BÄCKLUND TRANSFORMATIONS OF PAINLEVÉ EQUATIONS AND DISCRETE EQUATIONS OF PAINLEVÉ TYPE

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ABSTRACT

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With the help of the Schlesinger transformations, we obtain the Bäcklund transformations of the classical continuous Painlevé equations (PII-PVI). Then using these Bäcklund transformations we derived the corresponding discrete equations. The main idea in obtaining these equations is to eliminate y' from the Bäcklund transformations, T_i and T_j, in such a way that $T_i \circ T_j = T_j \circ T_i = I$. Then we obtain an algebraic relation between y_i, y and y_j. This algebraic relation can be considered as as a discrete equation of the Painlevé type.

Keywords: Painlevé Equation, Bäcklund Transformation, Schlesinger Transformation, Discrete Painlevé Equations.

ÖZET

PAINLEVÉ DENKLEMLERININ BÄCKLUND DÖNÜŞÜMLERI VE AYRıK PAINLEVÉ DENKLEMLERI

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Schlesinger dönüşümlerini kullanarak sürekli Painlevé denklemlerinin Bäcklund dönüşümlerini bulduk ve bu Bäcklund dönüşümlerini kullanarak ayrık denklemleri elde ettik. Birbirinin tersi olan iki Bäcklund dönüşümünün içindeki y' 'lı terimleri elimine ederek y^{+}, y ve y^{-} arasında bir ilişki bulduk. Bu ilişki Painlevé tipi denklemlerin ayrık denklemleri olarak düşünülür.

Anahtar sözcükler: Painlevé Denklemleri, Bäcklund Dönüşümleri, Schlesinger Dönüşümleri, Ayrık Painlevé Denklemleri.

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Contents

Chapter 1

Introduction

The Painlevé equations were discovered by Painlevé [1], Gambier [2] and their colleagues, while studying a problem posed by Picard [3]. Problem was about the second-order ordinary differential equations of the form

$$y'' = F(t, y, y') \ , \qquad ' \equiv d/dt \tag{0.1}$$

where F is rational in y', algebraic in y and analytic in t, with the Painlevé property, i.e. the singularities other than poles of the solutions are independent of the integrating constant and so are dependent only on the equation. The differential equations, possessing Painlevé property are called equations of Painlevé type. Painlevé and his school showed that, within a Möbius transformation, there are fifty canonical equations [4] of the form (0.1). Among the fifty equations, the following six were well known:

$$PI \quad : \quad y'' = 6y' + t$$

$$PII \quad : \quad y'' = 2y^3 + ty + \alpha$$

$$PIII \ : \quad y'' = \frac{(y')^2}{y} - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$PIV \quad : \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4ty^2 + 2y(t^2 - \alpha) + \frac{\beta}{y}$$

$$PV \quad : \quad y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{1}{t}y' + \frac{1}{t^2}(y-1)^2(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}$$

$$PVI \quad : \quad y'' = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t})(y')^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}[\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}]$$

Remaining forty-four are integrable in terms of known elementary functions and reducible to one of these six equation. These six equations are known as Painlevé equations and denoted by PI-PVI.

Although the Painlevé equations were discovered from mathematical considerations, they occur in many physical situations; plasma physics, statistical mechanics, nonlinear waves, quantum gravity, general relativity, quantum field theory, nonlinear optics and fibre optics. Painlevé equations have attracted much interest as reduction of the soliton equations which are solvable by inverse scattering transformation (see [5, 6, 7] and references therein).

1.1 Properties of Painlevé equations

The general solution of Painlevé equations are called Painlevé transcendent. However, for certain values of parameters, PII-PVI posses rational solutions and solutions expressible in terms of special functions: Airy [2, 8], Bessel [9], Weber-Hermite [10], Whittaker [11] and Hypergeometric [12] respectively.

PII-PVI admit Bäcklund transformations which relate one solution to another solution of the same equation but with different values of parameters [13, 14, 15].

Painlevé equations can be written as Hamiltonian systems [16, 17].

Painlevé equations appear on the compatibility conditions of linear system of equation, Lax-pairs, possessing irregular singular points. By using Lax-pairs, one can find the general solution of a given Painlevé equation as the Fredholm integral equation.

1.2 Discrete Painlevé Equations

The discrete Painlevé equations (dPI-dPVI) have the form

$$x_{n+1} = \frac{f_1(x_n, n) + x_{n-1} f_2(x_n, n)}{f_3(x_n, n) + x_{n-1} f_4(x_n, n)} \tag{2.2}$$

where $f_i(x_n, n)$ are polynomials of degree at most four in x_n. The discrete Painlevé equations appear in a variety of physical applications and indeed dPI and dPII were discovered in physical situations [18, 19, 20]. For historical reasons the basic forms of the discrete Painlevé equations are:

$$dPI \quad : \quad x_{n+1} + x_n + x_{n-1} = \frac{an + b}{x_n} + c$$

$$dPII \quad : \quad x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c}{1 - x_n^2}$$

$$dPIII \quad : \quad x_{n+1} x_{n-1} = \frac{cd(x_n - aq^{2n})(x_n - bq^{2n})}{(x_n - c)(x_n - d)}$$

$$dPIV \quad : \quad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n + a + b)(x_n - a - b)(x_n + a - b)(x_n - a + b)}{(x_n + cn + d + e)(x_n + cn + d - e)}$$

$$dPV \quad : \quad (x_{n+1} x_n - 1)(x_n x_{n-1} - 1) = \frac{cd(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(x_n - c)(x_n - d)}$$

$$dPVI \quad : \quad \frac{(x_n x_{n+1} - z_n z_{n+1})(x_n x_{n-1} - z_n z_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - a z_n)(x_n - z_n/a)(x_n - b z_n)(x_n - z_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}$$

where $z_n = z_0 \lambda^n$ and a,b,c,d constants.

There are many similarities between the properties of the discrete Painlevé equations and their continuous counterparts. For example both posses Lax pairs, both have solutions related though Bäcklund and Miura transformations, both have particular solutions expressible in terms of special functions or rational solutions for certain parameter values, both can be written into bilinear forms. The fundamental difference between the discrete Painlevé equations and the continuous Painlevé equations is the canonical form. Although the continuous Painlevé equations have a unique canonical form, the discrete Painlevé equations do not.

Discrete Painlevé equations should yield their respective continuous Painlevé equation in the continuous limit. Clarkson and Webster showed that dPIII yields PIII in the continuous limit [27]. Consider dPIII

$$x_{n+1}x_{n-1} = \frac{cd(x_n - aq^{2n})(x_n - bq^{2n})}{(x_n - c)(x_n - d)}. \tag{2.3}$$

By setting

$$a = \frac{1}{4}\sqrt{-\delta}\varepsilon + \Big(\frac{\mu}{8} - \frac{\beta}{16}\Big)\varepsilon^2, \qquad c = \varepsilon^{-1} + \Big(-\frac{\alpha}{4} - \frac{\mu}{2}\Big),$$
$$b = -\frac{1}{4}\sqrt{-\delta}\varepsilon + \Big(-\frac{\mu}{8} - \frac{\beta}{16}\Big)\varepsilon^2, \qquad d = -\varepsilon^{-1} + \Big(-\frac{\alpha}{4} + \frac{\mu}{2}\Big),$$

in the continuous limit and taking $x_n = y,\ n\varepsilon = t,\ q^2 = 1 + \varepsilon$ and expand x_{n+1} and x_{n-1} using Taylor series to give

$$x_{n\pm1} = y \pm \varepsilon\frac{dy}{dt} + \frac{1}{2}\varepsilon^2\frac{d^2y}{dt^2} + O(\varepsilon^3). \tag{2.4}$$

In the limit as $\varepsilon \to 0$, (2.3) yield the following differential equation

$$yy'' = (y')^2 + \frac{1}{2}\alpha y^3 + \frac{1}{8}\beta e^t y + y^4 + \frac{1}{16}e^{2t}. \tag{2.5}$$

Then applying the transformation $T = e^{t/2}$ and $Y = 2y/T$ into (2.4) gives PIII with $\gamma = 1$.

1.3 Derivation of discrete Painlevé equations

There are several methods to derive the discrete Painlevé equations.

i) Singularity confinement method, dPIII-dPVI have been derived using this method [21, 22].

ii) Similarity reduction of lattices, e.g. Nijhoff and Papageorgiou derived dPII as a similarity reduction of the discrete mKdV equation.

iii) Derivation of linear problem associated to a discrete Painlevé equation, i.e. Lax pair method that is related to orthogonal polynomial method, discrete AKNS method, etc.

iv) From Bäcklund transformations of the continuous Painlevé equations [23, 24, 25, 26].

In this thesis we derived the discrete equations of the Painlevé type from the Bäcklund transformations of the continuous Painlevé equations. Suppose that there are two Bäcklund transformations for a given Painlevé equation in the form

$$T^{+}(y(t;\alpha)) = y^{+}(t;\alpha^{+}) = F^{+}(y(t;\alpha), y'(t;\alpha), t) \tag{3.6}$$

$$T^{-}(y(t;\alpha)) = y^{-}(t;\alpha^{-}) = F^{-}(y(t;\alpha), y'(t;\alpha), t) \tag{3.7}$$

where $y(t;\alpha)$ is a solution corresponding to the parameter set $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ and $y^{\pm}(t;\alpha^{\pm})$ are solutions corresponding to the parameter set $\alpha^{\pm} = (\alpha_1^{\pm}, \alpha_2^{\pm}, ..., \alpha_k^{\pm})$.

Eliminating $y'(t;\alpha)$ between (3.6) and (3.7) yields an algebraic relation, which is recurrence relation, between $y^{+}(t;\alpha^{+})$, $y(t;\alpha)$ and $y^{-}(t;\alpha^{-})$. This algebraic relation can be considered as a nonlinear superposition principle for solutions of the Painlevé equations or as a discrete equation of the Painlevé type.

Each one of T^{+} and T^{-} is the inverse of the other. The solutions and parameters are linked as follows:

$$y^{+}(t;\alpha^{+}) \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} y(t;\alpha) \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} y^{-}(t;\alpha^{-}), \tag{3.8}$$

$$\alpha^{+} \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} \alpha \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} \alpha^{-}. \tag{3.9}$$

By setting $y(t;\alpha) = x_n$ and $y^{\pm}(t;\alpha^{\pm}) = x_{n\pm1}$ with $\alpha = a_n$ and $\alpha^{\pm} = a_{n\pm1}$, we obtain

$$x_{n+1} \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} x_n \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} x_{n-1}, \tag{3.10}$$

$$\alpha_{n+1} \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} \alpha_n \underset{T^{+}}{\overset{T^{-}}{\rightleftarrows}} \alpha_{n-1}. \tag{3.11}$$

Chapter 2

The Second Painlevé Equation

The second Painlevé equation (PII),

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \tag{0.1}$$

where α is a complex parameter, can be obtained as the compatibility condition of the following linear system of equations

$$Y_z(z,t) = A(z,t)Y(z,t), \qquad Y_t(z,t) = B(z,t)Y(z,t), \tag{0.2}$$

where

$$A(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2 + \begin{pmatrix} 0 & u \\ -\frac{2v}{u} & 0 \end{pmatrix} z + \begin{pmatrix} v + \frac{t}{2} & -uy \\ -\frac{2}{u}(\theta + yv) & -(v + \frac{t}{2}) \end{pmatrix},$$

$$B(z,t) = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \tfrac{1}{2} \begin{pmatrix} 0 & u \\ -2\frac{v}{u} & 0 \end{pmatrix}, \tag{0.3}$$

u,v and y are functions of t and θ is a constant. The compatibility condition $Y_{zt} = Y_{tz}$ yields

$$\frac{dv}{dt} = -2yv - \theta \ , \quad \frac{du}{dt} = -uy \ , \quad \frac{dy}{dt} = v + y^2 + \frac{t}{2} \ . \tag{0.4}$$

If one differentiate (0.4.c) with respect to t and using (0.4.a), y satisfies PII with parameter $\alpha = \frac{1}{2} - \theta$.

(0.2.a) has an irregular singular point at $z = \infty$.

2.1 Solution About $z = \infty$

The two linearly independent formal solutions $\tilde{Y}_\infty(z) = (\tilde{Y}_\infty^{(1)}(z), \tilde{Y}_\infty^{(2)}(z))$ of (0.2) have the expansions [30]

$$\tilde{Y}_\infty^{(1)}(z) = \left(\tfrac{1}{z}\right)^{\theta} e^{q(z)} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -K \\ \frac{v}{u} \end{pmatrix} \tfrac{1}{z} + ... \right\} = \left(\tfrac{1}{z}\right)^{\theta} e^{q(z)}, \hat{Y}_\infty^{(1)}(z)$$

$$\tilde{Y}_\infty^{(2)}(z) = \left(\tfrac{1}{z}\right)^{-\theta} e^{-q(z)} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{u}{2} \\ K \end{pmatrix} \tfrac{1}{z} + ... \right\} = \left(\tfrac{1}{z}\right)^{-\theta} e^{-q(z)} \hat{Y}_\infty^{(2)}(z), \tag{1.5}$$

where

$$K = \frac{1}{2}v^2 + (y + \frac{t}{2})v + \theta y, \qquad q(z) = \frac{z^3}{3} + \frac{t}{2}z.$$

Let $Y_j(z),\ j = 1, ..., 6$ be solutions of (0.2.a) analytic in the certain sector S_j such that $\det Y_j(z) = 1$ and $Y_j(z) \sim \tilde{Y}_\infty(z)$ as $|z| \to \infty$ in the sector S_j , and the sectors S_j are given by

$$S_1 : -\frac{\pi}{6} \le \arg z < \frac{\pi}{6}, \qquad S_2 : \frac{\pi}{6} \le \arg z < \frac{\pi}{2}, \qquad S_3 : \frac{\pi}{2} \le \arg z < \frac{5\pi}{6},$$

$$S_4 : \frac{5\pi}{6} \le \arg z < \frac{7\pi}{6}, \qquad S_5 : \frac{7\pi}{6} \le \arg z < \frac{3\pi}{2}, \qquad S_6 : \frac{3\pi}{2} \le \arg z < \frac{11\pi}{2}. \tag{1.6}$$

The solutions $Y_j(z)$ are related by the Stokes matrices G_j

$$Y_{j+1}(z) = Y_j(z) G_j, \quad j = 1, ..., 5, \qquad Y_1(z) = Y_6(ze^{2i\pi}) G_6 e^{2i\pi\theta\sigma_3}, \tag{1.7}$$

where

$$G_1 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

$$G_4 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.8}$$

and $a, b, c, d, e, f,$ are constants with respect to z . Monodromy data, $MD = \{a, b, c, d, e, f\}$ satisfy the consistency condition

$$\prod_{j=1}^{6} G_j e^{2i\pi\theta\sigma_3} = I. \tag{1.9}$$

2.2 Bäcklund Transformations

Consider the transformation given by the Schlesinger transformation matrix $R(z)$

$$Y'(z) = R(z)Y(z), \tag{2.10}$$

which leaves the monodromy data of the solution matrix $Y(z)$ the same but shifts the exponent θ as $\theta \to \theta' = \theta + \kappa$. In other words the transformation leaves the consistency condition of the monodromy data (1.9) the same. Eq.(1.9) is invariant under the transformation if θ is shifted by an integer.

Let $R(z) = R_j(z)$ when z in S_j, then the definition of the Stokes matrices (1.7) imply that the transformation matrix $R(z)$ satisfies the the following Riemann-Hilbert problem:

$$\begin{aligned} &R_{j+1}(z) = R_j(z) \qquad \text{on } C_{j+1}, \qquad j = 1, ..., 5, \\ &R_1(z) = R_6(ze^{2i\pi}) \qquad \text{on } C_1, \end{aligned} \tag{2.11}$$

with the boundary condition,

$$R_j(z) \sim \hat{Y}'_\infty(z) \left(\frac{1}{z}\right)^{n\sigma_3} \hat{Y}_\infty^{-1}(z), \quad \text{as} \quad z \to \infty, \quad z \text{ in } S_j. \tag{2.12}$$

(2.11) implies that the transformation matrix $R(z)$ is analytic everywhere in complex z-plane and can be determined explicitly by using the boundary conditions (2.12). It is enough to consider the following two cases [30] :

$$\begin{aligned} &\theta' = \theta + 1, \qquad R_{(1)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0 & -\frac{u}{v} \\ \frac{v}{u} & -\frac{\theta}{v} - y \end{pmatrix}, \\ &\theta' = \theta - 1, \qquad R_{(2)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} y & \frac{u}{2} \\ -\frac{2}{u} & 0 \end{pmatrix}. \end{aligned} \tag{2.13}$$

Successive application of the transformation matrices $R_{(i)}(z,t), \quad i=1,2$ map to θ to $\theta'=\theta+n, \quad n\in Z$. If, $y',u',v',\theta'=\theta+1$ are the transformed quantities of y,u,v,θ under the transformation given by $R_{(1)}(z,t)$, i.e.

$$Y'(z;t,y',u',v',\theta')=R_{(1)}(z;t,y,u,v,\theta)Y(z;t,y,u,v,\theta), \tag{2.14}$$

and if $y'',u'',v'',\theta''=\theta'-1$ are the transformed quantities of y',u',v',θ' under the transformation given by $R_{(2)}(z,t)$, i.e.

$$Y''(z;t,y'',u'',v'',\theta'')=R_{(2)}(z;t,y',u',v',\theta')Y(z;t,y',u',v',\theta'), \tag{2.15}$$

then,

$$R_{(2)}(z;t,y'(y,u,v,\theta),...)R_{(1)}(z;t,y,u,v,\theta)=I. \tag{2.16}$$

The linear equation (0.2.a) is transformed under the Schlesinger transformation as follows:

$$Y'_z(z,t)=A'(z,t)Y'(z,t), \qquad A'(z,t)=[R(z,t)A(z,t)+R_z(z,t)]R^{-1}(z,t). \tag{2.17}$$

Hence, y,u,v,θ are the transformed under the transformation matrix $R_{(1)}(z,t)$ as follows:

$$\begin{aligned} &\theta_1=\theta+1\ , \quad y_1=-y-\tfrac{\theta_1}{v}\ , \\ &u_1=\tfrac{2u}{v}\ , \qquad v_1=-v-2(y+\tfrac{\theta}{v})^2-t\ , \end{aligned} \tag{2.18}$$

which guarantees the integrability condition of the linearization (0.2). That is, if $y(t)$ solves the second Painlevé equation with parameter $\alpha=\frac{1}{2}-\theta$, then $y_1(t)$ solves the second Painlevé equation with parameter $\alpha_1=\alpha-1$. If we use (0.4) and (2.18) then the following Bäcklund transformation for the second Painlevé equation can be obtained [26],

$$y_1(t;\alpha_1)=-y-\frac{2\alpha-1}{2y^2-2y'+t}\ , \qquad \alpha_1=\alpha-1, \qquad y'=\frac{dy}{dt}. \tag{2.19}$$

y,u,v,θ are the transformed under the transformation matrix $R_{(2)}(z,t)$ as follows:

$$\begin{aligned} &\theta_2=\theta-1\ , \quad y'=-y+\tfrac{\theta_2}{2y^2+v+t}\ , \\ &u_2=\tfrac{u}{2}v'\ , \qquad v_2=-v-2y^2-t\ , \end{aligned} \tag{2.20}$$

and we obtain Bäcklund transformation for the second Painlevé equation can be obtained [30],

$$y_2(t;\alpha_2) = -y - \frac{2\alpha+1}{2y^2+2y'+t}\ , \qquad \alpha_2 = \alpha + 1\ . \tag{2.21}$$

2.3 Discrete Equation

Eliminating y' from (2.19) and (2.21) , and setting

$$x_{n+1} = y_1 \qquad x_n = y \qquad x_{n-1} = y_2\ , \tag{3.22}$$

and

$$a_{n+1} = \alpha_1 = \alpha - 1\ , \qquad a_n = \alpha\ , \qquad a_{n-1} = \alpha_2 = \alpha + 1 \tag{3.23}$$

we obtain the discrete equation [23],

$$\frac{2a_n - 1}{x_{n+1} + x_n} + \frac{2a_n + 1}{x_{n-1} + x_n} + 4y^2 + 2t = 0, \tag{3.24}$$

where $a_n = \kappa - n$ is the solution of the difference equation (3.23), with κ being an arbitrary constant. This equation is another discrete form of PI, known as *alternative-dPI*.

Chapter 3

The Third Painlevé Equation

The third Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \tag{0.1}$$

can be obtained as the compatibility condition of the following linear systems of equations

$$Y_z(z,t) = A(z,t)Y(z,t), \qquad Y_t(z,t) = B(z,t)Y(z,t), \tag{0.2}$$

where,

$$A(z,t) = \tfrac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -\theta_\infty/2 & u \\ v & \theta_\infty/2 \end{pmatrix}\tfrac{1}{z} + \begin{pmatrix} s-\frac{t}{2} & -ws \\ \frac{1}{w}(s-t) & -(s-\frac{t}{2}) \end{pmatrix}\tfrac{1}{z^2},$$

$$B(z,t) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}z + \tfrac{1}{t}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} - \tfrac{1}{t}\begin{pmatrix} s-\frac{t}{2} & -ws \\ \frac{1}{w}(s-t) & -(s-\frac{t}{2}) \end{pmatrix}\tfrac{1}{z}, \tag{0.3}$$

u,v,w and s are functions of t and θ_∞ is a constant. The compatibility condition $Y_{zt} = Y_{tz}$ implies that;

$$\begin{aligned} \frac{du}{dt} &= \frac{u\theta_\infty}{t} - 2sw\ , \\ \frac{dv}{dt} &= -\frac{v\theta_\infty}{t} - \frac{2(s-t)}{w}\ , \end{aligned}$$

$$\begin{aligned} \frac{ds}{dt} &= \frac{2su}{tw} - \frac{2u}{w} + \frac{s(2vw+1)}{t} \ , \\ \frac{dw}{dt} &= -\frac{2vw^2}{t} - \frac{\theta_\infty w}{t} + \frac{2u}{t} \ . \end{aligned} \tag{0.4}$$

If we let $y = -\frac{u}{sw}$, equations (0.4) yield,

$$ty' = 2y\theta_\infty + 2t + 4sy^2 - 2ty^2 - y \ . \tag{0.5}$$

Differentiating (0.5) with respect to t, give PIII with parameters $\alpha = 4\theta_0, \beta = 4(1-\theta_\infty), \gamma = 4, \delta = -4$ where θ_0 is given as,

$$\tfrac{\theta_0}{2} = -\tfrac{s-t}{wt}(u - \tfrac{\theta_\infty}{2}w) + \tfrac{s}{t}(wv + \tfrac{\theta_\infty}{2}). \tag{0.6}$$

3.1 Bäcklund Transformations

Let $R(z)$ be the transformation matrix which leaves the monodromy data of the solution matrix $Y(z)$ of (0.2) the same but changes the exponents θ_0 and θ_∞ .

$$Y'(z,t) = R(z,t)Y(z,t), \tag{1.7}$$

where $Y(z,t) \equiv Y(z,t;y,u,v,w,s,\theta_0,\theta_\infty)$, and $Y'(z,t) \equiv Y'(z,t;y',u',v',w',s',$ ${\theta_0}' = \theta_0 \pm 1, {\theta_\infty}' = \theta_\infty \pm 1)$. The explicit form of the Schlesinger transformation matrix $R(z,t)$ are [30],

$$\begin{cases} {\theta_0}' = \theta_0 - 1 \\ {\theta_\infty}' = \theta_\infty + 1, \end{cases} \qquad R_{(1)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & -\frac{ws}{s-t} \\ -\frac{v}{t} & \frac{v}{t}\frac{ws}{s-t} \end{pmatrix} z^{-1/2},$$

$$\begin{cases} {\theta_0}' = \theta_0 + 1 \\ {\theta_\infty}' = \theta_\infty + 1, \end{cases} \qquad R_{(2)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & -w \\ -\frac{v}{t} & \frac{wv}{t} \end{pmatrix} z^{-1/2},$$

$$\begin{cases} {\theta_0}' = \theta_0 - 1 \\ {\theta_\infty}' = \theta_\infty - 1, \end{cases} \qquad R_{(3)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} -\frac{u}{t}\frac{s-t}{ws} & \frac{u}{t} \\ -\frac{s-t}{ws} & 1 \end{pmatrix} z^{-1/2},$$

$$\begin{cases} {\theta_0}' = \theta_0 + 1 \\ {\theta_\infty}' = \theta_\infty - 1, \end{cases} \qquad R_{(4)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} -\frac{u}{tw} & \frac{u}{t} \\ -\frac{1}{w} & 1 \end{pmatrix} z^{-1/2}. \tag{1.8}$$

The transformation matrices $R_{(i)}(z,t)$, $i = 1, ..., 4$ generate all the possible transformation matrices. Since, if

$$Y_k(z,t;y_k,u_k,v_k,w_k,s_k,(\theta_\infty)_k,(\theta_0)_k) = R_{(k)}(z,t;y,...,\theta_0)Y(z,t;y,...,\theta_0), \quad (1.9)$$

and

$$Y_l(z,t;y_l,u_l,v_l,w_l,s_l,(\theta_\infty)_l,(\theta_0)_l) = R_{(l)}(z,t;y_k,...,(\theta_0)_k)Y_k(z,t;y_k,...,(\theta_0)_k), \quad (1.10)$$

then

$$R_{(k)}(z,t;y'(y,u,...\theta_0),...)R_{(l)}(z,t;y,...,\theta_0) = I, \quad (1.11)$$

for $k,l = 2,3$ and $k,l = 1,4$. The linear equation (0.2) is transformed under the Schlesinger transformation as follows:

$$Y'_z(z,t) = A'(z,t)Y'(z,t) \ , \qquad A'(z,t) = [R(z,t)A(z,t) + R_z(z,t)]R^{-1}(z,t) \ . \quad (1.12)$$

Transformation I: Quantities y,u,w,s,θ are the transformed under the transformation matrix $R_{(1)}(z,t)$ as;

$$\begin{aligned} u_1 &= \frac{stw}{s-t} \\ -w_1 s_1 &= u - \frac{sw}{s-t}\left(\theta_\infty + \frac{svw}{s-t}\right) \ , \end{aligned} \quad (1.13)$$

which guarantees the integrability condition of the linearization (0.2). That is, if $y(t)$ solves the third Painlevé equation with parameters $\alpha = 4\theta_0$, $\beta = 4(1-\theta_\infty)$, $\gamma = 4$ and $\delta = -4$ then $y_1(t)$ solves the third Painlevé equation with parameter $\alpha_1 = \alpha - 4$, $\beta_1 = \beta - 4$, $\gamma_1 = \gamma = 4$ and $\delta_1 = \delta = -4$. If we use (0.5) and (1.13) then the following Bäcklund transformation for the third Painlevé equation can be obtained,

$$T_1 : \qquad y_1(y,t;\alpha_1,\beta_1,4,-4) = \frac{2ty' - 4ty^2 + (\beta-2)y - 4t}{y[2ty' - 4ty^2 + (-\alpha+2)y - 4t]}. \quad (1.14)$$

$$\alpha_1 = \alpha - 4 \ , \qquad \beta_1 = \beta - 4 \ .$$

Transformation II: If we use transformed quantities under the transformation matrix $R_{(2)}(z,t)$

$$\begin{aligned} u_2 &= tw \\ -w_2 s_2 &= u - \theta_\infty w - vw^2 \ , \end{aligned} \quad (1.15)$$

and (0.5) then the following Bäcklund transformation for the third Painlevé equation can be obtained,

$$T_2 : \qquad y_2(y,t;\alpha_2,\beta_2,4,-4) = -\frac{2ty' + 4ty^2 + (\beta-2)y - 4t}{y[2ty' + 4ty^2 + (\alpha+2)y - 4t]} \ . \tag{1.16}$$
$$\alpha_2 = \alpha + 4 \ , \qquad \beta_2 = \beta - 4 \ .$$

Transformation III: If we use transformed quantities under the transformation matrix $R_{(3)}(z,t)$

$$\begin{aligned} u_3 &= \frac{u}{t}\Big(\theta_\infty - 1\Big) + \frac{u^2}{w}\left(\frac{1}{s} - \frac{1}{t}\right) - sw \\ -w_3 s_3 &= \frac{u^3}{s^2w^2t^2}\Big(s-t\Big)^2 - \frac{\theta_\infty u^2}{swt^2}\Big(s-t\Big) - \frac{u^2 v}{t^2} - u \ , \end{aligned} \tag{1.17}$$

and (0.5) then the following Bäcklund transformation for the third Painlevé equation can be obtained,

$$T_3 : \qquad y_3(y,t;\alpha_3,\beta_3,4,-4) = -\frac{2ty' - 4ty^2 - (\beta+2)y + 4t}{y[2ty' - 4ty^2 + (-\alpha+2)y + 4t]} \ . \tag{1.18}$$
$$\alpha_3 = \alpha - 4 \ , \qquad \beta_3 = \beta + 4 \ .$$

Transformation IV: If we use transformed quantities under the transformation matrix $R_{(4)}(z,t)$

$$\begin{aligned} u_4 &= -\frac{u}{t}(1 - \theta_\infty + \frac{u}{w}) - sw \\ -w_4 s_4 &= u + \frac{u^2}{t^2w^2}(u - \theta_\infty w - vw^2) \ , \end{aligned} \tag{1.19}$$

and (0.5) then the following Bäcklund transformation for the third Painlevé equation can be obtained,

$$T_4 : \qquad y_4(y,t;\alpha_4,\beta_4,4,-4) = \frac{2ty' + 4ty^2 - (\beta+2)y + 4t}{y[2ty' + 4ty^2 + (\alpha+2)y + 4t]} \ . \tag{1.20}$$
$$\alpha_4 = \alpha + 4 \ , \qquad \beta_4 = \beta + 4 \ .$$

Moreover the following Lie-point discrete symmetries of the third Painlevé equation are known [28].

$$\hat{T} : \qquad \hat{y}(t;\hat{\alpha},\hat{\beta},4,-4) = -y(\alpha,\beta,4,-4) \ , \qquad \hat{\alpha} = -\alpha \ , \qquad \hat{\beta} = -\beta \ . \tag{1.21}$$

$$\tilde{T} : \qquad \tilde{y}(\tilde{\alpha},\tilde{\beta},4,-4) = 1/y(\alpha,\beta,4,-4) \ , \qquad \tilde{\alpha} = -\beta \ , \qquad \tilde{\beta} = -\alpha \ . \tag{1.22}$$

We can consider the following combinations

$$\hat{T}_j \equiv \hat{T} \circ T_j \ , \qquad \tilde{T}_j \equiv \tilde{T} \circ T_j \ , \qquad T_j{}^* \equiv \hat{T} \circ \tilde{T} \circ T_j \ .$$

3.2 Discrete Equations

The Bäcklund transformations that are inverse of each other, i.e. $T_1 \circ T_4 = I$, $T_2 \circ T_3 = I$, ${T_1}^* \circ {T_4}^* = I$ and $\tilde{T}_2 \circ \tilde{T}_3 = I$, will be used to get the discrete equations. The pairs of transformations $\hat{T}_j \equiv \hat{T} \circ T_j$ do not give us any discrete equation. Because we couldn't find any transformations which are the inverse of the other. Also ${T_2}^*$, ${T_3}^*$ and $\tilde{T}_1$, $\tilde{T}_4$ do not give us any discrete equation.

Discrete Equation from T_1 and T_4 :

After eliminating y' from (1.14) and (1.20) , and setting

$$x_{n+1} = y_1 \qquad x_n = y \qquad x_{n-1} = y_4 \ , \tag{2.23}$$

and

$$\begin{aligned} a_{n+1} = \alpha_1 = \alpha - 4 \ , \qquad a_n = \alpha \ , \qquad a_{n-1} = \alpha_4 = \alpha + 4 \ , \\ b_{n+1} = \beta_1 = \beta - 4 \ , \qquad b_n = \beta \ , \qquad b_{n-1} = \beta_4 = \beta + 4 \ , \end{aligned} \tag{2.24}$$

we obtain the discrete equation,

$$2\left(\frac{4t}{x_n} + 4tx_n + a_n\right) - \frac{4 - a_n - b_n}{x_n x_{n+1} - 1} + \frac{4 + a_n + b_n}{x_n x_{n-1} - 1} = 0 \ , \tag{2.25}$$

where

$$a_n = \kappa - 4n \qquad b_n = \mu - 4n \tag{2.26}$$

are the solutions of the difference equation (2.24), with κ and μ being arbitrary constants.

Discrete Equation from T_2 and T_3 :

After eliminating y' from (1.16) and (1.18) , and setting

$$x_{n+1} = y_2 \qquad x_n = y \qquad x_{n-1} = y_3 \ , \tag{2.27}$$

and

$$a_{n+1} = \alpha_2 = \alpha + 4 \ , \qquad a_n = \alpha \ , \qquad a_{n-1} = \alpha_3 = \alpha - 4 \ ,$$
$$b_{n+1} = \beta_2 = \beta - 4 \ , \qquad b_n = \beta \ , \qquad b_{n-1} = \beta_3 = \beta + 4 \ , \tag{2.28}$$

we obtain the discrete equation,

$$2\left(4tx_n + a_n - \frac{4t}{x_n}\right) - \frac{4 + a_n - b_n}{x_n x_{n+1} + 1} + \frac{4 - a_n + b_n}{x_n x_{n-1} + 1} = 0 \tag{2.29}$$

where

$$a_n = \kappa + 4n \qquad b_n = \mu - 4n \tag{2.30}$$

are the solutions of the difference equation (2.28) with κ and μ being arbitrary constants.

Discrete Equation from $T_1{}^*$ and $T_4{}^*$:

By considering transformations $T_1{}^*$ and $T_4{}^*$

$$T_1{}^* : \qquad y_1{}^*(y, t; \alpha_1{}^*, \beta_1{}^*, 4, -4) = -\frac{y(2ty' - 4ty^2 + (-\alpha + 2)y - 4t)}{2ty' - 4ty^2 + (\beta - 2)y - 4t} \tag{2.31}$$

$$\alpha_1{}^* = \beta - 4 \qquad \beta_1{}^* = \alpha - 4$$

$$T_4{}^* : \qquad y_4{}^*(y, t; \alpha_4{}^*, \beta_4{}^*, 4, -4) = -\frac{y(2ty' + 4ty^2 + (\alpha + 2)y + 4t)}{2ty' + 4ty^2 - (\beta + 2)y + 4t} \tag{2.32}$$

$$\alpha_4{}^* = \beta + 4 \qquad \beta_4{}^* = \alpha + 4$$

one can obtain a new discrete equation. After eliminating y' from (2.31) and (2.32) , and setting

$$x_{n+1} = y_1{}^* \qquad x_n = y \qquad x_{n-1} = y_4{}^*$$

and

$$a_{n+1} = \alpha_1{}^* = \beta - 4 \qquad a_n = \alpha \qquad a_{n-1} = \alpha_4{}^* = \beta + 4$$
$$b_{n+1} = \beta_1{}^* = \alpha - 4 \qquad b_n = \beta \qquad b_{n-1} = \beta_4{}^* = \alpha + 4 \tag{2.33}$$

we obtain the discrete equation, given by

$$2\left(4t-\frac{b_n}{x_n}+\frac{4t}{{x_n}^2}\right)+\frac{a_n+b_n-4}{x_n+x_{n+1}}+\frac{a_n+b_n+4}{x_n+x_{n-1}}=0 \tag{2.34}$$

where

$$a_n=\kappa-4n+\mu(-1)^n \qquad b_n=\kappa-4n-\mu(-1)^n \tag{2.35}$$

are the solutions of the difference equation (2.33). κ and μ being an arbitrary constants.

Discrete Equation from $\tilde{T_2}$ and $\tilde{T_3}$:

Similarly, by considering transformations $\tilde{T_2}$ and $\tilde{T_3}$

$$\tilde{T_2}: \qquad \tilde{y_2}(y,t;\tilde{\alpha_2},\tilde{\beta_2},4,-4)=-\frac{y(2ty'+4ty^2+(\alpha+2)y-4t)}{2ty'+4ty^2+(\beta-2)y-4t} \tag{2.36}$$

$$\tilde{\alpha_2}=-\beta+4 \qquad \tilde{\beta_2}=-\alpha-4$$

$$\tilde{T_3}: \qquad \tilde{y_3}(y,t;\tilde{\alpha_3},\tilde{\beta_3},4,-4)=-\frac{y(2ty'-4ty^2+(-\alpha+2)y+4t)}{2ty'-4ty^2-(\beta+2)y+4t} \tag{2.37}$$

$$\tilde{\alpha_3}=-\beta-4 \qquad \tilde{\beta_3}=-\alpha+4$$

we can obtain the new discrete equation. After eliminating y' from (2.36) and (2.37) , and setting

$$x_{n+1}=\tilde{y_1} \qquad x_n=y \qquad x_{n-1}=\tilde{y_4} \tag{2.38}$$

and

$$\begin{aligned} a_{n+1}&=\tilde{\alpha_2}=-\beta+4 \qquad a_n=\alpha \qquad a_{n-1}=\tilde{\alpha_3}=-\beta-4\\ b_{n+1}&=\tilde{\beta_2}=-\alpha-4 \qquad b_n=\beta \qquad b_{n-1}=\tilde{\alpha_3}=-\alpha+4 \end{aligned} \tag{2.39}$$

we obtain the discrete equation,

$$2\left(4t+\frac{b_n}{x_n}-\frac{4t}{{x_n}^2}\right)+\frac{a_n-b_n+4}{x_n+x_{n+1}}+\frac{a_n-b_n-4}{x_n+x_{n-1}}=0 \tag{2.40}$$

where

$$a_n=\kappa+4n+\mu(-1)^n \qquad b_n=-\kappa-4n+\mu(-1)^n \tag{2.41}$$

are the solutions of the difference equation (2.39). κ being an arbitrary constant.

Chapter 4

The Forth Painlevé Equation

The forth Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}, \tag{0.1}$$

is the compatibility condition of the linear problem

$$Y_z(z,t) = A(z,t)Y(z,t), \qquad Y_t(z,t) = B(z,t)Y(z,t), \tag{0.2}$$

where,

$$\begin{aligned} A(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} t & u \\ \frac{2}{u}(v-\theta_0-\theta_\infty) & -t \end{pmatrix} \\ + \begin{pmatrix} \theta_0 - v & -\frac{uy}{2} \\ 2\frac{v}{uy}(v-2\theta_0) & -(\theta_0 - v) \end{pmatrix} \frac{1}{z}, \end{aligned} \tag{0.3}$$

$$B(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} 0 & u \\ \frac{2}{u}(v-\theta_0-\theta_\infty) & 0 \end{pmatrix},$$

The parameters α, β are given as,

$$\alpha = 2\theta_\infty - 1, \qquad \beta = -8{\theta_0}^2. \tag{0.4}$$

u,v and y are functions of t, and θ_∞ and θ_0 are the constants. The compatibility condition $Y_{zt} = Y_{tz}$ yields.

$$-4v = y' - 4\theta_0 - 2yt - y^2. \tag{0.5}$$

where $y' = \frac{dy}{dt}$

4.1 Bäcklund Transformations

Let $R(z,t)$ be the transformation matrix which leaves the monodromy data of the solution matrix $Y(z,t)$ of (0.2) the same but changes the exponents θ_0 and θ_∞ .

$$Y'(z,t) = R(z,t)Y(z,t), \tag{1.6}$$

where, $Y(z,t) \equiv Y(z,t;y,u,v,\theta_0,\theta_\infty)$, and $Y'(z,t) \equiv Y'(z,t;y_i,u_i,v_i,{\theta_0}' = \theta_0 \pm \frac{1}{2}, {\theta_\infty}' = \theta_\infty \pm \frac{1}{2})$.

All the possible Schlesinger transformations are [30],

$$\begin{aligned}
&\begin{cases} {\theta_0}' = \theta_0 - \frac{1}{2} \\ {\theta_\infty}' = \theta_\infty + \frac{1}{2}, \end{cases}\\
&\qquad R_{(1)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & \frac{uy}{2(v-2\theta_0)} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{y(v-\theta_0-\theta_\infty}{2(v-2\theta_0)} \end{pmatrix} z^{-1/2},\\
&\begin{cases} {\theta_0}' = \theta_0 + \frac{1}{2} \\ {\theta_\infty}' = \theta_\infty - \frac{1}{2}, \end{cases}\\
&\qquad R_{(2)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} \frac{v}{y} & \frac{u}{2} \\ \frac{2v}{uy} & 1 \end{pmatrix} z^{-1/2},\\
&\begin{cases} {\theta_0}' = \theta_0 + \frac{1}{2} \\ {\theta_\infty}' = \theta_\infty + \frac{1}{2}, \end{cases}\\
&\qquad R_{(3)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & \frac{uy}{2v} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{y(v-\theta_0-\theta_\infty)}{2v} \end{pmatrix} z^{-1/2},\\
&\begin{cases} {\theta_0}' = \theta_0 - \frac{1}{2} \\ {\theta_\infty}' = \theta_\infty - \frac{1}{2}, \end{cases}\\
&\qquad R_{(4)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} \frac{v-2\theta_0}{y} & \frac{u}{2} \\ \frac{2}{uy}(v-2\theta_0) & 1 \end{pmatrix} z^{-1/2}.
\end{aligned} \tag{1.7}$$

If $y_1,u_1,v_1,{\theta_0}' = \theta_0 - 1/2, {\theta_\infty}' = \theta_\infty + 1/2$ are the transformed quantities of $y,u,v,\theta_0,\theta_\infty$ under the transformation given by $R_{(1)}$, and if $y_2,u_2,v_2,{\theta_0}'' =$

${\theta_0}' + 1/2, {\theta_\infty}'' = {\theta_\infty}' - 1/2$ are the transformed quantities of $y_1, u_1, v_1, {\theta_0}', {\theta_\infty}'$ under the transformation given by $R_{(2)}$ then

$$R_{(2)}(z,t;y_1(y,u,...),...)R_{(1)}(z,t;y,...) = I. \tag{1.8}$$

Similarly,

$$R_{(3)}(z,t;y_4(y,u,...),...)R_{(4)}(z,t;y,...) = I. \tag{1.9}$$

The linear equation (0.2) is transformed under the Schlesinger transformation as follows:

$$Y'_z(z,t) = A'(z,t)Y'(z,t), \qquad A'(z,t) = [R(z,t)A(z,t) + R_z(z,t)]R^{-1}(z,t). \tag{1.10}$$

Transformation I: Quantities y, u, θ_∞ and θ_0 are the transformed under the transformation matrix $R_{(1)}(z,t)$ as;

$$\begin{aligned} u_1 &= -\frac{uy}{v-2\theta_0}, \\ -\frac{u_1y_1}{2} &= \frac{uy^2(\theta_0+\theta_\infty-v)}{2(v-2\theta_0)^2} - \frac{tuy}{v-2\theta_0} + u, \end{aligned} \tag{1.11}$$

which guarantees the integrability condition of the linearization (0.2). That is, if $y(t)$ solves the fourth Painlevé equation with parameters $\alpha = 2\theta_\infty - 1$ and $\beta = -8{\theta_0}^2$ then $y_1(t)$ solves the fourth Painlevé equation with parameter $\alpha_1 = \alpha + 1$ and $\beta_1 = -\frac{1}{2}(-2+\epsilon\sqrt{-2\beta})$. If we use (0.5) and (1.11) then the following Bäcklund transformation for the fourth Painlevé equation can be obtained,

$$T_1: \quad y_1(y,t;\alpha_1,\beta_1) = -\frac{1}{2y}\Big(y' + y^2 + 2ty + \epsilon\sqrt{-2\beta}\Big) - \frac{y(\epsilon\sqrt{-2\beta} - 2\alpha - 2)}{-y' + y^2 + 2ty - \epsilon\sqrt{-2\beta}}. \tag{1.12}$$

$$\alpha_1 = \alpha + 1\,, \qquad \beta_1 = -\frac{1}{2}\Big(-2+\epsilon\sqrt{-2\beta}\Big)^2.$$

where $\epsilon = \pm 1$ and $y' = dy/dt$ in the transformations.

Transformation II: If we use transformed quantities under the transformation matrix $R_{(2)}(z,t)$

$$\begin{aligned} u_2 &= u\Big(-t + \frac{v}{y} - \frac{y}{2}\Big), \\ -\frac{u_2y_2}{2} &= -\frac{u}{2} - \frac{uv}{2} + \frac{uv^2}{y^2} - \frac{tuv}{y} - \frac{u(\theta_0-\theta_\infty)}{2}, \end{aligned} \tag{1.13}$$

and (0.5) then the following Bäcklund transformation for the fourth Painlevé equation can be obtained,

$$T_2: \quad y_2(y,t;\alpha_2,\beta_2) = -\frac{1}{2y}\Big(-y'+y^2+2ty+\epsilon\sqrt{-2\beta}\Big) - \frac{y(\epsilon\sqrt{-2\beta}-2\alpha+2)}{y'+y^2+2ty-\epsilon\sqrt{-2\beta}}. \tag{1.14}$$
$$\alpha_2 = \alpha - 1\ , \qquad \beta_2 = -\frac{1}{2}\Big(2+\epsilon\sqrt{-2\beta}\Big)^2.$$

Transformation III: If we use transformed quantities under the transformation matrix $R_{(3)}(z,t)$

$$\begin{aligned} u_3 &= -\frac{uy}{v}, \\ -\frac{u_3 y_3}{2} &= u\Big[1-\frac{ty}{v}+\frac{y^2}{2v^2}\Big(\theta_0+\theta_\infty-v\Big)\Big], \end{aligned} \tag{1.15}$$

and (0.5) then the following Bäcklund transformation for the fourth Painlevé equation can be obtained,

$$T_3: \quad y_3(y,t;\alpha_3,\beta_3) = -\frac{1}{2y}\Big(y'+y^2+2ty-\epsilon\sqrt{-2\beta}\Big) + \frac{y(\epsilon\sqrt{-2\beta}+2\alpha+2)}{-y'+y^2+2ty+\epsilon\sqrt{-2\beta}}. \tag{1.16}$$
$$\alpha_3 = \alpha + 1\ , \qquad \beta_3 = -\frac{1}{2}\Big(2+\epsilon\sqrt{-2\beta}\Big)^2.$$

Transformation IV: If we use transformed quantities under the transformation matrix $R_{(3)}(z,t)$

$$\begin{aligned} u_4 &= u\Big[\frac{v-2\theta_0}{y}-\frac{1}{2}\Big(y+2t\Big)\Big], \\ -\frac{u_4 y_4}{2} &= u\Big[\frac{(v-2\theta_0)^2}{y^2}-\frac{t}{y}\Big(v-2\theta_0\Big)-\frac{v}{2}+\frac{3\theta_0}{2}-\frac{1}{2}+\frac{\theta_\infty}{2}\Big], \end{aligned} \tag{1.17}$$

and (0.5) then the following Bäcklund transformation for the fourth Painlevé equation can be obtained,

$$T_4: \quad y_4(y,t;\alpha_4,\beta_4) = -\frac{1}{2y}\Big(-y'+y^2+2ty-\epsilon\sqrt{-2\beta}\Big) + \frac{y(\epsilon\sqrt{-2\beta}+2\alpha-2)}{y'+y^2+2ty+\epsilon\sqrt{-2\beta}}. \tag{1.18}$$
$$\alpha_4 = \alpha - 1\ , \qquad \beta_4 = -\frac{1}{2}\Big(-2+\epsilon\sqrt{-2\beta}\Big)^2.$$

Transformation V:

$$T_5: \quad y_5(y,t;\alpha_5,\beta_5) = \frac{(y'-y^2-2ty)^2+2\beta}{2y[y'-y^2-2ty+2(\alpha+1)]}, \quad \alpha_5=\alpha+2, \quad \beta_5=\beta. \tag{1.19}$$

Transformation VI:

$$T_7: \quad y_7(y,t;\alpha_7,\beta_7) = -\frac{(y'+y^2+2ty)^2+2\beta}{2y[y'+y^2+2ty-2(\alpha-1)]}, \quad \alpha_7=\alpha-2, \quad \beta_7=\beta. \tag{1.20}$$

transformations T_5 and T_7 derived by Fokas, Mugan and Ablowitz [29].

There are known Bäcklund transformations [15] of the compact form as follows:

$$\tilde{T}: \quad y(t,\alpha,\beta) \to \tilde{y}(t,\tilde{\alpha},\tilde{\beta}) \tag{1.21}$$

$$\tilde{y}(t,\tilde{\alpha},\tilde{\beta}) = \frac{1}{2\mu y}\Big(y'-\mu y^2-2\mu t y-q\Big) \tag{1.22}$$

$$\tilde{\alpha} = \frac{1}{4}\Big(2\mu-2\alpha+3\mu q\Big), \qquad \tilde{\beta} = -\frac{1}{2}\Big(1+\alpha\mu+\frac{1}{2}q\Big)^2 \tag{1.23}$$

where $\mu^2=1$, $q^2=-2\beta$.

That form gives us four transformation:

$$\tilde{T}_1: \qquad \tilde{y}_1(t,\alpha_1,\beta_1) = \frac{1}{2y}\Big(y'-y^2-2ty-\sqrt{-2\beta}\Big), \tag{1.24}$$

$$\begin{aligned} \tilde{\alpha}_1 &= \frac{1}{4}\Big(2-2\alpha+3\sqrt{-2\beta}\Big), \\ \tilde{\beta}_1 &= -\frac{1}{2}\Big(1+\alpha+\frac{1}{2}\sqrt{-2\beta}\Big)^2. \end{aligned} \tag{1.25}$$

$$\tilde{T}_2: \qquad \tilde{y}_2(t,\alpha_2,\beta_2) = -\frac{1}{2y}\Big(y'+y^2+2ty+\sqrt{-2\beta}\Big), \tag{1.26}$$

$$\begin{aligned} \tilde{\alpha}_2 &= -\frac{1}{4}\Big(2+2\alpha-3\sqrt{-2\beta}\Big), \\ \tilde{\beta}_2 &= -\frac{1}{2}\Big(1-\alpha-\frac{1}{2}\sqrt{-2\beta}\Big)^2. \end{aligned} \tag{1.27}$$

$$\tilde{T}_3: \qquad \tilde{y_3}(t,\alpha_3,\beta_3) = \frac{1}{2y}\Big(y' - y^2 - 2ty + \sqrt{-2\beta}\Big), \tag{1.28}$$

$$\begin{aligned} \tilde{\alpha_3} &= \frac{1}{4}\Big(2 - 2\alpha - 3\sqrt{-2\beta}\Big), \\ \tilde{\beta_3} &= -\frac{1}{2}\Big(1 + \alpha - \frac{1}{2}\sqrt{-2\beta}\Big)^2. \end{aligned} \tag{1.29}$$

$$\tilde{T}_4: \qquad \tilde{y_4}(t,\alpha_4,\beta_4) = -\frac{1}{2y}\Big(y' + y^2 + 2ty - \sqrt{-2\beta}\Big), \tag{1.30}$$

$$\begin{aligned} \tilde{\alpha_4} &= -\frac{1}{4}\Big(2 + 2\alpha + 3\sqrt{-2\beta}\Big), \\ \tilde{\beta_4} &= -\frac{1}{2}\Big(1 - \alpha + \frac{1}{2}\sqrt{-2\beta}\Big)^2. \end{aligned} \tag{1.31}$$

4.2 Discrete Equations

The Bäcklund transformations that are inverse of each other,
i.e. $T_5 \circ T_7 = I,\ T_i \circ T_{i+1} = I$ and $\tilde{T}_i \circ \tilde{T}_{i+1} = I$, i=1,3. will be used to get the discrete equations.

Discrete Equation from T_1 and T_2 :

T_1 and T_2 are the inverse of each other if $\beta < -\frac{1}{2}$.

Setting $a_{n+1} = \alpha_1,\ a_n = \alpha,\ a_{n-1} = \alpha_2,\ c_{n+1} = \sqrt{-2\beta_1},\ c_n = \sqrt{-2\beta}$ and $c_{n-1} = \sqrt{-2\beta_2}$ in the parameter relations yield the difference equations

$$\begin{aligned} a_{n+1} &= a_n + 1, \qquad & a_{n-1} &= a_n - 1, \\ \pm c_{n+1} &= \pm c_n - 2, \qquad & \pm c_{n-1} &= \pm c_n + 2, \end{aligned} \tag{2.32}$$

which have the solutions $a_n = \kappa + n$ and $c_n = \mu \mp 2n$, where κ and μ are arbitrary constants. Setting

$$x_{n+1} = y_1, \qquad x_n = y, \qquad x_{n-1} = y_2, \tag{2.33}$$

and eliminating y' from (1.12) and (1.14) yields the discrete equation

$$\begin{aligned}&2x_n^4\{-4-x_{n-1}[(a_n+1)x_{n-1}+2(2t+x_n)]+x_{n+1}[2(2t+x_n)+x_{n-1}(-2a_n\\&+(2t+x_n)(2t+x_{n-1}+x_n))]+x_{n+1}^2[1-a_n+x_{n-1}(2t+x_n)]\}+c_nx_n^3(x_{n-1}\\&+x_{n+1})\{-8a_n+2(2t+x_n)^2+x_{n-1}(4t+3x_n-2x_{n+1})+(4t+3x_n)x_{n+1}\}\\&+2c_n^2x_n^2\{-4a_n-x_{n-1}^2+(2t+x_n)^2+x_{n-1}(2t+3x_n-3x_{n+1})+(2t+3x_n)x_{n+1}\\&-x_{n+1}^2\}-4c_n^3x_n(x_{n-1}-x_n+x_{n+1})-2c_n^4=0.\end{aligned} \tag{2.34}$$

Discrete Equation from T_3 and T_4 :

T_3 and T_4 are the inverse of each other if $\beta<-\frac{1}{2}$. After eliminating y' from (1.16) and (1.18) , and setting

$$\begin{aligned}&x_{n+1}=y_3, && x_n=y, && x_{n-1}=y_4,\\&a_{n+1}=\alpha_3, && a_n=\alpha, && a_{n-1}=\alpha_4,\\&c_{n+1}=\sqrt{-2\beta_3}, && c_n=\sqrt{-2\beta}, && c_{n-1}=\sqrt{-2\beta_4},\end{aligned} \tag{2.35}$$

we obtain the following discrete equation,

$$\begin{aligned}&2x_n^4\{-4-x_{n-1}[(1+a_n)x_{n-1}+2(2t+x_n)]+x_{n+1}[2(2t+x_n)+x_{n-1}(-2a_n\\&+(2t+x_n)(2t+x_{n-1}+x_n))]+x_{n+1}^2[1-a_n+x_{n-1}(2t+x_n)]\}-c_nx_n^3(x_{n-1}\\&+x_{n+1})\{-8a_n+2(2t+x_n)^2+x_{n-1}(4t+3x_n-2x_{n+1})+(4t+3x_n)x_{n+1}\}\\&+2c_n^2x_n^2\{-4a_n-x_{n-1}^2+(2t+x_n)^2+x_{n-1}(2t+3x_n-3x_{n+1})+(2t+3x_n)x_{n+1}\\&-x_{n+1}^2\}+4c_n^3x_n(x_{n-1}-x_n+x_{n+1})-2c_n^4=0\end{aligned} \tag{2.36}$$

where $a_n=\kappa+n$ and $c_n=\mu\pm 2n$ are the solutions of the difference equations

$$\begin{aligned}a_{n+1}&=a_n+1, & a_{n-1}&=a_n-1,\\ \pm c_{n+1}&=\pm c_n+2, & \pm c_{n-1}&=\pm c_n-2.\end{aligned} \tag{2.37}$$

κ and μ are arbitrary constants.

Discrete Equation from T_5 and T_7 :

After eliminating y' from (1.19) and (1.20) , and setting

$$\begin{array}{lll} x_{n+1} = y_5, & x_n = y, & x_{n-1} = y_7, \\ a_{n+1} = \alpha_5, & a_n = \alpha, & a_{n-1} = \alpha_7, \\ b_{n+1} = \beta_5, & b_n = \beta, & b_{n-1} = \beta_7, \end{array} \tag{2.38}$$

we obtain the following discrete equation,

$$\begin{aligned} & b_n(4t + x_{n-1} + 2x_n + x_{n+1})^2 + 2\{x_{n-1}^2[(1 - a_n + 2tx_n + x_n^2)^2 \\ & + x_n x_{n+1}(x_n(2t + x_n) - 2a_n)] + x_{n-1}[2x_n(2t + x_n)^2(1 - a_n + 2tx_n + x_n^2) \\ & + x_{n+1}(2 - 2a_n^2 - 4a_n x_n(2t + x_n) + 3x_n^2(2t + x_n)^2) + x_n x_{n+1}^2(x_n(2t + x_n) \\ & - 2a_n)] + [(a_n + 1)x_{n+1} - x_n(2t + x_n)(2t + x_n + x_{n+1})]^2\} = 0 \end{aligned} \tag{2.39}$$

where $a_n = \kappa + 2n$ and $b_n = \mu$ are the solutions of the difference equation

$$\begin{array}{rcll} a_{n+1} & = & a_n + 2, & a_{n-1} = a_n - 2, \\ b_{n+1} & = & b_n, & b_{n-1} = b_n. \end{array} \tag{2.40}$$

κ and μ are arbitrary constants.

Discrete Equation from $\tilde{T}_1$ and $\tilde{T}_2$:

$\tilde{T}_1$ and $\tilde{T}_2$ are the inverse of each other if $\beta < -2(1-\alpha)^2$ and $\beta < -2(1+\alpha)^2$. After eliminating y' from (1.24) and (1.26) , and setting

$$\begin{array}{lll} x_{n+1} = \tilde{y}_1, & x_n = y, & x_{n-1} = \tilde{y}_2, \\ c_{n+1} = \sqrt{-2\tilde{\beta}_1}, & c_n = \sqrt{-2\beta}, & c_{n-1} = \sqrt{-2\tilde{\beta}_2}, \end{array} \tag{2.41}$$

we obtain the following discrete equation,

$$c_n + 2tx_n + x_n^2 + 2x_n(x_{n-1} + x_{n+1}) = 0 \tag{2.42}$$

where $c_n = n + C$ is the solution of the difference equation

$$c_{n+1} \;=\; 1 + \alpha + \frac{c_n}{2}, \qquad c_{n-1} = -1 + \alpha + \frac{c_n}{2}. \tag{2.43}$$

C being an arbitrary constant.

Discrete Equation from $\tilde{T}_3$ and $\tilde{T}_4$:

$\tilde{T}_3$ and $\tilde{T}_4$ are the inverse of each other if $\beta < -2(1-\alpha)^2$ and $\beta < -2(1+\alpha)^2$. After eliminating y' from (1.28) and (1.30) , and setting

$$\begin{array}{lll} x_{n+1} = \tilde{y}_3, & x_n = y, & x_{n-1} = \tilde{y}_4, \\ b_{n+1} = \sqrt{-2\tilde{\beta}_3}, & b_n = \sqrt{-2\beta}, & b_{n-1} = \sqrt{-2\tilde{\beta}_4}, \end{array} \tag{2.44}$$

we obtain the following discrete equation,

$$-b_n + 2tx_n + x_n^2 + 2x_n(x_{n-1} + x_{n+1}) = 0 \tag{2.45}$$

where $b_n = B - n$ is the solution of the difference equation

$$b_{n+1} = -1 - \alpha + \frac{b_n}{2}, \qquad b_{n-1} = 1 - \alpha + \frac{b_n}{2}. \tag{2.46}$$

B being an arbitrary constant.

Chapter 5

The Fifth Painlevé Equation

The fifth Painlevé equation

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{0.1}$$

is the compatibility condition of the linear problem,

$$Y_z(z,t) = A(z,t)Y(z,t), \qquad Y_t(z,t) = B(z,t)Y(z,t), \tag{0.2}$$

where,

$$A(z,t) = \tfrac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} v + \frac{\theta_0}{2} & -u(v+\theta_0) \\ \frac{v}{u} & -(v + \frac{\theta_0}{2}) \end{pmatrix}\frac{1}{z} + \begin{pmatrix} -w & uy(w - \frac{\theta_1}{2}) \\ -\frac{1}{uy}(w + \frac{\theta_1}{2}) & w \end{pmatrix}\frac{1}{z-1},$$

$$B(z,t) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \tfrac{1}{t}\begin{pmatrix} 0 & -u[v + \theta_0 - y(w - \frac{\theta_1}{2})] \\ \frac{1}{u}[v - \frac{1}{y}(w + \frac{\theta_1}{2})] & 0 \end{pmatrix}.$$

$$w = v + \tfrac{1}{2}\ (\theta_0 + \theta_\infty). \tag{0.3}$$

The parameters $\alpha, \beta, \gamma, \delta$ are,

$$\alpha = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2, \quad \beta = -\frac{1}{2}\left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2}\right)^2, \quad \gamma = 1-\theta_0-\theta_1, \quad \delta = -\frac{1}{2}. \tag{0.4}$$

u,v and y are functions of t, and θ_0 , θ_1 and θ_∞ are the constants. The compatibility condition $Y_{zt} = Y_{tz}$ is satisfied if

$$v = \frac{1}{4(y-1)^2}(-3\theta_0+4y\theta_0-y^2\theta_0-\theta_1+2ty+y^2\theta_1-\theta_\infty+2y\theta_\infty-y^2\theta_\infty-2ty') \,. \tag{0.5}$$

where $y' = \frac{dy}{dt}$.

5.1 Bäcklund Transformations

Let $R(z,t)$ be the transformation matrix which transforms the solution matrix $Y(z,t)$ but leaves the monodromy data of $Y(z,t)$ the same .i.e, $Y'(z,t) = R(z,t)Y(z,t)$, and $y_i,\ u_i,\ v_i,\ {\theta_0}' = \theta_0 + \kappa_0,\ {\theta_1}' = \theta_1 + \kappa_1,\ {\theta_\infty}' = \theta_\infty + \kappa_\infty$, $i = 1, 2, ..., 8$ are the transformed quantities of $y,\ u,\ v,\ \theta_0,\ \theta_1,\ \theta_\infty$. The monodromy data or equivalent the consistency condition of the monodromy data is invariant under the transformation if the exponents $\theta_0,\ \theta_1,\ \theta_\infty$ are shifted as follows;

$$a. \begin{cases} {\theta_0}' = \theta_0 + n \\ {\theta_1}' = \theta_0 \\ {\theta_\infty}' = \theta_\infty + m, \end{cases} \qquad b. \begin{cases} {\theta_0}' = \theta_0 \\ {\theta_1}' = \theta_1 + n \\ {\theta_\infty}' = \theta_\infty + m, \end{cases} \qquad c. \begin{cases} {\theta_0}' = \theta_0 + n \\ {\theta_1}' = \theta_1 + m \\ {\theta_\infty}' = \theta_\infty \end{cases}, \tag{1.6}$$

where n and m are either even or odd integers.

It is enough to determine the Schlesinger transformation matrix $R(z,t)$ for $n, m = \pm 1$. The explicit form of $R(z)$ can be listed as follows [30]:

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$
$$R_{(1)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & -\frac{u}{v}(v+\theta_0) \\ -\frac{1}{tu}\left[v - \frac{1}{y}(w+\theta_1/2)\right] & \frac{1}{tv}(v+\theta_0)\left[v - \frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] \end{pmatrix} z^{-1/2}, \tag{1.7}$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$
$$R_{(2)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} \frac{1}{t}\left[v+\theta_0 - y\left(w-\frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v+\theta_0 - y\left(w-\frac{\theta_1}{2}\right)\right] \\ -\frac{1}{u} & 1 \end{pmatrix} z^{-1/2}, \tag{1.8}$$

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$
$$R_{(3)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{1/2} + \begin{pmatrix} \frac{1}{t}\left[v+\theta_0 - y\left(w-\frac{\theta_1}{2}\right)\right]\frac{v}{v+\theta_0} & -\frac{u}{t}\left[v+\theta_0 - y\left(w-\frac{\theta_1}{2}\right)\right] \\ -\frac{v}{u(v+\theta_0)} & 1 \end{pmatrix} z^{-1/2}, \tag{1.9}$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$
$$R_{(4)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 1 & -u \\ -\frac{1}{tu}\left[v - \frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] & \frac{1}{t}\left[v - \frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] \end{pmatrix} z^{-1/2}, \tag{1.10}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + 1, \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$

$$R_{(5)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z-1)^{1/2} + \begin{pmatrix} 1 & -\frac{uy}{w_1} \\ -\frac{1}{tu}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] & \frac{y}{t}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right]\frac{1}{w_1} \end{pmatrix} (z-1)^{-1/2}, \tag{1.11}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 - 1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$

$$R_{(6)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (z-1)^{1/2} + \begin{pmatrix} \frac{1}{ty}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_0}{2}\right)\right] \\ -\frac{1}{uy} & 1 \end{pmatrix} (z-1)^{-1/2}, \tag{1.12}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + 1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases}$$

$$R_{(7)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (z-1)^{1/2} + \begin{pmatrix} \frac{w_1}{ty}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v + \theta_0 - y\left(w - \frac{\theta_1}{2}\right)\right] \\ -\frac{1}{uy}w_1 & 1 \end{pmatrix} (z-1)^{-1/2}, \tag{1.13}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 - 1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases}$$

$$R_{(8)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z-1)^{1/2} + \begin{pmatrix} 1 & -uy \\ -\frac{1}{tu}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] & \frac{y}{t}\left[v - \frac{1}{y}\left(w + \frac{\theta_1}{2}\right)\right] \end{pmatrix} (z-1)^{-1/2}, \tag{1.14}$$

where,

$$w_1 = \frac{w + \theta_1/2}{w - \theta_1/2}. \tag{1.15}$$

The transformation matrices $R_{(i)}(z,t)$, $i = 1, 2, ..., 8$, are sufficient to obtain the transformation matrix $R(z,t)$ which shifts the exponents $\theta_0, \theta_1, \theta_\infty$ to ${\theta_0}', {\theta_1}', {\theta_\infty}'$ with any integer differences. If,

$$Y'(z,t;y',u',v',{\theta_0}',{\theta_1}'{\theta_\infty}') = R_{(j)}(z,t;y,...,\theta_\infty)Y(z,t;y,...,\theta_\infty), \tag{1.16}$$

and

$$Y''(z,t;y'',u'',v'',{\theta_0}'',{\theta_1}',{\theta_\infty}') = R_{(k)}(z,t;y',...,{\theta_\infty}')Y(z,t;y',...,{\theta_\infty}'). \tag{1.17}$$

Then

$$R_{(k)}(z,t;y'(y,u,...\theta_\infty),...)R_{(j)}(z,t;y,...,\theta_\infty) = I, \tag{1.18}$$

for $k = j+1$, $j = 1, 3, 5, 7$.

The linear equation (0.2) is transformed under the Schlesinger transformation as follows:

$$Y'_z(z,t) = A'(z,t)Y'(z,t), \qquad A'(z,t) = [R(z,t)A(z,t) + R_z(z,t)]R^{-1}(z,t). \tag{1.19}$$

Transformation I: If we use transformed quantities under the transformation matrix $R_{(1)}(z,t)$

$v_1 = -\frac{1}{4tv^2y^2}(-4v^2w^2 + 4tv^2wy + 4v^3wy + 8v^2w^2y + 4tv^2y^2 - tv^3y^2 - 8v^3wy^2 - 4v^2w^2y^2 + 4v^3wy^3 - 8vw^2\theta_0 + tvwy\theta_0 + 8v^2wy\theta_0 + 8vw^2y\theta_0 - 8v^2wy^2\theta_0 - w^2\theta_0^2 + 4vwy\theta_0^2 - 4v^2w\theta_1 + 2tv^2y\theta_1 + 2v^3y\theta_1 + v^2wy\theta_1 - 2v^3y^3\theta_1 - 8vw\theta_0\theta_1 + 2tvy\theta_0\theta_1 + v^2y\theta_0\theta_1 + 4vwy\theta_0\theta_1 - 4w\theta_0^2\theta_1 + 2vy\theta_0^2\theta_1 - v^2\theta_1^2 + v^2y^2\theta_1^2 - 2v\theta_0\theta_1^2 - \theta_0^2\theta_1^2)$

$u_1 = -(2tuy)/(-2w + 2vy - \theta_1)$

$y_1 = -[(-v + vy - \theta_0)(-2w + 2vy - \theta_1)]/(-2vw + 2tvy + 2v^2y + 2vwy - 2v^2y^2 - 2w\theta_0 + vy\theta_0 - v\theta_1 + vy\theta_1 - \theta_0\theta_1)$

and (0.5) , then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$T_1 : y_1 = 1 - \frac{A_1}{B_1}. \tag{1.20}$$

$$
\begin{aligned}
A_1 &= 2ty[t(y-y') + (y-1)(1+\epsilon\sqrt{2\alpha} - \epsilon y\sqrt{2\alpha} - \gamma)],\\
B_1 &= -t^2(y')^2 - 2\epsilon\sqrt{2\alpha}t(y-1)yy' + t^2y^2 - 2(y-1)^2(y^2\alpha+\beta)\\
&\quad + 2ty(y-1)(1+\epsilon\sqrt{2\alpha}-\gamma).
\end{aligned} \tag{1.21}
$$

$$
\alpha_1 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2, \qquad \beta_1 = \beta, \qquad \gamma_1 = \gamma - 1
$$

where $y' = dy/dt$ and $\epsilon = \pm 1$ in the transformations.

Transformation II: If we use transformed quantities under the transformation matrix $R_{(2)}(z,t)$

$v_2 = -\frac{1}{4ty}(4vw - 4ty - 4tvy - 8vwy - 4w^2y + 4twy^2 + 4vwy^2 + 8w^2y^2 - 4w^2y^3 + 4w\theta_0 - 8wy\theta_0 + 4wy^2\theta_0 + 2v\theta_1 - 2ty^2\theta_1 - 2vy^2\theta_1 - 4wy^2\theta_1 + 4wy^3\theta_1 + 2\theta_0\theta_1 - 2y^2\theta_0\theta_1 + y\theta_1^2 - y^3\theta_1^2)$

$u_2 = \frac{u}{2t}(4vw - 4ty - 4tvy - 8vwy - 4w^2y + 4twy^2 + 4vwy^2 + 8w^2y^2 - 4w^2y^3 + 4w\theta_0 - 8wy\theta_0 + 4wy^2\theta_0 + 2v\theta_1 - 2ty^2\theta_1 - 2vy^2\theta_1 - 4wy^2\theta_1 + 4wy^3\theta_1 + 2\theta_0\theta_1 - 2y^2\theta_0\theta_1 + y\theta_1^2 - y^3\theta_1^2)/(-2w + 2ty + 4wy - 2wy^2 - \theta_1 + y^2\theta_1)$

$y_2 = \frac{1}{(y-1)}[(2w - 2ty - 4wy + 2wy^2 + \theta_1 - y^2\theta_1)(2v - 2ty - 2vy - 2wy + 2wy^2 + 2\theta_0 - 2y\theta_0 + y\theta_1 - y^2\theta_1)]/(-4vw + 4ty + 4tvy + 8vwy + 4w^2y - 4twy^2 - 4vwy^2 - 8w^2y^2 + 4w^2y^3 - 4w\theta_0 + 8wy\theta_0 - 4wy^2\theta_0 - 2v\theta_1 + 2ty^2\theta_1 + 2vy^2\theta_1 + 4wy^2\theta_1 - 4wy^3\theta_1 - 2\theta_0\theta_1 + 2y^2\theta_0\theta_1 - y\theta_1^2 + y^3\theta_1^2)$

and (0.5), then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$
T_2 : y_2 = 1 - \frac{A_2}{B_2}. \tag{1.22}
$$

$$
\begin{aligned}
A_2 &= 2ty[t(y+y') + (y-1)(-1+\epsilon\sqrt{2\alpha} - \epsilon y\sqrt{2\alpha} - \gamma)],\\
B_2 &= -t^2(y')^2 + 2\epsilon\sqrt{2\alpha}t(y-1)yy' + t^2y^2 - 2(y-1)^2(y^2\alpha+\beta)\\
&\quad + 2t(y-1)y(-1+\epsilon\sqrt{2\alpha}-\gamma)\}.
\end{aligned} \tag{1.23}
$$

$$
\alpha_2 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2, \qquad \beta_2 = \beta, \qquad \gamma_2 = \gamma + 1.
$$

Transformation III: If we use transformed quantities under the transformation matrix $R_{(3)}(z,t)$

$v_3 = -\frac{(2v-2wy+2\theta_0+y\theta_1)}{4ty(v+\theta_0)^2}(2v^2w - 2tv^2y - 4v^2wy + 2v^2wy^2 + 4vw\theta_0 - 2tvy\theta_0 - 4vwy\theta_0 + 2w\theta_0^2 + v^2\theta_1 - v^2y^2\theta_1 + 2v\theta_0\theta_1 + \theta_0^2\theta_1)$

$u_3 = -\frac{u}{2t}(2v - 2wy + 2\theta_0 + y\theta_1)$

$y_3 = (-2v^2 + 2tvy + 2v^2y + 2vwy - 2vwy^2 - 4v\theta_0 + 2ty\theta_0 + 2vy\theta_0 + 2wy\theta_0 - 2\theta_0^2 - vy\theta_1 + vy^2\theta_1 - y\theta_0\theta_1)/[(-v + vy - \theta_0)(2v - 2wy + 2\theta_0 + y\theta_1)]$

and (0.5), then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$T_3 : y_3 = 1 - \frac{A_3}{B_3}. \tag{1.24}$$

$$\begin{aligned} A_3 &= 2ty[t(y-y') + (y-1)(y - \epsilon\sqrt{-2\beta} + \epsilon y\sqrt{-2\beta} - y\gamma)], \\ B_2 &= t^2y'^2 - 2t[ty - \epsilon\sqrt{-2\beta}(y-1)]y' + t^2y^2 - \epsilon 2\sqrt{-2\beta}t(y-1)y \\ &\quad - 2(y-1)^2(y^2\alpha + \beta)\}. \end{aligned} \tag{1.25}$$

$$\alpha_3 = \alpha, \qquad \beta_3 = -\frac{1}{2}\Big(\epsilon\sqrt{-2\beta} + 1\Big)^2, \qquad \gamma_3 = \gamma - 1.$$

Transformation IV: If we use transformed quantities under the transformation matrix $R_{(4)}(z,t)$

$v_4 = \frac{1}{4ty^2}(-2w + 2vy - \theta_1)(-2w + 2ty + 4wy - 2wy^2 - \theta_1 + y^2\theta_1)$

$u_4 = -[2tuy(-2w + 2ty + 4wy - 2wy^2 - \theta_1 + y^2\theta_1)]/(4w^2 - 4twy - 4vwy - 8w^2y - 4ty^2 + 4tvy^2 + 8vwy^2 + 4w^2y^2 - 4vwy^3 + 4ty^2\theta_0 + 4w\theta_1 - 2ty\theta_1 - 2vy\theta_1 - 4wy\theta_1 + 2vy^3\theta_1 + \theta_1^2 - y^2\theta_1^2)$

$y_4 = [(-1+y)(-4w^2 + 4twy + 4vwy + 8w^2y + 4ty^2 - 4tvy^2 - 8vwy^2 - 4w^2y^2 + 4vwy^3 - 4ty^2\theta_0 - 4w\theta_1 + 2ty\theta_1 + 2vy\theta_1 + 4wy\theta_1 - 2vy^3\theta_1 - \theta_1^2 + y^2\theta_1^2)]/[(2w - 2ty - 2vy - 2wy + 2vy^2 + \theta_1 - y\theta_1)(2w - 2ty - 4wy + 2wy^2 + \theta_1 - y^2\theta_1)]$

and (0.5), then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$T_4 : y_4 = 1 - \frac{A_4}{B_4}. \tag{1.26}$$

$$\begin{aligned} A_4 &= 2ty[t(y+y') + (y-1)(-y - \epsilon\sqrt{-2\beta} + \epsilon y\sqrt{-2\beta} - y\gamma)], \\ B_4 &= t^2y'^2 + 2t[ty - \epsilon\sqrt{-2\beta}(y-1)]y' + t^2y^2 - 2\epsilon\sqrt{-2\beta}t(y-1)y \\ &\quad - 2(y-1)^2(y^2\alpha + \beta). \end{aligned} \tag{1.27}$$

$$\alpha_4 = \alpha, \qquad \beta_4 = -\frac{1}{2}\Big(\epsilon\sqrt{-2\beta} - 1\Big)^2, \qquad \gamma_4 = \gamma + 1.$$

Transformation V: If we use transformed quantities under the transformation matrix $R_{(5)}(z,t)$

$v_5 = -\frac{1}{2ty(2w+\theta_1)^2}[(-2w + 2wy - \theta_1 - y\theta_1)(-4vw^2 + 4tvwy + 4v^2wy + 4vw^2y - 4v^2wy^2 - 4w^2\theta_0 + 4vwy\theta_0 - 4vw\theta_1 + 2tvy\theta_1 + 2v^2y\theta_1 + 2v^2y^2\theta_1 - 4w\theta_0\theta_1 + 2vy\theta_0\theta_1 - v\theta_1^2 - vy\theta_1^2 - \theta_0\theta_1^2)]$

$u_5 = [2tuy(-2w + 2wy - \theta_1 - y\theta_1)]/(-4w^2 + 4twy + 4vwy + 4w^2y - 4vwy^2 - 4w\theta_1 + 2ty\theta_1 + 2vy\theta_1 + 2vy^2\theta_1 - \theta_1^2 - y\theta_1^2)$

$y_5 = -(-4w^2 + 4twy + 4vwy + 4w^2y - 4vwy^2 - 4w\theta_1 + 2ty\theta_1 + 2vy\theta_1 + 2vy^2\theta_1 - \theta_1^2 - y\theta_1^2)/[(-2w + 2vy - \theta_1)(-2w + 2wy - \theta_1 - y\theta_1)]$

and (0.5) then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$T_5 : y_5 = 1 - \frac{A_5}{B_5}. \tag{1.28}$$

$$\begin{aligned} A_5 &= 2ty[t(y-y') + (y-1)(y + \epsilon\sqrt{-2\beta} - \epsilon y\sqrt{-2\beta} - y\gamma)], \\ B_5 &= t^2y'^2 - 2t[ty + \epsilon\sqrt{-2\beta}(y-1)]y' + t^2y^2 + 2\epsilon\sqrt{-2\beta}t(y-1)y \\ &\quad - 2(y-1)^2(y^2\alpha + \beta). \end{aligned} \tag{1.29}$$

$$\alpha_5 = \alpha, \qquad \beta_5 = -\frac{1}{2}\Big(\epsilon\sqrt{-2\beta} - 1\Big)^2, \qquad \gamma_5 = \gamma - 1.$$

Transformation VI: If we use transformed quantities under the transformation matrix $R_{(6)}(z,t)$

$v_6 = -\frac{(v - vy + \theta_0)}{2ty^2}(-2v + 2ty + 2vy + 2wy - 2wy^2 - 2\theta_0 + 2y\theta_0 - y\theta_1 + y^2\theta_1)$

$u_6 = -\frac{u}{2t(-1+y)}(-2v + 2ty + 2vy + 2wy - 2wy^2 - 2\theta_0 + 2y\theta_0 - y\theta_1 + y^2\theta_1)$

$y_6 = -[(y-1)(2v^2 - 2tvy - 4v^2y - 2vwy - 2ty^2 + 2v^2y^2 + 2twy^2 + 4vwy^2 - 2vwy^3 + 4v\theta_0 - 2ty\theta_0 - 6vy\theta_0 - 2wy\theta_0 + 2vy^2\theta_0 + 2wy^2\theta_0 + 2\theta_0^2 - 2y\theta_0^2 + vy\theta_1 + ty^2\theta_1 - 2vy^2\theta_1 + vy^3\theta_1 + y\theta_0\theta_1 - y^2\theta_0\theta_1)]/[(-v + ty + 2vy - vy^2 - \theta_0 + y\theta_0)(-2v + 2ty + 2vy + 2wy - 2wy^2 - 2\theta_0 + 2y\theta_0 - y\theta_1 + y^2\theta_1)]$

and (0.5) then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$T_6 : y_6 = 1 - \frac{A_6}{B_6}. \tag{1.30}$$

$$\begin{aligned} A_6 &= 2ty[t(y + y') + (y-1)(-y + \epsilon\sqrt{-2\beta} - \epsilon y\sqrt{-2\beta} - y\gamma)], \\ B_6 &= t^2y'^2 + 2t[ty + \epsilon\sqrt{-2\beta}(y-1)]y' + t^2y^2 + 2\epsilon\sqrt{-2\beta}t(y-1)y \\ &\quad - 2(y-1)^2(y^2\alpha + \beta). \end{aligned} \tag{1.31}$$

$$\alpha_6 = \alpha, \qquad \beta_6 = -\frac{1}{2}\Big(\epsilon\sqrt{-2\beta} + 1\Big)^2, \qquad \gamma_6 = \gamma + 1.$$

Transformation VII: If we use transformed quantities under the transformation matrix $R_{(7)}(z,t)$

$v_7 = -[(2vw - 2vwy + 2w\theta_0 + v\theta_1 + vy\theta_1 + \theta_0\theta_1)(-4vw + 4twy + 4vwy + 4w^2y - 4w^2y^2 - 4w\theta_0 + 4wy\theta_0 - 2v\theta_1 - 2ty\theta_1 - 2vy\theta_1 + 4wy^2\theta_1 - 2\theta_0\theta_1 - 2y\theta_0\theta_1 - y\theta_1^2 - y^2\theta_1^2)]/[2ty^2(2w - \theta_1)^2]$

$u_7 = -\frac{u}{2t}(-4vw + 4twy + 4vwy + 4w^2y - 4w^2y^2 - 4w\theta_0 + 4wy\theta_0 - 2v\theta_1 - 2ty\theta_1 - 2vy\theta_1 + 4wy^2\theta_1 - 2\theta_0\theta_1 - 2y\theta_0\theta_1 - y\theta_1^2 - y^2\theta_1^2)/(-2w + 2wy - \theta_1 - y\theta_1)$

$y_7 = [(-2v + 2vy - 2\theta_0 + y\theta_0 - y\theta_1 + y\theta_\infty)(-2v + 2vy - \theta_0 + y\theta_0 - \theta_1 - y\theta_1 - \theta_\infty + y\theta_\infty)]/(4v^2 - 4tvy - 8v^2y + 4v^2y^2 + 6v\theta_0 - 2ty\theta_0 - 10vy\theta_0 + 4vy^2\theta_0 + 2\theta_0^2 - 3y\theta_0^2 + y^2\theta_0^2 + 2v\theta_1 + 2ty\theta_1 + 2vy\theta_1 - 4vy^2\theta_1 + 2\theta_0\theta_1 + 2y\theta_0\theta_1 - 2y^2\theta_0\theta_1 + y\theta_1^2 + y^2\theta_1^2 + 2v\theta_\infty - 2ty\theta_\infty - 6vy\theta_\infty + 4vy^2\theta_\infty + 2\theta_0\theta_\infty - 4y\theta_0\theta_\infty + 2y^2\theta_0\theta_\infty - 2y^2\theta_1\theta_\infty - y\theta_\infty^2 + y^2\theta_\infty^2)$

and (0.5) then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$T_7 : y_7 = 1 - \frac{A_7}{B_7}. \tag{1.32}$$

$$
\begin{aligned}
A_7 &= 2ty[t(y-y') + (y-1)(1-\epsilon\sqrt{2\alpha} + \epsilon y\sqrt{2\alpha} - \gamma)], \\
B_7 &= -t^2y'^2 + 2\epsilon\sqrt{2\alpha}t(y-1)yy' + t^2y^2 - 2(y-1)^2(y^2\alpha+\beta) \\
&\quad - 2t(y-1)y(-1+\epsilon\sqrt{2\alpha}+\gamma).
\end{aligned} \tag{1.33}
$$

$$
\alpha_7 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2, \qquad \beta_7 = \beta, \qquad \gamma_7 = \gamma - 1.
$$

Transformation VIII: If we use transformed quantities under the transformation matrix $R_{(8)}(z,t)$

$v_8 = -\frac{y-1}{2ty}(-2vw+2tvy+2v^2y+2vwy-2v^2y^2-2w\theta_0+2vy\theta_0-v\theta_1+vy\theta_1-\theta_0\theta_1)$

$u_8 = \frac{2tu(y-1)y}{-2w+2ty+2vy+2wy-2vy^2-\theta_1+y\theta_1}$

$y_8 = -[(-v+ty+2vy-vy^2-\theta_0+y\theta_0)(-2w+2ty+2vy+2wy-2vy^2-\theta_1+y\theta_1)]/[(y-1)(2vw-2ty-2v^2y-2twy-4vwy+2tvy^2+4v^2y^2+2vwy^2-2v^2y^3+2w\theta_0-2vy\theta_0-2wy\theta_0+2vy^2\theta_0+v\theta_1+ty\theta_1-2vy\theta_1+vy^2\theta_1+\theta_0\theta_1-y\theta_0\theta_1)]$

and (0.5) then the following Bäcklund transformation for the fifth Painlevé equation can be obtained,

$$
T_8 : y_8 = 1 - \frac{A_8}{B_8}. \tag{1.34}
$$

$$
\begin{aligned}
A_8 &= 2ty[t(y+y') + (y-1)(-1-\epsilon\sqrt{2\alpha} + \epsilon y\sqrt{2\alpha} - \gamma)], \\
B_8 &= -t^2y'^2 - 2\epsilon\sqrt{2\alpha}t(y-1)yy' + t^2y^2 - 2(y-1)^2(y^2\alpha+\beta) \\
&\quad - 2t(y-1)y(1+\epsilon\sqrt{2\alpha}+\gamma).
\end{aligned} \tag{1.35}
$$

$$
\alpha_8 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2, \qquad \beta_8 = \beta, \qquad \gamma_8 = \gamma + 1.
$$

There are known Bäcklund transformations [15] in the compact form as follows:

$$
\tilde{y}_i(t, \tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, -\frac{1}{2}) = 1 - \frac{2kty}{F_1} \tag{1.36}
$$

where $F_1(t) = ty' - cy^2 + (c-a+kt)y + a \neq 0$, $c^2 = 2\alpha$, $a^2 = -2\beta$ and $k^2 = 1$.

and the parameters change

$$
\tilde{\alpha}_i = \frac{1}{8}\Big[\gamma + k(1-a-c)\Big]^2
$$

$$\begin{aligned}\tilde{\beta}_i &= -\frac{1}{8}\Big[\gamma - k(1-a-c)\Big]^2 \\ \tilde{\gamma}_i &= k(a-c)\end{aligned} \tag{1.37}$$

By considering all the possibilities of k, a and c we get the following transformations:

If $k = 1,\ a = \sqrt{-2\beta},\ c = \sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_1: \qquad \tilde{y_1} = 1 - \frac{2ty}{ty' - \sqrt{2\alpha}y^2 + (\sqrt{2\alpha} - \sqrt{-2\beta} + t)y + \sqrt{-2\beta}}. \tag{1.38}$$

$$\begin{aligned}\tilde{\alpha_1} &= \frac{1}{8}\Big(\gamma + 1 - \sqrt{-2\beta} - \sqrt{2\alpha}\Big)^2 \\ \tilde{\beta_1} &= -\frac{1}{8}\Big(\gamma - 1 + \sqrt{-2\beta} + \sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma_1} &= \sqrt{-2\beta} - \sqrt{2\alpha}\end{aligned} \tag{1.39}$$

If $k = 1,\ a = -\sqrt{-2\beta},\ c = -\sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_2: \qquad \tilde{y_2} = 1 - \frac{2ty}{ty' + \sqrt{2\alpha}y^2 - (\sqrt{2\alpha} - \sqrt{-2\beta} - t)y - \sqrt{-2\beta}}. \tag{1.40}$$

$$\begin{aligned}\tilde{\alpha_2} &= \frac{1}{8}\Big(\gamma + 1 + \sqrt{-2\beta} + \sqrt{2\alpha}\Big)^2 \\ \tilde{\beta_2} &= -\frac{1}{8}\Big(\gamma - 1 - \sqrt{-2\beta} - \sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma_2} &= -\sqrt{-2\beta} + \sqrt{2\alpha}\end{aligned} \tag{1.41}$$

If $k = 1,\ a = -\sqrt{-2\beta},\ c = \sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_3: \qquad \tilde{y_3} = 1 - \frac{2ty}{ty' - \sqrt{2\alpha}y^2 + (\sqrt{2\alpha} + \sqrt{-2\beta} + t)y - \sqrt{-2\beta}}. \tag{1.42}$$

$$\begin{aligned}\tilde{\alpha_3} &= \frac{1}{8}\Big(\gamma + 1 + \sqrt{-2\beta} - \sqrt{2\alpha}\Big)^2 \\ \tilde{\beta_3} &= -\frac{1}{8}\Big(\gamma - 1 - \sqrt{-2\beta} + \sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma_3} &= -\sqrt{-2\beta} - \sqrt{2\alpha}\end{aligned} \tag{1.43}$$

If $k=1,\ a=\sqrt{-2\beta},\ c=-\sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_4: \quad \tilde{y}_4 = 1-\frac{2ty}{ty'+\sqrt{2\alpha}y^2+(-\sqrt{2\alpha}-\sqrt{-2\beta}+t)y+\sqrt{-2\beta}}. \tag{1.44}$$

$$\begin{aligned} \tilde{\alpha}_4 &= \frac{1}{8}\Big(\gamma+1-\sqrt{-2\beta}+\sqrt{2\alpha}\Big)^2 \\ \tilde{\beta}_4 &= -\frac{1}{8}\Big(\gamma-1+\sqrt{-2\beta}-\sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma}_4 &= \sqrt{-2\beta}+\sqrt{2\alpha} \end{aligned} \tag{1.45}$$

If $k=-1,\ a=-\sqrt{-2\beta},\ c=-\sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_5: \quad \tilde{y}_5 = 1+\frac{2ty}{ty'+\sqrt{2\alpha}y^2-(\sqrt{2\alpha}-\sqrt{-2\beta}+t)y-\sqrt{-2\beta}}. \tag{1.46}$$

$$\begin{aligned} \tilde{\alpha}_5 &= \frac{1}{8}\Big(\gamma-1-\sqrt{-2\beta}-\sqrt{2\alpha}\Big)^2 \\ \tilde{\beta}_5 &= -\frac{1}{8}\Big(\gamma+1+\sqrt{-2\beta}+\sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma}_5 &= \sqrt{-2\beta}-\sqrt{2\alpha} \end{aligned} \tag{1.47}$$

If $k=-1,\ a=\sqrt{-2\beta},\ c=\sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_6: \quad \tilde{y}_6 = 1+\frac{2ty}{ty'-\sqrt{2\alpha}y^2+(\sqrt{2\alpha}-\sqrt{-2\beta}-t)y+\sqrt{-2\beta}}. \tag{1.48}$$

$$\begin{aligned} \tilde{\alpha}_6 &= \frac{1}{8}\Big(\gamma-1+\sqrt{-2\beta}+\sqrt{2\alpha}\Big)^2 \\ \tilde{\beta}_6 &= -\frac{1}{8}\Big(\gamma+1-\sqrt{-2\beta}-\sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma}_6 &= -\sqrt{-2\beta}+\sqrt{2\alpha} \end{aligned} \tag{1.49}$$

If $k=-1,\ a=-\sqrt{-2\beta},\ c=\sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_7: \quad \tilde{y}_7 = 1+\frac{2ty}{ty'-\sqrt{2\alpha}y^2+(\sqrt{2\alpha}+\sqrt{-2\beta}-t)y-\sqrt{-2\beta}}. \tag{1.50}$$

$$\begin{aligned} \tilde{\alpha}_7 &= \frac{1}{8}\Big(\gamma-1-\sqrt{-2\beta}+\sqrt{2\alpha}\Big)^2 \\ \tilde{\beta}_7 &= -\frac{1}{8}\Big(\gamma+1+\sqrt{-2\beta}-\sqrt{2\alpha}\Big)^2 \\ \tilde{\gamma}_7 &= \sqrt{-2\beta}+\sqrt{2\alpha} \end{aligned} \tag{1.51}$$

If $k = -1,\ a = \sqrt{-2\beta},\ c = -\sqrt{2\alpha}$ in eq. (1.36) then we obtain,

$$\tilde{T}_8: \qquad \tilde{y_8} = 1 + \frac{2ty}{ty' + \sqrt{2\alpha}y^2 - (\sqrt{2\alpha} + \sqrt{-2\beta} + t)y + \sqrt{-2\beta}}. \tag{1.52}$$

$$\begin{aligned} \tilde{\alpha_8} &= \frac{1}{8}\left(\gamma - 1 + \sqrt{-2\beta} - \sqrt{2\alpha}\right)^2 \\ \tilde{\beta_8} &= -\frac{1}{8}\left(\gamma + 1 - \sqrt{-2\beta} + \sqrt{2\alpha}\right)^2 \\ \tilde{\gamma_8} &= -\sqrt{-2\beta} - \sqrt{2\alpha} \end{aligned} \tag{1.53}$$

5.2 Discrete Equations

The Bäcklund transformations that are inverse of each other, i.e. $T_i \circ T_j = I$, $i = 1, 3, 5, 7$ *and* $j = i + 1$ and $\tilde{T}_k \circ \tilde{T}_l = I$, $k = 1, 2, 3, 4$ *and* $l = k + 4$ will be used to get the discrete equations.

Discrete Equation from T_1 and T_2 :

T_1 and T_2 are the inverse of each other if $\alpha > \frac{1}{2}$. Setting $a_{n+1} = \sqrt{2\alpha_1}$, $a_n = \sqrt{2\alpha}$, $a_{n-1} = \sqrt{2\alpha_2}$, $b_{n+1} = \beta_1$, $b_n = \beta$, $b_{n-1} = \beta_2$, $c_{n+1} = \gamma_1$, $c_n = \gamma$ and $c_{n-1} = \gamma_2$ in the parameter relations yield the difference equations

$$\begin{aligned} \pm a_{n+1} &= \pm a_n + 1, \qquad & \pm a_{n-1} &= \pm a_n - 1, \\ b_{n+1} &= b_n, & b_{n-1} &= b_n, \\ c_{n+1} &= c_n - 1, & c_{n-1} &= c_n + 1, \end{aligned} \tag{2.54}$$

which have the solutions $a_n = \kappa \pm n$, $b_n = \mu$ and $c_n = \psi - n$, where κ , μ and ψ are arbitrary constants. Hence, setting

$$x_{n+1} = y_1, \qquad x_n = y, \qquad x_{n-1} = y_2, \tag{2.55}$$

and eliminating y' from (1.20) and (1.22) yields the discrete equation;

$$\begin{aligned}
&\{x_n^2[t(x_{n-1}+x_{n+1}-2)+2a_n(x_n-1)(1-x_{n-1}-x_{n+1}+x_{n-1}x_{n+1})]^2\\
&-t^2x_n^2(x_{n-1}^2-2x_{n-1}^2x_{n+1}+x_{n+1}^2-2x_{n-1}x_{n+1}^2+2x_{n-1}^2x_{n+1}^2)\\
&+4b_n(x_{n-1}-1)^2(x_n-1)^2(x_{n+1}-1)^2-2t(x_{n-1}-1)(x_n-1)x_n(x_{n+1}-1)\\
&\langle x_{n-1}-x_{n+1}+(c_n-a_n)(x_{n-1}+x_{n+1}-2x_{n-1}x_{n+1})\rangle\}^2\\
&-4(x_{n-1}-1)^2(x_{n+1}-1)^2\{t^2x_{n-1}^2x_n^2-2(x_{n-1}-1)(x_n-1)\\
&\langle b_n(1-x_{n-1}-x_n)+x_{n-1}x_n(t-ta_n+b_n+tc_n)\rangle\}\{t^2x_n^2x_{n+1}^2\\
&-2(x_n-1)(x_{n+1}+1)[b_n(1-x_n-x_{n+1})-x_nx_{n+1}(t+ta_n-b_n-tc_n)]\}=0.
\end{aligned} \tag{2.56}$$

Discrete Equation from T_3 and T_4 :

T_3 and T_4 are the inverse of each other if $\beta < -\frac{1}{2}$. After eliminating y' from (1.24) and (1.26) , and setting

$$\begin{aligned}
x_{n+1}&=y_3, & x_n&=y, & x_{n-1}&=y_4,\\
a_{n+1}&=\alpha_3 & a_n&=\alpha, & a_{n-1}&=\alpha_4,\\
b_{n+1}&=\sqrt{-2\beta_3}, & b_n&=\sqrt{-2\beta}, & b_{n-1}&=\sqrt{-2\beta_4},\\
c_{n+1}&=\gamma_3, & c_n&=\gamma, & c_{n-1}&=\gamma_4,
\end{aligned} \tag{2.57}$$

we obtain the following discrete equation,

$$\begin{aligned}
&\{[tx_n(x_{n-1}+x_{n+1}-2x_{n-1}x_{n+1})+2b_n(x_n-1)(1-x_{n-1}-x_{n+1}+x_{n-1}x_{n+1})]^2\\
&-x_n^2[t^2(2-2x_{n-1}+x_{n-1}^2-2x_{n+1}+x_{n+1}^2)-2t(x_{n-1}-1)(x_n-1)(x_{n+1}-1)\\
&(x_{n-1}-x_{n+1}+(b_n-c_n)(-2+x_{n-1}+x_{n+1}))+4a_n(x_{n-1}-1)^2(x_n-1)^2\\
&(x_{n+1}-1)^2]\}^2-4x_n^4(x_{n-1}-1)^2(x_{n+1}-1)^2\\
&\{t^2+2(x_{n-1}-1)(x_n-1)[t+a_n-tb_n+tc_n-a_n(x_{n-1}+x_n-x_{n-1}x_n)]\}\\
&\{t^2-2(x_n-1)(x_{n+1}-1)[t-a_n+tb_n-tc_n+a_n(x_n+x_{n+1}-x_nx_{n+1})]\}=0,
\end{aligned} \tag{2.58}$$

where $a_n=\kappa$, $b_n=\mu\pm n$ and $c_n=\psi-n$ are the solutions of the difference equations

$$\begin{aligned}
a_{n+1}&=a_n, & a_{n-1}&=a_n,\\
\pm b_{n+1}&=\pm b_n+1, & \pm\, b_{n-1}&=\pm b_n-1,\\
c_{n+1}&=c_n-1, & c_{n-1}&=c_n+1,
\end{aligned} \tag{2.59}$$

κ , μ and ψ being an arbitrary constants.

Discrete Equation from T_5 and T_6 :

T_5 and T_6 are the inverse of each other if $\beta < -\frac{1}{2}$. After eliminating y' from (1.28) and (1.30) , and setting

$$\begin{array}{lll} x_{n+1} = y_5, & x_n = y, & x_{n-1} = y_6, \\ a_{n+1} = \alpha_5, & a_n = \alpha, & a_{n-1} = \alpha_6, \\ b_{n+1} = \sqrt{-2\beta_5}, & b_n = \sqrt{-2\beta}, & b_{n-1} = \sqrt{-2\beta_6}, \\ c_{n+1} = \gamma_5, & c_n = \gamma, & c_{n-1} = \gamma_6, \end{array} \tag{2.60}$$

we obtain the following discrete equation,

$$\begin{aligned} &\{[2b_n(x_n-1)(1-x_{n-1}-x_{n+1}+x_{n-1}x_{n+1})+tx_n(-x_{n-1}-x_{n+1}+2x_{n-1}x_{n+1})]^2 \\ &-x_n^2[t^2(2-2x_{n-1}+x_{n-1}^2-2x_{n+1}+x_{n+1}^2)+2t(x_{n-1}-1)(x_n-1)(x_{n+1}-1) \\ &(x_{n+1}-x_{n-1}+(b_n+c_n)(x_{n-1}+x_{n+1}-2))+4a_n(x_{n-1}-1)^2(x_n-1)^2 \\ &(x_{n+1}-1)^2]\}^2-4(x_{n-1}-1)^2x_n^4(x_{n+1}-1)^2 \\ &\{t^2+2(x_{n-1}-1)(x_n-1)[t+a_n+tb_n+tc_n-a_n(x_{n-1}+x_n-x_{n-1}x_n)]\} \\ &\{t^2-2(x_n-1)(x_{n+1}-1)[t-a_n-tb_n-tc_n+a_n(x_n+x_{1+n}-x_nx_{1+n})]\}=0, \end{aligned} \tag{2.61}$$

where $a_n = \kappa$, $b_n = \mu \mp n$ and $c_n = \psi - n$ are the solutions of the difference equations

$$\begin{array}{ll} a_{n+1} = a_n, & a_{n-1} = a_n, \\ \pm b_{n+1} = \pm b_n - 1, & \pm\, b_{n-1} = \pm b_n + 1, \\ c_{n+1} = c_n - 1, & c_{n-1} = c_n + 1, \end{array} \tag{2.62}$$

κ , μ and ψ being an arbitrary constants.

Discrete Equation from T_7 and T_8 :

T_7 and T_8 are the inverse of each other if $\alpha > \frac{1}{2}$. After eliminating y' from (1.32) and (1.34) , and setting

$$\begin{array}{lll} x_{n+1} = y_7, & x_n = y, & x_{n-1} = y_8, \\ a_{n+1} = \sqrt{2\alpha_7}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_8}, \\ b_{n+1} = \beta_7, & b_n = \beta, & b_{n-1} = \beta_8, \\ c_{n+1} = \gamma_7, & c_n = \gamma, & c_{n-1} = \gamma_8, \end{array} \tag{2.63}$$

we obtain the following discrete equation,

$$
\begin{aligned}
&\{x_n^2[t(-2+x_{n-1}+x_{n+1})+2a_n(x_n-1)(-1+x_{n-1}+x_{n+1}-x_{n-1}x_{n+1})]^2\\
&-t^2x_n^2(x_{n-1}^2-2x_{n-1}^2x_{n+1}+x_{n+1}^2-2x_{n-1}x_{n+1}^2+2x_{n-1}^2x_{n+1}^2)+2tx_n(x_{n-1}-1)\\
&(x_n-1)(x_{n+1}-1)[x_{n+1}-x_{n-1}+(a_n+c_n)(2x_{n-1}x_{n+1}-x_{n-1}-x_{n+1})]\\
&+4b_n(x_{n-1}-1)^2(x_n-1)^2(x_{n+1}-1)^2\}^2\\
&-4(x_{n-1}-1)^2(x_{n+1}-1)^2\{t^2x_{n-1}^2x_n^2-2(x_{n-1}-1)(x_n-1)[b_n(1-x_{n-1}-x_n)\\
&+x_{n-1}x_n(t+ta_n+b_n+tc_n)]\}\{t^2x_n^2x_{n+1}^2-2(x_n-1)(x_{n+1}-1)[b_n(1-x_n-x_{n+1})\\
&-x_nx_{n+1}(t-ta_n-b_n-tc_n)]\}=0,
\end{aligned}
\tag{2.64}
$$

where $a_n = \kappa \mp n$, $b_n = \mu$ and $c_n = \psi - n$ are the solutions of the difference equations

$$
\begin{aligned}
\pm a_{n+1} &= \pm a_n - 1, & \pm a_{n-1} &= \pm a_n + 1,\\
b_{n+1} &= b_n, & b_{n-1} &= b_n,\\
c_{n+1} &= c_n - 1, & c_{n-1} &= c_n + 1,
\end{aligned}
\tag{2.65}
$$

κ , μ and ψ being an arbitrary constants.

Discrete Equation from $\tilde{T}_1$ and $\tilde{T}_5$: After eliminating y' from (1.38) and (1.46) , and setting

$$
\begin{aligned}
x_{n+1} &= \tilde{y_1}, & x_n &= y, & x_{n-1} &= \tilde{y_5},\\
a_{n+1} &= \sqrt{2\tilde{\alpha_1}}, & a_n &= \sqrt{2\alpha}, & a_{n-1} &= \sqrt{2\tilde{\alpha_5}},\\
b_{n+1} &= \sqrt{-2\tilde{\beta}_1}, & b_n &= \sqrt{-2\beta}, & b_{n-1} &= \sqrt{-2\tilde{\beta}_5},
\end{aligned}
\tag{2.66}
$$

we obtain the following discrete equation,

$$
x_n + \frac{x_n}{x_{n-1}-1} + \frac{x_n}{x_{n+1}-1} - \frac{(x_n-1)(b_n+a_nx_n)}{t} = 0,
\tag{2.67}
$$

where $a_n = \frac{n}{2} + \mu$ and $b_n = -\frac{n}{2} + \nu$ are the solutions of the difference equations

$$
\begin{aligned}
a_{n+1} &= \tfrac{1}{2}(\gamma + 1 - b_n - a_n), & a_{n-1} &= \tfrac{1}{2}(\gamma - 1 - b_n - a_n),\\
b_{n+1} &= \tfrac{1}{2}(\gamma - 1 + b_n + a_n), & b_{n-1} &= \tfrac{1}{2}(\gamma + 1 + b_n + a_n),
\end{aligned}
\tag{2.68}
$$

μ and ν being an arbitrary constants.

Discrete Equation from $\tilde{T_2}$ and $\tilde{T_6}$: After eliminating y' from (1.40) and (1.48) , and setting

$$\begin{array}{lll} x_{n+1} = \tilde{y_2}, & x_n = y, & x_{n-1} = \tilde{y_6}, \\ a_{n+1} = \sqrt{2\tilde{\alpha_2}}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\tilde{\alpha_6}}, \\ b_{n+1} = \sqrt{-2\tilde{\beta_2}}, & b_n = \sqrt{-2\beta}, & b_{n-1} = \sqrt{-2\tilde{\beta_6}}, \end{array} \tag{2.69}$$

we obtain the following discrete equation,

$$x_n + \frac{x_n}{x_{n-1}-1} + \frac{x_n}{x_{n+1}-1} + \frac{(x_n-1)(b_n+a_nx_n)}{t} = 0, \tag{2.70}$$

where $a_n = \frac{n}{2} + \phi$ and $b_n = -\frac{n}{2} + \varphi$ are the solutions of the difference equations

$$\begin{array}{ll} a_{n+1} = \frac{1}{2}(\gamma + 1 + b_n + a_n), & a_{n-1} = \frac{1}{2}(\gamma - 1 + b_n + a_n), \\ b_{n+1} = \frac{1}{2}(\gamma - 1 - b_n - a_n), & b_{n-1} = \frac{1}{2}(\gamma + 1 - b_n - a_n), \end{array} \tag{2.71}$$

ϕ and φ being an arbitrary constants.

Discrete Equation from $\tilde{T_3}$, $\tilde{T_7}$ and $\tilde{T_4}$,$\tilde{T_8}$:

After eliminating y' between (1.42), (1.50)and between (1.44), (1.52), and setting

$$x_{n+1} = \tilde{y_3}, \qquad x_n = y, \qquad x_{n-1} = \tilde{y_7}, \tag{2.72}$$

$$x_{n+1} = \tilde{y_4}, \qquad x_n = y, \qquad x_{n-1} = \tilde{y_8}, \tag{2.73}$$

we obtain the following discrete equation

$$2x_n\left(1 + \frac{1}{x_{n-1}-1} + \frac{1}{x_{n+1}-1}\right) = 0. \tag{2.74}$$

Chapter 6

The Sixth Painlevé Equation

The sixth Painlevé equation

$$\begin{aligned}\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt} \\ +\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right),\end{aligned} \tag{0.1}$$

can be obtained as the compatibility condition of the following linear system of equations [22]

$$Y_z(z,t) = A(z,t)Y(z,t), \qquad Y_t(z,t) = B(z,t)Y(z,t), \tag{0.2}$$

where

$$\begin{aligned}
&A(z,t) = \frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t} = \begin{pmatrix} a_{11}(z,t) & a_{12}(z,t) \\ a_{21}(z,t) & a_{22}(z,t)\end{pmatrix}, \\
&A_0 = \begin{pmatrix} u_0+\theta_0 & -w_0u_0 \\ {w_0}^{-1}(u_0+\theta_0) & -u_0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} u_1+\theta_1 & -w_1u_1 \\ {w_1}^{-1}(u_1+\theta_1) & -u_1 \end{pmatrix}, \\
&A_t = \begin{pmatrix} u_t+\theta_t & -w_tu_t \\ {w_t}^{-1}(u_t+\theta_t) & -u_t \end{pmatrix}, \qquad B(z,t) = -A_t\frac{1}{z-t}.
\end{aligned} \tag{0.3}$$

Schlesinger transformations and the closed form of the Bäcklund transformations derived by Mugan and Sakka [31]. Setting,

$$A_\infty = -(A_0 + A_1 + A_t) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

$$\begin{aligned}
&\kappa_1 + \kappa_2 = -(\theta_0 + \theta_1 + \theta_t), \qquad \kappa_1 - \kappa_2 = \theta_\infty, \\
&a_{12}(z,t) = -\frac{w_0 u_0}{z} - \frac{w_1 u_1}{z-1} - \frac{w_t u_t}{z-t} = \frac{k(z-y)}{z(z-1)(z-t)}, \\
&u = a_{11}(y) = \frac{u_0 + \theta_0}{y} + \frac{u_1 + \theta_1}{y-1} + \frac{u_t + \theta_t}{y-t}, \\
&\bar{u} = -a_{22}(y) = u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t}{y-t}.
\end{aligned} \tag{0.4}$$

Then

$$\begin{aligned}
&u_0 + u_1 + u_t = \kappa_2, \qquad w_0 u_0 + w_1 u_1 + w_t u_t = 0, \\
&\frac{u_0 + \theta_0}{w_0} + \frac{u_1 + \theta_1}{w_1} + \frac{u_t + \theta_t}{w_t} = 0, \\
&(t+1) w_0 u_0 + t w_1 u_1 + w_t u_t = k, \qquad t w_0 u_0 = k(t) y,
\end{aligned} \tag{0.5}$$

which are solved as,

$$\begin{aligned}
w_0 &= \frac{ky}{tu_0}, \qquad w_1 = -\frac{k(y-1)}{u_1(t-1)}, \qquad w_t = \frac{k(y-t)}{t(t-1)u_t}, \\
u_0 &= \frac{y}{t\theta_\infty}\{y(y-1)(y-t)\bar{u}^2 + [\theta_1(y-t) + t\theta_t(y-1) - 2\kappa_2(y-1)(y-t)]\bar{u} \\
&\quad + {\kappa_2}^2(y-t-1) - \kappa_2(\theta_1 + t\theta_t)\}, \\
u_1 &= -\frac{y-1}{(t-1)\theta_\infty}\{y(y-1)(y-t)\bar{u}^2 + [(\theta_1 + \theta_\infty)(y-t) + t\theta_t(y-1) \\
&\quad -2\kappa_2(y-1)(y-t)]\bar{u} + {\kappa_2}^2(y-t) - \kappa_2(\theta_1 + t\theta_t) - \kappa_1\kappa_2\}, \\
u_t &= \frac{y-t}{t(t-1)\theta_\infty}\{y(y-1)(y-t)\bar{u}^2 + [\theta_1(y-t) + t(\theta_t + \theta_\infty)(y-1) - \\
&\quad 2\kappa_2(y-1)(y-t)]\bar{u} + {\kappa_2}^2(y-1) - \kappa_2(\theta_1 + t\theta_t) - t\kappa_1\kappa_2\}.
\end{aligned} \tag{0.6}$$

The equation $Y_{zt} = Y_{tz}$ implies

$$\begin{aligned}
\frac{dy}{dt} &= \frac{y(y-1)(y-t)}{t(t-1)}\left(2u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t - 1}{y-t}\right), \\
\frac{du}{dt} &= \frac{1}{t(t-1)}\{[-3y^2 + 2(1+t)y - t]u^2 \\
&\quad + [(2y-1-t)\theta_0 + (2y-t)\theta_1 + (2y-1)(\theta_t - 1)]u - \kappa_1(\kappa_2 + 1)\}, \\
\frac{1}{k}\frac{dk}{dt} &= (\theta_\infty - 1)\frac{y-t}{t(t-1)}.
\end{aligned} \tag{0.7}$$

Thus y satisfies the sixth Painlevé equation (0.1), with the parameters

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \qquad \beta = -\frac{1}{2}\theta_0^2, \qquad \gamma = \frac{1}{2}\theta_1^2, \qquad \delta = \frac{1}{2}(1 - \theta_t^2). \tag{0.8}$$

6.1 Bäcklund Transformations

Let $R(z,t)$ be the transformation matrix which transforms the solution of the linear problem (0.2) as;

$$Y'(z,t) = R(z,t)Y(z,t), \tag{1.9}$$

but leaves the monodromy data associated with $Y(z)$ the same. Let u_i', w_i', $\theta_i' = \theta_i + \lambda_i$ be the transformed quantities of u_i, w_i, θ_i, $i = 0, 1, t, \infty$. The consistency condition of the monodromy data is invariant under the transformation if $\lambda_1 + \lambda_0 = k$, $\lambda_1 - \lambda_0 = l$,
$\lambda_\infty + \lambda_t = m$, $\lambda_\infty - \lambda_t = n$, where k, l, m, n are either odd or even integers. It is enough to consider the following three cases;

$$a: \begin{cases} \theta_0' = \theta_0 + \lambda_0 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty + \lambda_\infty, \end{cases} \qquad b: \begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + \lambda_1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty + \lambda_\infty, \end{cases} \qquad c: \begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t + \lambda_t \\ \theta_\infty' = \theta_\infty + \lambda_\infty, \end{cases} \tag{1.10}$$

All possible Schlesinger transformations admitted by the linear problem (0.2) may be generated by the following transformation matrices :

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty + 1, \end{cases} \qquad R_{(1)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 1 & -w_0 \\ -r_1 & w_0 r_1 \end{pmatrix}, \tag{1.11}$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty - 1, \end{cases} \qquad R_{(2)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{u_0+\theta_0}{u_0 w_0} r_2 & -r_2 \\ -\frac{u_0+\theta_0}{u_0 w_0} & 1 \end{pmatrix} \frac{1}{z}, \tag{1.12}$$

$$\begin{cases} \theta_0' = \theta_0 - 1 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty + 1, \end{cases} \qquad R_{(3)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{u_0 w_0}{u_0+\theta_0} \\ -r_1 & \frac{u_0 w_0}{u_0+\theta_0} r_1 \end{pmatrix} \frac{1}{z}, \tag{1.13}$$

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty - 1, \end{cases} \qquad R_{(4)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} \frac{r_2}{w_0} & -r_2 \\ -\frac{1}{w_0} & 1 \end{pmatrix}, \tag{1.14}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + 1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty + 1, \end{cases} \qquad R_{(5)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (z-1) + \begin{pmatrix} 1 & -w_1 \\ -r_1 & w_1 r_1 \end{pmatrix}, \tag{1.15}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 - 1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty - 1, \end{cases} \qquad R_{(6)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{u_1+\theta_1}{u_0 w_0} r_2 & -r_2 \\ -\frac{u_1+\theta_1}{u_1 w_1} & 1 \end{pmatrix} \frac{1}{z-1}, \tag{1.16}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 - 1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty + 1, \end{cases} \qquad R_{(7)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{u_1 w_1}{u_1+\theta_1} \\ -r_1 & \frac{u_1 w_1}{u_1+\theta_1} r_1 \end{pmatrix} \frac{1}{z-1}, \tag{1.17}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 + 1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty - 1, \end{cases} \qquad R_{(8)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(z-1) + \begin{pmatrix} \frac{r_2}{w_1} & -r_2 \\ -\frac{1}{w_1} & 1 \end{pmatrix}, \tag{1.18}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t + 1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases} \qquad R_{(9)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(z-t) + \begin{pmatrix} 1 & -w_t \\ -r_1 & w_t r_1 \end{pmatrix}, \tag{1.19}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t - 1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases} \qquad R_{(10)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{u_t+\theta_t}{u_t w_t} r_2 & -r_2 \\ -\frac{u_t+\theta_t}{u_t w_t} & 1 \end{pmatrix}\frac{1}{z-t}, \tag{1.20}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t - 1 \\ \theta_\infty' = \theta_\infty + 1, \end{cases} \qquad R_{(11)}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{u_t w_t}{u_t+\theta_t} \\ -r_1 & \frac{u_t w_t}{u_t+\theta_t} r_1 \end{pmatrix}\frac{1}{z-t}, \tag{1.21}$$

$$\begin{cases} \theta_0' = \theta_0 \\ \theta_1' = \theta_1 \\ \theta_t' = \theta_t + 1 \\ \theta_\infty' = \theta_\infty - 1, \end{cases} \qquad R_{(12)}(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(z-t) + \begin{pmatrix} \frac{r_2}{w_t} & -r_2 \\ -\frac{1}{w_t} & 1 \end{pmatrix}, \tag{1.22}$$

where

$$r_1 = -\frac{1}{1+\theta_\infty}\left(\frac{u_1+\theta_1}{w_1} + \frac{u_t+\theta_t}{w_t}t\right), \qquad r_2 = \frac{1}{1-\theta_\infty}(w_1u_1 + tw_tu_t), \tag{1.23}$$

and u_i, w_i, $i = 0, 1, t$ are given in (0.6). The transformation matrices $R_{(k)}(z,t)$, $k = 1, 2, ..., 12$ are sufficient to obtain the transformation matrix $R(z,t)$ which shifts the exponents $\theta_0, \theta_1, \theta_t, \theta_\infty$ to $\theta_0', \theta_1', \theta_t', \theta_\infty'$ with any integer differences. If,

$$Y'(z,t;u_0',u_1',u_t',w_0',w_1',w_t',\theta_0',\theta_1,\theta_t',\theta_\infty') = R_{(j)}(z,t;u_0,...,\theta_\infty)Y(z,t;u_0,...,\theta_\infty), \tag{1.24}$$

and

$$Y''(z,t;u_0'',u_1'',u_t'',w_0'',w_1'',w_t'',\theta_0'',\theta_1'',\theta_t'',\theta_\infty'') = R_{(k)}(z,t;u_0',...,\theta_\infty')Y(z,t;u_0',...,\theta_\infty'), \tag{1.25}$$

Then

$$R_{(k)}(z,t;u_0'(u_0,...\theta_\infty),...)R_{(j)}(z,t;u_0,...,\theta_\infty) = I, \tag{1.26}$$

for $k = j+1, \quad j = 1,3,5,7,9,11.$

Transformation I:

$$T_1: \qquad y_1(y,t;\alpha_1,\beta_1,\gamma_1,\delta_1) = \left(\frac{t}{y}\right)\left[1+(t-y)(y-1)\frac{A}{B}\right] \tag{1.27}$$

$$\alpha_1 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2, \quad \gamma_1 = \gamma \ , \quad \beta_1 = -\frac{1}{2}\Big(\epsilon\sqrt{-2\beta}+1\Big)^2, \quad \delta_1 = \delta \ . \tag{1.28}$$

where

$$\begin{aligned} A = {} & 2t(t-1)(1+\epsilon\alpha_0+\epsilon\beta_0)y' + 2(\alpha_0+\beta_0+\epsilon)(\alpha_0 y^2-\beta_0 t) \\ & - y[2(t-1)(\gamma+\delta)+\alpha_0(\alpha_0+t\alpha_0+2t\epsilon)-\beta_0(\beta_0+\beta_0 t+2\epsilon)] \ , \end{aligned} \tag{1.29}$$

$$\begin{aligned} B = {} & t^2(t-1)^2(y')^2 - 2t(t-1)(y-1)(t+\epsilon\alpha_0 t-\epsilon\alpha_0 y)y' + 2(t-1)y \\ & [y\gamma - t(\gamma+\delta-y\delta)] + (t-y)(y-1)[\epsilon\alpha_0^2(1+t-y)y+\epsilon\beta_0 t(2+\epsilon\beta_0) \\ & + 2\epsilon\alpha_0 t(y+\epsilon\beta_0)] \ . \end{aligned} \tag{1.30}$$

where $\alpha_0 = \sqrt{2\alpha}$, $\beta_0 = \sqrt{-2\beta}$, $\gamma_0 = \sqrt{2\gamma}$, $\delta_0 = \sqrt{1-2\delta}$, $y' = \frac{dy}{dt}$ and $\epsilon = \pm 1$ in the transformations.

Transformation II:

$$T_2: \qquad y_2(y,t;\alpha_2,\beta_2,\gamma_2,\delta_2) = \left(\frac{t}{y}\right)\left[1-(t-y)(y-1)\frac{C}{D}\right] , \tag{1.31}$$

$$\alpha_2 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2, \quad \beta_2 = -\frac{1}{2}\Big(\epsilon\sqrt{-2\beta}-1\Big)^2, \quad \gamma_2 = \gamma \ , \quad \delta_2 = \delta \ . \tag{1.32}$$

where

$C = 2(t-1)t(\epsilon\alpha_0+\epsilon\beta_0-1)y' - 2(\alpha_0+\beta_0-\epsilon)(\alpha_0 y^2-\beta_0 t) + y[2(t-1)(\gamma+\delta)+\alpha_0(\alpha_0+t\alpha_0-2t\epsilon)-\beta_0(\beta_0+\beta_0 t-2\epsilon)]$

$D = (t-1)^2t^2(y')^2 - 2(t-1)t(y-1)(t-\epsilon\alpha_0 t+\epsilon\alpha_0 y)y' + 2(t-1)y[y\gamma - t(\gamma + \delta - y\delta)] + (t-y)(y-1)[\epsilon\alpha_0^2(1+t-y)y + \beta_0 t(\beta_0 - 2\epsilon) - 2\alpha_0 t(\epsilon y - \beta_0)]$

Transformation III:

$$T_3: \qquad y_3(y,t;\alpha_3,\beta_3,\gamma_3,\delta_3) = \left(\frac{t}{y}\right)\left[1+(t-y)(y-1)\frac{E}{F}\right] \tag{1.33}$$

$$\alpha_3 = \frac{1}{2}\left(\epsilon\sqrt{2\alpha}+1\right)^2, \quad \beta_3 = -\frac{1}{2}\left(\epsilon\sqrt{-2\beta}-1\right)^2, \quad \gamma_3 = \gamma, \quad \delta_3 = \delta\ . \tag{1.34}$$

where

$E = 2(t-1)t(\epsilon\alpha_0 - \epsilon\beta_0 + 1)y' + 2(\alpha_0 - \beta_0 + \epsilon)(\alpha_0 y^2 + \beta_0 t) - y[2(t-1)(\gamma + \delta) + \alpha_0(\alpha_0 + \alpha_0 t + 2\epsilon t) - \beta_0(\beta_0 + \beta_0 t - 2\epsilon)]$

$F = (t-1)^2t^2(y')^2 - 2(t-1)t(y-1)(t+\epsilon\alpha_0 t - \epsilon\alpha_0 y)y' + 2(t-1)y[y\gamma - t(\gamma + \delta - y\delta)] + (t-y)(y-1)[\epsilon\alpha_0^2 y(1+t-y) + 2\alpha_0 t(\epsilon y - \beta_0) + \beta_0 t(\beta_0 - 2\epsilon)]$

Transformation IV:

$$T_4: \qquad y_4(y,t;\alpha_4,\beta_4,\gamma_4,\delta_4) = \left(\frac{t}{y}\right)\left[1-(t-y)(y-1)\frac{G}{H}\right] \tag{1.35}$$

$$\alpha_4 = \frac{1}{2}\left(\epsilon\sqrt{2\alpha}-1\right)^2, \quad \beta_4 = -\frac{1}{2}\left(\epsilon\sqrt{-2\beta}+1\right)^2, \quad \gamma_4 = \gamma, \quad \delta_4 = \delta \tag{1.36}$$

where

$G = 2(t-1)t(\epsilon\alpha_0 - \epsilon\beta_0 - 1)y' - 2(\alpha_0 - \beta_0 - \epsilon)(\alpha_0 y^2 + \beta_0 t)$
$+ y[2(t-1)(\gamma+\delta) + \alpha_0(\alpha_0 + \alpha_0 t - 2\epsilon t) - \beta_0(\beta_0 + \beta_0 t + 2\epsilon)]$

$H = (t-1)^2t^2(y')^2 + 2(t-1)t(y-1)(\epsilon\alpha_0 t - \epsilon\alpha_0 y - t)y'$
$+ 2(t-1)y[y\gamma - t(\gamma + \delta - y\delta)] + (t-y)(y-1)[\epsilon\alpha_0^2 y(1+t-y) + \beta_0 t(2\epsilon + \beta_0) - 2\alpha_0 t(\epsilon y + \beta_0)]$

Transformation V:

$$T_5: \qquad y_5(y,t;\alpha_5,\beta_5,\gamma_5,\delta_5) = \left(\frac{y-t}{y-1}\right)\left[1+(t-1)y\frac{I}{J}\right] \tag{1.37}$$

$$\alpha_5 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2\ , \quad \beta_5 = \beta\ , \quad \gamma_5 = \frac{1}{2}\Big(\epsilon\sqrt{2\gamma}+1\Big)^2\ , \quad \delta_5 = \delta \tag{1.38}$$

where

$I = 2(t-1)t(\epsilon\alpha_0 + \epsilon\gamma_0 + 1)y' + 2\alpha_0(\alpha_0 + \gamma_0 + \epsilon)y^2 - y[2(\alpha_0+\gamma_0)(\epsilon + \alpha_0 + \gamma_0) + \epsilon t(2\epsilon\delta + 2\alpha_0 + \alpha_0^2 - 2\epsilon\beta - \gamma_0^2)] + t[(\alpha_0+\gamma_0)^2 + 2(\delta + \epsilon\alpha_0 + \epsilon\gamma_0) - 2\beta]$

$J = (t-1)^2t^2(y')^2 - 2(t-1)ty(t + \epsilon\alpha_0 t - \epsilon\alpha_0 y - 1)y' + 2(t-1)t(y-1)y\delta + (t-y)[\epsilon\alpha_0^2 y(t-y-1)(y-1) - 2t(y-1)\beta + 2\epsilon\alpha_0(t-1)y(y-\epsilon\gamma_0-1) - \gamma_0 y(t-1)(2\epsilon+\gamma_0)]$

Transformation VI:

$$T_6: \qquad y_6(y,t;\alpha_6,\beta_6,\gamma_6,\delta_6) = \Big(\frac{y-t}{y-1}\Big)\Big(1 - y\frac{K}{L}\Big) \tag{1.39}$$

$$\alpha_6 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2\ , \quad \beta_6 = \beta\ , \quad \gamma_6 = \frac{1}{2}\Big(\epsilon\sqrt{2\gamma}-1\Big)^2\ , \quad \delta_6 = \delta\ . \tag{1.40}$$

where

$K = 2(t-1)^2t(\epsilon\alpha_0 + \epsilon\gamma_0 - 1)y' + \epsilon\alpha_0^2(t-1)(t-2y)(y-1) + 2\epsilon\alpha_0\{(t-y)(y-1)(1-t+4t\beta_1) + \epsilon\gamma_0[t^2(3-4y) + t(1+y(3y-2)) + y(y-2)]\} + (t-1)\{2t(y-1)(\delta-\beta) + 2\epsilon\gamma_0(t-y) - \epsilon\gamma_0^2[t + (t-2)y]\}$

$L = (t-1)^2t^2(y')^2 + 2(t-1)ty(1 - t + \epsilon\alpha_0 t - \epsilon\alpha_0 y)y' + 2(t-1)t(y-1)y\delta + (t-y)\{\epsilon\alpha_0^2 y(t-y-1)(y-1) - 2t(y-1)\beta - \gamma_0(t-1)y(\gamma_0 - 2\epsilon) + 2\epsilon\alpha_0 y[t + y - ty - 1 + 4\beta_1 t(y-1) + \epsilon\gamma_0(1 + 3t - 4ty)]\}$

Transformation VII:

$$T_7: \qquad y_7(y,t;\alpha_7,\beta_7,\gamma_7,\delta_7) = \Big(\frac{y-t}{y-1}\Big)\Big[1 + y(t-1)\frac{M}{N}\Big] \tag{1.41}$$

$$\alpha_7 = \frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2\ , \quad \beta_7 = \beta\ , \quad \gamma_7 = \frac{1}{2}\Big(\epsilon\sqrt{2\gamma}-1\Big)^2\ , \quad \delta_7 = \delta\ . \tag{1.42}$$

where

$M = 2(t-1)t(\epsilon\gamma_0 - \epsilon\alpha_0 - 1)y' + (y-1)[2\epsilon\alpha_0(t-y) + \epsilon\alpha_0^2(t-2y) + 2t(\delta - \beta)] + 2\epsilon\gamma_0\{t - y + \epsilon\alpha_0[t + y(y-2)]\} - \epsilon\gamma_0^2[t + y(t-2)]$

$N=(t-1)^2t^2(y')^2-2(t-1)ty(t+\epsilon\alpha_0 t-\epsilon\alpha_0 y-1)y'+2(t-1)t(y-1)y\delta+(t-y)[\epsilon\alpha_0^2 y(t-y-1)(y-1)-2t(y-1)\beta-\gamma_0(t-1)y(\gamma_0-2\epsilon)+2\epsilon\alpha_0(t-1)y(y+\epsilon\gamma_0-1)]$

Transformation VIII:

$$T_8: \qquad y_8(y,t;\alpha_8,\beta_8,\gamma_8,\delta_8)=\Big(\frac{y-t}{y-1}\Big)\Big[1-y(t-1)\frac{P}{R}\Big] \tag{1.43}$$

$$\alpha_8=\frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2\ , \quad \beta_8=\beta\ , \quad \gamma_8=\frac{1}{2}\Big(\epsilon\sqrt{2\gamma}+1\Big)^2\ , \quad \delta_8=\delta\ . \tag{1.44}$$

where

$P=2(t-1)t(\epsilon\gamma_0-\epsilon\alpha_0+1)y'+2t(y-1)(\beta-\delta)+\alpha_0(y-1)[2\epsilon(t-y)-(t-2y)\alpha_0]-2\epsilon\gamma_0\{y-t+\epsilon\alpha_0[t+(y-2)y]\}+\epsilon\gamma_0^2[t+(t-2)y]$

$R=(t-1)^2t^2(y')^2+2(t-1)ty(1-t+\epsilon\alpha_0 t-\epsilon\alpha_0 y)y'+2t(y-1)(y\beta-t\beta-y\delta+ty\delta)+(t-y)y[\epsilon\alpha_0^2(t-y-1)(y-1)-\gamma_0(t-1)(2\epsilon+\gamma_0)+2\epsilon\alpha_0(t-1)(1-y+\epsilon\gamma_0)]$

Transformation IX:

$$T_9: \qquad y_9(y,t;\alpha_9,\beta_9,\gamma_9,\delta_9)=t^2\Big(\frac{y-1}{y-t}\Big)\Big[1-(t-1)y\frac{S}{T}\Big] \tag{1.45}$$

$$\alpha_9=\frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2\ , \quad \beta_9=\beta\ , \quad \gamma_9=\gamma\ , \quad \delta_9=\frac{1}{2}-\frac{1}{2}\Big(\epsilon\sqrt{1-2\delta}+1\Big)^2 \tag{1.46}$$

where

$S=2(t-1)t(\epsilon\alpha_0+\epsilon\delta_0+1)y'+2\alpha_0 y^2(\epsilon+\alpha_0+\delta_0)+y[(1+\epsilon\delta_0)^2-2t(\alpha_0+\delta_0)(2\epsilon+\alpha_0+\delta_0)-2(t-\gamma)-\epsilon\alpha_0^2+2\beta]+t[1-2\beta+2(\epsilon\alpha_0-\gamma+\epsilon\delta_0)+(\alpha_0+\delta_0)^2]$

$T=(t-1)^2t^2(y')^2+2\epsilon\alpha_0(t-1)ty(y-1)y'+\epsilon\alpha_0^2y^3(y-2)-y^2\{(t-1)[t(\alpha_0+\delta_0)^2-2(\gamma-\epsilon\alpha_0 t-\epsilon\delta_0 t)+t]-\epsilon\alpha_0^2-2t\beta\}+ty\{(t-1)[(\alpha_0+\delta_0)^2-2(\gamma-\epsilon\alpha_0-\epsilon\delta_0)+1]-2(t+1)\beta\}+2t^2\beta$

Transformation X:

$$T_{10}: \qquad y_{10}(y,t;\alpha_{10},\beta_{10},\gamma_{10},\delta_{10})=t\Big(\frac{y-1}{y-t}\Big)\Big[1+(t-1)y\frac{U}{V}\Big] \tag{1.47}$$

$$\alpha_{10}=\frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2\ ,\ \ \beta_{10}=\beta\ ,\ \ \gamma_{10}=\gamma\ ,\ \ \delta_{10}=\frac{1}{2}-\frac{1}{2}\Big(\epsilon\sqrt{1-2\delta}-1\Big)^2\ . \tag{1.48}$$

where

$U=2(t-1)t(\epsilon\alpha_0+\epsilon\delta_0-1)y'-2\alpha_0y^2(\alpha_0+\delta_0-\epsilon)+y[2t(\epsilon\alpha_0+\epsilon\delta_0-1)^2-(\epsilon\delta_0-1)^2-2(\beta+\gamma)+\epsilon\alpha_0^2]+t[2(\beta+\gamma+\epsilon\alpha_0+\epsilon\delta_0)-(\alpha_0+\delta_0)^2-1]$

$V=(t-1)^2t^2(y')^2-2\epsilon\alpha_0(t-1)ty(y-1)y'+\epsilon\alpha_0^2y^3(y-2)$
$-y^2\{(t-1)[t(\alpha_0+\delta_0)^2-2(\gamma+t\epsilon\alpha_0+t\epsilon\delta_0)+t]-\epsilon\alpha_0^2-2t\beta\}+ty\{(t-1)[(\alpha_0+\delta_0)^2-2(\gamma+\epsilon\alpha_0+\epsilon\delta_0)+1]-2(1+t)\beta\}+2t^2\beta$

Transformation XI:

$$T_{11}:\qquad y_{11}(y,t;\alpha_{11},\beta_{11},\gamma_{11},\delta_{11})=t\Big(\frac{y-1}{y-t}\Big)\Big[1-(t-1)y\frac{Z}{X}\Big] \tag{1.49}$$

$$\alpha_{11}=\frac{1}{2}\Big(\epsilon\sqrt{2\alpha}+1\Big)^2\ ,\ \ \beta_{11}=\beta\ ,\ \ \gamma_{11}=\gamma\ ,\ \ \delta_{11}=\frac{1}{2}-\frac{1}{2}\Big(\epsilon\sqrt{1-2\delta}-1\Big)^2 \tag{1.50}$$

where

$Z=2(t-1)t(\epsilon\alpha_0-\epsilon\delta_0+1)y'+2y^2\alpha_0(\alpha_0-\delta_0+\epsilon)-y[2t(\epsilon\alpha_0-\epsilon\delta_0+1)^2-(\epsilon\delta_0-1)^2-2\gamma-2\beta+\epsilon\alpha_0^2]+t[(\alpha_0-\delta_0)^2-2\beta-2(\gamma-\epsilon\alpha_0+\epsilon\delta_0)+1]$

$X=(t-1)^2t^2(y')^2+2\epsilon\alpha_0(t-1)ty(y-1)y'+\epsilon\alpha_0^2y^3(y-2)-y^2\{(t-1)[t(\alpha_0-\delta_0)^2-2(\gamma-\epsilon\alpha_0t+\epsilon\delta_0t)+t]-\epsilon\alpha_0^2-2t\beta\}+ty\{(t-1)[(\alpha_0-\delta_0)^2-2(\gamma-\epsilon\alpha_0+\epsilon\delta_0)+1]-2(t+1)\beta\}+2t^2\beta$

Transformation XII:

$$T_{12}:\qquad y_{12}(y,t;\alpha_{12},\beta_{12},\gamma_{12},\delta_{12})=t\Big(\frac{y-1}{y-t}\Big)\Big[1+(t-1)y\frac{\zeta}{\eta}\Big] \tag{1.51}$$

$$\alpha_{12}=\frac{1}{2}\Big(\epsilon\sqrt{2\alpha}-1\Big)^2\ ,\ \ \beta_{12}=\beta\ ,\ \ \gamma_{12}=\gamma\ ,\ \ \delta_{12}=\frac{1}{2}-\frac{1}{2}\Big(\epsilon\sqrt{1-2\delta}+1\Big)^2\ . \tag{1.52}$$

where

$\zeta=2(t-1)t(\epsilon\alpha_0-\epsilon\delta_0-1)y'-2\alpha_0y^2(\alpha_0-\delta_0-\epsilon)+y[2t(\epsilon\alpha_0-\epsilon\delta_0-1)^2-(\epsilon\delta_0+1)^2-2(\beta+\gamma)+\epsilon\alpha_0^2]-t[(\alpha_0-\delta_0)^2-2(\beta+\gamma+\epsilon\alpha_0-\epsilon\delta_0)+1]$

$$\eta = (t-1)^2t^2(y')^2 - 2\epsilon\alpha_0(t-1)ty(y-1)y' + \epsilon\alpha_0^2y^3(y-2) - y^2\{(t-1)[t(\alpha_0 - \delta_0)^2 - 2(\gamma + \epsilon\alpha_0 t - \epsilon\delta_0 t) + t] - 2t\beta - \epsilon\alpha_0^2\}$$
$$+ ty\{(t-1)[(\alpha_0 - \delta_0)^2 - 2(\gamma + \epsilon\alpha_0 - \epsilon\delta_0) + 1] - 2(t+1)\beta\} + 2t^2\beta$$

There are sixteen known Bäcklund transformations [15] in the compact form as follows:

$$\tilde{y} = y - \frac{\sigma y(y-1)(y-t)}{t(t-1)y' + y^2(\eta_2 + \eta_3 + \eta_4 - 1) - y(t\eta_2 + t\eta_3 + \eta_2 + \eta_4 - 1) + t\eta_2} \tag{1.53}$$

where ${\eta_1}^2 = 2\alpha$, ${\eta_2}^2 = -2\beta$, ${\eta_3}^2 = 2\gamma$, ${\eta_4}^2 = 1 - 2\delta$ and $\sigma = -1 + \sum_{k=1}^4 \eta_k$.

The parameters change

$$\tilde{\alpha} = \frac{\hat{\eta}_1^2}{2}, \qquad \tilde{\beta} = -\frac{\hat{\eta}_2^2}{2}, \qquad \tilde{\gamma} = \frac{\hat{\eta}_3^2}{2}, \qquad \tilde{\delta} = \frac{1 - \hat{\eta}_4^2}{2} \tag{1.54}$$

where $\quad \hat{\eta}_j = \frac{1}{2} + \eta_j - \frac{1}{2}\sum_{k=1}^4 \eta_k$.

We use the notations a, b, c, d for simplicity, which are $a = \sqrt{2\alpha}$, $b = \sqrt{-2\beta}$, $c = \sqrt{2\gamma}$, $d = \sqrt{1-2\delta}$.

$$\tilde{T}_1: \quad \tilde{y}_1 = y - \frac{(a+b+c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b+c+d-1) - y(tb+tc+b+d-1) + tb} \tag{1.55}$$

$$\begin{aligned} \tilde{\alpha}_1 &= \tfrac{(\frac{1}{2}+a-\frac{1}{2}(a+b+c+d))^2}{2}, & \tilde{\beta}_1 &= -\tfrac{(\frac{1}{2}+b-\frac{1}{2}(a+b+c+d))^2}{2}, \\ \tilde{\gamma}_1 &= \tfrac{(\frac{1}{2}+c-\frac{1}{2}(a+b+c+d))^2}{2}, & \tilde{\delta}_1 &= \tfrac{1-(\frac{1}{2}+d-\frac{1}{2}(a+b+c+d))^2}{2} \end{aligned} \tag{1.56}$$

$$\tilde{T}_2: \quad \tilde{y}_2 = y - \frac{(a+b+c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b+c-d-1) - y(tb+tc+b-d-1) + tb} \tag{1.57}$$

$$\begin{aligned} \tilde{\alpha}_2 &= \frac{(\frac{1}{2}+a-\frac{1}{2}(a+b+c-d))^2}{2}, \qquad & \tilde{\beta}_2 &= -\frac{(\frac{1}{2}+b-\frac{1}{2}(a+b+c-d))^2}{2}, \\ \tilde{\gamma}_2 &= \frac{(\frac{1}{2}+c-\frac{1}{2}(a+b+c-d))^2}{2}, \qquad & \tilde{\delta}_2 &= \frac{1-(\frac{1}{2}-d-\frac{1}{2}(a+b+c-d))^2}{2} \end{aligned} \tag{1.58}$$

$$\tilde{T}_3: \quad \tilde{y}_3 = y - \frac{(a+b-c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b-c+d-1) - y(tb-tc+b+d-1) + tb} \tag{1.59}$$

$$\begin{aligned} \tilde{\alpha}_3 &= \frac{(\frac{1}{2}+a-\frac{1}{2}(a+b-c+d))^2}{2}, \qquad & \tilde{\beta}_3 &= -\frac{(\frac{1}{2}+b-\frac{1}{2}(a+b-c+d))^2}{2}, \\ \tilde{\gamma}_3 &= \frac{(\frac{1}{2}-c-\frac{1}{2}(a+b-c+d))^2}{2}, \qquad & \tilde{\delta}_3 &= \frac{1-(\frac{1}{2}+d-\frac{1}{2}(a+b-c+d))^2}{2} \end{aligned} \tag{1.60}$$

$$\tilde{T}_4: \quad \tilde{y}_4 = y - \frac{(a-b+c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b+c+d-1) - y(-tb+tc-b+d-1) - tb} \tag{1.61}$$

$$\begin{aligned} \tilde{\alpha}_4 &= \frac{(\frac{1}{2}+a-\frac{1}{2}(a-b+c+d))^2}{2}, \qquad & \tilde{\beta}_4 &= -\frac{(\frac{1}{2}-b-\frac{1}{2}(a-b+c+d))^2}{2}, \\ \tilde{\gamma}_4 &= \frac{(\frac{1}{2}+c-\frac{1}{2}(a-b+c+d))^2}{2}, \qquad & \tilde{\delta}_4 &= \frac{1-(\frac{1}{2}+d-\frac{1}{2}(a-b+c+d))^2}{2} \end{aligned} \tag{1.62}$$

$$\tilde{T}_5: \quad \tilde{y}_5 = y - \frac{(a+b-c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b-c-d-1) - y(tb-tc+b-d-1) + tb} \tag{1.63}$$

$$\begin{aligned} \tilde{\alpha}_5 &= \frac{(\frac{1}{2}+a-\frac{1}{2}(a+b-c-d))^2}{2}, \qquad & \tilde{\beta}_5 &= -\frac{(\frac{1}{2}+b-\frac{1}{2}(a+b-c-d))^2}{2}, \\ \tilde{\gamma}_5 &= \frac{(\frac{1}{2}-c-\frac{1}{2}(a+b-c-d))^2}{2}, \qquad & \tilde{\delta}_5 &= \frac{1-(\frac{1}{2}-d-\frac{1}{2}(a+b-c*d))^2}{2} \end{aligned} \tag{1.64}$$

$$\tilde{T}_6 : \quad \tilde{y}_6 = y - \frac{(a-b+c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b+c-d-1) - y(-tb+tc-b-d-1) - tb} \tag{1.65}$$

$$\begin{aligned} \tilde{\alpha}_6 &= \tfrac{(\frac{1}{2}+a-\frac{1}{2}(a-b+c-d))^2}{2}, \qquad & \tilde{\beta}_6 &= -\tfrac{(\frac{1}{2}-b-\frac{1}{2}(a-b+c-d))^2}{2}, \\ \tilde{\gamma}_6 &= \tfrac{(\frac{1}{2}+c-\frac{1}{2}(a-b+c-d))^2}{2}, \qquad & \tilde{\delta}_6 &= \tfrac{1-(\frac{1}{2}-d-\frac{1}{2}(a-b+c-d))^2}{2} \end{aligned} \tag{1.66}$$

$$\tilde{T}_7 : \quad \tilde{y}_7 = y - \frac{(a-b-c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b-c+d-1) - y(-tb-tc-b+d-1) - tb} \tag{1.67}$$

$$\begin{aligned} \tilde{\alpha}_7 &= \tfrac{(\frac{1}{2}+a-\frac{1}{2}(a-b-c+d))^2}{2}, \qquad & \tilde{\beta}_7 &= -\tfrac{(\frac{1}{2}-b-\frac{1}{2}(a-b-c+d))^2}{2}, \\ \tilde{\gamma}_7 &= \tfrac{(\frac{1}{2}-c-\frac{1}{2}(a-b-c+d))^2}{2}, \qquad & \tilde{\delta}_7 &= \tfrac{1-(\frac{1}{2}+d-\frac{1}{2}(a-b-c+d))^2}{2} \end{aligned} \tag{1.68}$$

$$\tilde{T}_8 : \quad \tilde{y}_8 = y - \frac{(a-b-c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b-c-d-1) - y(-tb-tc-b-d-1) - tb} \tag{1.69}$$

$$\begin{aligned} \tilde{\alpha}_8 &= \tfrac{(\frac{1}{2}+a-\frac{1}{2}(a-b-c-d))^2}{2}, \qquad & \tilde{\beta}_8 &= -\tfrac{(\frac{1}{2}-b-\frac{1}{2}(a-b-c-d))^2}{2}, \\ \tilde{\gamma}_8 &= \tfrac{(\frac{1}{2}-c-\frac{1}{2}(a-b-c-d))^2}{2}, \qquad & \tilde{\delta}_8 &= \tfrac{1-(\frac{1}{2}-d-\frac{1}{2}(a-b-c-d))^2}{2} \end{aligned} \tag{1.70}$$

$$\tilde{T}_9 : \quad \tilde{y}_9 = y - \frac{(-a+b+c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b+c+d-1) - y(tb+tc+b+d-1) + tb} \tag{1.71}$$

$$\tilde{\alpha}_9 = \frac{(\frac{1}{2}-a-\frac{1}{2}(-a+b+c+d))^2}{2}, \qquad \tilde{\beta}_9 = -\frac{(\frac{1}{2}+b-\frac{1}{2}(-a+b+c+d))^2}{2},$$
$$\tilde{\gamma}_9 = \frac{(\frac{1}{2}+c-\frac{1}{2}(-a+b+c+d))^2}{2}, \qquad \tilde{\delta}_9 = \frac{1-(\frac{1}{2}+d-\frac{1}{2}(-a+b+c+d))^2}{2} \tag{1.72}$$

$$\tilde{T}_{10}: \quad \tilde{y}_{10} = y - \frac{(-a+b+c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b+c-d-1) - y(tb+tc+b-d-1) + tb} \tag{1.73}$$

$$\tilde{\alpha}_{10} = \frac{(\frac{1}{2}-a-\frac{1}{2}(-a+b+c-d))^2}{2}, \qquad \tilde{\beta}_{10} = -\frac{(\frac{1}{2}+b-\frac{1}{2}(-a+b+c-d))^2}{2},$$
$$\tilde{\gamma}_{10} = \frac{(\frac{1}{2}+c-\frac{1}{2}(-a+b+c-d))^2}{2}, \qquad \tilde{\delta}_{10} = \frac{1-(\frac{1}{2}-d-\frac{1}{2}(-a+b+c-d))^2}{2} \tag{1.74}$$

$$\tilde{T}_{11}: \quad \tilde{y}_{11} = y - \frac{(-a+b-c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b-c+d-1) - y(tb-tc+b+d-1) + tb} \tag{1.75}$$

$$\tilde{\alpha}_{11} = \frac{(\frac{1}{2}-a-\frac{1}{2}(-a+b-c+d))^2}{2}, \qquad \tilde{\beta}_{11} = -\frac{(\frac{1}{2}+b-\frac{1}{2}(-a+b-c+d))^2}{2},$$
$$\tilde{\gamma}_{11} = \frac{(\frac{1}{2}-c-\frac{1}{2}(-a+b-c+d))^2}{2}, \qquad \tilde{\delta}_{11} = \frac{1-(\frac{1}{2}+d-\frac{1}{2}(-a+b-c+d))^2}{2} \tag{1.76}$$

$$\tilde{T}_{12}: \quad \tilde{y}_{12} = y - \frac{(-a-b+c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b+c+d-1) - y(-tb+tc-b+d-1) - tb} \tag{1.77}$$

$$\tilde{\alpha}_{12} = \frac{(\frac{1}{2}-a-\frac{1}{2}(-a-b+c+d))^2}{2}, \qquad \tilde{\beta}_{12} = -\frac{(\frac{1}{2}-b-\frac{1}{2}(-a-b+c+d))^2}{2},$$
$$\tilde{\gamma}_{12} = \frac{(\frac{1}{2}+c-\frac{1}{2}(-a-b+c+d))^2}{2}, \qquad \tilde{\delta}_{12} = \frac{1-(\frac{1}{2}+d-\frac{1}{2}(-a-b+c+d))^2}{2} \tag{1.78}$$

$$\tilde{T}_{13}: \quad \tilde{y}_{13} = y - \frac{(-a+b-c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(b-c-d-1) - y(tb-tc+b-d-1)+tb} \tag{1.79}$$

$$\begin{aligned} \tilde{\alpha}_{13} &= \frac{(\frac{1}{2}-a-\frac{1}{2}(-a+b-c-d))^2}{2}, \qquad \tilde{\beta}_{13} = -\frac{(\frac{1}{2}+b-\frac{1}{2}(-a+b-c-d))^2}{2}, \\ \tilde{\gamma}_{13} &= \frac{(\frac{1}{2}-c-\frac{1}{2}(-a+b-c-d))^2}{2}, \qquad \tilde{\delta}_{13} = \frac{1-(\frac{1}{2}-d-\frac{1}{2}(-a+b-c-d))^2}{2} \end{aligned} \tag{1.80}$$

$$\tilde{T}_{14}: \quad \tilde{y}_{14} = y - \frac{(-a-b+c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b+c-d-1) - y(-tb+tc-b-d-1)-tb} \tag{1.81}$$

$$\begin{aligned} \tilde{\alpha}_{14} &= \frac{(\frac{1}{2}-a-\frac{1}{2}(-a-b+c-d))^2}{2}, \qquad \tilde{\beta}_{14} = -\frac{(\frac{1}{2}-b-\frac{1}{2}(-a-b+c-d))^2}{2}, \\ \tilde{\gamma}_{14} &= \frac{(\frac{1}{2}+c-\frac{1}{2}(-a-b+c-d))^2}{2}, \qquad \tilde{\delta}_{14} = \frac{1-(\frac{1}{2}-d-\frac{1}{2}(-a-b+c-d))^2}{2} \end{aligned} \tag{1.82}$$

$$\tilde{T}_{15}: \quad \tilde{y}_{15} = y - \frac{(-a-b-c+d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b-c+d-1) - y(-tb-tc-b+d-1)-tb} \tag{1.83}$$

$$\begin{aligned} \tilde{\alpha}_{15} &= \frac{(\frac{1}{2}-a-\frac{1}{2}(-a-b-c+d))^2}{2}, \qquad \tilde{\beta}_{15} = -\frac{(\frac{1}{2}-b-\frac{1}{2}(-a-b-c+d))^2}{2}, \\ \tilde{\gamma}_{15} &= \frac{(\frac{1}{2}-c-\frac{1}{2}(-a-b-c+d))^2}{2}, \qquad \tilde{\delta}_{15} = \frac{1-(\frac{1}{2}+d-\frac{1}{2}(-a-b-c+d))^2}{2} \end{aligned} \tag{1.84}$$

$$\tilde{T}_{16}: \quad \tilde{y}_{16} = y - \frac{(-a-b-c-d-1)y(y-1)(y-t)}{t(t-1)y' + y^2(-b-c-d-1) - y(-tb-tc-b-d-1)-tb} \tag{1.85}$$

$$\tilde{\alpha}_{16} = \frac{(\frac{1}{2}-a+\frac{1}{2}(a+b+c+d))^2}{2}, \qquad \tilde{\beta}_{16} = -\frac{(\frac{1}{2}-b+\frac{1}{2}(a+b+c+d))^2}{2},$$
$$\tilde{\gamma}_{16} = \frac{(\frac{1}{2}-c+\frac{1}{2}(a+b+c+d))^2}{2}, \qquad \tilde{\delta}_{16} = \frac{1-(\frac{1}{2}-d+\frac{1}{2}(a+b+c+d))^2}{2} \tag{1.86}$$

6.2 Discrete Equations

The Bäcklund transformations that are inverse of each other, i.e. $T_i \circ T_j = I$, $j = i+1$, $i = 1,3,5,7$ will be used to get the discrete equations.

Discrete Equation from T_1 and T_2 :

T_1 and T_2 are the inverse of each other if $\alpha > \frac{1}{2}$ and $\beta < -\frac{1}{2}$. Setting $a_{n+1} = \sqrt{2\alpha_1}$, $a_n = \sqrt{2\alpha}$, $a_{n-1} = \sqrt{2\alpha_2}$, $b_{n+1} = \sqrt{-2\beta_1}$, $b_n = \sqrt{-2\beta}$, $b_{n-1} = \sqrt{-2\beta_2}$, $c_{n+1} = \gamma_1$, $c_n = \gamma$, $c_{n-1} = \gamma_2$, $d_{n+1} = \delta_1$, $d_n = \delta$, $d_{n-1} = \delta_2$ in the parameter relations yield the difference equations

$$\begin{aligned} \pm a_{n+1} &= \pm a_n + 1, & \pm a_{n-1} &= \pm a_n - 1, \\ \pm b_{n+1} &= \pm b_n + 1, & \pm b_{n-1} &= \pm b_n - 1, \\ c_{n+1} &= c_n, & c_{n-1} &= c_n, \\ d_{n+1} &= d_n, & d_{n-1} &= d_n, \end{aligned} \tag{2.87}$$

which have the solutions $a_n = \kappa \pm n$, $b_n = \mu \pm n$, $c_n = \psi$ and $d_n = \phi$, where κ, μ, ϕ and ψ are arbitrary constants. Hence, setting

$$x_{n+1} = y_1, \qquad x_n = y, \qquad x_{n-1} = y_2 \tag{2.88}$$

and eliminating y' from (1.27) and (1.31) yields the discrete equation;

$$\begin{aligned} &B_1(x_n x_{n+1} - t) - A_1 x_n^2(x_{n-1}x_n - t) - 4x_n(x_n - t)(x_n - 1)[2t^2 b_n \\ &+ t x_{n-1}x_n(a_n - b_n - 2) + t x_n x_{n+1}(a_n - b_n) + 2x_{n-1}x_n^2 x_{n+1}(1 - a_n)] = 0 \end{aligned} \tag{2.89}$$

where,

$$A_1 = 4\{t(a_n + b_n)^2(t - x_n)(x_n - 1)(t - x_{n+1})(x_{n+1} - 1)$$
$$+ (t-1)(t - x_n x_{n+1})[t(tc_n^2 - d_n^2) + t(d_n^2 - c_n^2)(x_n + x_{n+1}) + x_n x_{n+1}(c_n^2 - td_n^2)]\}^{1/2},$$

$$B_1 = 4x_n^2\{t(a_n + b_n - 2)^2(t - x_{n-1})(x_{n-1} - 1)(t - x_n)(x_n - 1)$$
$$+ (t-1)(t - x_{n-1}x_n)[t(tc_n^2 - d_n^2) + t(d_n^2 - c_n^2)(x_{n-1} + x_n) + x_{n-1}x_n(c_n^2 - td_n^2)]\}^{1/2}.$$

Discrete Equation from T_3 and T_4 :

T_3 and T_4 are the inverse of each other if $\alpha > \frac{1}{2}$ and $\beta < -\frac{1}{2}$. After eliminating y' from (1.33) and (1.35) , and setting

$$\begin{array}{lll} x_{n+1} = y_3, & x_n = y, & x_{n-1} = y_4, \\ a_{n+1} = \sqrt{2\alpha_3}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_4}, \\ b_{n+1} = \sqrt{-2\beta_3}, & b_n = \sqrt{-2\beta}, & b_{n-1} = \sqrt{-2\beta_4}, \\ c_{n+1} = \gamma_3, & c_n = \gamma, & c_{n-1} = \gamma_4, \\ d_{n+1} = \delta_3, & d_n = \delta, & d_{n-1} = \delta_4, \end{array} \tag{2.90}$$

we obtain the following discrete equation,

$$A_2(x_{n-1}x_n - t) - B_2 x_n^2(x_n x_{n+1} - t) - 4x_n(x_n - t)(x_n - 1)[2t^2 b_n$$
$$+ t x_{n-1}x_n(2 - a_n - b_n) - t x_n x_{n+1}(a_n + b_n) + 2x_{n-1}x_n^2 x_{n+1}(a_n - 1)] = 0 \tag{2.91}$$

where,

$$A_2 = 4x_n^2\{t(a_n - b_n)^2(t - x_n)(x_n - 1)(t - x_{n+1})(x_{n+1} - 1)$$
$$+ (t-1)(t - x_n x_{n+1})[t(tc_n^2 - d_n^2) + t(d_n^2 - c_n^2)(x_{n+1} + x_n) + x_{n+1}x_n(c_n^2 - td_n^2)]\}^{1/2},$$

$$B_2 = 4\{t(a_n - b_n - 2)^2(t - x_{n-1})(x_{n-1} - 1)(t - x_n)(x_n - 1)$$
$$+ (t-1)(t - x_{n-1}x_n)[t(tc_n^2 - d_n^2) + t(d_n^2 - c_n^2)(x_{n-1} + x_n) + x_{n-1}x_n(c_n^2 - td_n^2)]\}^{1/2}.$$

where $a_n = \kappa \pm n$,$b_n = \mu \mp n$,$c_n = \psi$ and $d_n = \phi$ are the solutions of the difference equations

$$\begin{array}{ll} \pm a_{n+1} = \pm a_n + 1 & \pm a_{n-1} = \pm a_n - 1, \\ \pm b_{n+1} = \pm b_n - 1 & \pm b_{n-1} = \pm b_n + 1, \\ c_{n+1} = c_n & c_{n-1} = c_n, \\ d_{n+1} = d_n & d_{n-1} = d_n. \end{array} \tag{2.92}$$

κ , μ ,ψ and ϕ being arbitrary constants.

Discrete Equation from T_5 and T_6 :

T_5 and T_6 are the inverse of each other if $\alpha > \frac{1}{2}$ and $\gamma > \frac{1}{2}$. After eliminating y' from (1.37) and (1.39) , and setting

$$\begin{array}{lll} x_{n+1} = y_5, & x_n = y, & x_{n-1} = y_6, \\ a_{n+1} = \sqrt{2\alpha_5}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_6}, \\ b_{n+1} = \beta_5, & b_n = \beta, & b_{n-1} = \beta_6, \\ c_{n+1} = \sqrt{2\gamma_5}, & c_n = \sqrt{2\gamma}, & c_{n-1} = \sqrt{2\gamma_6}, \\ d_{n+1} = \delta_5, & d_n = \delta, & d_{n-1} = \delta_6, \end{array} \tag{2.93}$$

we obtain the following discrete equation,

$$\begin{aligned} & A_3[(x_n - t)(x_n - 1)^2 - x_{n-1}(x_n - 1)^3] + B_3[x_n(x_{n+1} - 1) - x_{n+1} + t] \\ & -4x_n(x_n - t)(x_n - 1)\{(a_n + ta_n + c_n - tc_n)[2(x_n - t) \\ & -(x_n - 1)(x_{n-1} + x_{n+1})] + 2(t + 1)x_n(x_{n-1} - 1) - 2a_n x_n(x_{n-1} + x_{n+1}) \\ & +2(t + t^2 a_n - tx_{n-1} - x_{n+1} + 2x_n x_{n+1}) - 2(a_n - 1) \\ & (x_n^2 - x_{n-1}x_n^2 + x_{n-1}x_{n+1} - 2x_{n-1}x_n x_{n+1} - x_n^2 x_{n+1} + x_{n-1}x_n^2 x_{n+1})\} = 0 \end{aligned} \tag{2.94}$$

where,

$$A_3 = 4\{(t-x_n)(t-x_{n+1})[tb_n^2(t-x_n-x_{n+1}+x_n x_{n+1})-a_n^2 x_n x_{n+1}(t-1)]-(t-1) x_n x_{n+1}[2a_n c_n(t-x_n)(t-x_{n+1})+t(c_n^2-d_n^2)(t-x_n-x_{n+1})+x_n x_{n+1}(c_n^2-td_n^2)]\}^{1/2},$$

$$B_3 = 4(x_n - 1)^2\{(t - x_n)[tb_n(t - x_{n-1} - x_n + x_{n-1}x_n)(b_n(t - x_{n-1}) - 8x_{n-1}x_n (a_n-1))-x_{n-1}x_n(t-x_{n-1})(t-1)(a_n-2)^2]+x_{n-1}x_n[(t-1)td_n^2(t-x_{n-1}-x_n+x_{n-1}x_n) +c_n(t-x_n)((t-x_{n-1})(-4-4t+2a_n+6ta_n+c_n-tc_n)+8t(x_{n-1}-1)x_n(a_n-1))]\}^{1/2}.$$

where $a_n = \kappa \pm n$, $b_n = \mu$, $c_n = \psi \pm n$ and $d_n = \phi$ are the solutions of the difference equations

$$\begin{array}{ll} \pm a_{n+1} = \pm a_n + 1 & \pm a_{n-1} = \pm a_n - 1, \\ b_{n+1} = b_n & b_{n-1} = b_n, \\ \pm c_{n+1} = \pm c_n + 1 & \pm c_{n-1} = \pm c_n - 1, \\ d_{n+1} = d_n & d_{n-1} = d_n. \end{array} \tag{2.95}$$

κ , μ ,ψ and ϕ being arbitrary constants.

Discrete Equation from T_7 and T_8 :

T_7 and T_8 are the inverse of each other if $\alpha > \frac{1}{2}$ and $\gamma > \frac{1}{2}$. After eliminating y' from (1.41) and (1.43) , and setting

$$\begin{array}{lll} x_{n+1} = y_7, & x_n = y, & x_{n-1} = y_8, \\ a_{n+1} = \sqrt{2\alpha_7}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_8}, \\ b_{n+1} = \beta_7, & b_n = \beta, & b_{n-1} = \beta_8, \\ c_{n+1} = \sqrt{2\gamma_7}, & c_n = \sqrt{2\gamma}, & c_{n-1} = \sqrt{2\gamma_8}, \\ d_{n+1} = \delta_7, & d_n = \delta, & d_{n-1} = \delta_8, \end{array} \tag{2.96}$$

we obtain the following discrete equation,

$$\begin{aligned} &A_4[x_n(x_{n-1}-1) - x_{n-1} + t] + B_4[(x_n - t)(x_n-1)^2 - x_{n+1}(x_n-1)^3] \\ &+4x_n(x_n-t)(x_n-1)\{(a_n + ta_n - c_n + tc_n)[2(x_n-t) \\ &-(x_n-1)(x_{n-1}+x_{n+1})]2(t+1)x_n(x_{n-1}-1) - 2a_nx_n(x_{n-1}+x_{n+1}) \\ &+2(t+t^2a_n - tx_{n-1} - x_{n+1} + 2x_nx_{n+1}) - 2(a_n-1) \\ &(x_n^2 - x_{n-1}x_n^2 + x_{n-1}x_{n+1} - 2x_{n-1}x_nx_{n+1} - x_n^2x_{n+1} + x_{n-1}x_n^2x_{n+1})\} = 0 \end{aligned} \tag{2.97}$$

where,

$$A_4 = 4(x_n-1)^2\{(t-x_n)(t-x_{n+1})[tb_n^2(t-x_n-x_{n+1}+x_nx_{n+1}) - a_n^2x_nx_{n+1}(t-1)] + (t-1)x_nx_{n+1}[t(c_n^2-d_n^2-2a_nc_n)(x_n+x_{n+1}-t) - x_nx_{n+1}(c_n^2-td_n^2-2a_nc_n)]\}^{1/2},$$

$$B_4 = 4\{(t-x_{n-1})(t-x_n)[tb_n^2(t-x_{n-1}-x_n)+(4-4t-4a_n+ta_n+a_n^2-ta_n^2+tb_n^2) x_{n-1}x_n] - (t-1)x_{n-1}x_n[t(4c_n+c_n^2-d_n^2-2a_nc_n)(t-x_{n-1}-x_n)+(4c_n+c_n^2-td_n^2-2a_nc_n) x_{n-1}x_n]\}^{1/2}.$$

and $a_n = \kappa \pm n$,$b_n = \mu$,$c_n = \psi \mp n$ and $d_n = \phi$ are the solutions of the difference equations

$$\begin{array}{ll} \pm a_{n+1} = \pm a_n + 1 & \pm a_{n-1} = \pm a_n - 1, \\ b_{n+1} = b_n & b_{n-1} = b_n, \\ \pm c_{n+1} = \pm c_n - 1 & \pm c_{n-1} = \pm c_n + 1, \\ d_{n+1} = d_n & d_{n-1} = d_n. \end{array} \tag{2.98}$$

κ , μ ,ψ and ϕ being arbitrary constants.

Discrete Equation from T_9 and T_{10} :

T_9 and T_{10} are the inverse of each other if $\alpha > \frac{1}{2}$ and $\delta < 0$. After eliminating y' from (1.45) and (1.47) , and setting

$$\begin{array}{lll} x_{n+1} = y_9, & x_n = y, & x_{n-1} = y_{10}, \\ a_{n+1} = \sqrt{2\alpha_9}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_{10}}, \\ b_{n+1} = \beta_9, & b_n = \beta, & b_{n-1} = \beta_{10}, \\ c_{n+1} = \gamma_9, & c_n = \gamma, & c_{n-1} = \gamma_{10}, \\ d_{n+1} = \sqrt{1-2\delta_9}, & d_n = \sqrt{1-2\delta}, & d_{n-1} = \sqrt{1-2\delta_{10}}, \end{array} \tag{2.99}$$

we obtain the following discrete equation,

$$\begin{aligned} &B_5(t^2 - t^2x_n - tx_{n+1} + x_nx_{n+1}) - 4A_5[t(x_n-1)(x_n-t)^2 + x_{n-1}(x_n+t)^3] \\ &+4x_n(t-x_n)(x_n-1)\{t(a_n + ta_n - d_n + td_n)[2t^2(x_n-1) \\ &-(x_n-t)(tx_{n-1}+x_{n+1})] - 2t^2(t+1)x_n(t-x_{n-1}) - 2t^2a_nx_n(tx_{n-1}+x_{n+1}) \\ &+2t^2(t^2 + ta_n - tx_{n-1} - tx_{n+1} + 2x_nx_{n+1}) - 2(a_n-1) \\ &(t^3x_n^2 - t^2x_{n-1}x_n^2 + t^2x_{n-1}x_{n+1} - 2tx_{n-1}x_nx_{n+1} - tx_n^2x_{n+1} + x_{n-1}x_n^2x_{n+1})\} = 0 \end{aligned} \tag{2.100}$$

where,

$$\begin{aligned} A_5 = \{&x_nx_{n+1}t(t-1)(a_n+d_n)^2(x_n-1)(x_{n+1}-t) + (t^2x_n - t^2 + tx_{n+1} \\ &- x_nx_{n+1})[tb_n^2(x_n-1)(t-x_{n+1}) + c_n^2x_nx_{n+1}(t-1)]\}^{1/2} \end{aligned}$$

$$\begin{aligned} B_5 = 4(t-x_n)^2\{&x_nx_{n-1}t(t-1)(a_n+d_n-2)^2(x_{n-1}-1)(x_n-1) + (t - tx_{n-1} - tx_n \\ &+ x_{n-1}x_n)[tb_n^2(x_n-1)(x_{n-1}-1) - c_n^2x_{n-1}x_n(t-1)]\}^{1/2} \end{aligned}$$

where $a_n = \kappa \pm n$,$b_n = \mu$,$c_n = \psi$ and $d_n = \phi \pm n$ are the solutions of the difference equations

$$\begin{array}{ll} \pm a_{n+1} = \pm a_n + 1 & \pm a_{n-1} = \pm a_n - 1, \\ b_{n+1} = b_n & b_{n-1} = b_n, \\ c_{n+1} = c_n & c_{n-1} = c_n, \\ \pm d_{n+1} = \pm d_n + 1 & \pm d_{n-1} = \pm d_n - 1. \end{array} \tag{2.101}$$

κ , μ ,ψ and ϕ being arbitrary constants.

Discrete Equation from T_{11} and T_{12} :

T_{11} and T_{12} are the inverse of each other if $\alpha > \frac{1}{2}$ and $\delta < 0$. After eliminating y' from (1.49) and (1.51) , and setting

$$\begin{array}{lll} x_{n+1} = y_{11}, & x_n = y, & x_{n-1} = y_{12}, \\ a_{n+1} = \sqrt{2\alpha_{11}}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_{12}}, \\ b_{n+1} = \beta_{11}, & b_n = \beta, & b_{n-1} = \beta_{12}, \\ c_{n+1} = \gamma_{11}, & c_n = \gamma, & c_{n-1} = \gamma_{12}, \\ d_{n+1} = \sqrt{1-2\delta_{11}}, & d_n = \sqrt{1-2\delta}, & d_{n-1} = \sqrt{1-2\delta_{12}}, \end{array} \tag{2.102}$$

we obtain the following discrete equation,

$$\begin{aligned} &A_6(t - tx_{n-1} - tx_n + x_{n-1}x_n) + B_6[t(x_n - t)^2(x_n - 1) - (x_n - t)^3 x_{n+1}] \\ &+4(t - x_n)(x_n - 1)x_n\{t(a_n + ta_n + d_n - td_n)[(x_n - t)(x_{n-1} + x_{n+1}) - 2t(x_n - 1)] \\ &+2t(t+1)x_n(t - x_{n-1}) + 2t^2 a_n x_n(x_{n-1} + x_{n+1}) \\ &-2t^2(t + a_n - x_{n-1} - tx_{n+1}2x_n x_{n+1}) + 2(a_n - 1) \\ &(t^2x_n^2 - tx_{n-1}x_n^2 + t^2x_{n-1}x_{n+1} - 2tx_{n-1}x_nx_{n+1} - tx_n^2x_{n+1} + x_{n-1}x_n^2x_{n+1})\} = 0 \end{aligned} \tag{2.103}$$

where,

$$A_6 = 4(t - x_n)^2\{x_nx_{n+1}t(t-1)(a_n - d_n)^2(x_n - 1)(x_{n+1} - 1) + (t - tx_n - tx_{n+1} + x_nx_{n+1})(tb_n^2 - tb_n^2x_n - tb_n^2x_{n+1} + tb_n^2x_nx_{n+1} + c_n^2x_nx_{n+1} - tc_n^2x_nx_{n+1})\}^{1/2},$$

$$B_6 = 4\{x_nx_{n-1}t(t-1)(a_n - d_n - 2)^2(x_{n-1} - 1)(x_n - 1) + (t - tx_{n-1} - tx_n + x_{n-1}x_n)(tb_n^2 - tb_n^2x_{n-1} - tb_n^2x_n + tb_n^2x_{n-1}x_n + c_n^2x_{n-1}x_n - tc_n^2x_{n-1}x_n)\}^{1/2}.$$

where $a_n = \kappa \pm n$,$b_n = \mu$,$c_n = \psi$ and $d_n = \phi \mp n$ are the solutions of the difference equations

$$\begin{array}{ll} \pm a_{n+1} = \pm a_n + 1 & \pm a_{n-1} = \pm a_n - 1, \\ b_{n+1} = b_n & b_{n-1} = b_n, \\ c_{n+1} = c_n & c_{n-1} = c_n, \\ \pm d_{n+1} = \pm d_n - 1 & \pm d_{n-1} = \pm d_n + 1. \end{array} \tag{2.104}$$

κ , μ ,ψ and ϕ being arbitrary constants.

Discrete Equation from $\tilde{T}_1$ and $\tilde{T}_{16}$: After eliminating y' from (1.55) and (1.85) , and setting

$$\begin{array}{lll} x_{n+1} = y_1, & x_n = y, & x_{n-1} = y_{16}, \\ a_{n+1} = \sqrt{2\alpha_1}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_{16}}, \\ b_{n+1} = \sqrt{-2\beta_1}, & b_n = \sqrt{-2\beta}, & b_{n-1} = \sqrt{-2\beta_{16}}, \\ c_{n+1} = \sqrt{2\gamma_1}, & c_n = \sqrt{2\gamma}, & c_{n-1} = \sqrt{2\gamma_{16}}, \\ d_{n+1} = \sqrt{1-2\delta_1}, & d_n = \sqrt{1-2\delta}, & d_{n-1} = \sqrt{1-2\delta_{16}}, \end{array} \tag{2.105}$$

we obtain the following discrete equation,

$$\begin{aligned} &\frac{1}{t(t-1)}\Bigg\{\frac{1}{(x_{n-1}-x_n)}\{-b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1) \\ &-c_n(t-x_n)(1-2x_{n-1}+x_n)+(x_n-1)[t-x_n+d_n(t-2x_{n-1}+x_n)]\Big]\} \\ &+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n+b_n+c_n+d_n)(t-x_n)(x_n-1)x_n\Big]\Bigg\}=0 \end{aligned} \tag{2.106}$$

where

$$\begin{aligned} a_n &= \kappa\cos(\frac{n\pi}{2})+\mu\sin(\frac{n\pi}{2})+\frac{1}{2}, & b_n &= \phi\cos(\frac{n\pi}{2})+\varphi\sin(\frac{n\pi}{2})+\frac{1}{2}, \\ c_n &= \lambda\cos(\frac{n\pi}{2})+\tau\sin(\frac{n\pi}{2})+\frac{1}{2}, & d_n &= \chi\cos(\frac{n\pi}{2})+\psi\sin(\frac{n\pi}{2})+\frac{1}{2}, \end{aligned} \tag{2.107}$$

are the solutions of the difference equations

$$\begin{aligned} a_{n+1}+a_{n-1} &= 1, & b_{n+1}+b_{n-1} &= 1, \\ c_{n+1}+c_{n-1} &= 1, & d_{n+1}+d_{n-1} &= 1, \end{aligned} \tag{2.108}$$

κ, λ, μ, τ, ψ, φ, χ and ϕ being arbitrary constants.

Discrete Equation from $\tilde{T}_2$ and $\tilde{T}_{15}$: After eliminating y' from (1.57) and (1.83) , and setting

$$\begin{array}{lll} x_{n+1} = y_2, & x_n = y, & x_{n-1} = y_{15}, \\ a_{n+1} = \sqrt{2\alpha_2}, & a_n = \sqrt{2\alpha}, & a_{n-1} = \sqrt{2\alpha_{15}}, \\ b_{n+1} = \sqrt{-2\beta_2}, & b_n = \sqrt{-2\beta}, & b_{n-1} = \sqrt{-2\beta_{15}}, \\ c_{n+1} = \sqrt{2\gamma_2}, & c_n = \sqrt{2\gamma}, & c_{n-1} = \sqrt{2\gamma_{15}}, \\ d_{n+1} = \sqrt{1-2\delta_2}, & d_n = \sqrt{1-2\delta}, & d_{n-1} = \sqrt{1-2\delta_{15}}, \end{array} \tag{2.109}$$

we obtain the following discrete equation,

$$\frac{1}{t(t-1)}\Big\{\frac{1}{(x_{n-1}-x_n)}\{-b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1)$$
$$-c_n(t-x_n)(1-2x_{n-1}+x_n)-(x_n-1)[-t+x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}$$
$$+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n+b_n+c_n-d_n)(t-x_n)(x_n-1)x_n\Big]\Big\}=0 \tag{2.110}$$

where a_n, b_n, c_n and d_n are the same as (2.107).

Discrete Equation from $\tilde{T}_3$ and $\tilde{T}_{14}$: After eliminating y' from (1.59) and (1.81) , and setting

$$\begin{array}{lll} x_{n+1}=y_3, & x_n=y, & x_{n-1}=y_{14} \\ a_{n+1}=\sqrt{2\alpha_3}, & a_n=\sqrt{2\alpha}, & a_{n-1}=\sqrt{2\alpha_{14}}, \\ b_{n+1}=\sqrt{-2\beta_3}, & b_n=\sqrt{-2\beta}, & b_{n-1}=\sqrt{-2\beta_{14}}, \\ c_{n+1}=\sqrt{2\gamma_3}, & c_n=\sqrt{2\gamma}, & c_{n-1}=\sqrt{2\gamma_{14}}, \\ d_{n+1}=\sqrt{1-2\delta_3}, & d_n=\sqrt{1-2\delta}, & d_{n-1}=\sqrt{1-2\delta_{14}} \end{array} \tag{2.111}$$

we obtain the following discrete equation,

$$\frac{1}{t(t-1)}\Big\{\frac{1}{(x_{n-1}-x_n)}\{-b_n(t-x_n)(x_n-1)(-2x_{n-1}+x_n)+x_n\Big[a_n(t-x_n)(x_n-1)$$
$$+c_n(t-x_n)(1-2x_{n-1}+x_n)+(x_n-1)[t-x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}$$
$$+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n+b_n-c_n+d_n)(t-x_n)(x_n-1)x_n\Big]\Big\}=0 \tag{2.112}$$

where a_n, b_n, c_n and d_n are the same as (2.107).

Discrete Equation from $\tilde{T}_4$ and $\tilde{T}_{13}$: After eliminating y' from (1.61) and (1.79) , and setting

$$\begin{array}{lll} x_{n+1}=y_4, & x_n=y, & x_{n-1}=y_{13}, \\ a_{n+1}=\sqrt{2\alpha_4}, & a_n=\sqrt{2\alpha}, & a_{n-1}=\sqrt{2\alpha_{13}}, \\ b_{n+1}=\sqrt{-2\beta_4}, & b_n=\sqrt{-2\beta}, & b_{n-1}=\sqrt{-2\beta_{13}}, \\ c_{n+1}=\sqrt{2\gamma_4}, & c_n=\sqrt{2\gamma}, & c_{n-1}=\sqrt{2\gamma_{13}}, \\ d_{n+1}=\sqrt{1-2\delta_4}, & d_n=\sqrt{1-2\delta}, & d_{n-1}=\sqrt{1-2\delta_{13}}, \end{array} \tag{2.113}$$

we obtain the following discrete equation,

$$\frac{1}{t(t-1)}\Big\{\frac{1}{(x_{n-1}-x_n)}\{b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1)$$
$$-c_n(t-x_n)(1-2x_{n-1}+x_n)+(x_n-1)[t-x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}$$
$$+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n-b_n+c_n+d_n)(t-x_n)(x_n-1)x_n\Big]\Big\}=0 \tag{2.114}$$

where a_n, b_n, c_n and d_n are the same as (2.107).

Discrete Equation from $\tilde{T}_5$ and $\tilde{T}_{12}$: After eliminating y' from (1.63) and (1.77) , and setting

$$\begin{array}{lll} x_{n+1}=y_5, & x_n=y, & x_{n-1}=y_{12}, \\ a_{n+1}=\sqrt{2\alpha_5}, & a_n=\sqrt{2\alpha}, & a_{n-1}=\sqrt{2\alpha_{12}}, \\ b_{n+1}=\sqrt{-2\beta_5}, & b_n=\sqrt{-2\beta}, & b_{n-1}=\sqrt{-2\beta_{12}}, \\ c_{n+1}=\sqrt{2\gamma_5}, & c_n=\sqrt{2\gamma}, & c_{n-1}=\sqrt{2\gamma_{12}}, \\ d_{n+1}=\sqrt{1-2\delta_5}, & d_n=\sqrt{1-2\delta}, & d_{n-1}=\sqrt{1-2\delta_{12}}, \end{array} \tag{2.115}$$

we obtain the following discrete equation,

$$\frac{1}{t(t-1)}\Big\{\frac{1}{(x_{n-1}-x_n)}\{-b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1)$$
$$+c_n(t-x_n)(1-2x_{n-1}+x_n)-(x_n-1)[-t+x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}$$
$$+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n+b_n-c_n-d_n)(t-x_n)(x_n-1)x_n\Big]\Big\}=0 \tag{2.116}$$

where a_n, b_n, c_n and d_n are the same as (2.107).

Discrete Equation from $\tilde{T}_6$ and $\tilde{T}_{11}$: After eliminating y' from (1.65) and (1.75) , and setting

$$\begin{array}{lll} x_{n+1}=y_6, & x_n=y, & x_{n-1}=y_{11}, \\ a_{n+1}=\sqrt{2\alpha_6}, & a_n=\sqrt{2\alpha}, & a_{n-1}=\sqrt{2\alpha_{11}}, \\ b_{n+1}=\sqrt{-2\beta_6}, & b_n=\sqrt{-2\beta}, & b_{n-1}=\sqrt{-2\beta_{11}}, \\ c_{n+1}=\sqrt{2\gamma_6}, & c_n=\sqrt{2\gamma}, & c_{n-1}=\sqrt{2\gamma_{11}}, \\ d_{n+1}=\sqrt{1-2\delta_6}, & d_n=\sqrt{1-2\delta}, & d_{n-1}=\sqrt{1-2\delta_{11}}, \end{array} \tag{2.117}$$

we obtain the following discrete equation,

$$\begin{aligned}&\frac{1}{t(t-1)}\Bigg\{\frac{1}{(x_{n-1}-x_n)}\{b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1)\\&-c_n(t-x_n)(1-2x_{n-1}+x_n)-(x_n-1)[-t+x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}\\&+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n-b_n+c_n-d_n)(t-x_n)(x_n-1)x_n\Big]\Bigg\}=0\end{aligned} \tag{2.118}$$

where a_n, b_n, c_n and d_n are the same as (2.107).

Discrete Equation from $\tilde{T}_7$ and $\tilde{T}_{10}$: After eliminating y' from (1.67) and (1.73) , and setting

$$\begin{array}{lll} x_{n+1}=y_7, & x_n=y, & x_{n-1}=y_{10},\\ a_{n+1}=\sqrt{2\alpha_7}, & a_n=\sqrt{2\alpha}, & a_{n-1}=\sqrt{2\alpha_{10}},\\ b_{n+1}=\sqrt{-2\beta_7}, & b_n=\sqrt{-2\beta}, & b_{n-1}=\sqrt{-2\beta_{10}},\\ c_{n+1}=\sqrt{2\gamma_7}, & c_n=\sqrt{2\gamma}, & c_{n-1}=\sqrt{2\gamma_{10}},\\ d_{n+1}=\sqrt{1-2\delta_7}, & d_n=\sqrt{1-2\delta}, & d_{n-1}=\sqrt{1-2\delta_{10}}, \end{array} \tag{2.119}$$

we obtain the following discrete equation,

$$\begin{aligned}&\frac{1}{t(t-1)}\Bigg\{\frac{1}{(x_{n-1}-x_n)}\{b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1)\\&+c_n(t-x_n)(1-2x_{n-1}+x_n)+(x_n-1)[t-x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}\\&+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n-b_n-c_n+d_n)(t-x_n)(x_n-1)x_n\Big]\Bigg\}=0\end{aligned} \tag{2.120}$$

where a_n, b_n, c_n and d_n are the same as (2.107).

Discrete Equation from $\tilde{T}_8$ and $\tilde{T}_9$: After eliminating y' from (1.69) and (1.71) , and setting

$$\begin{array}{lll} x_{n+1}=y_8, & x_n=y, & x_{n-1}=y_9,\\ a_{n+1}=\sqrt{2\alpha_8}, & a_n=\sqrt{2\alpha}, & a_{n-1}=\sqrt{2\alpha_9},\\ b_{n+1}=\sqrt{-2\beta_8}, & b_n=\sqrt{-2\beta}, & b_{n-1}=\sqrt{-2\beta_9},\\ c_{n+1}=\sqrt{2\gamma_8}, & c_n=\sqrt{2\gamma}, & c_{n-1}=\sqrt{2\gamma_9},\\ d_{n+1}=\sqrt{1-2\delta_8}, & d_n=\sqrt{1-2\delta}, & d_{n-1}=\sqrt{1-2\delta_9}, \end{array} \tag{2.121}$$

we obtain the following discrete equation,

$$
\begin{aligned}
&\frac{1}{t(t-1)}\Big\{\frac{1}{(x_{n-1}-x_n)}\{b_n(t-x_n)(x_n-1)(x_n-2x_{n-1})+x_n\Big[a_n(t-x_n)(x_n-1)\\
&+c_n(t-x_n)(1-2x_{n-1}+x_n)-(x_n-1)[-t+x_n+d_n(t-2x_{n-1}+x_n)]\Big]\}\\
&+\frac{1}{(x_{n+1}-x_n)}\Big[(-1+a_n-b_n-c_n-d_n)(t-x_n)(x_n-1)x_n\Big]\Big\}=0
\end{aligned}
\tag{2.122}
$$

where a_n, b_n, c_n and d_n are the same as (2.107).

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