PHASE-EQUIVALENT SUPERSYMMETRIC DEEP AND SHALLOW POTENTIALS

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Okan ÖZER
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Approval of the Graduate School of Natural and Applied Science.

Assoc. Prof. Dr Ali Rıza TEKİN

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr Refik KAYALI

Chairman of the Department

I certify that I have read this thesis and that my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr Bülent GÖNÜL

Supervisor

Examining Committee in Charge :

Prof. Dr Ramazan SEVER (Chairman)

Assoc. Prof. Dr Bülent GÖNÜL

Assist. Prof. Dr Ramazan KOÇ

ABSTRACT

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ÖZER Okan
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In this work we review theoretical formulation of the Supersymmetric Quantum Mechanics, which has received a lot of attention in the literature recently, and apply it to the determination of a shallow potential which is phase-equivalent to a deep potential, whose bound spectrum is identical except for the ground state which is suppressed. With this method, the non-physical Pauli forbidden bound states encountered in the spectrum of any potential can be eliminated, preserving the phase shift. The present approach is exact and provides differential relations between the wavefunctions of the two types of potentials.

The method is applied to the α+α scattering, the deuteron and ^{11}Be systems. We show that the deep potentials involving unphysical bound states and its supersymmetric, shallow, phase-equivalent partner excluding Pauli forbidden states give very similar physical quantities, although the corresponding eigenstates differ

in the node-number. This lends support to two nearly equivalent treatments of the Pauli principle. The supersymmetry aspect of nuclear interactions in the light of the calculation results obtained is discussed.

Keywords : Supersymmetric Quantum Mechanics, Phase-Equivalent Potentials.

ÖZET

FAZ-EŞDEĞERLİ SÜPERSİMETRİK DERİN VE SIĞ POTANSİYELLER

ÖZER Okan
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Bu çalışmada biz, son zamanlarda çok fazla dikkat çeken, Supersimetrik Kuantum Mekaniğinin teorik formülasyonunu gözden geçirdik ve bu metodu, düşürülmüş taban durumu haricinde spektrumu derin bir potansiyelinkine denk, faz- eşdeğerli sığ bir potansiyelin elde edilmesinde kullandık.Söz konusu teknikle, herhangi bir potansiyelin spektrumunda karşılaşılan fiziksel olmayan Pauli yasaklanmış bağlı durumlar, faz kayması korunarak, elenebilir. Sunulan yaklaşım kesindir ve iki potansiyelin dalga fonksiyonları arasında diferansiyel ilişki sağlamaktadır.

Süpersimetrik metod α+α saçılması, döteron ve ^{11}Be sistemlerine uygulandı. Karşılık gelen dalga fonksiyonlarındaki düğüm sayılarının farklı olmasına rağmen; fiziksel olmayan bağlı durumlar içeren derin potansiyeller ve

onun süpersimetrik, sığ, Pauli yasaklanmış durumları düşürülmüş faz-eşdeğerli eşinin çok benzer fiziksel nicelikler verdiğini gösterdik. Bu, Pauli prensibinin hemen hemen iki denk davranışını göstermektedir. Bulunan sonuçların ışığı altında nükleer etkileşimlerin süpersimetri görünüşü tartışıldı.

Anahtar Kelimeler : Süpersimetrik Kuantum Mekaniği, Faz-Eşdeğerli Potansiyeller.

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CHAPTER 1

INTRODUCTION

Physicists have long striven to obtain a unified description of all basic interactions of nature, i.e. strong, electroweak, and gravitational interactions. Several ambitious attempts have been made in the last two decades, and it is now widely felt that supersymmetry (SUSY) is a necessary ingredient in any unifying approach [1]. SUSY relates bosonic and fermionic degrees of freedom. The algebra involved in SUSY is a graded Lie algebra which closes under a combination of commutation and anti-commutation relations.

Despite the beauty of the unifying properties of SUSY theories, there has so far been no experimental evidence of SUSY being realized in nature. In other words, it is a fact that bosons and fermions are clearly distinguishable and SUSY is apparently broken in our present environment. One of the important predictions of unbroken SUSY theories is the existence of SUSY partners of quarks, leptons and gauge bosons which have the same masses as their SUSY counterparts. The fact that no such particles however have been seen implies that SUSY must be spontaneously broken.

Nevertheless, the SUSY idea has led to new insights in the studies of nuclear physics, condensed matter physics, statistical physics and mathematical physics [2]. In particular, supersymmetric quantum mechanics (SUSYQM), which originally introduced by Nicolai in 1976 [3] and re-discovered in 1981 by Witten [4], nowadays attracts much attention [5].

It has been shown by Andrianow et al. [6] and Sukumar [7] independently that all one-dimensional quantum systems have supersymmetric partners. This implies the existence of a hierarchy of Hamiltonians with a special relationship between the eigenvalues and eigenfunctions of the different members of the hierarchy. The Gelfand-Levitan procedure [8] underlying the relation between scattering data and potentials has been re-examined by Sukumar [9] within the framework of SUSYQM and shown that supersymmetric quantum mechanics links two Hamiltonians with the same scattering (phase equivalence) but different number of bound states.

Phase-equivalent potentials (PEP) have been discussed by many authors in a variety of contexts [10]. The approaches such as Gelfand-Levitan procedure and the others to construct PEP are technically much more complicated than the supersymmetric approach described in this work. Indeed, the advent of SUSYQM has produced a revival of interest in the determination of PEP [5]. In connection with this development, SUSYQM has then been applied to the determination of a shallow potential which is phase-equivalent to a deep potential, and whose bound spectrum is identical except for the ground state which is suppressed.

The deep or shallow nature of nucleus-nucleus potentials has been a controversial question for a long time [11]. How can so different potentials, with so different bound spectra, share the same phase shifts? This problem may seem academic if both types of potentials fit accurately the same set of existing experimental data, which is one of the subjects of the present study. Using the supersymmetric quantum mechanical methods we first will show that, throughout the work described in this thesis, the Pauli forbidden bound states encountered in the spectrum of theoretically constructed deep potentials can be eliminated. We will then make clear that the resulting shallow potentials present for small r values an r^{-2} singularity in accord with generalized Levinson theorem for the treatment of the Pauli principle. These phenomenological singular potentials derived within the framework of SUSYQM are to be shown as phase-equivalent to deep potentials via the application of the model to different systems in nuclear physics.

The long-standing dichotomy of choosing a shallow or deep effective local potential to describe nucleus-nucleus elastic scattering is to greatly clarified by our detailed investigation of $\alpha+\alpha$ scattering problem by SUSYQM.

For the analysis of elastic scattering, such drastic differences observed in the character of the potentials are immaterial since only phase shifts are required. However, when one has to choose one of these potentials to be used in nuclear structure studies in which wave functions are explicitly involved, the implication of a deep or shallow potential may be immense. At present, there is an increasing general interest in supersymmetric potentials, not only in the field of nuclear physics. To understand further the wave function-sensitive properties of these two different type partner potentials, we study the weakly bound deuteron and ^{11}Be nuclei, by the phase-equivalent deep and shallow potential description for the treatment of ground state wave functions of such nuclei. As the supersymmetric procedure produces two sets of wave functions for weakly bound systems, which coincide at large distances but differ at small distances by the additional node appearing inside the core by use of the deep potential, we calculate the matter radii of the deuteron and ^{11}Be, as a physical observable, to see the consequence of using different wave functions/potentials. These additional calculations will lead us to the conclusion on the supersymmetry aspect of nuclear interactions.

In the next section, we begin with a brief presentation of the mathematical language of the SUSYQM, together with the factorization of the radial Schrödinger equation and generation of the phase-equivalent potentials. We apply the supersymmetric quantum mechanics to $\alpha+\alpha$ scattering in Section 3 to make a connection between PEP. Sections 4 and 5 discusses the application methods to the deuteron system and ^{11}Be nucleus, respectively. The analysis of the results obtained in these sections will focus on to which extent observables are sensitive to the nodal structure of the wave functions. In addition, the connection between the exclusion of deep-lying Pauli forbidden bound states from a deep potential and supersymmetry is to be discussed in Section 4. Finally, Section 6 contains a summary and the conclusion.

CHAPTER 2

SUPERSYMMETRIC QUANTUM MECHANICS

Since our argument is framed in the language of SUSYQM, we begin by reviewing it briefly. One of the key ingredients in solving exactly for the spectrum of one dimensional potential problems is the connection between the bound state wavefunction and the potential. Let us choose the ground state energy for the moment to be zero. Then one has from the Schrödinger radial equation that the ground state wave function, for the l th partial wave, $\Psi_0^l(r)$ obeys

$$H_0^l\Psi_0^l = \left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_0^l\right)\Psi_0^l = 0 \quad , \tag{1}$$

where μ is the reduced mass. For simplicity, the orbital angular momentum superscript l appearing in (1) will be dropped through out the present work. So that

$$V_0(r) = \frac{\hbar^2}{2\mu}\frac{\ddot{\Psi}_0(r)}{\Psi_0(r)} \quad . \tag{2}$$

where $\ddot{\Psi}_0(r)$ is the second derivative of the wave function with respect to r. This allows a global reconstruction of the potential $V_0(r)$ from a knowledge of its ground state wave function which has no nodes (we will discuss the excited wave

functions later in this section). Once we realize this, it is now very simple to factorize the Hamiltonian using the following ansatz :

$$H_0 = A_0^+ A_0^- = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_0(r) \quad , \tag{3}$$

where the first-order differential operators A_0^+ and A_0^- are defined by [4]

$$A_0^\pm = \pm\frac{\hbar}{\sqrt{2\mu}}\frac{d}{dr} + W_0(r) \quad . \tag{4}$$

This allows us to identify

$$V_0(r) = W_0{}^2(r) + \frac{\hbar}{\sqrt{2\mu}}\dot{W}_0(r) \quad . \tag{5}$$

Here, $\dot{W}_0(r)$ is the first derivative of $W_0(r)$ and this equation is the well-known Riccati equation. The quantity $W_0(r)$ is generally referred to as the "Superpotential" in SUSYQM literature. The solution for $W_0(r)$ in terms of the ground state wavefunction is

$$\mathrm{W_0(r)} = \frac{\hbar}{\sqrt{2\mu}}\frac{\dot{\Psi}_0(r)}{\Psi_0(r)} \quad . \tag{6}$$

This solution is obtained by recognizing that once we satisfy $A_0^-\Psi_0 = 0$, we automatically have a solution: $H_0\Psi_0 = A_0^+ A_0^-\Psi_0 = 0$.

The next step in constructing the SUSY theory related to the original Hamiltonian H_0 is to define a partner operator

$$H_1 = A_0^- A_0^+ = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_1(r) \quad ,$$

$$V_1(r) = W_0^{\,2}(r) - \frac{\hbar}{\sqrt{2\mu}}\dot{W}_0(r) \quad . \tag{7}$$

The potentials $V_0(r)$ and $V_1(r)$ known as supersymmetric partner potentials. The energy eigenvalues and the wavefunction of H_0 and H_1 are related. To that end notice that the energy eigenvalues of both H_0 and H_1 are positive ($E_m^{(n)} \geq 0$, $m = 0,1$, where m denotes m th Hamiltonian while n refers to energy levels). For $n > 0$, the Schrödinger equation for H_0

$$H_0 \Psi_0^{(n)} = A_0^+ A_0^- \Psi_0^{(n)} = E_0^{(n)} \Psi_0^{(n)} \quad . \tag{8}$$

Multiplying Eq.(8) from the left by A_0^- one gets,

$$A_0^- A_0^+ A_0^- \Psi_0^{(n)} = H_1 (A_0^- \Psi_0^{(n)}) = E_0^{(n)} (A_0^- \Psi_0^{(n)}) \quad . \tag{9}$$

Similarly the Schrödinger equation for H_1

$$H_1 \Psi_1^{(n)} = A_0^- A_0^+ \Psi_1^{(n)} = E_1^{(n)} \Psi_1^{(n)} \quad , \tag{10}$$

multiplying Eq.(10) from the left by A_0^+,

$$A_0^+ A_0^- A_0^+ \Psi_1^{(n)} = H_0 (A_0^+ \Psi_1^{(n)}) = E_1^{(n)} (A_0^+ \Psi_1^{(n)}) \quad . \tag{11}$$

From Eqs.(8-11) and the fact that $E_0^{(0)} = 0$, it is clear that the eigenvalues and eigenfunctions of the two Hamiltonians H_0 and H_1 are related by ($n = 0,1,2,...$)

$$E_1^{(n)} = E_0^{(n+1)} \quad , \quad E_0^{(0)} = 0 \quad ,$$

$$\Psi_1^{(n)} = \left[E_0^{(n+1)}\right]^{-1/2} A_0^- \Psi_0^{(n+1)} \quad , \tag{12}$$

$$\Psi_0^{(n+1)} = \left[E_1^{(n)}\right]^{-1/2} A_0^+ \Psi_1^{(n)} \quad .$$

Notice that if $\Psi_0^{(n+1)}$ ($\Psi_1^{(n)}$) of H_0 (H_1) is normalized then the wavefunction $\Psi_1^{(n)}$ ($\Psi_0^{(n+1)}$) in Eq.(12) is also normalized. Further, the operator A_0^- (A_0^+) not only converts an eigenfunction of H_0 (H_1) into an eigenfunction of H_1 (H_0) with the same energy, but it also destroys (creates) an extra node in the eigenfunction. Since the ground state wavefunction of H_0 is annihilated by the operator A_0^-, this state has no SUSY partner. Thus the picture we get is that knowing all the eigenfunctions of H_0 we can determine the eigenfunctions of H_1 using the operator A_0^-, and vice versa.

The underlying reason for the degeneracy of the spectra of H_0 and H_1 can be understood most easily from the properties of the SUSY algebra. That is one can consider a matrix SUSY Hamiltonian of the form

$$H = \begin{bmatrix} H_0 & 0 \\ 0 & H_1 \end{bmatrix} \quad , \tag{13}$$

which contains partner Hamiltonians. This matrix Hamiltonian is part of a closed algebra which contains both bosonic and fermionic operators with commutation and anti-commutation relations. We consider the operators

$$Q^- = \begin{bmatrix} 0 & 0 \\ A_0^- & 0 \end{bmatrix} \quad , \qquad Q^+ = \begin{bmatrix} 0 & A_0^+ \\ 0 & 0 \end{bmatrix} \quad , \tag{14}$$

in connection with H. The following commutation and anti-commutation relations then describe the closed super algebra

$$\left[H,Q^{-}\right]=\left[H,Q^{+}\right]=0 \quad ,$$

$$\left\{Q^{-},Q^{+}\right\}=H \quad , \quad \left\{Q^{-},Q^{-}\right\}=\left\{Q^{+},Q^{+}\right\}=0 . \qquad (15)$$

The fact that the supercharges Q^{-} and Q^{+} commute with H is responsible for the degeneracy. The operators Q^{-} and Q^{+} can be interpreted as operators which change bosonic degrees of freedom into fermionic ones and vice versa.

2.1 Factorization And The Hierarchy Of Hamiltonians

In the previous section we have seen that once we know the ground state wavefunction corresponding to a Hamiltonian H_0, we can find the superpotential $W_0(r)$ from Eq.(6). The resulting operators $A_0^{\pm}$ including $W_0(r)$, can be used to factorize Hamiltonian H_0. We also know that the ground state wavefunction of the partner Hamiltonian H_1 is determined from the first excited state of H_0 via the application of the operator A_0^{-}. This allows a re-factorization of the second Hamiltonian in terms of $W_1(r)$. The partner of this re-factorization is now another Hamiltonian H_2. Each of the new Hamiltonians has one fewer bound state, so that this process can be continued until the number of unphysical bound states is exhausted. Thus if one has an exactly solvable potential problem for H_0, one can solve for the energy eigenvalues and wavefunctions for the entire hierarchy of Hamiltonians created by repeated re-factorizations. Conversely, if we know then ground state wavefunctions for all the Hamiltonians in this hierarchy, we can reconstruct the solutions of the original problem.

Let us now be more specific. From the last section we have seen that if the ground state energy of a Hamiltonian H_0 is zero then it can always be written in a factorizable form as a product of a pair of linear differential operators. It is then clear that if the ground state energy of a Hamiltonian H_0 is $E_0^{(0)}$ with eigenfunction $\Psi_0^{(0)}$ then in view of Eq. (3), it can always be written in the form (where for clarity the units are chosen in such a way that $\hbar^2/2\mu=1$):

$$H_0 = A_0^+ A_0^- + E_0^{(0)} = -\frac{d^2}{dr^2} + V_0(r) \quad , \tag{16}$$

The SUSY partner Hamiltonian is then given by

$$H_1 = A_0^- A_0^+ + E_0^{(0)} = -\frac{d^2}{dr^2} + V_1 \quad , \tag{17}$$

where

$$V_1 = W_0^2(r) - \dot{W}_0(r) + E_0^{(0)} = V_0 - 2\dot{W}_0(r) \quad ,$$

$$= V_0 - 2\frac{d^2}{dr^2}\ln\Psi_0^{(0)}(r) \quad . \tag{18}$$

From the previous section, the energy eigenvalues and eigenfunctions of the two Hamiltonians H_0 and H_1 are related by

$$E_0^{(n+1)} = E_1^{(n)} \quad , \quad \Psi_1^{(n)} = \left[E_0^{(n+1)} - E_0^{(0)}\right]^{-1/2} A_0^- \Psi_0^{(n+1)} \quad . \tag{19}$$

Now starting from H_1 whose ground state energy is $E_1^{(0)} = E_0^{(1)}$, we can similarly generate a second Hamiltonian H_2 as a SUSY partner of H_1 since we can write H_1 in the form :

$$H_1 = A_0^- \; A_0^+ + E_0^{(0)} = A_1^+ \; A_1^- + E_0^{(1)} \qquad , \qquad (20)$$

where

$$A_1^{\pm} = \pm \frac{d}{dr} + W_1(r) \qquad , \qquad \mathrm{W_1(r)} = \frac{d}{dr} \ln \Psi_1^{(0)} \; . \qquad (21)$$

Continuing in this manner we obtain

$$H_2 = A_1^- \; A_1^+ + E_0^{(1)} = -\frac{d^2}{dr^2} + V_2 \qquad , \qquad (22)$$

where

$$V_2 = W_1^2(r) - \dot{W}_1(r) + E_0^{(1)} = V_1 - 2\frac{d^2}{dr^2} \ln \Psi_1^{(0)}(r) \qquad ,$$

$$= V_0 - 2\frac{d^2}{dr^2} \ln\left[\Psi_0^{(0)}(r)\Psi_1^{(0)}(r)\right] \qquad . \qquad (23)$$

Furthermore,

$$E_2^{(n)} = E_1^{(n+1)} = E_0^{(n+2)} \qquad ,$$

$$\Psi_2^{(n)} = \left[E_1^{(n+1)} - E_1^{(0)}\right]^{-1/2} \; A_1^- \Psi_1^{(n+1)} \qquad ,$$

$$\Psi_2^{(n)} = \left[E_0^{(n+2)} - E_0^{(1)}\right]^{-1/2} \left[E_0^{(n+2)} - E_0^{(0)}\right]^{-1/2} A_1^- A_0^- \Psi_0^{(n+2)} . \qquad (24)$$

In this way, it is clear that if the original Hamiltonian H_0 has $p(\geq 1)$ bound states with eigenvalues $E_0^{(n)}$, and eigenfunctions $\Psi_0^{(n)}$ with $0 \leq n \leq (p-1)$, then we can always generate a hierarchy of $(p-1)$ Hamiltonians $H_1,\ldots,H_p$ such that the m th member of the hierarchy of Hamiltonians (H_m) has the same eigenvalue spectrum as H_0 except that the first $(m-1)$ eigenvalues of H_0 are missing in H_m. In particular, we can always write $(m = 1,2,\ldots,p)$:

$$H_m = A_m^+ A_m^- + E_m^{(0)} = -\frac{d^2}{dr^2} + V_m(r) \quad , \tag{25}$$

where

$$A_m^- = -\frac{d}{dr} + W_m(r) \ , \ W_m(r) = \frac{d}{dr} \ln \Psi_m^{(0)} \quad . \tag{26}$$

One also has

$$E_m^{(n)} = E_{m-1}^{(n+1)} = E_0^{(n+m)} \quad ,$$

$$\Psi_m^{(n)} = \left[E_0^{(n+m)} - E_0^{(m-1)}\right]^{-1/2} \ldots \left[E_0^{(n+m)} - E_0^{(0)}\right]^{-1/2} A_{m-1}^- \ldots A_0^- \Psi_0^{(n+m)},$$

$$V_m(r) = V_0(r) - 2\frac{d^2}{dr^2} \ln\left[\prod_{j=1}^{m} \Psi_{m-j}^{(0)}\right] \quad . \tag{27}$$

CHAPTER 3

APPLICATION TO ALPHA-ALPHA SCATTERING

Before the application of SUSYQM, we first deal with the usual quantum mechanical treatment of the α+α interaction in Section 3.1 to have a theoretical background on the problem under consideration. Section 3.2 discusses the connection between the shallow and deep PEP considering the α+α scattering within the frame of SUSYQM, which is one of the main interests of this thesis work.

3.1 Theoretical Background On Alpha-Alpha Scattering

A. Formulation

Solving the equation of Schrödinger equation is the central problem of non-relativistic quantum mechanics. A simple case is the study of the motion of a particle without spin in an external potential. The time-independent Schrödinger equation in this case reads

$$H\Psi(\mathbf{r}) = E\Psi(\mathbf{r}) \quad , \tag{28}$$

with the Hamilton operator

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) \quad . \tag{29}$$

If the particle is scattered by the potential, the energy E may be given any positive value. If he particle is bound by the potential, E becomes negative and can only take discrete values.

Some complicated problems can be reduced to solving the one-particle Schrödinger equation. Solutions of the one-particle Schrödinger equation accordingly form the basis of states of the shell models in atomic physics and nuclear physics. The mutual scattering of two particles also leads to a Schrödinger equation of type (28). In this case **r** is the separation vector of the two particles, m becomes μ ,the reduced mass, and E is the energy in the center of mass system, i.e. in that inertial system in which the center of mass of the two particles is at rest.

Analytical solutions of (28) are known only for a few special forms of potential. In general the equation has to be solved numerically. It is fairly easy to do this for a spherically symmetric potential, i.e. when

$$V(\mathbf{r}) = V(r) \quad . \tag{30}$$

We shall assume in what follows that this condition is fulfilled.

Restriction to central potentials makes the introduction of spherical coordinates (r,θ,ϕ) appropriate. The Schrödinger equation is then separable, i.e. one obtains one equation for the radial motion and one for the angular motion.

In spherical coordinates the Laplacian operator ∇^2 may be written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}L^2(\theta,\phi) \tag{31}$$

where

$$L^2(\theta,\phi) = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] . \qquad (32)$$

The operator L^2 given by Eq.(32) is just

$$\mathbf{L}^2 = (\mathbf{r}\times\mathbf{p})\cdot(\mathbf{r}\times\mathbf{p}) = -\hbar^2(\mathbf{r}\times\nabla)\cdot(\mathbf{r}\times\nabla) \qquad (33)$$

and represents the square of the orbital angular momentum. The spherical harmonics $Y_{lm}(\theta,\phi)$ are defined to be eigenfunctions of L^2 and the z component of **L**,

$$L_z = (r\times p)\cdot\hat{z} = -i\hbar\frac{\partial}{\partial\phi} \quad , \qquad (34)$$

such that

$$L^2\,Y_{lm}(\theta,\phi) = l(l+1)\hbar^2\,Y_{lm}(\theta,\phi) \quad , \qquad (35)$$

and

$$L_z\,Y_{lm}(\theta,\phi) = m\hbar\,Y_{lm}(\theta,\phi) \quad . \qquad (36)$$

The function

$$\Psi_{lm}(\mathbf{r}) \equiv r^{-1}\,u_l(r)\,Y_{lm}(\theta,\phi) \quad , \qquad (37)$$

is then a solution of Eq.(28), provided $u_l(r)$ is a solution of the radial equation

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_l(r)+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}u_l(r)+V(\mathbf{r})u_l(r)=Eu_l(r)\ . \qquad (38)$$

The quantity l occurring in Eqs.(35-38) is the angular momentum quantum number. The symbol m is the magnetic quantum number appearing in Eqs.(35-37).

It is evident that any linear combination of the functions $\Psi_{lm}(\mathbf{r})$, for different l and m, is also a solution of Eq.(28). The physical situation is symmetric about the z axis, so we shall seek a solution of Eq.(28) which has this symmetry and is therefore independent of ϕ. We may write such a solution as

$$\Psi(\mathbf{r})=\sum_{l,m}\frac{1}{r}u_l(r)\,c_{l,m}Y_{l,m}(\theta,\phi) \qquad , \qquad (39)$$

where the wave function $u_l(r)$ satisfies the radial equation, Eq.(38).

The decomposition of the total wave function into angular momentum partial waves brings certain advantages. Bound states contain only one partial wave. For scattering states one must take account of more than one partial wave, but usually only a small number. The reason for this is as follows: Apart from the Coulomb potential most physical potentials are of short range. Particles with finite energy and large angular momentum cannot enter into the interaction region. In the radial Eq.(38) this is expressed by the second term, which contains the centrifugal barrier $\hbar^2 l(l+1)/(2\mu r^2)$. The centrifugal barrier acts like a potential which is so strong for large values of l that $V(r)$ becomes negligible by comparison. When $V(r)$ is neglected Eq.(38) becomes the free Schrödinger equation, whose analytic solutions are known. We therefore have to solve the radial Eq.(38) for only a few values of l.

The radial Schrödinger Eq.(38) is a differential equation of the second order. Accordingly we need two boundary conditions to get a unique solution. One

of the conditions is obtained from the requirement that the probability density $|\Psi(r)|^2$ must remain finite at the origin of coordinates. The probability density is given by

$$|\Psi(r)|^2 = \left|\sum_{l,m} \frac{u_l(r)}{r} c_{l,m} Y_{l,m}(\theta,\phi)\right|^2 \quad . \tag{40}$$

The probability density can remain finite at the origin for arbitrary expansion coefficients $c_{l,m}$ only if

$$u_l(0) = 0 \tag{41}$$

This is our first boundary condition.

The second boundary condition arises in connection with the normalization of the wave function $u_l(r)$. We must be sure here between bound state solutions $(E<0)$, and scattering solutions $(E>0)$.

In bound states the motion of the particle (or the relative motion of the particles) is restricted to a finite space, i.e. the wave function must be normalizable to unity. As the second boundary condition we can tentatively set the wave function equal to any chosen number, e.g. unity, at an arbitrarily chosen distance $r_0(\neq 0)$:

$$u_l(r_0) = 1 \quad . \tag{42}$$

The solution of the radial equation is thereby determined. We shall find it numerically and see that, for values of r greater than the range of the potential, it has the form

$$u_l(r) \to a_l e^{-\kappa r} + b_l e^{+\kappa r} \quad , \tag{43}$$

with

$$\kappa = \left(\frac{2\mu(-E)}{\hbar^2} \right) \quad . \tag{44}$$

The condition for normalizability,

$$\int_0^\infty |u_l(r)|^2 \, dr = \text{finite}, \tag{45}$$

is however, fulfilled only if b_l vanishes, since otherwise the function $u_l(r)$ would increase without the limit for large values of r. We shall see that b_l is a function of the energy E introduced by us into the radial equation. For a binding energy E_i, b_l will pass through zero as a function of E:

$$b_l(E_i) = 0 \quad . \tag{46}$$

We shall recognize this zero from the fact that for large of r the wave function $u_l(E,r)$ flips between large negative values and large positive values for slight variations in the energy E. The energy at which the change occurs is one of the binding energies E_i. For $E = E_i$ the wave function $u_l(E,r)$ becomes normalizable. For practical computation we must replace the upper integration limit in the integral (45) by a value R which is not excessively large, since our solution becomes numerically unstable for large r -values. Because of the arbitrary second boundary condition (42) the bound state function obtained is normalized, but not to unity. We can, however, compute the integral

$$\int_0^R |u_l(E_i,r)|^2 dr = N^2 \quad , \tag{47}$$

and then with

$$\hat{u}_l(E_i,r) = \frac{1}{N} u_l(E_i,r) \quad , \tag{48}$$

we get a bound state function normalized to unity. Because of the linearity of the Schrödinger equation, also $\hat{u}_l(E_i,r)$ satisfies the equation.

For scattering solutions the energy E has the physical significance of an energy imparted by an accelerator, and may take any positive value. With the boundary conditions (41) and (42) we again obtain a uniquely defined solution $u_l(E_i,r)$ which now, however, does not vanish asymptotically. If the potential $V(r)$ has no long range component, then $u_l(E_i,r)$ has the asymptotic form

$$u_l(E,r) \to N \sin\left(kr - \frac{1}{2} l\pi + \delta_l\right) \quad , \tag{49}$$

where $k = \left(\frac{2\mu}{\hbar^2} E\right)^{1/2}$.

The second boundary condition (42) will give us some value for N. If we wish, we can normalize $u_l(E_i,r)$ by multiplying it by $1/N$. Normalization means in this case that the wave has an asymptotic amplitude equal to unity. The quantity δ_l is called the scattering phase. It is a function of the energy E. The quantity $l\pi/2$ is a phase shift produced by the angular momentum barrier.

As a physical example we consider the mutual scattering of two α-particles with the Coulomb interaction, which is switched off at initial. The α-α scattering is rather interesting, since α-particles are in reality not point masses, but the nuclei of Helium atoms. They consist of two protons and two neutrons each. Additionally, we have chosen the example of α-α collision because it is more straightforward to construct a potential model for collision between spin-zero nuclei. If two α-particles are so close to one another that they interpenetrate, then the Pauli principle has a strong influence on the interaction. Nevertheless the mutual scattering of two α-particles can be described very well by means of a simple potential [12] which is

$$V(r) = -V_0 \exp(-\beta r^2) \quad , \tag{50a}$$

with

$$V_0 = 122.694 \text{ MeV} \quad \text{and} \quad \beta = 0.22 \text{ fm}^{-2} \quad . \tag{50b}$$

For a realistic computation one would also have to take account of the Coulomb potential which will be considered in our full calculations later, in this section.

We shall solve first the radial Schrödinger equation (38) using quantum mechanics with the potential (50) and examine the solutions in detail. As angular momentum quantum numbers we choose $l = 0$, 2, 4, and 6. Odd angular momentum quantum numbers are forbidden for two α-particles by the Pauli principle. Our unit of energy is 1 MeV, the unit of length is 1 fm.

In the region of negative energies it is well known that there are four states [12-14], namely three bound states for $l = 0$ and one for $l = 2$. The two lowest bound states for $l = 0$ and the bound state for $l = 2$ are unphysical. They are Pauli-forbidden, as they correspond to too dense packing of the 8 nucleons. Because of these unphysical bound states the potential (50) reproduces the scattering wave functions well, for these are orthogonal to the bound state functions. The third bound state for $l = 0$ is also unphysical, but for a different reason. We have omitted

the Coulomb interaction for initial calculations, and thereby arises a weakly bound state of two α-particles. When the Coulomb potential is included this bound state becomes the well known sharp resonance at 0.092 MeV, which one can also regard as the short-lived ground state of the ^{8}Be nucleus.

Since we are dealing with the relative motion, we use the reduced mass of two α-particles instead of the mass of one in the radial equation (38) and also in (44) and (49). It occurs only in the combination $\hbar^2 / 2\mu$. We use the value

$$\frac{\hbar^2}{2\mu} = 10.375\ MeV\,fm^2 \quad . \tag{51}$$

B. Programming

In order to solve Eq.(38) using the Fox-Goodwin recursion, we write a subroutine entitled FOX, which solves the differential equation

$$u''(r) + w(r)u(r) = 0 \quad , \tag{52}$$

where we have used the abbreviation

$$w(r) = \frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2} \quad . \tag{53}$$

We introduce the arrays (r_i), (u_i) and (w_i), with

$$r_i = ih \quad , \quad u_i = u(r_i) \quad , \quad w_i = w(r_i) \quad , \quad i = 0,1,2,.....,i_b \quad . \tag{54}$$

The radial wave function is to be computed in the interval $[0,b]$, i.e. the lower interval boundary lies at $r = 0$, the upper at $r = b$. The interval $[0,b]$ is divided into i_b subintervals of length

$$h = \frac{b}{i_b} \quad . \tag{55}$$

The number of mesh points is therefore equal to $(i_b + 1)$. With the notations, we can rewritten the Eq.(38) in the following form using the Fox-Goodwin technique,

$$u_{i+1} = \frac{2u_i - u_{i-1} - (h^2/12)(10w_i u_i + w_{i-1}u_{i-1})}{1 + (h^2/12)w_{i+1}} \quad . \tag{56}$$

Physical title	FORTRAN title	Physical title	FORTRAN title
i_b	IB	u_i	U (I)
B	B	w_i	W (I)
H	H		

Figure 3.1 Notation used in the subroutine FOX.

Figure 3.2 shows the subroutine FOX; the notation used is to be found in Figure 3.1. The calling program transfers to the subroutine FOX the upper boundary b of the integration interval, the number of integration steps i_b and the values w_i of the function $w(r)$. It receives back from FOX the values u_i of the solution function $u(r)$. In lines 104 and 105 the boundary conditions (41) and (42) are specified. The recursion (56) is carried out in the DO-loop from line 106.

```
      SUBROUTINE FOX  (IB,B,W,U)                                        100
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)                              101
      DIMENSION W(0:IB), U(0:IB)                                        102
      H=B/IB                                                            103
      U(0)=0                                                            104
      U(1)=1                                                            105
      DO10 I=1,IB-1                                                     106
      U(I+1)=(2.D0*U(I-1)-H*H/12.D0*(10.D0*W(I)*U(I)+W(I-1)*U(I-1)))    107
     &        /(1.D0+H*H/12.D0*W(I+1))                                   108
   10 CONTINUE                                                          109
      END                                                               110
```

Figure 3.2 Subroutine FOX.

In order to be able to change the potential function easily, we introduce the FUNCTION subprogram V, Figure (3.3-3.4). In our case V calculates the α-α interaction potential for the separation distance r from Eq.(50), line 205. The potential parameters are fixed, line 202. The IF-block of lines 204 to 208 prevents a possible exponent overflow and ensures that the exponential function is not computed unnecessarily often.

Physical title	FORTRAN title	Physical title	FORTRAN title
V_0	Vo	R	R
β	BETA		

Figure 3.3 Notation used in the FUNCTION subprogram V.

```
DOUBLE PRECISION FUNCTION  V(R)                          200
IMPLICIT DOUBLE PRECISION (A-H,O-Z)                     201
PARAMETER (V0=122.694D0, BETA=0.22D0)                   202
PARAMETER (EXPMAX=50.D0)                                203
IF (BETA*R*R.LE.EXPMAX) THEN                            204
 V=-V0*EXP(-BETA*R*R)                                   205
ELSE                                                    206
 V=0.D0                                                 207
ENDIF                                                   208
END                                                     209
```

Figure 3.4 FUNCTION subprogram V(R).

Physical title	FORTRAN title	Physical title	FORTRAN title
i_b	IB	$\hbar^2 / 2m$	H2M
b	B	L	L
h	H	E	E
r_i	R (I)	u_i	U (I)
		w_i	W (I)

Figure 3.5 Notation used in main program.

Figure 3.6 shows the numerical portion of the main program, the notation used is listed in Figure 3.5. In the DO-loop from line 307 the mesh points r_i are computed from Eq. (54), and the function values $w(r_i) = w_i$ from Eq.(53). In line 306 we set $w(r_i) = 0$. This needs a justification since, for non-vanishing angular

momentum, $w(r)$ diverges as $r \to 0$. In Eq.(52) $w(r)$ occurs only in the product $w(r)u(r)$, and we have set $u(r)$ equal to zero according to the boundary condition (41). The quantity " infinity times zero ", however, is undefined. With our choice $w(0) = 0$ we arbitrarily set $w(0)u(0)$ equal to zero. The justification for this will be obtained numerically: Whenever the angular momentum barrier dominates $V(r)$, our solution $u_l(r)$ transforms into the known analytic solution of the free Schrödinger equation.

```
      PROGRAM ALPHA                                                    300

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)                              301
      PARAMETER (IMAX=10000, H2M=10.375D0, BMAX=30.D0)                 302
      DIMENSION R(0:IMAX), U(0:IMAX), W(0:IMAX)                        303

*     (Input : E, L, B, IB)

      H=B/IB                                                           304
      R(0)=0.D0                                                        305
      W(0)=0.D0                                                        306
      DO 10 I=1,IB                                                     307
      R(I)=I*H                                                         308
      W(I)=(E-V(R(I)))/H2M-L*(L+1)/R(I)*R(I)                           309
10    CONTINUE                                                         310
      CALLFOX (IB, B, W, U)                                            311

*     (Output : U)

      END                                                              312
```

Figure 3.6 Numerical portion of main program ALPHA.

The value for $\hbar^2 / 2\mu$ is fixed by Eq.(51), line 302, whereas the energy E, the angular momentum l, the upper boundary b of the desired integration interval $[0,b]$ and the number i_b of integration steps are input. As output we obtain the radial wave function $u_l(r)$ in graphics form and optionally also in numerical form.

C. Analysis And Discussion

We have investigated for $l = 0$ the stability of the zeros of the wave function for small number of integration steps. We have carried out this investigation with energies in the region 1 MeV, 10 MeV and 30 MeV for the number of integration steps taken the values 10, 20, 40, 80, 160, Figure 3.7. For $l = 0$, E=10 MeV, $b = 5$ fm and 10 integration intervals the third zero lies at 3.6711 fm. This value is found by linearly interpolating the numerical printout of the wave function between 3,5 and 4.0 fm. By raising the number of integration intervals to 20, 40, 80, 160 caused the position of the zero to converge very rapidly to 3.770122 fm. The error in the position of the third zero decreases, for each doubling of the number of integration intervals, by a factor of about 1/18 to 1/20.

We have found the bound states by determining the energies for which the radial wave function at large distances changes sign. For $l = 0$ the lowest bound state lies at -76.9036145 MeV. Figure 3.8 shows it in the region 0 to 5 fm. The maximum lies at about 0.84 fm. The experimental radius of the α-particle amounts to about 1.6 fm, i.e. in this lowest bound state the two α-particles would lie so close together that they would overlap almost completely. It is clear that here the Pauli principle regarding the neutrons and protons play a crucial role.

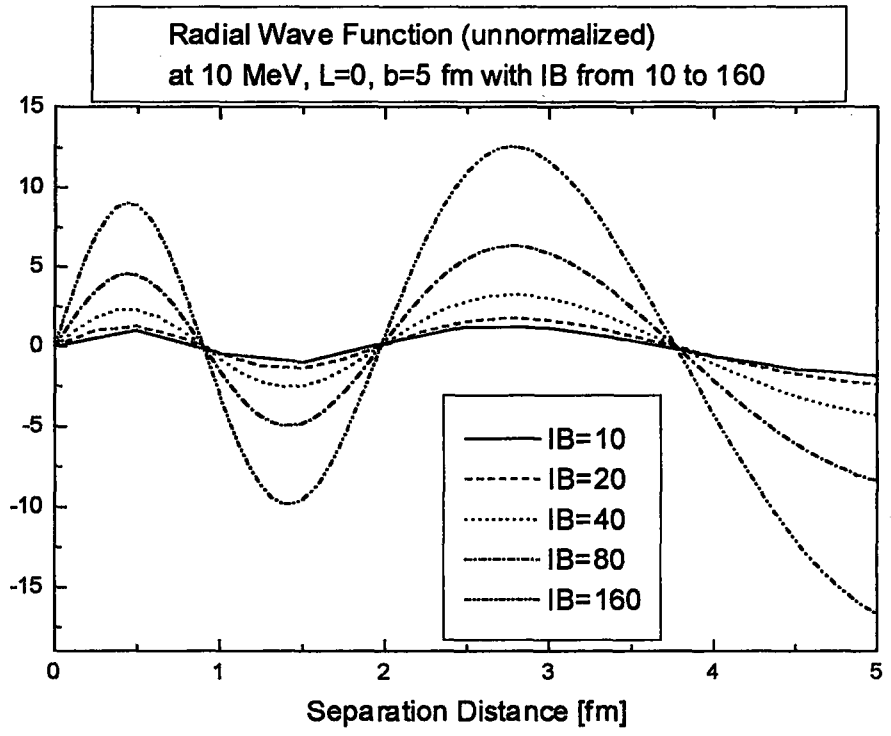

Figure 3.7 Radial wave function (unnormalized).

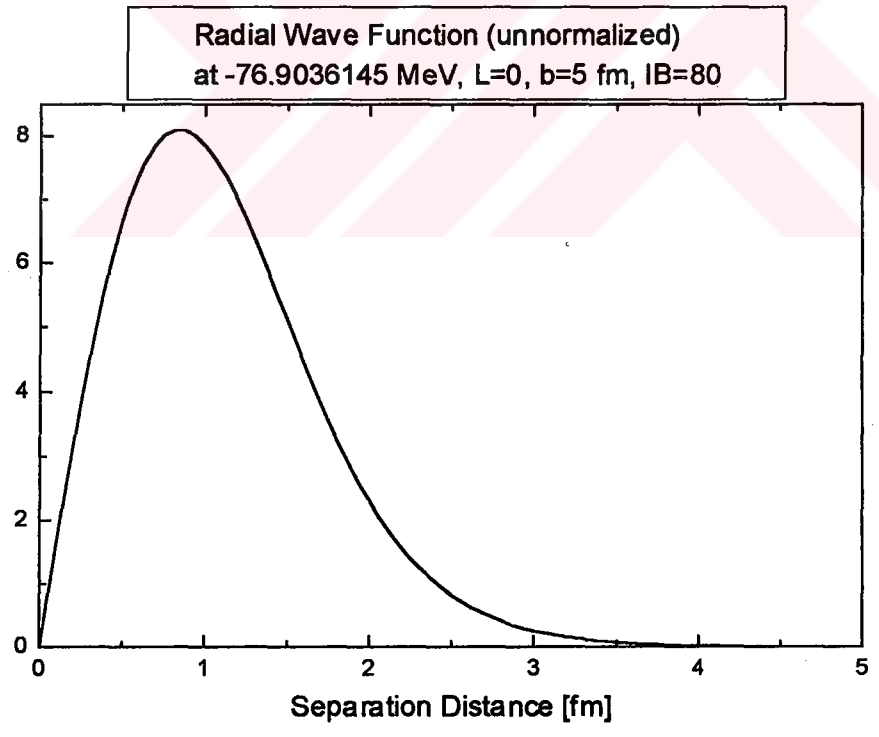

Figure 3.8 Relative wave function of the lowest bound state of two α-particles (Pauli forbidden).

The second bound state with $l = 0$ has the energy -29.00048 MeV and it is shown in Figure 3.9. The wave function has a zero and is orthogonal to the wave function of the first bound state. The bound state with $l = 2$ is shown in Figure 13.10. Notice the different type of behavior of the wave function near the origin.

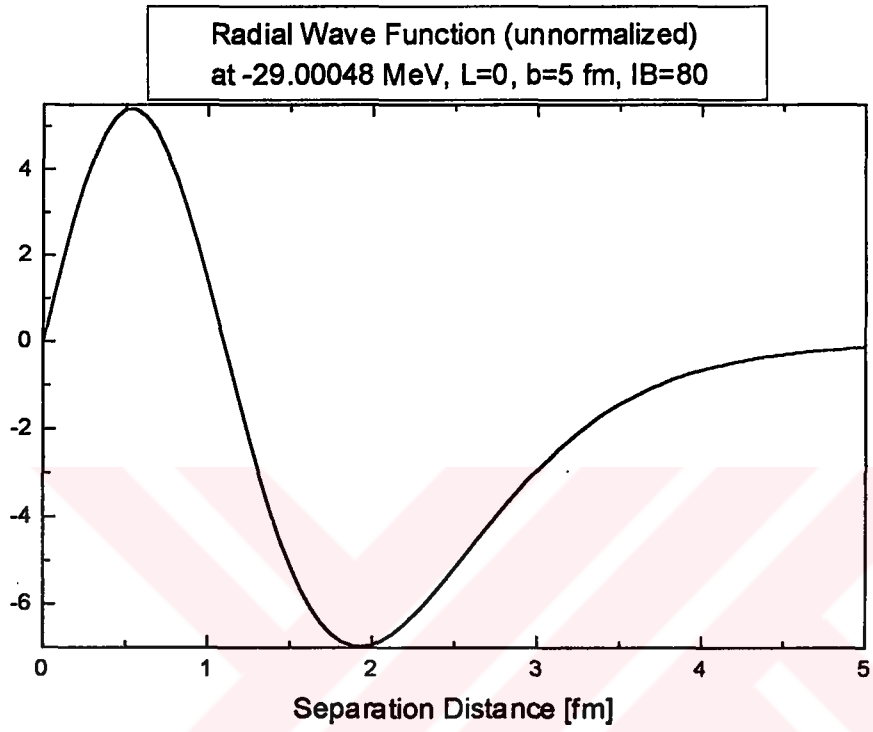

Figure 3.9 Relative wave function of the second bound state of two α-particles (Pauli forbidden).

A scattering wave for $l = 0$ and 1 MeV is shown in Figure 3.11; a scattering wave for l=0 and 20 MeV is shown in Figure 3.12. At small distances our potential $V(r)$ is so deep that the term $E\,u(r)$ in Eq.(38) is scarcely important compared with the term $V(r)u(r)$. For this reason the first two zeros of the wave function are fairly insensitive to a variation of the energy E. An equally appropriate explanation, which of course is linked with that just given, is that the scattering wave functions must be orthogonal to the bound state functions, and that this condition has a dominant influence on the form of the scattering wave functions at short distances.

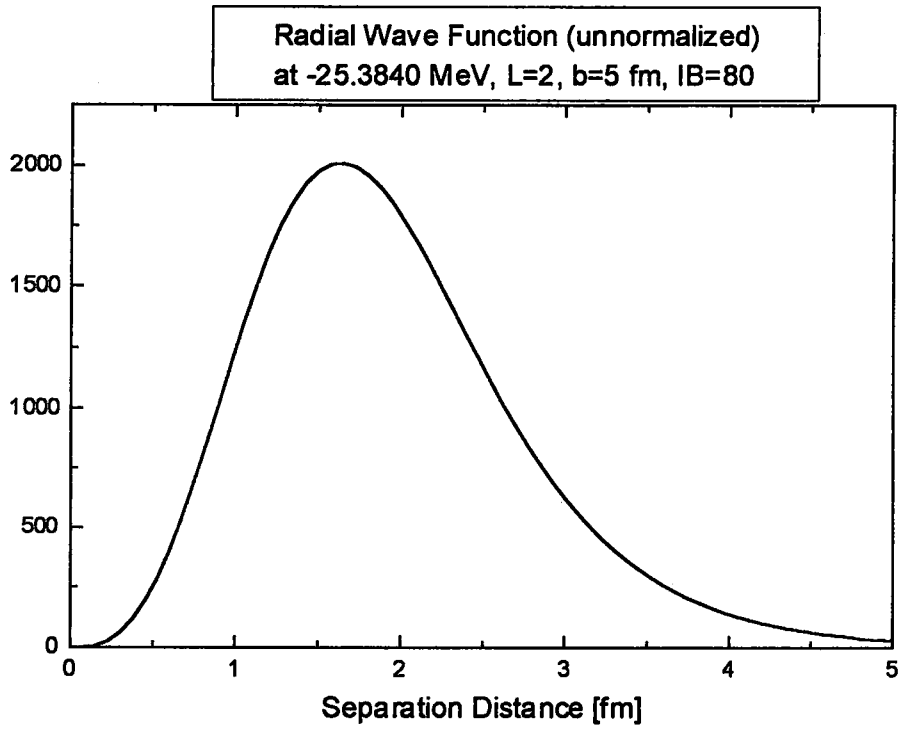

Figure 3.10 Relative wave function of the second bound state of two α-particles (Pauli forbidden).

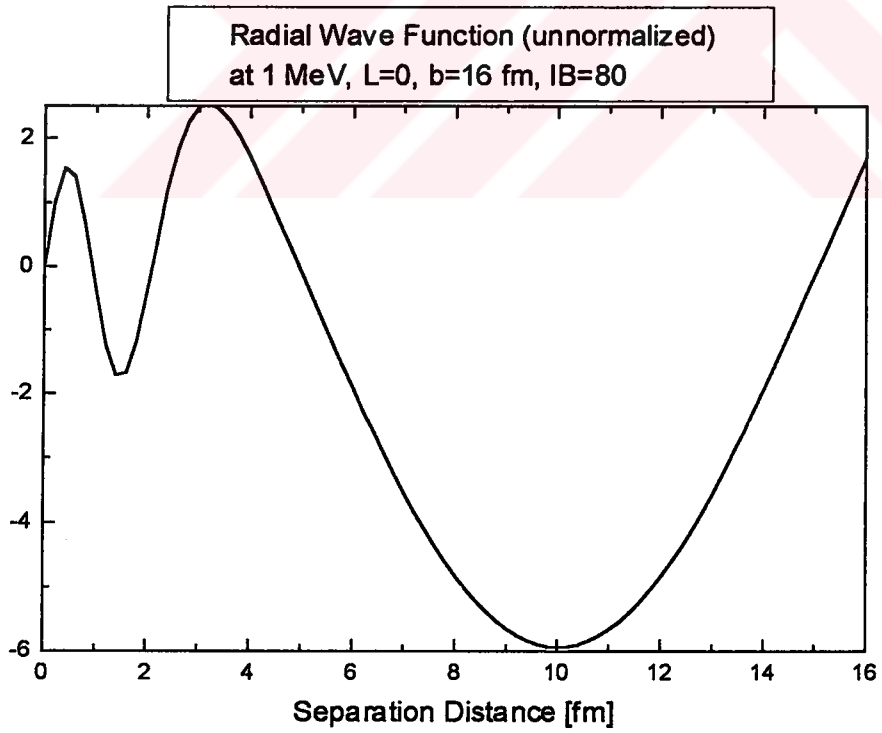

Figure 3.11 Scattering wave function for $l = 0$ and $E = 1$ MeV.

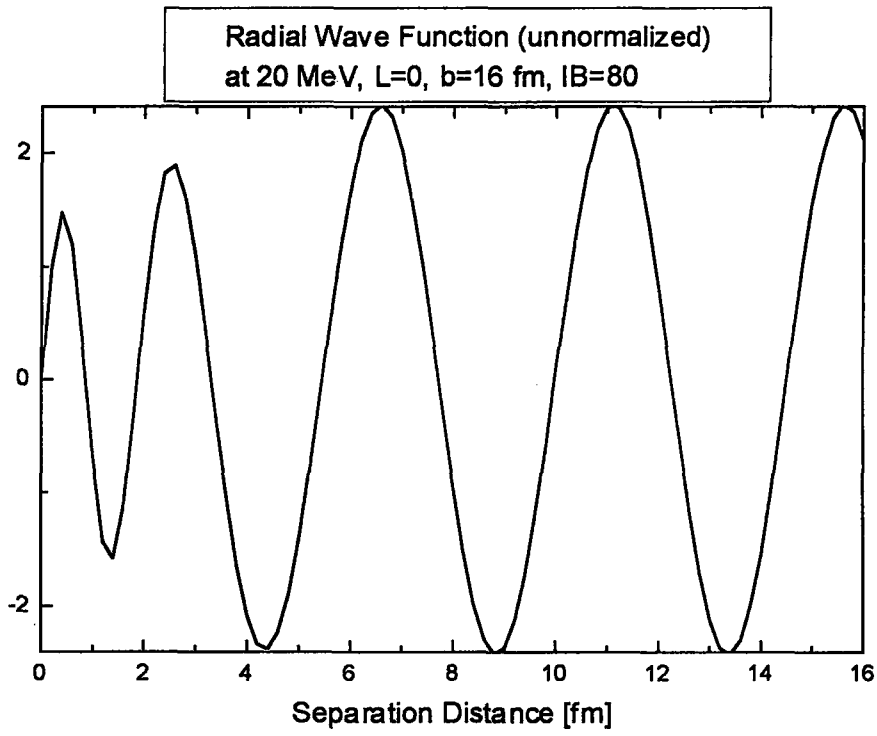

Figure 3.12 Scattering wave function for $l = 0$ and $E = 20$ MeV.

Figure 3.13 shows a scattering wave function for $l = 4$, $E = 5$ MeV, and Figure 3.14 shows for comparison a scattering wave function with $l = 10$ and $E = 10$ MeV. We have noticed the strong influence of the centrifugal barrier, in that the wave function with $l = 10$ is suppressed still more strongly in the neighborhood of the origin than the wave function with $l = 4$. From analytical investigations one knows that $u_l(r)$ behaves in the neighborhood of the origin like r^{l+1}, i.e. $u_4(r)$ goes to zero like r^5 and $u_{10}(r)$ like r^{11}.

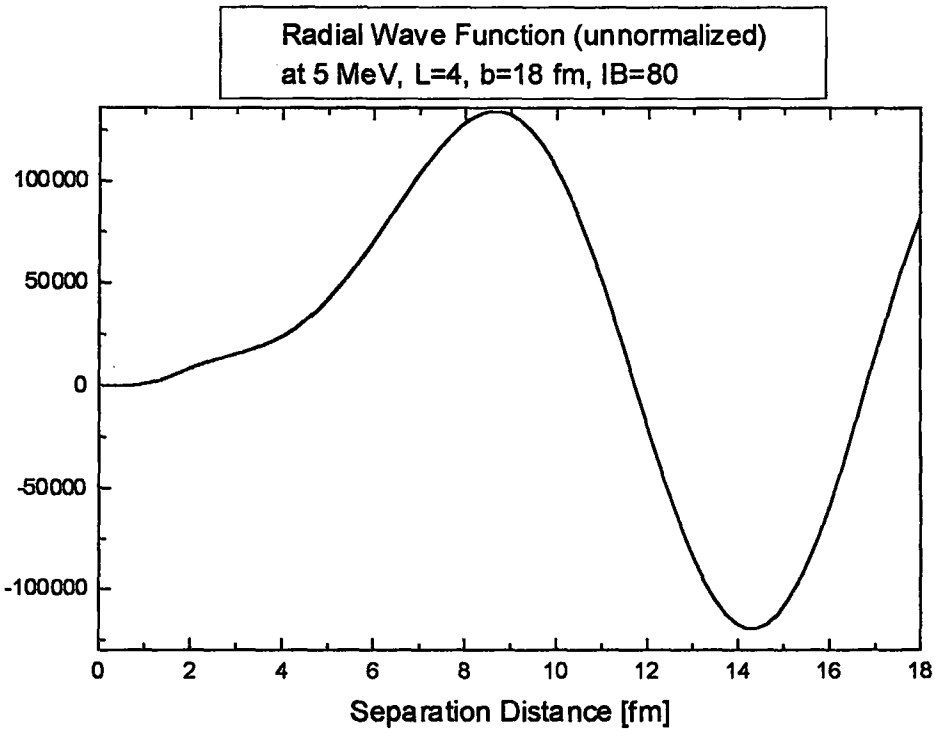

Figure 3.13 Scattering wave function for $l = 4$ and $E = 5$ MeV.

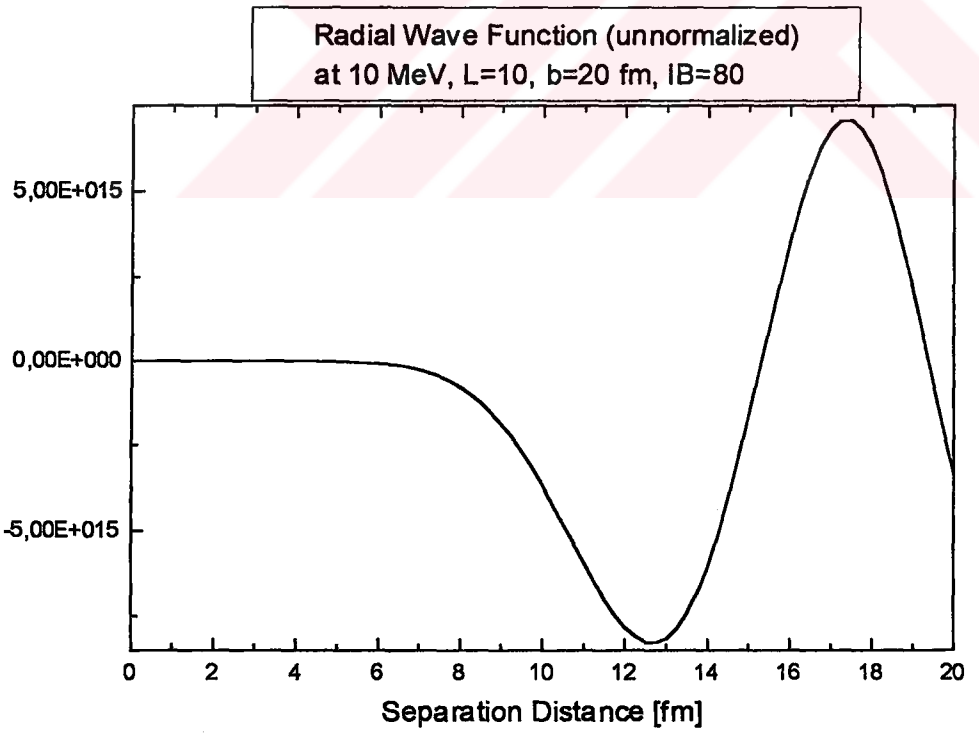

Figure 3.14 Scattering wave function for $l = 10$ and $E = 10$ MeV.

In order to incorporate the Coulomb potential into the program, we only extend the subroutine V so that the potential becomes :

$$V(r) = -V_0 \exp(-\beta r^2) + V_C(r) \quad , \tag{57}$$

where $V_C(r)$ describes the Coulomb interaction of two α-particles:

$$V_C(r) = \frac{4e^2}{r} erf(\gamma r) \quad . \tag{58}$$

As we know that the Coulomb potential $4e^2 / r$ is valid for point particles. The finite extend of the charge distribution in α-particles is approximately taken into account by the factor $erf(\gamma r)$, where the parameter γ is adjusted to the radius of the α-particle.

We have used the following numerical values

$$e = 1.2\ (MeV\ fm)^{1/2} \quad \text{and} \quad \gamma = 0.75\ fm^{-1} \tag{59}$$

and $erf(\gamma r)$ in Eq.(58) is the Gaussian error function as;

$$erf(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5)\exp(-x^2) \ , \tag{60a}$$

with

$$\begin{aligned} &t = \frac{1}{1+px} \ , p = 0.3275911 \ , a_1 = 0.254829592 \ , \\ &a_2 = -0.284496736 \ , a_3 = 1.421413741 \ , \\ &a_4 = -1.453152027 \ , a_5 = 1.061405429 \ . \end{aligned} \tag{60b}$$

To take into account of the Coulomb potential, as mentioned above, we only need to follow the IF-block in the FUNCTION subprogram V, i.e. line 208 of Figure 3.4, with the instruction

$$V = V + 4.D0 \times 1.2D0 \times 1.2D0 / R \times ERF(0.75D0 \times R) \quad . \qquad (61)$$

For the Gaussian error function $erf(x)$ a further FUNCTION subprogram ERF must be available, Figure 3.15.

```
      DOUBLE PRECISION FUNCTION ERF(R)
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      PARAMETER (A1=0.254829592D0, A2= -0.284496736D0,
      A3=1.421413741D0, A4= -1.453152027D0)
      PARAMETER (A5=1.06140542D0,P=0.3275911D0)
      T(Z)=1.D0/(2.D0+P*Z)
      Y=ABS(X)
      IF (Y.LT.25.D0) THEN
       ERF=1.D0-(A1*T(Y)+A2* T(Y)* T(Y)+A3* T(Y)**3+A4*
     &       T(Y)**4+A5*T(Y)**5)*EXP(-Y*Y)
      ELSE
       ERF=1.D0
      ENDIF
      IF (X.LT.0.D0) ERF= - ERF
      END
```

Figure 3.15 FUNCTION subprogram ERF.

The rearranged program for our proposed approximation is to be written as in Figure 3.16.

```
DOUBLE PRECISION FUNCTION V(R)
IMPLICIT DOUBLE PRECISION (A-H, O-Z)
PARAMETER (V0=122.694D0, BETA=0.22D0)
PARAMETER (EXPMAX=50.D0)
IF (BETA*R*R.LE.EXPMAX) THEN
IF (R.LE.15D0) THEN
  V=-V0*EXP(-BETA*R*R)+ERF(R)
ELSE
  V=ERF(R)
ENDIF
END
```

Figure 3.16 FUNCTION subprogram V, including the Coulomb potential.

Because the resonance is so sharp we choose $i_b = 10000$, $b = 160$ fm, $l = 0$ and show the resonance wave function (Figure 3.19) of ^{8}Be system, together with the wrong behavior of the system (See Figure 3.17 and 3.18) at resonance neighborhood energies, at 0.091 and 0.093 MeV. This means that the probability of the two α-particles being close together is large at the energy of the resonance and small at all other energies. This becomes strongly apparent in the scattering cross-section.

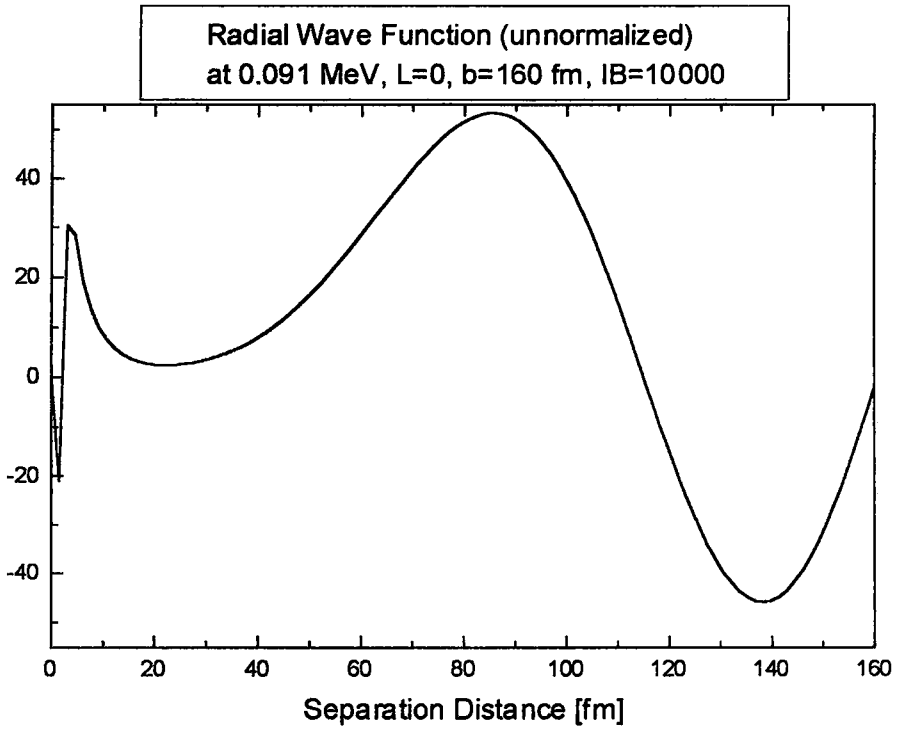

Figure 3.17 Scattering wave function for $l = 0$ and $E = 0.091$ MeV.

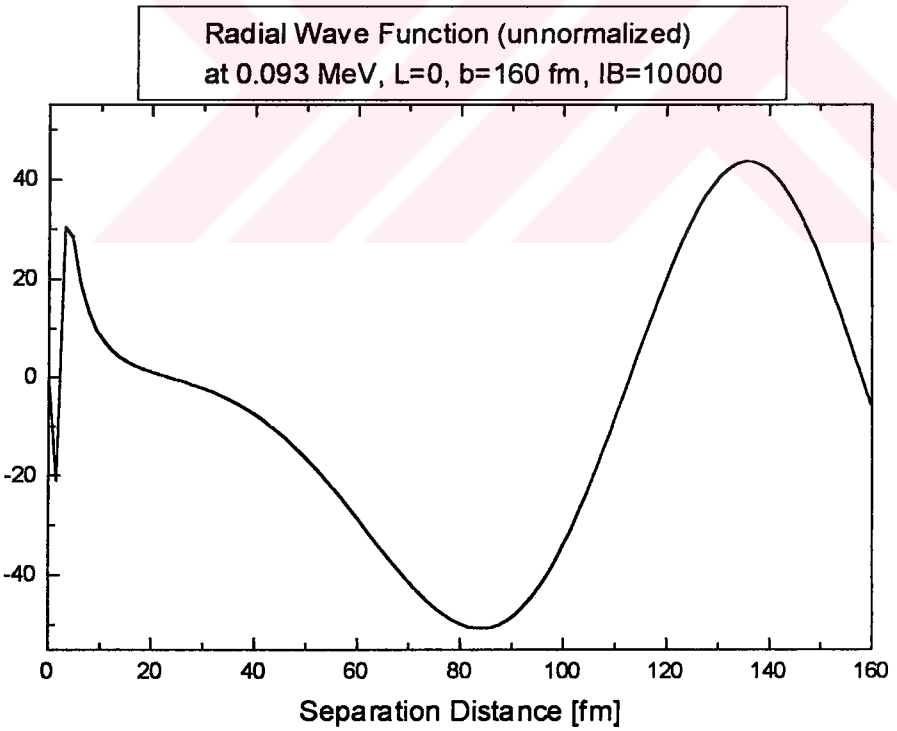

Figure 3.18 Scattering wave function for $l = 0$ and $E = 0.093$ MeV.

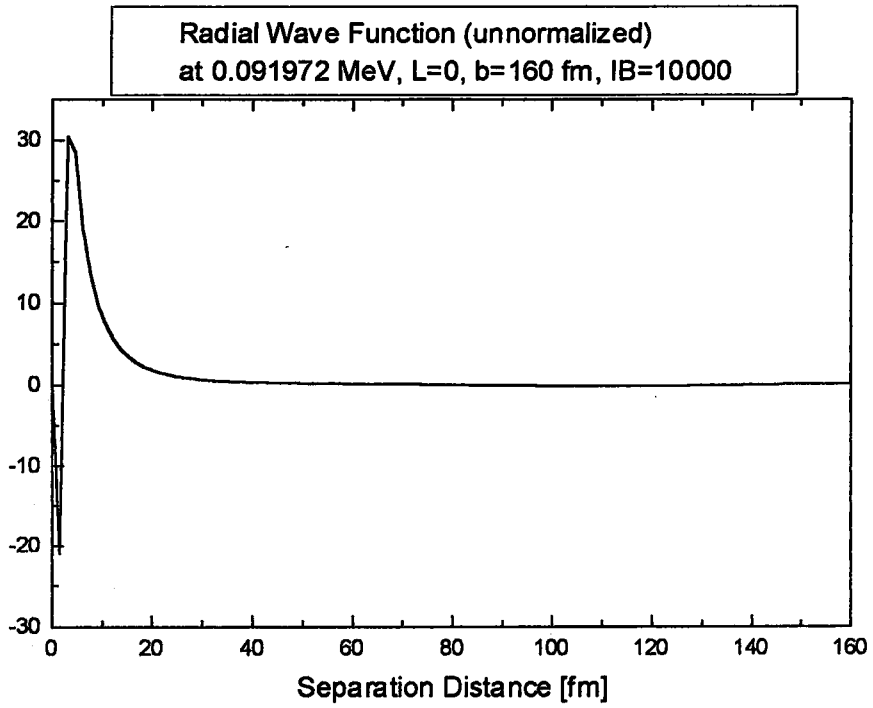

Figure 3.19 The resonance wave function for $l = 0$ and $E = 0.091972$ MeV.

3.2 Supersymmetric Partner Potentials For The Alpha-Alpha Interaction

Supersymmetric Quantum Mechanics allows one to transform a Hamiltonian to its partner such that they possess identical spectra or differ at most from each other by having the lowest eigenstate eliminated in its partner's spectrum [5-7,9]. Repeated application of the supersymmetric transform could yield a Hamiltonian which has a prescribed number of eigenstates less than the starting Hamiltonian. By the same token, a supersymmetric transform can also add an eigenstate of the desired energy to the starting spectrum. The supersymmetric transform, apart from eliminating the lowest eigenstate, induces in the first instance a change in the phase shifts of the continuum states of the starting spectrum. Baye showed [13] that by a judicious choice of repeated supersymmetric transforms, the partner Hamiltonian can be made to be phase-equivalent. Thus, in the case of α-α scattering, where the starting Hamiltonian has a deep effective local potential [12] known to possess bound states which are unphysical because of the Pauli principle, as investigated earlier. Baye eliminated these unphysical states by successive supersymmetric transform while preserving the phase equivalence. In the course of this process, the original potential is transformed to be singular and progressively shallower. Moreover, the supersymmetric transforms are performed for each orbital angular momentum (l) separately. Therefore, at the final stage, a set of shallow, energy-independent and l-dependent effective local potentials is obtained which is completely phase-equivalent to the original deep, energy-independent and l-independent potential. The set of shallow potentials bears a remarkable resemblance to the l-dependent α-α potentials documented in the literature [14].

This dual picture of the nucleus-nucleus interaction arises most probably because the many-body description of the system has been simplified in different ways to a two-body interaction between two structureless particles. For the analysis of the elastic scattering, such drastic difference in the character of the potentials is immaterial since only phase shifts are required and these potentials are phase-equivalent or in the main so. However, when one has to choose one of these potentials to be used in nuclear-structure studies in which wave functions are explicitly involved, the implication of a deep or shallow potential may be immense.

Therefore, it is important to have a quantitative criteria to discriminate which potential has the correct wave function sensitive properties, which is the subject of Chapters 4 and 5.

Our present work belongs to the category of potential-model investigation. The phase-equivalent supersymmetric partner potentials (SUSY PEP) are eminently suitable for this purpose because they yield identical phase shifts. The chain of phase-equivalent potentials is illustrated by the α-α scattering example. In this case, the number of forbidden states is two for $l = 0$, one for $l = 2$ and zero for $l \geq 4$. The starting point is the accurate two-parameter deep potential of Buck, Friedrich, and Wheatley [12]. The s-wave potential (denoted as V) contains two nonphysical bound states as discussed in Section 3.1, and the ^{8}Be ground state at ~0.092 MeV.

We employ the relations of Eq. (18) and Eq. (23) to determine the potentials V_1 and V_2 which are illustrated in Figure 3.20 . The original deep nucleus-nucleus potential $V(r)$ includes nuclear, Coulomb, and centrifugal terms. In order to simplify the discussion, we assume the nuclear term to be real, local, and regular. The wave functions Ψ_0 and Ψ_1 are computed with a modified version of the computer code discussed in the previous section. The computer program used for calculating the SUSY partner potentials is given in Appendix A. The second-order derivatives are calculated with a five-point Lagrange derivation formula. The auxiliary potential V_1 contains only one bound state, which corresponds to the first excited state of V, but provides different phase shift, which can be understood by the following discussion: If we apply the asymptotic form of A_0^- $(= -\frac{d}{dr} - \gamma_0)$, where for clarity $\hbar = 2\mu = 1$, and $\gamma_0 = (-E_0^{(0)})^{1/2}$, to the asymptotic form of a scattering wave function, then we have

$$\lim_{r\to\infty} A_0^- \sin(kr - \frac{1}{2}l\pi - \eta \ln 2kr + \delta_0^l)$$

$$\sim \lim_{r\to\infty}\left(-\frac{d}{dr}-\gamma_0\right)\sin(kr-\frac{1}{2}l\pi-\eta\ln 2kr+\delta_0^l)$$

$$\sim \sin\left[kr-\frac{1}{2}l\pi-\eta\ln 2kr+\delta_0^l+\tan^{-1}(k/\gamma_0)\right] . \quad (62)$$

which shows that the phase shift is increased by $\tan^{-1}(k/\gamma_0)$.

Potential V_1, which does not reproduce ^{8}Be ground state resonance at a correct energy, tends very slowly towards V as r becomes large. In contrast, V_2 has the same spectrum as V_1 but the phase shifts -and resonance- as V. The phase-equivalent potential V_2 is very close to V above 2.5 fm. Notice the repulsive cores of V_1 and V_2 which are due to their singular behavior.

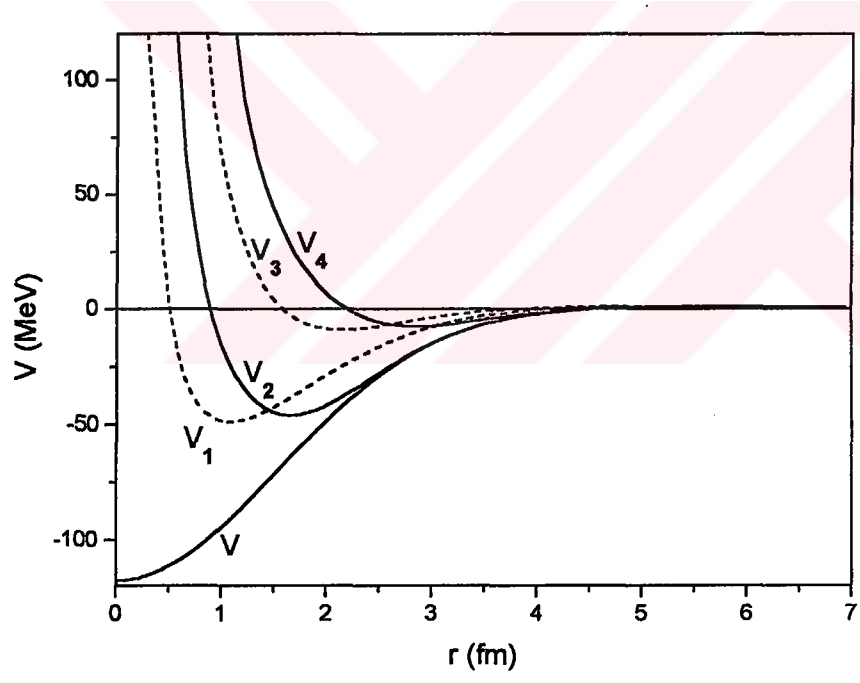

Figure 3.20 Supersymmetric partner potentials.

A second iteration eliminates the remaining bound state and provides the auxiliary potential V_3 (which leads to the same phase shift as V_1) and the potential V_4 which is very similar to V above 5 fm shown in Figure 3.20. Within the

numerical accuracy of the computation, the phase shifts provided by V, V_2, and V_4 are identical at all energies.

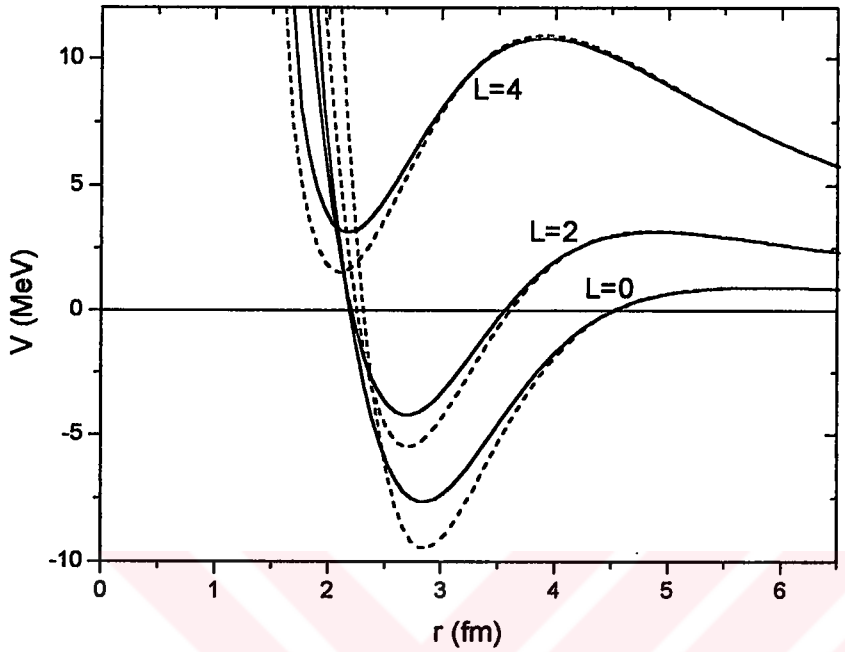

Figure 3.21 Comparison between the phase-equivalent shallow potentials (solid lines) and the α+α potentials of Ali and Bodmer (dotted lines) for $l = 0, 2, 4$.

In Figure 3.21, we have also compared the potentials V_4 ($l = 0$), V_2 ($l = 2$), V ($l = 4$) of the $l = 0$, 2, and 4 partial waves with the shallow α+α potentials of Ali & Bodmer [14] which is briefly described below.

For their potentials, Ali & Bodmer used four parameters, such that

$$V_{\alpha\alpha} = V_{\alpha\alpha}^{(N)} + V_C \quad , \tag{63}$$

where

$$V_{\alpha\alpha}^{(N)}(r) = V_r \exp(-\mu_r^2 r^2) - V_a \exp(-\mu_a^2 r^2) \qquad (64)$$

is the nuclear part of the α+α potential; V_r and V_a are the strengths of the repulsive and attractive parts, respectively, μ_r and μ_a the corresponding inverse ranges. For the Coulomb potential V_C they used just

$$V_C = \frac{4e^2}{r} \quad , \qquad (65)$$

but they had also made the calculations with

$$V_C = \frac{4e^2}{r} erf(\frac{\sqrt{3r}}{2R_\alpha}) \quad . \qquad (66)$$

with an rms radius $R_\alpha = 1.44$ fm (other parameter's values can be seen in the computer code in the Appendix A).

From Figure 3.21, the similarity of two sets of potentials are quite striking. The differences around the minima are most likely due to the fact that the Ali&Bodmer potentials are not singular but only possess regular repulsive cores. Ali&Bodmer potentials can be considered as approximate supersymmetric partners of the Buck at al potential [12].

We have shown the phase relation of these potentials in Fig. 3.22. From the figure it is seen that, as clarified by Eq.(62), although the potential V_1 is the supersymmetric partner potential of the deep potential V, it is however not phase-equivalent (non-PEP), unlike V_4 that is PEP to the original deep potential V of Ref. [12].

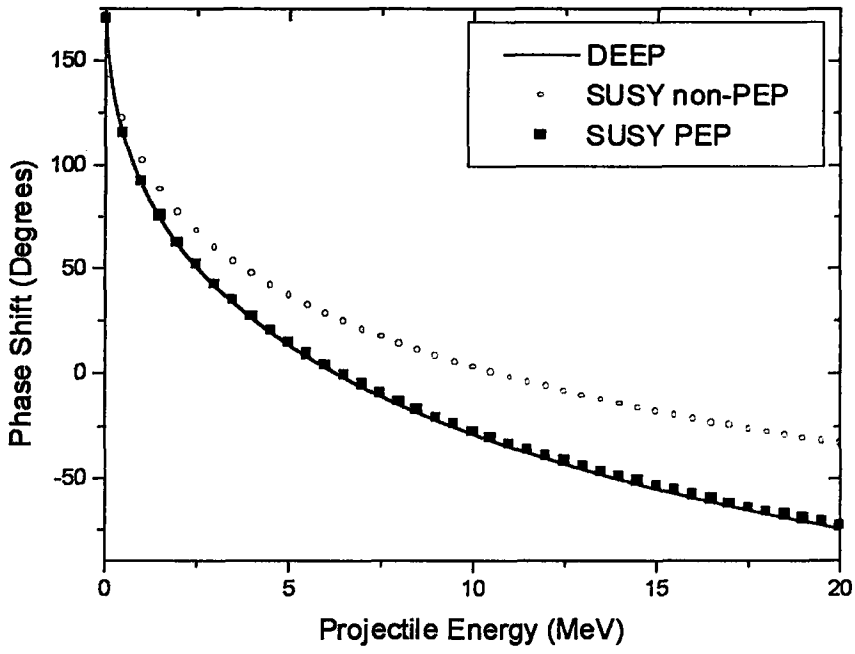

Figure 3.22 Comparison of s-wave phase shifts for V (DEEP), V_1 (SUSY non-PEP), and V_4 (SUSY PEP).

In summary, here we have shown that the co-existence of deep l-independent and shallow l-dependent potentials can be explained using supersymmetric transformations. Supersymmetry allows one to remove all bound states from a real potential without affecting the phase shifts. The deep potential of Ref. [12] is transformed into a phase-equivalent shallow potential very close to the potential of Ref. [14], which was the controversial question for a long time. However, the transformed potential presents an r^{-2} singularity at the origin. The shallow potential of Ref. [14] can be considered as a regular approximation of the singular potential (V_4) equivalent to the deep potential of Ref. [12].

CHAPTER 4

APPLICATION TO THE DEUTERON SYSTEM

In Ref.[15], the on-shell equivalence of the deep quantum-chromodynamically motivated realistic nucleon-nucleon interaction proposed by Kukulin at al. [16] *with more conventional repulsive-core forces* has been investigated by eliminating its unphysical deeply bound states, while preserving its scattering properties and the binding energy of the deuteron. Using the spirit of this work, here we use an alternative superdeep potential to investigate supersymmetry aspect of nucleus-nucleus interaction. In addition as the eigenstates of a superdeep potential and of its supersymmetric partner differ in the node-number, the work presented in this section will also provide us an insight into the wave function-sensitive properties of the supersymmetric potentials via the calculation of the model deuteron radius.

4.1 Introduction

All the available "realistic" nucleon-nucleon forces are characterized by a relatively weak central attractive part and by the presence of a hard or soft repulsive core at small distances; the first feature reflects the loose binding of the neutron-proton system, while the introduction of a repulsive core is required by the negative values assumed by the low-l experimental phase-shifts when energy increases.

However the feasibility of a description of comparable quality of the low energy (E_{cm}<250 MeV) properties of the two-nucleon system in the $^1S_0$ and $^3S_1$ – $^3D_1$ channels (including deuteron properties), in terms of a *deep, purely attractive*

interaction called Moscow potential has recently been demonstrated by Kukulin et al. [16]. Their potential differs from those obtained in the more traditional approaches by *the existence of an additional deeply-bound state in each channel,* and by an increase of π of the absolute singlet and triplet phase-shifts due to this extra unphysical bound state. This idea of using very deep realistic nucleon-nucleon interactions can be traced back to a paper by Neudatchin et al. [17].

It is well known from cluster nuclear physics that these seemingly contradictory features-that is, repulsive core versus deep potential descriptions-are two ways to stimulate the effects of the Pauli principle in a local potential model description when the two interacting particles are composed of *identical fermions* which will be explained in detail later when discussing and analyzing the results obtained in this section. The best understood example in this context is undoubtedly the system composed of two alpha-particles which has been discussed in detail in the previous Chapter. From this earlier discussion, we know that the empirical local Ali&Bodmer potential [14] extracted from the experimental alpha-alpha phase-shifts is l-dependent and presents a *repulsive* core for $l \leq 2$, while the deep, l-independent potentials ($V(r=0) \cong 120$ MeV), binding a number of unphysical bound states, can prove as successful in reproducing the data [12]. Moreover, it has also been realized [18] that the deep potential description leads to more satisfactory relative motion wave functions when compared to those of repulsive core potentials, which will be investigated here in this section for the weakly bound deuteron system by the consideration of the *node* number in these two different type wave functions.

There have been quite a few attempts to derive the features of the two-nucleon interaction from a quark picture of the nucleon. The most recent non-relativistic quark model calculations [19] have led to an effective nucleon-nucleon interaction with a strong repulsive core and an intermediate range attraction similar to those displayed by the empirical potentials. On the other hand, the work described in Ref.[16,20] has indicated, for the nucleon-nucleon scattering, that *the relative s-wavefunction has to have a node at small distance.* The existence of a node in the relative motion wavefunction can readily incorporated in a local

potential description, provided interaction is *deep* enough to accommodate one (nodeless) deeply bound state.

The work described here will, in addition to the other investigations undertaken, demonstrate explicitly the equivalence of such deep potentials with the more orthodox *repulsive* core empirical interactions, by constructing the phase-equivalent supersymmetric partner of the deep potential freed from unphysical bound states but still binding the deuteron with correct energy. The resulting supersymmetric potential for the deuteron case are to be shown to have a short range repulsive core followed by a shallow attractive part, which are very similar to those displayed by realistic interactions such as the Reid soft core potential [21].

As the main ingredients of the supersymmetric method aiming at the elimination of some bound states of a given potential without changing the rest of the spectrum has been presented in the previous chapters, we here directly apply this technique to the system of interest without introducing the formal theory used.

4.2 Superdeep And Supersymmetric Potentials

Now, superdeep potentials such as the Moscow potential introduced by Kukulin and his co-workers give deuteron wavefunctions which also have a node at short distances as required. The node arises because there is an additional bound state which is Pauli-forbidden for the actual neutron-proton (n-p) system. The latest version of Moscow potential [16] includes both central and a tensor component, together with the central and tensor one-pion exchange contributions (OPEP). In addition, the equations in Ref.[16] to solve both for the bound state (S- and D-wave) and the scattering problem in case of triplet potential are more complicated, one needs to work out the usual coupled equations.

For simplicity, in our calculations presented here, we use an alternative superdeep potential which produces a model deuteron wavefunction that has a node like the case of Kukulin et al. However, this simple potential does not have the required OPEP tail and do not consider D-wave related to the tensor potential.

So the physical observables calculated by this potential, such as the radius of the deuteron which we will deal with later in this section, should not be compared with the experimental value. We note at this point that our aim here is not the rigorous reproduction of experimental data but to test the reliability of a deep potential involving some unphysical bound states, and in particular to search for the wavefunction sensitivity features of the deep and of its supersymmetric partner-shallow potentials. For this reason, the use of an appropriate simple potential in the present analysis does not cause any physical problem.

A. The Two-Parameter Potential

Bearing in mind that the following discussion will be restricted to the s-wave n-p relative motion for simplicity, we consider here the two-parameter superdeep potential [22]

$$V(r) = -\frac{\hbar^2}{2\mu} A(A+1)\beta^2 \,\mathrm{sech}^2(\beta r) \tag{67}$$

where A and β are constants. Here, explicitly $\mathrm{sech}^2(\beta r) = \cosh^{-2}(\beta r)$ $= \left\{\left(e^{\beta r} + e^{-\beta r}\right)/2\right\}^{-2}$. The energy spectrum for this potential is given by [22]

$$E_n = -\frac{\hbar^2}{2\mu}(A-2n-1)^2\beta^2 \quad , \quad n = 0,1,2,\ldots \tag{68}$$

It is clear that far from the origin this potential decays exponentially, i.e. $\mathrm{sech}^2(\beta; r \to \infty) \approx 4e^{-2\beta r}$, whereas at the origin it has a parabolic behavior, such as $\mathrm{sech}^2(\beta; r \to 0) \approx 1-(\beta r)^2$. Throughout our calculations, the arbitrary constants A and β, for the Moscow-type binary potential introduced above, are taken as 3.146 and 1.585 fm^{-1} respectively, which give the correct *rms* and separation energy for the system considered. In this case, the ground state has a

binding energy of about 480 MeV, which is unphysical and needs to be suppressed. The physically meaningful deuteron bound state for this superdeep potential corresponds to the first excited state having a binding energy of about 2.225 MeV.

B. Partner Potentials For The Deuteron System

For building the partner of a given potential admitting the same eigenvalues except for that of the missing ground state has been discussed in Chapters 2 and 3. The method relies on a factorization property of the Hamiltonian, and makes possible the (exact) construction of the partner potential starting from the original potential ground state wavefunction. It actually requires two steps, the intermediate potential (V_1) having the same negative energy spectrum as the original *superdeep* potential V (except for the ground state of the latter) but a different phase-shift; V_1 is given by

$$V_1(r) = V(r) - 2\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}\ln\Psi_0(E_0) \quad , \tag{69}$$

where $\Psi_0(E_0)$ denotes the original ground state wavefunction. The second step provides the final phase-equivalent potential V_2 as

$$V_2(r) = V(r) - 2\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}\ln\left[\Psi_0(E_0)\Psi_1(E_0)\right] \quad , \tag{70}$$

where $\Psi_1(E_0)$ stands for the wavefunction at the same energy, but calculated with the intermediate potential V_1. Elimination of more than one state is accomplished by iterating this two-step procedure.

The two-parameter superdeep potential-$V(r)$ and its phase-equivalent supersymmetric partner-$V_2(r)$ are shown in Figure 4.1, and their corresponding wavefunctions in Figure 4.2.

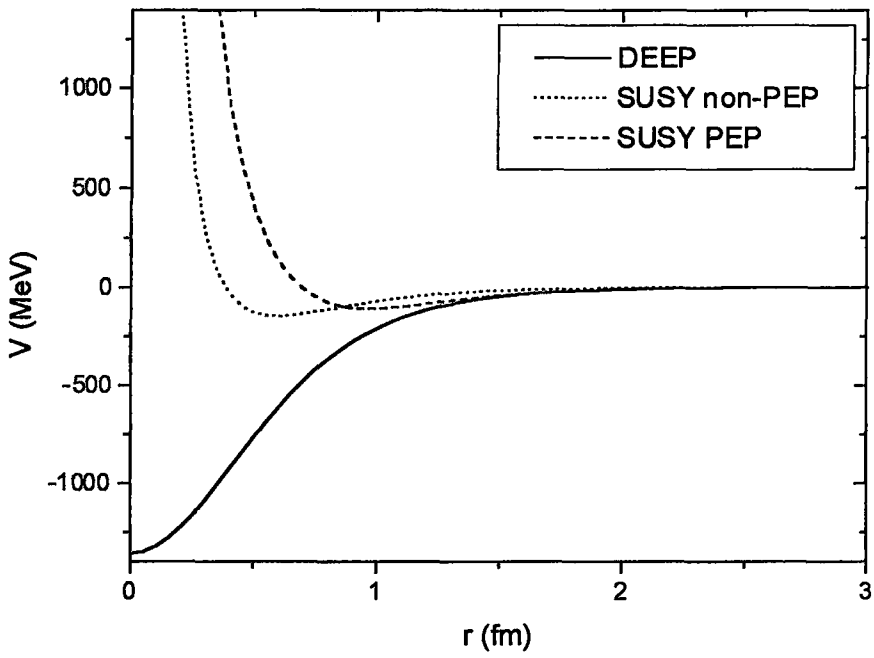

Figure 4.1 Superdeep potential $V(r)$ (solid line) and its supersymmetric partners $V_1(r)$ (non-PEP, dotted line) and $V_2(r)$ (PEP, dashed line) as a function of radius *r*.

As a result of the presence of one spurious bound state, the deuteron wavefunction for the superdeep potential possesses one node near the origin. This result supports the model deuteron wavefunction with a node obtained by Moscow potential. However, it is worth stressing that the transition from the two-parameter superdeep potential to our repulsive core interaction (V_2) does not significantly alter the outer part of the deuteron wavefunction, nor the deuteron *rms* radius which will be discussed later in this section.

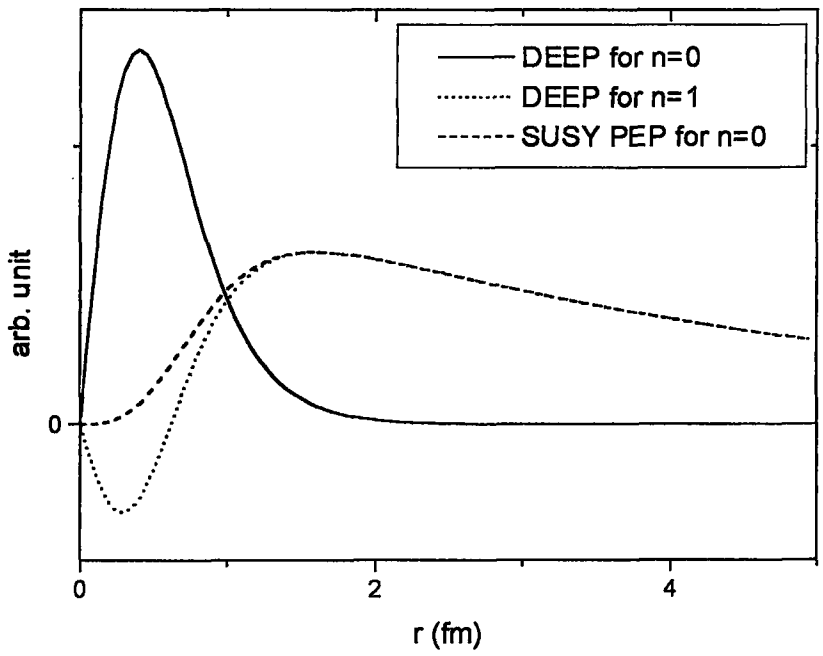

Figure 4.2 The first two eigenstates, n=0 for the ground state (solid line) and n=1 for the first excited state (dotted line) of the Hamiltonian with the superdeep two-parameter potential. The wave function illustrated by dashed line represents the ground state of the SUSY PEP, $V_2(r)$.

Figure 4.3 compares the superdeep potential considered with our repulsive r^{-2} core interaction. The strong central attraction of the former is seen to have replaced by a shallow attraction at intermediate distance; the general features of the reconstructed potential ($V_2(r)$), such as the radius of the repulsive core and the strength of the attractive part, are seen to be quite similar to those of the usual realistic interactions such as the Reid soft core central potential [21], which has also been plotted on the same figure.

In order to check the accuracy of the supersymmetric quantum mechanical methods used in constructing the phase-equivalent potential, we have carried out additional calculations on the phase-shift. The resulting phase-shift curve is compared with the original phase-shift in Figure 4.4 and seen an excellent agreement with the initial phase-shift.

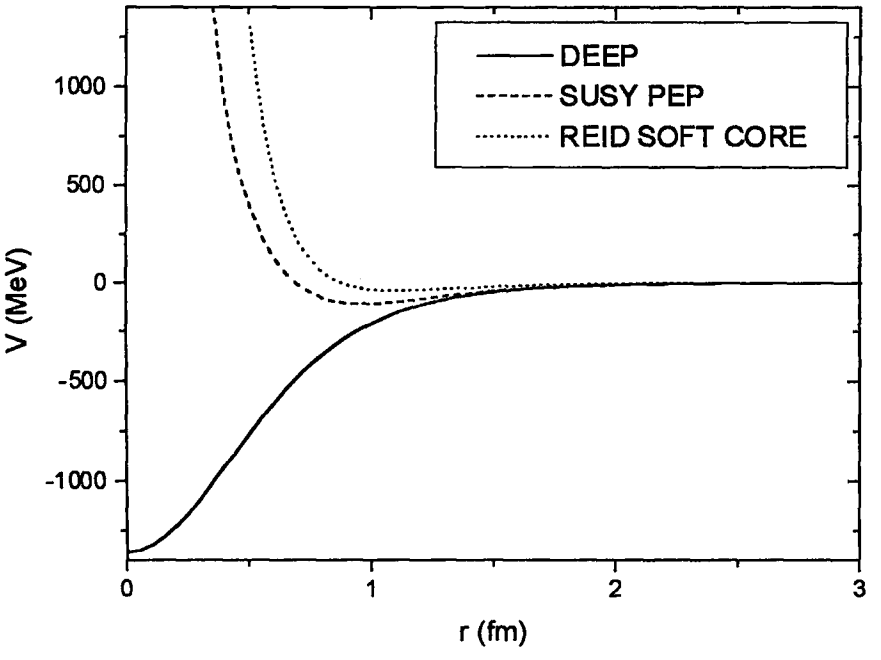

Figure 4.3 Comparison of the nucleon-nucleon central potential of the superdeep *sech-squared* $V(r)$ (solid line) potential with its PEP potential $V_2(r)$ (dashed line) and with the central Reid Soft Core (dotted line) interaction.

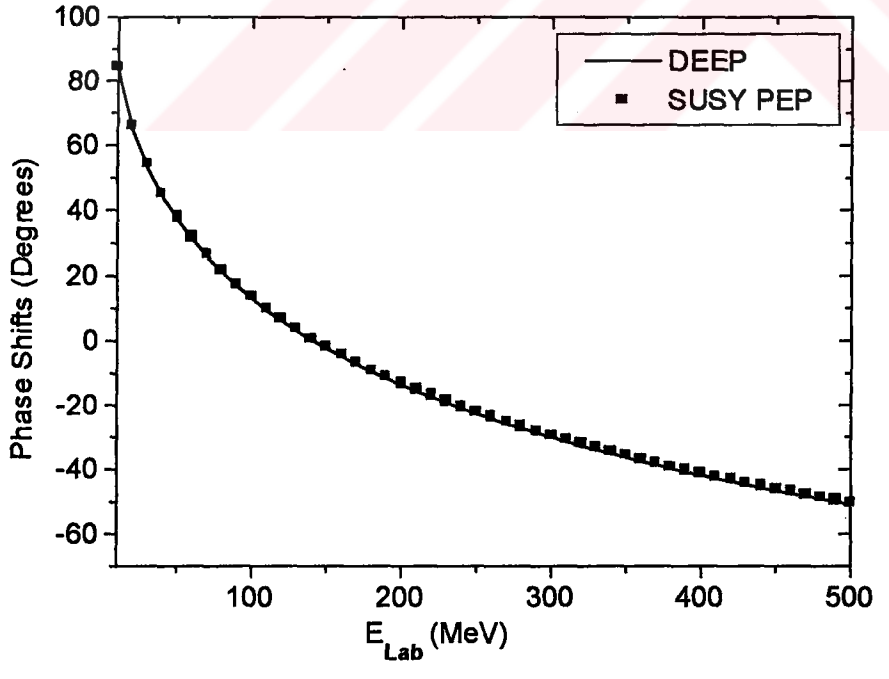

Figure 4.4 Nucleon-nucleon phase shift with the interaction of the two-parameter superdeep potential (solid line) and its phase-equivalent supersymmetric partner $V_2(r)$ (solid square). The full line has been shifted downwards by π.

The low energy behavior of the phase-shift is not affected by the singular nature of the repulsive core, and the usual formulation of Levinson's theorem [23] remains valid. This theorem relates the difference

$$\delta_l(E=0)-\delta_l(E=\infty)=N_l\pi \ , \tag{71}$$

of the phase-shifts of partial waves for the relative motion of the colliding particles at zero and infinite energies E to the number of bound states N_l of these potentials at a given angular momentum l. But our supersymmetric potentials, in spite of having different numbers of bound states, are nevertheless phase-equivalent. Now it must be admitted that this violation of the Levinson theorem is obtained at the cost of introducing a highly singular potential. That means if the original potential which we seek to modify is regular, i.e., if its first two moments are finite, the reconstructed potential cannot share this property because of the Levinson theorem. To make clear, we recall the idea of Swan [24] based on the property of singular r^{-2} repulsive potentials to lead to (negative) constant absolute phase-shifts at large energies, $E \to \infty$. In this case, $\delta_l(E)$ behaves like

$$\delta_l(E \to \infty) \approx -M_l\pi \quad . \tag{72}$$

This therefore opens the possibility to preserve the scattering properties of the original potential for angular momentum l while canceling M_l of its bound states provided the singular core potential behavior is introduced at small distances.

4.3 Matter Radius Of Deuteron

From Figure 4.2 it can easily be seen that our reconstructed phase-equivalent supersymmetric partner potential has lead to relative motion wavefunctions very similar to this generated by the deep potential outside the core region, but which lack the small distance radial node resulting from the suppression of the unphysical bound state. To clarify that if these wavefunctions lead to quantitatively different

results, we investigate the dependence of radius calculations, as an observable, on the wavefunction properties.

We have another reason for undertaking these additional calculations. If there is a node in the wavefunction and the wavefunction is reasonably large at small distances, then one might expect that because of the normalization the wavefunction at large distances would be reduced. In other words, the asymptotic function $\Psi_{r\to\infty} \approx Ae^{-ar}$ will have a smaller value of the asymptotic amplitude A and hence the radius would be reduced. This expectation is also to be clarified in this section.

We can calculate the deuteron matter radius r_m numerically from the formula

$$r_m^2 = \frac{1}{4}\int_0^\infty r^2\ \Psi^2(r)\,dr \quad , \tag{73}$$

for either $\Psi(r) = \Psi_0^{(1)}(r)$ or $\Psi(r) = \Psi_2^{(0)}(r)$, the bound state wavefunctions for the superdeep and transformed partner potential. The numerical calculations of *rms value* (root-mean-square radius) show that the radius of deuteron is 1.9538 fm and 1.9554 fm for the superdeep potential that has bound states at about 2.225 MeV (for n = 1) and for the shallow supersymmetric partner (V_2), respectively. The two values are so close and certainly do not differ by 0.002 fm. That means as a physical quantity, the radius calculated using the wavefunctions having one/no node reflect the nodal structure rather weakly. It can also be concluded that it is possible to find smaller deuteron radii for potentials like the sech-squared potential used here. Thus, in case of experiment requiring a reduction in r_m then short range contribution to the potential appears, such as the superdeep potential, to be necessary.

4.4 Discussion And Analysis On The Supersymmetric Behavior Of The Deuteron System

We briefly discuss here the supersymmetry aspect of nucleon-nucleon interaction in the light of the calculation results presented in the previous section. This very interesting aspect is connected with very deep interrelation between many-body and potential treatment for the composite particle interaction in case of the particles composed of elementary *fermions.*

Formal consideration based on the potential model has been discussed in this thesis work in the earlier chapter. This method makes it possible to establish explicit relation between initial potential having a deep-lying state and exactly phase-shift equivalent to the initial one, which does not comprising such deep-lying state with a singularity near the origin.

It is also clear from Figure 4.3 that in spite of *different* analytical behavior near the origin for both Reid Soft core interaction (behaves at $r \to 0$ as e^{-mx}/x) and the transformed phase-equivalent shallow potential $(V_2(r)_{r\to 0} \approx (const/r^2))$, we can observe in the figure almost complete coincidence of both interactions. This means we have very tight interrelation between superdeep two-parameter nucleon-nucleon model and the standard Reid soft core model.

However there is one more thing which is more interesting. This is just the supersymmetry treatment of the interrelation found between both models. Witten [4] was likely one of the first who suggested a general algebraic framework for supersymmetrical quantum mechanics which may be naturally used for the interpretation of such phenomena. As a simple illustrating example of his much more general supersymmetry considerations in field theory, Witten constructed very simple superalgebra joining commutators and anticommutators of some abstract charge operators and then he defined a super Hamiltonian from these charge operators presented in Chapter 2. After this, identifying some first-order

differential operators with the abstract charge operators he formulated some concrete realization of the suggested superalgebra. It was shown by Sukumar [9] that the simple Witten's superalgebra results in completely the same formulation as the factorization method discussed earlier in Chapter 2. Both Hamiltonians, i.e., the supersymmetric factorized Hamiltonian and initial (with extra bound state) appear to be two diagonal operators of a unified matrix supersymmetric Hamiltonian. This derivation, thus, is establishing much deeper connection between two alternative models, such as a superdeep and Reid soft core interaction potentials.

It makes it possible to proceed further and to ask about the deep reason of such supersymmetry. One possible answer may be as follows. The superalgebra, as it stands, is joining in one unified algebra the boson and fermion degrees of freedom. And accordingly, it can describe the interaction between bosonic and fermionic degrees of freedom.

Now the question is arising: *Which connection might exist between the exclusion of deep-lying bound states from some potential and supersymmetry?* The reply to this question we suggest is as follows. On the formal level, the interrelation may be the formal one only and it, likely, does not lead to a fruitful development. However, from physical point of view this interrelation means the existence of a deeply hidden relation between the relative motion of composite nucleus comprised from fermions and the internal excitations of the composites.

In fact, in non-relativistic quantum mechanics which is used for the treatment of interaction of the composite particles, the relative motion of the composites is treated as a *bosonic degree of freedom* (i.e., no Pauli principle constrains are put to the relative motion). On the other hand, the internal excitations of the quark degrees of freedom inside the composites in the process of the mutual collision of the composites should be treated as the manifestation of fermionic degrees of freedom. The main problem in description of composite particle interaction is the complicated interrelation between relative motion of the composites and their inner excitations. And from this point of view, the existence of the above supersymmetry aspect could mean that the collision of such

composites should be described correctly only within the framework of supersymmetrical quantum mechanics and the potential model is only the projection of this nontrivial picture onto mutual relative motion of the composites. It is evident at least, that the development along this direction may result in many new interesting ideas.

4.5 Conclusion On The Supersymmetric Solution Of The Deuteron System

We have shown that, for the deuteron system, the phase-equivalent central potentials have to be repulsive and singular at small distances, as the alpha-alpha scattering discussed in the previous Chapter, in order to preserve the energy behavior of the phase-shifts, and they present a shallow attractive part of intermediate range, in good agreement with the existing realistic repulsive- core nucleon-nucleon potentials. In particular our phase-equivalent partner potential is very similar to its Reid soft core counterpart. We stress however that the motivation of the present study is not to advocate the use of repulsive core potentials in many-body calculations, but rather to display in a transparent way the on-shell equivalence of superdeep potentials, such as the one considered here, with more conventional interactions.

In addition, by means of supersymmetric quantum mechanics we have found that the two-parameter superdeep potential and its supersymmetric partner, which is phase-equivalent to the former and freed from the unphysical deeply bound states, give very similar *rms* values for the deuteron system. Although the corresponding eigenstates differ in the node-number, our investigations have shown that also other physical quantities, such as the radius, calculated using these wavefunctions reflect the nodal structure rather weakly. This lends support to two nearly equivalent treatments of the Pauli principle by choosing the physical solution either by node-number criteria or by inclusion of a repulsive part of the potential at the origin. However, the similar *rms* results obtained do not automatically imply that other observables like transition probabilities, momentum distributions, or differential cross sections calculated using these wavefunctions have to coincide. Clearly considerable additional work is still needed to test further the virtues of a deep potential description of the nucleon-nucleon interaction.

CHAPTER 5

APPLICATION TO ONE-NEUTRON HALO SYSTEMS

Research with radioactive nuclear beams is currently one of the most active areas in nuclear physics. As one of the successful applications of such nuclear beams, exotic structures have been observed in nuclei near to the driplines, which are called, as the most interesting discoveries, *the neutron halo* [25]. The nucleus ^{11}Li is the first observed case of a neutron halo. Other neutron halo nuclei are ^{6}He, ^{11}Be, ^{14}Be and ^{17}B. These nuclei have opened studies of weakly bound nuclear systems, which has not freely accessible before. The current investigations on halo nuclei might shed a light on some unsolved problems related to the early evolution of the Universe, the Big Bang nucleosynthesis.

The term *neutron halo* derives from quantum mechanics, in which the size of a nucleus really means the extent of the probability density, given by the square modulus of the wave function. We can think of a *nucleus as a potential well* in which the nucleons are confined, and the wave function of the nucleus then decays exponentially outside of the well. *Normal nuclei*, which are strongly bound, have nucleons that occupy bound states *deep in the well.* For these nuclei the probability density falls of sharply outside the core, making it possible to picture them as classical objects. However, the neutrons in some neutron dripline nuclei -called neutron halo- are very loosely bound and occupy states close to the *top of the well.* These nuclei have *long-range* wave functions and are characterized by a cloud, or

halo, of neutron probability *that extends far beyond the dense core.* According to classical physics such nuclei should not exist at all because the strong nuclear force (the glue that binds neutrons and protons together) has too short to hold the far off neutrons in the halo. Instead, they owe their existence to quantum theory which describes the location of subatomic particles by a mathematical cloud of probability.

5.1 Introduction

In [26] and references therein essential features of loosely bound systems, having an unusually large size, are discussed. In this Chapter we examine the ground state of ^{11}Be as an example to halo systems, using our expertise gained from the works presented in the previous chapters. ^{11}Be consists of a single neutron halo with a ^{10}Be core nucleus and has a lifetime of 166 ± 15 fs before it decays to ^{11}B.

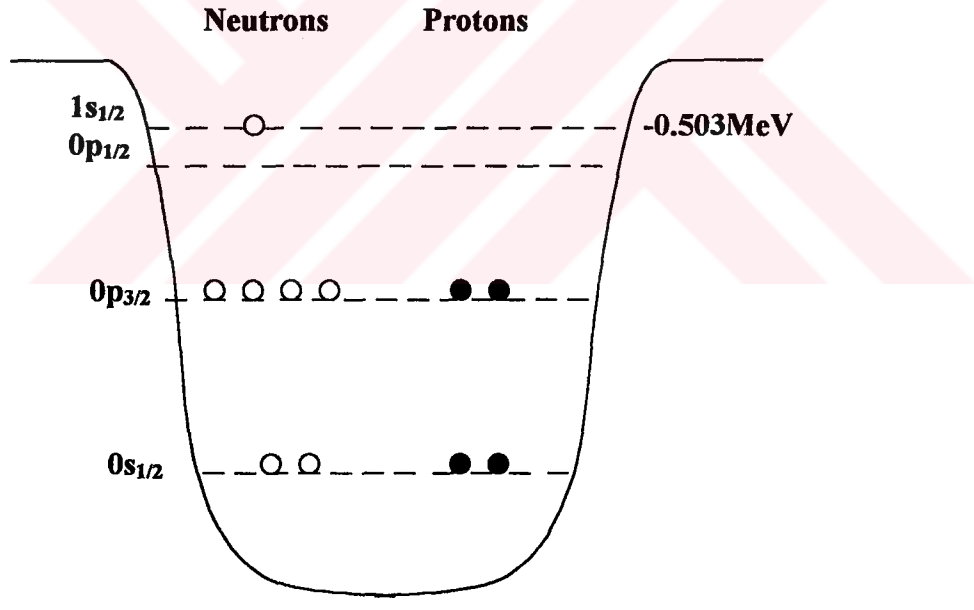

Figure 5.1 ^{11}Be shell structure.

The ratio of neutrons and protons is nearly 2. The naive picture of this state is that the core neutrons occupy $0s_{1/2}$ and $0p_{3/2}$ completely, while $0p_{1/2}$ holds

just one neutron. The ground state J^{π} is thus *expected* to be $\frac{1}{2}^{-}$, whereas *experimentally it is known* to be $\frac{1}{2}^{+}$ having the single neutron separation energy of 0.503 MeV [27]. Thus, the dominant component of the ^{11}Be ground state is produced by the coupling of a $1s_{1/2}$ neutron to a ^{10}Be core. This nucleus is of particular theoretical interest because the ground state parity is exactly opposite to what one naively would expect from the spherical shell model.

There is one more crucial parameter of ^{11}Be which will be used later on in the calculations; the matter root-mean-square radius R_{RMS} with value measured to be 2.73 ± 0.05 fm [28] or 3.10 ± 0.38 fm [29]. One can relate the R_{RMS} to the single neutron root-mean-square radius R_{rms} by the simple formula [26]:

$$R^2_{RMS}(matter) = \frac{A}{A+1} R^2_{RMS}(core) + \frac{A}{(A+1)^2} R^2_{rms} \ , \qquad (74)$$

where A is the mass number of the *core*, here the ^{10}Be nucleus. Using $R_{RMS}(core) = 2.3$ fm extracted in [28] from experiment, it is easy to check that the value $R_{rms} = 6.70$ fm gives roughly the average of the measured values of $R_{RMS}(matter)$.

5.2 Phase-Equivalent Potentials For The ^{11}Be System

In this analysis, which will be restricted to the $s-motion$ only, we make use of the sech-squared deep potential introduced in the previous Chapter, due to the success in analyzing the deuteron with such a potential. To determine the free parameters A and β for this potential in analyzing ^{11}Be system we have to solve the following system equations:

$$E_n = -\frac{\hbar^2}{2\mu}(A-2n-1)^2\beta^2 = -0.503\,MeV \quad ;$$

$$R_{rms}^2 = \int_0^\infty dr\, r^2\, \Psi^{(n)}(A,\beta,r)^2 = (6.70\,fm)^2 \quad , \tag{75}$$

where μ is the reduced mass of the ^{10}Be+n system.

We start with the corresponding wavefunctions $\Psi_m^{(n)}$ of the ground and first excited state for the potential of interest, which have the following analytical expressions[30]:

$$\Psi_0^{(0)}(r) = N_0 \frac{\sinh(\beta r)}{\cosh^A(\beta r)} \quad ,$$

$$\Psi_0^{(1)}(r) = N_1 \frac{\sinh(\beta r)}{\cosh^A(\beta r)}\left[2(2-A)\cosh^2(\beta r) + 2A - 1\right] \quad , \tag{76}$$

with normalization constants N_0 and N_1 given by

$$N_0^2 = 2\beta \frac{\Gamma(A+1/2)}{\Gamma(3/2)\Gamma(A-1)} \quad \text{and} \quad N_1^2 = \frac{\beta(A-3)\Gamma(A+1/2)}{3\Gamma(3/2)\Gamma(A-1)} \quad . \tag{77}$$

The physical solution is chosen using the node-number and parity quantum number criteria. As we take into account that, see Figure 5.1, the $0s_{1/2}$ orbit is completely occupied and put the single neutron in $1s_{1/2}$ state (having one node in the wave function) to have $J^\pi = \frac{1}{2}^+$ suggested by the experiment, in the calculations the case with node number $n = 1$ is selected that corresponds to $\Psi_0^{(1)}(r)$.

Now we are ready to apply the supersymmetric technique to create the phase-equivalent potentials (PEP) for the sech-squared superdeep potential given. For the ^{11}Be one-nucleon halo system, with the calculated values of $A = 3.2113$ and $\beta = 0.7007$ fm^{-1}, using the formulae given by the previous chapters, the supersymmetric partner potentials and corresponding eigenstates are calculated. Fig.5.2 illustrates the superdeep and related partner potentials while Figure 5.3 gives their eigenstates respectively. It is seen from Figure 5.3 that the eigenfunction corresponding the PEP is a nodeless ground state. In this case the Pauli principle is taken into account by the repulsive part of the potential (see Figure 5.2), repulsive up to 1.5 fm approximately.

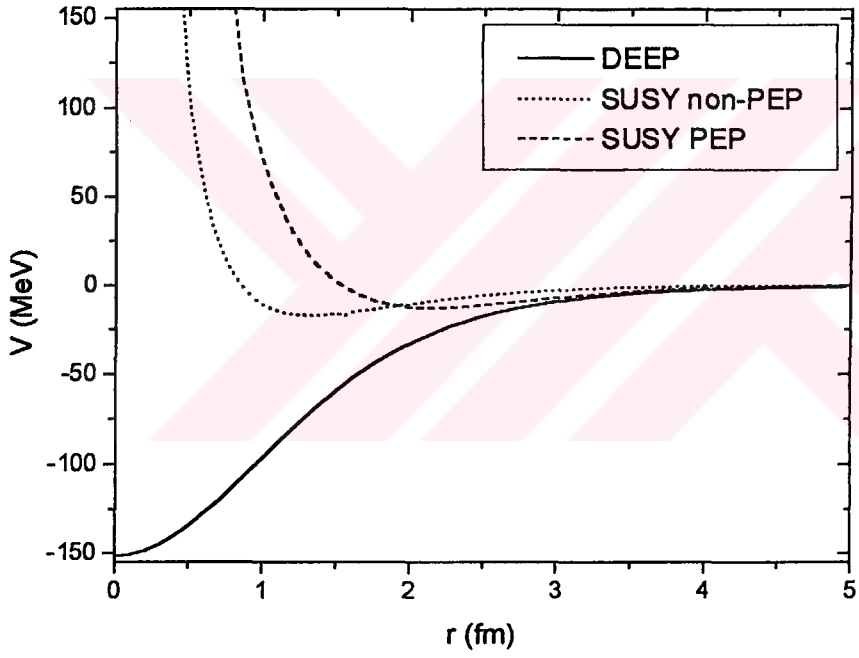

Figure 5.2 Potential $V(A,\beta;r)$ (solid line) and its supersymmetric partners $V_1(A,\beta;r)$ (non-PEP, dotted line) and $V_2(A,\beta;r)$ (PEP, dashed line) as a functions of radius r.

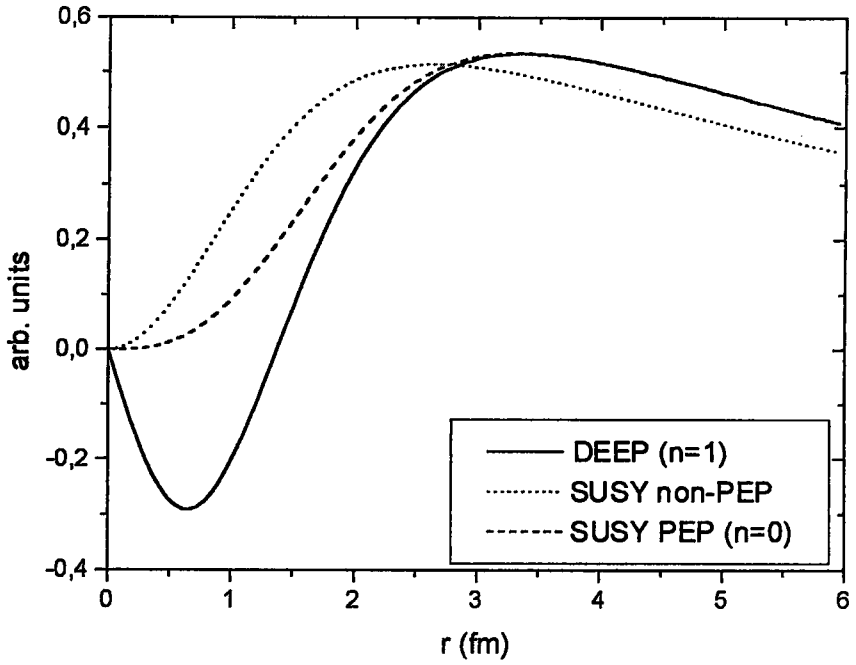

Figure 5.3 Radial wave functions $\Psi_0^{(1)}(A,\beta;r)$ (solid curve), $\Psi_1^{(0)}(A,\beta;r)$ (dotted curve) and $\Psi_2^{(0)}(A,\beta;r)$ (dashed curve) all normalized to one.

If one calculates the R_{rms} using the nodeless eigenstate of PEP, again the value of 6.70 fm is obtained. Even the wave functions differ in the node number, like in the deuteron case discussed in detail in the previous chapter, the R_{rms} is nearly the same for the phase-equivalent potentials, namely for the attractive - superdeep sech-squared and its phase-equivalent repulsive-shallow potentials. We have tabulated below the calculated *rms* radii of a valance neutron with respect to the ^{10}Be core:

Table 5.1: Calculated *rms* radii of a valence neutron with respect to the ^{10}Be core.

^{11}Be	$V(A,\beta;r)$	$V_1(A,\beta;r)$	$V_2(A,\beta;r)$
Neutron *rms* radius, fm	6.7099	6.1613	6.7826

One should not forget that the non-PEP potential does not belong to the PEP family, that is why the corresponding R_{rms} for this particular case differs approximately 10% from the other two values. Moreover, the corresponding eigenfunction (see Figure 5.3) of this potential has a different asymptotic behavior as well, as was discussed earlier in this thesis work.

We have also calculated the s-wave phase shifts in case of ^{10}Be(n, n)^{10}Be elastic scattering up to 20 MeV for the three potentials, see Figure 5.4. The results clearly exhibit the difference between PEP and non-PEP.

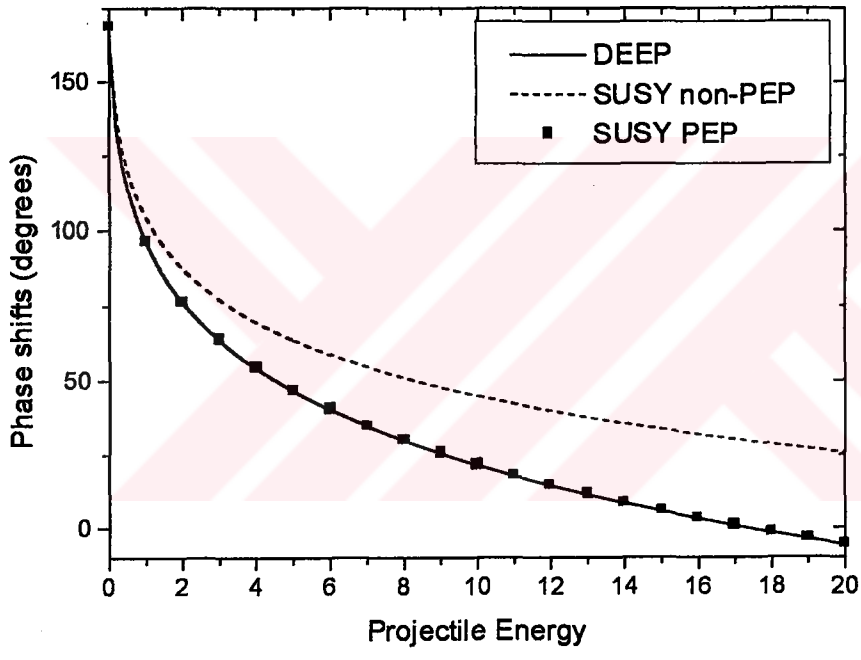

Figure 5.4 Comparison of the s-wave phase shifts for the potential $V(A,\beta;r)$ and its supersymmetric partners $V_1(A,\beta;r)$ (non-PEP) and $V_2(A,\beta;r)$ (PEP).

5.3 Conclusion

We have discussed a halo nucleus, ^{11}Be, in the frame of supersymmetric quantum mechanics. A halo system includes both the weak binding and huge *rms* of the valance neutron. Taking into account both these quantities mentioned above, the two-parameter sech-squared superdeep potential is constructed and its phase-equivalent partner is found. Our calculations have shown that both the two-parameter deep potential and its phase-equivalent partner give radial wave functions with essentially the same R_{rms} in spite of the fact that they differ in node-number. This observation strongly suggests that the Pauli principle in two-body dynamics can be approximated in essentially two equivalent ways; either selection according to node number criteria or by using a repulsive core.

CHAPTER 6

SUMMARY and CONCLUSION

Using the supersymmetric quantum mechanics, the properties of the deep nuclear interaction have been investigated by constructing explicitly phase-equivalent potentials freed from the unphysical deeply bound states of the former, by the consideration of α+α interaction, the deuteron and ^{11}Be nuclei. The resulting central potentials have to be repulsive and singular at small distance in order to preserve the energy behaviour of the phase shifts, and they present a shallow attractive part of intermediate range.

The supersymmetric formulations used through the present calculations have focused in general to the Pauli principle for the weakly bound systems. Assuming that the two-body potentials have Pauli forbidden states, one can then construct easily and use the phase-equivalent partners without these forbidden states. At small distances the lowest levels of the original deep potentials correspond to identical fermions occupying the same states. Removal of these terms therefore forces the particles to occupy higher-lying orbits and thereby introducing the necessary repulsion preventing violation of the Pauli principle. The supersymmetric method used in this work to exclude the Pauli forbidden states in the weakly bound systems has firm mathematical and numerical foundations. It is a practical and accurate alternative to the other existing methods.

Our reconstructed potentials (PEP) for weakly bound systems, such as the deuteron and ^{11}Be nuclei, have led to relative motion wave functions very similar to

those generated by the deep potentials outside the core region, but which lack the small distance radial node. Both types of potentials are therefore expected to display rather different behaviours, and presumably lead to qualitatively different results. However, we have shown in the present work that a deep potential and its phase-equivalent supersymmetric partner give very similar *rms* values for the weakly bound nuclei considered . This lends support to two nearly equivalent treatments of the Pauli principle, which means that Pauli principle in two-body dynamics can be approximated in essentially two equivalent ways; either selection according to node number criteria or by using a repulsive core at the origin.

However, the similar *rms* results obtained here do not automatically imply that other observables calculated using the supersymmetric wavefunctions with one-/no- node have to coincide. Clearly considerable additional work is still needed to test further the virtues of a deep potential description of the nuclear interactions. The work along this line is in progress.

REFERENCES

[1] J. Wess and B. Zumino, Nucl. Phys. **B 70** (1974) 39; M. F. Sohnius, Phy. Rep. **128** (1985) 39.

[2] V. A. Kostelecky and D. K. Campbell , *Supersymmetry in Physics*, (North Holland, 1985).

[3] H. Nicolai, J. Phys. **A 9** (1976) 1497.

[4] E. Witten, Nucl. Phys. **B 202** (1982) 253; E. Witten, Nucl. Phys. **B 188** (1985) 513.

[5] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251** (1995) 267, and references therein.

[6] A. A. Andrianov, N. V. Borisov and M. V. Ioffe, Phys. Lett. **A 105** (1985) 19.

[7] C. V. Sukumar, J. Phys. **A 18** (1985) L57.

[8] I. M. Gelfand and B. M. Levitan, Izv. Akad. Nauk. SSSR, Ser. Mat. **15** (1951) 309.

[9] C. V. Sukumar, J. Phys. **A 18** (1985) 2917; 2937.

[10] A. Khare and U. Sukhatme, J. Phys. **A 22** (1989) 2847, and references therein.

[11] P. E. Hodgson, *Nuclear Heavy Ion Reactions*, (Clarendon, Oxford, 1978).

[12] B. Buck, H. Friedrich, and c. Wheatley, Nucl. Phys. **A 275** (1977) 246.

[13] D. Baye, Phys. Rev. Lett. **58** (1987) 2738.

[14] S. Ali and A.R. Bodmer, Nucl.Phys. **80** (1966) 99.

[15] F. Michel and G. Reidemeister, Z. Phys. A **329** (1988) 385.

[16] V. I. Kukulin, V. M. Krasnopol'skii, V. N. Pomerantsev, P. B. Sazonov, Sov. J. Nucl. Phys. **43** (1986) 355 [Yad. Fiz. **43** (1986) 559]; V. I. Kukulin, V. N. Pomerantsev, V. M. Krasnopol'skii, P. B. Sazonov, Phys. Lett. **135** (1984) 20.

[17] V.G. Neudatchin, V.I. Kukulin, V.L. Korotkikh, V.P. Korennoy, Phys. Lett. **B 34** (1971) 581.

[18] Q.Q.K. Liu, Nucl. Phys. **A 550** (1992) 263.

[19] K. Maltman, N. Isgun, Phys. Rev. **D 29** (1984) 952.

[20] A. Faessler, in : Proceedings Of The Fourth International Conference On Clustering Aspects Of Nuclear Structure And Nuclear Reactions, Chester, U. K. (1985) p. 319.

[21] R. V. Reid, Ann. Phys., (N.Y.) **50** (1968) 411.

[22] R. Dutt, A. Khare, and U. Sukhatme, Am. J. Phys. **56** (1988) 163.

[23] R.G. Newton, *Scattering theory of waves and particles.* 2nd Edn., Berlin, Heidelberg, New York: Springer 1982;
B.N. Zakhariev and A.A. Suzko, *Direct and Inverse Problems* (Potentials in Quantum Mechanics) (Springer-Verlag Berlin Heidelberg 1990).

[24] P. Swan, Ann. Phys., (N.Y) **48** (1968) 455; P. Swan. Nucl. Phys. **46** (1963) 669.

[25] I. Tanihata, J. Phys. G **22** (1996) 157.

[26] M. V. Zhukov, B. V. Danilin, D. V. Ferodov, J. M. Bang, I. J. Thompson and J. S. Vaagen, Phys. Rep. **231** (1993) 151; P. G. Hansen, A. S. Jensen and B. Jonson, Ann. Rev. Nucl. Part. Sciences **45** (1995) 591.

[27] F. Ajzenberg-Selove, Nucl. Phys. **A 506** (1990) 1.

[28] I. Tanihata, T. Kobayashi, O. Yamakawa, S. Shimoura, K. Ekuni, K. Sugimoto, N. Takahashi, T. Shimoda and H. Sato, Phys. Lett. **B 206** (1998) 592.

[29] E. Liatard, J. F. Glasser, S. Kox, Tsan Ung Chan, G. J. Costa, C. Heitz, Y. El Masri, F. Hanappe, R. Bimbot, D. Guillemaud-Mueller and A. C. Mueller, Europhys. Lett. **401** (1990) 13.

[30] D. Ridikas, J.S. Vaagen, and J.M. Bang. NP **A 609** (1996) 21.

PUBLICATIONS

1) B. Gönül, O. Özer and M. Yılmaz, Investigation Of Phase-Equivalent Potentials By A Halo Transfer Reaction, Submitted To the Few-Body Systems Journal.

APPENDICES

APPENDIX A

The Fortran Program Used for the Calculations of Supersymmetric Partner Potentials

```
      PROGRAM SUPER
*-----------------------------------------------------------------
*     PROGRAM 'SUPER.FOR' CALCULATES THE SUPERSYMMETRIC PARTNER
*      POTENTIALS FOR ALPHA-ALPHA SCATTERING AND EXACTLY REPRODUCES
*     D.BAYE'S RESULTS.
*     SEE REF: PHYS.REV.LETT., VOL.58, (1987) 2738, FIGURE 1 AND 2
*     SEE REF: NUCL.PHYS. A550 (1992) 263, FIGURE 2
*-----------------------------------------------------------------
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (IB=4000, H2M=10.375D0, B=7.D0)
      DIMENSION R(IB),W(IB),S(IB),SDER1(IB),RATIO(IB),ADDPOT(IB)
      DIMENSION V01(IB),V02(IB),V03(IB),V04(IB),V00(IB),W00(IB)
      DIMENSION VALBOD(IB)
*-----------------------------------------------------------------
*     CHANGE THIS DATA IF NECESSARY
*-----------------------------------------------------------------

      DATA L,ITERM/0,2/
*-----------------------------------------------------------------
         H=B/IB
*-----------------------------------------------------------------
*     DEFINE THE STARTING DEEP GAUSSIAN TYPE POTENTIAL
*     FOR THE SOLUTION OF SCHRODINGER EQUATION
*-----------------------------------------------------------------
        DO 10 I=1,IB
        R(I)=I*H
        V00(I)=V(R(I))
        W(I)=(V00(I)/H2M)+L*(L+1)/(R(I)*R(I))
        W00(I)=W(I)
10      CONTINUE
*-----------------------------------------------------------------
*       START FOR THE CALCULATION OF PARTNER POTENTIALS
*-----------------------------------------------------------------

         DO 15 IJ=1,ITERM
*-----------------------------------------------------------------
*        SET G.S ENERGY ZERO
*-----------------------------------------------------------------
         E=0.D0
         S2=0.D0
         EPS=0.001D0
```

```
*-----------------------------------------------------------------
*         EPS=ACCURACY
*-----------------------------------------------------------------
*         DEFINE NUMBER OF NODES, ZERO FOR G.S
-----------------------------------------------------------------

          NEL=0
*-----------------------------------------------------------------
*      CALCULATE G.S WAVEFUNCTION,S(R),USING D.BAYE'S NUMEROV
       TECHNIQUE REF. RAYNAL-1972
*-----------------------------------------------------------------

          CALL NUMEROV(IB,H,E,S2,W,S,NEL,EPS)
*-----------------------------------------------------------------
*         CALCULATE FIRST DERIVATIVE OF S(R)
*-----------------------------------------------------------------

          CALL PDERIV(S,SDER1,IB,H)
*-----------------------------------------------------------------
*      CALCULATE S'/S FOR EVALUATING V1 AND V3 PARTNER
*      POTENTIALS
*-----------------------------------------------------------------

           DO 5002 I=1,IB
5002       RATIO(I)=SDER1(I)/S(I)
*-----------------------------------------------------------------
*         TAKE DERIVATIVE OF RATIO,(d/dr)(s'/s)
*-----------------------------------------------------------------

          CALL PDERIV(RATIO,ADDPOT,IB,H)
*-----------------------------------------------------------------
          DO 5003 I=1,IB
*-----------------------------------------------------------------
*         CALCULATES ONLY V1 AND V3 WHICH
*         ARE NOT PHASE EQUIVALENT TO STARTING DEEP POTENTIAL
*-----------------------------------------------------------------
          IF(IJ.EQ.1) THEN
          V01(I)=H2M*W(I)-2.D0*H2M*ADDPOT(I)
          ELSE
          V03(I)=H2M*W(I)-2.D0*H2M*ADDPOT(I)
          ENDIF
5003      CONTINUE
*-----------------------------------------------------------------
*         CALCULATES V2 AND V4 WHICH ARE PHASE EQUIVALENT TO
*         THE STARTING DEEP POTENTIAL. WE USE HERE D. BAYE'S
*         TECHNIQUE:THE INTEGRAL,INT(0-R)(S_g.s)**2 dr IS
*         DETERMINED AT EACH MESH POINT WITH THE ADAM'S
*         INTERPOLATION FORMULA (ABRAMOWITZ AND STEGUN
*         1972.EQ.25.5.5)
*-----------------------------------------------------------------
       ins=1
       s1=s(3)
       s2=s(2)
       s3=s(1)
       k=2*ins+1
       s(1)=log(h*s(1)**2/k)
       ss=2*h*s(2)**2/k
       s(2)=log(ss)
       ss=ss+h/24*(9*s(3)**2+19*s2**2-5*s3**2)
       s(3)=log(ss)
```

```
      do3 j=4,ib
      ss=ss+h/24*(9*s(j)**2+19*s1**2-5*s2**2+s3**2)
      s3=s2
      s2=s1
      s1=s(j)
    3 s(j)=log(ss)
*-----------------------------------------------------------------
*     5-point second derivative of the logarithm of the integral
*     (Abramowicz and Stegun: 25.3.24)
*-----------------------------------------------------------------
      s2=s(1)
      s1=s(2)
      s(1)=-k/h**2
      s(2)=s(1)/4
      do4 j=5,ib
      s3=s2
      s2=s1
      s1=s(j-2)
    4 s(j-2)=(-(s(j)+s3)+16*(s(j-1)+s2)-30*s1)/(12*h**2)
      s(ib-1)=0
      s(ib)=0
      do5 j=1,ib
      w(j)=w(j)-s(j)*2
*-----------------------------------------------------------------
      if(ij.eq.1) then
      v02(j)=h2m*w(j)
      else
      v04(j)=h2m*w(j)
      endif
*-----------------------------------------------------------------
    5 continue

*-----------------------------------------------------------------
*     ALI AND BODMER POTENTIAL IS
*-----------------------------------------------------------------

      DO I=1,IB
       VALBOD(I)=VV(R(I),L)
      END DO
*-----------------------------------------------------------------

       IOMAX=105
       IS=(IB-1)/IOMAX+1
*-----------------------------------------------------------------
       OPEN(20,file='V01-V02.DAT',STATUS='UNKNOWN')
       OPEN(21,FILE='V03-V04.DAT',STATUS='UNKNOWN')

*-----------------------------------------------------------------
*      COMPARISON OF PHASE EQUIVALENT POTENTIALS WITH
*      THOSE OF ALI&BODMER POTENTIALS FOR DIFFERENT
*      'L' VALUES
*-----------------------------------------------------------------

*       DO 8000 I=1,IB,IS
*      IF(L.EQ.0.AND.IJ.EQ.2)WRITE(20,200)R(I),V04(I),VALBOD(I)
*      IF(L.EQ.2.AND.IJ.EQ.1)WRITE(20,200)R(I),V02(I),VALBOD(I)
*      IF(L.EQ.4.AND.IJ.EQ.1)WRITE(20,200)R(I),H2M*W00(I),VALBOD(I)
*200     FORMAT(2X,F10.5,2X,E13.5,2X,E13.5)
*8000    CONTINUE
*-----------------------------------------------------------------
```

```
*******************PHASE-EQUIVALENT POTENTIALS*******************
        DO 8000 I=1,IB,IS
         IF (IJ.EQ.1) THEN
          WRITE(20,200) R(I),H2M*W00(I),V01(I),V02(I)
         ELSE
          WRITE(21,200) R(I),V03(I),V04(I)
         ENDIF
200     FORMAT(2X,F10.5,2X,E13.5,2X,E13.5,2X,E13.5,2X,E13.5)
8000    CONTINUE

*---------------------------------------------------------------------
*       RETURN BACK FOR THE NEXT ITERATION IN CALCULATING V3 AND V4
*---------------------------------------------------------------------

15      CONTINUE

        END

*---------------------------------------------------------------------
*      DEFINE THE DEEP POTENTIAL FOR ALPHA-ALPHA INTERACTION
*      INCLUDES COULOMB AND ERF CORRECTION TERM
*---------------------------------------------------------------------

       DOUBLE PRECISION FUNCTION V(R)
       IMPLICIT DOUBLE PRECISION (A-H,O-Z)
       PARAMETER (V0=122.694D0, BETA=0.22D0)
        V=-V0*EXP(-BETA*R*R)+4.D0*1.2D0*1.2D0/R*(ERF(0.75D0*R))
       END
*/////////////////////////////////////////////////////////////////

*---------------------------------------------------------------------
*      DEFINE ALI&BODMER POTENTIAL FOR ALPHA-ALPHA INTERACTION
*      INCLUDING COULOMB AND ERF CORRECTION TERM WRT 'L'
*---------------------------------------------------------------------

       DOUBLE PRECISION FUNCTION VV(R,L)
       IMPLICIT DOUBLE PRECISION (A-H,O-Z)
       PARAMETER (H2M=10.375D0)
       IF(L.EQ.0) THEN
          CR=0.7D0
          CA=0.475D0
          VR=500.D0
          VA=130.D0
       ELSE IF(L.EQ.2)THEN
          CR=0.7D0
          CA=0.475D0
          VR=320.D0
          VA=130.D0
       ELSE
          CR=0.D0
          CA=0.475D0
          VR=0.D0
          VA=130.D0
       ENDIF
         VV=VR*EXP(-CR*CR*R*R)-VA*EXP(-A*CA*R*R)+4.D0*1.2D0*1.2D0/R*
      &   (ERF((SQRT(3.D0)/(2.D0*1.44D0))*R))+H2M*L*(L+1)/(R*R)
       END
*/////////////////////////////////////////////////////////////////
```

```
*-----------------------------------------------------------------
      DOUBLE PRECISION FUNCTION ERF(X)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER(A1=0.254829592D0,A2=-0.284496736D0,A3=1.42141374D0)
      PARAMETER(A4=-1.453152027D0,A5=1.061405429D0,P=0.3275911D0)
      T(Z)=1.D0/(1.D0+P*Z)
      Y=ABS(X)
      IF(Y.LE.3.2.D0)THEN
      ERF=1.D0-(A1*T(Y)+A2*T(Y)*T(Y)+A3*T(Y)**3+A4*T(Y)**4
     & +A5*T(Y)**5)*EXP(-Y*Y)
      ELSE
       ERF=1.D0
      ENDIF
      IF(X.LT.0.D0) ERF=-ERF
      END
*/////////////////////////////////////////////////////////////////

*-----------------------------------------------------------------
*     SUBROUTINE TAKES DERIVATIVES
*-----------------------------------------------------------------
      SUBROUTINE PDERIV(V,VP,MAXJ,H)
*-----------------------------------------------------------------
      implicit real*8 (a-h,o-z)
      dimension v(maxj),vp(maxj),x(5)
      j2=maxj-2
      v1=1.e0/(h*12.e0)
      do 10 i=1,5
10     x(i)=v(i)
      vp(1)=v1*(48.e0*x(2)-25.e0*x(1)-36.e0*x(3)+16.e0*x(4)-
     %      3.e0*x(5))
      vp(2)=v1*(18.e0*x(3)-3.e0*x(1)-10.e0*x(2)-6.e0*x(4)+x(5))
      vp(3)=v1*(8.e0*(x(4)-x(2))+x(1)-x(5))
      do 30 j=4,j2
      do 20 i=1,4
20      x(i)=x(i+1)
        x(5)=v(j+2)
30     vp(j)=v1*(8.e0*(x(4)-x(2))+x(1)-x(5))
      vp(maxj-1)=v1*(6.e0*x(2)-x(1)-
*           18.e0*x(3)+10.e0*x(4)+3.e0*x(5))
      vp(maxj)=(3.e0*x(1)-16.e0*x(2)+36.e0*x(3)-
*           48.e0*x(4)+25.e0*x(5))
      vp(maxj)=vp(maxj)*v1
      return
      end
*/////////////////////////////////////////////////////////////////

*-----------------------------------------------------------------
*    USE PROF D. BAYE S VERSION OF NUMEROV METHOD FOR
*    THE CALCULATION OF SUPERSYMMETRIC PARTNER WAVEFUNCTIONS
*-----------------------------------------------------------------

      SUBROUTINE NUMEROV(N,H,E,S2,U,S,NO,EPS)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION U(1),S(1)
      DATA RAP1,RAP2/0,0/
      H12=H*H/12
C*****CONTROLE DES CONDITIONS ASYMPTOTIQUES
      IF(E.GT.0) E=0
      DEI=0
      EPSS=1.0D-11
```

```
      IF(U(N-1).GT.EPSS) GO TO 10
      DEI=U(N-1)-EPSS
      DO 8 K=1,N
    8 U(K)=U(K)-DEI
   10 U(N)=U(N-1)
C*****CALCUL DU NOMBRE D"ETATS LIES PAR INTEGRATION A ENERGIE NULLE
      S(1)=1.0D-10
      B0=0
      AA=H12*U(1)
      IF (S2) 16,18,16
   16 B0=-S(1)*AA
   18 B1=S(1)*(1-AA)
      DO 38 K=2,N
      B2=12*S(K-1)-10*B1-B0
      IF (ABS(B2).LT.1.E+10) GO TO 22
      B2=B2*1.0D-20
      B1=B1*1.0D-20
   22 AA=H12*U(K)
      S(K)=B2/(1-AA)
      B0=B1
   38 B1=B2
      DO 42 K=5,N
      N0=K
      IF(U(K).LT.0) GO TO 44
   42 CONTINUE
   44 NEL=0
      DO 52 K=N0,N
      IF (S(K-1)*S(K)) 46,50,52
   46 NEL=NEL+2
      GO TO 52
   50 NEL=NEL+1
   52 CONTINUE
      NEL=NEL/2
      IF(NEL.GT.NO) GO TO 64
      IF(NEL.EQ.NO) GO TO 60
   62 NO=-1
      RETURN
   60 RAP1=S(N-1)/S(N)
      RAP2=EXP(H*SQRT(U(N-1)-E))
      IF(RAP1.LT.RAP2) GO TO 62
C*****CALCUL DE EMIN ET EMAX ENTRE LESQUELLES SE TROUVE L"ENERGIE
C*****PROPRE
   64 UMIN=U(1)
      DO 70 K=2,N
      IF(U(K).LT.UMIN) UMIN=U(K)
   70 CONTINUE
      EMIN=UMIN
      EMAX=0
C*****DEBUT DE LA RECHERCHE DE L"ENERGIE PROPRE DANS L"INTERVALLE
C*****MAXIMU
      TE=EMAX-EMIN
C*****REJET DE L"ENERGIE D"ESSAI E PROPOSEE SI ELLE EST A
C*****L"EXTERIEUR DE
C*****BORNES (EMIN,EMAX)
      IF((E.LT.EMIN).OR.(E.GT.EMAX)) E=EMIN+TE/2
      E1=EMIN
      E2=EMAX
      J=2
      I=1
      GO TO 102
```

```
C*****REDUCTION DES BORNES EMIN ET EMAX
   90 EMIN=E1
      EMAX=E2
      TE=EMAX-EMIN
      J=2
   98 I=1
  100 E=TE/J
      E=EMIN+E*I
  102 DE=0
  104 E=E+DE
      IF(E.GT.0) GO TO 204
      S(N)=1.0D-10
      N1=N-1
      RAP2=EXP(H*SQRT((U(N-1)+U(N))/2-E))
      S(N1)=S(N)*RAP2
      AA=H12*(U(N1)-E)
      B0=S(N)*(1-AA)
      B1=S(N1)*(1-AA)
      N1=N-2
      DO 138 KAUX=1,N1
      K=N1-KAUX+1
      B2=12*S(K+1)-10*B1-B0
      AA=H12*(U(K)-E)
      S(K)=B2/(1-AA)
      B0=B1
      B1=B2
      IF(U(K).LT.E) GO TO 140
  138 CONTINUE
  140 N1=K
C*****NORMALISATION DE LA FONCTION D"ONDE A S(N1)
      DO 146 KAUX=N1,N
      K=N-KAUX+N1
  146 S(K)=S(K)/S(N1)
C*****DEBUT DE L"INTEGRATION VERS L"EXTERIEUR JUSQU"A N1
      S(1)=1.0D-10
      B0=0
      AA=H12*(U(1)-E)
      IF(S2) 156,158,156
  156 B0=-S(1)*AA
  158 B1=S(1)*(1-AA)
      DO 170 K=2,N1
      B2=12*S(K-1)-10*B1-B0
      AA=H12*(U(K)-E)
      S(K)=B2/(1-AA)
      B0=B1
  170 B1=B2
C*****NORMALISATION DE LA FONCTION A S(N1)
      DO 174 K=1,N1
  174 S(K)=S(K)/S(N1)
C*****CALCUL DE LA CORECTION D"ENERGIE
      SOM=0
      DO 180 K=1,N
  180 SOM=SOM+S(K)*S(K)
      DE=((-S(N1-1)+2-S(N1+1))/(H*H)+U(N1)-E)/SOM
      IF(ABS(DE).GT.EPS) GO TO 104
C*****CALCUL DU NOMBRE DE NOEUDS DE L"ETAT PROPRE TROUVE
      DO 182 K=5,N
      IF(U(K).LT.E) GO TO 184
  182 CONTINUE
  184 N0=K
```

```
      NEL=0
      DO 192 K=N0,N1
      IF(S(K-1)*S(K)) 186,190,192
  186 NEL=NEL+2
      GO TO 192
  190 NEL=NEL+1
  192 CONTINUE
      NEL=NEL/2
C*****L"ETAT PROPRE TROUVE EST-IL LE BON
      IF(NEL-NO) 198,214,202
  198 IF(E.GT.E1) E1=E
      GO TO 204
  202 IF(E.LT.E2) E2=E
  204 I=I+2
      IF (I.LE.J)  GO TO 100
      J=2*J
      IF(ABS(E1-EMIN).GT.EPS.OR.ABS(EMAX-E2).GT.EPS) GO TO 90
      GO TO 98
C*****NORMALISATION DE LA FONCTION PROPRE
  214 SOM=1/SQRT(SOM*H)
      DO 218 K=1,N
  218 S(K)=S(K)*SOM
      E=E+DEI
      RETURN
C*****DEBUT FORMATS
 2000 FORMAT(/,35X,56HL"ETAT DEMANDE N"EST PAS LIE. RETOUR DE NUM1L
C*****AVEC
     1 NO=-1,/)
C*****FIN FORMATS
      END
```

APPENDIX B

A Discussion on the Phase-Shift Calculations

A simple case is the study of the motion of a particle without spin in an external spherically symmetric potential. The time-independent Schrödinger equation in this case reads

$$H\Psi(r) = E\Psi(r) \tag{B.1}$$

with the Hamilton operator

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r) \tag{B.2}$$

The solution of Eq. (B.1) can be written as, providing $u_l(r)$ is a solution of the radial equation,

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_l(r) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}u_l(r) + V(r)u_l(r) = E\,u_l(r) \quad . \tag{B.3}$$

If Eq. (B.3) is rearranged, one gets

$$\frac{d^2\,u_l(r)}{dr^2} + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]u_l(r) = 0 \quad , \tag{B.4}$$

where $k^2 = \left(\frac{2\mu}{\hbar^2}\right)E$ and $U(r) = \left[\frac{2\mu}{\hbar^2}\right]V(r)$.

The quantity l occurring in Eqs. (B.3) and (B.4) is the angular momentum quantum number. The symbol μ is the reduced mass appearing in the above equations.

The solution of Eq. (B.4) is well-known Bessel functions and the behavior of the wave function, asymptotically, is

$$u_l^{phy}(r) = e^{i\delta_l}\left[\cos\delta_l F_l(kr) + \sin\delta_l G_l(kr)\right] \qquad \text{(B.5)}$$

where $u_l^{phy}(r)$ is the physical solution.

To obtain an equation for numerical calculations, we will chose two random points where $r_2 > r_1 > R$ so that our solutions, according to Eq.(B.5), will be

$$A\,u_l^{num}(r_1) = u_l^{phy}(r_1) = e^{i\delta_l}\left[\cos\delta_l F_l(kr_1) + \sin\delta_l G_l(kr_1)\right], \qquad \text{(B.6)}$$

and

$$A\,u_l^{num}(r_2) = u_l^{phy}(r_2) = e^{i\delta_l}\left[\cos\delta_l F_l(kr_2) + \sin\delta_l G_l(kr_2)\right]. \qquad \text{(B.7)}$$

If we take the ratio of Eq. (B.6) to Eq. (B.7), we obtain

$$\frac{u_l^{num}(r_1)}{u_l^{num}(r_2)} = \frac{u_l^{phy}(r_1)}{u_l^{phy}(r_2)} = \frac{\left[\cos\delta_l F_l(kr_1) + \sin\delta_l G_l(kr_1\right]}{\left[\cos\delta_l F_l(kr_2) + \sin\delta_l G_l(kr_2\right]}. \qquad \text{(B.8)}$$

Eq. (B.8) is reduced to the form

$$\frac{u_l^{num}(r_1)}{u_l^{num}(r_2)} = \frac{[F_l(kr_1) + \tan\delta_l G_l(kr_1]}{[F_l(kr_2) + \tan\delta_l G_l(kr_2]} \quad , \tag{B.9}$$

and for the numerical calculations of the phase shift, Eq. (B.9) is rearranged as

$$\tan\delta_l = \frac{u_l^{num}(r_1)F_l(kr_2) - u_l^{num}(r_2)F_l(kr_1)}{u_l^{num}(r_2)G_l(kr_1) - u_l^{num}(r_1)G_l(kr_2)} \quad . \tag{B.10}$$

The related computer program considering a gaussian type n-p interaction, as an example, is attached.

```
*-----------------------------------------------------------------------
*     Program 'Phase.for' calculates the phase shift for a gaussian
*      type n-p interaction.
*-----------------------------------------------------------------------
      implicit real*8(a-h,o-z)
      dimension wfn(201)
      data conno1/0.2196079d0/
      data ndata,sep/201,0.1d0/
*-----------------------------------------------------------------------
      pai=4.d0*atan(1.d0)
*-----------------------------------------------------------------------
*     fmu is a changable value and shows the reduced mass of the
*     system, and gets 0.5 for deuteron reduced mass.
*-----------------------------------------------------------------------
      fmu=0.5d0
      h2mu=1.d0/(fmu*conno1*conno1)
*-----------------------------------------------------------------------
      ll=0
      rjval=0.0
      spin=0.0
      print *,'input rk '
      read *,rk
      ener=rk*rk/conno1/conno1/fmu
      call wx(wfn,ndata,sep,fmu,ener,ll,rjval,spin)
      stop
      end
*************************************************************************
      subroutine wx(wf,nx,drx,fmu,e1,lval,rjval,spin)
*-----------------------------------------------------------------------
*     calculates relative wavefunction assuming gaussian
*     n-p interaction
*-----------------------------------------------------------------------
      implicit real*8 (a-h,o-z)
      dimension wf(201)
      dimension potr(201)
      data rc0,conno1,conno2/3.0d0,0.2196079d0,0.15805086d0/
      data r11,v11/1.08d0,-123.696496785d0/
      pai=4.d0*atan(1.d0)
      ll=lval+1
      sowt=-0.5*(rjval*(rjval+1.)-spin*(spin+1)-lval*(lval+1))
      zt=0.d0
      do 33 i=2,nx
      rrr=(i-1)*drx
      ex1=dexp(-(rrr/r11)**2)
      potr(i)=v11*ex1
   33 continue
      wf(1)=0.d0
      rk=conno1*dsqrt(e1*fmu)
      drhod=rk*drx
      constd=(conno1)**2*fmu/(rk**2)
      hhpd=(5.d0/6.d0)*drhod**2
      hhmd=-drhod**2/12.d0
      pai=4.d0*datan(1.d0)
      xmd=0.d0
      f1d=0.d0
      fd=drhod**ll
      ii=1
      wf(ii+1)=fd
      con=ll*(ll-1)
      f2d=con/drhod**2+constd*(potr(ii+1)-e1)
```

```
      do 778 ii=1,nx-1
      if(ii.eq.1) go to 688
      rr=ii*drx
      rhod=rr*rk
      con=ll*(ll-1)
      f3d=con/rhod**2+constd*(potr(ii+1)-e1)
      gpd=fd
      fd=(2.d0+hhpd*f2d)*fd-(1.d0+hhmd*f1d)*xmd
      fd=fd/(1.d0+hhmd*f3d)
      wf(ii+1)=fd
      xmd=gpd
      f1d=f2d
      f2d=f3d
  688 continue
  778 continue
      i2=nx-1
      rho2=(i2-1)*rk*drx
      i1=i2-10
      rho1=(i1-1)*rk*drx
      rj1=sin(rho1)
      rn1=-cos(rho1)
      rj2=sin(rho2)
      rn2=-cos(rho2)
      det=-rj1*rn2+rj2*rn1
      aval=(-wf(i1)*rn2+wf(i2)*rn1)/det
      bval=( wf(i2)*rj1-wf(i1)*rj2)/det
      amp=sqrt(1.d0/(aval**2+bval**2))
      do 222 i=2,nx
*-----------------------------------------------------------------
*     wave function norm is sqrt(2/pai)*sin(kr+...) as r-> infinity
*-----------------------------------------------------------------
      wf(i)=wf(i)*amp*sqrt(2/pai)
  222 continue
      if(wf(5).gt.0.d0) go to 77
      do 223 i=1,nx
  223 wf(i)=-wf(i)
      aval=-aval
      bval=-bval
   77 continue
      phase=atan2(bval,aval)
*-----------------------------------------------------------------
*     calculated phase value is converted into Degree by dividing
*     (pai*180.0)
*-----------------------------------------------------------------
      phased=phase/pai*180.
      print 734,e1,rk,phased,exp(2*(0.d0,1.d0)*phase)
  734 format(1h ,5f10.5)
      return
      end
```