

**Vibrational Energy Harvesting with Piezoelectric Patches
Integrated on Plate Structures**

by

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ABSTRACT

Vibrational energy harvesting has received considerable attention in recent years to develop self-powered devices that can offer many advantages for a wide variety of medical and aerospace applications. As a part of this trend, piezoelectric resonant beams have been extensively studied over the past decade. As an alternative to harvester beams, patch-based piezoelectric energy harvesters integrated to thin plates can be more convenient and practical for use on the surfaces of aerospace, automotive, and marine applications. However, piezoelectric energy harvesting on two-dimensional lightweight and flexible structures has been rarely reported in the existing literature and accurate mathematical models are required for the design and analysis of patch-based harvesters.

Considering these needs, for the first time in the literature, this dissertation introduces analytical electroelastic models of patch-based piezoelectric energy harvesters attached on thin plates along with experimental validations and performance investigations for periodic and random vibrations. At first, distributed-parameter electroelastic model of a single patch-based piezoelectric harvester attached on a thin plate is developed. Exact solutions for steady-state electrical and structural responses are derived and closed-form expressions for electroelastic frequency response functions (FRFs) are obtained. Experimental measurements are carried out on a piezoceramic patch harvester structurally integrated on an all-four-edges clamped aluminum plate. Analytical electroelastic model is validated by comparing analytical predictions and experimental results. Power generation performance of the harvester patch from multiple vibration modes is investigated for a wide range of resistive loads. The single-patch model is extended to multiple patches connected in series and parallel. Coupled electroelastic models of multiple harvesters are developed and closed-form steady-state response expressions are derived for the series and parallel connection configurations. Experimental and analytical case studies of series- and parallel-connected double patch harvesters are given for validating predictions of multiple-patch

models. Finally, patch-based harvesting from broadband and band-limited random vibrations of thin plates is studied. Analytical and numerical solutions for the expected power output and mean-square vibration response are obtained based on multiple-patch electroelastic models. Experiments for random vibration energy harvesting are conducted with the experimental setup having triple patch-based harvesters connected in series. It is shown that analytical and numerical predictions of expected structural and electrical responses agree well with experimental measurements for a wide range of resistive loads. Finally, random vibrational energy harvesting scenarios are studied to explore the effects of the excitation bandwidth on the expected power output and the optimal position of the piezoelectric energy harvester.

The studies presented in this dissertation demonstrate that the developed analytical models successfully predict the electroelastic behavior of patch-based harvesters attached on thin plates. The analytical and experimental results also show that patch-based harvesting is an effective interface for vibrational energy harvesting from two-dimensional thin structures. Therefore, this dissertation contributes to the energy harvesting field by presenting accurate and compact analytical models of patch-based piezoelectric harvesters.

ÖZETÇE

Titreşimden elektrik enerjisi üretimi, medikal ve havacılık uygulamalarında kullanılabilir ve kendi enerjisini üreten cihazların geliştirilmesi amacıyla son yıllarda önemli derecede ilgi çeken bir araştırma alanı olmuştur. Bu araştırma eğiliminin bir parçası olarak, son on yılda, titreşen piezoelektrik kirişler üzerine yoğun araştırma faaliyetleri yürütülmüştür. Kiriş üreteçlere alternatif olarak, plaka ve paneller üzerine yerleştirilebilen yama-tipi piezoelektrik enerji üreteçler otomobil, gemi ve havacılık yapıları için daha uygun olanaklar sunmaktadır. Ancak, esnek ve hafif yapıların titreşiminden piezoelektrik malzeme ile elektrik enerjisi üretimi çok az sayıda araştırmada raporlanmıştır ve yama-tipi piezoelektrik enerji üreteçlerin tasarım ve analizinde kullanılmak üzere doğru matematiksel modellere ihtiyaç bulunmaktadır.

Bu tez, ince bir plaka yapının titreşiminden yama-tipi piezoelektrik enerji üreteçlerle elektrik enerjisi üretiminin analitik yöntemlerle modellenmesini, deneysel doğrulama çalışmalarını ve performans analizlerini ilk kez sunmaktadır. İlk olarak, ince bir plaka üzerinde bulunan yama-tipi piezoelektrik enerji üreticinin dağıtılmış-parametre elektroelastik modeli geliştirilmiştir. Yapısal ve elektriksel çıktılarının kararlı durumları için analitik çözümleri bulunmuş ve elektroelastik frekans tepki fonksiyonlarının (FTF) kapalı formda formülleri elde edilmiştir. Deneysel ölçümler dört-kenarından tutturulmuş alüminyum bir plaka üzerinde bulunan piezoelektrik yama-tipi üreteçle gerçekleştirilmiştir. Analitik elektroelastik model, deneysel ve analitik sonuçlar karşılaştırılarak doğrulanmış ve plakanın farklı titreşim modlarındaki elektriksel güç çıktısı geniş bir direnç aralığı için incelenmiştir.

Daha sonra, tek-yama elektroelastik modeli, çok sayıda seri veya paralel bağlı yamalar için geliştirilmiştir. Böylelikle, çok sayıda piezoelektrik enerji üreteçler için elektroelastik model elde edilmiş ve yapısal ile elektriksel çıktılarının kararlı durumları için analitik çözümler elde edilmiştir. Deneysel ve analitik durum çalışmaları iki adet seri ve paralel

bağlı yama-tipi piezoelektrik üreteçler için sunulmuş ve analitik model tahminlerinin deneysel ölçümlerle çokça tutarlı olduğunu gösterilmiştir.

Çalışmanın devamında, geniş ve kısıtlı bantta gelişigüzel (random) titreşimlerden elektrik enerjisi üretilmesi incelenmiştir. Beklenen elektrik güç değeri ile titreşim hızının ortalama-kare değerinin analitik ve sayısal çözümleri, bu tez kapsamında geliştirilen elektroelastik modeller kullanılarak türetilmiştir. Gelişigüzel titreşim deneyleri plaka üzerine yapıştırılan üç adet seri bağlı yama-tipi üreteç için gerçekleştirilmiştir. Deneysel sonuçlar, analitik ve sayısal tahminlerin tutarlı ve benzer olduğunu göstermiştir. Son olarak, gelişigüzel titreşimden elektrik enerjisi üretimi senaryoları, kuvvet bant aralığının beklenen elektrik gücü ile yama-tipi enerji üreteçlerin eniyileştirilmiş pozisyonu üzerindeki etkilerini incelemek için sunulmuştur.

Bu doktora tezi kapsamında sunulan çalışmalar, yama-tipi üreteçlerin elektroelastik davranışlarını geliştirilen analitik modellerin doğru olarak tahmin ettiğini göstermiştir. Ayrıca, analitik ve deneysel sonuçlar yama-tipi enerji üreteçlerin iki boyutlu yapılar üzerinde başarılı bir şekilde titreşimden elektrik enerjisi üretimi için kullanılabilir olduğunu göstermektedir. Bu sebeple, bu doktora tezi enerji üretimi araştırma alanına kullanılabilir ve doğru analitik modeller sunarak katkıda bulunmaktadır.

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DEDICATION

To my mother, Tlay Arıdođan,

and

In the loving memory of my father, İsmet Arıdođan,

January 15, 1943 – July 18, 2012

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Chapter 1

INTRODUCTION

Clean energy technologies, efficient energy use and energy recovery are the right and effective ways to combat climate change, save the environment, and reach the sustainable world. Therefore, energy related problems at large- and small- scales have become a research focus of scientists from all disciplines. Within this focus, energy harvesting can be regarded as energy recovery operation aiming to generate useful electric power from available ambient energy that is otherwise wasted as movement, heat, light, sound and vibration.

Vibrational energy harvesting research has grown significantly over the past decade and targets to generate useful electrical power for small electronic components and eliminate the need of battery replacement and disposal [1-7]. For instance, distributed wireless sensor networks are the main components of structural health and condition monitoring systems that are becoming more widely used in aerospace, automotive, and marine applications. It is worth pointing out that the average level power consumption of a wireless sensor network node is in the order of milliwatts (i.e., low-powered [8-10]) and new wireless technologies lead a power consumption level below microwatts (i.e., ultra-low powered [11-13]), which are reasonable amounts to be provided by vibrational energy harvesting technologies [14].

Over the last two decades, several methods for vibration-to-electric energy conversion have been demonstrated ranging from fundamental transduction principles to smart materials. The fundamental principles include electromagnetic [15, 16] and electrostatic [17, 18]) mechanisms, whereas the successive examples of smart materials are piezoelectric [4,

19], magnetostrictive [20], electroactive [21], and metamaterials [22]. Among all conversion these alternatives, piezoelectric transduction has become the most preferable technique since piezoelectric materials generate relatively large output voltages together with high power densities compared to other available conversion mechanisms [5]. In addition to these, the key advantage of piezoelectric materials among others is the manufacturability in a wide variety of geometries and sizes [4, 19].

The literature of piezoelectric energy harvesting has mostly focused on analytical, numerical and experimental investigations of unimorph and bimorph piezoelectric cantilevered beams excited by base vibrations [4, 6, 23-26]. However, a cantilevered beam harvester usually needs an external structural support, which may not be preferred in weight critical systems (e.g., aircraft and spacecraft applications). It is also essential to note that broadband energy harvesting capabilities of cantilevered harvester beams are limited due to the large spacing between the vibration modes for a given frequency range unless combined with a nonlinear system.

As an alternative to using harvester beam arrangements, patch-based piezoelectric energy harvesters can be more practical and effective by directly attaching to plate-like structures while enabling more compact and broadband energy harvesters. It is worthy to mention that the first vibration modes of plate-like host structures are much denser than that of cantilevered harvester beam configurations. However, power output investigations of piezoelectric energy harvesters on lightweight and flexible structures have been rarely reported in the existing literature. The common use of plate structures in marine, aerospace, and transportation systems results in the need for accurate models to be used in energy harvesting performance evaluation and optimization.

1.1 Objectives of the Dissertation

The main goal of this dissertation is to develop a complete modelling and analysis framework for piezoelectric energy harvesting from vibrations of flexible thin plates. A reliable analytical model can be used in design, performance prediction, and power output optimization of patch-based piezoelectric energy harvesters. Compared to finite element modelling techniques, analytical solutions are much more computationally efficient while enabling better and deeper understanding of coupled electroelastic behavior of piezoelectric energy harvesting on host structures.

Therefore, the first objective of this dissertation is to develop an analytical model of a single patch-based piezoelectric energy harvester structurally integrated to a thin plate. Distributed-parameter electroelastic modelling approach is used based on Kirchhoff's plate theory assumptions and the solution is obtained using the modal analysis procedure. Closed-form exact solutions for the steady-state voltage output and vibration response are derived in the scope of this objective. Investigation of the importance of positioning of the patch-based harvester is also of interest and is carried out through an electroelastic analysis of mode-dependent dynamic strain distribution. Experimental validation of the single-patch electroelastic model is performed with a case study. Comparisons of experimental measurements and analytical predictions validate the electroelastic model over a wide range of resistive loads. The case study also demonstrate that the electrical power can be generated from multiple vibration modes of the thin plate using a single patch-based harvester.

The second objective is to extend the single-patch electroelastic model to multiple patch-based harvesters connected in series and parallel. The main purpose of this objective is to show that series and parallel connections of patch-based harvesters can yield effective broadband energy harvesting by combining the electrical outputs of harvesters from different vibration modes of thin plates. For each connection configuration, the

electroelastic model is developed and closed-form expressions are derived. After obtaining closed-form solutions, model validations are provided with experimental case studies comprising series and parallel connections of patch-based harvesters.

The third objective is to study random vibrational energy harvesting with patch-based harvesters. In many practical applications, the nature of vibrational excitations is non-harmonic and random, and the ambient energy is distributed over a wide frequency spectra. Thus, the primary focus of the third objective is placed on derivation of analytical and numerical solutions for the expected power output and mean-square vibration response for the case of random force inputs. The analytical and numerical predictions are validated for random vibrational energy harvesting by comparing with experimental measurements. The effects of excitation bandwidth of the force input on the expected power output and the optimal position of the harvester are explored to maximize the power output of the harvester.

1.2 Contributions of the Dissertation

The main contribution of this dissertation is to develop analytical models that can be used in the design and analysis of patch-based piezoelectric energy harvesters integrated to thin plate-like structures. While presenting analytical electroelastic models and closed-form exact solutions, this dissertation provides experimental validations of the proposed analytical models with several case studies. Most of the material given in this dissertation is based on recent publications of Aridogan et al. [27-31]. It is important to note that the analytical electroelastic modelling of piezoelectric energy harvesting on two-dimensional structures is the extension of the electroelastic beam model developed by Erturk and Inman [32].

The motivation for choosing flexible two-dimensional plates as host structures is their widespread use in engineering structures and systems. In this dissertation, experimental

measurements are conducted on lightweight and flexible aluminum panels in the laboratory environment and electrical power generations of the patch-based piezoelectric energy harvesters are reported analytically and experimentally. Closed-form exact solutions derived by this dissertation can be utilized for optimal positioning as well as material selection and analysis of the harvesters and host structures.

The other major contribution of this dissertation is to present modelling and analysis of series and parallel connection configurations of multiple patch-based energy harvesters attached on thin plates. Experimental measurements and analytical predictions presented in this dissertation illustrate the effectiveness of multiple patch-based harvesters to generate the electrical power outputs from multiple vibration modes of the host structures.

Final contribution of this dissertation is the formulation and demonstration of random vibrational energy harvesting with patch-based harvesters.

Overall, this dissertation provides a complete analytical and experimental framework for electroelastic modelling and analysis of patch-based harvesters and demonstrates that patch-based harvesting is a promising approach to be used on thin vibrating flexible plate-like structures.

1.3 Organization of the Dissertation

This dissertation is organized as follows. Chapter 2 provides a background of piezoelectric materials and presents their role in energy harvesting and vibration control. A brief history of piezoelectricity is provided and then a comprehensive literature review of piezoelectric vibration control and energy harvesting applications is presented. The main advantages and challenges associated with these applications are discussed.

Chapter 3 presents analytical electroelastic modeling and experimental validations of a single patch-based energy harvester integrated on a thin plate. At first, the distributed-parameter electroelastic model of the thin plate with the patch-based harvester is derived

based on Kirchhoff's plate theory assumptions in physical coordinates. Then, by following a modal analysis procedure for all-four-edges clamped rectangular plate, closed-form expressions for electrical and structural steady-state responses are obtained for the transverse force excitations. After that, electroelastic frequency response functions (FRFs) are extracted and generalized for different boundary conditions of thin plates. For a case study, experimental measurements and analytical predictions are given for a wide range of load resistances to show the accuracy of the analytical model.

Chapter 4 extends the single-patch electroelastic model introduced in Chapter 3 to multiple patch-based energy harvesters that are connected in series and parallel configurations. For each connection configuration, electroelastic models of multiple patch-based harvesters are obtained and closed-form steady-state electrical and structural response expressions are derived by taking into account multiple vibration modes of the host plate. Model validations for series and parallel connection configurations are also presented with experimental case studies.

In Chapter 5, analytical modelling and experimental investigations of patch-based energy harvesting from broadband and band-limited random vibrations of thin plates are presented. In this chapter, the frequency and time domain solutions are obtained for the expected power output and mean-square vibration response. After that, experimental investigations are presented to validate the analytical and numerical predictions for a wide range of resistive loads. At the end of this chapter, the effects of excitation bandwidth and the optimal position of the harvester patch are discussed through random energy harvesting simulations.

Chapter 6 draws conclusions from the chapters and concludes the dissertation with outcomes and discussions of future work. At the end of this dissertation, appendices are provided for additional details of the main chapters that could be of interest to piezoelectric energy harvesting and vibration control community.

Chapter 2

BACKGROUND: APPLICATIONS OF PIEZOELECTRIC MATERIALS

This chapter gives a brief overview of piezoelectric materials and reviews their role in energy harvesting and vibration control applications. Research on piezoelectric energy harvesting has followed the research on piezoelectric vibration control of flexible structures. During the last four decades, piezoelectric materials have been employed as sensors and actuators in vibration and noise control systems as an alternative to passive treatment techniques, whereas piezoelectric energy harvesting for electrical power generation has attracted the interest of researchers over the last two decades. Noting that vibration control and energy harvesting are associated with the aims of dissipating and extracting mechanical energy, it is worthy to mention that piezoelectric energy harvesting and vibration control research fields are linked with challenges of piezoelectric modelling and instrumentation, and the motivations for real-life applications. Thus, both of energy harvesting and control communities can benefit from each other. In line with this purpose, this chapter reviews the studies focused on energy harvesting and vibration control applications.

2.1 Piezoelectricity and Piezoelectric Materials

In the year of 1880, Jacques and Pierre Curie (as cited by Cady [33] and Mason [34]) discovered an electromechanical phenomenon in certain classes of crystals as transformation of mechanical strain energy to electrical energy. One year later, based on thermodynamic principals, Lippman [34, 35] theoretically showed that there also exists a converse effect of this electromechanical phenomenon as a conversion of electrical energy to mechanical energy. Following to Lippman's theoretical analysis, the Curie brothers experimentally verified two-way coupling of electromechanical phenomena in 1881. It is worth pointing out that Hankel (as cited by Ballato [35]) named this phenomena as “piezoelectricity” from the Greek word “piezein”, which means to squeeze.

Almost a century after the discovery of natural piezoelectric materials, the concise mathematical formulation was provided in the form of linear constitutive equations by IEEE Standard on Piezoelectricity [36] as

$$D_i = e_{ikl} S_{kl} + \varepsilon_{ij}^S E_k \quad (1.1)$$

$$T_{ij} = c_{ijkl}^E S_{kl} - e_{kij} E_k \quad (1.2)$$

where c_{ijkl}^E , e_{kij} and ε_{ij}^S are elastic, piezoelectric, and permittivity parameters, respectively. Superscripts E and S denote constant electrical field and constant strain where the respective elastic and permittivity parameters are obtained. The electrical displacement tensor is D_i , the stress tensor is T_{ij} , the electrical field tensor is E_k and the strain tensor is S_{kl} . These electro-elasto-mechanical parameters are expressed in Voigt notation [37].

The electromechanical coupling described by equation (1.1) is referred as direct piezoelectric effect and transforms mechanical strain to electrical charge while enabling piezoelectric materials to be used as sensors [38-40] and energy harvesters [32, 41, 42]. On the other hand, the converse piezoelectric effect given by equation (1.2) provides

transformation of electrical charge to mechanical strain and offers the ability to be used as electromechanical strain actuators [43-45]. Readers are referred to Cady [33], Yang [46], IEEE Standard on Piezoelectricity [36] for more aspects of piezoelectricity.

Starting from 1950s, synthetic piezoelectric materials have been manufactured in a wide range of sizes and shapes, and PZT ceramics (lead-zirconate-titanate) has become the most popular synthetic piezoelectric material [37]. The piezoelectric coupling of synthetic materials is many times higher than the natural materials; therefore, synthetic piezoelectric materials have been most widely used and researched for vibration control, structural health monitoring, and energy harvesting applications.

2.2 Piezoelectric Vibration Control

In one of the first studies on piezoelectric vibration control, Forward [47] introduced the concepts of passive and active vibration suppression with piezoelectric patches. In his pioneer work, experimental demonstrations of passive and active vibration suppression with piezoelectric actuators were presented for a metal bar and a complex optical system. For the case of passive vibration suppression, piezoelectric actuator patch was connected to a passive circuit (e.g., shunt circuit), whereas for the case of active vibration suppression, high voltage signal was applied to piezoelectric patch by using an audio amplifier in the control circuit. Forward's experimental results [47] showed the feasibility and practicality of active and passive vibration suppression systems with piezoelectric materials and opened the era of piezoelectric vibration control. From then on, piezoelectric vibration control has attracted significant attention of researchers.

Based on two concepts introduced by Forward [47], piezoelectric vibration control research field can be divided into two areas: piezoelectric shunt damping and active vibration control. In line with the objectives of this dissertation, the focus of the review in this subsection is on piezoelectric vibration control of the plate-like structures with

different boundary conditions, which can be regarded as a basis for piezoelectric energy harvesting research on plate-like structures. It is worth recalling that the objective of vibration control is to dissipate mechanical energy, whereas energy harvesting aims to extract electrical power output from mechanical vibration.

2.2.1 Piezoelectric Shunt Damping

Having introduced and experimentally demonstrated by Forward [47], piezoelectric shunt damping is a passive feedback control method used for vibration suppression of structures. Piezoelectric shunt damping of a host structure can be achieved by connecting piezoelectric layer (attached on a host structure) to a passive circuit, which may include components such as capacitors, inductors, and resistors combined in series and/or parallel. The direct and the converse piezoelectric effects simultaneously provide sensing and actuation for vibration suppression with the help of passive electrical circuit. Thus, a part of mechanical energy of the host structure is dissipated due to feedback sent from the circuit components (i.e., electrical domain) to piezoelectric layer, which results in an increase in mechanical damping and leads vibration suppression.

Almost a decade after Forward [48], Hagood and von Flotow [49] investigated piezoelectric shunt damping of a cantilever beam with analytical modelling and experimental measurements for the fundamental vibration mode. Hollkamp [50] further developed Hagood and von Flotow's approach for multimodal vibration suppression by adding parallel circuits. After that, Wu [51] introduced the current-blocking technique where each branch of shunt circuit network is functional around the targeted natural frequency and behaves like open-circuit at all other natural frequencies. The improvement of the multi-mode vibration suppression with the current-blocking technique was demonstrated by presenting experimental measurements on a cantilever beam [51].

Saravanos [52] extended piezoelectric shunt damping of beam-like structures to plate-like structures and derived equations of motion for a simply-supported piezoelectric laminated composite (piezolaminate) plate coupled with a passive electrical network. Having obtained numerical solutions with state-space formulation, Saravanos investigated the optimal shunting resistance for the maximum modal damping of the first vibration modes of the plate. Behrens et al. [53] showed that a pair of collocated piezoelectric patches can effectively suppress multimodal vibration of the simply-supported plate when the current-flowing technique is used. In the current-flowing technique, which is similar to the current-blocking introduced by Wu [51], each branch of shunt circuit network is short-circuited around the targeted natural frequency and behaves like open-circuited at all other natural frequencies. Prior to that study [53], Behrens et al. [54] theoretically and experimentally investigated the negative capacitance controller (i.e., active shunting) similar to the passive shunt damping and presented that it is capable of broadband damping of modally dense structures. Fein [55] studied the optimal positioning of piezoelectric patches on a fully-clamped plate by numerical analysis and experimental validations to investigate the effect of shunt damping.

However, as pointed out by Moheimani [56], the large inductance requirement limits the usage of piezoelectric shunt damping at low frequencies and the vibration suppression performance significantly depends on environmental conditions. In order to overcome the challenges of shunt damping, several research groups have explored different methods for patch-based piezoelectric shunt damping: synthetic impedance [57, 58], online impedance adaptation [59], synchronized switch damping [60], simultaneous use of passive and active piezoelectric actuators [61, 62] and periodic arrangement of shunted piezoelectric patches [63, 64]. It is worth to note that energy-harvesting capabilities of integrated patch-based piezoelectric actuators have not been addressed in those studies.

More recently, a piezoelectric corrugated device for concurrent vibration suppression and energy harvesting has been modelled, manufactured and tested by Harne [65]. The corrugated device includes a distributed piezoelectric spring layer below a top plate and can be attached on a host structure [66]. The performance of corrugated device for vibration suppression and energy harvesting was investigated by attaching on a simply-supported plate [65] and on a structural panel of a typical bus [67]. Numerical predictions [66] and experimental results [65, 67] showed that the corrugated device significantly attenuates the vibration of the host structure while providing reasonable performance on energy harvesting.

2.2.2 Piezoelectric Active Vibration Control

Although piezoelectric shunt damping significantly attenuates vibrations of beam-like and plate-like structures in laboratory environments, in practice, it is not very possible to obtain optimal, adaptive, and robust vibration suppression with passive electrical circuits in real-life applications. Hence, researchers have focused on active vibration control techniques to take advantage of external power sources and digital acquisition systems as well as a wide range of different sensors and actuators.

A half dozen years after Forward's initiation [47], Bailey and Hubbard [68] designed and demonstrated active vibration suppression system for a cantilever beam with piezoelectric actuators. Analytical model and experimental validations of piezoelectric active vibration control of a cantilever beam were presented by Crawley and Luis [69] in 1987. Afterwards, research studies on piezoelectric active vibration control for simple and complex structures have been published extensively.

Several review articles discussing fundamental aspects of piezoelectric active vibration control of mechanical structures and systems can be found in the existing literature. For instance, Rao and Sunar's review paper [70] on employment of piezoelectric sensors and

actuators for vibration control of flexible structures was published almost two decades ago. Key applications and modelling approaches were also reviewed by Loewy [71] and Chee et al. [72] in 1990s. The reader is referred to Gupta et al. [73] for the optimal positioning studies focused on piezoelectric sensors and actuators on beam-like and plate-like structures. In addition to these, controller design procedures were reviewed in depth, and advantages of different controllers were given along with several application examples by Alkhatib and Golnaraghi [74]. However, positioning of sensors and actuators has been the major problem in the design and implementation of active vibration control systems for real-life applications up to today.

In general, active control systems are designed to achieve one of the following objectives: tracking a trajectory and rejecting disturbances. The objective of active vibration control is to reject external disturbances and attenuate vibration levels. This objective can be accomplished using feedforward and feedback controllers. Feedforward controllers operate based on measurable disturbances, and they are more commonly used in noise cancellation systems (e.g., filtered-X LMS [75]), whereas feedback algorithms generate control signals based on output response signals. The feedback controllers are mostly used in active vibration suppression systems, and divided into two categories [76]: active damping systems and model-based controllers.

An active damping system is a low-authority controller which uses vibration response signal (e.g., in terms of acceleration, velocity and/or displacement) and does not require accurate modeling [77]. The main advantages of active damping systems are their simple architectures, model independence and guaranteed stability conditions for the case of collocated dual sensor and actuator pairs [74, 78]. Shields et al. [79] experimentally demonstrated the effectiveness of the active damping system on vibration suppression of a fully-clamped aluminum square plate (298 x 298 x 0.4 mm). A piezoelectric layer was used as the actuator, whereas laser displacement sensor and microphone were used for vibration

and noise measurements. Later, Gardonio et al. [80] studied multiple velocity feedback loops for vibration suppression and noise radiation attenuation of a fully-clamped aluminum plate (414 x 314 x 1 mm). The decentralized system was composed of sixteen collocated accelerometers and piezoceramic actuator pairs. Similar to these studies, Hamer et al. [81] investigated positive position feedback controller with bonded piezoceramic actuators and sensors on a fully-clamped flexible aluminum plate (210 x 160 x 1 mm). More recently, Mahmoodi and Ahmadian [82] designed an active damping system based on acceleration feedback for a fully-clamped steel plate (500 x 400 x 0.9 mm). Their controller system was composed of a second order resonant compensator for stiffness control and a first order compensator for damping as an extension of position feedback control. The bonded piezoelectric patches were used as actuators while vibration of the plate was monitored using accelerometers. The experimental results of these reviewed studies [79-82] illustrate that active damping systems with piezoelectric actuators can effectively reduce the vibration of flat metal plates around resonance frequencies.

As a strong alternative to active damping systems, model-based controllers provide vibration suppression of host structures within the frequency bandwidths of the controllers, sensors, and actuators. In the design and implementation of model-based controllers, system identification with piezoelectric materials is an essential step since the controller parameters are determined based on the models in hand. Model-based controllers can be divided into four groups: optimal and robust controllers [83-86], intelligent controllers [87], nonlinear controllers [88-90] and hybrid methodologies [91].

Noting that piezoelectric materials have limited sensing and actuation capabilities, they may exhibit nonlinearities under extreme electrical and mechanical conditions [92, 93]. Model-based controllers can be designed such that optimal performance, stability, and robustness can be guaranteed. However, there is a trade-off between performance and robustness, and this tradeoff process is a fundamental task for the controller design [94].

Larger the robustness to the uncertainties, smaller the performance is. Therefore, the aim of controller design is to find the achievable performance in the presence of uncertainties and constraints. In the control engineering literature, tradeoff between robustness and performance has been widely studied [94-96].

A comprehensive framework for an optimal and robust controller was firstly presented by Zames [97] by the application of theory of Hardy spaces to the unstable systems. This controller design approach became widespread in the literature as robust controller design methodology. Readers can also refer to the review paper [98] which specifically includes robust controller applications of piezoelectric smart structures.

Additionally, control systems based on fuzzy logic, neural network and genetic algorithms have been investigated for piezoelectric active vibration suppression of flexible structures [99-102]. These controllers are referred as model-based intelligent control systems due to their emulations of human intelligence (e.g., adaptation and learning) [103-105].

Like in the other fields, there are challenges in implementing active piezoelectric vibration control to real-life applications. The reliability, cost, system integration, and high power requirements of piezoelectric materials are questionable when real-life applications are of interest. In a recent review article, Hudson and Reynolds [106] reported practical challenges of active vibration controller systems as (1) addition of complexity due to magnitude of control forces and uncertainties in the loading environments, (2) implementation limitations of sensors and actuators to host systems, (3) fail-safe requirements especially for civil, marine, and aerospace systems, (4) higher operating and maintenance costs. Readers can also refer to Hudson and Reynolds's paper [106] for environmental and economic impacts of vibration and noise control systems.

Recently, the two research areas of energy harvesting and active vibration control have been also considered together for the purpose of self-powered vibration control systems,

which are referred as energy recycling [107]. For instance, in a recent study, Makiyara et al. [108] proposed a digital self-powered multimodal vibration suppression system powered by recycled energy. Wang and Inman [109] reviewed and compared the power consumptions of vibration controller algorithms to assess the feasibility of energy recycling systems. Thus, piezoelectric energy harvesting is considered as a partner of piezoelectric active vibration control systems.

2.3 Piezoelectric Energy Harvesting

Williams and Yates [110] proposed the idea of vibration-based piezoelectric energy harvesting together with electromagnetic and electrostatic transduction mechanisms almost two decades ago. Afterwards, research on piezoelectric energy harvesting has heavily focused on cantilever piezoelectric beams due to their ease of implementation and use [4, 5]. Analytical and numerical electroelastic models of cantilever beam harvesters have been presented by several research groups [32, 41, 111-113]. Distributed-parameter modeling [32], Rayleigh-Ritz solutions [112], and assumed-modes modeling [114] were investigated along with experimental verifications for deterministic and random vibrations [115]. Figure 2.1 presents a typical configuration of a linear cantilevered piezoelectric energy harvester beam that generates usable electrical power due to base excitations.

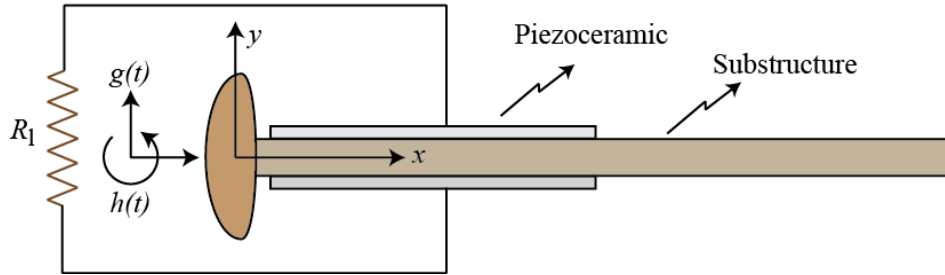


Figure 2.1 Illustration of a cantilevered piezoelectric energy harvester beam [32]. Here, $g(t)$ and $h(t)$ are translational and rotational base excitations and R_l is a load resistance connected to electrodes of the piezoceramic layers

Major modeling issues [23] and methods for performance enhancement [116-118] have also been reported for cantilever harvester beams. For example, requirement of matching the resonance frequency with the ambient dominant frequency is a bottleneck for achieving best performance [118] and broadband energy harvesting is limited due to the relatively large spacing between resonance frequencies of the first vibrational modes [117]. Thus, the performance improvement of cantilever beam harvesters has been widely studied with different structural designs and configurations. Size and shape optimization of piezoceramic layers [42], effects of strain distribution on electrical outputs [119], use of highly coupled materials [120] were studied. Novel mechanisms such as initial energy injection [121], array of harvesters [122], multi-dimensional operation [26] and buckled configurations [25, 123] have been investigated.

In many real-life applications, the nature of ambient mechanical energy is distributed over a broadband frequency spectrum, and vibrational excitations are usually non-harmonic and have stochastic characteristics. As opposed to the considerable research on harmonic excitations [6], it is worth pointing out that limited research has been conducted for non-harmonic or random excitations of cantilever beam harvesters. Broadband random energy

harvesting has been studied with lumped-parameter [24, 124-126] and distributed-parameter [115] modelling, while random vibrations of nonlinear resonant beams with higher complexities have been modelled and analyzed as Duffing oscillators [127, 128].

For simpler and more efficient broadband random vibrational energy harvesting, design and analysis of new harvesting devices are of current interest. Patch-based energy harvesters structurally integrated to thin plates can be conveniently used as an alternative to cantilevered harvester beams. Thin plate structures are commonly used in large engineering systems that can definitely benefit from energy harvesting. However, the literature on patch-based piezoelectric energy harvesting on thin plates is rather limited. De Marqui et al. [129] presented an electromechanical finite element model (FEM) for a piezoelectric energy harvester embedded in a cantilever plate, and later on extended their numerical model to airflow excitation problems by electroaeroelastic coupling [130, 131]. Rupp et al. [132] conducted topology optimization studies for the distribution of piezoelectric material on cantilever plate and shell structures using numerical techniques to maximize the harvested power output. Erturk [133] derived analytical formulation for energy harvesting with piezoceramic patches from surface strain fluctuations of large structures through one-way coupling. As a part of this dissertation, analytical model of a single patch-based piezoelectric harvester has been developed for transverse vibrations of thin plates by Aridogan et al. [29]. Figure 2.2 and 2.3 presents the schematic of a patch-based harvester on a thin plate and experimental setup for model validations, respectively. Recently, Aridogan et al. [31] extended the single-patch model to multiple patch-based harvesters connected in series and parallel configurations. Broadband and band-limited random vibration energy harvesting has been also investigated using a patch-based harvester attached on a thin plate along with analytical modelling and case studies [30].

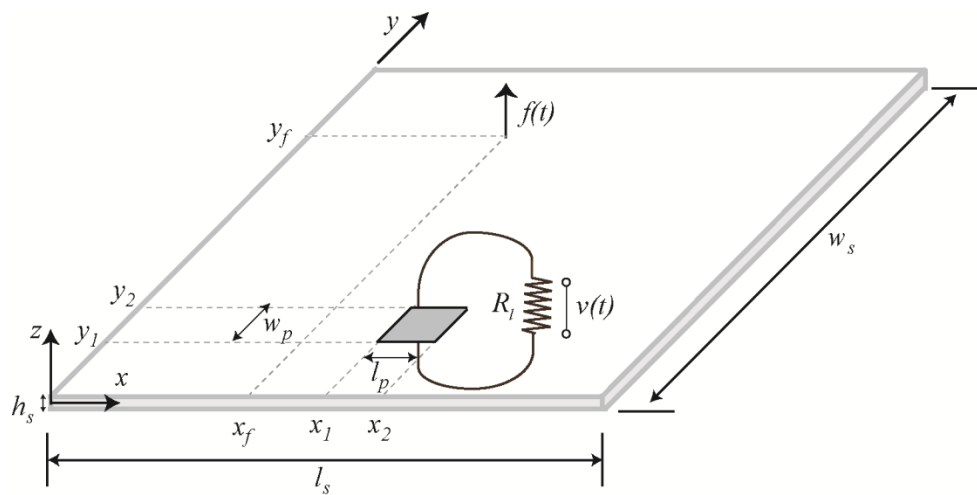


Figure 2.2: Schematic of a patch-based harvester attached on a thin plate substructure and excited by a point force [30] (referred from Chapter 3)



Figure 2.3: Experimental setup for model validations (referred from Chapter 3)

2.4 Summary and Conclusion

This chapter presented history of piezoelectricity and reviewed piezoelectric energy harvesting and vibration control applications. Piezoelectric energy harvesting research follows the path of piezoelectric vibration control. Despite the fact that limited effort has been devoted for piezoelectric energy harvesting on thin plates, significant literature exists on piezoelectric vibration control of thin plates. While the purpose of vibration control is to suppress the vibration levels, piezoelectric energy harvesting aims the extraction of usable electrical energy. In addition to these, energy harvesting and active vibration control research areas have also been considered together for the aim of energy-recycled and autonomous vibration control systems. Therefore, in line with the objectives of this dissertation, the focus of the review was on vibration control and energy harvesting research studies.

Chapter 3

A PATCH-BASED PIEZOELECTRIC ENERGY HARVESTER ON A THIN PLATE

This chapter presents analytical electroelastic modeling and experimental validations of a single patch-based piezoelectric energy harvester structurally integrated on a thin plate. In the following sections, distributed-parameter electroelastic model of the thin plate with the patch-based harvester is developed based on Kirchhoff's plate theory. By following modal analysis procedure, closed-form steady-state exact solutions for electrical and structural responses are obtained for harmonic excitations. Multi-mode and single-mode analytical frequency response functions (FRFs) between voltage output-to-force input and displacement-to-force input are derived and generalized for different boundary conditions of thin plates. A case study is presented for the experimental model validation and the dynamic strain distribution of an all-four-edges clamped (CCCC) plate is discussed with an emphasis on vibration mode-dependent preferred locations of patch-based harvester for effective energy harvesting. Experimental and analytical FRFs are then compared to show the accuracy of the analytical predictions for a wide range of resistive loads. Finally, energy harvesting performance from multiple vibration modes of thin plate is analyzed using analytical and experimental results.

3.1 Distributed-parameter Electroelastic Modeling

3.1.1 Mechanical Equations of Motion in Physical Coordinates

A piezoelectric energy harvester in the form of a thin piezoceramic patch (small plate) structurally integrated to a large host plate is presented in figure 3.1. The host plate is excited by a transverse point force $f(t)$ acting at the position coordinates (x_0, y_0) . The host plate is assumed as an isotropic and geometrically uniform thin plate. The effects of transverse shear deformation are neglected based on the Kirchhoff plate theory assumptions. A perfectly bonded transversely isotropic piezoceramic patch with a length of l_p and width of w_p covers a rectangular region with two corners at (x_1, y_1) and (x_2, y_2) . The length and width of the host plate are a and b , respectively. Thicknesses of the host plate and the piezoceramic patch are denoted by h_s and h_p , respectively.

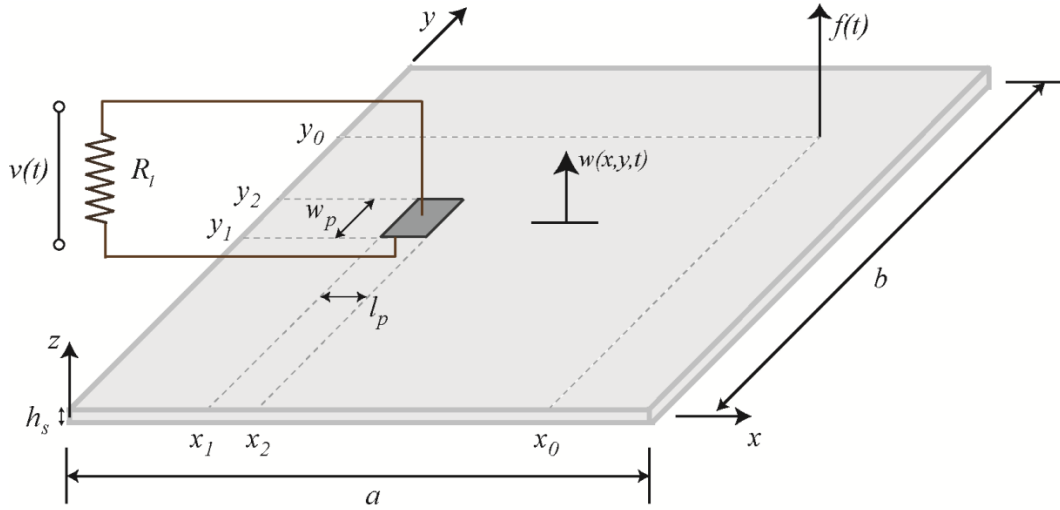


Figure 3.1: A patch-based piezoelectric energy harvester on a thin plate under transverse force excitation

A load resistance (R_l) is considered as an external electrical load connected to the perfectly conductive electrode layers of negligible thickness covering the top and bottom surfaces of the piezoceramic patch. It is assumed that the piezoelectric volume is much smaller than the host plate so the patch is coupled to the host only electromechanically (with negligible mass and stiffness contribution).

For a thin piezoceramic patch attached on the plate, the linear piezoelectric constitutive equations can be written as follows [37]:

$$\begin{bmatrix} T_1^P \\ T_2^P \\ T_6^P \\ D_3 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11}^E & \bar{c}_{12}^E & 0 & -\bar{e}_{31} \\ \bar{c}_{12}^E & \bar{c}_{22}^E & 0 & -\bar{e}_{31} \\ 0 & 0 & \bar{c}_{66}^E & 0 \\ \bar{e}_{31} & \bar{e}_{31} & 0 & \bar{\epsilon}_{33}^S \end{bmatrix} \begin{bmatrix} S_1^P \\ S_2^P \\ S_6^P \\ E_3 \end{bmatrix} \quad (3.1)$$

where T_1^P and T_2^P are the normal stress components along x and y axes respectively, T_6^P is the shear stress in the x - y plane. D_3 is the electric displacement in z direction (poling direction of piezoceramic patch). S_1^P and S_2^P are the normal strain components along the x and y axes, S_6^P is the shear strain component in the x - y plane. E_3 is the electric field in the thickness direction. The effective piezoelectric stress constant is $\bar{e}_{31}$, elastic stiffness components are $\bar{c}_{11}^E$, $\bar{c}_{12}^E$ and $\bar{c}_{66}^E$ (note that $\bar{c}_{11}^E = \bar{c}_{22}^E$), and $\bar{\epsilon}_{33}^S$ represents the permittivity component. Superscript P represents the piezoceramic patch, while E and S denote that the respective parameters are evaluated at constant electric field and constant strain, respectively. The over-bar denotes that the respective parameter is reduced to two-dimensional (2-D) form from three-dimensional (3-D) electroelasticity components based on the following relations

$$\begin{aligned}
\bar{c}_{11}^E &= \frac{s_{11}^E}{(s_{11}^E + s_{12}^E)(s_{11}^E - s_{12}^E)} & \bar{c}_{12}^E &= \frac{-s_{12}^E}{(s_{11}^E + s_{12}^E)(s_{11}^E - s_{12}^E)} & \bar{c}_{66}^E &= \frac{1}{s_{66}^E} \\
\bar{e}_{31} &= \frac{d_{31}}{s_{11}^E + s_{12}^E} & \bar{\epsilon}_{33}^S &= \epsilon_{33}^T - \frac{2d_{31}^2}{s_{11}^E + s_{12}^E}
\end{aligned} \tag{3.2}$$

where s_{11}^E , s_{12}^E and s_{66}^E are elastic compliance parameters at constant electric field E , d_{31} is the piezoelectric strain constant, and ϵ_{33}^T is the permittivity component at constant stress T .

The partial differential equation governing the forced vibrations of a thin plate with a small piezoceramic patch can be obtained as [29]

$$\begin{aligned}
&\frac{\partial^2 (M_1^S + M_1^P)}{\partial x^2} + 2\frac{\partial^2 (M_6^S + M_6^P)}{\partial x \partial y} + \frac{\partial^2 (M_2^S + M_2^P)}{\partial y^2} - c\frac{\partial w(x, y, t)}{\partial t} \\
&- \rho_s h_s \frac{\partial^2 w(x, y, t)}{\partial t^2} + f(t)\delta(x - x_0)\delta(y - y_0) = 0
\end{aligned} \tag{3.3}$$

where $w(x, y, t)$ is transverse deflection of the plate at position (x, y) and time t . The internal bending moments are M_1 , M_2 and M_6 while superscripts S and P stand for the host structure and the piezoceramic patch, respectively. The mass density of the plate is ρ_s and c is the viscous damping coefficient. The transverse force $f(t)$ is acting at (x_0, y_0) . The Dirac delta functions are $\delta(x)$ and $\delta(y)$ along the x and y directions. The internal bending moments of the host structure can be expressed in terms of curvatures as follows:

$$M_1^S = -D \left(\frac{\partial^2 w(x, y, t)}{\partial x^2} + \nu_s \frac{\partial^2 w(x, y, t)}{\partial y^2} \right) \quad (3.4)$$

$$M_2^S = -D \left(\frac{\partial^2 w(x, y, t)}{\partial y^2} + \nu_s \frac{\partial^2 w(x, y, t)}{\partial x^2} \right) \quad (3.5)$$

$$M_6^S = -D(1 - \nu_s) \frac{\partial^2 w(x, y, t)}{\partial x \partial y} \quad (3.6)$$

where the bending stiffness of the plate is $D = Y_s h_s^3 / (12 - 12\nu_s^2)$, while its Young's modulus and Poisson's ratio are Y_s and ν_s . The internal moments of the piezoceramic patch are

$$M_1^P = [H(x - x_1) - H(x - x_2)][H(y - y_1) - H(y - y_2)] \int_p T_1^P z dz \quad (3.7)$$

$$M_2^P = [H(x - x_1) - H(x - x_2)][H(y - y_1) - H(y - y_2)] \int_p T_2^P z dz \quad (3.8)$$

$$M_6^P = [H(x - x_1) - H(x - x_2)][H(y - y_1) - H(y - y_2)] \int_p T_6^P z dz \quad (3.9)$$

where $H(x)$ and $H(y)$ are the Heaviside functions and the integrals are over the thickness of piezoceramic patch. Substituting internal moments of the host plate and the piezoceramic patch in equations (3.4-3.6) and (3.7-3.9) into equation (3.3), the governing partial differential equation of the plate with piezoelectric coupling is

$$\begin{aligned}
& D \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + c \frac{\partial w(x, y, t)}{\partial t} + \rho_s h_s \frac{\partial^2 w(x, y, t)}{\partial t^2} \\
& - \theta v(t) \left\{ \left[\frac{d \delta(x - x_1)}{dx} - \frac{d \delta(x - x_2)}{dx} \right] [H(y - y_1) - H(y - y_2)] \right. \\
& \left. + \left[\frac{d \delta(y - y_1)}{dy} - \frac{d \delta(y - y_2)}{dy} \right] [H(x - x_1) - H(x - x_2)] \right\} \\
& = f(t) \delta(x - x_0) \delta(y - y_0)
\end{aligned} \tag{3.10}$$

where $v(t)$ is the voltage across the external resistive load and the electromechanical term θ is defined as $\theta = \bar{e}_{31} h_{pc}$, which is the multiplication of effective piezoelectric constant $\bar{e}_{31}$ and reference distance h_{pc} of the center layer of the piezoceramic patch to the reference surface (i.e., the neural axis level of the plate) at the location of the patch. It is important to note that the volume of the harvester patch is assumed significantly smaller than the host plate while inertial and stiffness effects of the patch-based harvester are neglected. Equation (3.10) governs the mechanical motion of the host plate and the harvester patch with electrical coupling.

3.1.2 Electrical Circuit Equations in Physical Coordinates

The governing differential equation of the electrical circuit is derived in the following. The electric displacement can be written using equation (3.1) as

$$D_3 = \bar{e}_{31} S_1^p + \bar{e}_{31} S_2^p + \bar{e}_{33}^s E_3 \tag{3.11}$$

where the electric field E_3 can be defined in terms of the electrical potential difference $v(t)$, as $E_3 = -v(t)/h_p$, and the axial strain components (S_1^p and S_2^p along the x and y axes) at the center layer of the piezoceramic patch are

$$S_1^p(x, y, t) = -h_{pc} \frac{\partial^2 w(x, y, t)}{\partial x^2} \quad S_2^p(x, y, t) = -h_{pc} \frac{\partial^2 w(x, y, t)}{\partial y^2} \quad (3.12)$$

The electric current flowing to the resistive load can be obtained using

$$\frac{d}{dt} \int_A \mathbf{D} \cdot \mathbf{n} dA = \frac{v(t)}{R_l} \quad (3.13)$$

where $\mathbf{n}$ is the unit vector outward from electrode surface, $\mathbf{D}$ is the electric displacement vector and integral is over the electrode's surface area A . The inner product between the unit vector $\mathbf{n}$ and the electric displacement $\mathbf{D}$ vector results in electric displacement term D_3 , which is in the normal direction of the electrode surface along the z-axis. Hence, substituting equation (3.11) and (3.12) into equation (3.13), one can obtain

$$\frac{d}{dt} \int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} \left[-h_{pc} \bar{\epsilon}_{31} \left(\frac{\partial^2 w(x, y, t)}{\partial x^2} + \frac{\partial^2 w(x, y, t)}{\partial y^2} \right) - \bar{\epsilon}_{33}^S \frac{v(t)}{h_p} \right] dx dy = \frac{v(t)}{R_l} \quad (3.14)$$

and defining the capacitance of the piezoceramic patch as $C_p = (\bar{\epsilon}_{33}^S w_p l_p) / h_p$ and electromechanical coupling term as $\theta = \bar{\epsilon}_{31} h_{pc}$, the following equation governs coupled electrical circuit dynamics:

$$C_p \frac{dv(t)}{dt} + \frac{v(t)}{R_l} + \theta \left\{ \int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} \left[\frac{\partial^3 w(x, y, t)}{\partial x^2 \partial t} + \frac{\partial^3 w(x, y, t)}{\partial y^2 \partial t} \right] dx dy \right\} = 0 \quad (3.15)$$

Therefore, equations (3.10) and (3.15) represent the distributed-parameter electroelastic model of patch-based harvester in physical coordinates, and these equations can be solved following modal analysis procedure for two-dimensional structures.

3.1.3 Electroelastic Model in Modal Coordinates

Based on the standard modal analysis procedure, the vibration of a thin plate can be represented as

$$w(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn}(x, y) \eta_{mn}(t) \quad (3.16)$$

where mass-normalized eigenfunction is $\phi_{mn}(x, y)$ and the modal time response is $\eta_{mn}(t)$ for the m th vibration mode. The eigenfunctions of the undamped CCCC rectangular plate are obtained as [134]

$$\begin{aligned} \phi_{mn}(x, y) = & \Theta_{mn} \left(\cos \lambda_{mn}^x x - \sigma_{mn}^x \sin \lambda_{mn}^x x - \cosh \mu_{mn}^x x + \frac{\lambda_{mn}^x}{\mu_{mn}^x} \sigma_{mn}^x \sinh \mu_{mn}^x x \right) \\ & \times \left(\cos \lambda_{mn}^y y - \sigma_{mn}^y \sin \lambda_{mn}^y y - \cosh \mu_{mn}^y y + \frac{\lambda_{mn}^y}{\mu_{mn}^y} \sigma_{mn}^y \sinh \mu_{mn}^y y \right) \end{aligned} \quad (3.17)$$

where σ_{mn}^x and σ_{mn}^y are

$$\begin{aligned} \sigma_{mn}^x &= \frac{\cos \lambda_{mn}^x a - \cosh \mu_{mn}^x a}{\sin \lambda_{mn}^x a - \frac{\lambda_{mn}^x}{\mu_{mn}^x} \sinh \mu_{mn}^x a} & \sigma_{mn}^y &= \frac{\cos \lambda_{mn}^y b - \cosh \mu_{mn}^y b}{\sin \lambda_{mn}^y b - \frac{\lambda_{mn}^y}{\mu_{mn}^y} \sinh \mu_{mn}^y b} \end{aligned} \quad (3.18)$$

and Θ_{mn} is the modal amplitude constant, which can be evaluated by normalizing the eigenfunctions according to the following orthogonality conditions:

$$\int_0^b \int_0^a \rho_s h_s \phi_{mn}(x, y) \phi_{rs}(x, y) dx dy = \delta_{mr} \delta_{ns} \quad (3.19)$$

$$\int_0^b \int_0^a D \left(\frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} \frac{\partial^2 \phi_{rs}(x, y)}{\partial x^2} + 2 \frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} \frac{\partial^2 \phi_{rs}(x, y)}{\partial y^2} + \frac{\partial^2 \phi_{mn}(x, y)}{\partial y^2} \frac{\partial^2 \phi_{rs}(x, y)}{\partial y^2} \right) dx dy = \omega_{mn}^2 \delta_{mr} \delta_{ns} \quad (3.20)$$

where δ_{mr} and δ_{ns} are Kronecker delta functions, δ_{mr} is equal to unity for $m=r$, and zero for $m \neq r$. Similarly δ_{ns} is equal to unity for $n=s$ and zero for $n \neq s$. The undamped natural frequency ω_{mn} for the m th vibration mode of the plate in short circuit conditions (as $R_l \rightarrow 0$) is then

$$\omega_{mn} = \sqrt{\frac{\kappa_{mn}^4}{\rho_s h_s} D} \quad (3.21)$$

where κ_{mn} is the frequency parameter of the undamped plate. The eigenvalues ($\lambda_{mn}^x, \lambda_{mn}^y, \mu_{mn}^x, \mu_{mn}^y$) can be obtained from the solution of the transcendental characteristic equations [134]:

$$\frac{1 - \cosh \mu_{mn}^x a \cos \lambda_{mn}^x a}{\sinh \mu_{mn}^x a \sin \lambda_{mn}^x a} = \frac{(\lambda_{mn}^x)^2 - (\mu_{mn}^x)^2}{2 \lambda_{mn}^x \mu_{mn}^x} \quad (3.22)$$

$$\frac{1 - \cosh \mu_{mn}^y b \cos \lambda_{mn}^y b}{\sinh \mu_{mn}^y b \sin \lambda_{mn}^y b} = \frac{(\lambda_{mn}^y)^2 - (\mu_{mn}^y)^2}{2\lambda_{mn}^y \mu_{mn}^y} \quad (3.23)$$

$$\kappa_{mn} = \sqrt{(\lambda_{mn}^x)^2 + (\lambda_{mn}^y)^2} \quad (3.24)$$

$$\mu_{mn}^y = \sqrt{2\kappa_{mn}^2 - (\lambda_{mn}^y)^2} \quad (3.25)$$

$$\mu_{mn}^x = \sqrt{2\kappa_{mn}^2 - (\lambda_{mn}^x)^2} \quad (3.26)$$

for the intervals of

$$\begin{aligned} \lambda_{mn}^x a &\in \left[m\pi, m\pi + \frac{\pi}{2} \right], & m = 1, 2, \dots \\ \lambda_{mn}^y b &\in \left[n\pi, n\pi + \frac{\pi}{2} \right], & n = 1, 2, \dots \end{aligned} \quad (3.27)$$

The reader is referred to Appendix A for the steps of separation of variables solution procedure to obtain the transcendental characteristic equations given by equations (3.22-3.27) of an all-four-edges clamped thin plate.

By following the modal analysis procedure [135] for a two-dimensional structure (multiplying partial differential equations in physical coordinates by eigenfunction $\phi_{rs}(x, y)$ and integrating over the surface area of the plate), the following electromechanically coupled ordinary differential equations for the modal time response η_{mn} can be obtained:

$$\frac{d^2 \eta_{mn}(t)}{dt^2} + 2\zeta_{mn} \omega_{mn} \frac{d\eta_{mn}(t)}{dt} + \omega_{mn}^2 \eta_{mn}(t) - \tilde{\theta}_{mn} v(t) = f_{mn}(t) \quad (3.28)$$

Here, ω_{mn} is the undamped natural frequency for the m th vibration mode. The modal damping ratio ζ_{mn} can be obtained by using the standard damping identification techniques [136]. The modal forcing in equation (3.28) is

$$f_{mn} = \int_0^b \int_0^a f(t) \delta(x - x_0) \delta(y - y_0) \phi_{mn}(x, y) dx dy = f(t) \phi_{mn}(x_0, y_0) \quad (3.29)$$

while the electromechanical coupling term $\tilde{\theta}_{mn}$ can be given by

$$\tilde{\theta}_{mn} = \theta \left[\int_{y_1}^{y_2} \left. \frac{\partial \phi_{mn}(x, y)}{\partial x} \right|_{x_1}^{x_2} dy + \int_{x_1}^{x_2} \left. \frac{\partial \phi_{mn}(x, y)}{\partial y} \right|_{y_1}^{y_2} dx \right] \quad (3.30)$$

Hence, the slope of mode shapes ϕ_{mn} along the edges of the piezoceramic patch together with the region covered determines the electromechanical coupling term. The modal expansion given by equation (3.16) can be substituted into the first-order differential equation governing electrical circuit (see equation (3.15)) to obtain

$$C_p \frac{dv(t)}{dt} + \frac{v(t)}{R_l} + \theta \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} + \frac{\partial^2 \phi_{mn}(x, y)}{\partial y^2} \right) \frac{d\eta_{mn}(t)}{dt} dx dy \right\} = 0 \quad (3.31)$$

Finally, the governing equation of electrical circuit can be written in modal coordinates as

$$C_p \frac{dv(t)}{dt} + \frac{v(t)}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tilde{\theta}_{mn} \frac{d\eta_{mn}(t)}{dt} = 0 \quad (3.32)$$

where $v(t)$ is the voltage output across the resistive load R_l .

3.2 Steady-state Response to Harmonic Force Excitation

3.2.1 Closed-form Voltage Output and Vibration Response Expressions

The electroelastic equations for a piezoelectric energy harvester attached on a thin plate are given in physical coordinates in equations (3.10 and 3.15) and in modal coordinates in equations (3.28 and 3.32). If the transverse force acting on the surface of the plate is assumed to be harmonic in the form $f(t) = F_0 e^{j\omega t}$ (where F_0 is the amplitude of the force and ω is the excitation frequency) and assuming linear oscillations, the steady-state expressions for modal response $\eta_{mn}(t)$ and voltage response $v(t)$ across the resistive load can be expressed as

$$\eta_{mn}(t) = H_{mn} e^{j\omega t} \quad v(t) = V e^{j\omega t} \quad (3.33)$$

where H_{mn} and V are the complex amplitudes. Substituting these steady-state expressions into governing equations in modal coordinates (equations (3.28) and (3.32)) gives

$$-\omega^2 H_{mn} e^{j\omega t} + j2\zeta_{mn}\omega_{mn}\omega H_{mn} e^{j\omega t} + \omega_{mn}^2 H_{mn} e^{j\omega t} - \tilde{\theta}_{mn} V e^{j\omega t} = F_0 e^{j\omega t} \phi_{mn}(x_0, y_0) \quad (3.34)$$

$$j\omega C_p V e^{j\omega t} + \frac{V e^{j\omega t}}{R_l} + e^{j\omega t} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} j\omega \tilde{\theta}_{mn} H_{mn} = 0. \quad (3.35)$$

and by eliminating $e^{j\omega t}$ in equation (3.34), H_{mn} can be expressed as

$$H_{mn} = \frac{F_0 \phi_{mn}(x_0, y_0) + \tilde{\theta}_{mn} V}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \quad (3.36)$$

Then, substitution of H_{mn} into equation (3.35) gives the voltage amplitude V as

$$V = \frac{-j\omega \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{F_0 \phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega C_p + \frac{1}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega \tilde{\theta}_{mn}^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \quad (3.37)$$

Hence, the steady-state voltage response $v(t)$ across the resistive load is

$$v(t) = Ve^{j\omega t} = \frac{-j\omega \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{F_0 \phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega C_p + \frac{1}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega \tilde{\theta}_{mn}^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} e^{j\omega t} \quad (3.38)$$

Using the closed-form steady-state expression of the voltage across the resistive load in equation (3.38), the current $i(t)$ and instantaneous power output $P(t)$ generated by harvester patch can be calculated by employing

$$i(t) = \frac{v(t)}{R_l} \quad P(t) = \frac{v^2(t)}{R_l} \quad (3.39)$$

For the transverse deflection of the plate, the voltage amplitude V in equation (3.37) can be substituted into equation (3.36) and the expressions in equations (3.16) and (3.33) can be used to obtain

$$w(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\phi_{mn}(x_0, y_0) - \frac{j\omega \tilde{\theta}_{mn} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega C_p + \frac{1}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega \tilde{\theta}_{mn}^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \right. \\ \left. \times \frac{F_0 \phi_{mn}(x, y) e^{j\omega t}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \right] \quad (3.40)$$

Closed-form steady-state expressions for voltage output and vibration responses are derived for the CCCC plate; however, it should be noted that these analytical expressions are applicable to different boundary conditions of host plates using the respective modal parameters (natural frequencies ω_{mn} , eigenfunctions ϕ_{mn} and damping ratios ζ_{mn}).

3.2.2 Multi-mode and Single-mode Electroelastic FRFs

Closed-form steady-state expressions for electrical voltage output and vibration response are derived in the former section. Using these steady-state expressions, two multi-mode FRFs can be extracted between two outputs and one input (voltage output-to-transverse force excitation and displacement-to-transverse force excitation). The voltage FRF relating the voltage output to point force input can be given by

$$\alpha(\omega) = \frac{v(t)}{F_0 e^{j\omega t}} = \frac{-j\omega \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega C_p + \frac{1}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega \tilde{\theta}_{mn}^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \quad (3.41)$$

and the displacement FRF relating the transverse displacement to point force input is

$$\beta(x, y, \omega) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\phi_{mn}(x_0, y_0) - \frac{j\omega \tilde{\theta}_{mn} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega C_p + \frac{1}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega \tilde{\theta}_{mn}^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \right) \phi_{mn}(x, y) \quad (3.42)$$

For excitation close to a resonance frequency, these two FRFs can be also simplified for a single mode as

$$\hat{\alpha}(\omega) = \frac{-j\omega R_l \phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}}{(j\omega R_l C_p + 1)(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega) + j\omega R_l \tilde{\theta}_{mn}^2} \quad (3.43)$$

$$\hat{\beta}(x, y, \omega) = \frac{(j\omega R_l C_p + 1)\phi_{mn}(x_0, y_0)\phi_{mn}(x, y)}{(j\omega R_l C_p + 1)(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega) + j\omega R_l \tilde{\theta}_{mn}^2} \quad (3.44)$$

where a hat ($\hat{\cdot}$) symbol denotes the single-mode frequency response function. This single-mode expression is only valid for excitation frequencies close to the natural frequency ($\omega_{mn} \approx \omega$). Note that these multi-mode and single-mode FRF expressions can be used for different boundary conditions of thin host plates by using their respective modal parameters.

3.3 Electroelastic Analysis

The dynamic strain distribution under the piezoceramic patch has significant effects on the performance of the collected charge output on the electrodes of the harvester. If the sign of the strain distribution changes under the area of piezoceramic patch, strong cancellations may occur in a continuous electrode due to the integral of strain on the continuous electrode surface. Erturk et al. [119] pointed out that segmented electrodes should be used in order to avoid cancellations. In case of a patch-based piezoelectric energy harvester attached on a thin plate, the cancellation of voltage output is also an issue since the host structure has vibration modes with certain mode shapes (as given by equation (3.17)) and therefore mode-dependent dynamic strain distributions. Therefore, in this section, mode shapes and strain distributions of the thin plate are analyzed with a case study, which can be used a guideline for the positioning of piezoelectric energy harvester on the thin plate. The case study includes a patch-based harvester structurally integrated on a CCCC aluminum plate as seen in figure 3.2. The material, geometric, dielectric and electroelastic properties of the aluminum plate and piezoceramic patch are given in Table 3.1.

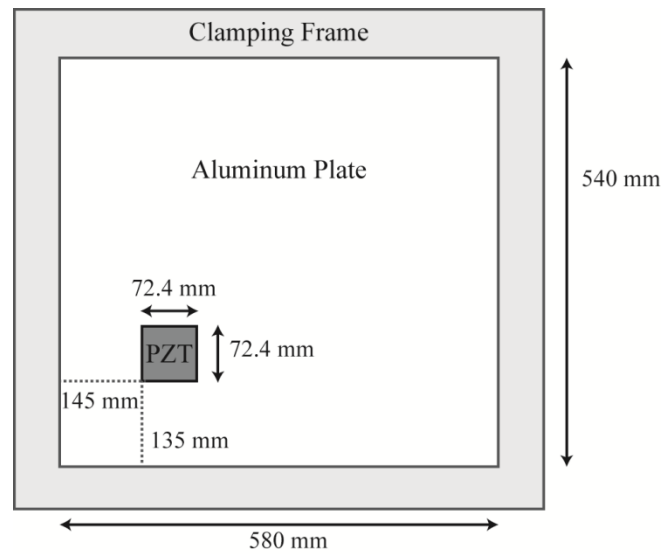


Figure 3.2: Position of the patch-based piezoelectric energy harvester on the host plate with fully clamped boundary conditions

Table 3.1: Geometric, material and electroelastic properties

Property	Aluminum	Piezoceramic
Length (mm)	580	72.4
Width (mm)	540	72.4
Thickness (mm)	1.96	0.267
Young's modulus (GPa)	70	66
Mass density (kg m^{-3})	2700	7800
Piezoelectric constant d_{31} (pm V^{-1})	-	-190
Permittivity constant $\bar{\epsilon}_{33}^S$ (nF m^{-1})	-	10.38

The first four undamped natural frequencies and corresponding normalized mode shapes of the host plate in the case study are obtained and displayed in figure 3.3. Note that the location of the piezoceramic patch is shown in the figure with a gray square. The analytical fundamental ($m = 1$ and $n = 1$) natural frequency is 55.34 Hz and the corresponding mode shape is in-phase on the overall surface with a maximum deflection at the center in figure 3.3(a). However, as can be seen in figure 3.3(b)-3.3(d), higher vibration modes have several in-phase and out-of-phase regions across the surface. The second mode ($m = 2$ and $n = 1$) has an undamped natural frequency of 107.83 Hz, and has a nodal line along the center of the x axis (length of the plate). For the third mode ($m = 1$ and $n = 2$), the natural frequency is 117.83 Hz, and has a nodal line along the center of the y axis (width of the plate). The fourth mode ($m = 2$ and $n = 2$) has a natural frequency of 165.94 Hz and the nodal lines of this mode are along the center lines of the x and y axes.

To investigate the cancellation of voltage output, strain mode shapes can be obtained using the Laplace operator ∇ with the undamped mode shape $\phi_{mn}(x, y)$ for m th vibration mode of the plate:

$$\nabla \phi_{mn}(x, y) = \frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} + \frac{\partial^2 \phi_{mn}(x, y)}{\partial y^2} \quad (3.45)$$

Using equation (3.45), these strain mode shapes (i.e., normalized strain fields) are obtained and presented for the first four modes of the case study in figure 3.4. Strain mode shapes have in-phase and out-of-phase regions even for the fundamental mode of the CCCC plate. The number of the in-phase and out-of-phase regions of strain mode shapes can be obtained via the derived formulation as $m \times n + 2(m + n)$. For instance, the first mode has five distinct in-phase and out-of-phase regions while the second mode has eight such regions.

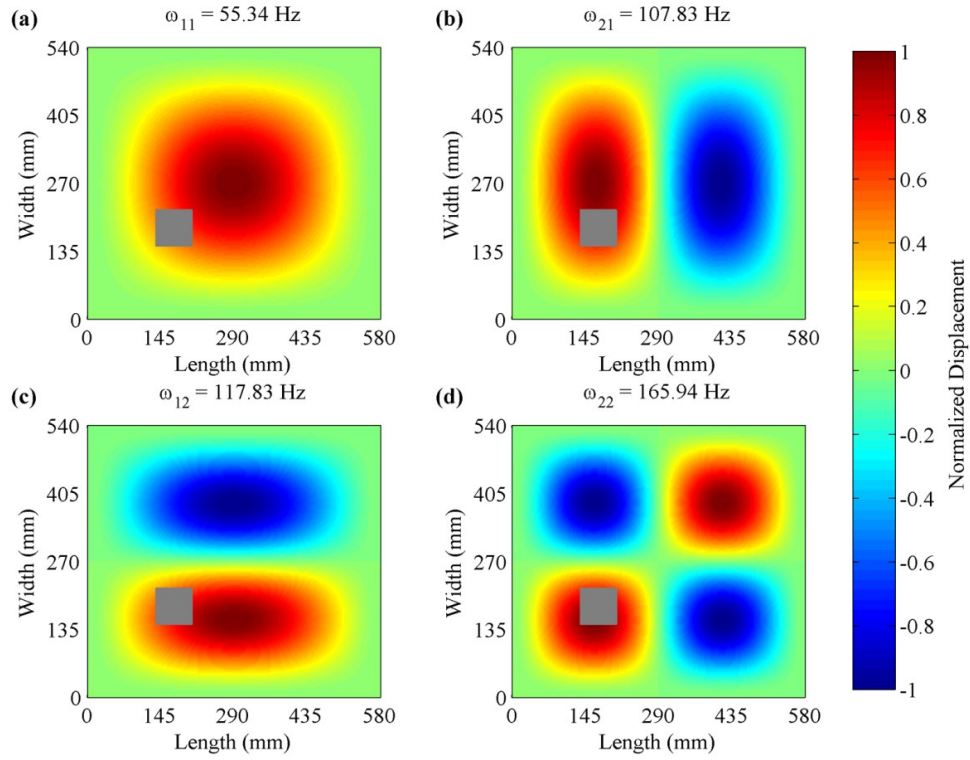


Figure 3.3: Normalized displacement fields for the (a) first, (b) second, (c) third, and (d) fourth modes of the fully-clamped plate analyzed in this chapter

If a continuous electrode is placed on the locations where strain changes sign, strong cancellations may occur, yielding significant reduction of voltage output. Therefore, a careful investigation is required prior to bonding the patch-based harvester to the plate. In this case, the piezoceramic patch is placed at the left-lower quadrant of the plate to generate electrical output from multiple vibration modes of the plate. It should also be recalled that the piezoelectric patch is assumed small and therefore its effect on the strain distribution and mode shape characteristics is negligible.

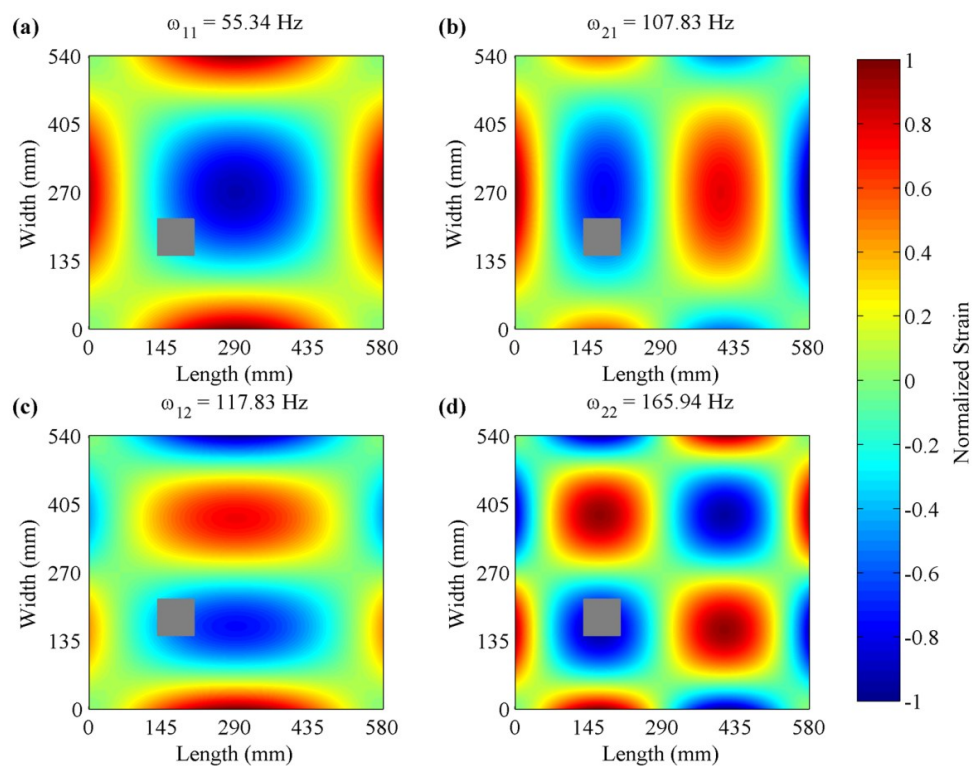


Figure 3.4: Normalized strain fields for the (a) first, (b) second, (c) third, and (d) fourth modes of the fully clamped plate analyzed in this chapter

3.4 Experimental Validations

3.4.1 Experimental Setup

In the experimental setup presented in figure 3.5, the aluminum host plate is clamped at all four edges to thick aluminum structural bars. Screws in two rows are tightened for perfect clamped boundary condition with a torque wrench to establish zero deflection and zero slope symmetrically along all four edges of the host plate. An off-the-shelf piezoceramic patch (T105-A4E-602 from Piezo Systems, Inc.) is attached on the host plate. The aluminum plate is excited by an electromechanical shaker through a sine-sweep signal generated by a signal generator while a force transducer (PCB 208C01) is placed between the shaker's rod and plate to measure the force input. The transverse vibration of the plate is measured with a laser vibrometer (Polytec PDV 100) by targeting the center of the piezoceramic patch. Resistive loads ranging from short-circuit to open-circuit conditions are connected to the electrode's terminals of the piezoceramic patch. The signal analyzer (in figure 3.5) is used to collect and analyze the signals from the force transducer, the laser vibrometer, and the voltage across the resistive load. By defining the force transducer's output as the reference channel in the signal analyzer, the experimental FRFs including the velocity response-to-force input (velocity FRF – also known as the mobility FRF) and voltage output-to-force input (voltage FRF) are obtained.

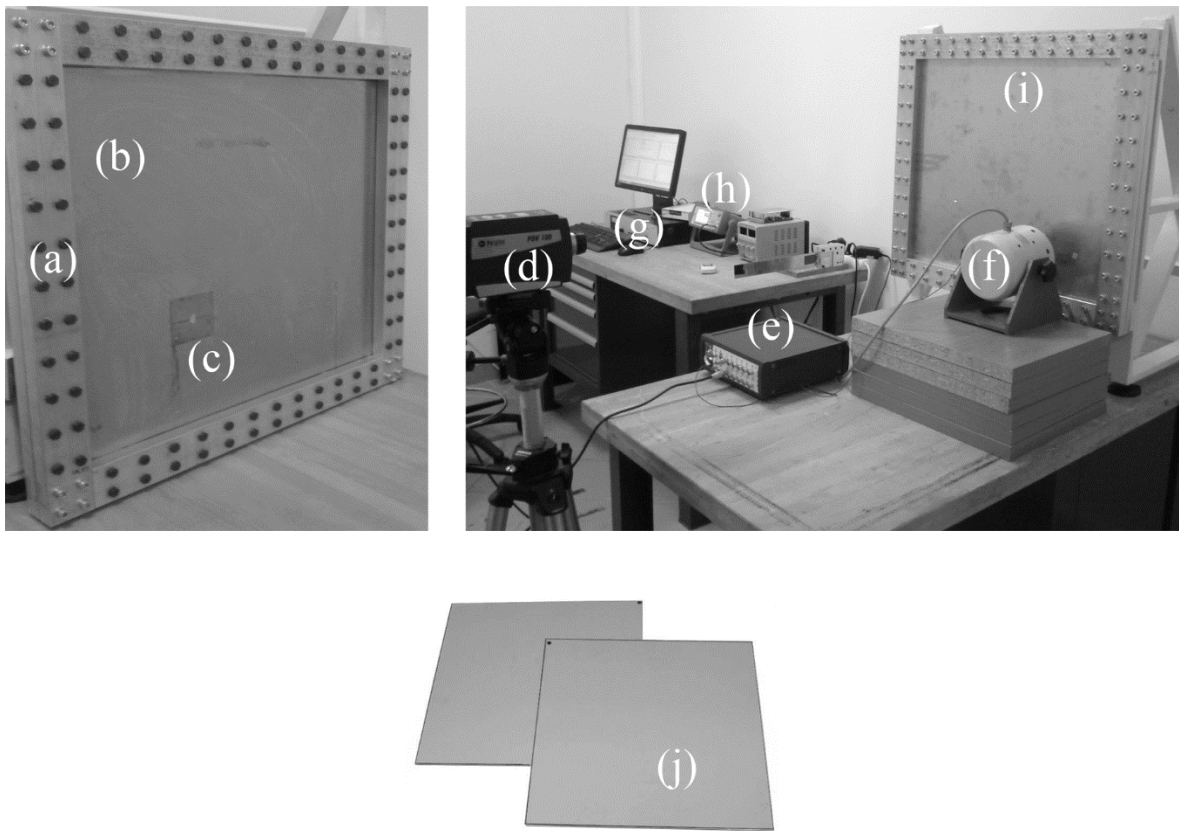


Figure 3.5: Details of the experimental setup: (a) clamping frame; (b) aluminum plate with patch-based harvester; (c) patch-based piezoelectric harvester; (d) laser vibrometer; (e) signal analyzer; (f) shaker; (g) amplifier; (h) signal generator; (i) aluminum plate without patch-based harvester; (j) PZT-5A Piezoceramic Patch [137]

3.4.2 Experimental and Analytical Electroelastic FRFs

The analytical and experimental FRFs are presented in the following to demonstrate the accuracy of the analytical model. Note that the analytical FRFs are obtained between displacement of the plate and force input, whereas the velocity of the plate is measured using the laser vibrometer. Experimental velocity measurements can be integrated or analytical displacement FRFs can be differentiated for comparison. Here, the analytical velocity FRFs are obtained simply multiplying analytical displacement FRFs with $j\omega$.

The mechanical damping ratios used in the analytical model are extracted from the experimental voltage FRF by applying the half-power point method at the resonance frequencies. Figures 3.6 and 3.7 present the analytical and experimental voltage and velocity FRFs for the moderate load resistance of 1.18 k Ω . As can be seen in these figures, the analytical velocity and voltage FRFs exhibit very good agreement with experimental measurements near the resonance and anti-resonance frequencies. The analytical model accurately predicts the velocity and voltage amplitudes for the first four modes of the host plate. At higher modes, there are small shifts at the resonance and anti-resonance frequencies, which may be due to experimental imperfections related to boundary conditions as well as material imperfections, however, the resonance and anti-resonance trends are well predicted.

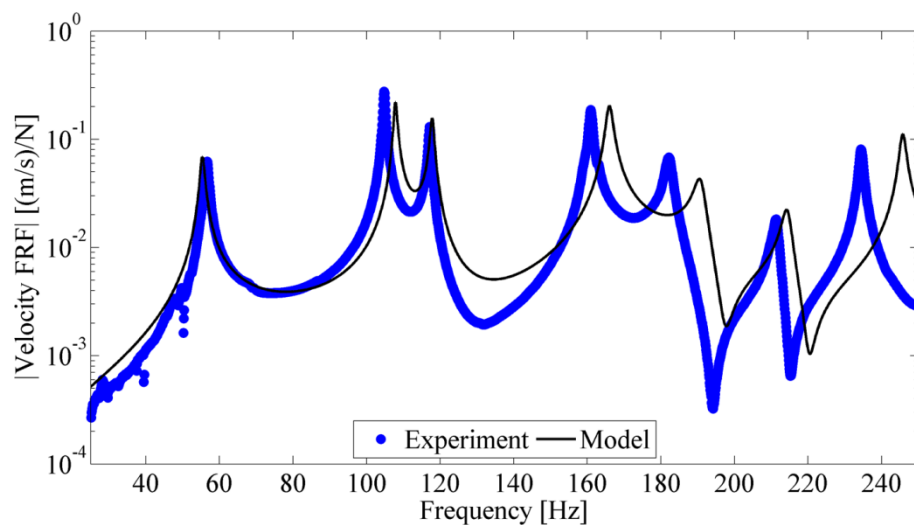


Figure 3.6: Comparison of the experimental and analytical velocity FRFs for a load resistance of 1.18 k Ω

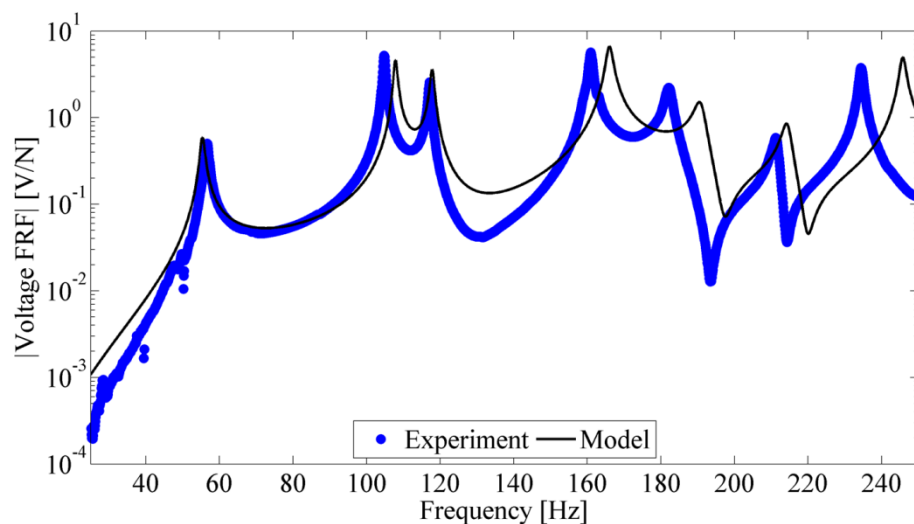


Figure 3.7: Comparison of the experimental and analytical voltage FRFs for a load resistance of 1.18 k Ω

Having shown the accuracy of the analytical model with experimental FRF measurements for a moderate load resistance 1.18 k Ω , analytical and experimental voltage FRFs are obtained for a set of resistive loads consisting of 0.012, 0.058, 0.22, 0.464, 1.18, 2.83, 3.3, 5.49, 9.88, 14.6, 21.94, 32.48, 47.4, 100, 334.4, 479, 693, and 991 k Ω . Figure 3.8 presents the analytical voltage FRFs for these load resistance values. The voltage amplitude increases with increased load resistance up to a certain maximum voltage value around each resonance frequency. The maximum voltage amplitude differs for different resonance frequencies due to the positioning of the piezoelectric energy harvester on the plate (based on the strain mode shapes formerly discussed with figure 3.4).

The experimental voltage FRFs for the set of resistive loads are presented in figure 3.9. The overall agreement between model predictions and experimental measurements is very good in figures 3.8 and 3.9.

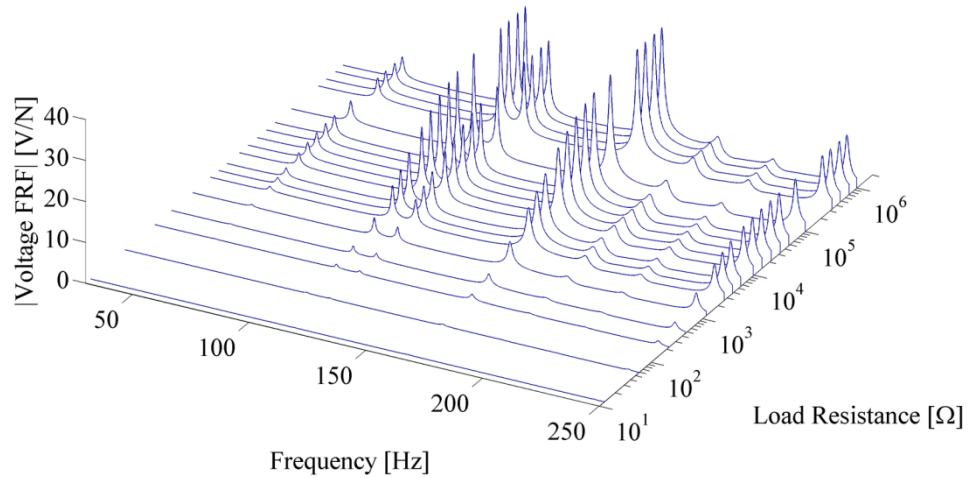


Figure 3.8: Analytical voltage FRFs for a set of resistive loads

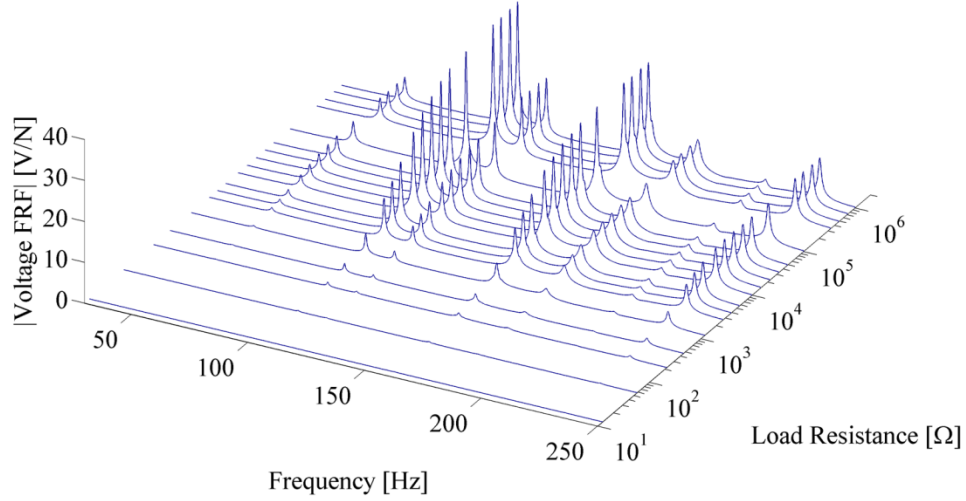


Figure 3.9: Experimental voltage FRFs for a set of resistive loads

3.4.3 Experimental and Analytical Peak Electrical Outputs

The peak electrical outputs (voltage, current, and power) versus load resistance values are investigated. For this peak-amplitude analysis, the experimental voltage FRFs are gathered for eighteen different resistive loads between 0.012 and 991 k Ω while the analytical voltage FRFs are obtained for one hundred resistive loads in the same range to have continuous curves. Note that, for each vibration mode, the analytical and experimental resonances frequencies move from short-circuit resonance frequency (ω_{mn}^{sc} for $R_l \rightarrow 0$) to open-circuit resonance frequency (ω_{mn}^{oc} for $R_l \rightarrow \infty$). The first four short-circuit and open-circuit resonance frequencies of the CCCC plate are listed in Table 3.2. Using these analytical and experimental resonance frequencies, it can be noticed that the maximum relative error of the analytical resonance frequency predictions is less than 3.2% in the first four modes.

Table 3.2: First four short-circuit and open-circuit resonance frequencies of the plate

Resonance frequency	Experiment	Analytical
ω_{11}^{sc} (short circuit) (Hz)	56.54	55.34
ω_{11}^{oc} (open circuit) (Hz)	57.03	55.40
ω_{21}^{sc} (short circuit) (Hz)	104.69	107.83
ω_{21}^{oc} (open circuit) (Hz)	105.47	108.10
ω_{12}^{sc} (short circuit) (Hz)	116.99	117.83
ω_{12}^{oc} (open circuit) (Hz)	117.68	118.10
ω_{22}^{sc} (short circuit) (Hz)	160.84	165.94
ω_{22}^{oc} (open circuit) (Hz)	161.52	166.60

The peak amplitudes versus resistive load trends are analyzed at the open-circuit resonance frequencies of the plate. Figure 3.10 presents the peak voltage amplitude versus load resistance graphs for the first four open-circuit resonance frequencies. As can be seen in this figure, the voltage across the resistive load at each resonance frequency increases monotonically up to a certain level and stays at this level in the limit $R \rightarrow \infty$ (open-circuit condition). These peak voltage results show that the analytical voltage amplitudes have very good agreement with the experimental results for a wide range of load resistance values.

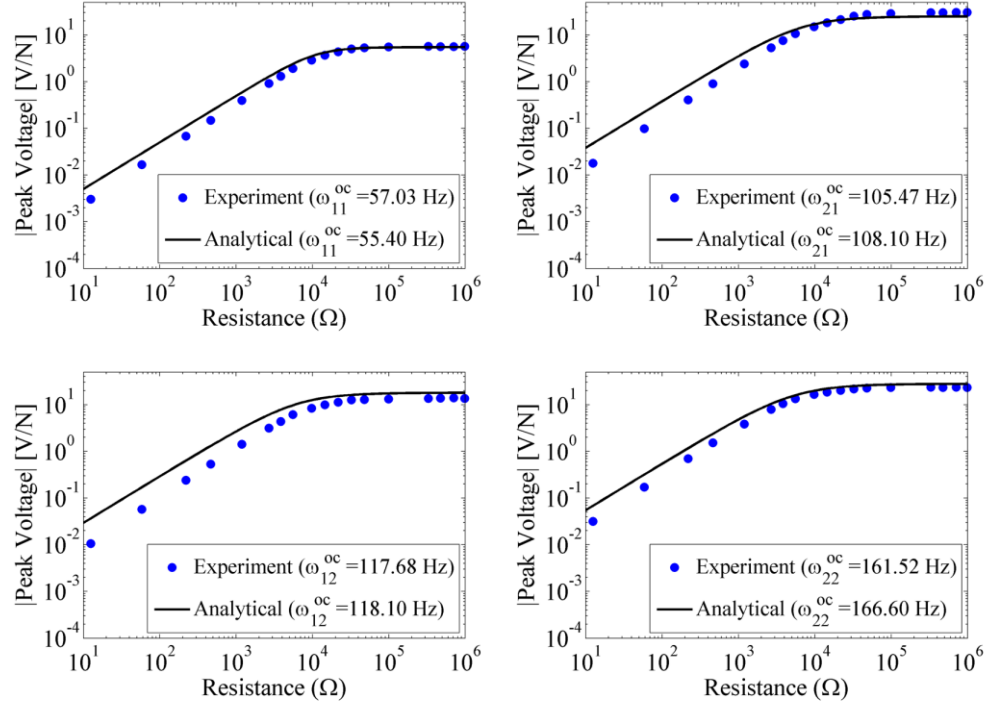


Figure 3.10: Variations of the voltage amplitude with load resistance for the first four vibration modes

The current output flowing to the resistor is obtained by dividing the peak voltage amplitudes to load resistance values. Figure 3.11 shows the trend in the current generated by the patch-based harvester for the first four resonance frequencies. The current values monotonically decrease to a zero value at open-circuit conditions from a certain level (maximum level) at short-circuit condition ($R_l \rightarrow 0$). The experimental trend in the current amplitude with varying load resistance is well captured by the analytical model. Mismatch in the low resistance region can be attributed to the level of noise in the measured electrical response for those resistive loads close to short-circuit conditions.

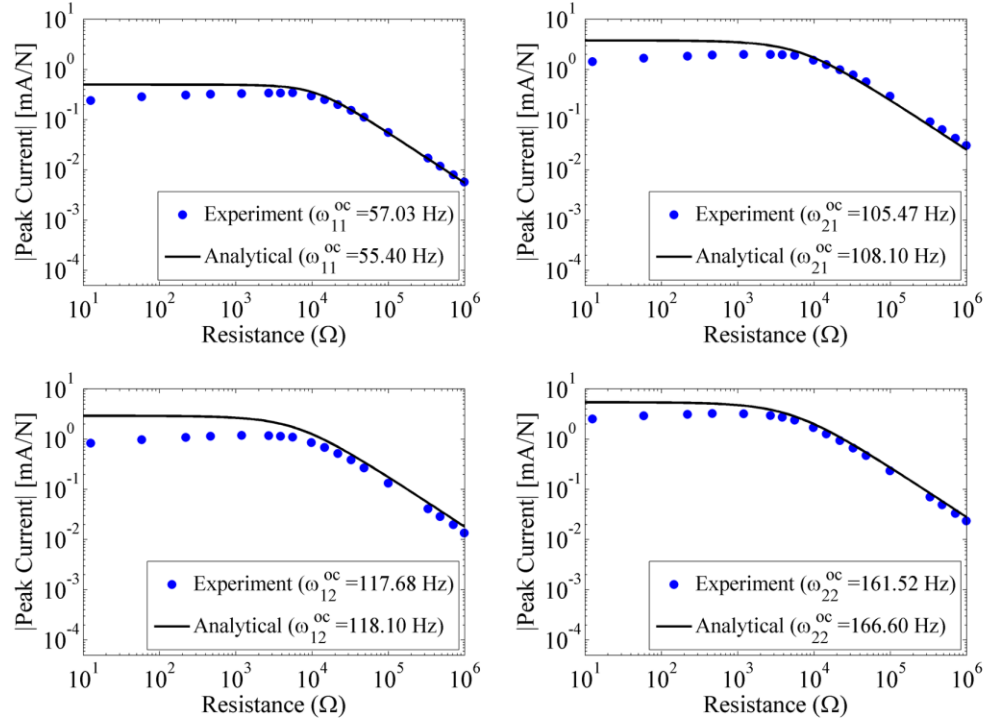


Figure 3.11: Variations of the current amplitude with load resistance for the first four vibration modes

Each resistive load results in different peak voltage and current amplitudes. Using the current and voltage amplitudes, the electrical power output can be calculated for power output analysis. Figure 3.12 shows the trend of maximum power generated by the piezoelectric energy harvester with changing resistive load for the first four resonance frequencies. The variation of power output with changing load resistance yields a similar trend for each resonance frequency with different resistance values for the maximum output. It is useful to compare figure 3.12 and 3.4 with respect to the location of the piezoelectric patch relative to the strain nodes. In particular, the patch is located near a

strain node curve in the fundamental vibration mode, and this mode indeed yields the lowest power output. For vibration modes that yield homogenous strain field under the patch, much larger power output can be extracted from the harvester patch.

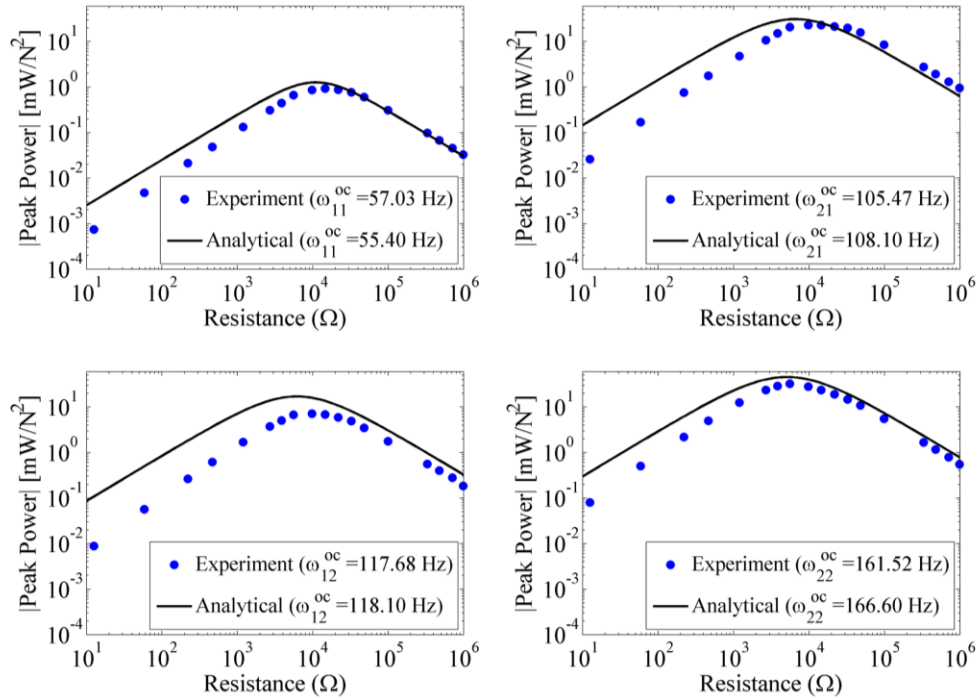


Figure 3.12: Variations of the peak power output with load resistance for the first four vibration modes

3.5 Summary and Conclusions

In this chapter, the distributed-parameter electroelastic model of the single patch-based piezoelectric energy harvester structurally integrated to the thin plate is developed and presented. Closed-form steady-state solutions for the electrical output and structural response are derived for transverse force excitations. Although the modeling and validations have been given for fully clamped boundary conditions, the solution can be extended to other boundary conditions of the thin plates. Voltage and velocity FRFs are presented and model predictions are experimentally verified with a case study for a range of external electrical load changing from short- to open-circuit conditions. The positioning of the patch-based harvester is discussed based on the mode-shape dependent dynamic strain distribution. The single-patch electroelastic model presented herein can be utilized for optimal positioning as well as material selection of the patch-based piezoelectric harvester and the host plate.

Chapter 4

MULTIPLE PATCH-BASED PIEZOELECTRIC ENERGY HARVESTERS ON A THIN PLATE

This chapter extends the single-patch electroelastic model given by Chapter 3 to series and parallel connections of multiple patch-based energy harvesters attached on thin plates for broadband energy harvesting. Coupled electroelastic models of multiple patch-based harvesters attached on a thin plate are introduced in physical coordinates and transformed into modal coordinates for an all-four-edges clamped rectangular (CCCC) plate. Exact steady-state coupled voltage and vibration response expressions are derived considering the multiple vibration modes of the plate for harmonic force excitations. For series and parallel connection configurations, closed-form expressions for electroelastic FRFs are derived for the vibration response to force input and voltage output to force input. Experimental and analytical case studies are then presented for validating the closed-form solutions. Finally, broadband nature of energy harvesting using piezoelectric patches integrated to thin plates is discussed. The analytical and experimental voltage and power output curves are reported over a wide range of electrical loads for the first eight vibration modes of the plate.

4.1 Distributed-parameter Electroelastic Modeling in Physical Coordinates

4.1.1 Coupled Mechanical Equation

Series and parallel connection configurations of multiple patch-based piezoelectric energy harvesters attached on a host plate are shown in figure 4.1(a) and 4.1(b). Patch-based harvesters are in the form of transversely isotropic and thin piezoceramic patches. The length and the width of the host plate are a and b , respectively. The thickness ($h_{\bar{s}}$) of the plate is assumed to be uniform and much smaller than the total length and width of the plate. In this chapter, the subscripts $\bar{s}$ and $\bar{p}$ stand for the parameters related to the host structure and patch-based harvesters, respectively. Each patch-based harvester covers a rectangular area with two corners at $(x_{k,1}, y_{k,1})$ and $(x_{k,2}, y_{k,2})$ where $k = 1, 2, \dots, K$ and K is the number of patch-based harvesters. The thickness of the k th patch is $(h_{\bar{p}})_k$ and each patch is poled in thickness direction (z -axis). The top and bottom faces of the patches are covered by negligibly thin and highly conductive electrode layers. In order to enable the series and parallel connection configurations, it is assumed that the surfaces of the plate are electrically insulated. A load resistance R_l is considered as an external electrical load connected to the resultant terminals of the series and parallel configurations of patch-based harvesters. The plate is persistently excited by transverse point force $f(t)$ at (x_0, y_0) , hence the voltage output across the load resistance can be extracted continuously.

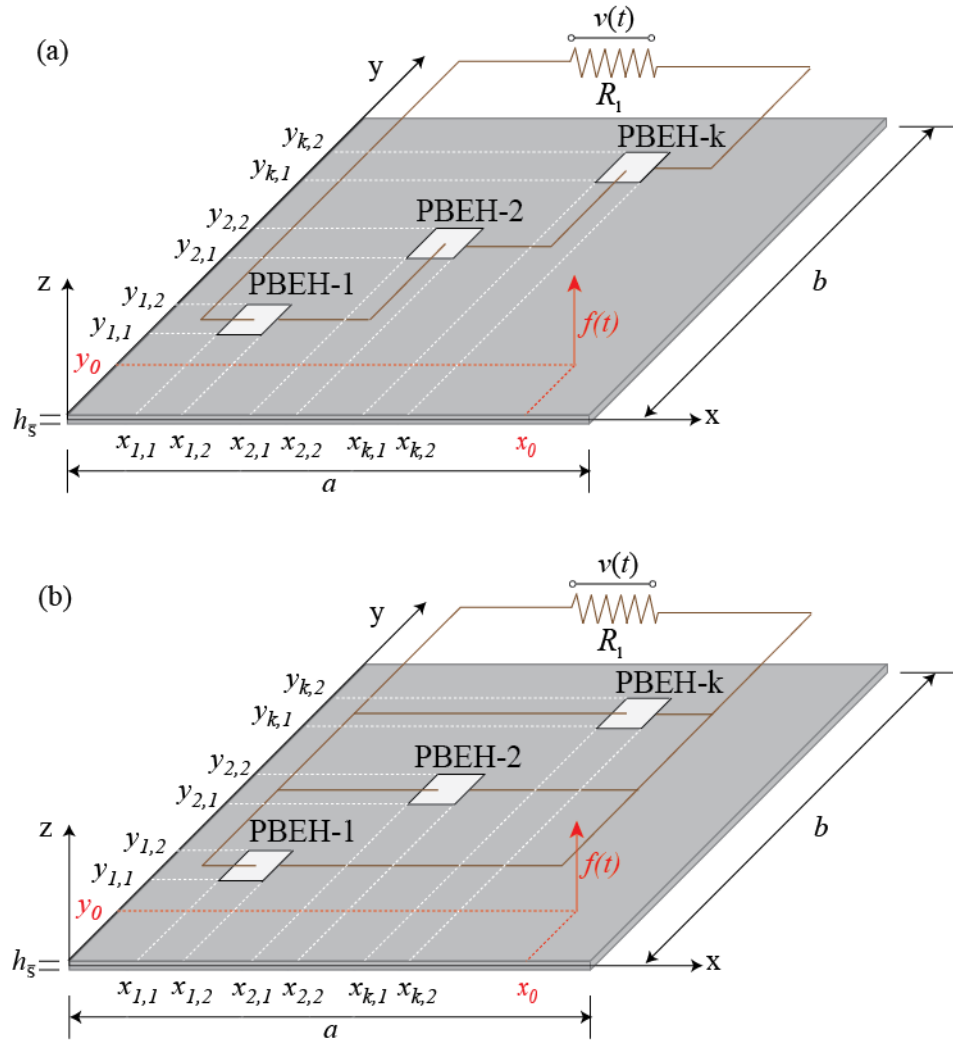


Figure 4.1: Multiple patch-based piezoelectric energy harvester configurations on a thin plate: (a) series and (b) parallel connection of patch-based energy harvesters (PBEH)

The partial differential equation governing the forced vibration of the thin plate with structurally-integrated patch-based harvesters depicted in figure 4.1 can be written as [31]

$$\begin{aligned}
 & D_{\bar{s}} \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + c \frac{\partial w(x, y, t)}{\partial t} + \rho_{\bar{s}} h_{\bar{s}} \frac{\partial^2 w(x, y, t)}{\partial t^2} \\
 & - \sum_{k=1}^K \theta_k v_k(t) \left\{ \left[\frac{d\delta(x - x_{k,1})}{dx} - \frac{d\delta(x - x_{k,2})}{dx} \right] [H(y - y_{k,1}) - H(y - y_{k,2})] \right. \\
 & \left. + \left[\frac{d\delta(y - y_{k,1})}{dy} - \frac{d\delta(y - y_{k,2})}{dy} \right] [H(x - x_{k,1}) - H(x - x_{k,2})] \right\} \\
 & = f(t) \delta(x - x_0) \delta(y - y_0)
 \end{aligned} \tag{4.1}$$

where $w(x, y, t)$ is the transverse deflection of the reference surface (neutral surface) at position (x, y) and time t . The voltage across the electrodes of the k th harvester patch is $v_k(t)$ and the voltage across the resistive load is $v(t)$ as shown in figures 4.1(a) and 4.1(b). The mass density of the plate is denoted by $\rho_{\bar{s}}$ and the flexural rigidity of the plate is $D_{\bar{s}} = Y_{\bar{s}} h_{\bar{s}}^3 / (12(1 - \nu_{\bar{s}}^2))$, where $Y_{\bar{s}}$ and $\nu_{\bar{s}}$ are Young's modulus and Poisson's ratio, respectively. Furthermore, $\delta(x)$ and $\delta(y)$ are the Dirac delta functions; while $H(x)$ and $H(y)$ are the Heaviside functions. The viscous damping coefficient is c and the electromechanical coupling θ_k for the k th harvester patch is

$$\theta_k = (h_{pc})_k (\bar{e}_{31})_k \tag{4.2}$$

where $(\bar{e}_{31})_k$ is the effective piezoelectric constant for the k th harvester patch in two-dimensional form as given by [37] and $(h_{pc})_k$ is the reference distance (between the neutral surface and the center of the patch thickness) for the k th harvester patch. It is important to note that the piezoceramic patches' inertial and stiffness effects are neglected (by assuming the total amount of piezoelectric material to be much smaller than the host plate dimensions) while the piezoelectrically induced moments are taken into account within equation (4.1) because of the piezoelectric converse effect on the thin plate.

In the following, coupled electrical circuit equations are derived for series and parallel configurations. Note that, hereafter, the superscripts s and p stand for the series and parallel connections of patch-based harvesters.

4.1.2 Coupled Electrical Circuit Equation for Series Connection

A simple electrical circuit in figure 4.2 represents the series connection of harvester patches connected to a load resistance R_l . Each harvester patch is represented as a dependent current source $i_k^s(t)$ in parallel with its internal capacitance $(C_{\bar{p}})_k$.

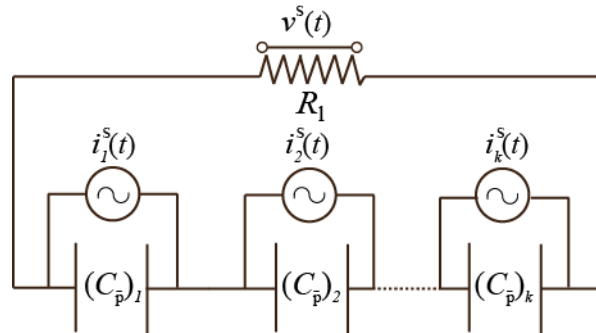


Figure 4.2: Electrical circuit for series connection of multiple harvester patches to a resistive load (current sources depend on the voltage output due to backward coupling)

Using Kirchhoff's laws, the governing electrical circuit equation for each harvester patch can be derived as

$$(C_{\bar{p}})_k \frac{d v_k^s(t)}{dt} + \frac{1}{R_l} v^s(t) = i_k^s(t) \quad k = 1, 2, \dots, K \quad (4.3)$$

where $v_k^s(t)$ is the voltage across the electrodes of the k th harvester patch and $v^s(t)$ is the voltage across the resistive load. The number of governing circuit equations is equal to the number of harvester patches (K). The internal capacitance of the k th harvester patch $(C_{\bar{p}})_k$ can be expressed as

$$(C_{\bar{p}})_k = (\bar{\epsilon}_{33}^S)_k \frac{(l_{\bar{p}})_k (w_{\bar{p}})_k}{(h_{\bar{p}})_k} \quad (4.4)$$

where the length, width and thickness of the k th harvester patch are $(l_{\bar{p}})_k$, $(w_{\bar{p}})_k$ and $(h_{\bar{p}})_k$ and $(\bar{\epsilon}_{33}^S)_k$ is the permittivity of the k th patch at constant strain S . The current produced by the k th harvester patch can be obtained as [29]

$$i_k^s(t) = -\theta_k \int_{y_{k,1}}^{y_{k,2}} \int_{x_{k,1}}^{x_{k,2}} \left[\frac{\partial^3 w(x, y, t)}{\partial x^2 \partial t} + \frac{\partial^3 w(x, y, t)}{\partial y^2 \partial t} \right] dx dy \quad (4.5)$$

Equation (4.5) shows that the current output depends on the rate of change of strain distribution on the plate's surface covered by the harvester patch. The area integral of the time derivative of strain distribution determines the collected charge output on the electrode layer of the harvester patch. Therefore, depending on the location of the harvester patches on the thin plate, charge cancellations may occur. The reader is referred to Chapter 3 for detailed discussions regarding the relationship between the strain distribution and the power output of a single patch-based harvester attached on a thin plate. It should be noted that the electrical outputs of multiple patches are combined in the cases of series and

parallel connections, and consequently, enhancement or cancellation of the electrical output can take place. Therefore, the relative positioning of the harvester patches and proper combination of their terminals are very important.

The distributed-parameter electroelastic model of multiple harvesters combined in series configuration consists of a mechanical equation (equation (4.1)) and K number of electrical circuit equations (equation (4.3)). Once these electromechanically coupled $K + 1$ number of equations are simultaneously solved for a given force input, vibration response of the plate and voltage output of each harvester patch can be calculated. Then, the voltage output across the resistor can be calculated using

$$v^s(t) = \sum_{k=1}^K v_k^s(t) \quad (4.6)$$

4.1.3 Coupled Electrical Circuit Equation for Parallel Connection

The electrical circuit in figure 4.3 illustrates the coupled electrical dynamics of harvester patches combined in parallel. The harvester patches are all connected in parallel to a resistor and each harvester patch is again modeled as a dependent current source with a parallel internal capacitance. Note that, in the parallel configuration, the voltage across the electrodes of each harvester patch $v_k^p(t)$ is equal to the voltage $v^p(t)$ across the resistor.

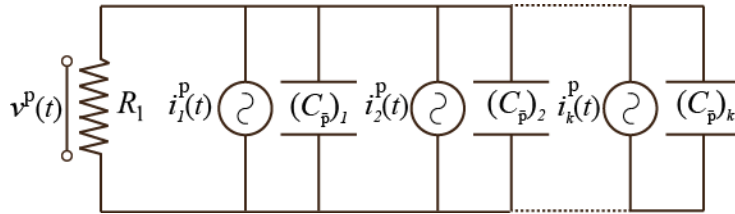


Figure 4.3: Electrical circuit for parallel connection of multiple harvester patches to a resistive load (current sources depend on the voltage output due to backward coupling)

The governing electrical equation for harvester patches connected in parallel can be obtained as

$$\frac{d v^p(t)}{dt} \sum_{k=1}^K (C_{\bar{p}})_k + \frac{v^p(t)}{R_l} = \sum_{k=1}^K i_k^p(t) \quad (4.7)$$

where the capacitance and the current output of the k th harvester are $(C_{\bar{p}})_k$ and $i_k^p(t)$, which are identical to equation (4.4) and (4.5), respectively. Thus, one circuit equation governs the electrical part of the electroelastic model of multiple patches connected in parallel. Therefore, two differential equations (mechanical part in equation (4.1) and electrical part in equation (4.7)) form the electroelastic model of parallel-connected harvester patches attached on a thin plate. For a given force input, voltage across the load resistance and shunted vibration response of the plate can be calculated by solving these two electromechanically coupled differential equations simultaneously.

4.2 Distributed-parameter Electroelastic Modeling in Modal Coordinates

4.2.1 Modal Analysis of a Thin Plate

The vibration response of a thin plate can be represented as

$$w^s(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn}(x, y) \eta_{mn}^s(t) \quad (4.8a)$$

$$w^p(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn}(x, y) \eta_{mn}^p(t) \quad (4.8b)$$

where $\phi_{mn}(x, y)$ is the mass-normalized eigenfunction of the m th vibration mode in x and y axes, and $\eta_{mn}^s(t)$ and $\eta_{mn}^p(t)$ are the generalized modal responses for the series and parallel connections of harvester patches, respectively.

The all-four-edges clamped boundary conditions (CCCC) of a thin rectangular plate ensure zero deflections and zero slopes along all four edges. For the series connection of harvester patches, the boundary conditions of a CCCC plate with a length of a and a width of b can be written as

$$\begin{aligned} w^s(0, y, t) = 0 \quad w^s(a, y, t) = 0 \quad w^s(x, 0, t) = 0 \quad w^s(x, b, t) = 0 \\ \left. \frac{\partial w^s(x, y, t)}{\partial x} \right|_{x=0} = 0 \quad \left. \frac{\partial w^s(x, y, t)}{\partial x} \right|_{x=a} = 0 \quad \left. \frac{\partial w^s(x, y, t)}{\partial y} \right|_{y=0} = 0 \quad \left. \frac{\partial w^s(x, y, t)}{\partial y} \right|_{y=b} = 0 \end{aligned} \quad (4.9)$$

and expectedly the boundary conditions are the same in parallel connections.

The eigenfunctions satisfying the boundary conditions of a CCCC plate can be derived by using the separation of spatial variables [134] as follows:

$$\begin{aligned} \phi_{mn}(x, y) = \Theta_{mn} & \left(\cos \lambda_{mn}^x x - \sigma_{mn}^x \sin \lambda_{mn}^x x - \cosh \mu_{mn}^x x + \frac{\lambda_{mn}^x}{\mu_{mn}^x} \sigma_{mn}^x \sinh \mu_{mn}^x x \right) \\ & \times \left(\cos \lambda_{mn}^y y - \sigma_{mn}^y \sin \lambda_{mn}^y y - \cosh \mu_{mn}^y y + \frac{\lambda_{mn}^y}{\mu_{mn}^y} \sigma_{mn}^y \sinh \mu_{mn}^y y \right) \end{aligned} \quad (4.10)$$

where σ_{mn}^x and σ_{mn}^y are

$$\begin{aligned} \sigma_{mn}^x = \frac{\cos \lambda_{mn}^x a - \cosh \mu_{mn}^x a}{\sin \lambda_{mn}^x a - \frac{\lambda_{mn}^x}{\mu_{mn}^x} \sinh \mu_{mn}^x a} \quad \sigma_{mn}^y = \frac{\cos \lambda_{mn}^y b - \cosh \mu_{mn}^y b}{\sin \lambda_{mn}^y b - \frac{\lambda_{mn}^y}{\mu_{mn}^y} \sinh \mu_{mn}^y b} \end{aligned} \quad (4.11)$$

and the modal amplitude constant Θ_{mn} can be calculated by applying the following orthogonality condition

$$\int_0^b \int_0^a \rho_{\bar{s}} h_{\bar{s}} \phi_{mn}(x, y) \phi_{rs}(x, y) dx dy = \delta_{mr} \delta_{ns} \quad (4.12)$$

where δ_{mr} and δ_{ns} are Kronecker delta functions; δ_{mr} equals to unity when $m=r$ and it is zero when $m \neq r$, similarly, δ_{ns} equals to unity only for $n=s$ and zero for $n \neq s$. The undamped natural frequency ω_{mn} for the m th vibration mode of the plate in short circuit conditions (as $R_1 \rightarrow 0$) is given by

$$\omega_{mn} = \sqrt{\frac{\kappa_{mn}^4}{\rho_{\bar{s}} h_{\bar{s}}}} D_{\bar{s}} \quad (4.13)$$

where κ_{mn} is the frequency parameter of the undamped plate and eigenvalues ($\lambda_{mn}^x, \lambda_{mn}^y, \mu_{mn}^x, \mu_{mn}^y$) can be calculated in the intervals of

$$\begin{aligned} \lambda_{mn}^x a &\in \left[m\pi, m\pi + \frac{\pi}{2} \right], & m = 1, 2, \dots \\ \lambda_{mn}^y b &\in \left[n\pi, n\pi + \frac{\pi}{2} \right], & n = 1, 2, \dots \end{aligned} \quad (4.14)$$

by using a transcendental solver for the algebraic equations, which are presented in equations (4.15a) to (4.15e) as follows [134]:

$$\frac{1 - \cosh \mu_{mn}^x a \cos \lambda_{mn}^x a}{\sinh \mu_{mn}^x a \sin \lambda_{mn}^x a} = \frac{(\lambda_{mn}^x)^2 - (\mu_{mn}^x)^2}{2 \lambda_{mn}^x \mu_{mn}^x} \quad (4.15a)$$

$$\frac{1 - \cosh \mu_{mn}^y b \cos \lambda_{mn}^y b}{\sinh \mu_{mn}^y b \sin \lambda_{mn}^y b} = \frac{(\lambda_{mn}^y)^2 - (\mu_{mn}^y)^2}{2\lambda_{mn}^y \mu_{mn}^y} \quad (4.15b)$$

$$\kappa_{mn} = \sqrt{(\lambda_{mn}^x)^2 + (\lambda_{mn}^y)^2} \quad (4.15c)$$

$$\mu_{mn}^x = \sqrt{2\kappa_{mn}^2 - (\lambda_{mn}^x)^2} \quad (4.15d)$$

$$\mu_{mn}^y = \sqrt{2\kappa_{mn}^2 - (\lambda_{mn}^y)^2} \quad (4.15e)$$

These modal parameters (eigenvalues and eigenfunctions) are derived for the short-circuit condition ($R_l \rightarrow 0$) of series and parallel connections of harvester patches since the voltage term in the mechanical part of the electroelastic model in equation (4.1) becomes zero ($v_k \rightarrow 0$) in short circuit condition. Therefore, the electromechanical coupling between the harvester patches and the plate vanishes in short-circuit conditions and standard modal analysis is valid. For finite load resistance values, the electromechanical coupling between the harvester patches and the plate takes place due to the induced moments created along the edges of the harvester patch. In this case, modal response of the plate depends on the voltage of each harvester patch and differs in series and parallel configurations. In the following sections, differential equations governing the modal responses of the series and parallel connections are obtained by transforming the coupled mechanical and electrical equations of harvester patches into modal coordinates.

4.2.2 Modal Transformation of Electroelastic Equations for Series Connection

The mechanical equation of motion given by equation (4.1) in the physical coordinates can be transformed into modal coordinates by following the modal analysis process [29].

This procedure includes substituting equation (4.8a) into equation (4.1) and applying the orthogonality conditions, yielding

$$\frac{d^2 \eta_{mn}^s(t)}{dt^2} + 2\zeta_{mn} \omega_{mn} \frac{d\eta_{mn}^s(t)}{dt} + \omega_{mn}^2 \eta_{mn}^s(t) - \sum_{k=1}^K (\tilde{\theta}_{mn})_k v_k^s(t) = f_{mn}(t) \quad (4.16)$$

where ω_{mn} is the undamped natural frequency (in short-circuit condition) and ζ_{mn} is the modal damping ratio for the mn th vibration mode of the thin plate. Note that, the modal damping ratio can be obtained by using the standard damping identification techniques [136] based on experimental measurements. The transverse force in modal coordinates $f_{mn}(t)$ is

$$f_{mn}(t) = f(t) \phi_{mn}(x_0, y_0) \quad (4.17)$$

and the electromechanical coupling term for the k th harvester in modal coordinates is given by

$$(\tilde{\theta}_{mn})_k = \theta_k \int_{y_{k,1}}^{y_{k,2}} \int_{x_{k,1}}^{x_{k,2}} \left[\frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} + \frac{\partial^2 \phi_{mn}(x, y)}{\partial y^2} \right] dx dy \quad (4.18)$$

To obtain the electrical circuit equation of the electroelastic model in modal coordinates, first, equation (4.8a) is substituted into the current output expression in equation (4.5) to give

$$i_k^s(t) = -\theta_k \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{y_{k,1}}^{y_{k,2}} \int_{x_{k,1}}^{x_{k,2}} \left[\frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} + \frac{\partial^2 \phi_{mn}(x, y)}{\partial y^2} \right] \frac{d\eta_{mn}^s(t)}{dt} dx dy \quad (4.19)$$

and then equation (4.19) can be reduced to

$$i_k^s(t) = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k \frac{d\eta_{mn}^s(t)}{dt} \quad (4.20)$$

by substituting electromechanical coupling term given by equation (4.18) into equation (4.19). Then, using equation (4.20) and equation (4.3), the electrical part of the electroelastic model can be obtained as

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k \frac{d\eta_{mn}^s(t)}{dt} + (C_p^-)_k \frac{dv_k^s(t)}{dt} + \frac{1}{R_1} \sum_{k=1}^K v_k^s(t) = 0 \text{ where } k = 1, 2, \dots, K \quad (4.21)$$

Here, the electrical part of the electroelastic model is constituted by K number of equations in modal coordinates. As a result, the distributed-parameter electroelastic model of the harvester patches connected in series is expressed in modal coordinates by $K + 1$ number of ordinary differential equations given by equations (4.16) and (4.21).

4.2.3 Modal Transformation of Electroelastic Equations for Parallel Connection

The electromechanically coupled equation of motion can be obtained in modal coordinates as

$$\frac{d^2 \eta_{mn}^p(t)}{dt^2} + 2\zeta_{mn}\omega_{mn} \frac{d\eta_{mn}^p(t)}{dt} + \omega_{mn}^2 \eta_{mn}^p(t) - v^p(t) \sum_{k=1}^K (\tilde{\theta}_{mn})_k = f_{mn}(t) \quad (4.22)$$

For the electrical part of the electroelastic model, equation (4.8b) can be substituted into equation (4.5) to obtain

$$i_k^p(t) = -\theta_k \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{y_{k,1}}^{y_{k,2}} \int_{x_{k,1}}^{x_{k,2}} \left[\frac{\partial^2 \phi_{mn}(x, y)}{\partial x^2} + \frac{\partial^2 \phi_{mn}(x, y)}{\partial y^2} \right] \frac{d\eta_{mn}^p(t)}{dt} dx dy \quad (4.23)$$

Equation (4.23) is then simplified to

$$i_k^p(t) = -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k \frac{d\eta_{mn}^p(t)}{dt} \quad (4.24)$$

by using the electromechanical coupling term $(\tilde{\theta}_{mn})_k$ given by equation (4.18). Then, substitution of equation (4.24) into equation (4.7) gives the electrical part of the electroelastic model in modal coordinates as

$$\sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k \frac{d\eta_{mn}^p(t)}{dt} + \frac{dv^p(t)}{dt} \sum_{k=1}^K (C_{\bar{p}})_k + \frac{v^p(t)}{R_l} = 0 \quad (4.25)$$

Consequently, the distributed-parameter electroelastic model for the parallel configuration of harvester patches in modal coordinates consists of two differential equations as given by equation (4.22) and (4.25).

4.3 Steady-state Response to Harmonic Force Excitation

4.3.1 Multi-mode Electroelastic FRFs for Series Connection

The coupled electroelastic equations forming the distributed-parameter model of series-connected harvester patches in modal coordinates are given by equations (4.16) and (4.21). If the transverse point force acting on the surface of the plate is assumed to be harmonic of the form $f(t) = F_0 e^{j\omega t}$ (where F_0 is the amplitude of the force and ω is the excitation

frequency), the mechanical and electrical responses are assumed to be harmonic based on the linear system assumption as follows:

$$\eta_{mn}^s(t) = H_{mn}^s e^{j\omega t}, \quad v_k^s(t) = V_k^s e^{j\omega t} \quad (4.26)$$

where H_{mn}^s and V_k^s are complex valued amplitudes. The coupled mechanical and electrical equations of the distributed-parameter model for series configuration then yield the following forms

$$(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega)H_{mn}^s - \sum_{k=1}^K (\tilde{\theta}_{mn})_k V_k^s = F_0 \phi_{mn}(x_0, y_0) \quad (4.27)$$

and

$$j\omega \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k H_{mn}^s + j\omega (C_{\bar{p}})_k V_k^s + \frac{1}{R_1} \sum_{k=1}^K V_k^s = 0 \text{ where } k = 1, 2, \dots, K \quad (4.28)$$

Then, the complex amplitude H_{mn}^s can be extracted from equation (4.27) as

$$H_{mn}^s = \frac{F_0 \phi_{mn}(x_0, y_0) + \boldsymbol{\theta}_{mn}^T \mathbf{V}^s}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \quad (4.29)$$

Here, the voltage amplitude vector $\mathbf{V}^s$ with the dimensions of $K \times 1$ includes the voltage amplitude of each harvester patch as $\mathbf{V}^s = [V_1^s \ V_2^s \ \dots \ V_K^s]^T$, where superscript T stands for the transpose, and the electromechanical coupling vector $\boldsymbol{\theta}_{mn}$ with the dimensions of $K \times 1$ is $\boldsymbol{\theta}_{mn} = [(\tilde{\theta}_{mn})_1 \ (\tilde{\theta}_{mn})_2 \ \dots \ (\tilde{\theta}_{mn})_K]^T$. The voltage amplitude vector $\mathbf{V}^s$ can be solved using the following matrix form

$$\mathbf{V}^s = F_0 \mathbf{Q}^{-1} \mathbf{P} \quad (4.30)$$

where $\mathbf{Q}$ is a square matrix with the dimensions of $K \times K$ and $\mathbf{P}$ is a column vector with the dimensions of $K \times 1$, which can be written as

$$\mathbf{Q} = \begin{bmatrix} \frac{1 + j\omega(C_{\bar{p}})_1 R_1}{R_1} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega[(\tilde{\theta}_{mn})_1]^{\bar{p}}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} & \dots & \dots & \frac{1}{R_1} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_1(\tilde{\theta}_{mn})_K}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{R_1} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_K(\tilde{\theta}_{mn})_1}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} & \dots & \dots & \frac{1 + j\omega(C_{\bar{p}})_K R_1}{R_1} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega[(\tilde{\theta}_{mn})_K]^{\bar{p}}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \end{bmatrix} \quad (4.31)$$

$$\mathbf{P} = \begin{bmatrix} -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_1 \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \\ \vdots \\ -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_K \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \end{bmatrix}$$

Therefore, the steady-state voltage output $v^s(t)$ across the resistive load can be obtained as

$$v^s(t) = \sum_{k=1}^K V_k^s e^{j\omega t} = \mathbf{u}^T F_0 \mathbf{Q}^{-1} \mathbf{P} e^{j\omega t} \quad (4.32)$$

where $\mathbf{u}$ is a unit vector with the dimensions of $K \times 1$. The steady-state vibration response can be written as

$$w^s(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn}(x, y) \frac{F_0 \phi_{mn}(x_0, y_0) + \boldsymbol{\theta}_{mn}^T F_0 \mathbf{Q}^{-1} \mathbf{P}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} e^{j\omega t} \quad (4.33)$$

After formulating the steady-state responses, two multi-mode FRFs can be extracted between two outputs and one input: voltage output-to-force input ratio (i.e., voltage FRF $\alpha^s(\omega)$) and the displacement-to-force input ratio (i.e., displacement FRF $\beta^s(x, y, \omega)$).

The voltage FRF is given by

$$\alpha^s(\omega) = \frac{v^s(t)}{F_0 e^{j\omega t}} = \mathbf{u}^T \mathbf{Q}^{-1} \mathbf{P} \quad (4.34)$$

Similarly, the displacement FRF at position x and y on the plate is

$$\beta^s(x, y, \omega) = \frac{w^s(x, y, t)}{F_0 e^{j\omega t}} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn}(x, y) \frac{\phi_{mn}(x_0, y_0) + \boldsymbol{\theta}_{mn}^T \mathbf{Q}^{-1} \mathbf{P}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \quad (4.35)$$

4.3.2 Multi-mode Electroelastic FRFs for Parallel Connection

Similar to the series connection case, when the transverse point force acting on the surface of the plate is assumed to be harmonic of the form $f(t) = F_0 e^{j\omega t}$, the steady-state mechanical and electrical responses are

$$\eta_{mn}^p(t) = H_{mn}^p e^{j\omega t}, \quad v^p(t) = V^p e^{j\omega t} \quad (4.36)$$

where H_{mn}^p and V^p are complex valued amplitudes. Recall that, in the parallel connection case, the voltage output of each harvester patch is equal to the voltage $v^p(t)$ across the resistor (see figure 4.3). Using the harmonic forms of steady-state structural and electrical responses in equation (4.36), the coupled mechanical and electrical equations for parallel configuration can be written as

$$(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega)H_{mn}^p - V^p \sum_{k=1}^K (\tilde{\theta}_{mn})_k = F_0 \phi_{mn}(x_0, y_0) \quad (4.37)$$

and

$$\sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} j\omega (\tilde{\theta}_{mn})_k H_{mn}^p + \left(\sum_{k=1}^K j\omega (C_{\bar{p}})_k + \frac{1}{R_l} \right) V^p = 0 \quad (4.38)$$

Then, the complex amplitude H_{mn}^p can be extracted from equation (4.37) as

$$H_{mn}^p = \frac{F_0 \phi_{mn}(x_0, y_0) + V^p \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \quad (4.39)$$

and substituting equation (4.39) into equation (4.38) gives the steady-state voltage $v^p(t)$ across the resistor as

$$v^p(t) = V^p e^{j\omega t} = \frac{- \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k F_0 \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega \sum_{k=1}^K (C_{\bar{p}})_k + \frac{1}{R_1} + \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} e^{j\omega t} \quad (4.40)$$

and the transverse deflection of the plate at any point (x, y) and time t can be obtained by using equations (4.8b), (4.36), (4.39) and (4.40) as follows:

$$w^p(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\phi_{mn}(x, y)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \left\{ F_0 \phi_{mn}(x_0, y_0) \right. \quad (4.41)$$

$$\left. - \frac{\left[\sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k F_0 \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \right] \times \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{j\omega \sum_{k=1}^K (C_{\bar{p}})_k + \frac{1}{R_1} + \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \right\} e^{j\omega t}$$

The closed-form expression for the voltage FRF $\alpha^p(\omega)$ is given by

$$\alpha^p(\omega) = \frac{v^p(t)}{F_0 e^{j\omega t}} = \frac{-\sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}}{j\omega \sum_{k=1}^K (C_{\bar{p}})_k + \frac{1}{R_1} + \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \quad (4.42)$$

and the displacement FRF $\beta^p(x, y, \omega)$ is

$$\beta^p(x, y, \omega) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\phi_{mn}(x, y)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \left\{ \phi_{mn}(x_0, y_0) \right. \quad (4.43)$$

$$\left. - \frac{\left[\sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \right] \times \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{j\omega \sum_{k=1}^K (C_{\bar{p}})_k + \frac{1}{R_1} + \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_k \sum_{k=1}^K (\tilde{\theta}_{mn})_k}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega}} \right\}$$

Note that these multi-mode expressions can be used for thin plates with different boundary conditions using the respective modal parameters (natural frequencies and mass-normalized mode shapes).

4.4 Electroelastic Analysis and Experimental Validations

4.4.1 Experimental Setup

The experimental setup used for model validation and experimental demonstrations is shown in figure 4.4. A rectangular aluminum sheet plate is used as a thin host plate and all four edges are clamped symmetrically to the thick aluminum bars by tightening the screws along each edge. The transversely isotropic piezoceramic square patches (T105-A4E-602 manufactured by Piezo Systems Inc.) are employed as patch-based harvesters by bonding on the aluminum plate. The geometric, material and electroelastic properties of the plate and the patch-based harvesters are given in Table 4.1.

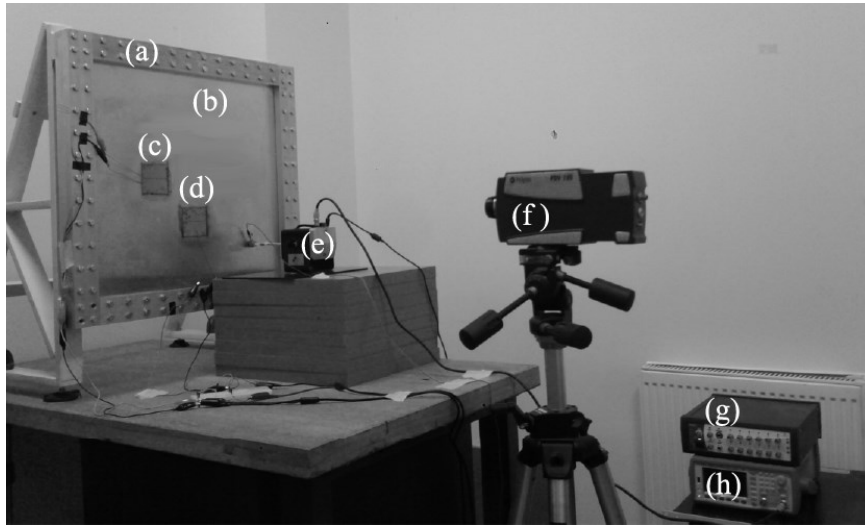


Figure 4.4: Experimental setup: (a) clamping frames; (b) aluminum plate; (c) PBEH-1; (d) PBEH-2; (e) shaker with a force transducer; (f) laser vibrometer; (g) data acquisition unit; (h) signal generator

Table 4.1: Geometric, material and electroelastic properties

Property	Aluminum	Piezoceramic
Length (mm)	580	72.4
Width (mm)	540	72.4
Thickness (mm)	1.9	0.267
Young's modulus (GPa)	65.1	66
Mass density (kg m ⁻³)	2575	7800
Piezoelectric constant d_{31} (pm V ⁻¹)	-	-190
Permittivity constant $\bar{\epsilon}_{33}^S$ (nF m ⁻¹)	-	10.38

Two harvester patches (labeled as PBEH-1 and PBEH-2) are attached on the front surface of the aluminum plate as shown in figure 4.4 and 4.5(a). The patches come from the manufacturer with highly conductive electrode layers and the attachment regions on the aluminum plate are electrically isolated with 3M Scotch 1601 insulating spray. The center of the PBEH-1 is located at 0.175 m and 0.272 m away from the bottom left corner of the plate, whereas the center of the PBEH-2 is at 0.287 m and 0.165 m away from the bottom left corner as can be seen in figure 4.5(a). For the series connection of the two harvester patches displayed in figure 4.5(b), the top electrode of PBEH-1 is connected to one end of the resistor, and the other end of the resistor is connected to the bottom electrode of PBEH-2, while the bottom electrode of PBEH-1 and the top electrode of PBEH-2 are connected together. In the case of the parallel connection, the top electrodes of PBEH-1 and PBEH-2 are directly connected to the one end of the resistor and the bottom electrodes are connected to the other end of the resistor as displayed in figure 4.5(c). The aluminum plate is excited by a linear sine-sweep signal over the bandwidth of 1-260 Hz through a modal shaker during the experimental FRF measurements. The shaker's rod is attached to 0.105 m and 0.1 m away from the right bottom of the plate as shown in figure 4.5(a). The dynamic force loading on the plate is measured by a force transducer (PCB 208C01) mounted

between the shaker's rod and the plate surface. The transverse velocity of the plate is measured by the LDV (laser Doppler vibrometer - Polytec PDV 100) and its target location is provided in figure 4.5(a). A data acquisition system is used to record and analyze the signals coming from the force transducer, the LDV and the voltage across the resistor. The experimental FRFs as the ratio of the velocity response-to-force input (velocity FRF) and the voltage across the resistive load-to-force input (voltage FRF) are obtained.

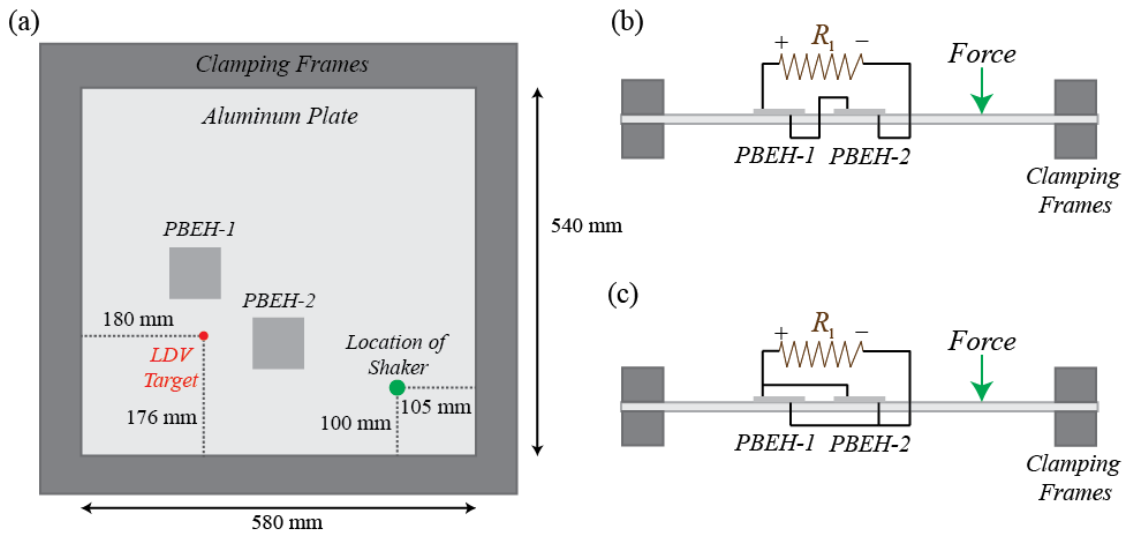


Figure 4.5: (a) Velocity measurement point (target of the LDV) and the excitation point (by the shaker); (b) series and (c) parallel connection configurations of the harvester patches

4.4.2 Electroelastic Analysis

Current output of each harvester patch depends on the dynamic strain induced in the harvester patch as given by equation (4.5). In order to investigate the placement of harvester patches on the basis of vibration-mode dependent strain distribution (i.e., strain mode shape), the first eight undamped natural frequencies and displacement mode shapes of the aluminum plate are first calculated according to the properties given in Table 4.1.

Then, the strain mode shapes are obtained by evaluating the second derivative (curvature) of the displacement mode shapes. Figure 4.6 displays the normalized strain mode shapes along with undamped natural frequencies for the first eight modes of the uniform plate (for simplicity, by excluding the effects of piezoelectric patches since they are small). The gray shaded areas illustrate the regions covered by harvester patches PBEH-1 and PBEH-2.

As can be seen in figure 4.6(a), PBEH-1 and PBEH-2 are located at moderate strain regions in the same phase for the first vibration mode of the plate. Besides, in the second vibration of the plate as displayed in figure 4.6(b), PBEH-1 is located over the high strain region while PBEH-2 is located on the neighborhood of a strain node. Therefore, at the excitation frequency close to the second mode, one can expect more electrical output from PBEH-1 as compared to PBEH-2. For the case of the third vibration mode shown in figure 4.6(c), PBEH-1 is located over the neighborhood of the strain node region whereas PBEH-2 is at the center of the high strain region of the third vibration mode as opposed to the second mode. Hence, PBEH-2 can generate more electrical output at the excitation frequency close to the third vibration mode. The electrical outputs of PBEH-1 and PBEH-2 can be combined properly for excitation frequencies covering second and third vibration modes, yielding a single output and broadband performance. For higher vibration modes, depending on the number of the vibration mode, PBEH-1 and PBEH-2 are located in the high and low strain regions of the same or opposite phases. Note that, at excitation frequencies around the fifth and the sixth vibration modes, the combination of current outputs can lead to cancellations due to out-of-phase behavior.

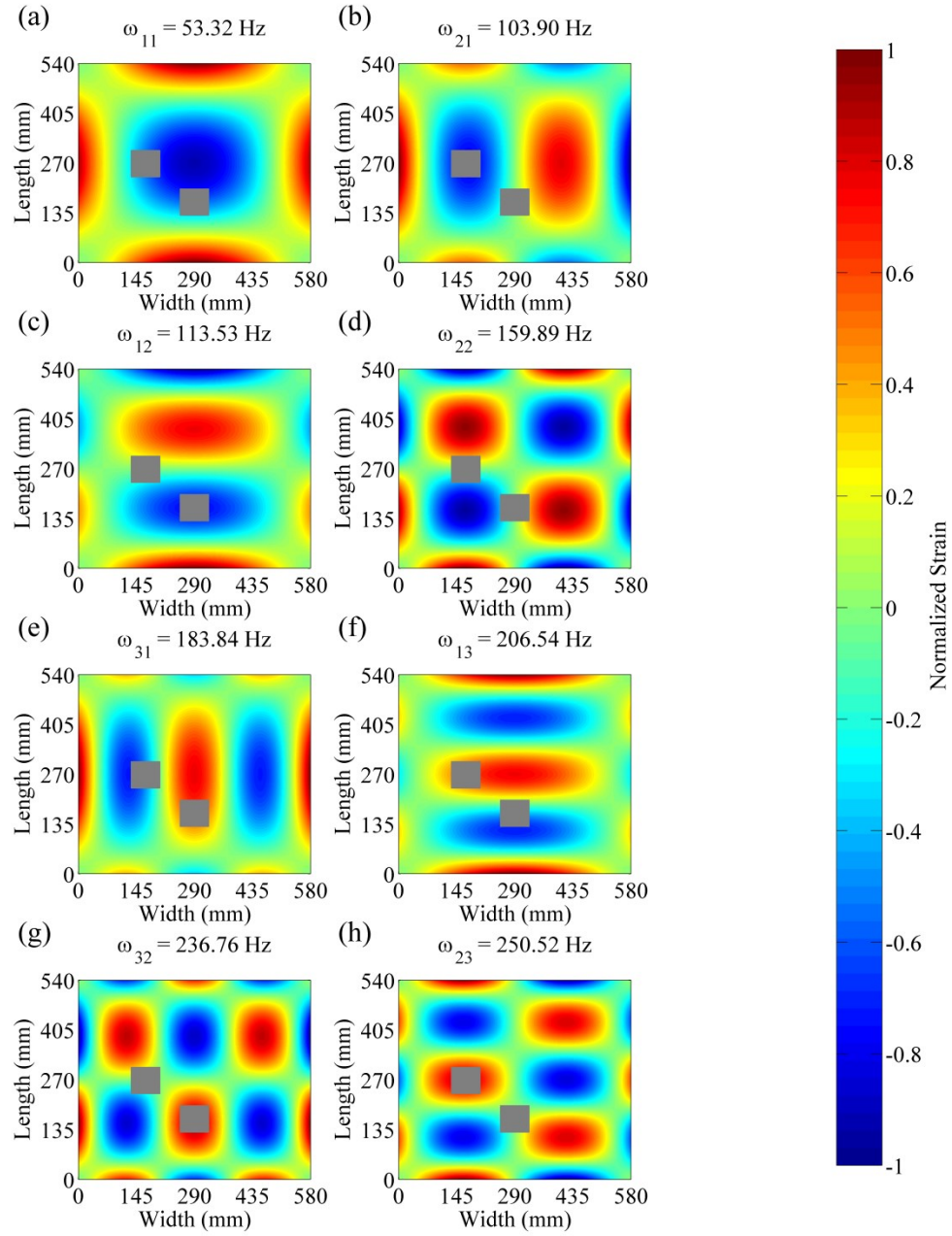


Figure 4.6: Normalized strain mode shapes: (a) first mode; (b) second mode; (c) third mode; (d) fourth mode; (e) fifth mode; (f) sixth mode; (g) seventh mode; (h) eighth mode

4.4.3 Experimental and Analytical Electroelastic FRFs

Before discussing the series and parallel connections of harvester patches, the analytical voltage FRF of each harvester patch close to open-circuit condition is simulated for the configuration illustrated in figure 4.5(a) with the parameters provided in Table 4.1. Figure 4.7 presents the voltage FRFs with solid and dashed lines corresponding to PBEH-1 and PBEH-2, respectively. For excitation frequencies around the first vibration mode of the plate, the generated voltage amplitude of each harvester is similar because of comparable strain levels at the regions covered by the harvester patches for this mode in figure 4.6(a). Additionally, as expected from strain distribution analysis in figure 4.6(b), PBEH-1 generates significant amount of voltage output at excitation frequencies around the second vibration mode (103.9 Hz) and does not generate voltage at excitation frequencies near the third vibration mode (113.5 Hz). Likewise, the voltage output of PBEH-2 is not observed at the second mode but significantly larger amount of voltage output is generated at excitation frequencies around the third vibration mode by PBEH-2. Similar trends in voltage outputs of PBEH-1 and PBEH-2 can be also observed in the seventh and eighth vibration modes in the voltage FRFs, which agree strain distribution analysis provided in figure 4.6(g) and figure 4.6(h).

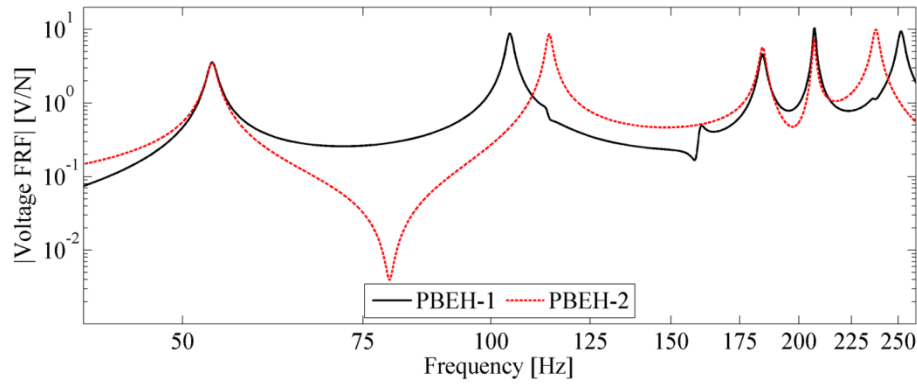


Figure 4.7: Comparison of the analytical voltage FRFs of harvester patches PBEH-1 and PBEH-2 in open-circuit condition

Having discussed the independent open-circuit voltage FRF of each harvester patch, analytical simulations and experimental measurements are carried out for the series connection (figure 4.5(b)) of PBEH-1 and PBEH-2. Figure 4.8 presents the experimental and analytical voltage FRFs close to open-circuit conditions (for a resistive load of 1 M Ω). The analytical prediction for the voltage FRF is in very good agreement with experimental data for the first eight vibration modes of the plate. For the analytical predictions, the electroelastic model explained in the previous sections is used along with modal damping ratio values extracted from the experimental data points.

By examining figures 4.7 and 4.8 together, it can be observed that the series connection of PBEH-1 and PBEH-2 enables the combination of the voltage outputs at the second, third, seventh and eighth vibration modes into a single voltage output, enabling energy harvesting from all vibration modes of interest. It is also worth mentioning that PBEH-1 and PBEH-2 are both located on the neighborhood of nodal strain region of the fourth vibration mode (see figure 4.6(d)) and therefore the voltage output at the fourth vibration mode is significantly smaller than that at other resonance frequencies.

During the experimental voltage FRF measurements, the vibration of the plate is also monitored at the target point with LDV as shown in figure 4.4 and 4.5(a). Experimental velocity FRF is obtained from the outputs of LDV and force transducer while the analytical velocity FRF is simulated for the same target point using the coupled electroelastic model. Figure 4.9 displays the experimental and analytical velocity FRFs in open-circuit conditions and it can be seen that analytical model predicts the vibration response of the plate very well for the first eight modes of the plate.

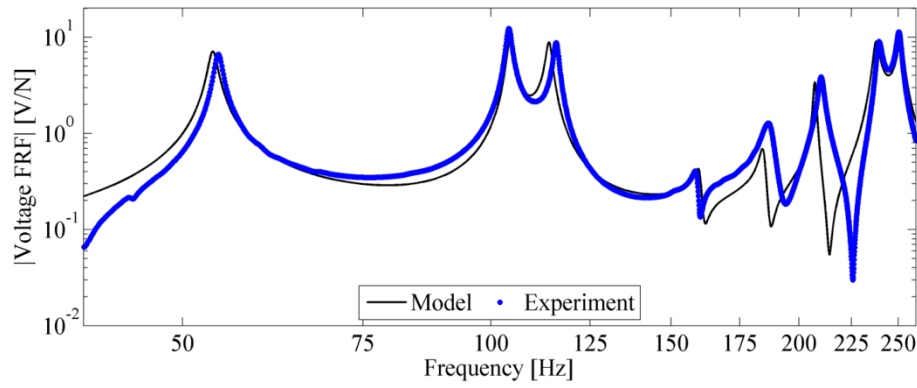


Figure 4.8: Comparison of the experimental and analytical voltage FRFs for series connection of harvesters in open-circuit conditions

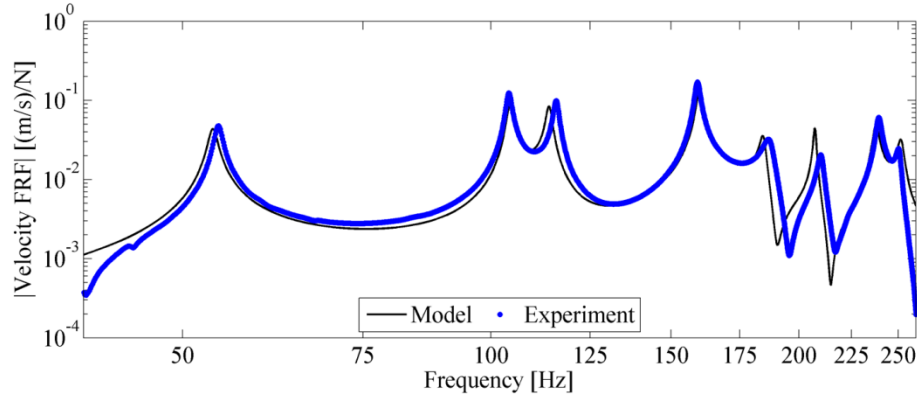


Figure 4.9: Comparison of the experimental and analytical velocity FRFs for series connection of harvesters in open-circuit conditions

For the case of parallel connection of harvester patches, PBEH-1 and PBEH-2 are configured as displayed in figure 4.5(c). The experimentally measured and analytically predicted voltage and velocity FRFs are shown in figures 4.10 and 4.11, respectively. The experimental voltage and velocity FRFs are very well predicted by the analytical model. Similar to the results of series connection, using two parallel-connected patches located at different positions on the host plate yielded combined energy harvesting from multiple vibration modes of the thin plate.

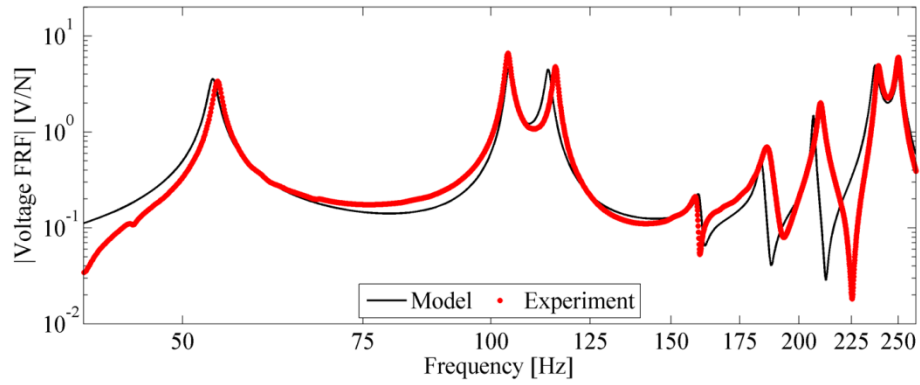


Figure 4.10: Comparison of the experimental and analytical voltage FRFs for parallel connection of harvesters in open-circuit conditions

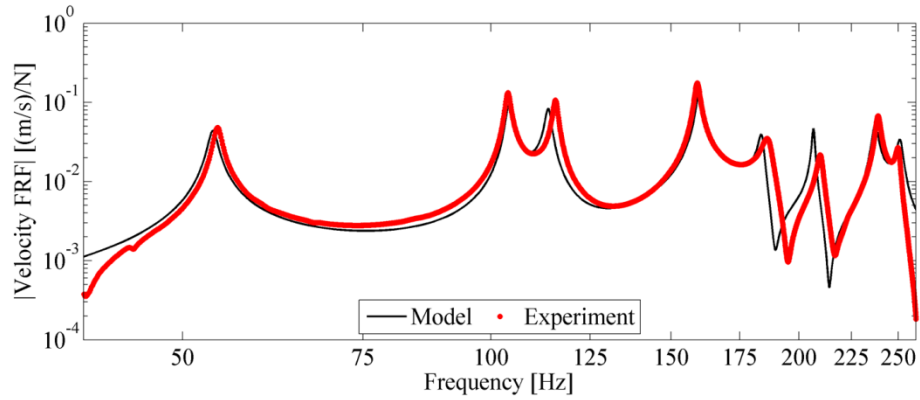


Figure 4.11: Comparison of the experimental and analytical velocity FRFs for parallel connection of harvesters in open-circuit conditions

4.4.4 Experimental and Analytical Peak Electrical Outputs

To investigate the variation of electrical outputs with different load resistance values, experimental measurements and analytical simulations of voltage FRFs are carried out for a set of resistive loads listed in Table 4.2. The resistive loads used in the test range from 0.012 k Ω (near short-circuit condition) to 991 k Ω (near open-circuit condition). Having obtained the voltage FRFs analytically and experimentally, the peak voltage outputs at the open-circuit resonance frequencies of the first eight vibration modes are picked from the experimental and analytical data points corresponding to the each load resistance value. Figure 4.12 presents the variation of peak voltage output with load resistance for the first eight vibration modes of the aluminum plate. The peak voltage amplitudes monotonically increase up to a certain level near open-circuit conditions and then saturate as a common trend for all vibration modes of interest. Expectedly, the voltage output of series connection is larger than that of the parallel connection near open-circuit condition, whereas close to short-circuit condition; the voltage amplitudes of parallel connection case are larger than that of series connection counterpart. Overall, the analytical results exhibit very good agreement with the experimental data for both the series and parallel connections of harvester patches.

Table 4.2: Set of resistive loads used in the experiments

Resistance Order	Resistance Value (k Ω)	Resistance Order	Resistance Value (k Ω)
1	0.012	10	14.7
2	0.057	11	21.7
3	0.219	12	32.5
4	0.464	13	48.1
5	1.18	14	100
6	2.68	15	330
7	3.83	16	472
8	5.56	17	695
9	9.75	18	991

The experimental and analytical power FRFs are then calculated for each resistive load by using the voltage FRFs. Similar to the previous discussion on peak voltage amplitudes, the peak power outputs are extracted at the open-circuit resonance frequencies of the first eight vibration modes of the plate. Variations of peak power amplitudes for excitations at the first eight vibration modes are plotted versus load resistance values in figure 4.13. The power curves have similar trends for each vibration mode, exhibiting an optimum resistive load for the maximum peak power output. It can be observed that the optimal load resistance value of parallel connection configuration is smaller than that of series connection case. Note that strain mode shape related cancellations might differ in the outputs values for the series and parallel connection cases. The modal power output trends are in agreement with the modal strain distributions given in figure 4.6. The analytical predictions agree well with the experimental data in figure 4.13 for a wide range of resistive loads and vibration modes.

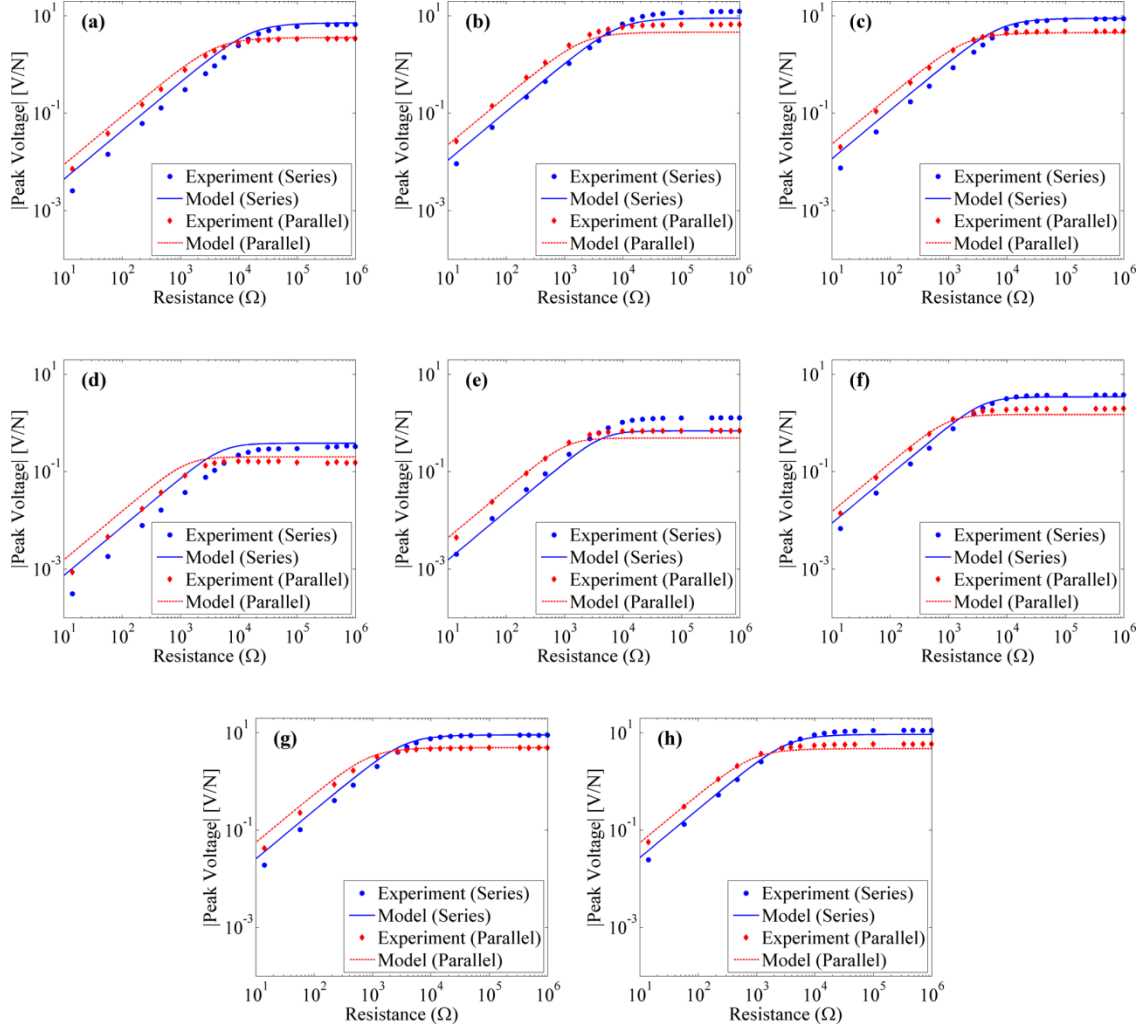


Figure 4.12: Variations of resonant voltage amplitudes of series and parallel connections of harvesters with load resistance for excitations at the open circuit resonance frequencies: (a) first mode; (b) second mode; (c) third mode; (d) fourth mode; (e) fifth mode; (f) sixth mode; (g) seventh mode; (h) eighth mode

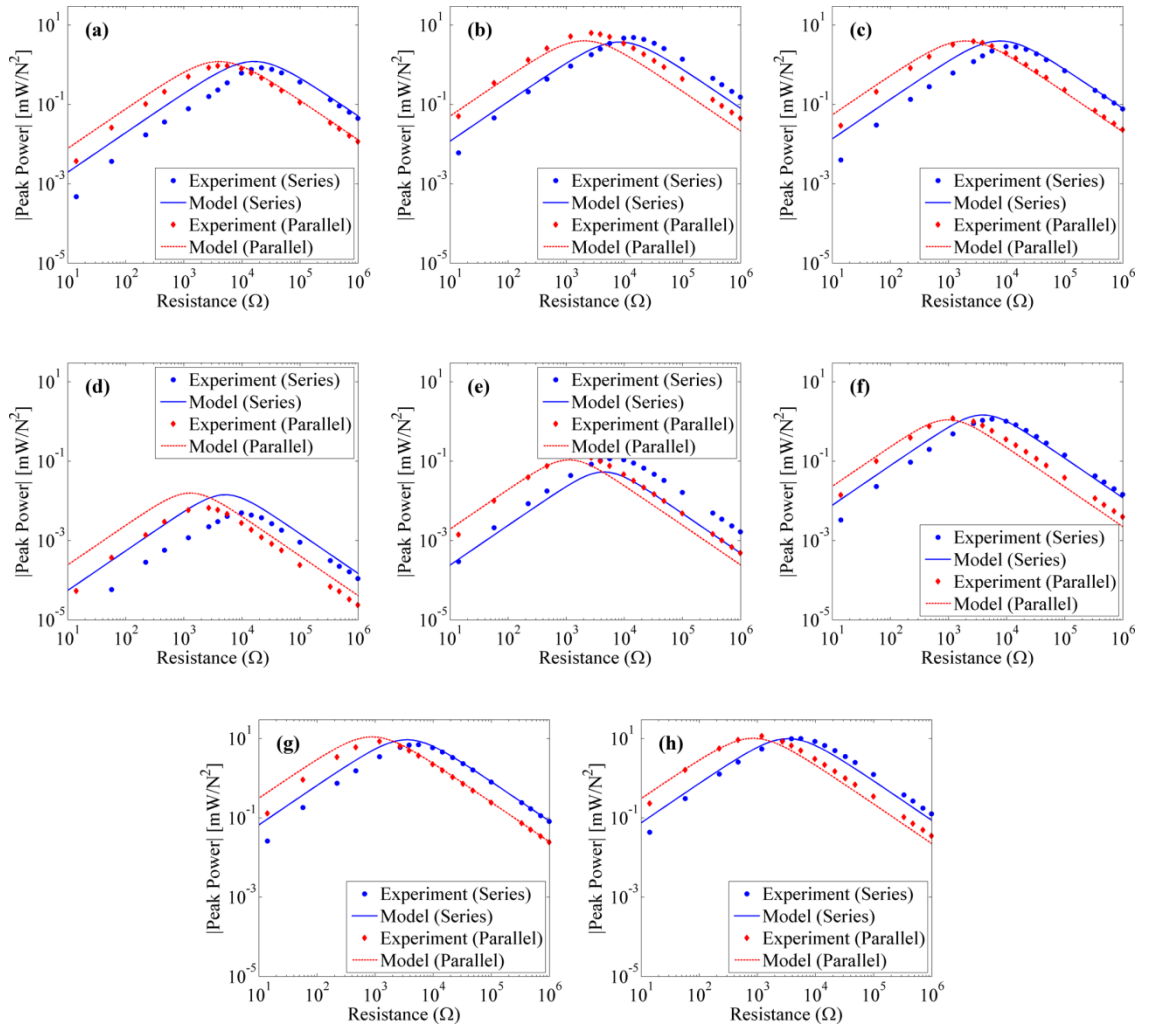


Figure 4.13: Variations of resonant power amplitudes of series and parallel connections of harvesters with load resistance for excitations at the open circuit resonance frequencies: (a) first mode; (b) second mode; (c) third mode; (d) fourth mode; (e) fifth mode; (f) sixth mode; (g) seventh mode; (h) eighth mode

4.5 Summary and Conclusions

This chapter presented electroelastic modeling, experimental validation, and performance investigation of multiple patch-based energy harvesters integrated to thin plates. In this chapter, electroelastic models of series and parallel connections of harvester patches attached on thin plates are presented along with experimental investigations. The electroelastic model of harvester patches together with the host plate is derived in physical coordinates based on a distributed-parameter modelling approach. By transforming the coupled electroelastic equations from physical coordinates to modal coordinates for a fully-clamped thin plate, analytical solutions are obtained. Steady-state coupled structural and electrical response expressions are formulated for the transversely applied point force excitations in the cases of series and parallel connections of harvester patches in the presence of a resistive load. Electroelastic frequency response functions (FRFs) relating the voltage output and vibration response to force input are derived for multiple vibration modes of thin plates for the series and parallel connection cases. Model validation experiments are presented with an experimental case study and it is shown that the analytical predictions and experimental measurements of voltage and velocity FRFs are in very good agreement for multiple vibration modes of the thin plate. Using the experimental measurements and analytical simulations, the harvested power and voltage outputs across the resistive load are explored for the first eight vibration modes. The experimental and analytical results presented in this chapter show that using multiple-patch based energy harvesters can lead to effective broadband energy harvesting on thin plates.

Chapter 5

PATCH-BASED PIEZOELECTRIC ENERGY HARVESTING FROM RANDOM VIBRATIONS

In this chapter, analytical modelling, analytical and numerical solutions, and experimental investigations of patch-based piezoelectric energy harvesting from broadband and band-limited random vibrations of thin plates are presented. Analytical and numerical solutions for the expected power output and mean-square vibration response are obtained using multiple-patch electroelastic models given in Chapter 4. The frequency-domain solutions are derived based on electroelastic frequency response functions (FRFs) and the power spectral density (PSD) of the random force excitation. For the time-based numerical solutions, electroelastic models are represented in state-space forms and solved in a deterministic fashion by considering the Fourier series expansion of the random force input. After that, an experimental case study is given to validate the analytical and numerical predictions for a wide range of resistive loads. Finally, to explore the effects of excitation bandwidth and position of the harvester patch on the expected power output, random vibrational energy harvesting scenarios are provided with a final remark on optimal positioning of a single patch-based piezoelectric harvester for the case of broadband and band-limited force excitations.

5.1 Distributed-parameter Electroelastic Model

This chapter revisits the distributed-parameter electroelastic models of multiple patch-based harvesters given by Chapter 4. Figure 5.1 presents series and parallel connections of multiple patch-based harvesters attached on a host plate. It is worth recalling that the distributed-parameter electroelastic models in physical coordinates are obtained for these two connection configurations in Section 4.1.

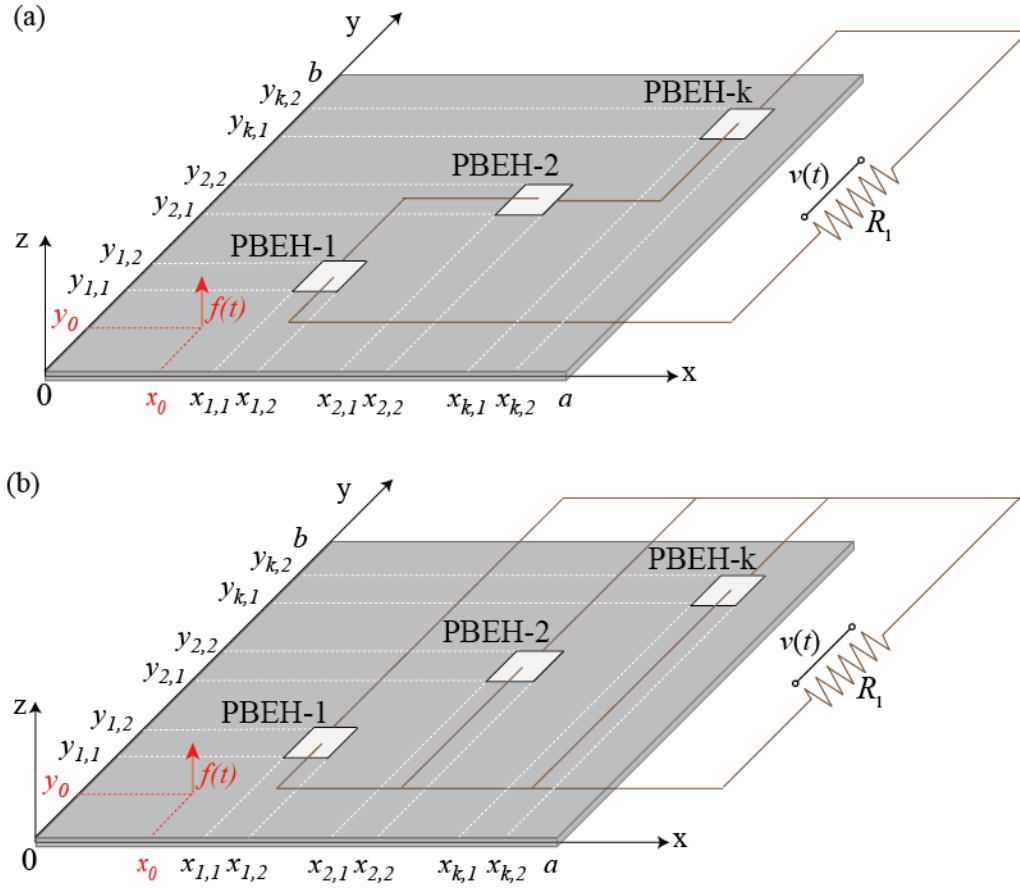


Figure 5.1: Patch-based piezoelectric energy harvesters on a thin plate: (a) series and (b) parallel connection configuration

By following the modal analysis procedure as described in Section 4.2 for K number of patch-based harvesters connected in series and parallel, electromechanically coupled second-order differential equation for the modal time response $\eta_{mn}(t)$ of m th vibration mode can be obtained as

$$\frac{d^2 \eta_{mn}}{dt^2} + 2\zeta_{mn}\omega_{mn} \frac{d\eta_{mn}}{dt} + \omega_{mn}^2 \eta_{mn}(t) - \sum_{k=1}^K \tilde{\theta}_{mn}^k v_k(t) = f_{mn}(t) \quad (5.1)$$

where ω_{mn} is the undamped natural frequency, ζ_{mn} is the modal damping ratio, $\tilde{\theta}_{mn}^k$ is the electromechanical coupling term for the k th harvester patch (given by equation 4.18) and $f_{mn}(t)$ is the modal forcing term which is $f_{mn}(t) = f(t)\phi_{mn}(x_0, y_0)$.

For the case of K number of harvesters connected in series, K number of electrical circuit equations can be written as

$$(C_p)_k \frac{dv_k(t)}{dt} + \frac{v(t)}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^k = 0 \quad k = 1, 2, \dots, K \quad (5.2)$$

whereas, if K patch-based harvesters are connected in parallel, there is only one circuit equation governing the electrical dynamics, which is given by

$$\frac{dv(t)}{dt} \sum_{k=1}^K (C_p)_k + \frac{v(t)}{R_l} + \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^k = 0 \quad (5.3)$$

Thus, the electroelastic model of patch-based harvesters connected in series is formed by $K+1$ differential equations given by equations (5.1) and (5.2), and the electroelastic model for parallel configuration is constituted by two differential equations given by equations (5.1) and (5.3).

5.2 Frequency Domain Analytical Solutions

5.2.1 Broadband Random Vibrations

If the transverse point-force excitation $f(t)$ acting on the plate is assumed to be ideal Gaussian white noise excitation having a constant power spectral density (PSD) value of S_0 in the entire frequency band, the multi-mode solution for the expected (mean) power output across the resistive load R_l is given by

$$E[P(t)] = \int_{-\infty}^{\infty} \frac{S_0}{R_l} |\alpha(\omega)|^2 d\omega \quad (5.4)$$

which is based on $v^2(t)/R_l$, and $\alpha(\omega)$ is the unified representation of the voltage FRF for the series and the parallel connection cases and includes all vibration modes of the host plate in the entire frequency spectrum. These specific voltage FRFs for series and parallel connection configurations can be found in Section 4.3.1 and 4.3.2, respectively. One can substitute corresponding voltage FRF into equation (5.4) and compute the expected power output by carrying out analytical and/or numerical integration over the entire frequency range.

To compute the expected power output of a single k th patch-based piezoelectric harvester for the m th vibration mode, the single-mode solution can be written as

$$E[\hat{P}(t)] = \int_{-\infty}^{\infty} \frac{S_0}{R_l} |\hat{\alpha}_{mn}^k(\omega)|^2 d\omega \quad (5.5)$$

where $\hat{\alpha}_{mn}^k(\omega)$ is the single-mode voltage FRF given by equation (3.43). Substitution of equation (3.43) into equation (5.4) gives

$$E[\hat{P}(t)] = \int_{-\infty}^{\infty} \frac{S_0}{R_l} \left| \frac{-j\omega R_l \phi_{mn}(x_0, y_0) \tilde{\theta}_{mn}^k}{(j\omega R_l (C_p)_k + 1)(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega) + j\omega R_l (\tilde{\theta}_{mn}^k)^2} \right|^2 d\omega \quad (5.6)$$

Then, with the help [138], equation (5.6) can be solved as

$$E[\hat{P}(t)] = \frac{\pi S_0 R_l \phi_{mn}^2(x_0, y_0) (\tilde{\theta}_{mn}^k)^2}{R_l (\tilde{\theta}_{mn}^k)^2 + 2\zeta_{mn}\omega_{mn} \left\{ 1 + R_l^2 (C_p)_k (\tilde{\theta}_{mn}^k)^2 + \omega_{mn} R_l (C_p)_k [\omega_{mn} R_l (C_p)_k + 2\zeta_{mn}] \right\}} \quad (5.7)$$

which is analogous to the expressions derived for a cantilever beam harvester by Zhao and Erturk [115]. After that, the optimal resistive load (R_l^*) is identified for the particular mn th vibration mode as

$$\frac{\partial}{\partial R_l} \{E[\hat{P}(t)]\}_{R_l=R_l^*} = 0 \rightarrow R_l^* = \frac{1}{\sqrt{(C_p)_k \{ (C_p)_k \omega_{mn}^2 + (\tilde{\theta}_{mn}^k)^2 \}}} \quad (5.8)$$

and, back substitution of equation (5.8) into equation (5.7) gives the expected power output at the optimal resistive load:

$$E[\hat{P}(t)]_{R_l=R_l^*} = \frac{\pi S_0 \phi_{mn}^2(x_0, y_0) (\tilde{\theta}_{mn}^k)^2 \sqrt{(C_p)_k [(C_p)_k \omega_{mn}^2 + (\tilde{\theta}_{mn}^k)^2]}}{4\zeta_{mn}\omega_{mn}^3 (C_p)_k^2 + [4\zeta_{mn}^2 \omega_{mn}^2 (C_p)_k + (\tilde{\theta}_{mn}^k)^2] \sqrt{(C_p)_k [(C_p)_k \omega_{mn}^2 + (\tilde{\theta}_{mn}^k)^2]} + 4\zeta_{mn}\omega_{mn} (C_p)_k (\tilde{\theta}_{mn}^k)^2} \quad (5.9)$$

Likewise, the multi-mode analytical solution for the mean-square vibration response at position x and y is

$$E[w^2(x, y, t)] = \int_{-\infty}^{\infty} S_0 |\beta(x, y, \omega)|^2 d\omega \quad (5.10)$$

when the transverse point-force excitation has a flat PSD value of S_0 in the entire frequency band. Note that $\beta(\omega)$ is the unified representation of the displacement FRF for

the series and the parallel connection cases, and considers all vibration modes of the host plate in the entire frequency spectrum. The reader is referred to Section 4.3.1 and 4.3.2 for displacement FRFs for each connection configuration.

For a single k th patch-based piezoelectric harvester, the single-mode solution of the mean-square vibration response can be written as

$$E[\hat{w}^2(x, y, t)] = \int_{-\infty}^{\infty} S_0 |\hat{\beta}_{mn}(x, y, \omega)|^2 d\omega \quad (5.11)$$

which can be used to explore the contribution of the m th vibration mode to the overall mean-square vibration response, and the single-mode displacement FRF $\hat{\beta}(x, y, \omega)$ is given by equation (3.44). Substituting equation (3.44) into equation (5.11) leads the following integral:

$$E[\hat{w}^2(x, y, t)] = \int_{-\infty}^{\infty} \frac{S_0}{R_l} \left| \frac{(j\omega R_l (C_p)_k + 1) \phi_{mn}(x_0, y_0) \phi_{mn}(x, y)}{(j\omega R_l (C_p)_k + 1)(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn} \omega_{mn} \omega) + j\omega R_l (\tilde{\theta}_{mn}^k)^2} \right|^2 d\omega \quad (5.12)$$

and the result can be obtained as

$$E[\hat{w}^2(x, y, t)] = \frac{\pi S_0 \phi_{mn}^2(x_0, y_0) \phi_{mn}^2(x, y) [1 + \omega_{mn}^2 R_l^2 (C_p)_k^2 + 2\zeta_{mn} \omega_{mn} R_l (C_p)_k]}{\omega_{mn}^2 \{2\zeta_{mn} \omega_{mn} [1 + \omega_{mn}^2 R_l^2 (C_p)_k^2 + 2\zeta_{mn} \omega_{mn} R_l (C_p)_k + R_l^2 (C_p)_k (\tilde{\theta}_{mn}^k)^2] + R_l (\tilde{\theta}_{mn}^k)^2\}} \quad (5.13)$$

by using integration tables in [138]. After that, the optimum resistive load (R_l^{**}) for the maximum shunt damping of the m th vibration mode can be obtained as

$$\frac{\partial}{\partial R_l} \{E[\hat{w}^2(x, y, t)]\}_{R_l=R_l^{**}} = 0 \rightarrow R_l^{**} = \frac{1}{\sqrt{\omega_{mn} (C_p)_k (1 - 2\zeta_{mn})}} \quad (5.14)$$

and back substitution leads to maximum shunt damping of the m th vibration mode as

$$E[\hat{w}^2(x, y, t)]_{R_l=R_l^{**}} = \frac{2\pi S_0 \phi_{mn}^2(x_0, y_0) \phi_{mn}^2(x, y) (C_p)_k (1 - \zeta_{mn})}{\omega_{mn} \left[(\tilde{\theta}_{mn}^k)^2 + 4\zeta_{mn} \omega_{mn}^2 (C_p)_k - 4\omega_{mn}^2 \zeta_{mn}^2 (C_p)_k \right]} \quad (5.15)$$

It is worth recalling that the closed-form single-mode solutions for the mean-square vibration response, optimal resistive load, and maximum shunt damping are also comparable to those for the cantilever harvester beam given by Zhao and Erturk [115].

5.2.2 Band-limited Random Vibrations

For the case of band-limited random excitation over a frequency range $[-\bar{\omega}, \bar{\omega}]$, the expected power output can be computed by solving the definite integral given by

$$E[P(t)] = \int_{-\bar{\omega}}^{\bar{\omega}} \frac{S_f(\omega)}{R_l} |\alpha(\omega)|^2 d\omega \quad (5.16)$$

and the mean-square vibration response can be calculated from

$$E[w^2(x, y, t)] = \int_{-\bar{\omega}}^{\bar{\omega}} S_f(\omega) |\beta(x, y, \omega)|^2 d\omega \quad (5.17)$$

where $S_f(\omega)$ is the frequency-dependent PSD. It should be noted that the upper frequency limit $\bar{\omega}$ must be set to a high value such that the vibration modes contributing to the power output and vibration response are totally covered.

5.3 Time Domain Numerical Solutions

If the time history of a stationary random force excitation over a time period T is described in a deterministic fashion using its Fourier series representation

$$f(t) \cong p_0 + \sum_{r=1}^{N_r} \left[p_r \cos\left(\frac{2\pi t r}{T}\right) + q_r \sin\left(\frac{2\pi t r}{T}\right) \right] \quad (5.18)$$

where N_r is the number of terms, p_0 is the mean value, while p_r and q_r are the Fourier coefficients [115], the expected structural and electrical responses can be predicted over the time period T .

In the following, state-space representations of electroelastic ordinary differential equations (ODEs) are obtained for series and parallel connections of multiple harvester patches, and then the expressions for the expected power output and the mean-square vibration response are formulated for each connection configuration.

5.3.1 State-space Representation for Series Connection

At first, electroelastic model equations of K number of patch-based harvesters connected in series given by equations (5.1) and (5.2) are expanded as follows:

$$\begin{aligned} \frac{d^2 \eta_{11}}{dt^2} + 2\zeta_{11}\omega_{11} \frac{d\eta_{11}}{dt} + \omega_{11}^2 \eta_{11}(t) - \sum_{k=1}^K \tilde{\theta}_{11}^k v_k(t) &= \phi_{11}(x_0, y_0) f(t) \\ \frac{d^2 \eta_{21}}{dt^2} + 2\zeta_{21}\omega_{21} \frac{d\eta_{21}}{dt} + \omega_{21}^2 \eta_{21}(t) - \sum_{k=1}^K \tilde{\theta}_{21}^k v_k(t) &= \phi_{21}(x_0, y_0) f(t) \\ &\vdots \end{aligned} \quad (5.19)$$

$$\begin{aligned} (C_p)_1 \frac{dv_1(t)}{dt} + \frac{v_1(t)}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^1 &= 0 \\ &\vdots \\ (C_p)_K \frac{dv_K(t)}{dt} + \frac{v_K(t)}{R_l} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^K &= 0 \end{aligned}$$

where there are infinite number of second-order ordinary differential equations along with K number of first-order different equations.

For the time-based simulation of a given force history $f(t)$, finite number of vibration modes can be taken into account as

$$\begin{aligned}
& \frac{d^2 \eta_{11}}{dt^2} + 2\zeta_{11}\omega_{11} \frac{d\eta_{11}}{dt} + \omega_{11}^2 \eta_{11}(t) - \sum_{k=1}^K \tilde{\theta}_{11}^k v_k(t) = \phi_{11}(x_0, y_0) f(t) \\
& \frac{d^2 \eta_{21}}{dt^2} + 2\zeta_{21}\omega_{21} \frac{d\eta_{21}}{dt} + \omega_{21}^2 \eta_{21}(t) - \sum_{k=1}^K \tilde{\theta}_{21}^k v_k(t) = \phi_{21}(x_0, y_0) f(t) \\
& \vdots \\
& \frac{d^2 \eta_{MN}}{dt^2} + 2\zeta_{MN}\omega_{MN} \frac{d\eta_{MN}}{dt} + \omega_{MN}^2 \eta_{MN}(t) - \sum_{k=1}^K \tilde{\theta}_{MNk}^k v_k(t) = \phi_{MN}(x_0, y_0) f(t) \quad (5.20) \\
& (C_p)_1 \frac{dv_1(t)}{dt} + \frac{v_1(t)}{R_1} + \sum_{n=1}^N \sum_{m=1}^M \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^1 = 0 \\
& \vdots \\
& (C_p)_K \frac{dv_K(t)}{dt} + \frac{v_K(t)}{R_1} + \sum_{n=1}^N \sum_{m=1}^M \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^K = 0
\end{aligned}$$

by truncating the computation of the modal time responses at M number of modes along the x -axis and N number of modes along the y -axis. After that, the vibration response (in terms of velocity) can be approximated as a finite series of the eigenfunctions and the modal time responses as:

$$\dot{w}(x, y, t) \equiv \sum_{n=1}^N \sum_{m=1}^M \phi_{mn} \dot{\eta}_{mn} = \phi_{11} \dot{\eta}_{11} + \phi_{21} \dot{\eta}_{21} + \phi_{12} \dot{\eta}_{12} + \dots + \phi_{MN} \dot{\eta}_{MN} \quad (5.21)$$

For convenience, electroelastic state variables for the m th vibration mode are defined in the same vein of Erturk and Inman [37] as follows:

$$u_{mn}^{(1)} = \eta_{mn}(t) \quad u_{mn}^{(2)} = \dot{\eta}_{mn}(t) \quad u^{(k+2)} = v_k(t) \quad (5.22)$$

where $u_{mn}^{(1)}$ is the modal displacement, $u_{mn}^{(2)}$ is the modal velocity, and $u^{(k+2)}$ is the individual voltage output of the k th harvester patch. The voltage across the resistor can be calculated as

$$v(t) = \sum_{k=1}^K u^{(k+2)} \quad (5.23)$$

By introducing the electroelastic state variables into equations (5.20), and assuming zero initial conditions, $2MN+K$ number of first order differential equations (ODEs) are obtained as

$$\begin{aligned} \frac{du_{11}^{(1)}}{dt} &= u_{11}^{(2)} \\ \frac{du_{11}^{(2)}}{dt} &= -\omega_{11}^2 u_{11}^{(1)} - 2\zeta_{11}\omega_{11}u_{11}^{(2)} + [\tilde{\theta}_{11}^1 u^{(3)} + \dots + \tilde{\theta}_{11}^K u^{(K+2)}] + \phi_{11}(x_0, y_0)f(t) \\ \frac{du_{21}^{(1)}}{dt} &= u_{21}^{(2)} \\ \frac{du_{21}^{(2)}}{dt} &= -\omega_{21}^2 u_{21}^{(1)} - 2\zeta_{21}\omega_{21}u_{21}^{(2)} + [\tilde{\theta}_{21}^1 u^{(3)} + \dots + \tilde{\theta}_{21}^K u^{(K+2)}] + \phi_{21}(x_0, y_0)f(t) \\ &\vdots \\ \frac{du_{MN}^{(1)}}{dt} &= u_{MN}^{(2)} \\ \frac{du_{MN}^{(2)}}{dt} &= -\omega_{MN}^2 u_{MN}^{(1)} - 2\zeta_{MN}\omega_{MN}u_{MN}^{(2)} + [\tilde{\theta}_{MN}^1 u^{(3)} + \dots + \tilde{\theta}_{MN}^K u^{(K+2)}] + \phi_{MN}(x_0, y_0)f(t) \end{aligned} \quad (5.24)$$

$$\begin{aligned}
\frac{du^{(3)}}{dt} &= \frac{-1}{(C_p)_1} \left[\frac{u^{(3)} + u^{(4)} + \dots + u^{(K+2)}}{R_1} + \left(\tilde{\theta}_{11}^1 u_{11}^{(2)} + \tilde{\theta}_{21}^1 u_{21}^{(2)} + \dots + \tilde{\theta}_{MN}^1 u_{MN}^{(2)} \right) \right] \\
&\vdots \\
\frac{du^{(K+2)}}{dt} &= \frac{-1}{(C_p)_K} \left[\frac{u^{(3)} + u^{(4)} + \dots + u^{(K+2)}}{R_1} + \left(\tilde{\theta}_{11}^K u_{11}^{(2)} + \tilde{\theta}_{21}^K u_{21}^{(2)} + \dots + \tilde{\theta}_{MN}^K u_{MN}^{(2)} \right) \right]
\end{aligned}$$

and then equation (5.24) can be written in compact-matrix form as follows:

$$\begin{bmatrix} \dot{u}_{11}^{(1)} & \dot{u}_{21}^{(1)} & \dots & \dot{u}_{MN}^{(1)} & \dot{u}_{11}^{(2)} & \dot{u}_{21}^{(2)} & \dots & \dot{u}_{MN}^{(2)} & \dot{u}^{(3)} & \dots & \dot{u}^{(K+2)} \end{bmatrix}^T = \begin{bmatrix} u_{11}^{(2)} \\ u_{21}^{(2)} \\ \vdots \\ u_{MN}^{(2)} \\ -2\zeta_{11}\omega_{11}u_{11}^{(2)} - \omega_{11}^2 u_{11}^{(1)} + (\tilde{\theta}_{11}^1 u^{(3)} + \tilde{\theta}_{11}^2 u^{(4)} + \dots + \tilde{\theta}_{11}^K u^{(K+2)}) + \phi_{11}(x_0, y_0)f(t) \\ -2\zeta_{21}\omega_{21}u_{21}^{(2)} - \omega_{21}^2 u_{21}^{(1)} + (\tilde{\theta}_{21}^1 u^{(3)} + \tilde{\theta}_{21}^2 u^{(4)} + \dots + \tilde{\theta}_{21}^K u^{(K+2)}) + \phi_{21}(x_0, y_0)f(t) \\ \vdots \\ -2\zeta_{MN}\omega_{MN}u_{MN}^{(2)} - \omega_{MN}^2 u_{MN}^{(1)} + (\tilde{\theta}_{MN}^1 u^{(3)} + \tilde{\theta}_{MN}^2 u^{(4)} + \dots + \tilde{\theta}_{MN}^K u^{(K+2)}) + \phi_{MN}(x_0, y_0)f(t) \\ \frac{-1}{(C_p)_1} \left[\frac{1}{R_1} (u^{(3)} + u^{(4)} + \dots + u^{(K+2)}) + (\tilde{\theta}_{11}^1 u_{11}^{(2)} + \tilde{\theta}_{21}^1 u_{21}^{(2)} + \dots + \tilde{\theta}_{MN}^1 u_{MN}^{(2)}) \right] \\ \vdots \\ \frac{-1}{(C_p)_K} \left[\frac{1}{R_1} (u^{(3)} + u^{(4)} + \dots + u^{(K+2)}) + (\tilde{\theta}_{11}^K u_{11}^{(2)} + \tilde{\theta}_{21}^K u_{21}^{(2)} + \dots + \tilde{\theta}_{MN}^K u_{MN}^{(2)}) \right] \end{bmatrix} \quad (5.25)$$

where superscript T stands for the transpose. Using an ODE solver (e.g., explicit Runge-Kutta method), the time histories of electroelastic states can be computed. Then, the mean-square velocity response can be calculated by using

$$E[\dot{w}^2(x, y, t)] = \frac{1}{T} \int_0^T \dot{w}^2(x, y, t) dt = \sigma_{\dot{w}}^2 \quad (5.26)$$

where $\sigma_{\dot{w}}^2$ is the standard deviation of the velocity of the plate at position x and y . The expected value of the power output can be obtained with

$$E[P(t)] = \frac{1}{T} \int_0^T \frac{v^2(t)}{R_l} dt = \frac{\sigma_v^2}{R_l} \quad (5.27)$$

where σ_v^2 is the standard deviation of the voltage output across the resistor.

5.3.2 State-space Representation for Parallel Connection

Similar to the series-configuration, electroelastic model equations of K number of patch-based harvesters connected in parallel given by equation (5.1) and (5.3) can be expanded as

$$\begin{aligned} \frac{d^2 \eta_{11}}{dt^2} + 2\zeta_{11}\omega_{11} \frac{d\eta_{11}}{dt} + \omega_{11}^2 \eta_{11}(t) - \sum_{k=1}^K \tilde{\theta}_{11}^k v_k(t) &= \phi_{11}(x_0, y_0) f(t) \\ \frac{d^2 \eta_{21}}{dt^2} + 2\zeta_{21}\omega_{21} \frac{d\eta_{21}}{dt} + \omega_{21}^2 \eta_{21}(t) - \sum_{k=1}^K \tilde{\theta}_{21}^k v_k(t) &= \phi_{21}(x_0, y_0) f(t) \\ &\vdots \end{aligned} \quad (5.28)$$

$$\frac{dv(t)}{dt} \sum_{k=1}^K (C_p)_k + \frac{v(t)}{R_l} + \sum_{k=1}^K \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^k = 0$$

In equation (5.28), there is an infinite set of second-order ordinary differential equations together with a first-order ordinary differential equation. If finite number of vibration modes is considered for the time simulation of a given time history of $f(t)$, equation (5.28) can be rewritten as

$$\begin{aligned}
\frac{d^2\eta_{11}}{dt^2} + 2\zeta_{11}\omega_{11}\frac{d\eta_{11}}{dt} + \omega_{11}^2\eta_{11}(t) - \sum_{k=1}^K \tilde{\theta}_{11}^k v_k(t) &= \phi_{11}(x_0, y_0)f(t) \\
\frac{d^2\eta_{21}}{dt^2} + 2\zeta_{21}\omega_{21}\frac{d\eta_{21}}{dt} + \omega_{21}^2\eta_{21}(t) - \sum_{k=1}^K \tilde{\theta}_{21}^k v_k(t) &= \phi_{21}(x_0, y_0)f(t) \\
&\vdots \\
\frac{d^2\eta_{MN}}{dt^2} + 2\zeta_{MN}\omega_{MN}\frac{d\eta_{MN}}{dt} + \omega_{MN}^2\eta_{MN}(t) - \sum_{k=1}^K \tilde{\theta}_{MN}^k v_k(t) &= \phi_{MN}(x_0, y_0)f(t) \\
\frac{dv(t)}{dt} \sum_{k=1}^K (C_p)_k + \frac{v(t)}{R_1} + \sum_{k=1}^K \sum_{n=1}^N \sum_{m=1}^M \frac{d\eta_{mn}(t)}{dt} \tilde{\theta}_{mn}^k &= 0
\end{aligned} \tag{5.29}$$

after truncating the computation at M number of modes along the x -axis and N number of modes along the y -axis. After that, the vibration response (in terms of velocity) can be approximated as a finite series of the eigenfunctions with the corresponding modal time responses as:

$$\dot{w}(x, y, t) \cong \sum_{n=1}^N \sum_{m=1}^M \phi_{mn} \dot{\eta}_{mn} = \phi_{11} \dot{\eta}_{11} + \phi_{21} \dot{\eta}_{21} + \phi_{12} \dot{\eta}_{12} + \dots + \phi_{MN} \dot{\eta}_{MN} \tag{5.30}$$

Electroelastic state variables for the parallel configuration are defined as

$$u_{mn}^{(1)} = \eta_{mn}(t) \quad u_{mn}^{(2)} = \dot{\eta}_{mn}(t) \quad u^{(3)} = v(t) \tag{5.31}$$

where $u_{mn}^{(1)}$ is the modal displacement, $u_{mn}^{(2)}$ is the modal velocity, and $u^{(3)}$ is the voltage output across the resistive load. By introducing the electroelastic state variables into the governing electroelastic equations given by equations (5.29), $2MN+1$ number of first order differential equations are obtained for the electroelastic states as:

$$\begin{aligned}
\frac{du_{11}^{(1)}}{dt} &= u_{11}^{(2)} \\
\frac{du_{11}^{(2)}}{dt} &= -\omega_{11}^2 u_{11}^{(1)} - 2\zeta_{11}\omega_{11} u_{11}^{(2)} + [\tilde{\theta}_{11}^1 u^{(3)} + \dots + \tilde{\theta}_{11}^K u^{(K+2)}] + \phi_{11}(x_0, y_0)f(t)
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
 \frac{du_{21}^{(1)}}{dt} &= u_{21}^{(2)} \\
 \frac{du_{21}^{(2)}}{dt} &= -\omega_{21}^2 u_{21}^{(1)} - 2\zeta_{21} \omega_{21} u_{21}^{(2)} + \left[\tilde{\theta}_{21}^1 u^{(3)} + \dots + \tilde{\theta}_{21}^K u^{(K+2)} \right] + \phi_{21}(x_0, y_0) f(t) \\
 &\vdots \\
 \frac{du_{MN}^{(1)}}{dt} &= u_{MN}^{(2)} \\
 \frac{du_{MN}^{(2)}}{dt} &= -\omega_{MN}^2 u_{MN}^{(1)} - 2\zeta_{MN} \omega_{MN} u_{MN}^{(2)} + \left[\tilde{\theta}_{MN}^1 u^{(3)} + \dots + \tilde{\theta}_{MN}^K u^{(K+2)} \right] + \phi_{MN}(x_0, y_0) f(t) \\
 \frac{du^{(3)}}{dt} &= \frac{-1}{(C_p)_1 + \dots + (C_p)_K} \left\{ \frac{u^{(3)}}{R_1} + \left[\begin{aligned} &u_{11}^{(2)} (\tilde{\theta}_{11}^1 + \tilde{\theta}_{11}^2 + \dots + \tilde{\theta}_{11}^K) + \\ &u_{21}^{(2)} (\tilde{\theta}_{21}^1 + \tilde{\theta}_{21}^2 + \dots + \tilde{\theta}_{21}^K) + \dots + \\ &u_{MN}^{(2)} (\tilde{\theta}_{MN}^1 + \tilde{\theta}_{MN}^2 + \dots + \tilde{\theta}_{MN}^K) \end{aligned} \right] \right\}
 \end{aligned}$$

and then equation (5.32) can be written in compact-matrix form as follows:

$$\begin{bmatrix} \dot{u}_{11}^{(1)} & \dot{u}_{21}^{(1)} & \dots & \dot{u}_{MN}^{(1)} & \dot{u}_{11}^{(2)} & \dot{u}_{21}^{(2)} & \dots & \dot{u}_{MN}^{(2)} & \dot{u}^{(3)} \end{bmatrix}^T =$$

$$\begin{bmatrix} u_{11}^{(2)} \\ u_{21}^{(2)} \\ \vdots \\ u_{MN}^{(2)} \\ -2\zeta_{11}\omega_{11}u_{11}^{(2)} - \omega_{11}^2u_{11}^{(1)} + (\tilde{\theta}_{11}^1 + \tilde{\theta}_{11}^2 + \dots + \tilde{\theta}_{11}^K)u^{(3)} + \phi_{11}(x_0, y_0)f(t) \\ -2\zeta_{21}\omega_{21}u_{21}^{(2)} - \omega_{21}^2u_{21}^{(1)} + (\tilde{\theta}_{21}^1 + \tilde{\theta}_{21}^2 + \dots + \tilde{\theta}_{21}^K)u^{(3)} + \phi_{21}(x_0, y_0)f(t) \\ \vdots \\ -2\zeta_{MN}\omega_{MN}u_{MN}^{(2)} - \omega_{MN}^2u_{MN}^{(1)} + (\tilde{\theta}_{MN}^1 + \tilde{\theta}_{MN}^2 + \dots + \tilde{\theta}_{MN}^K)u^{(3)} + \phi_{MN}(x_0, y_0)f(t) \\ \frac{-1}{(C_p)_1 + \dots + (C_p)_K} \left\{ \begin{aligned} & \frac{u^{(3)}}{R_1} + \left[u_{11}^{(2)}(\tilde{\theta}_{11}^1 + \tilde{\theta}_{11}^2 + \dots + \tilde{\theta}_{11}^K) + u_{21}^{(2)}(\tilde{\theta}_{21}^1 + \tilde{\theta}_{21}^2 + \dots + \tilde{\theta}_{21}^K) + \right. \\ & \left. \dots + u_{MN}^{(2)}(\tilde{\theta}_{MN}^1 + \tilde{\theta}_{MN}^2 + \dots + \tilde{\theta}_{MN}^K) \right] \end{aligned} \right\} \end{bmatrix} \quad (5.33)$$

By numerically solving these electroelastic ODEs using the time history of the random force excitation over a time period T , the voltage output across the resistive load is given by

$$v(t) = u^{(3)} \quad (5.34)$$

and the expected (mean) power and the mean-square vibration response can be calculated as in the series configuration case given by equations (5.26) and (5.27).

5.4 Experimental Validations

5.4.1 Experimental Setup

The experimental setup used for the frequency and time measurements is presented in figure 5.2. An aluminum plate is used as the host plate by fully clamping symmetrically along all-four edges by thick bars and transversely isotropic piezoceramic patches (T105-A4E-602 by Piezo Systems, Inc.) are employed as patch-based piezoelectric energy harvesters by attaching them on the plate. Geometric, material and electroelastic properties of the aluminum plate and harvester patches are given in Table 5.1. The three patch-based piezoelectric harvesters (labeled as PBEH-1, PBEH-2, and PBEH-3) having identical polarities are perfectly bonded on the surface of the aluminum plate as shown in figures 5.2 and 5.3. The bottom (negative) and top (positive) surfaces of the patch-based piezoelectric harvesters are covered by thin vacuum sputtered nickel electrodes and the bonding areas on the aluminum plate are electrically insulated with 3M Scotch 1601 spray.

Table 5.1: Geometric, material and electroelastic properties

Property	Aluminum	Piezoceramic Patch
Length (mm)	580	72.4
Width (mm)	540	72.4
Thickness (mm)	1.9	0.267
Young's modulus (GPa)	65.1	66
Mass density (kg m^{-3})	2575	7800
Piezoelectric constant d_{31} (pm V^{-1})	-	-190
Permittivity constant $\bar{\epsilon}_{33}^S$ (nF m^{-1})	-	10.38

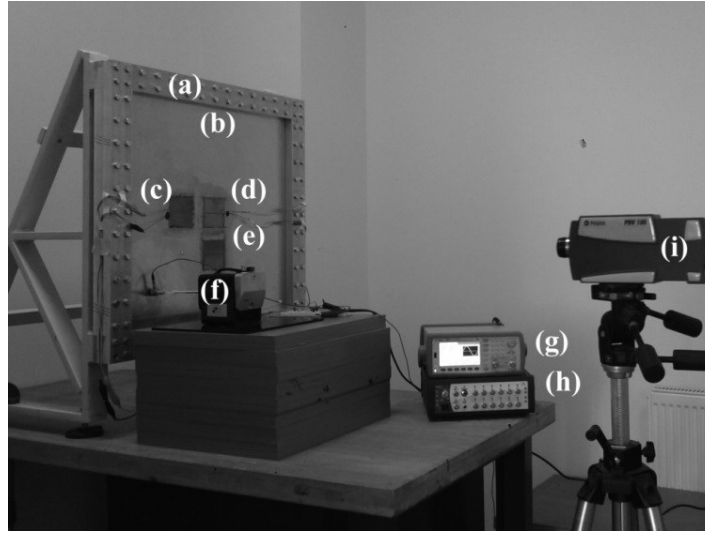


Figure 5.2: Experimental setup: (a) clamping frame (b) aluminum plate (c) PBEH-I (d) PBEH-II (e) PBEH-III (f) modal shaker (g) signal generator (h) data acquisition system (i) laser vibrometer

Series connection of the three patch-based piezoelectric harvesters is established by connecting the bottom and top electrodes sequentially and the electrical load (i.e., simply a resistor) is connected to the resultant terminals as illustrated in figure 5.3. A modal shaker is used to excite the plate and the shaker's attachment point can be seen in both figures 5.2 and 5.3. The applied force is monitored by mounting a force transducer (PCB 208C01) between the shaker's armature and the plate surface. A laser vibrometer (Polytec PDV 100) is used to measure the transverse velocity of the plate at the target point, which is shown in figure 5.3. The signals coming from the force transducer, the laser vibrometer, and the voltage across the resistive load are sent to the data acquisition system for the frequency and time domain analysis.

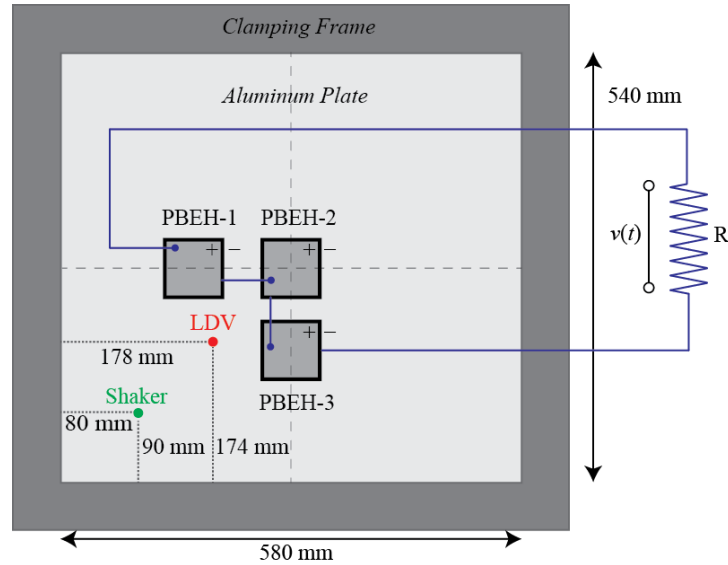


Figure 5.3: Series configuration of patch-based piezoelectric energy harvesters attached on the thin aluminum plate and the locations of LDV target and shaker attachment point

5.4.2 Experimental and Analytical Electroelastic FRFs

Before carrying out random vibration analysis, the experimental frequency response functions (FRFs) are measured and compared with analytical predictions to demonstrate the accuracy of the electroelastic model for triple patch-based harvesters connected in series. Using the experimental setup in figure 5.2, velocity FRF (i.e., between the velocity response-to-force input) and voltage FRF (i.e., between the voltage output-to-force input) are obtained by exciting the plate with a modal shaker (as in figures 5.2 and 5.3). Figure 5.4 presents the experimental and analytical velocity FRFs for a resistive load of $1\text{ M}\Omega$ close to open-circuit conditions). It is clear from this figure that the analytical velocity FRF accurately predicts the experimental velocity FRF results near resonance and anti-resonance frequencies up to 250 Hz. Note that, the modal damping ratios used for the analytical predictions are identified from the experimental velocity FRF using half-power point method. The experimental measurements and analytical predictions of voltage FRF are given for a resistive load of $1\text{ M}\Omega$ in figure 5.5. It can be seen from figure 5.5 that the analytical voltage FRF is in very good agreement with experimental data points for the first three resonance modes. At higher modes, there are minor amplitude and frequency wise inaccuracies between analytical predictions and experimental data points, which may be related to the bonding and positioning imperfections of patch-based piezoelectric harvesters as well as the material defects. However, the overall results of the analytical predictions and experiment results agree reasonably well. Therefore, it can be concluded that the analytical electroelastic model can be used to predict the vibration and velocity responses in the bandwidth up to 250 Hz.

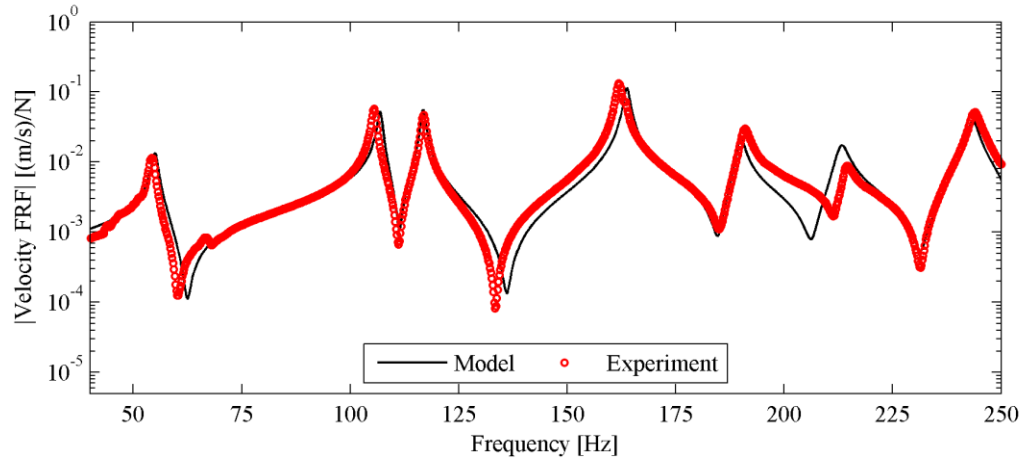


Figure 5.4: Comparison of the experimental and analytical velocity FRFs for a resistive load of $1\text{ M}\Omega$

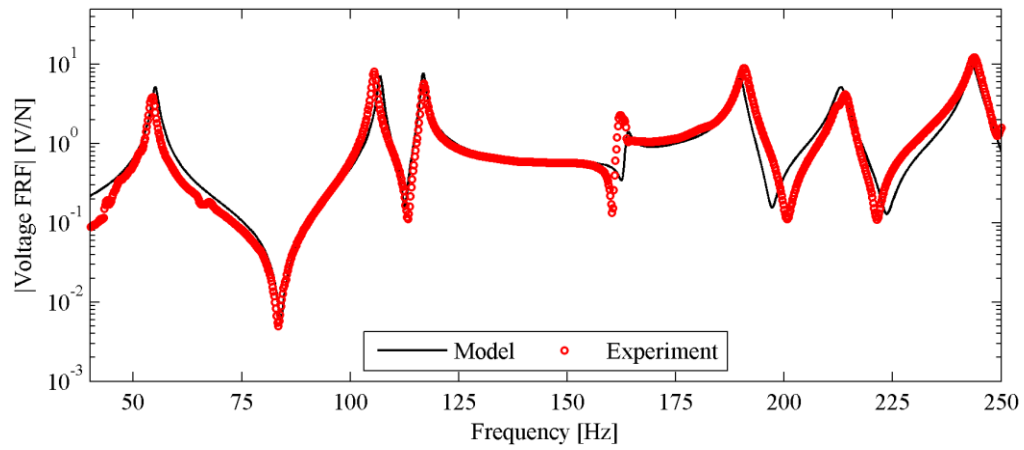


Figure 5.5: Comparison of the experimental and analytical voltage FRF for a resistive load of $1\text{ M}\Omega$

5.4.3 Experimental Random Excitations

Random vibration experiments are carried out by exciting the plate with a random force through the modal shaker with the configuration shown in figures 5.2 and 5.3. In the experiments, the force input, the voltage across the resistor, and the transverse velocity of the plate are measured and recorded by using the data acquisition system. The excitation signal driving the modal shaker is a band-limited noise signal produced by the signal generator in a bandwidth of 350 Hz. Figures 5.6 and 5.7 present a sample time history of the force input and its PSD, respectively. The sampling frequency is set to 12.8 kHz and the PSD is presented until 6400 Hz. It can be seen that the PSD of the excitation signal is as reasonably flat up to 250 Hz where the electroelastic model predicts the electromechanical behavior in a high accuracy. Because of the limitations of the experimental setup (mainly shaker armature's connection to the plate and overall experimental arrangement), there is significant decrease in the PSD above 250 Hz.

The standard deviation of the force input is computed as 0.473 N from the sample time history and the average value of flat portion of PSD is read from figure 5.7 as $S_0 = 5 \times 10^{-4} \text{ N}^2/\text{Hz}$.

A resistor sweep experiment is then performed for the fourteen resistive loads ranging from 500 Ω (close to short-circuit conditions) to 1 M Ω (close to open-circuit conditions). For each resistive load, experimental measurement is repeated five times and each measurement is taken for 3 seconds. Therefore, 70 different time series are recorded with total duration of 210 seconds and PSD profiles are obtained for 70 measurements. For each measurement of each resistor test, figures 5.8 and 5.9 present the standard deviations the average PSD levels, respectively. This comparison implies that the excitation signal provided by the modal shaker can be assumed as a stationary-ergodic process through all the measurements. Hence, a fair evaluation of different resistors is possible.

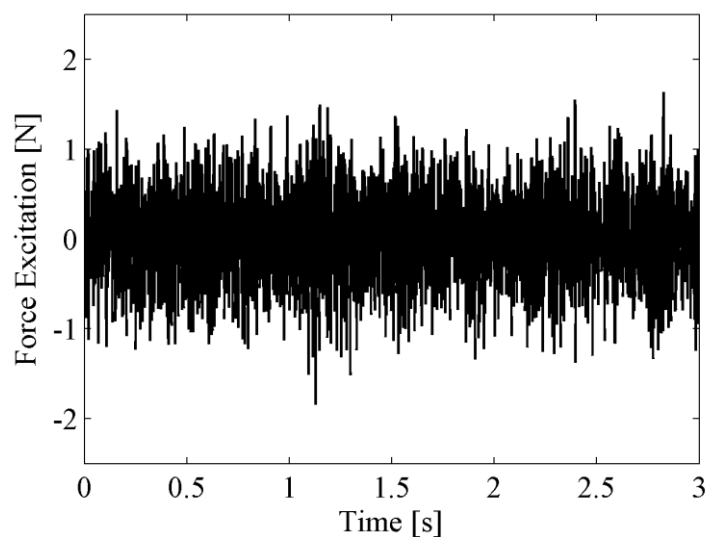


Figure 5.6: A sample time history of the measured random point force

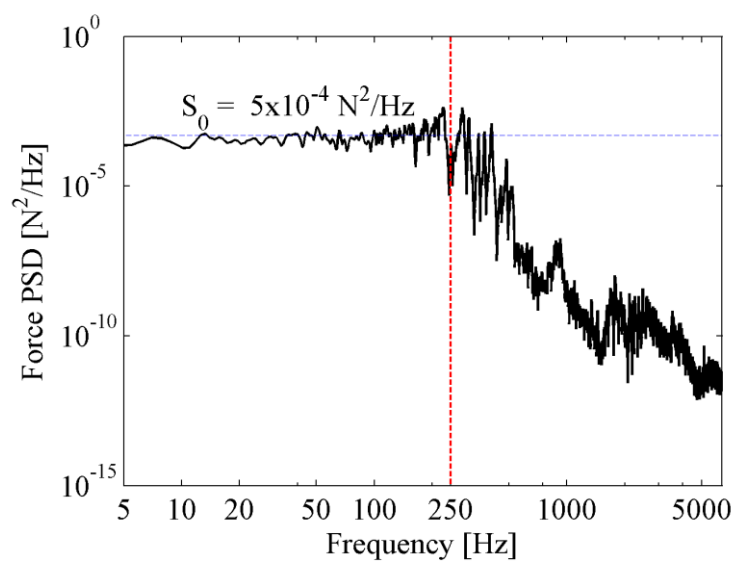


Figure 5.7: PSD of the measured random point force

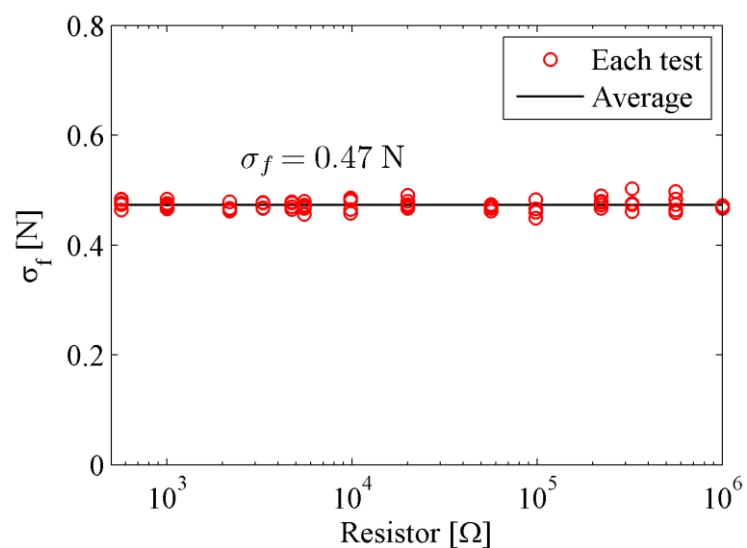


Figure 5.8: Standard deviations for all tests of the resistor sweep experiment

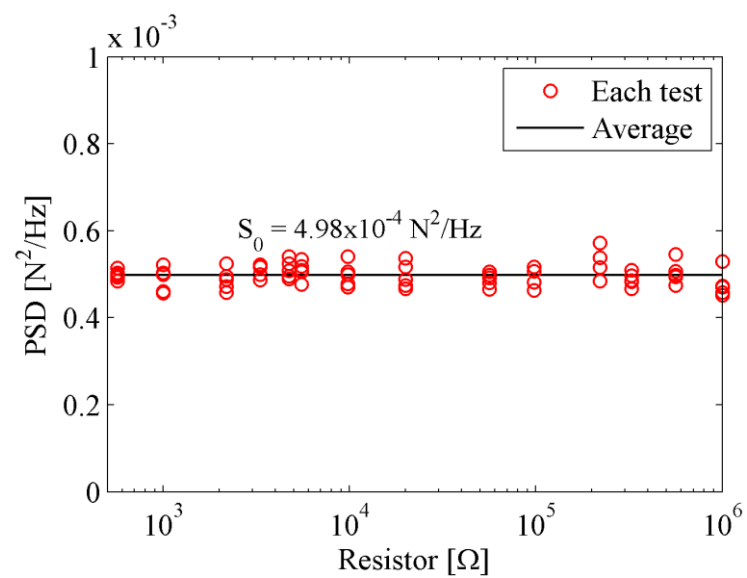


Figure 5.9: Average PSD levels for all tests of the resistor sweep experiment

5.4.4 Mean Power Output and Mean-square Velocity Response

The experimental measurements for expected power output and mean-square velocity responses are presented along with the analytical and numerical predictions in figures 5.10 and 5.11. It can be observed that the numerical and analytical predictions agree well with the experimental data points for a wide range of resistive load. For the frequency-domain (analytical) solution process, the mean power and mean-square velocity responses are calculated using equations (5.16) and (5.17). The frequency dependent PSD is obtained from the first experimental measurement (i.e., first measurement of a resistive load of 1 M Ω). The upper frequency limit ($\bar{\omega}$) of the frequency-domain integration is set to 250 Hz. In order to obtain numerical predictions of the mean power output and mean-square vibration response, explicit fourth-order Runge-Kutta algorithm is used to solve the electroelastic ODEs. Time simulations are performed with the individual time histories of force inputs for all 70 experimental measurements (5 repetitive tests for each 14 resistors) using the first order differential equations given by equation (5.25). As can be seen in figure 5.11, five different data points at each resistor are presented for the experimental measurements and time-domain predictions.

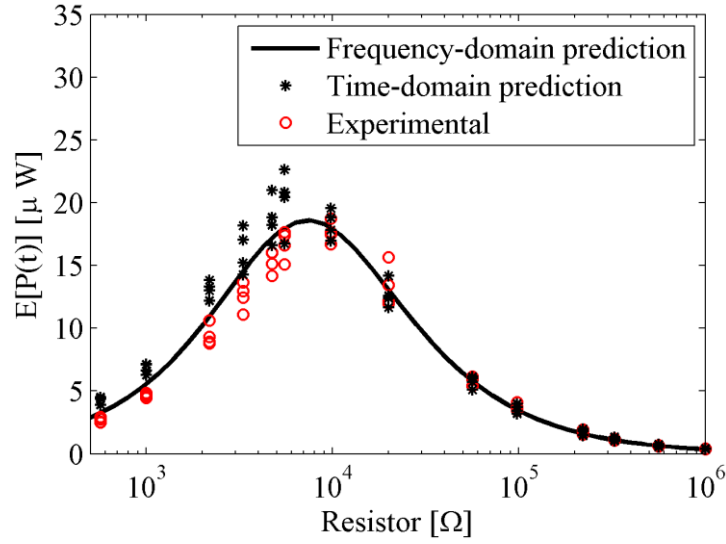


Figure 5.10: Comparison of the experimental measurements of the mean power output with frequency- and time-domain predictions for a set of resistive loads

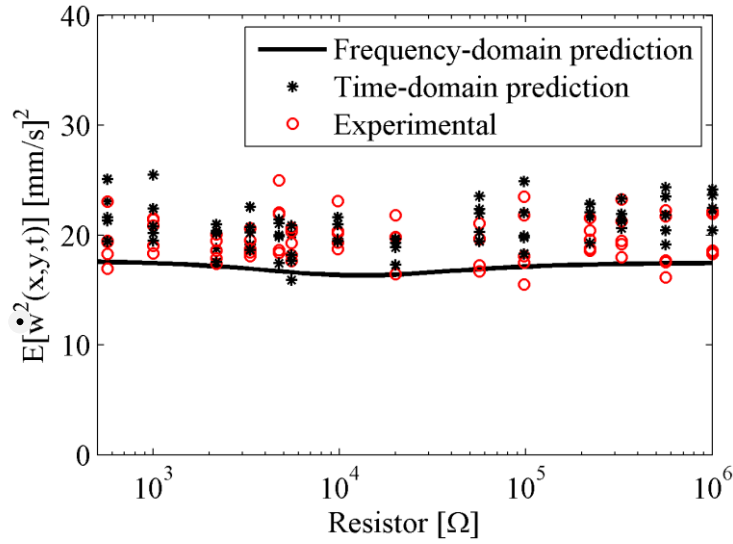


Figure 5.11: Comparison of the experimental measurements of the mean-square velocity response with frequency- and time-domain predictions for a set of resistive loads

Figures 5.12 and 5.13 present the average of five separate values at each resistor with error bars, indicating the deviations of five repeated measurements. It can be seen that the measurement deviations of mean power output are much wider around the optimum resistive loads, where more power output is extracted compared to short- and open- circuit conditions. It can be also observed from figure 5.13 that there are small amplitude-wise differences between the model predictions and experimental results. The particular reason of these inaccuracies might be high sensitivity of the velocity measurement to the orientation of the laser vibrometer and mounting of the shaker armature.

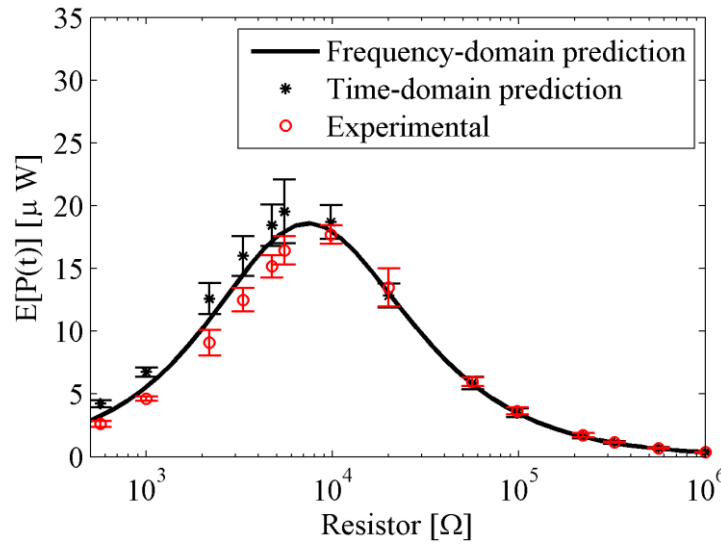


Figure 5.12: Comparison of the average experimental measurements of mean power output with frequency- and average time-domain predictions for a set of resistive loads

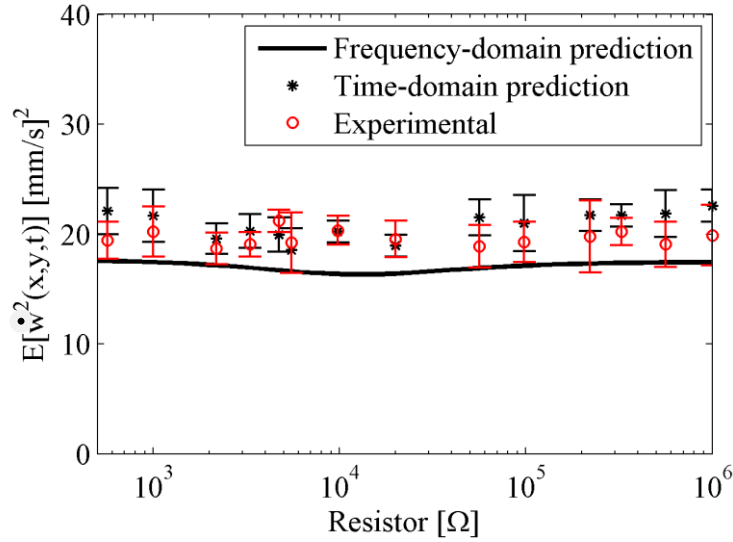


Figure 5.13: Comparison of the average experimental measurements of mean-square velocity response with frequency- and average time-domain predictions for a set of resistive loads

5.5 Band-limited Random Excitations

The effect of the excitation bandwidth on the mean power output is investigated with analytical simulations conducted for four different random force signals within the frequency range up to 250 Hz. A sample time history of each four-excitation signals and their PSD profiles are given in figure 5.14. It is important to note that standard deviations of four signals are identical and equal to 0.47 N while their mean values are zero. As can be observed from PSD profiles in figure 5.14, the first excitation signal covers the frequency range between 25 Hz to 75 Hz. The second and third signals have the flat PSD levels in the frequency ranges from 75 Hz to 125 Hz and from 125 Hz to 250 Hz, respectively. The PSD of the fourth signal is uniformly distributed over the whole 25-250 Hz band. The average

levels of flat portions of PSD profiles are also given in figure 5.14. In all cases, simulated configurations are identical to the previous case depicted in figure 5.2 and 5.3.

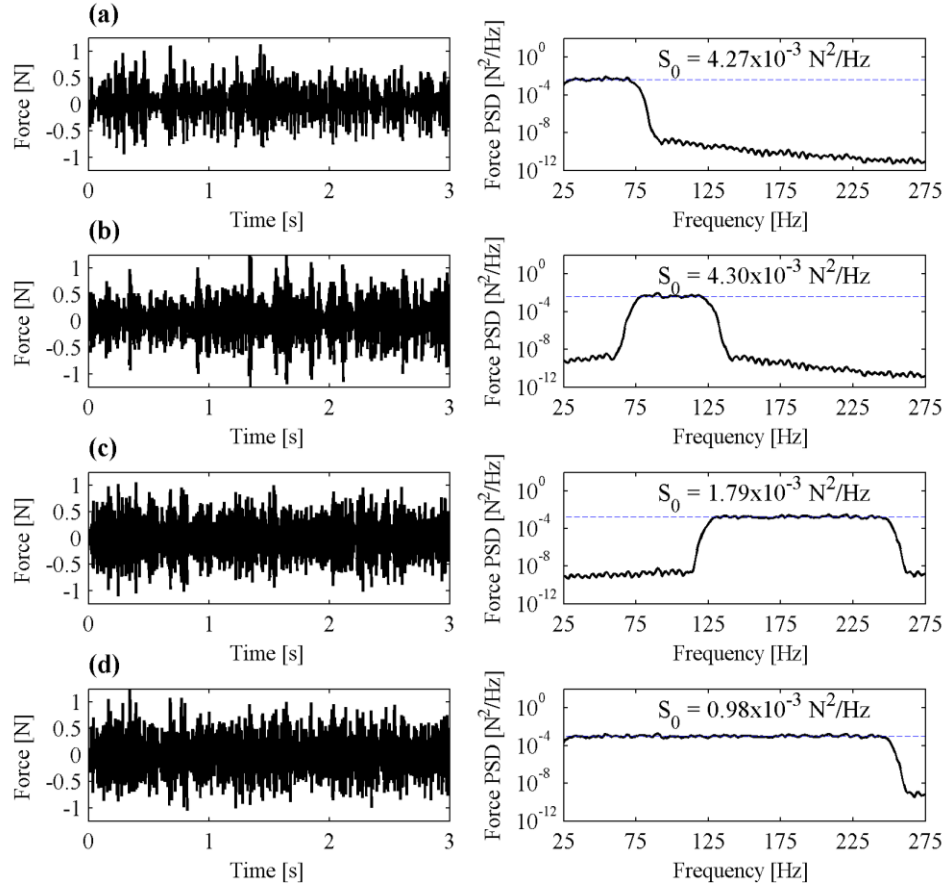


Figure 5.14: Sample time histories and corresponding PSD profiles of the four different force excitation signals in the bandwidth of (a) 25-75 Hz (b) 75-125 Hz (c) 125-250 Hz (d) 25-250 Hz (standard deviations of the time signals are identical)

5.5.1 Mean Power Output and Mean-square Velocity Response

The mean power output vs. resistive load curves are obtained and presented in figure 5.15 for the four cases of the force excitation signals presented formerly in figure 5.14. Comparison of the mean power output of four different band-limited excitations (i.e., with equal standard deviations) shows that the force excitation in the bandwidth from 125 Hz to 250 Hz produces significantly more mean power output. The reason for extracting more mean power in this band is due to the excitation of the four vibration modes of the thin plate in this corresponding frequency range and vibration-mode dependent performance of the patch-based piezoelectric harvesters. Noting that the fourth excitation signal's PSD is uniformly distributed over the total band from 25 Hz to 250 Hz, its maximum mean power is less than the one for the force excitation in the bandwidth of 125 Hz to 250 Hz. It should be noted that the input PSD levels are not identical in these simulations. Therefore, another comparison can be made by considering the band-limited PSD values (rather than equal standard deviations of time series).

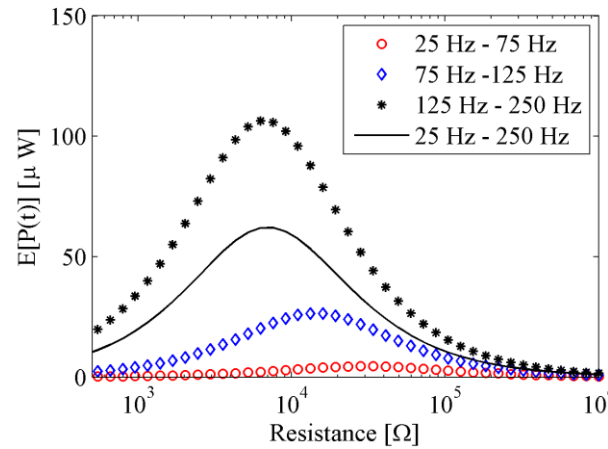


Figure 5.15: Comparison of the four different band-limited force excitation signals in terms of mean power output vs. resistive load (standard deviations of the time signals are identical)

For the PSD-based comparison of the four different band-limited excitation signals, the mean power outputs are normalized with respect to the flat PSD levels. The normalized power outputs are then presented in figure 5.16. As can be seen from this figure, the normalized mean power outputs of the excitation signals in the bandwidth of 25 Hz to 250 Hz and 125 Hz to 250 Hz are comparable and significantly larger than that belonging to other excitation bandwidths.

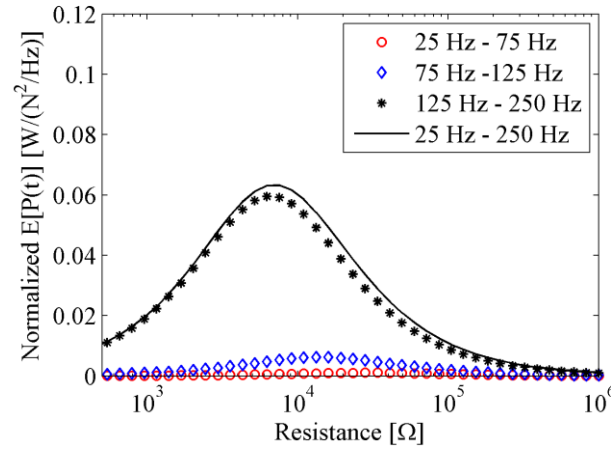


Figure 5.16: Comparison of the four different band-limited force excitation signals in terms of normalized mean power output vs. resistive load to PSD levels

5.5.2 Optimal Placement of a Patch-based Piezoelectric Harvester

The mean power output can be used as an optimization metric for the placement of a single patch-based piezoelectric harvester. In this section, analytical random vibrational energy harvesting simulations for the different positions of the patch-based harvester are performed to investigate the optimal placement of the harvester for the case of four different band-limited force excitations given by figure 5.14. The analytical simulations are carried out for a wide range of resistive loads and then the maximum expected power output is extracted by determining the optimum resistor for each location of the harvester. Note that, the center of the harvester (with the dimensions of 72.4 X 72.4 mm) is varied over an area 40 mm in from the clamped ends of the plate for the configuration described in Section 3.4.

Based on the simulation results, the maximum expected power output of the patch-based harvester at each position on the plate is determined and presented in figure 5.17. As can be seen from this figure, the optimal location depends on the bandwidth of the force excitation signal. For example, the bandwidth of 25 Hz to 75 Hz excites only the first vibration mode of the host plate and the optimal location is found to be the center of the host plate where the corresponding strain mode shape is the maximum as can be also seen in figure 4.6. The second excitation signal (in the bandwidth from 75 Hz to 125 Hz) excites the second and third vibration modes, and the optimal location is found to be near the quadrants of the host plate (it can be also depicted from figure 4.6). The signals (in the bandwidth from 125 Hz to 250 Hz and from 25 Hz to 250 Hz) excite the higher vibration modes of the host plate and the optimal location of the patch-based piezoelectric harvester moves to the intersection of intermediate strain fields, which can be observed by examining figure 5.17 and strain mode shapes presented in figure 4.6.

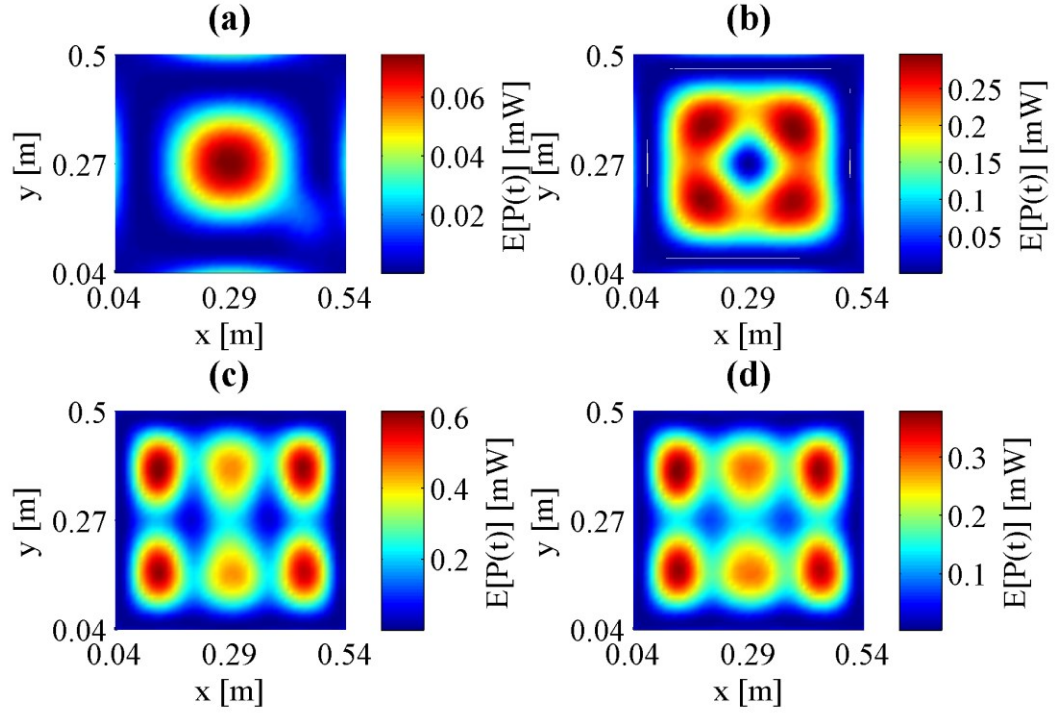


Figure 5.17: Expected power output at different patch-based harvester positions for the force excitation in the frequency range: (a) 25-75 Hz (b) 75-125 Hz (c) 125-250 Hz (d) 25-250 Hz

5.6 Summary and Conclusions

In this chapter, random vibrational energy harvesting with multiple patch-based harvesters is studied. For the case of the random force excitations of a thin plate, the frequency- and time-domain solutions are obtained for the expected power output and mean-square vibration response. The frequency-domain analytical solution is derived in terms of electroelastic FRFs and PSD of random force excitation whereas time-based numerical solutions are obtained by representing electroelastic models as first-order state-space equations. Therefore, if the PSD of random force excitation is known, the frequency-domain solution can be used to predict expected values of the structural and electrical responses, whereas one can also find the expected values of the responses using time-domain numerical solutions for a given time history. Experiment validations are provided with the experimental setup having triple patch-based harvesters connected in series. It is shown that analytical and numerical predictions of expected structural and electrical responses agree well with experimental measurements for a wide range of resistive load. Finally, the effects of excitation bandwidth and the positioning of the harvester patch are investigated with analytical random vibration scenarios. The simulation results show that the optimal location of the harvester depends on the bandwidth of the force excitation signal. Noting that the model predictions of power output values are the results of the selected case study, the location of force excitation can lead to a totally different power output levels.

Chapter 6

CONCLUSIONS

In this dissertation, experimentally validated analytical models and closed-form solutions are presented for patch-based piezoelectric energy harvesting from deterministic and random vibrations of thin plates.

In the first chapter, the scope, objectives, and contributions of this research work are provided. Chapter 2 gave the history of piezoelectricity and reviewed the applications of piezoelectric materials. Since piezoelectric materials can be used both for vibration control and energy harvesting, research outcomes and expertise in piezoelectric vibration control and energy harvesting fields can be shared. For that reason, this chapter reviewed the studies focused on energy harvesting and vibration control applications.

In Chapter 3, analytical model of a piezoelectric energy harvester in the form of a piezoceramic patch structurally integrated to a thin plate is developed for transverse force excitations. Closed-form exact steady-state solutions of structural and electrical responses are obtained by following modal analysis procedure for an all-four-edges clamped thin host plate. Electroelastic FRFs of the voltage output and the vibration response are then extracted and generalized for different boundary conditions. Experimental validation and extensive analytical analysis are then presented with a case study employing a thin PZT-5A piezoceramic patch attached on the surface of a fully clamped aluminum plate. Experimental and analytical FRF results showed that the maximum relative error of the resonance frequency predictions is less than 3.2% in the first four modes and the analytical voltage amplitudes agree very well with experimental results for a wide range of load resistances. Then, positioning of the patch-based harvester on a thin plate is discussed with mode-dependent strain distribution analysis and it is shown that if the mode-dependent

strain field is homogenous under the patch for a particular vibration mode, much larger power output can be harvested from that vibration mode.

Chapter 4 extended the single-patch electroelastic model to multiple patches attached on a thin plate. Distributed-parameter electroelastic models for multiple patch-based energy harvesters connected in series and parallel are developed in physical coordinates and transformed into modal coordinates for transverse force excitations. Closed-form expressions for structural and electrical responses are derived for multiple vibration modes of a fully clamped thin plate for the series and parallel connection configurations. Experimental and analytical case studies comprising both series and parallel connections are then presented for validating the analytical models. It is shown that voltage and velocity FRFs exhibit very good agreement with the experimental FRF measurements for the series and parallel connection cases. It is also observed that the series and parallel connections of harvester patches combine the voltage outputs of different vibration modes into a single voltage output and thus enable energy harvesting from all vibration modes of interest. This outcome can lead a substantial increase in the extracted power output when multiple patch-based harvesters are properly connected. Analytical and experimental power and voltage peak amplitudes are reported over a wide range of electrical loads for the first eight vibration modes of the plate in the frequency range of 0-250 Hz.

Chapter 5 presented analytical modelling of patch-based energy harvesting from broadband and band-limited random vibrations with numerical and experimental analyses. For the expected power output and mean-square vibration response, analytical and numerical solutions are derived. Frequency-based analytical solutions are obtained based on electroelastic FRFs (i.e., voltage and velocity FRFs) and PSD of broadband and band-limited random force inputs. Time-domain numerical solutions of the expected power output and mean-square vibration response are derived by representing the electroelastic models in the state-space form and considering the Fourier series expansion of the force

input. A case study with triple patch-based harvesters connected in series is investigated experimentally and analytically. It is shown that analytical and numerical predictions for the expected power output and mean-square velocity response are in very good agreement with the experimental measurements over a wide range of resistive loads. Finally, random vibrational energy harvesting scenarios are presented to explore the expected power output variation with the bandwidth of the force excitation and the position of the harvester. It is observed that if the bandwidth of the excitation signal only covers the first vibration mode of the host plate, the optimal position is found to be the center of the host plate where the corresponding strain mode shape is the maximum. On the other hand, it is also observed that if the excitation signals excite the higher vibration modes of the host plate, the optimal position moves to the intersection of intermediate strain fields. Note that, another significant factor that can affect the power output levels is the location of the force excitation, which can be studied using the electroelastic models developed in this work.

This dissertation presented the electroelastic models of single and multiple patch-based piezoelectric energy harvesters attached on a thin plate along with experimental validations and performance investigations. The limitations and future directions are discussed in the following. Electroelastic models developed in this dissertation are based on the assumption that the total volume of the patch-based harvesters is significantly smaller than the host plate; therefore, piezoceramic patches' stiffness and inertial effects are neglected. When this assumption is no longer applicable or valid, numerical finite element models can be built in multi-physics softwares. Therefore, in future studies, hybrid methodologies that take full advantage of analytical and finite element modelling approaches can be developed for real-life applications. For instance, by including fluid-structural interactions in finite element models, the performance of patch-based harvesters can be studied for the complex structural components of aerospace and marine applications. Topology optimizations can be also carried out to investigate the shape, size, and number of patch-based harvesters for

three-dimensional structures. For severe loading conditions, nonlinearities of host structures and patch-based harvesters can be studied and more general electroelastic models can be built. By considering the interface electronics, the direct current (DC) of patch-based harvesters connected in series and parallel can be investigated. The interface electronics can improve the broadband energy harvesting by switching between series and parallel connections. Patch-based energy harvesting can be combined with other available transduction mechanisms such as electrostatic, electromagnetism and even with solar panels and more electrical power output can be harvested from the available ambient energy. For all possible future research studies, this dissertation can serve as a guideline and a validation tool.

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Appendix A

SEPARATION OF SPATIAL VARIABLES SOLUTION

A.1 Transverse Vibrations of Thin Flat Plates

Figure A.1 shows a geometrically uniform and rectangular thin plate with a length of a , a width of b and a thickness of h . According to the Kirchhoff's plate theory, the equation of motion governing the undamped free vibrations of a thin plate is [139]

$$D \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0 \quad (\text{A.1})$$

where $w(x, y, t)$ is the transverse displacement of an arbitrary point (x, y) along the neutral surface of the plate at time t . The mass density of the plate is ρ and the flexural rigidity is D , which is $D = Eh^3 / (12 - 12\nu^2)$ where E and ν are Young's modulus and Poisson's ratio of the plate, respectively.

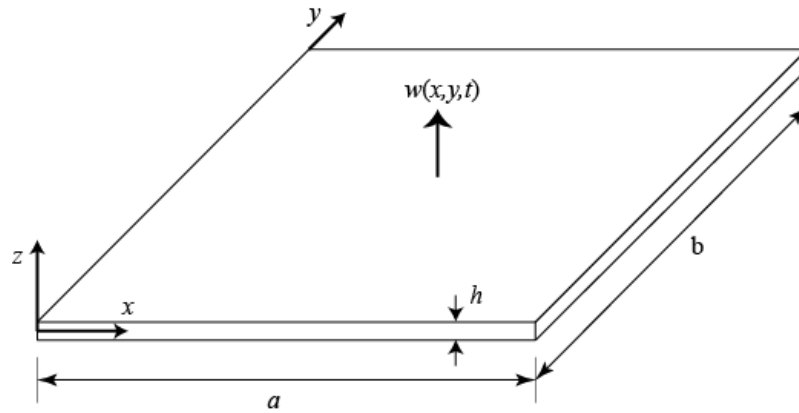


Figure A.1: A rectangular thin plate

The boundary conditions for an all-four-edges clamped plate can be written as

$$w(0, y, t) = 0 \quad w(a, y, t) = 0 \quad w(x, 0, t) = 0 \quad w(x, b, t) = 0 \quad (\text{A.2})$$

$$\left. \frac{\partial w(x, y, t)}{\partial x} \right|_{x=0} = 0 \quad \left. \frac{\partial w(x, y, t)}{\partial x} \right|_{x=a} = 0 \quad \left. \frac{\partial w(x, y, t)}{\partial y} \right|_{y=0} = 0 \quad \left. \frac{\partial w(x, y, t)}{\partial y} \right|_{y=b} = 0 \quad (\text{A.3})$$

where equations (A.2) are geometric boundary conditions and equations (A.3) are natural boundary conditions.

In the existing literature, exact modal solutions for free vibrations of thin plates are available for the boundary conditions of CCCC, SSSS, SCCC, and SSCC (where C stands for the clamped edge and S stands for the simply supported edge) [134, 139].

A.2 Separation of Variables Solution

The harmonic vibration of the plate given by equation (A.1) can be expressed in terms of the spatial and temporal functions as [134]

$$w(x, y, t) = \phi(x, y)e^{j\omega t} \quad (\text{A.4})$$

Here, $\phi(x, y)$ is mass-normalized eigenfunction that can be written as $\phi(x, y) = \xi(x)\psi(y)$ where $\xi(x)$ and $\psi(y)$ are eigenfunctions along x and y -axes, respectively. Substituting equation (A.4) into equation (A.1) gives

$$D \left(\frac{1}{\xi(x)} \frac{d^4 \xi(x)}{dx^4} + \frac{2}{\xi(x)\psi(y)} \frac{d^2 \xi(x)}{dx^2} \frac{d^2 \psi(y)}{dy^2} + \frac{1}{\psi(y)} \frac{d^4 \psi(y)}{dy^4} \right) - \delta^4 = 0 \quad (\text{A.5})$$

where the parameter δ is defined as

$$\delta^4 = \rho h \omega^2 \quad (\text{A.6})$$

Then, it is assumed that eigenfunctions $\xi(x)$ and $\psi(y)$ can be written as follows [134]:

$$\xi(x) = Ae^{\alpha x} \text{ and } \psi(y) = Be^{\beta y} \quad (\text{A.7})$$

where α and β are eigenvalues, and A and B are amplitudes along x and y axes, respectively. After that, equations (A.7) can be substituted into equation (A.5) to give the characteristic polynomial as

$$(\alpha^4 + 2\alpha^2\beta^2 + \beta^4) - \frac{\delta^4}{D} = 0 \quad (\text{A.8})$$

which can be simplified to

$$(\alpha^2 + \beta^2 - \kappa^2)(\alpha^2 + \beta^2 + \kappa^2) = 0 \quad (\text{A.9})$$

Here, κ is defined as the frequency parameter as $\kappa^4 = \delta^4 / D$.

Using equation (A.9), α can be solved in terms of β as follows:

$$\alpha^2 = \kappa^2 - \beta^2 \text{ and } \alpha^2 = -\kappa^2 - \beta^2 \quad (\text{A.10})$$

and the four solutions for α can be expressed as

$$\begin{aligned} \alpha_{1,2} &= \pm\sqrt{\kappa^2 - \beta^2} = \pm\lambda^x \\ \alpha_{3,4} &= \pm j\sqrt{\kappa^2 - \beta^2} = \pm j\mu^x \end{aligned} \quad (\text{A.11})$$

Similarly, β can be solved in terms of α as

$$\beta^2 = \kappa^2 - \alpha^2 \text{ and } \beta^2 = -\kappa^2 - \alpha^2 \quad (\text{A.12})$$

and the solutions for β are

$$\begin{aligned} \beta_{1,2} &= \pm\sqrt{\kappa^2 - \alpha^2} = \pm\lambda^y \\ \beta_{3,4} &= \pm j\sqrt{\kappa^2 - \alpha^2} = \pm j\mu^y \end{aligned} \quad (\text{A.13})$$

Substituting $\alpha_1 = \lambda^x$, $\alpha_3 = j\mu^x$, $\beta_1 = \lambda^y$, and $\beta_3 = j\mu^y$ into equation (A.10) yields the following three characteristic equations:

$$\kappa^2 = (\lambda^x)^2 + (\lambda^y)^2 \quad (\text{A.14})$$

$$2\kappa^2 = (\lambda^x)^2 + (\mu^x)^2 \quad (\text{A.15})$$

$$2\kappa^2 = (\lambda^y)^2 + (\mu^y)^2 \quad (\text{A.16})$$

It should be noted that there are five unknowns $(\kappa, \lambda^x, \lambda^y, \mu^x, \mu^y)$ and only three equations are obtained from the characteristic polynomial given by equation (A.10). For unique solution of five unknown variables, two more equations are needed. Additional two equations will be obtained from the fully clamped boundary conditions of the plate in the following.

The eigenfunctions $\xi(x)$ and $\psi(y)$ are formed using the solutions given by equations (A.11) and (A.13) as follows:

$$\xi(x) = A_1 \cos(\lambda^x x) + A_2 \sin(\lambda^x x) + A_3 \cosh(\mu^x x) + A_4 \sinh(\mu^x x) \quad (\text{A.17})$$

$$\psi(y) = B_1 \cos(\lambda^y y) + B_2 \sin(\lambda^y y) + B_3 \cosh(\mu^y y) + B_4 \sinh(\mu^y y) \quad (\text{A.18})$$

where $A_1, A_2, A_3, A_4, B_1, B_2, B_3$ and B_4 are unknown constants and the frequency parameter is

$$\kappa^4 = \omega^2 \frac{\rho h}{D} \quad (\text{A.19})$$

Equations (A.4) are substituted into equations (A.2) and (A.3) to obtain

$$\begin{array}{llll} \xi(0) = 0 & \xi(a) = 0 & \left. \frac{d\xi(x)}{dx} \right|_{x=0} = 0 & \left. \frac{d\xi(x)}{dx} \right|_{x=a} = 0 \end{array} \quad (\text{A.20})$$

$$\begin{array}{llll} \psi(b) = 0 & \psi(0) = 0 & \left. \frac{d\psi(y)}{dy} \right|_{y=0} = 0 & \left. \frac{d\psi(y)}{dy} \right|_{y=b} = 0 \end{array} \quad (\text{A.21})$$

Substituting equation (A.17) into equations (A.20) gives

$$\xi(0) = A_1 + A_3 = 0 \quad (\text{A.22})$$

$$\xi(a) = A_1 \cos(\lambda^x a) + A_2 \sin(\lambda^x a) + A_3 \cosh(\mu^x a) + A_4 \sinh(\mu^x a) = 0 \quad (\text{A.23})$$

$$\left. \frac{d\xi(x)}{dx} \right|_{x=0} = A_2 \lambda^x + A_4 \mu^x = 0 \quad (\text{A.24})$$

$$\left. \frac{d\xi(x)}{dx} \right|_{x=a} = -A_1 \lambda^x \sin(\lambda^x a) + A_2 \lambda^x \cos(\lambda^x a) + A_3 \mu^x \sinh(\mu^x a) + A_4 \mu^x \cosh(\mu^x a) = 0 \quad (\text{A.25})$$

Solving equations (A.22) and (A.24) for A_3 and A_4 in terms of A_1 and A_2 , respectively, gives the following two-by-two system of linear equations as

$$\begin{bmatrix} \cos(\lambda^x a) - \cosh(\mu^x a) & \sin(\lambda^x a) - \frac{\lambda^x}{\mu^x} \sinh(\mu^x a) \\ -\lambda^x \sin(\lambda^x a) - \mu^x \sinh(\mu^x a) & \lambda^x \cos(\lambda^x a) - \lambda^x \cosh(\mu^x a) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{A.26})$$

For non-trivial solutions of A_1 and A_2 , the coefficient matrix given by equation (A.26) has to be singular and thus the determinant of the coefficient matrix is to be zero. The determinant of the coefficient matrix can be written as

$$\begin{aligned} & [\cos(\lambda^x a) - \cosh(\mu^x a)] \times [\lambda^x \cos(\lambda^x a) - \lambda^x \cosh(\mu^x a)] \\ & + [\lambda^x \sin(\lambda^x a) + \mu^x \sinh(\mu^x a)] \left[\sin(\lambda^x a) - \frac{\lambda^x}{\mu^x} \sinh(\mu^x a) \right] = 0 \end{aligned} \quad (\text{A.27})$$

which yields the additional (i.e., the fourth) characteristic equation as

$$\frac{1 - \cos(\lambda^x a) \cosh(\mu^x a)}{\sin(\lambda^x a) \sinh(\mu^x a)} = \frac{(\lambda^x)^2 - (\mu^x)^2}{2\lambda^x \mu^x} \quad (\text{A.28})$$

in addition to the characteristics equations given by equations (A.14 - A.16).

Similarly, one can substitute equation (A.18) into equations (A.21) to obtain

$$\psi(0) = B_1 + B_3 = 0 \quad (\text{A.29})$$

$$\psi(b) = B_1 \cos(\lambda^y b) + B_2 \sin(\lambda^y b) + B_3 \cosh(\mu^y b) + B_4 \sinh(\mu^y b) = 0 \quad (\text{A.30})$$

$$\left. \frac{d\psi(y)}{dy} \right|_{y=0} = B_2 \lambda^y + B_4 \mu^y = 0 \quad (\text{A.31})$$

$$\left. \frac{d\psi(y)}{dy} \right|_{y=b} = -B_1 \lambda^y \sin(\lambda^y b) + B_2 \lambda^y \cos(\lambda^y b) + B_3 \mu^y \sinh(\mu^y b) + B_4 \mu^y \cosh(\mu^y b) = 0 \quad (\text{A.32})$$

Equations (A.29) and (A.31) can be solved for B_3 and B_4 to obtain the following two-by-two system of linear equations for B_1 and B_2 as follows:

$$\begin{bmatrix} \cos(\lambda^y b) - \cosh(\mu^y b) & \sin(\lambda^y b) - \frac{\lambda^y}{\mu^y} \sinh(\mu^y b) \\ -\lambda^y \sin(\lambda^y b) - \mu^y \sinh(\mu^y b) & \lambda^y \cos(\lambda^y b) - \lambda^y \cosh(\mu^y b) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{A.33})$$

The determinant of the coefficient matrix of B_1 and B_2 has to be zero for non-trivial solution, which can be written as

$$\begin{aligned} & [\cos(\lambda^y b) - \cosh(\mu^y b)] \times [\lambda^y \cos(\lambda^y b) - \lambda^y \cosh(\mu^y b)] \\ & + [\lambda^y \sin(\lambda^y b) + \mu^y \sinh(\mu^y b)] \left[\sin(\lambda^y b) - \frac{\lambda^y}{\mu^y} \sinh(\mu^y b) \right] = 0 \end{aligned} \quad (\text{A.34})$$

and the fifth characteristic equation can then be obtained as follows:

$$\frac{1 - \cos(\lambda^y b) \cosh(\mu^y b)}{\sin(\lambda^y b) \sinh(\mu^y b)} = \frac{(\lambda^y)^2 - (\mu^y)^2}{2\lambda^y \mu^y} \quad (\text{A.35})$$

Therefore, eigenvalues $\lambda^x, \lambda^y, \mu^x, \mu^y$ and the frequency parameter κ can be calculated for the m th vibration mode by solving the five characteristic equations given by equations (A.14-A.16, A.28, A.35) for the intervals of

$$\lambda^x a \in \left[m\pi, m\pi + \frac{\pi}{2} \right], \quad m = 1, 2, \dots \quad (\text{A.36})$$

$$\lambda^y b \in \left[n\pi, n\pi + \frac{\pi}{2} \right], \quad n = 1, 2, \dots$$

The undamped natural frequency of the plate for the m th vibration mode can be calculated by

$$\omega_{mn} = \sqrt{\frac{\kappa_{mn}^4}{\rho_s h_s}} D \quad (\text{A.37})$$

and the corresponding eigenfunction $\phi(x, y)$ of the m th vibration mode can be written as

$$\begin{aligned} \phi_{mn}(x, y) = \Theta_{mn} & \left(\cos \lambda_{mn}^x x - \sigma_{mn}^x \sin \lambda_{mn}^x x - \cosh \mu_{mn}^x x + \frac{\lambda_{mn}^x}{\mu_{mn}^x} \sigma_{mn}^x \sinh \mu_{mn}^x x \right) \\ & \times \left(\cos \lambda_{mn}^y y - \sigma_{mn}^y \sin \lambda_{mn}^y y - \cosh \mu_{mn}^y y + \frac{\lambda_{mn}^y}{\mu_{mn}^y} \sigma_{mn}^y \sinh \mu_{mn}^y y \right) \end{aligned} \quad (\text{A.38})$$

where Θ_{mn} is the modal amplitude constant that needs to be calculated by mass-normalization and σ_{mn}^x and σ_{mn}^y are

$$\begin{aligned} \sigma_{mn}^x &= \frac{\cos \lambda_{mn}^x a - \cosh \mu_{mn}^x a}{\sin \lambda_{mn}^x a - \frac{\lambda_{mn}^x}{\mu_{mn}^x} \sinh \mu_{mn}^x a} & \sigma_{mn}^y &= \frac{\cos \lambda_{mn}^y b - \cosh \mu_{mn}^y b}{\sin \lambda_{mn}^y b - \frac{\lambda_{mn}^y}{\mu_{mn}^y} \sinh \mu_{mn}^y b} \end{aligned} \quad (\text{A.39})$$

Appendix B

ELECTROELASTIC FINITE ELEMENT MODELLING OF A PATCH-BASED HARVESTER ON A THIN PLATE

B.1 Finite Element Model

A commercially available finite element analysis package can be used for electroelastic finite element modeling and analysis of the piezoceramic patch on a thin plate as shown in figure B.1 [28]. It is worth mentioning that ABAQUS, COMSOL, and ANSYS provide multi-physics environment for piezoelectric analysis.

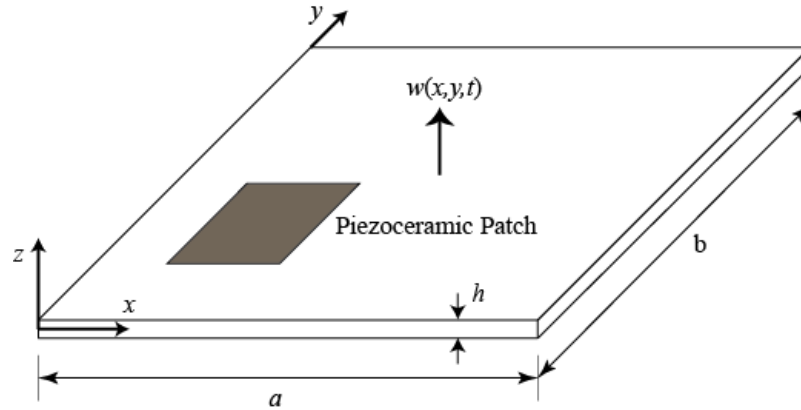


Figure B.1: Piezoceramic patch attached on a thin plate

In ABAQUS, a thin aluminum plate can be meshed with 8-node quadrilateral in-plane general-purpose continuum shell elements (named as SC8R) if the thickness is significantly smaller than the other dimensions, and the piezoceramic patch can be meshed with 20-node quadratic piezoelectric bricks (C3D20E) to couple the stress and electrical potential. Surface-based tie constraint can be used for perfect bonding of the piezoceramic patch and

the aluminum plate. Note that the modelling of highly conductive electrode layers of the harvester patches is very important for the accurate prediction of the voltage output (electrical potential) and the highly conductive electrode layer constraints the voltage output of each element such that each generates the identical voltage. For this purpose, in ABAQUS, master and slave nodes on the surfaces of the harvester patch can be defined as shown in Figure B.2, and using “EQUATION” constraint, the voltage output of master node can be assigned to voltage degree of freedom of each slave element.

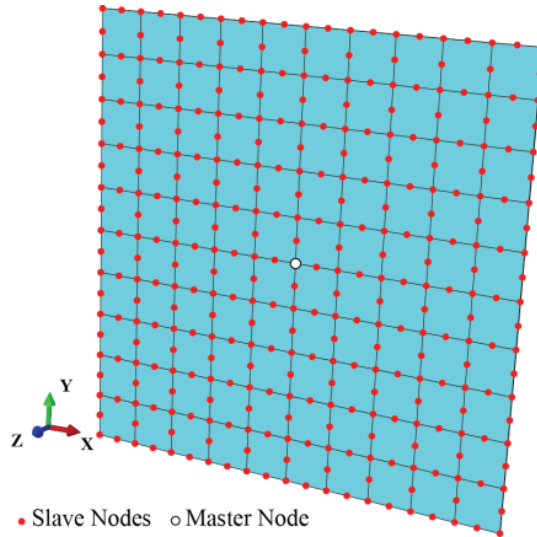


Figure B.2: Master and slave node definition for electrode layer of a harvester patch

The bottom electrode layer of the piezoceramic patch bonded to the plate can be assigned a zero electrical potential (grounded), while the voltage output can be monitored on the master node of the top electrode. Mass and stiffness proportional damping parameters (referred as damping parameters α and β in software packages) can be identified from experimental measurements and then the piezoelectric analysis can be carried out in frequency and time domains.

Appendix C

PATCH-BASED HARVESTERS CONNECTED TO SEPARATE RESISTIVE LOADS

C.1 Coupled Mechanical Equation

Figure C.1 presents separate resistor connection configuration of patch-based harvesters structurally integrated on a thin plate. Each patch-based harvester covers a rectangular area with two corners at $(x_{k,1}, y_{k,1})$ and $(x_{k,2}, y_{k,2})$ where $k = 1, 2, \dots, K$ and K is the number of patch-based harvesters. Recall from Chapter 4 that the patch-based harvesters are in the form of transversely isotropic piezoceramic patches which are perfectly bonded on the host plate with the length of a and the width of b .

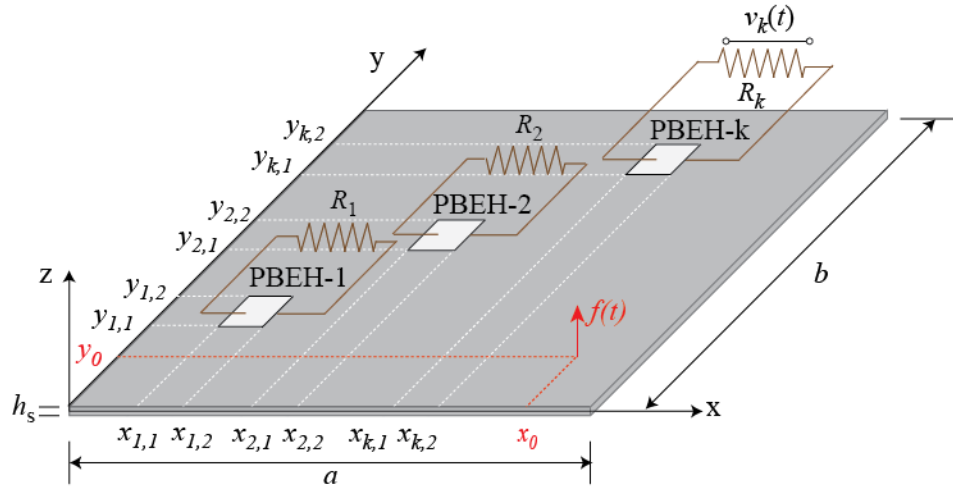


Figure C.1: Separate resistor connection configuration of multiple patch-based piezoelectric energy harvesters

The governing differential equation for the forced vibration of the thin plate with patch-based harvesters connected to separate loads can be written as

$$\begin{aligned}
 D_s \left(\frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + c \frac{\partial w(x, y, t)}{\partial t} + \rho_s h_s \frac{\partial^2 w(x, y, t)}{\partial t^2} \\
 - \sum_{k=1}^K \theta_k v_k(t) \left\{ \left[\frac{d\delta(x - x_{k,1})}{dx} - \frac{d\delta(x - x_{k,2})}{dx} \right] [H(y - y_{k,1}) - H(y - y_{k,2})] \right. \\
 \left. + \left[\frac{d\delta(y - y_{k,1})}{dy} - \frac{d\delta(y - y_{k,2})}{dy} \right] [H(x - x_{k,1}) - H(x - x_{k,2})] \right\} \\
 = f(t) \delta(x - x_0) \delta(y - y_0)
 \end{aligned} \tag{C.1}$$

which is identical to equation (4.1) and therefore the details can be found in Section 4.1.

C.2 Coupled Electrical Circuit Equation

The governing electrical circuit equation for each harvester patch shown in figure C.1 is given by

$$(C_{\bar{p}})_k \frac{dv_k(t)}{dt} + \frac{v_k(t)}{R_k} = i_k(t) \quad k = 1, 2, \dots, K \tag{C.2}$$

where $v_k(t)$ is the voltage across the electrodes of the k th harvester patch and R_k is the resistive load connected to the k th harvester patch. Similar to the series configuration, the number of governing circuit equations equals to the number of harvester patches (K) attached on the plate. The expressions for the capacitance $(C_{\bar{p}})_k$ and the current output $i_k(t)$ of the k th harvester are identical with the expressions given by equations (4.4) and (4.5), respectively.

C.3 Modal Transformation of Electroelastic Equations

By following the modal analysis procedure for an all-four edges clamped thin plate as explained in Sections 3.1.3 and 4.2, the equation of motion in equation (C.1) can be transformed into modal coordinates as follows:

$$\frac{d^2 \eta_{mn}(t)}{dt^2} + 2\zeta_{mn}\omega_{mn} \frac{d\eta_{mn}(t)}{dt} + \omega_{mn}^2 \eta_{mn}(t) - \sum_{k=1}^K (\tilde{\theta}_{mn})_k v_k(t) = f_{mn}(t) \quad (C.3)$$

where the modal time response is $\eta_{mn}(t)$, the undamped natural frequency (in short-circuit condition) is ω_{mn} and the modal damping ratio is ζ_{mn} for the m th vibration mode of the plate. The model forcing $f_{mn}(t)$ and electromechanical coupling term $(\tilde{\theta}_{mn})_k$ are given by equations (4.17) and (4.18), respectively.

The electrical circuit equations for the harvester patches can be then written in modal coordinates as

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k \frac{d\eta_{mn}(t)}{dt} + (C_{\bar{p}})_k \frac{dv_k(t)}{dt} + \frac{v_k(t)}{R_k} = 0 \quad \text{where } k = 1, 2, \dots, K \quad (C.4)$$

by following the steps given in Section 4.2.2.

C.4 Steady-state Response to Harmonic Force Excitation

Similar to the series and parallel connection cases, when the transverse point force acting on the surface of the plate is assumed to be harmonic of the form $f(t) = F_0 e^{j\omega t}$, the steady-state mechanical and electrical responses can be written as

$$\eta_{mn}(t) = H_{mn} e^{j\omega t}, \quad v_k(t) = V_k e^{j\omega t} \quad (C.5)$$

where H_{mn} and V_k are complex valued amplitudes.

The coupled mechanical and electrical equations of the distributed-parameter model in modal coordinates given by equations (C.3) and (C.4) can be then written as

$$(\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega)H_{mn} - \sum_{k=1}^K (\tilde{\theta}_{mn})_k V_k = F_0 \phi_{mn}(x_0, y_0) \quad (C.6)$$

and

$$j\omega \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\tilde{\theta}_{mn})_k H_{mn} + j\omega(C_{\bar{p}})_k V_k + \frac{V_k}{R_k} = 0 \text{ where } k = 1, 2, \dots, K \quad (C.7)$$

Then, H_{mn} can be obtained from equation (C.6) as

$$H_{mn} = \frac{F_0 \phi_{mn}(x_0, y_0) + \boldsymbol{\theta}_{mn}^T \mathbf{V}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \quad (C.8)$$

where the electromechanical coupling vector $\boldsymbol{\theta}_{mn}$ with the dimensions of $K \times 1$ is

$\boldsymbol{\theta}_{mn} = [(\tilde{\theta}_{mn})_1 \quad (\tilde{\theta}_{mn})_2 \quad \dots \quad (\tilde{\theta}_{mn})_K]^T$ and the voltage vector $\mathbf{V}$ with the dimensions of

$K \times 1$ includes the voltage amplitude of each harvester patch as $\mathbf{V} = [V_1 \quad V_2 \quad \dots \quad V_K]^T$,

where superscript T stands for the transpose. The voltage vector $\mathbf{V}$ is given by

$$\mathbf{V} = F_0 \mathbf{Q}^{-1} \mathbf{P} \quad (C.9)$$

where $\mathbf{Q}$ is a $K \times K$ square matrix and $\mathbf{P}$ is a $K \times 1$ vector, which can be written as

$$\mathbf{Q} = \begin{bmatrix} j\omega(C_{\bar{p}})_1 + \frac{1}{R_1} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega[(\tilde{\theta}_{mn})_1]^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} & \dots & \dots & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_1(\tilde{\theta}_{mn})_K}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_K(\tilde{\theta}_{mn})_1}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} & \dots & \dots & j\omega(C_{\bar{p}})_K + \frac{1}{R_K} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega[(\tilde{\theta}_{mn})_K]^2}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_1 \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \\ \vdots \\ -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{j\omega(\tilde{\theta}_{mn})_K \phi_{mn}(x_0, y_0)}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} \end{bmatrix} \quad (\text{C.10})$$

Therefore, the steady-state voltage output vector $\mathbf{v}(t)$ can be obtained as

$$\mathbf{v}(t) = \mathbf{u}^T F_0 \mathbf{Q}^{-1} \mathbf{P} e^{j\omega t} \quad (\text{C.11})$$

where $\mathbf{v}(t) = [v_1(t) \ v_2(t) \ \cdots \ v_K(t)]^T$ and the steady-state vibration response of an arbitrary point (x, y) can be written as

$$w(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn}(x, y) \frac{F_0 \phi_{mn}(x_0, y_0) + \boldsymbol{\theta}_{mn}^T F_0 \mathbf{Q}^{-1} \mathbf{P}}{\omega_{mn}^2 - \omega^2 + j2\zeta_{mn}\omega_{mn}\omega} e^{j\omega t}. \quad (\text{C.12})$$

which is identical to the expression given by equation (4.33).