization problem in (6.31) can be viewed as a joint mechanism that minimizes the total probability of symbol error composed of those that occur when modulation scheme is correctly identified and those that occur when a modulation classification error is made.

Noticing that the second term in (6.31) does not depend on ϕ , and based on the previous discussions (see (6.28)), it can be concluded that the MAP rule, i.e., $\phi_{\text{MAP}}^{(\ell)}(\cdot)$, solves the optimization problem in (6.31) for a fixed modulation classification rule δ . Next, after inserting the optimal MAP decoders, the optimization problem in (6.31) reduces to

$$\min_{\delta} \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) \left(C_{\text{p,b}}^{(\ell)}(\mathbf{x}) - 1 \right) d\mathbf{x}. \quad (6.33)$$

From (6.33), it can be seen that the optimal modulation classification rule is given by

$$\begin{aligned} \delta^{(j)}(\mathbf{x}) &= 1 \\ &\text{if } j = \arg \min_{\ell \in \{1, \dots, L\}} c_{\ell} f^{(\ell)}(\mathbf{x}) \left(C_{\text{p}}^{(\ell)}(\mathbf{x}) - 1 \right) \end{aligned} \quad (6.34)$$

and $\delta^{(j)}(\mathbf{x}) = 0$ otherwise, for $j = 1, \dots, L$.

Note that the optimal modulation classifier in the constrained problem contains a Lagrange multiplier term which essentially determines the tradeoff between the symbol decoding performance in the case of correct modulation classification and the average probability of misclassification. On the other hand, it is observed that the optimal modulation classifier derived in (6.34) is a special case of the constrained problem when $\lambda = 1$, which means that it does not permit tradeoffs between the considered performance measures of symbol decoding and modulation classification.

In the following, we provide another perspective to the unconstrained optimization problem in (6.31). Combining both terms in the objective function and

after some manipulation, the optimization problem in (6.31) can be expressed as

$$\begin{aligned}
& \max_{\boldsymbol{\delta}, \boldsymbol{\phi}} \sum_{\ell=1}^L c_{\ell} \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) \\
& \quad \left(\sum_{m_1=1}^{M^{(\ell)}} \mathbb{1}_{\{\phi^{(\ell)}(x_1)=m_1\}} p^{(\ell)}(\mu_{m_1}^{(\ell)} \mid x_1) \right) \\
& \quad \vdots \\
& \quad \left(\sum_{m_K=1}^{M^{(\ell)}} \mathbb{1}_{\{\phi^{(\ell)}(x_K)=m_K\}} p^{(\ell)}(\mu_{m_K}^{(\ell)} \mid x_K) \right) d\mathbf{x}. \tag{6.35}
\end{aligned}$$

Furthermore, the optimization problem in (6.35) can alternatively be written as

$$\begin{aligned}
& \max_{\ell \in \{1, \dots, L\}} \max_{m_k \in \mathcal{M}^{(\ell)}, k=1, \dots, K} c_{\ell} f^{(\ell)}(\mathbf{x}) \prod_{k=1}^K p^{(\ell)}(\mu_{m_k}^{(\ell)} \mid x_k) \\
& = \max_{\ell \in \{1, \dots, L\}} \max_{m_k \in \mathcal{M}^{(\ell)}, k=1, \dots, K} c_{\ell} \prod_{k=1}^K \frac{1}{M^{(\ell)}} f^{(\ell)}(x_k \mid \mu_{m_k}^{(\ell)}) \\
& = \max_{\ell \in \{1, \dots, L\}} \max_{m_k \in \mathcal{M}^{(\ell)}, k=1, \dots, K} \\
& \quad p(\mathbf{D} = \mathcal{S}^{(\ell)}, \mu_{m_1}^{(\ell)}, \dots, \mu_{m_K}^{(\ell)}, \mathbf{x}) \\
& = \max_{\ell \in \{1, \dots, L\}} \max_{m_k \in \mathcal{M}^{(\ell)}, k=1, \dots, K} \\
& \quad p(\mathbf{D} = \mathcal{S}^{(\ell)}, \mu_{m_1}^{(\ell)}, \dots, \mu_{m_K}^{(\ell)} \mid \mathbf{x}) \tag{6.36}
\end{aligned}$$

where $\mathcal{M}^{(\ell)} \triangleq \{1, \dots, M^{(\ell)}\}$. The final expression in (6.36) states that the optimal modulation classification and symbol decoding rules need to be such that the joint *a posteriori* probability of modulation format and symbol sequence are maximized given the observed signal sequence. Due to the fact that a maximization over the joint *a posteriori* probability is performed, this technique is referred to as *joint MAP*. All in all, using the cost function (6.26) in the objective function of (P1) and taking $\lambda = 1$ in the modulation classification rule, the *joint MAP* technique is obtained as a special case.

6.8 Numerical Results

In this section, numerical examples are presented to examine the performance of the proposed solutions to the joint modulation classification and symbol decoding problem. In particular, to illustrate the effectiveness of the proposed techniques, performance comparisons are conducted against the conventional techniques. Note that both the proposed techniques and the conventional techniques decode the symbols based on the MAP decoder for the recovered modulation scheme. The difference between the formulations is related to the modulation classification rules.⁵ Therefore, in the following, the proposed techniques in Section 6.5.1 and Section 6.5.2 are compared against the conventional techniques that classify the modulation scheme based on the Bayes modulation classifier and the minimax⁶ modulation classifier, respectively, which are described in Section 6.4.2.

The setup considered in the simulations is as follows: There exist three possible candidate modulation schemes which are 16-QAM, 32-QAM and 64-QAM, i.e., $M^{(1)} = 16$, $M^{(2)} = 32$ and $M^{(3)} = 64$ [155, 160, 164]. The modulation schemes are assumed to have unit average power.⁷ The noise samples η_k are taken as complex circular Gaussian random variables with variance $2\sigma_\ell^2$ under the ℓ th modulation scheme leading to an additive white Gaussian noise (AWGN) channel. Under this setting, the SNR for the ℓ th modulation scheme becomes $\text{SNR}_\ell = 1/(2\sigma_\ell^2)$. The aim of allowing for different noise levels under each modulation scheme is to model the behavior of adaptive modulation systems which select the modulation format to be employed according to the channel conditions.

⁵It is important to emphasize that although the resulting joint mechanisms of the proposed techniques lead to a change only in the modulation classifier compared to the conventional techniques, the proposed techniques are jointly optimal considering the introduced performance measures of both subproblems.

⁶In the numerical examples, since equal constraints on the individual probability of classification errors are assumed, i.e., $1 - \int \delta^{(\ell)}(\mathbf{x}) f^{(\ell)}(\mathbf{x}) d\mathbf{x} \leq \beta$ for all $\ell \in \{1, \dots, L\}$, the minimum of such β results in minimax decision rule [119].

⁷Different but known power levels can also be considered as there are no assumptions about the power levels in Section 6.5.

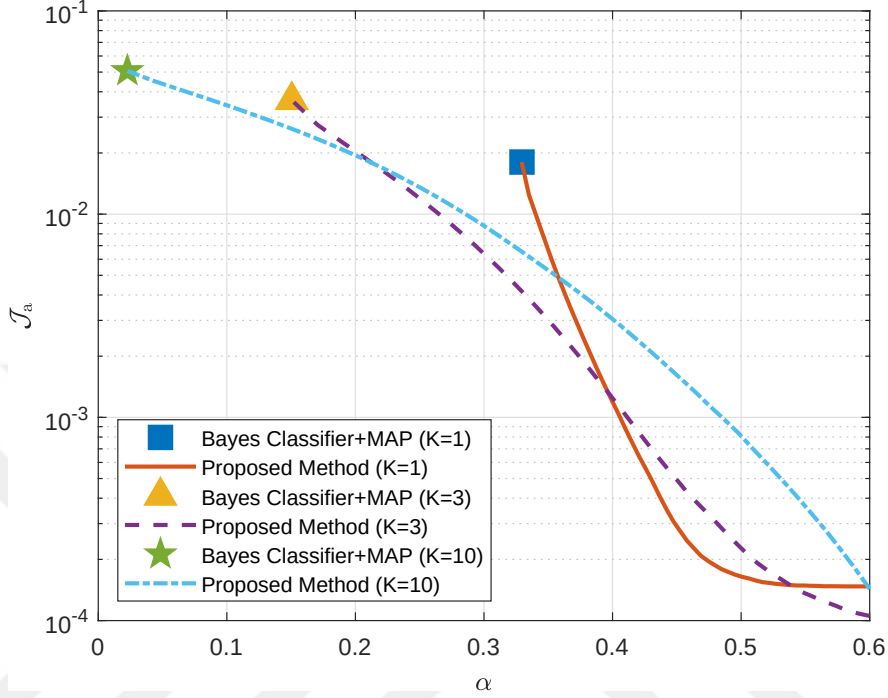


Figure 6.1: Comparison of the conventional and the proposed approaches in the case of known modulation scheme priors where $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = 18\text{dB}$.

6.8.1 Performance in the Case of Known Priors for the Modulation Schemes

We first investigate the case in which the prior probabilities of the modulation schemes are known by solving (P1) for the previously described setup. In particular, under each modulation scheme, SNR values are set to 18dB, and each modulation scheme is equally likely, i.e., $c_1 = c_2 = c_3 = 1/3$. For this scenario, Figure 6.1 plots the average symbol decoding performance in the case of correct modulation classification with respect to the constraint on the average probability of misclassification for the proposed technique for $K = 1, 3, 10$, together with the corresponding performances of the conventional technique, which are represented by the single points (as they yield a fixed performance for each K). It is observed that for each value of K , the symbol decoding performance can be improved by relaxing the constraint on the probability of modulation classification error. Therefore, it is verified that a tradeoff between the subproblems becomes possible via the proposed approach. Moreover, as the proposed technique is jointly

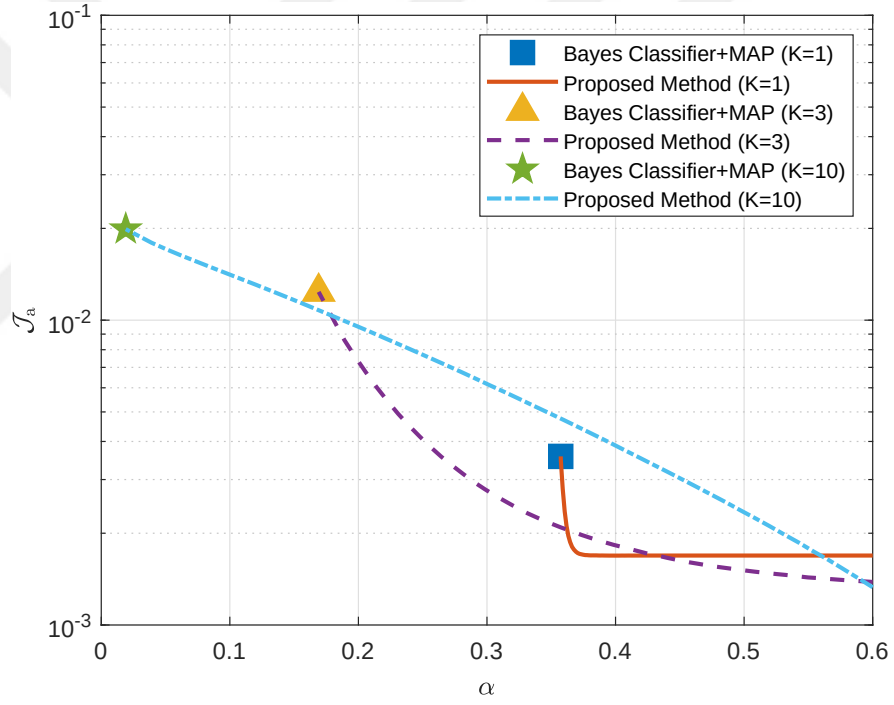


Figure 6.2: Comparison of the conventional and the proposed approaches in the case of known modulation scheme priors where $\text{SNR}_1 = 15\text{dB}$, $\text{SNR}_2 = 18\text{dB}$ and $\text{SNR}_3 = 21\text{dB}$.

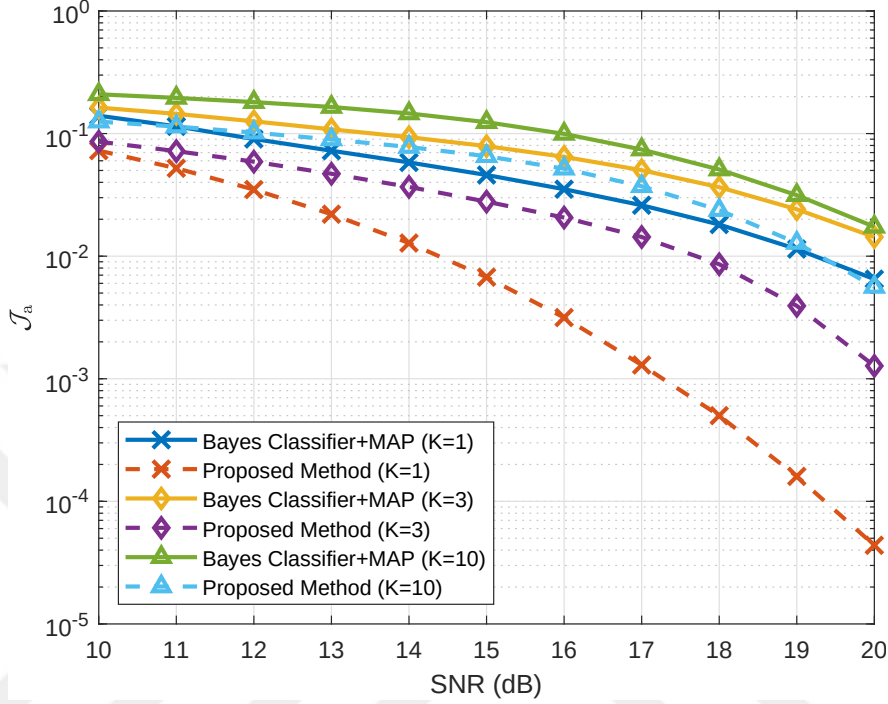


Figure 6.3: Symbol decoding performance versus SNR in the case of known modulation scheme priors for $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = \text{SNR}$ and $\alpha = 1 - 0.85(1 - \alpha_B)$.

optimal, no other method can operate below the plotted curves. Therefore, these plots provide performance benchmarks for any other joint mechanism. Furthermore, it is observed that when $K = 1$, a small relaxation (e.g., changing $\alpha = \alpha_B$ with $\alpha = 0.5$) in the constraint compared to the Bayesian classifier leads to a significant improvement in the symbol decoding performance. On the other hand, when further relaxation (e.g., changing $\alpha = 0.5$ with $\alpha = 0.6$) is applied, no significant improvement on the symbol decoding performance is obtained. On the other hand, when K is increased, the relaxation required for a certain order of magnitude reduction in the symbol decoding performance measure becomes larger. Therefore, if a significant improvement on the symbol decoding performance is desired in the case of a large K , the constraint should not be close to α_B . In fact, as can be observed from the plot, by taking the constraint as large as 0.5, it is possible to reduce the symbol error probability in the case of correct classification to a value lower than 10^{-3} , which is considered to be an acceptable level for a variety of applications, for $K = 10$.

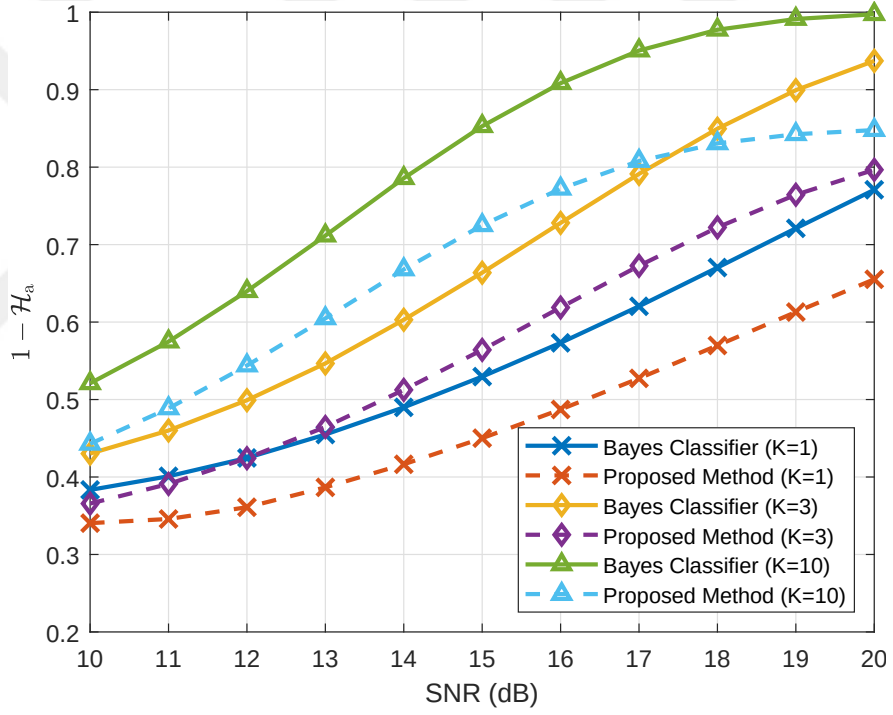


Figure 6.4: Average probability of correct modulation classification versus SNR in the case of known modulation scheme priors for $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = \text{SNR}$ and $\alpha = 1 - 0.85(1 - \alpha_B)$.

In Figure 6.2, we change the SNR values to $\text{SNR}_1 = 15 \text{ dB}$, $\text{SNR}_2 = 18 \text{ dB}$ and $\text{SNR}_3 = 21 \text{ dB}$ and plot the average symbol decoding performance when the modulation format is correctly identified versus the constraint on the probability of misclassification for $K = 1, 3, 10$. The reason for considering a different set of SNR values is to show the effectiveness of the proposed technique under a variety of conditions. It should also be emphasized that under the modulation format with a higher order, the channel is assumed to be less noisy (i.e., higher SNR), which is consistent with the operation of adaptive modulation systems. The first observation from Figure 6.2 is that by relaxing the constraint on the probability of misclassification, an improvement on the symbol decoding performance can be obtained compared to the conventional technique for different values of K . In other words, the tradeoff between the subproblems is enabled under a variety of conditions (i.e., for different SNR and K values). Similar to the previous case, for $K = 1$, a slight relaxation is sufficient to set the symbol decoding risk to a value that is close to the lowest possible value. On the other hand, for a large K , it is required to apply a further relaxation in order to reduce the symbol decoding risk, for instance to a value close to 10^{-3} . Moreover, the relaxation in the constraint in this case is not as effective as the relaxation in the previous case, especially for low K 's. The intuition behind this observation can be explained as follows: In the previous case, since SNR values are the same, the symbol decoding performance under 64-QAM is worse than that under 16-QAM and 32-QAM. Hence, the overall symbol decoding performance is mainly dominated by the performance of 64-QAM. This provides enhanced flexibility to reduce the overall symbol decoding risk by making more classification errors under 64-QAM. On the other hand, there is no such flexibility in the second case due to the fact that each modulation scheme has comparable symbol decoding performance.

We next investigate the performance of the proposed technique together with the conventional technique for a range of SNR values in which each modulation scheme operates under the same SNR condition. For a given SNR value, the conventional technique yields a fixed probability of modulation classification error, i.e., α_B . In order to obtain a tradeoff via the proposed technique,

a fixed percentage of correct modulation format decisions made by the conventional technique is allowed to be incorrect in the following simulations. In particular, we set α as $\alpha = 1 - 0.85(1 - \alpha_B)$; hence, the constraint in (P1) becomes $(1 - \mathcal{H}_a(\delta)) \geq 0.85(1 - \alpha_B)$. Under this scenario, Figure 6.3 plots symbol decoding performance in the case of correct classification versus SNR and Figure 6.4 plots probability of correct modulation classification versus SNR for each technique and for $K = 1, 3, 10$. From Figure 6.3, it can be observed that at low SNR values both techniques deliver similar symbol error rate, and thus relaxation is not very effective. On the other hand, in the high SNR regime, relaxation in the constraint becomes more effective, especially for small K values. Furthermore, an interesting observation is that symbol decoding performance, measured by $\mathcal{J}_a$ in the case of correct classification, degrades for the conventional method as the number of samples K increases. This can be explained by the fact that the modulation classification performance of the conventional method improves as K increases, and consequently, noisy samples that are prone to erroneous symbol decoding are included in the computation of $\mathcal{J}_a$.

6.8.2 Performance in the Case of Unknown Priors for the Modulation Schemes

We next investigate the case in which prior probabilities of the modulation schemes are unknown via solving (P2). First, the same SNR condition (i.e., $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = 18 \text{ dB}$) is examined. In Figure 6.5, we plot the worst case symbol decoding performance in the case of correct classification with respect to β where $\beta_1 = \beta_2 = \beta_3 = \beta$ for the proposed technique for $K = 1, 3, 10$, together with the performance of the conventional technique. It is observed that by increasing β , significant improvements on the symbol decoding performance can be obtained for $K = 1$. For $K = 3$ and $K = 10$ on the contrary, the improvements on the symbol decoding performance with respect to β become less significant compared to $K = 1$ case. Next, SNR values are set to $\text{SNR}_1 = 15 \text{ dB}$, $\text{SNR}_2 = 18 \text{ dB}$, and $\text{SNR}_3 = 21 \text{ dB}$. In Figure 6.6, we plot the worst case symbol error probability when the modulation type is correctly identified with respect to

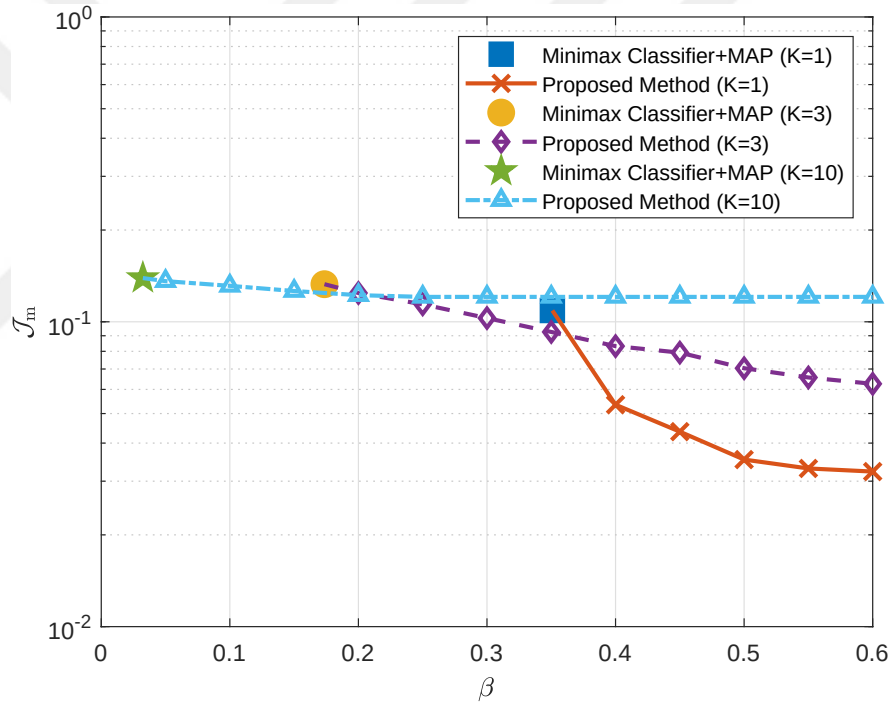


Figure 6.5: Comparison of the conventional and the proposed approaches in the case of unknown modulation scheme priors where $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = 18\text{dB}$.

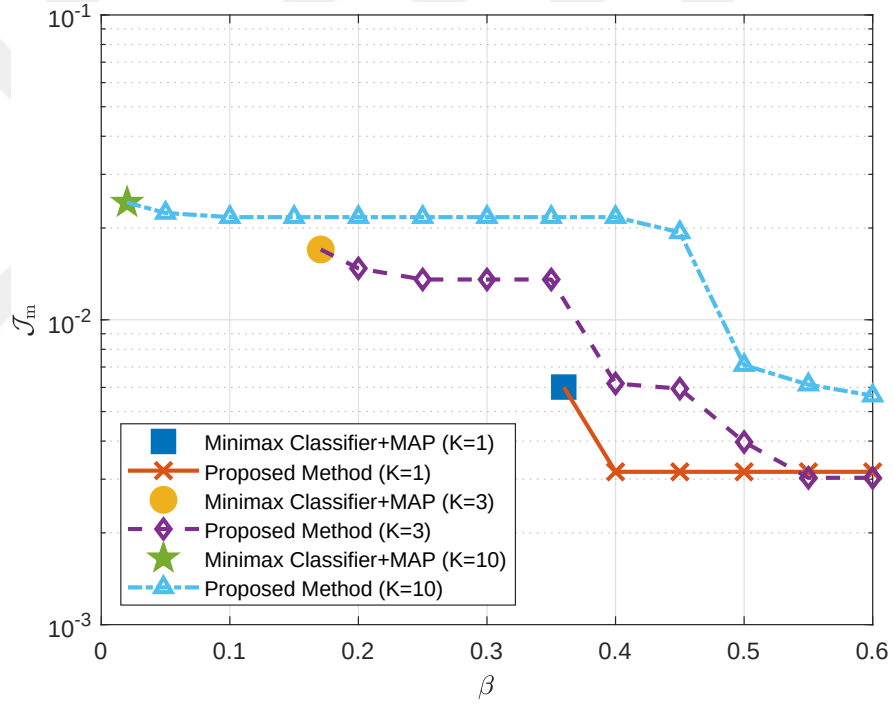


Figure 6.6: Comparison of the conventional and the proposed approaches in the case of unknown modulation scheme priors where $\text{SNR}_1 = 15\text{dB}$, $\text{SNR}_2 = 18\text{dB}$ and $\text{SNR}_3 = 21\text{dB}$.

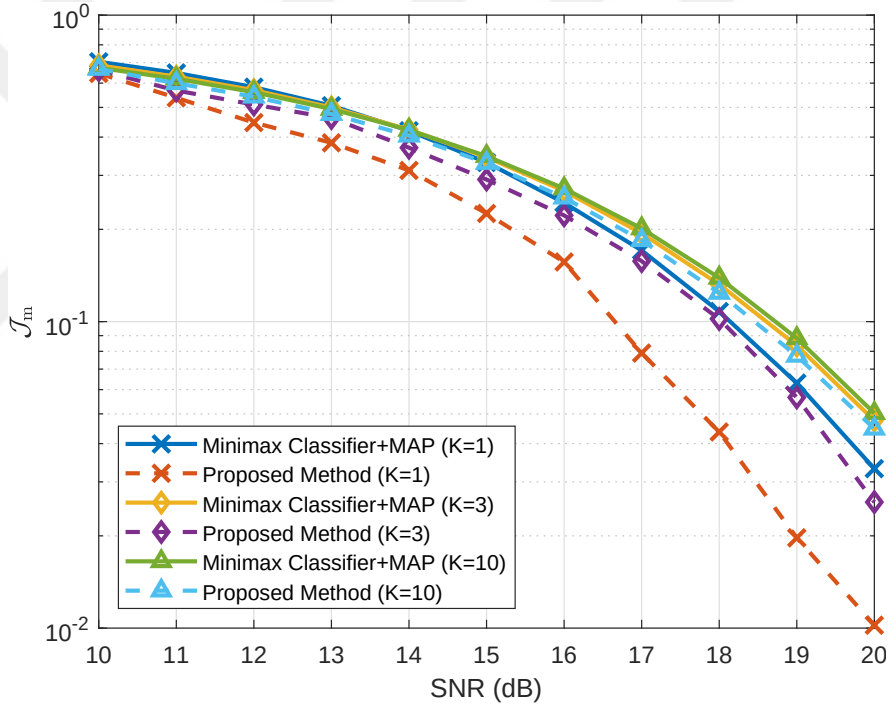


Figure 6.7: Symbol decoding performance versus SNR for the conventional and the proposed approaches in the case of unknown modulation scheme priors for $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = \text{SNR}$ and $\beta_1 = \beta_2 = \beta_3 = 1 - 0.85(1 - \beta_{\text{mm}})$.

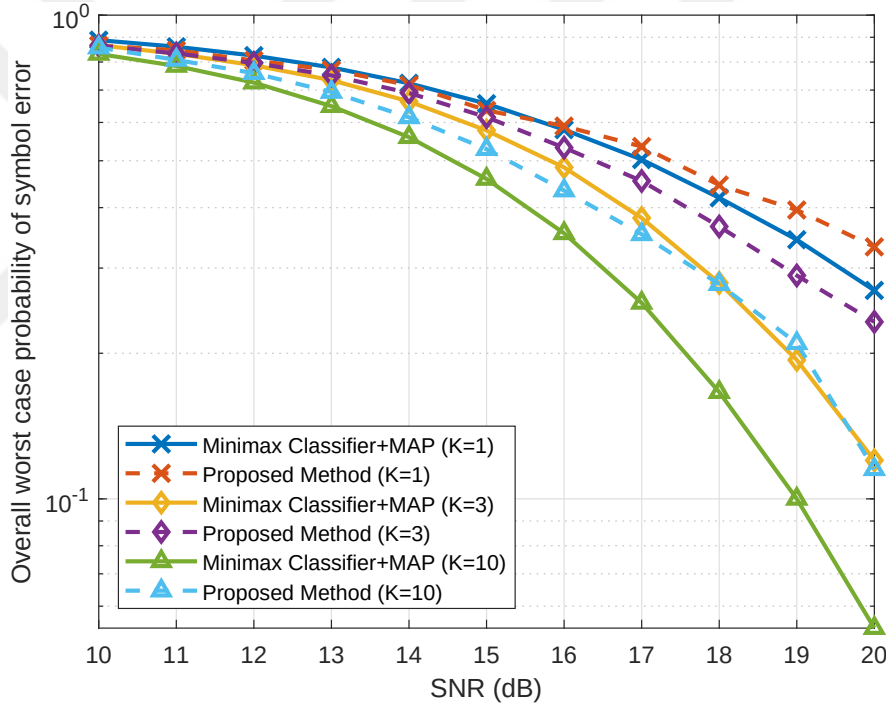


Figure 6.8: Overall symbol decoding performance versus SNR for the conventional and the proposed approaches in the case of unknown modulation scheme priors for $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}_3 = \text{SNR}$ and $\beta_1 = \beta_2 = \beta_3 = 1 - 0.85(1 - \beta_{\text{mm}})$.

β . It is seen that for $K = 1$, a slight relaxation in β improves the symbol decoding performance whereas further relaxation does not lead to any improvement. On the other hand, for $K = 3, 10$, multiple trends in the behavior of symbol decoding performance with respect to β are observed. This can intuitively be explained as follows: In this case, SNR levels are such that each modulation scheme has comparable symbol decoding performance. This makes it likely that the modulation format index that achieves the worst case symbol decoding performance, i.e., $\mathcal{J}_m$, changes while relaxing the constraints, which consequently yields multiple trends in the plot of $\mathcal{J}_m$ with respect to β .

In Figure 6.7, the worst case symbol decoding performance in the case of correct modulation classification for the conventional and the proposed techniques is illustrated by varying the SNR value under which each modulation format operates. For the proposed technique, all constraints are relaxed compared to the conventional technique as follows: $\beta_1 = \beta_2 = \beta_3 = 1 - 0.85(1 - \beta_{\text{mm}})$ where β_{mm} denotes the smallest β delivered by the minimax modulation classification rule. It is seen that for different K values, the conventional technique delivers close to worst case symbol error rate in the case of correct classification. On the other hand, an improvement on symbol decoding performance can be obtained with the proposed technique especially for small K and at high SNR values. Moreover, as SNR is increased, all individual probability of modulation classification error terms improve for the conventional technique since the conventional technique is optimized for modulation classification performance only. On the other hand, for the proposed technique, there is no such monotone behavior as the objective is to optimize for the symbol detection performance.

In Figure 6.8, we plot the overall worst case symbol decoding performance, in which a modulation classification error is regarded as a symbol error, versus SNR for $\beta_1 = \beta_2 = \beta_3 = 1 - 0.85(1 - \beta_{\text{mm}})$. This overall performance metric, which is different from $\mathcal{J}_m$ in (P2), is computed as follows: We first calculate the conditional symbol error probabilities given that the ℓ th modulation scheme is employed and then take the maximum over ℓ to account for the worst case scenario. It should be emphasized that as the proposed technique optimizes the symbol error rate when the modulation format is correctly identified, we do not

expect the proposed method to deliver the best performance in Figure 6.8. In fact, as the proposed technique performs a tradeoff between the modulation classification performance and the symbol decoding performance in the case of correct classification, its overall symbol error performance deteriorates as a result of the degraded classification performance. However, the proposed technique reduces the symbol error rate in the case of correct classification measured by $\mathcal{J}_m$ as depicted in Figure 6.7. In order to explain this phenomenon further, consider a particular scenario in which the SNR is 18 dB and $K = 3$. For the conventional technique, the conditional symbol error rates are 8.10×10^{-5} , 1.09×10^{-2} and 1.32×10^{-1} given that 16-QAM, 32-QAM and 64-QAM, respectively, are identified correctly. On the other hand, the corresponding symbol error rates for the proposed technique become 3.25×10^{-4} , 7.12×10^{-3} and 1.02×10^{-1} considering 16-QAM, 32-QAM and 64-QAM, respectively. Therefore, the improvement under QAM-64 yields reduced $\mathcal{J}_m$ with the proposed technique. However, the classification performance of the proposed technique is degraded, which consequently yields worse overall symbol decoding performance compared to the conventional technique.

6.8.3 Performance Considering the Cost Function in the Extension

In this part, performance of the conventional and the proposed techniques is presented when the cost function to measure symbol error rate is given by (6.26). More specifically, Figure 6.9 and Figure 6.10 consider the formulation in Section 6.5.1 and Section 6.5.2, respectively, when the objective functions are as in (6.26) instead of (6.8). As expected, block symbol error performance is poorer than average symbol error performance considered in the previous examples. From Figure 6.9 and Figure 6.10, it is verified that the proposed techniques corresponding to this cost function yield improved symbol decoding performance in the case of correct classification by relaxing the constraint(s) on average (or individual) probability of classification error(s) compared to the conventional techniques.

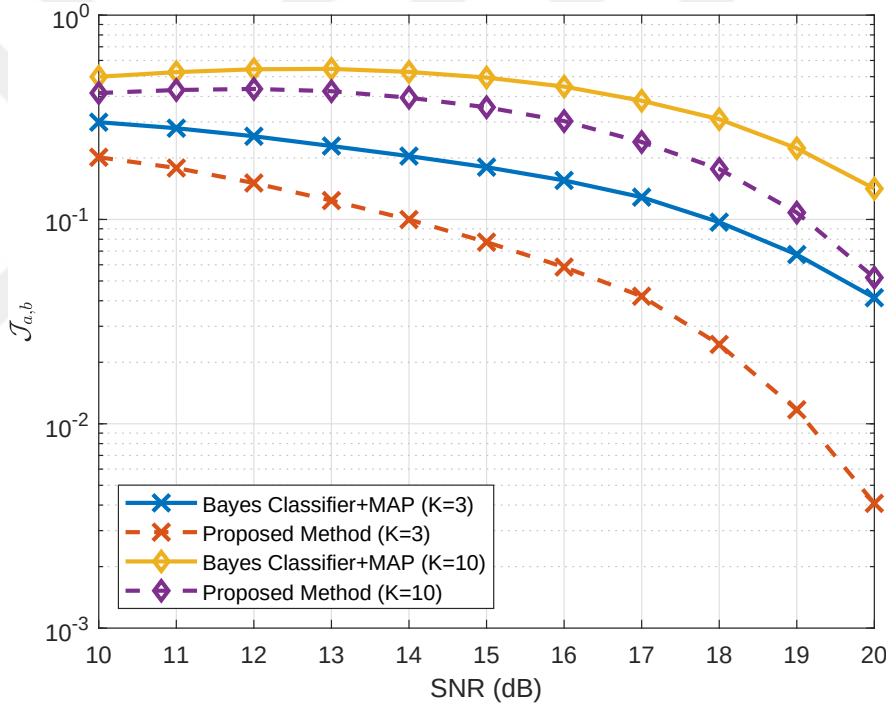


Figure 6.9: Performance of the conventional and the proposed approaches considering the cost function in (6.26) versus SNR in the case of known priors for the modulation schemes for $\alpha = 1 - 0.85(1 - \alpha_B)$.

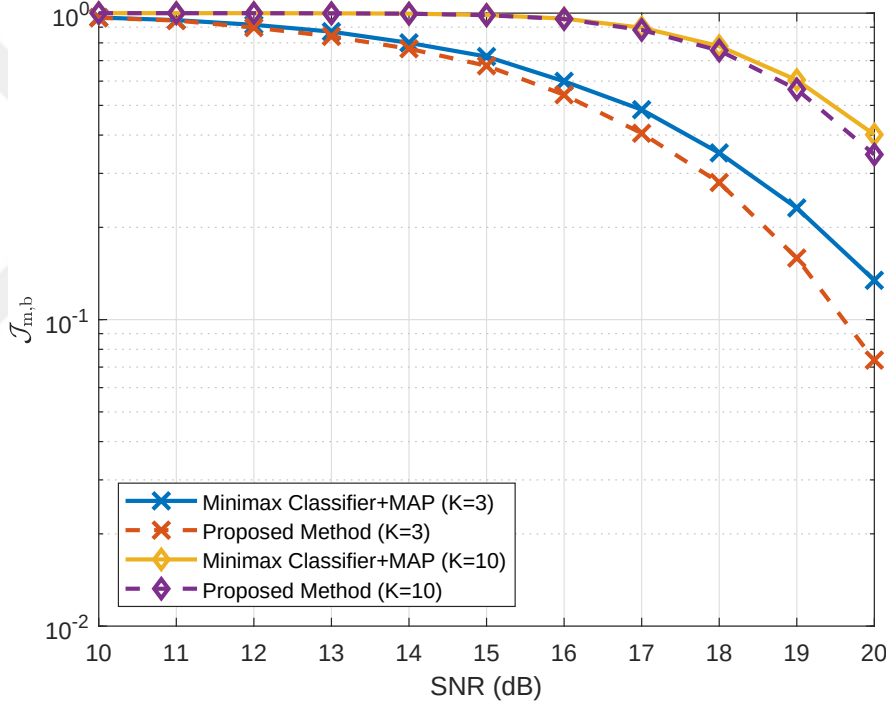


Figure 6.10: Performance of the conventional and the proposed approaches considering the cost function in (6.26) versus SNR in the case of unknown priors for the modulation schemes for $\beta_1 = \beta_2 = \beta_3 = 1 - 0.85(1 - \beta_{\text{mm}})$.

6.9 Conclusion

In this chapter, jointly optimal approaches for solving modulation classification and symbol decoding problems have been proposed. In particular, the scenarios with both known and unknown prior probabilities of modulation schemes have been addressed to provide a generic framework. In both cases, the introduced symbol decoding performance measure has been optimized under the constraint(s) on the modulation classification performance. In this way, a tradeoff between modulation classification and symbol decoding performance has been achieved in a jointly optimal manner.

Chapter 7

Conclusion and Future Work

In this dissertation, we have investigated signaling game and cheap talk problems involving communications between an encoder and a decoder. In these problems, given their policies, induced expected costs of the encoder and the decoder are misaligned due to cost criterion mismatch, privacy concerns of the encoder or prior mismatch. In such problems, it is required to analyze the problem using a suitable equilibrium concept. We have considered the Nash equilibrium or the Stackelberg equilibrium depending on the commitment assumption of the encoder to its policies. In addition, we have analyzed a communication theoretic problem of designing optimal decoder structures in the absence of modulation format information. This can be viewed as a signaling problem from an encoder to a decoder in a team theoretic framework where the encoder is fixed and the aim is to design the decoder.

In Chapter 2, we have investigated Crawford and Sobel's cheap talk formulation for scalar sources [12]. Rather than considering sources with a density supported on a bounded interval as investigated in [12], we have considered general sources that admit a log-concave density. We have investigated upper bounds on the number of bins in equilibrium as well as uniqueness of an equilibrium for a given number of bins. In particular, we have shown that for any finite N , there exists a unique equilibrium with N bins for sources with two-sided unbounded

support. A similar result has also been shown for sources with semi-unbounded support with the exception that depending on the sign of bias, there may be an upper bound on the number of bins in equilibrium. Furthermore, we have shown that an equilibrium with more bins leads to a more informative equilibrium, i.e., the expected costs of the encoder and the decoder in equilibrium improve as the number of bins increases. Moreover, we have investigated convergence behavior of best response iterations where the encoder and the decoder iteratively compute their best responses to a given strategy by the other player. We have shown that best response iterations starting from a given number of bins converge to the unique equilibrium with the corresponding number of bins for sources with strictly log-concave density and two-sided unbounded support. This result generalizes the classical convergence results in optimal quantization via Lloyd's Method I to a game theoretic setup.

In Chapter 3, we have investigated communication scenarios between a privacy concerned encoder and a decoder in a game theoretic framework where the objective functions involve mean squared errors for estimating a private parameter and a nonprivate parameter. We have modeled the private and nonprivate parameters as jointly Gaussian random vectors. We have shown that a payoff dominant Nash equilibrium is attained by linear policies. We have proven that these linear policies at a payoff dominant Nash equilibrium lead to a Stackelberg equilibrium as well. These results have been further generalized to the Gaussian noisy channel setting as well as a discrete noiseless channel setting for the special case of scalar sources. We have also formulated an information bottleneck problem for Gaussian random vectors with mean squared error cost criterion under the Stackelberg equilibrium concept. We have shown that the Stackelberg equilibria are attained by a set of characterized linear policies.

In Chapter 4, we have analyzed a signaling game problem under Stackelberg equilibria where an encoder and a decoder have subjective/inconsistent priors regarding the source distribution. We have proven that the Stackelberg equilibrium cost of the encoder is upper semi continuous around the point of identical priors as the encoder's prior approaches the decoder's prior. In the particular case of Gaussian priors, it is also lower semi continuous, which shows the robustness of

the Stackelberg equilibria around the team setup. Furthermore, we have proven that the Stackelberg equilibrium solution is in general nonlinear under Gaussian subjective priors. This shows that team theoretic optimality of affine/linear policies is not robust to perturbations. In other words, this is in contrast to the linearity property of team optimal solution where the Gaussian priors are identical from the encoder and decoder's perspectives. Finally, we have considered the Stackelberg equilibrium solution for Gaussian subjective priors under affine policy restriction. We have shown that depending on certain conditions stated explicitly, the affine Stackelberg equilibrium solution can be informative or noninformative.

In Chapter 5, we have investigated a multidimensional extension of Crawford and Sobel's cheap talk formulation [12]. Namely, rather than considering scalar sources as in [12], we have considered multidimensional sources. First, we have derived general necessary conditions that decoder actions in equilibrium need to satisfy considering any joint source distribution for the multidimensional observations. In particular, we have shown that decoder actions in equilibrium need to satisfy a geometry condition that determines how close two decoder actions can get in a certain direction. In addition, we have provided necessary conditions that an equilibrium with a continuum of decoder actions must satisfy. Then, by using these preliminary results, we have derived necessary and sufficient conditions under which a linear equilibrium exists considering i.i.d. sources. In particular, we have proven that if the bias vector has components which are not equal in magnitude, then there exists a linear equilibrium if and only if the source distribution is Gaussian. In the remaining cases, the existence of a linear equilibrium depends on the signs of bias components and whether the source distribution is symmetric or not. Finally, we have formulated a rate-distortion theoretic formulation of cheap talk which quantifies the tradeoff between the distortion and the rate of communication. We have obtained achievable rate and distortion pairs for the i.i.d. Gaussian setup.

In Chapter 6, we have considered a team theoretic problem in communication theory where the aim is to design optimal decoder structures. In particular, we have proposed jointly optimal approaches for solving modulation classification

and symbol decoding problems. If we view this setting as a communication between an encoder and a decoder, then the aim is to design the decoder while we do not have control over the encoder. In this setting, the decoder observes a sequence of signals whose modulation format is unknown while possible modulation formats are known. Given the sequence of received signals, the aim of the decoder is to classify the modulation format and to decode the transmitted symbols. We have addressed scenarios with both known and unknown prior probabilities of modulation schemes to provide a generic framework. The proposed framework enables optimal tradeoff between modulation classification and symbol decoding performance.

As future work, it is interesting to generalize the results presented in Chapter 2 considering the scalar cheap talk problem to the multidimensional cheap talk problem. As indicated in Chapter 5, we know that there may exist equilibria with a linear encoder for the multidimensional cheap talk problem with i.i.d. source components depending on certain explicit conditions. In the cases that there does not exist any equilibrium with a linear encoder, it is interesting to analyze the number of bins in equilibrium for the multidimensional cheap talk problem. Another interesting extension is to characterize payoff dominant equilibria especially when the equilibrium cannot involve a linear encoder. Another possible future research direction is to investigate whether best response iterations converge to an equilibrium in the multidimensional case.

Another future work is to analyze privacy metrics such as differential privacy [76, 77] and maximal leakage [84] and provide a game theoretic interpretation for these metrics. For this research direction, one may also consider joint discrete distributions instead of the Gaussian distribution for probabilistic models of the private and nonprivate parameters. It is also interesting to investigate probabilistic model mismatches that occur in a variety of settings [132–139] from a game theoretic perspective.

Regarding the results in Chapter 6, one interesting extension is to address the uncertainty in the prior probabilities of the modulation schemes by employing the restricted Bayesian framework [171, 181]. In addition, parameters such as fading

coefficient and time/frequency offset can be incorporated into the signal model to investigate a more general scenario in which these parameters are treated as unknown deterministic or random quantities. As another interesting future work, it is possible to study joint modulation classification and symbol detection problem considering multiple-input multiple-output (MIMO) systems.



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