

DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**IN – PLANE DYNAMIC STABILITY ANALYSIS
OF LAMINATED CURVED BEAMS**

by
Ali GÜNYAR

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İZMİR

IN – PLANE DYNAMIC STABILITY ANALYSIS OF LAMINATED CURVED BEAMS

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Graduate School of Natural and Applied Sciences of Dokuz Eylul University
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**by
Ali GÜNYAR**

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İZMİR**

M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled “**IN – PLANE DYNAMIC STABILITY ANALYSIS OF LAMINATED CURVED BEAMS**” completed by **ALİ GÜNYAR** under supervision of **PROF. DR. MUSTAFA SABUNCU** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

.....
Prof. Dr. Mustafa Sabuncu

Supervisor

.....

Jury Member

.....

Jury Member

Prof.Dr. Cahit HELVACI
Director

Graduate School of Natural and Applied Sciences

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IN – PLANE DYNAMIC STABILITY ANALYSIS OF LAMINATED CURVED BEAMS

ABSTRACT

In this study, the effects of variations of subtended angle, orientation angle and curvature of a laminated composite arc, having an in-plane curvature, on the natural frequencies, static and dynamic stability have been investigated by using the Finite Element Method. Sabir and Ashwell's displacement functions have been used to develop a finite element model to employ in this study. In-plane vibration and in-plane buckling analyses are also studied. In addition, the results obtained from this study are compared with the results obtained from Ansys for the fundamental natural frequency and critical buckling load. The effects of variations of subtended angle, orientation angle and curvature of curved beam and static and dynamic load parameters on the stability regions are shown in graphics. Moreover, analysis of natural frequencies, buckling and dynamic stability for curved composite beams are also compared.

Keywords : Dynamic stability, curved beam, composite, finite element method

KOMPOZİT EĞRİ ÇUBUKLARIN DÜZLEM İÇİ DİNAMİK STABİLİTE ANALİZİ

ÖZ

Bu çalışmada, düzlem içerisinde belirli bir eğriliğe sahip tabakalı kompozit dairesel bir yay parçasının merkez açısının, tabakalardaki fiber açılarının ve eğrilik yarıçapının değişiminin doğal frekans, statik ve dinamik stabiliteye etkileri sonlu elemanlar metodu kullanılarak araştırılmıştır. Bu analizde kullanılan sonlu eleman modelinin oluşturulması için Sabir ve Ashwell'in yerdeğiştirme fonksiyonları kullanılmıştır. Bunun yanında, düzlemiçi titreşim ve burkulma analizi yapılmıştır. Ayrıca birinci doğal frekans ve burkulma yükü değerleri Ansys'den elde edilen sonuçlarla karşılaştırılmış ve eğri çubuğun merkez açısının, tabakalardaki fiber açılarının, eğrilik yarıçapı değişimin ve, statik ve dinamik yük parametresinin kararlılık bölgeleri üzerindeki etkileri grafikler ile gösterilmiştir. Bunun yanında kompozit eğri çubuk için doğal frekans, kritik burkulma yükü ve dinamik kararlılık analizleri karşılaştırılmıştır.

Anahtar Sözcükler: Dinamik kararlılık, eğri çubuk, kompozit, sonlu elemanlar metodu.

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CHAPTER ONE

INTRODUCTION

The usage of curved beams at the high technology applications, especially in turbine blades, bridges and space industry it shows itself as a problem of elastic instability. Static and dynamic stability problems of laminated curved beams have been the subject of interest of several investigators due to its importance in many practical applications.

The problem of determining the natural frequencies and mode shapes of vibration of laminated curved beams is of importance in the design of bridge, automotive and outer space industry propellers. In recent years a new branch of the applied theory of elasticity has evolved, the theory of the dynamic stability of elastic systems. The problems which are examined in this branch of elasticity are related to those in the theory of vibrations and the stability of elastic systems.

The theory of dynamic stability has already opened the way for direct engineering applications. Parametrically excited vibrations are similar in appearance to the accompanying forced vibrations and can therefore qualify as ordinary resonance vibrations, by practical engineering standards. In a number of cases, however as in the presence of periodic vibrations the usual methods of damping on vibration isolation may break down and even bring about the opposite results. Although the vibrations may not threaten the structure or its normal operation, they can cause fatigue failure if they continue to act. Therefore, the study of the formation of parametric vibrations and the methods for the prevention of their occurrence is necessary in the various areas of mechanics, transportation and industrial construction.

Stability analysis of multiple degrees of freedom in parametric dynamic systems with periodic coefficients is a subject of recent interest, since the problem arises naturally in many physical situations; for example the dynamic stability of structures, under periodically varying loads, the stability of mechanisms with periodically

varying inertia and stiffness coefficients and the stability of the steady state response of nonlinear systems.

Many investigations about stability of curved beams have also been carried out.

Wu & Chiang (2003), investigated the natural frequencies and mode shapes for the radial bending vibrations of uniform circular arches by means of curved beam elements. The standard techniques were used to determine the natural frequencies and mode shapes for the curved beam with various boundary conditions and subtended angles.

Ken Susanto (2009), presented an analytical model of piezoelectric laminated slightly curved beams (PLSCB), which includes the computation of natural frequencies, mode shapes and transfer function formulation using the distributed transfer function method (DTFM).

Sabir and Ashwell (1971), discussed the natural frequency analysis of circular arches deformed in a plane. The finite elements developed by using different types of shape functions were employed in their analysis.

Petyt and Fleischer (1971), studied the free vibration of a curved beam under various boundary conditions.

Moshe and Efrahim (2001), investigated in-plane vibrations of shear-deformable curved beams. The exact dynamic stiffness matrix for a circular beam was used in this study.

Sabuncu and Erim (1987), studied in-plane vibration of a tapered curved beam. They presented a finite element model for the vibration analysis of a tapered curved beam. Linear and non-linear variations of cross-sections were considered in their analysis.

Kang, Bert and Striz (1996), discussed computation of the eigenvalues of the equations of motion governing the free in plane vibration including extensibility of the arch axis and the coupled out-of-plane twist-bending vibrations of circular arches using Differential Quadrature methods (DQM).

Kawakami, Sakiyama and Matsuda (1995), derived the characteristic equation by applying the discrete Green functions and using the numerical integration to obtain the eigenvalues for both the in-plane and out-of-plane free vibrations of the non-uniform curved beams, where the formulation is much complicated than that of the classical approaches.

Liu & Wu (2001), investigated in-plane free vibrations of circular arches using the generalized differential quadrature rule. The Kirchhoff assumptions for the these beams were considered and the neutral axis was taken as inextensible. They presented several examples of arches with uniform, continuously varying and stepped cross sections.

Yang & Kuo (1987), derived the nonlinear differential equations of equilibrium for a horizontally curved I-beam. Based on the principal of virtual displacements, the equilibrium of a bar was established for its deformed or buckled configuration using a Lagrangian approach.

Şakar, Öztürk & Sabuncu (2001), investigated the effects of subtended angle and curvature of an arch, variations on the in-plane natural frequencies, static and dynamic instability.

Chidamparam and Leissa (1993), organized and summarized the extensive published literature on the vibrations of curved bars, beams, rings and arches of arbitrary shape which lie in a plane. They also considered in-plane, out-of-plane and coupled vibration and examined various theories that have been developed to model curved beam vibration problems. In addition, they have studied the free vibrations of circular arches about a prestressed static equilibrium state.

X. Tong, N. Mrad and B. Tabarrok (1998), investigated free and forced in-plane vibrations of circular arches with variable cross-sections and various boundary conditions.

Sabuncu (1978), investigated the vibration analysis of thin curved beams. He used several types of shape functions to develop different curved beam finite elements and pointed out the effect of displacement functions on the natural frequencies by comparing the results.

Huang (2003), marked on in-plane free vibration and static stability of loaded and shear-deformable circular arches.

F. Yang, R. Sedaghati and E. Esmailzadeh (2008), derived the governing differential equations for the free in-plane vibration of uniform and non-uniform curved beams with variable curvatures, including the effects of the axis extensibility, shear deformation and the rotary inertia, by using the extended-Hamilton principle. These equations were then solved numerically utilizing the Galerkin finite element method and the curvilinear integral taken along the central line of the curvilinear beam.

Vlasov (1961), presented a dynamic theory of curved thin-walled beams, which was used successfully in several applications.

Snyder and Wilson (1992), used Vlasov's theory to study the dynamics of continuous curved thin-walled beams. They solved the equations by means of a closed-form solution, in order to provide numerical information for these structural members, which may be considered as a first approach in the design of highway, rail, rapid transit and guideway structures.

DiTaranto (1973), investigated free and forced vibrations of a three-layer damped ring.

Sagartz (1977), analysed Transient Response for three-layer rings.

Tatemichi A., Okazaki A. and Hikyama M. (1980), studied damping properties of curved sandwich beams with viscoelastic layer.

Qatu M. S. (1992), developed a consistent set of equations for slightly curved laminated beams and he obtained results for such beams having different boundary conditions.

P. Malekzadeh and A.R. Setoodeh (2009), presented a differential quadrature (DQ) solution for moderately thick laminated circular arches with general boundary conditions. The governing equations are based on the Reissner–Naghdi type shell theory, which include the effects of transverse shear deformation and rotary inertia.

C.-C. Lin and C.-S. Tseng (1998), dealt with the free vibration analysis of polar orthotropic laminated circular and annular plates. The first order shear deformation theory and the variational energy method are employed in the mathematical formulation, and an eight-node isoparametric finite element model in polar coordinates is used for finding natural frequencies. The effects of material property, stacking sequence, hole size, plate thickness to radius ratio and boundary conditions on natural frequencies are investigated.

Y. -P. Tseng, C. S. Huang and C. -J.- Lin (1997), studied a systematic approach to solve in-plane free vibrations of arches with variable curvature. The proposed approach basically introduces the concept of dynamic stiffness matrix into a series solution for in-plane vibrations of arches with variable curvature.

T. Sakiyama, M. Huang, H. Matsuda and C. Morita (2003), investigated approximate method for analyzing the free vibration of orthotropic right cantilever triangular plate. By adding an extremely thin part, a cantilever triangular plate can be translated into an equivalent rectangular plate with non-uniform thickness. Therefore, the free vibration characteristics of the triangular plate can be obtained by analyzing the equivalent rectangular plate.

Q. Lü and C.F. Lü (2008), discussed exact analysis on the in-plane free vibration of simply supported laminated circular arches based on the two-dimensional theory of elasticity.

P. Malekzadeh, A.R. Setoodeh and E. Barmshouri (2008), introduced an accurate and efficient solution procedure based on the two-dimensional elasticity theory for free vibration of arbitrary laminated thick circular deep arches with some combinations of classical boundary conditions.

Sudhakar R. Marur and Tarun Kant (2008), presented a higher-order refined model with seven degrees of freedom per node for the free vibration analysis of composite and sandwich arches. The strain field is modeled through cubic axial, cubic transverse shear and linear transverse normal strain components.

M. Ganapathi, B.P. Patel, J. Saravanan and M. Touratier (1997), studied the nonlinear free flexural vibrations of isotropic/laminated orthotropic straight/curved beams by using a cubic B-spline shear flexible curved element, based on the field consistency principle.

R. Emre Erkmén and Mark A. Bradford (2009), developed a novel 3D elastic total Lagrangian formulation for the numerical analysis of steel-concrete composite beams which are curved in-plan. Geometric nonlinearities are considered in the derivation of the strain expressions, and the partial interaction at the interface in the tangential direction as well as in the radial direction due to flexible shear connectors is incorporated in the unique proposed formulation, which is derived from considerations of fundamental engineering mechanics.

A. A. Khdeir and J. N. Reddy (1996), developed a model for the dynamic behavior of a laminated composite shallow arch from shallow shell theory. Linear equations of motion are derived for thin, moderately thick and thick arches. Free vibration of the arch is explored and exact natural frequencies of the third-order,

second-order, first-order and classical arch theories are determined for various boundary conditions.

Bazant and Cedolin (1991), discussed the buckling analysis of curved beams by using analytical and energy methods.

Timoshenko and Gere (1961), examined the buckling analysis of hinged-hinged Bernoulli-Euler curved beams by using the analytical method.

Yoo, Kang and Davidson (1996), performed buckling analysis of curved beams with the finite element method.

Loja, Barbosa ve Soares (1997), discussed static, buckling and dynamic behaviour of laminated composite beams by using a higher-order discrete model (HSDT) for static and dynamic analysis.

Öztürk, Yeşilyurt and Sabuncu (2006), investigated in-plane stability analysis of non-uniform cross-sectioned thin curved beams under uniformly distributed dynamic loads by using the Finite Element Method. In this study, two different finite element models, representing variations of cross-section, were developed by using simple strain functions in the analysis.

Banan, Karami and Farshad (1990), discussed Finite Elements analysis for the stability analysis of curved beams on elastic foundation.

Xin-Yan Tsai and Lien-Wen Chen (2002), discussed the dynamic stability problem of a shape memory alloy reinforced composite beam subjected to an axial periodic dynamic force.

Aditi Chattopadhyay and Adrian G. Radu (2000), used a higher order shear deformation theory to investigate the instability associated with composite plates subject to dynamic loads. Both transverse shear and rotary inertia effects are taken

into account. The procedure is implemented using the finite element approach. The natural frequencies and the critical buckling load are computed and compared with the results based on the classical laminate plate theory and the first-order shear deformation theory.

B. Kovacs (2001), investigated a new laminate model for the dynamic analysis of a laminated circular ring segment. The differential equations which govern the free vibrations of a circular ring segment and the associated boundary conditions are derived by Hamilton's principle considering bending and shear deformation of all layers.

In this thesis in-plane stability analysis of laminated curved beams is investigated by using the Finite Element Method. First unstable regions are examined. Since natural frequency and buckling load effect the determination of stability regions, the in-plane vibration and in-plane buckling analyses are also studied. In addition, the results obtained from this study are compared with the results obtained from Ansys for the fundamental natural frequency and critical buckling load. The effects of subtended angle and dynamic load parameter on the stability regions are shown in graphics.

Chapter 2 contains essential background on the composite materials.

Chapter 3 deals with the theories used to analyze the dynamic stability of elastic systems.

Chapter 4 is regarding the theories used to analyze the finite element method.

Chapter 5 contains the finite element model of a laminated curved beam. A thin beam theory is applied by neglecting shear deformation and rotary inertia effects. The geometric stiffness, elastic stiffness and mass matrices of curved beams, representing stability and vibrations in-plane direction, are obtained by using the energy equations.

Chapter 6 covers the effects of the subtended angle, orientation angle and curved beam dimensions on the vibration and, static (buckling analysis) and dynamic stability. All results are presented in tabular and/or graphical forms. The obtained results are discussed.

Finally in chapter 7, conclusions drawn from the study are presented.

CHAPTER TWO

COMPOSITE MATERIALS

2.1 Introduction

Traditional engineering materials (steel, aluminum, etc.) contain impurities that can represent different phases of the same material and fit the broad definition of a composite, but are not considered composites because the elastic modulus or strength of the impurity phase is nearly identical to that of the pure material. The definition of a composite material is flexible and can be augmented to fit specific requirements. In this text a composite material is considered to be one that contains two or more distinct constituents with significantly different macroscopic behavior and a distinct interface between each constituent (on the microscopic level). This includes the continuous fiber laminated composites of primary concern herein, as well as a variety of composites not specifically addressed.

2.2 Classification of Composite Materials

There are four types of composite materials; polymer (PMC), metal (MMC), ceramic (CMC), and carbon (CAMC) matrix composites. The carbon-carbon matrix composite (CCC) is the most important type of CAMCs. The matrix and fiber materials that can be mixed to compose composite material have shown in Table 2.1.

Table 2.1 Types of composite materials. (Harper, 2004)

Reinforcement				
Matrix	Polymer	Metal	Ceramic	Carbon
Polymer	√	√	√	√
Metal	√	√	√	√
Ceramic	√	√	√	√
Carbon	√	√	√	√

Polymer matrix composites (PCM) include thermoset (epoxy, polyimide, polyester) or thermoplastic (poly-ether-ether-ketone, polysulfone) resins reinforced with glass, carbon (graphite), aramid (Kevlar), or boron fibers. They are used primarily in relatively low temperature applications.

Metal matrix composites (MMC) consist of metals or alloys (aluminum, magnesium, titanium, copper) reinforced with boron, carbon (graphite), or ceramic fibers. Their maximum use temperature is limited by the softening or melting temperature of the metal matrix. The principal motivation was to dramatically extend the structural efficiency of metallic materials while retaining their advantages, including high chemical inertness, high shear strength, and good property retention at high temperatures.

Ceramic matrix composites (CMC) consist of ceramic matrices (silicon carbide, aluminum oxide, glass-ceramic, silicon nitride) reinforced with ceramic fibers. They are best suited for very high temperature applications. Ceramic-matrix composite development has continued to focus on achieving useful structural and environmental properties at the highest operating temperatures.

Carbon/carbon composites (CCC) consist of carbon or graphite matrix reinforced with graphite yarn or fabric. They have unique properties of relatively high strength at high temperatures coupled with low thermal expansion and low density. (Daniel & Ishai, 1994).

In another way, composite materials can be regrouped according to their appearance rather than their matrix. They are:

Fibrous composites: Obtained by putting long fiber groups or whiskers into a matrix with a predefined angle (Figure 2.1) or in random order, and curing at a specified temperature. The fibers can be straight or woven, and they can be continuous or discontinuous (Figure 2.2).

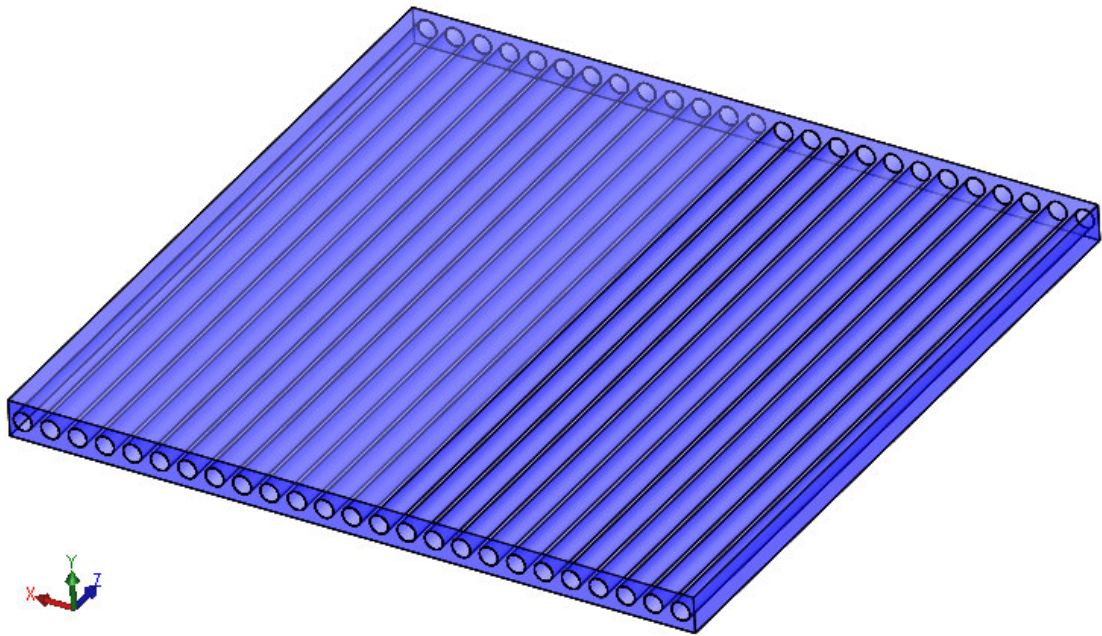


Figure 2.1 A lamina with longitudinal fibers.(Emre Erbil, 2008)

There is a difference between fibers and whiskers. Fiber is characterized geometrically not only by its very high length-to-diameter ratio but by its near-crystal-sized diameter. A whisker has essentially the same near-crystal-sized diameter as a fiber, but generally is very short and stubby, although the length-to-diameter ratio can be in the hundreds. Thus, a whisker is an even more obvious example of the crystal-bulk-material-property-difference paradox. That is, a whisker is even more perfect than a fiber and therefore exhibits even higher properties. Whiskers are obtained by crystallization on a very small scale resulting in a nearly perfect alignment of crystals. Materials such as iron have crystalline structures with a theoretical strength of 2900000 psi (20 GPa), yet commercially available structural steels, which are mainly iron, have strengths ranging from 75000 psi to about 100000 psi (570 to 690 MPa). (Jones, 1999)

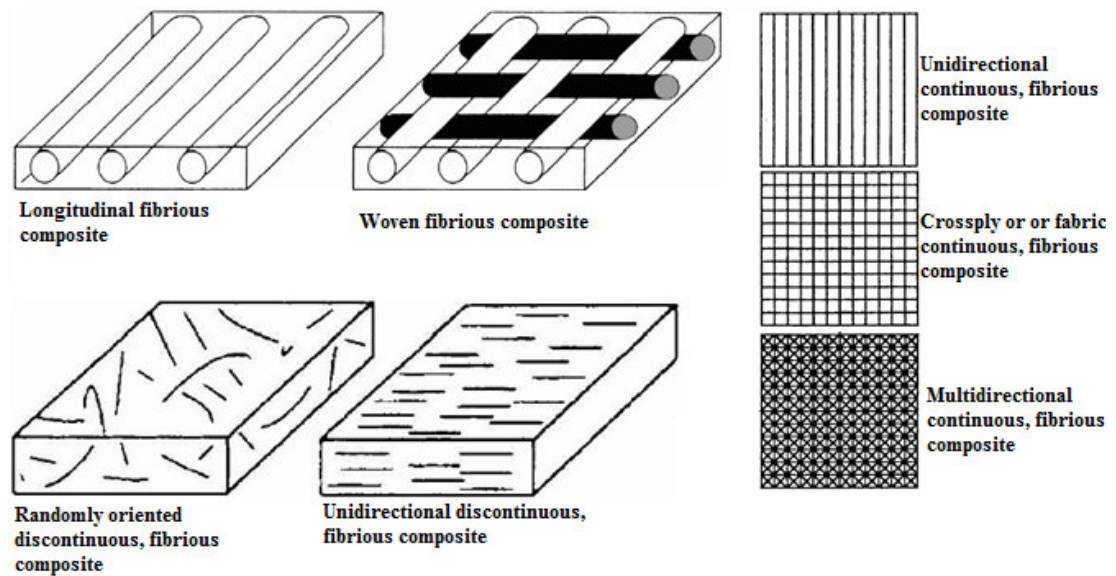


Figure 2.2 Types of fibrous composites.

Laminated composites: Two or more composite layers are bonded together. (Figure 2.3) Lamination achieves the mechanical properties in composite. Mechanical properties can be changed with the angle of fibers in laminate.

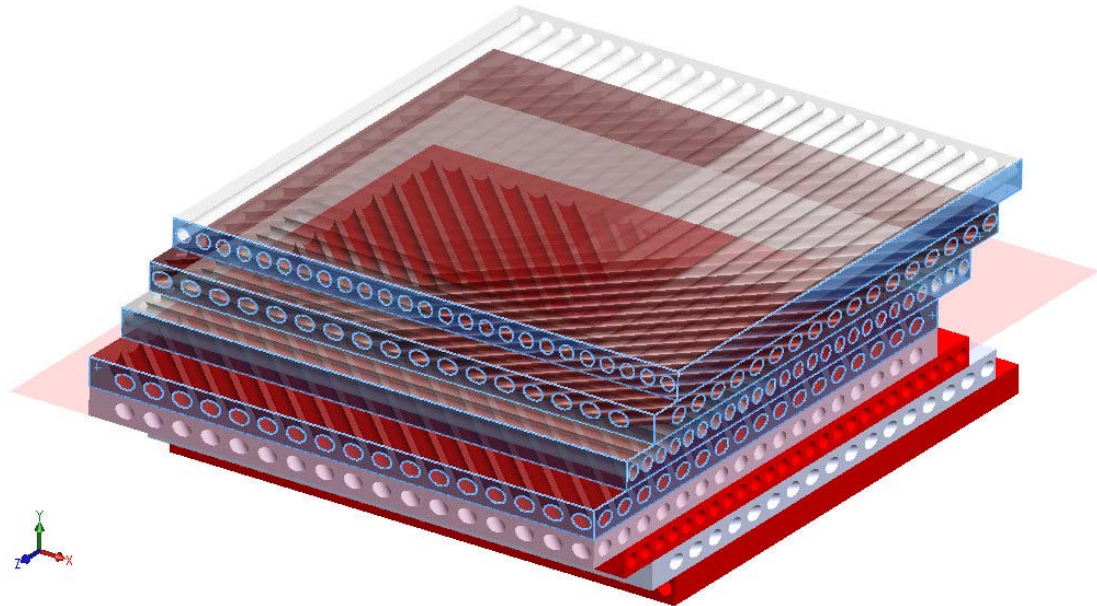


Figure 2.3 A laminated composite with the fiber angle of $[90/+45/0/-45]_s$. (Emre Erbil, 2008)

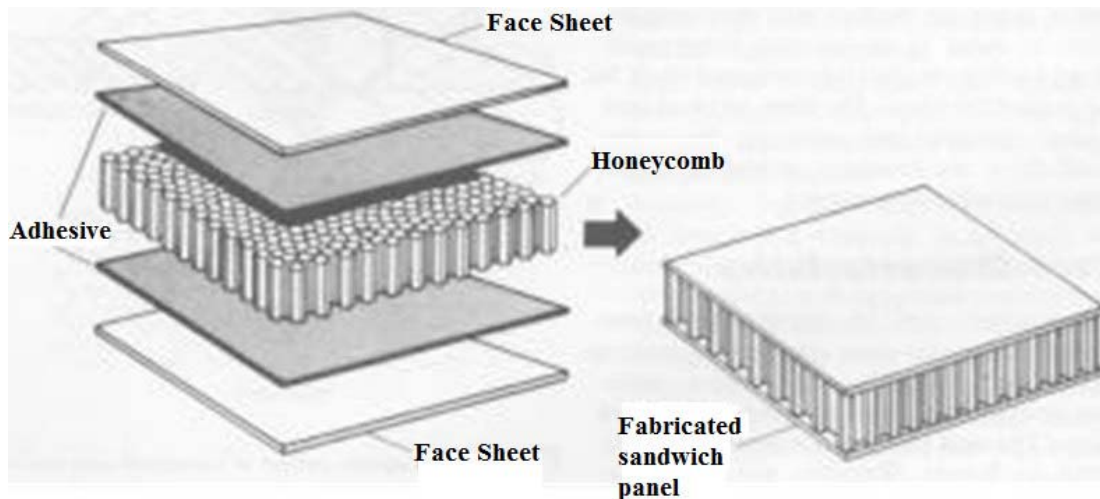


Figure 2.4 Honeycomb sandwich construction. (www.mdacomposites.org, 2008)

Laminated composites can be in the sandwich structural form. Structural sandwich is a layered composite formed by bonding two thin facings to a thick core (Figure 2.4). It is a type of stressed-skin construction in which the facings resist nearly all of the applied edgewise (in-plane) loads and flat wise bending moments. The thin spaced facings provide nearly all of the bending rigidity to the construction. The core spaces the facings and transmits shear between them so that they are effective about a common neutral axis. The core also provides most of the shear rigidity of the sandwich construction. By proper choice of materials for facings and core, constructions with high ratios of stiffness to weight can be achieved.

Laminated composites can be different from fiber-matrix form. For instance, bimetals, clad metals, laminated glass or plastic-based laminates are the laminated composites, too. Bimetals consists of two different types of metals that are bonded together. In this form, the advantage of thermal expansion coefficient differences between the metals that forms the bimetal can be used.

Clad metals obtained by sheathing a metal with another. For instance, copper wires can be clad with aluminum. So it becomes lighter and durable in fatigue loading. Other example is a laminated glass. Glass is brittle and can break into many sharp pieces. To solve this problem, a thin plastic film can clad with glass. So stiffness and durability in laminated glass can be gained. (Jones, 1999)

Particulate composites: They consist of particles immersed in matrices such as alloys and ceramics. Particulate composites can group into nonmetallic particles in nonmetallic matrix composites, metallic particles in nonmetallic matrix composites, metallic particles in metallic matrix composites and nonmetallic particles in metallic matrix composites. They are usually isotropic because the particles are added randomly. Particulate composites have advantages such as improved strength, increased operating temperature, oxidation resistance, etc. Typical examples include use of aluminum particles in rubber; silicon carbide particles in aluminum; and gravel, sand, and cement to make concrete. (Kaw, 2006) There are three types of particulate composites as shown in Figure 2.5.

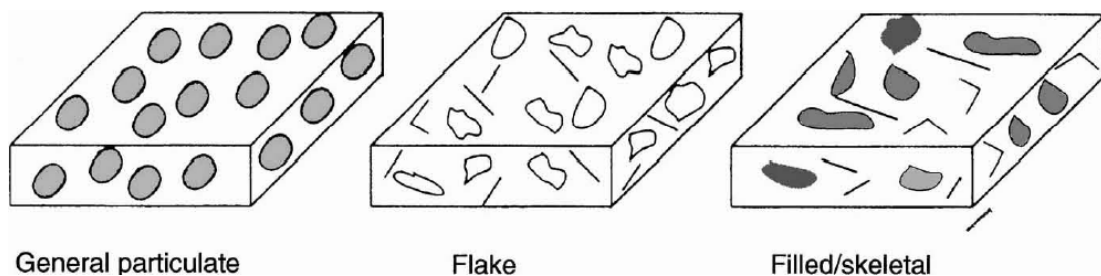


Figure 2.5 Types of particulate composites. (Staab, 1999)

Flake: A flake composite is generally composed of flakes with large ratios of platform area to thickness, suspended in a matrix material (particle board, for example).

Filled/Skeletal: A filled/skeletal composite is composed of a continuous skeletal matrix filled by a second material: for example, a honeycomb core filled with an insulating material. (Staab, 1999)

Combinations of composite materials: They are the mixture of fibrous, laminated, or particulate composites. In this classification method, this type can conflict with the two or three other types of composite classes. For instance, fiberreinforced concrete is both particulate (the composite is composed of gravel in a cement-paste binder) and fibrous (due to the steel reinforcement). And also, laminated fiber-reinforced composite materials are obviously both laminated and

fibrous composite materials. Thus, any classification system is arbitrary and imperfect.

2.3 Applications of Composite Materials

Composite materials have lots of applications in industry. Commercial and industrial applications of fiber-reinforced polymer composites are so varied that it is impossible to list them all. In this study, only the major structural application areas, which include aircraft, space, automotive, sporting goods, marine, and infrastructure will be discussed. Fiber-reinforced polymer composites are also used in electronics (e.g., printed circuit boards), building construction (e.g., floor beams), furniture (e.g., chair springs), power industry (e.g., transformer housing), oil industry (e.g., offshore oil platforms and oil sucker rods used in lifting underground oil), medical industry (e.g., bone plates for fracture fixation, implants, and prosthetics), and in many industrial products, such as stepladders, oxygen tanks, and power transmission shafts. Potential use of fiber-reinforced composites exists in many engineering fields. Putting them to actual use requires careful design practice and appropriate process development based on the understanding of their unique mechanical, physical, and thermal characteristics.

Aerospace Applications: The major structural applications for fiber-reinforced composites are in the field of military and commercial aircrafts, for which weight reduction is critical for higher speeds and increased payloads. With the introduction of carbon fibers in the 1970s, carbon fiber-reinforced epoxy has become the primary material in many wing, fuselage, and empennage components. The structural integrity and durability of these early components have built up confidence in their performance and prompted developments of other structural aircraft components, resulting in an increasing amount of composites being used in military aircrafts. The outer skin of B-2 (Figure 2.6) and other stealth aircrafts is almost all made of carbon fiber-reinforced polymers. The stealth characteristics of these aircrafts are due to the use of carbon fibers, special coatings, and other design features that reduce radar reflection and heat radiation.



Figure 2.6 The B-2 stealth bomber aircraft, which is made of advanced composite materials.
(Miracle & Donaldson, 2001)

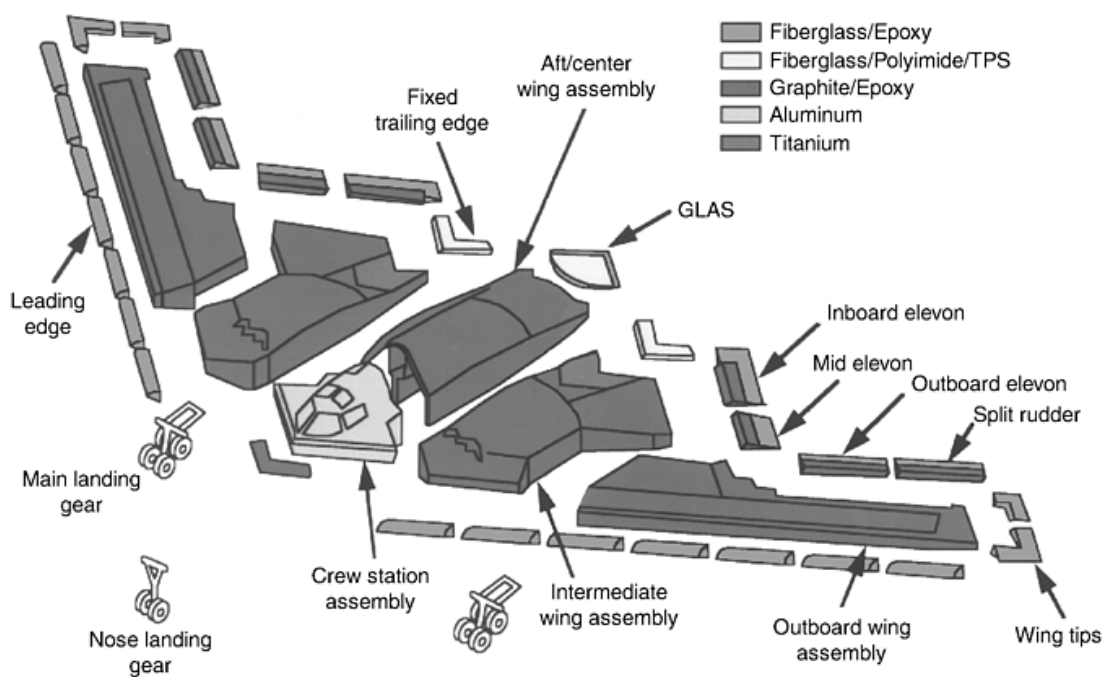


Figure 2.6 The B-2 stealth bomber aircraft, which is made of advanced composite materials.
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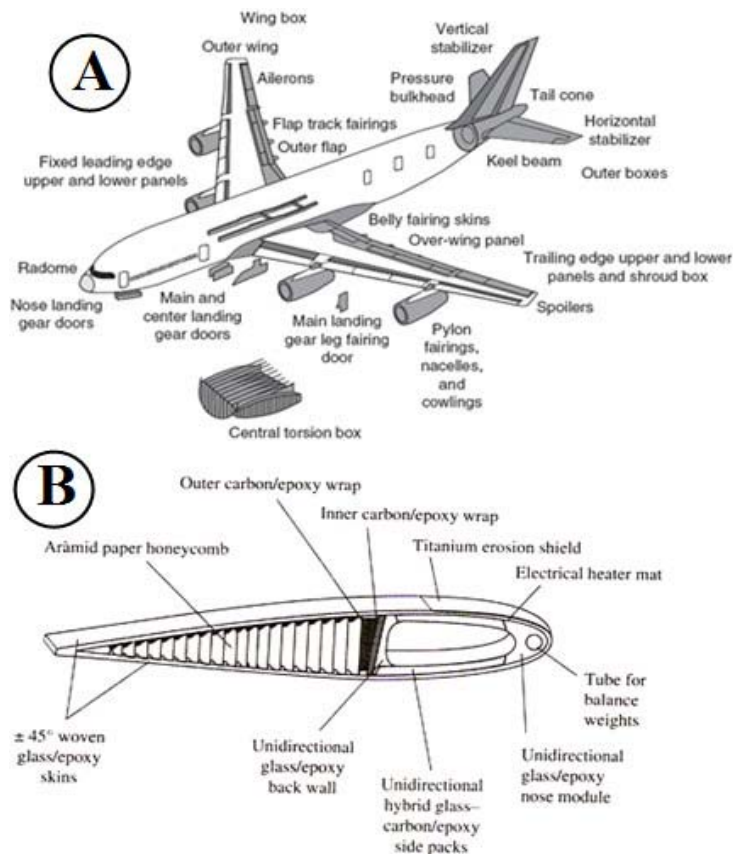


Figure 2.7 Use of fiber-reinforced polymer composites in Airbus 380 (A)(Mallick, 2007) and a helicopter rotor blade section (B) (Gibson,1994).

Advanced composites are used in air defense and civil applications that shown in Figure 2.7 (A). And in Figure 2.7 (B), composite construction of a helicopter rotor blade has shown.

Space applications: Among the various applications in the structure of space shuttles are the mid-fuselage truss structure (boron fiber-reinforced aluminum tubes), payload bay door (sandwich laminate of carbon fiber-reinforced epoxy face sheets and aluminum honeycomb core), remote manipulator arm (ultrahigh-modulus carbon fiber-reinforced epoxy tube), and pressure vessels (Kevlar 49 fiber-reinforced epoxy). Fiber-reinforced polymers are used for support structures for many smaller components, such as solar arrays, antennas, optical platforms, and so on. A major factor in selecting them for these applications is their dimensional stability over a wide temperature range. A space application of composites has shown in Figure 2.8.

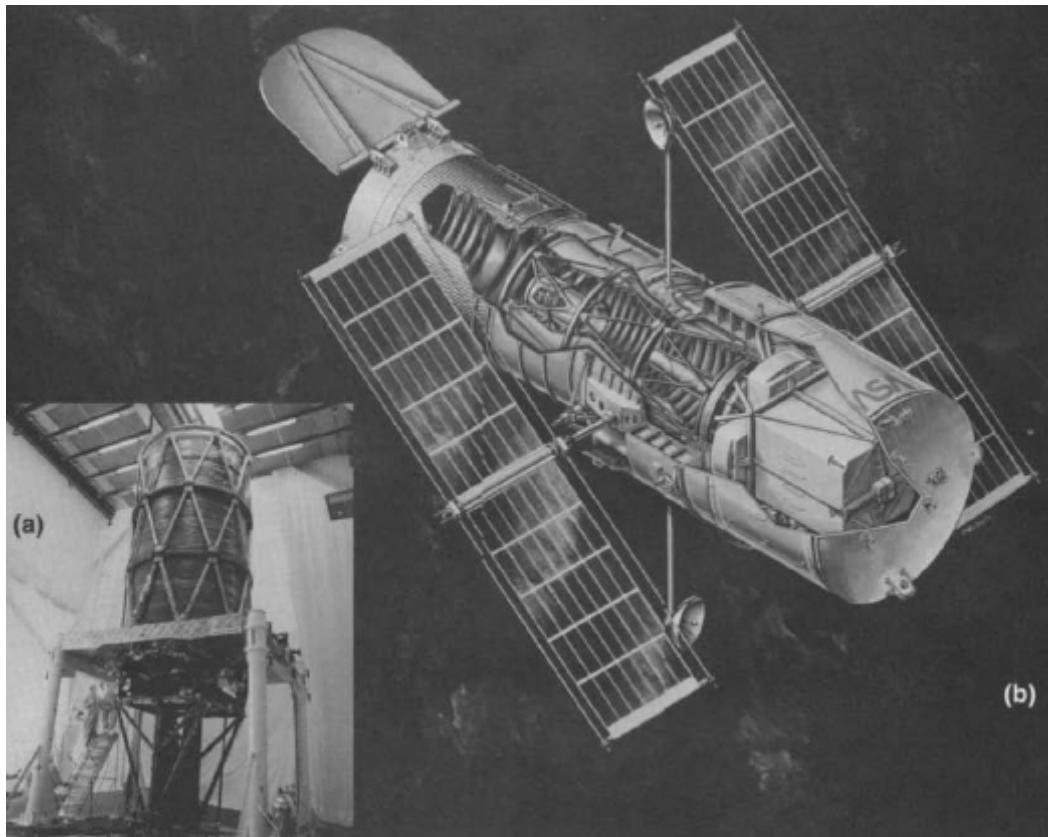


Figure 2.8 Composite structural truss (a) aboard Hubble Space - Telescope (b) aligns primary and secondary mirrors. (Peters, 1998)

Automotive applications: Applications of fiber-reinforced composites in the automotive industry can be classified into three groups; body components, chassis components, and engine components. Composites have proven to be very successful in a wide range of exterior body panels and are used in hundreds of vehicle applications. Exterior body components, such as the hood or door panels, require high stiffness and damage tolerance (dent resistance) as well as a “Class A” surface finish for appearance. Excellent surface finish, light weight, and a thermal coefficient of expansion near that of steel have made these applications successful. Customers appreciate the dent and corrosion resistance of composite panels. The composite material used for these components is E-glass fiber-reinforced sheet molding compound (SMC) composites. Another manufacturing process for making composite body panels in the automotive industry is called the structural reaction injection molding (SRIM). The fibers in these parts are usually randomly oriented discontinuous E-glass fibers and the matrix is a polyurethane or polyurea.

Sporting goods applications: Tennis rackets, golf club shafts, fishing rods, bicycle frames, snow and water skis helmets, athletic shoe soles and heels, and the most of the other sporting goods are made of composites. Fiber-reinforced polymers are extensively used in sporting goods and are selected over such traditional materials as wood, metals, and leather in many of these applications. The advantages of using fiber-reinforced polymers are weight reduction, vibration damping, and design flexibility.

Marine applications: Use of composites in marine applications is widespread. The two major advantages of fiber reinforced plastics over metals are resistance to the marine environment, particularly the elimination of galvanic corrosion and the ease of tailoring structures, which are fabricated by molding processes. In addition, composites have high strength-to-weight ratios. (Peters, 1998) Glass fiber-reinforced polyesters have been used in different types of boats (e.g., submarines, sail boats, fishing boats, dinghies, life boats, and yachts) ever since their introduction as a commercial material in the 1940s. Today, nearly 90% of all recreational boats are constructed of either glass fiber-reinforced polyester or glass fiber-reinforced vinyl ester resin.

Infrastructure: Fiber-reinforced polymers have a great potential for replacing reinforced concrete and steel in bridges (Figure 2.9), buildings, and other civil infrastructures. The principal reason for selecting these composites is their corrosion resistance, which leads to longer life and lower maintenance and repair costs.



Figure 2.9 An epoxy/glass fiber-reinforced construction of a bridge. (www.bceo.org, 2008)

CHAPTER THREE

THEORY OF STABILITY ANALYSIS

3.1 Static Stability

The modern use of steel and high-strength alloys in engineering structures, especially in bridges, ships and aircraft, has made elastic instability a problem of great importance. Urgent practical requirements have given rise in recent years to extensive theoretical investigations of the conditions governing the stability of beams, plates and shells.

The first problems of elastic instability, concerning lateral buckling of compressed members, were solved about 200 years ago by L. Euler. At that time the relatively low strength of materials necessitated stout structural members for which the question of elastic stability is not of primary importance. Thus Euler's theoretical solution, developed for slender bars, remained for a long time without a practical application. Only with the beginning of extensive steel constructions did the question of buckling of compression members become of practical importance. The use of steel led naturally to types of structures embodying slender compression members, thin plates and thin shells.

Stability problems can be treated in a general manner using the energy methods. As an introduction to such methods, the basic criteria for determining the stability of equilibrium is derived in this study for, conservative linearly elastic systems.

To establish the stability criteria, a function Π , called the potential of the system must be formulated. This function is expressed as the sum of the internal potential energy U (strain energy) and the potential energy Λ of the external forces that act on a system, i.e.,

$$\Pi = U + \Lambda \tag{3.1}$$

Disregarding a possible additive constant, $\Lambda = -W_e$, i.e., the loss of potential energy during the application of forces is equal to the work done on the system by external forces. Hence, equation (3.1) can be rewritten as

$$\Pi = U - W_e \quad (3.2)$$

As is known from classical mechanics, for equilibrium the total potential Π must be stationary, therefore its variation $\delta\Pi$ must equal zero,

$$\delta\Pi = \delta U - \delta W_e = 0 \quad (3.3)$$

For conservative, elastic systems this relation agrees with $\delta W_e = \delta W_{ei}$ equation (δW_{ei} : the external work on the internal elements of a body, δW_e : the total work), which states the virtual work principle. This condition can be used to determine the position of equilibrium. However, equation (3.3) cannot discern the type of equilibrium and there by establish the condition for the stability of equilibrium. Only by examining the higher order terms in the expression for increment in Π as given by Taylor's expansion must be examined. Such an expression is

$$\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta^2\Pi + \frac{1}{3!}\delta^3\Pi + \dots \quad (3.4)$$

Since for any type of equilibrium $\delta\Pi = 0$, it is the first nonvanishing term of this expansion that determines the types of equilibrium. For linear elastic systems the second term suffices. Thus, from equation (3.4), the stability criteria are

$$\delta^2\Pi > 0 \quad \text{for stable equilibrium}$$

$$\delta^2\Pi < 0 \quad \text{for unstable equilibrium}$$

$$\delta^2\Pi = 0 \quad \text{for neutral equilibrium associated with the critical load}$$

The meaning of these expressions may be clarified by examining the simple example shown in figure 3.1, where the shaded surfaces represent three different types of Π functions. It can be concluded at once that the ball on the concave spherical surface (a) is in stable equilibrium, while the ball on the convex spherical surface (b) is in unstable equilibrium. The ball on the horizontal plane (c) is said to be in different or neutral equilibrium. The type of equilibrium can be ascertained by considering the energy of the system. In the first case (figure 3.1(a)) any displacement of the ball from its position of equilibrium will raise the center of gravity. A certain amount of work is required to produce such a displacement; thus the potential energy of the system increases for any small displacement from the position of equilibrium. In the second case (figure 3.1 (b)), any displacement from the position of equilibrium will decrease the potential energy of the system. Thus in the case of stable equilibrium the energy of the system is a minimum and in the case of unstable equilibrium it is a maximum. If the equilibrium is indifferent (figure 3.1 (c)), there is no change in energy during a displacement.

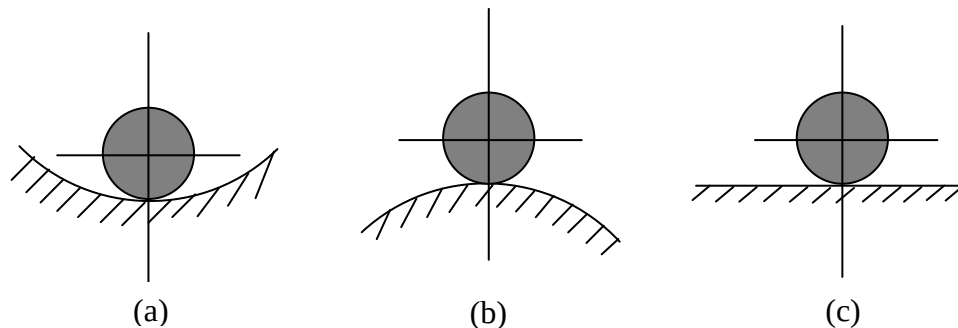


Figure 3.1 Three cases of equilibrium

For each of the systems shown in figure 3.1 stability depends only on the shape of the supporting surface and does not depend on the weight of the ball. In the case of a compressed column or plate it is found that the column or plate may be stable or unstable, depending on the magnitude of the axial load.

3.1.1 The Formulation of Static Stability

If the displacements are large, then the deformed geometry will obviously differ significantly from the undeformed geometry. This results in a nonlinear strain-displacement relationship. Large displacement problems of this type are said to be “geometrically nonlinear” which is a feature of elastic instability problems. From the design point of view calculation of the critical loads of structures is of considerable importance. In general case the strain energy of a system,

$$U = \frac{1}{2} \{q\}^T [K_e] \{q\} \quad (3.5)$$

The additional strain energy which is function of applied external load

$$U_g = \frac{1}{2} \{q\}^T [K_g] \{q\} \quad (3.6)$$

In which $[K_e]$ and $[K_g]$ are elastic stiffness and geometric stiffness matrices.

The total potential energy of a system in equilibrium is constant when small displacements are given to the system. So

$$\delta(U + U_g) = 0 \quad (3.7)$$

$(U + U_g)$ and δ define the total potential energy and the change of the virtual displacements. Applying the above formulation to equations (3.5) and (3.6)

$$[[K_e] - P[K_g]] \{q\} = 0 \quad (3.8)$$

The roots of the eigenvalue equation (3.8) gives the buckling loads and the eigenvectors of this equation are the buckling mode shapes.

3.2 Dynamic Stability

If the loading is nonconservative the loss of stability may not show up by the system going into another equilibrium state but by going into unbounded motion. To encompass this possibility we must consider the dynamic behavior of the system because stability is essentially a dynamic concept.

Whenever static loading of a particular kind causes a loss of static stability, vibrational loading of the same kind will cause a loss of dynamic stability. Such a loading is characterized by the fact that it is contained as a parameter on the left hand side of the equations of perturbed equilibrium (or motion). We will call such loading parametric; this term is more appropriate because it indicates the relation to the phenomenon of parametric resonance.

In the mechanical systems, parametric excitation occurs due to the following reasons;

- a) periodic change in rigidity
- b) periodic change in inertia
- c) periodic change in the loading of the system.

In this section firstly the differential equation related with dynamic stability is introduced and then, the determination of boundaries of the regions of instability and the amplitudes of parametrically excited vibrations for multi-degrees of freedom systems is presented.

An important special case of linear variational equations with variable coefficients occurs when the coefficient functions are periodic. Owing to their great practical importance in the theory of vibrations, a special theory has even been developed for the systems of differential equations with periodic coefficients are known as Mathieu-Hill differential equation. The Hill differential equation is in the following form,

$$\ddot{y} + [a - b f(t)]y = 0 \quad (3.9)$$

in which a and b are constant parameters, and $f(t)$ is a function having the period T . The prime denotes differentiation with respect to time. If $f(t) = 2 \cos 2t$ substituted into the Hill differential equation, the Mathieu differential equation which may be described a system that is subjected to parametric excitation is obtained in the standard form as

$$\ddot{y} + [a - 2b \cos 2t]y = 0 \quad (3.10)$$

The results of solving Mathieu's equation (3.10) for two different combinations of a and b are shown in figure 3.2. Although the parameter b of the system is the same in both cases ($b=0,1$), the vibrations are greatly different because of the difference between the values of the parameter a ($a=1$; $a=1,2$). In the first case, they increase, i.e., the system is dynamically unstable, while in the second case they remain bounded, i.e., the system is dynamically stable.

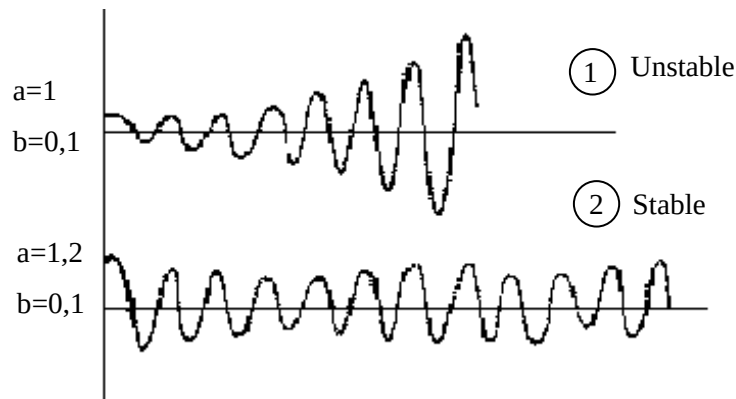


Figure 3.2 Two solutions of Mathieu's equation

The greatest importance, for practical purpose, is attached to the boundaries between the regions of stable and unstable solutions. This problem has been well studied, and the final results have been presented in the form of a diagram plotted in the plane of the parameters a and b . It is called the Haines-Strett diagram. Figure 3.3 shows part of a Haines-Strett diagram for small values of the parameter b . Any given system having the parameters a and b corresponds to the point with the co-ordinates

a and b on the Haines-Strett diagram. If the representative point is in the shaded parts of the diagram, the system is dynamically unstable, while stable systems correspond to representative points in the unshaded parts. The shaded regions are called the regions of dynamic instability.

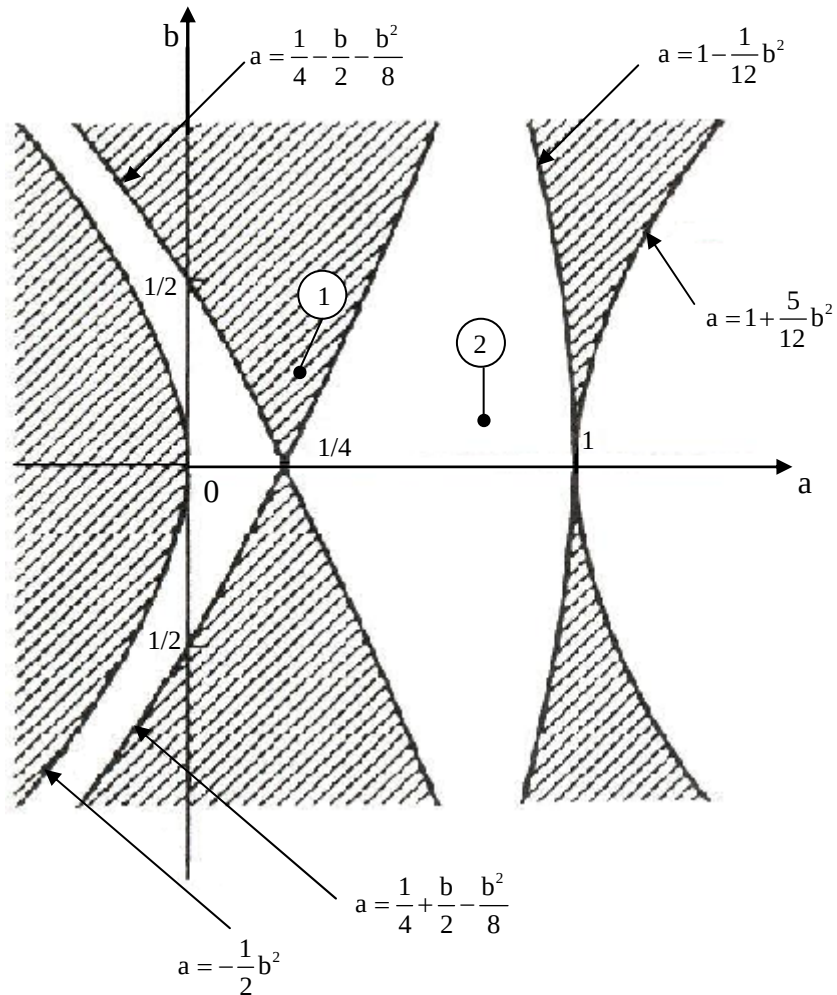


Figure 3.3 Part of Haines-Strett diagram the points ① and ② correspond to the solutions 1 and 2 in figure 3.2

As an example, the diagram in figure 3.3 shows the points 1 and 2 corresponding to the parameter $a_1=1$ and $b_1=0,1$, and $a_2=1,2$ and $b_2=0,1$. The point 1 is in the region of dynamic instability and the vibration occurs with increasing amplitude as shown in figure 3.2. The point 2 is in the stable region and it corresponds to motion with a limited amplitude.

3.2.1 The Formulation of Dynamic Stability

The matrix equation for the free vibration of an axially loaded system can be written as:

$$[M]\{\ddot{q}\} + [K_e]\{q\} - [K_g]\{q\} = 0 \quad (3.11)$$

where

$\{q\}$ is the generalized coordinates

$[M]$ is the inertia matrix

$[K_e]$ is the elastic stiffness matrix

$[K_g]$ is the geometric stiffness matrix, which is a function of the compressive axial load $P(t)$.

For a system subjected to a periodic force

$$P(t) = P_o + P_t f(t) \quad (3.12)$$

The static and time dependent components of the load can be represented as a fraction of the fundamental static buckling load P^* , in which $P_o = \alpha P^*$, $P_t = \beta_d P^*$. By writing $P = \alpha P^* + \beta_d P^* f(t)$ then the matrix equation K_g becomes

$$K_g = \alpha P^* [K_{gs}] + \beta_d P^* [K_{gt}] \quad (3.13)$$

where the matrices $[K_{gs}]$ and $[K_{gt}]$ reflect the influence of P_o and P_t respectively.

Substituting equation (3.13) into equation (3.11), the following system of n second order differential equations with a periodic coefficient of the known Mathieu-Hill type is obtained;

$$[M]\{\ddot{q}\} + [[K_e] - \alpha P^* [K_{gs}] - \beta_d P^* f(t) [K_{gt}]]\{q\} = 0 \quad (3.14)$$

$f(t)$ is a periodic function with period T . Therefore

$$f(t+T)=f(t) \quad (3.15)$$

Equation (3.14) is a system of n second order differential equations which may be written as

$$\{\ddot{q}(t)\} + [Z]\{q(t)\} = 0 \quad (3.16)$$

where

$$[Z] = [M]^{-1} \left[[K_e] - \alpha P * [K_{gs}] - \beta_d P * [K_{gt}] \right] \quad (3.17)$$

It is convenient to replace the n second order equations with $2n$ first order equations by introducing

$$\{h\} = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \quad (3.18)$$

and

$$[\phi] = \begin{bmatrix} 0 & -[I] \\ [Z] & 0 \end{bmatrix} \quad (3.19)$$

then, equation (3.16) becomes

$$\{\dot{h}(t)\} + [\phi(t)]\{h(t)\} = \begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} 0 & -[I] \\ [Z] & 0 \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} = 0 \quad (3.20)$$

Equation (3.19) needs not be solved completely in order to determine the stability of the system. It is merely necessary to determine whether the solution is bounded or unbounded.

It is assumed that the $2n$ linearly independent solutions of equation (3.20) are known over the interval $t = 0$ to $t = T$. Then they may be represented in matrix form as

$$[H(t)] = \begin{bmatrix} h_{1,1} & \cdot & \cdot & \cdot & h_{1,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{2n,1} & \cdot & \cdot & \cdot & h_{2n,2n} \end{bmatrix} \quad (3.21)$$

Since $f(t)$, and therefore $[\phi(t)]$ is periodic with period T , then the substitution $t = t + T$ will not alter the form of the equations, and the matrix solutions, at time $t + T$, $[H(t + T)]$ may be obtained from $[H(t)]$ by a linear transformation

$$[H(t + T)] = [R][H(t)] \quad (3.22)$$

where $[R]$ is the transformation matrix and is composed only of constant coefficients.

It is desirable to find a set of solutions for which the matrix $[R]$ can be diagonalized. Hence the i^{th} solution vector after period T , $\{\bar{h}(t + T)\}_i$ may be determined from $\{\bar{h}(t)\}_i$ using the simple expression

$$\{\bar{h}(t + T)\}_i = \rho_i \{\bar{h}(t)\}_i \quad (3.23)$$

The behavior of the solution is determined by ρ_i .

If $\rho_i > 1$, then the amplitude of vibration will increase with time. If $\rho_i < 1$, then the amplitude will decrease. For $\rho_i = 1$, the amplitude will remain unchanged, and this represents the stable boundary.

In order to diagonalize the matrix $[R]$, the characteristic equation

$$|[R] - \rho[I]| = 0 \quad (3.24)$$

must be solved for its $2n$ roots, where $[I]$ is the identity matrix. The roots of the equations, ρ_i , are eigenvalues, each having a corresponding eigenvector.

The $2n$ resulting eigenvectors are chosen as the $2n$ solutions to equation (3.20). They can be placed in a matrix, $[\bar{H}(t)]$, which will then satisfy the expression

$$[\bar{H}(t)] = [\bar{R}][\bar{H}(t+T)] \quad (3.25)$$

where

$$[\bar{R}] = \begin{bmatrix} \rho_1 & 0 & . & . & 0 \\ 0 & \rho_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & 0 & \rho_{2n} \end{bmatrix} \quad (3.26)$$

$[\bar{R}]$ is the diagonalized matrix of $[R]$ composed of the $2n$ eigenvalues of equation (3.24).

The periodic vector, $\{Z(t)\}_i$, with period T is introduced so that

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i e^{(t/T)\ln\rho_i} \quad (3.27)$$

For an even function of time like $[\phi(t)]$, it is true that

$$[\phi(t)] = [\phi(-t)] \quad (3.28)$$

Hence equation (3.27) can be written as

$$\{\bar{h}(-t)\}_i = \{Z(-t)\}_i e^{-(t/T)\ln\rho_i} \quad (3.29)$$

then

$$\{\bar{h}(-t)\}_i = \{Z(-t)\}_i e^{(t/T)\ln(1/\rho_i)} \quad (3.30)$$

It is clear from (3.30) that $1/\rho_i$ is also an eigenvalue. This property is not restricted to even functions, but is also preserved in the case of arbitrary periodic functions as shown by Bolotin, (1964).

In general, the eigenvalues ρ_i are complex numbers of the form

$$\rho_i = a_i + j b_i \quad (3.31)$$

and the natural logarithm of a complex number is given by

$$\ln \rho_i = \ln |\rho_i| + j (\text{argument } \rho) \quad (3.32)$$

or in this case

$$\ln \rho_i = \ln \sqrt{a_i^2 + b_i^2} + j \tan^{-1}(b_i / a_i) \quad (3.33)$$

where $j = \sqrt{-1}$

From equation (3.27), it is clear that if the real part of $\log \rho_i$ is positive for any of the solutions, then that solution will be unbounded with time. A negative real part

means that the corresponding solution will damp out with time. It therefore follows that the boundary case for a given solution is that for which the characteristic exponent has a zero real part. This is identical to saying that absolute value of ρ_i is unity. For the system to remain stable, every one of the solutions must remain bounded. If even one of the solutions has a characteristic exponent which is positive, then the corresponding solution is unbounded and therefore the system is unstable.

It has been shown that if ρ_i is a solution, then $1/\rho_i$ is also a solution. These two solutions can be written as

$$\rho_i = a_i + j b_i \quad (3.34)$$

$$\rho_{i+n} = (a_i - j b_i) / (a_i^2 + b_i^2) \quad (3.35)$$

Another restriction on the solutions of the characteristic equation is that the complex eigenvalues must occur in complex conjugate pairs. Hence it follows that ρ_{i+1} and ρ_{i+n+1} are also solutions where

$$\rho_{i+1} = a_i - j b_i \quad (3.36)$$

$$\rho_{i+n+1} = (a_i + j b_i) / (a_i^2 + b_i^2) \quad (3.37)$$

These solutions are presented in figure 3.4 which shows a unit circle in the complex plane. The area inside the unit circle represents stable or bounded solutions, while the area outside the unit circle represents unstable or unbounded solutions. For each stable solution which lies inside the circle, there corresponds an unstable solution outside the circle due to the reciprocity constraint. Therefore the only possible stable solutions must lie on the unit circle.

Points on this unit circle may be represented in polar co-ordinates by $r = 1$ and $\theta = \tan^{-1} b/a$ where $-\pi \leq \theta \leq \pi$. For each root on the upper semicircle, there is a

corresponding root on the lower semicircle due to the fact that the roots occur in complex conjugate pairs. The logarithm of ρ_i , when ρ_i lies on the unit circle will be

$$\ln \rho_i = j\theta \quad (3.38)$$

and equation (3.27) becomes

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i e^{j\left(\frac{\theta t}{T}\right)} \quad (3.39)$$

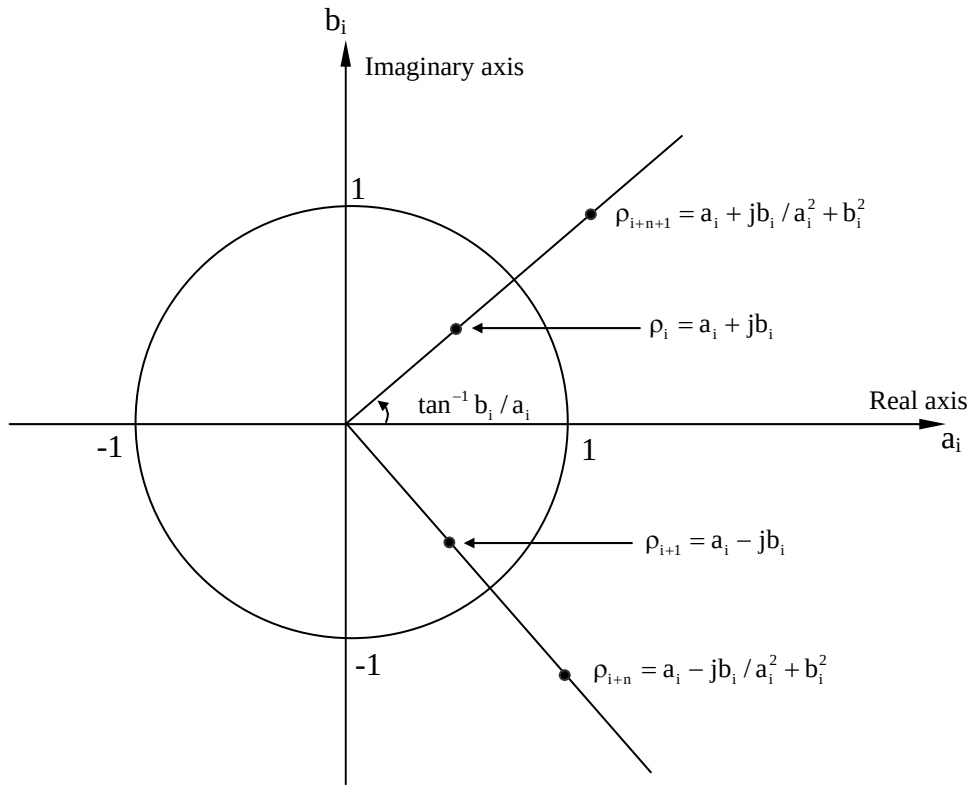


Figure 3.4 Unit circle in the complex plane

Since the eigenvalues occur in complex conjugate pairs, the limiting values of θ are zero and π .

When $\theta = 0$, equation (3.39) becomes

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i \quad (3.40)$$

and, therefore, the solution $\{\bar{h}(t)\}$ is periodic with period T when $\theta = \pi$, equation (3.39) becomes

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i e^{j\left(\frac{\pi t}{T}\right)} \quad (3.41)$$

$$\{\bar{h}(t+2T)\}_i = \{Z(t+2T)\}_i e^{j\left(\frac{\pi(t+2T)}{T}\right)} = \{\bar{h}(t)\}_i \quad (3.42)$$

It is clear from equation (3.42) that the solution $\{\bar{h}(t)\}$ is also periodic with a period $2T$.

It can be concluded that equation (3.11) has periodic solutions of period T and $2T$. Also the boundaries between stable and unstable regions are formed by periodic solutions of period T and $2T$.

For a system subjected to the periodic force

$$P = P_0 + P_t \cos \omega t \quad (3.43)$$

Where ω is the disturbing frequency, equation (3.11) becomes

$$[M]\{\ddot{q}\} + [[K_e] - \alpha P * [K_{gs}] - \beta_d P * \cos \omega t [K_{gt}]]\{q\} = 0 \quad (3.44)$$

Now we seek periodic solutions of period T and $2T$ of equation (3.44) where $T = 2\pi/\omega$.

When a solution of period $2T$ exists, it may be represented by the Fourier series

$$\{q\} = \sum_{k=1,3,5}^{\infty} \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (3.45)$$

Where $\{a\}_k$ and $\{b\}_k$ are time-independent vectors. Differentiating equation (3.45) twice with respect to time yields

$$\{\ddot{q}\} = \sum_{k=1,3,5}^{\infty} -\left(\frac{k\omega}{2}\right)^2 \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (3.46)$$

Substituting equations (3.45) and (3.46) into equation (3.44) and using the trigonometric relations

$$\begin{aligned} \sin A + \sin B &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\ \sin A - \sin B &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \\ \cos A + \cos B &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \\ \cos A - \cos B &= 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned} \quad (3.47)$$

and comparing the coefficients of $\sin \frac{k\omega t}{2}$ and $\cos \frac{k\omega t}{2}$ lead to the following matrix equations relating the vectors $\{a\}_k$ and $\{b\}_k$.

$$\begin{bmatrix} [K_e] - \alpha P^*[K_{gs}] + \frac{1}{2} \beta_d P^*[K_{gt}] - \frac{\omega^2}{4} [M] & -\frac{1}{2} \beta_d P^*[K_{gt}] & 0 \\ -\frac{1}{2} \beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - \frac{9\omega^2}{4} [M] & -\frac{1}{2} \beta_d P^*[K_{gt}] \\ 0 & -\frac{1}{2} \beta_d P^*[K_{gt}] & \left[K_e - \alpha P^*[K_{gs}] - \frac{25\omega^2}{4} [M] \right] \end{bmatrix} \cdot \begin{Bmatrix} \{a\}_1 \\ \{a\}_3 \\ \{a\}_5 \\ \vdots \end{Bmatrix} = 0 \quad (3.48)$$

and

$$\begin{bmatrix} [K_e] - \alpha P^*[K_{gs}] - \frac{1}{2}\beta_d P^*[K_{gt}] - \frac{\omega^2}{4}[M] & -\frac{1}{2}\beta_d P^*[K_{gt}] & 0 & \cdot \\ -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - \frac{9\omega^2}{4}[M] & -\frac{1}{2}\beta_d P^*[K_{gt}] & \cdot \\ 0 & -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - \frac{25\omega^2}{4}[M] & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = 0 \quad (3.49)$$

The orders of matrices in equations (3.48) and (3.49) are infinite. If solutions of period $2T$ exist, then the determinants of these matrices must zero. Combining these two determinants, the condition may be written as

$$\begin{bmatrix} [K_e - \alpha P^*[K_{gs}] \pm \frac{1}{2}\beta_d P^*[K_{gt}] - \frac{\omega^2}{4}[M]] & -\frac{1}{2}\beta_d P^*[K_{gt}] & 0 & \cdot \\ -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e - \alpha P^*[K_{gs}] - \frac{9\omega^2}{4}[M]] & -\frac{1}{2}\beta_d P^*[K_{gt}] & \cdot \\ 0 & -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - \frac{25\omega^2}{4}[M] & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} = 0 \quad (3.50)$$

If a solution to equation (3.44) exists with a period $T=2\pi/\omega$ then it may be expressed as Fourier series

$$\{q\} = \frac{1}{2}b_0 + \sum_{k=2,4,6}^{\infty} \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (3.51)$$

Differentiating equation (3.51) twice with respect to time yields

$$\{\ddot{q}\} = \sum_{k=2,4,6}^{\infty} -\left(\frac{k\omega}{2}\right)^2 \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (3.52)$$

Substituting equations (3.51) and (3.52) into equation (3.44), the following condition for the existence of solution with period T is obtained;

$$\begin{bmatrix} [K_e] - \alpha P^*[K_{gs}] - \omega^2[M] & -\frac{1}{2}\beta_d P^*[K_{gt}] & 0 & \cdot \\ -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 4\omega^2[M] & -\frac{1}{2}\beta_d P^*[K_{gt}] & \cdot \\ 0 & -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 9\omega^2[M] & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \{a\}_2 \\ \{a\}_4 \\ \{a\}_6 \\ \cdot \end{Bmatrix} = 0 \quad (3.53)$$

and

$$\begin{bmatrix} \frac{1}{2}([K_e] - \alpha P^*[K_{gs}]) & -\frac{1}{2}\beta_d P^*[K_{gt}] & 0 & 0 & \cdot \\ -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - \omega^2[M] & -\frac{1}{2}\beta_d P^*[K_{gt}] & 0 & \cdot \\ 0 & -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 4\omega^2[M] & -\frac{1}{2}\beta_d P^*[K_{gt}] & \cdot \\ 0 & \cdot & -\frac{1}{2}\beta_d P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 9\omega^2[M] & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \{b\}_0 \\ \{b\}_2 \\ \{b\}_4 \\ \{b\}_6 \\ \cdot \end{Bmatrix} = 0 \quad (3.54)$$

It has been shown by Bolotin (1964), that solutions with period 2T are the ones of the greatest practical importance and that as a first approximation the boundaries of the principal regions of dynamic instability can be determined from the equation

$$\left[[K_e] - \alpha P^*[K_{gs}] \pm \frac{1}{2}\beta_d P^*[K_{gt}] - \frac{\omega^2}{4}[M] \right] \{q\} = 0 \quad (3.55)$$

The two matrices $[K_{gs}]$ and $[K_{gt}]$ will be identical if the static and time dependent components of the loads are applied in the same manner. If $[K_{gs}] \equiv [K_{gt}] \equiv [K_g]$, then equation (3.56) becomes

$$\left[[K_e] - (\alpha \pm \frac{1}{2} \beta_d) P^* [K_g] - \frac{\omega^2}{4} [M] \right] \{q\} = 0 \quad (3.56)$$

Equation (3.56) represents solutions to three related problems

(i) Free vibration with $\alpha = 0$, $\beta_d = 0$ and $p = \omega/2$ the natural frequency

$$[[K_e] - p^2 [M]] \{q\} = 0 \quad (3.57)$$

(ii) Static stability with $\alpha = 1$, $\beta_d = 0$ and $\omega = 0$

$$[[K_e] - P^* [K_g]] \{q\} = 0 \quad (3.58)$$

(iii) Dynamic stability when all terms are present

$$\left[[K_e] - (\alpha \pm \frac{1}{2} \beta_d) P^* [K_g] - \frac{\omega^2}{4} [M] \right] \{q\} = 0 \quad (3.59)$$

CHAPTER FOUR

NUMERICAL METHOD

4.1 The Finite Element Method

4.1.1 A Brief History

Finite Element Analysis (FEA) was first developed in 1943 by R. Courant, who utilized the Ritz method of numerical analysis and minimization of variational calculus to obtain approximate solutions to vibration systems. Shortly thereafter, a paper published in 1956 by M. J. Turner, R. W. Clough, H. C. Martin, and L. J. Topp established a broader definition of numerical analysis. The paper centered on the "stiffness and deflection of complex structures".

By the early 70's, FEA was limited to expensive mainframe computers generally owned by the aeronautics, automotive, defense, and nuclear industries. Since the rapid decline in the cost of computers and the phenomenal increase in computing power, FEA has been developed to an incredible precision. Present day supercomputers are now able to produce accurate results for all kinds of parameters.

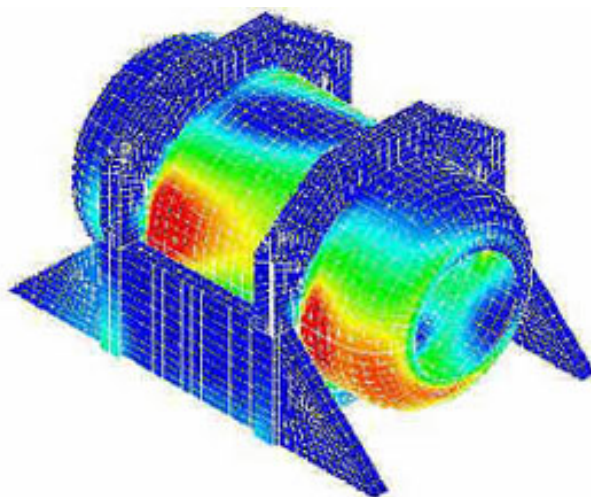


Figure 4.1 Heat transfer analysis model of an engine (<http://www.noraneng.com>)

4.1.2 What is Finite Element Analysis?

FEA consists of a computer model of a material or design that is stressed and analyzed for specific results. It is used in new product design, and existing product refinement. A company is able to verify a proposed design will be able to perform to the client's specifications prior to manufacturing or construction. Modifying an existing product or structure is utilized to qualify the product or structure for a new service condition. In case of structural failure, FEA may be used to help determine the design modifications to meet the new condition.

There are generally two types of analysis that are used in industry: 2-D modeling, and 3-D modeling. While 2-D modeling conserves simplicity and allows the analysis to be run on a relatively normal computer, it tends to yield less accurate results. 3-D modeling, however, produces more accurate results while sacrificing the ability to run on all but the fastest computers effectively. Within each of these modeling schemes, the programmer can insert numerous algorithms (functions) which may make the system behave linearly or non-linearly. Linear systems are far less complex and generally do not take into account plastic deformation. Non-linear systems do account for plastic deformation, and many also are capable of testing a material all the way to fracture.

4.1.3 How Does Finite Element Analysis Work?

FEA uses a complex system of points called nodes which make a grid called a mesh (Figure 4.2). This mesh is programmed to contain the material and structural properties which define how the structure will react to certain loading conditions. Nodes are assigned at a certain density throughout the material depending on the anticipated stress levels of a particular area. Regions which will receive large amounts of stress usually have a higher node density than those which experience little or no stress. Points of interest may consist of: fracture point of previously tested material, fillets, corners, complex detail, and high stress areas. The mesh acts like a spider web in that from each node, there extends a mesh element to each of the

adjacent nodes. This web of vectors is what carries the material properties to the object, creating many elements.

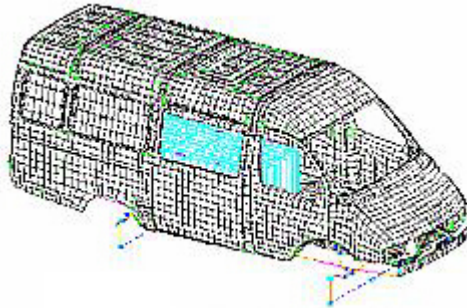


Figure 4.2 Model of a vehicle for FEA (<http://www.macsch.com:80/tech/mscworld/van.html>)

A wide range of objective functions (variables within the system) are available for minimization or maximization:

- Mass, volume, temperature
- Strain energy, stress strain
- Force, displacement, velocity, acceleration
- Synthetic (User defined)

There are multiple loading conditions which may be applied to a system. Next to Figure 4.3, some examples are shown:

- Point, pressure (Figure 4.3), thermal, gravity, and centrifugal static loads
- Thermal loads from solution of heat transfer analysis
- Enforced displacements
- Heat flux and convection
- Point, pressure and gravity dynamic loads

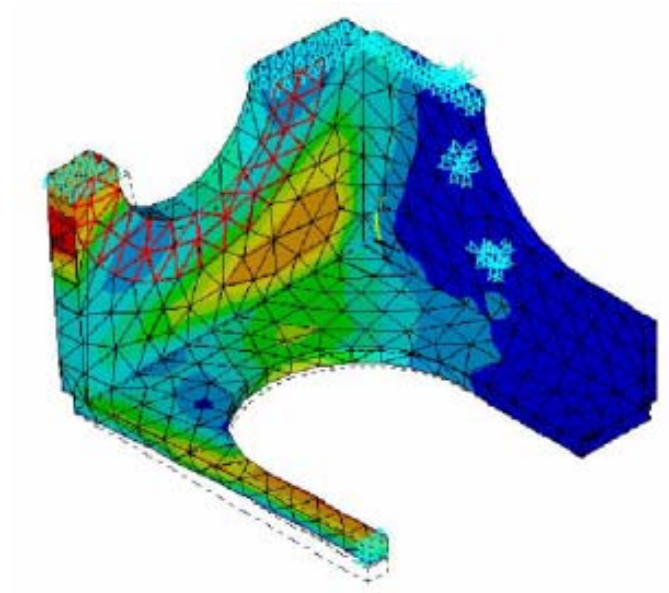


Figure 4.3 Pressure point on the structure

(<http://www.umass.edu/mie/labs/mda/fea/fealib/goldstein/PROJECT.html>)

Each FEA program may come with an element library, or one is constructed over time. Some sample elements are:

- Rod elements
- Beam elements
- Plate/Shell/Composite elements
- Shear panel
- Solid elements
- Spring elements
- Mass elements
- Rigid elements
- Viscous damping elements

Many FEA programs also are equipped with the capability to use multiple materials within the structure such as:

- Isotropic, identical throughout
- Orthotropic, identical at 90 degrees
- General anisotropic, different throughout

4.1.4 Types of Engineering Analysis

Structural analysis consists of linear and non-linear models. Linear models use simple parameters and assume that the material is not plastically deformed. Non-linear models consist of stressing the material past its elastic capabilities. The stresses in the material then vary with the amount of deformation as in Figure 4.4.



Figure 4.4 The stresses in the material (<http://www.macsch.com:80/products/nastran/bike.html>)

Vibrational analysis is used to test a material against random vibrations, shock, and impact. Each of these incidences may act on the natural vibrational frequency of the material which, in turn, may cause resonance and subsequent failure.

Fatigue analysis helps designers to predict the life of a material or structure by showing the effects of cyclic loading on the specimen. Such analysis can show the areas where crack propagation is most likely to occur. Failure due to fatigue may also show the damage tolerance of the material (Figure 4.5).

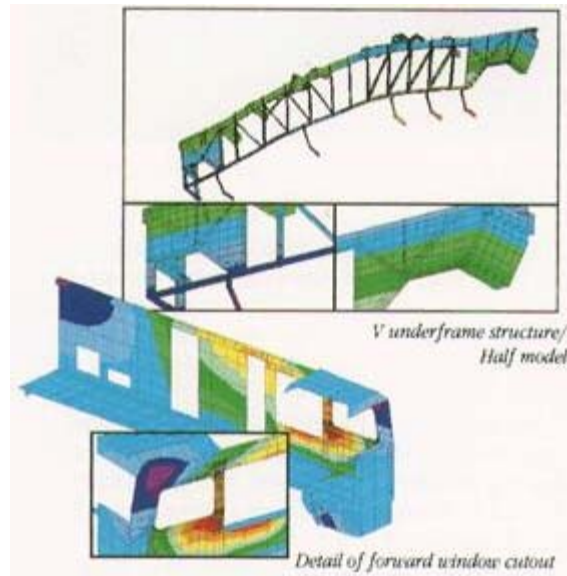


Figure 4.5 Failure due to fatigue in the material (<http://www.macsch.com:80/mscworld/train.html>)

Heat Transfer analysis models the conductivity or thermal fluid dynamics of the material or structure (Figure 4.1). This may consist of a steady-state or transient transfer. Steady-state transfer refers to constant thermoproperties in the material that yield linear heat diffusion.

4.1.5 Results of Finite Element Analysis

FEA has become a solution to the task of predicting failure due to unknown stresses by showing problem areas in a material and allowing designers to see all of the theoretical stresses within. This method of product design and testing is far superior to the manufacturing costs which would accrue if each sample was actually built and tested.

CHAPTER FIVE THEORETICAL ANALYSIS

5.1 The Finite Element Model of a Composite Curved Beam

5.1.1 Mathematical Model

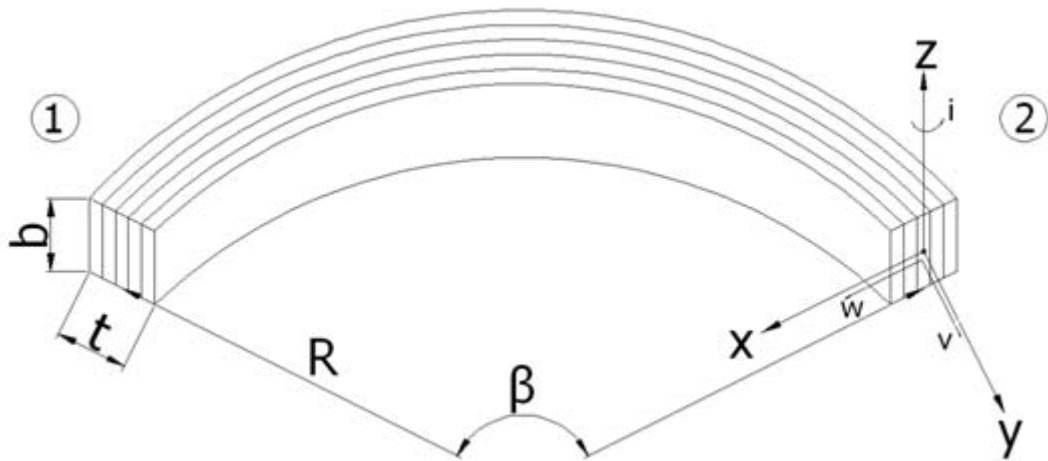


Figure 5.1 Coordinate system and displacements of the composite curved beam finite element

The in-plane vibration of a horizontally composite curved beam shown in figure 5.1 is considered. As seen from the figure that R denotes the radius of curvature at the centroid, b and t are the width and height of the cross-section and θ is the subtended angle. Axes x and z principal centroid axes of the beam cross-section and y is the tangent to the curved axis of this member.

5.1.2 Theoretical Consideration

As seen from Figure 5.1, that the composite curved beam has been investigated by using the Finite Element Method. In Figure 5.2, It is assumed that the width of the beam cross-section is fixed and radial force is $P(t) = P_{cr}(\alpha + \beta_d \cos \omega t)$ where ω is the disturbing frequency, the static- and time-dependent components of the load can

be represented as a fraction of the fundamental static buckling load P_{cr} . The radial and circumferential deflections in every node of the element are w , v , respectively.

Where;

$$P(t) = P_0 + P_t \cos \omega t$$

$$\alpha = \frac{P_0}{P_{cr}}, \quad \beta_d = \frac{P_t}{P_{cr}} \quad (5.1)$$

$$w = a_1 \cdot \cos \phi + a_2 \cdot \sin \phi + a_4 - a_6 \cdot \phi$$

$$v = -a_1 \cdot \sin \phi + a_2 \cdot \cos \phi + a_3 + a_5 \cdot \phi + a_6 \phi^2 / 2$$

$$i = \frac{dw}{dy} - \frac{v}{R}, \quad \phi = y/R \quad (5.2)$$

5.1.3 Strain Energy of In-plane Vibration of a Curved Beam

Strain energy of a curved beam element vibrating in its own plane is given by Belek, (1977) as,

$$V_l = \int_0^{l_{sh}} \left[E_s I_{xxs} \left(\ddot{w} - \frac{\dot{v}}{R} \right)^2 + E_s A_s \left(\frac{w}{R} + \dot{v} \right)^2 \right] dy \quad (5.3)$$

Equation (5.3) can be written as

$$V_l = \frac{1}{2} \{q_l\}^T [C_l]^{-T} [k_l] [C_l]^{-1} \{q_l\} \quad (5.4)$$

where

$$\{q_l\}^T = [v_i \quad w_i \quad i_i \quad v_{i+1} \quad w_{i+1} \quad i_{i+1}] \quad (5.5)$$

$$[k_I] = \frac{E_s I_{\text{XXS}}}{R^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & Q\beta & Q\beta & 0 \\ & & & & \beta(1+Q) & \frac{\beta^2}{2} \\ & & & & & \frac{\beta^3}{3} \end{bmatrix} \quad (5.6)$$

$$Q = \frac{12R^2}{t^2}$$

and the transformation matrix $[C_I]^{-1}$ is given by

$$[C_I]^{-1} = \begin{bmatrix} -B_4 & B_1 & -RB_4 & -B_5 & -B_1 & -RB_5 \\ -B_3 & B_2 & -RB_3 & -B_1 & -B_2 & -RB_1 \\ B_1 & -B_2 & RB_3 & B_1 & B_2 & RB_1 \\ B_4 & -B_3 & RB_4 & B_5 & B_1 & RB_5 \\ B_6 & -B_7 & RB_8 & B_6 & B_7 & RB_9 \\ -B_1 & B_2 & -RB_1 & -B_1 & -B_2 & -RB_1 \end{bmatrix} \quad (5.7)$$

$$D = 2 \cos \beta - 2 + \beta \sin \beta$$

$$B_1 = \frac{\cos \beta - 1}{D}$$

$$B_2 = \frac{\sin \beta}{D}$$

$$B_3 = \frac{1 - \beta \sin \beta - \cos \beta}{D}$$

$$B_4 = \frac{\sin \beta - \beta \cos \beta}{D}$$

$$B_5 = \frac{\beta - \sin \beta}{D}$$

$$B_6 = \frac{B_1 \beta}{2}$$

$$B_7 = \frac{B_2 \beta}{2}$$

$$B_8 = B_6 + \frac{1}{\beta}$$

$$B_9 = B_6 - \frac{1}{\beta}$$

5.1.4 External Work Done on a Curved Beam Under Uniform Compression Forces in the Plane of Curvature

External work will be done when the shroud is subjected to uniform compression force P in the plane of curvature. The additional energy V_{gl} due to a periodic force is (Papangelis & Trahair 1987)

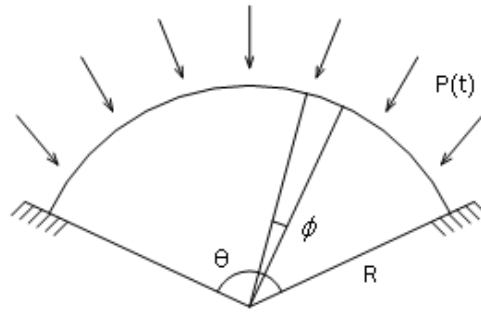


Figure 5.2 Curved beam subjected to uniform compression force

$$V_{gl} = \int_0^{l_{sh}} PR \left(\frac{dw}{dy} - \frac{v}{R} \right)^2 dy \quad (5.8)$$

It may be written in the matrix form as

$$V_{gl} = \frac{1}{2} \{q_I\}^T [C_I]^{-T} [k_{gl}] [C_I]^{-1} \{q_I\} \quad (5.9)$$

Where the corresponding degrees of freedom $\{q_I\}^T$ and transformation matrix $[C_I]^{-1}$ have been given in equations (5.5) and (5.7) respectively.

$$[k_{gl}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & \beta & 0 & \frac{\beta^2}{2} & \beta + \frac{\beta^3}{6} \\ & & & 0 & 0 & 0 \\ & & & & \frac{\beta^3}{3} & \frac{\beta^2}{2} + \frac{\beta^4}{8} \\ & & & & & \beta + \frac{\beta^3}{3} + \frac{\beta^5}{20} \end{bmatrix} \quad (5.10)$$

5.1.5 Kinetic Energy of In-plane Vibration of a Curved Beam

The kinetic energy expression of a curved beam element under combined bending-bending displacements is given by Belek, (1977) as,

$$T_I = \int_0^{l_{sh}} [\rho_s A_s (\dot{v}^2 + \dot{w}^2)] dy \quad (5.11)$$

$$T_I = \frac{1}{2} \{\dot{q}_I\}^T [C_I]^{-T} [m_I] [C_I]^{-1} \{\dot{q}_I\} \quad (5.12)$$

Where the corresponding degrees of freedom $\{q_I\}^T$ and transformation matrix $[C_I]^{-1}$ have been given in equations (5.5) and (5.7) respectively.

The mass matrix $[m_I]$ is ;

$$[m_I] = \rho_s A_s R \begin{bmatrix} \beta & 0 & \cos \beta - 1 & \sin \beta & \beta \cos \beta - \sin \beta & C_0 \\ 0 & \beta & \sin \beta & 1 - \cos \beta & \cos \beta + \beta \sin \beta - 1 & C_1 \\ 0 & 0 & \beta & 0 & \frac{\beta^2}{2} & \frac{\beta^3}{6} \\ 0 & 0 & 0 & \beta & 0 & -\frac{\beta^2}{2} \\ 0 & 0 & 0 & 0 & \frac{\beta^3}{3} & \frac{\beta^4}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta^5}{20} + \frac{\beta^3}{3} \end{bmatrix} \quad (5.13)$$

$$C_0 = (\beta^2 - 4) \cos \frac{\beta}{2} - 2\beta \sin \beta + 2$$

$$C_1 = (\beta^2 - 4) \sin \frac{\beta}{2} + 2\beta \cos \beta$$

5.2 The Effective Flexural Modulus of the Laminated Composite Beam

The beam is considered to have the bending of symmetrically laminated beams according to the classical laminated plate theory (CPLT). For symmetric laminates, the equations for bending deflection are uncoupled from those of the displacements in longitudinal. The in-plane forces are zero only for small deformations, and not necessary for stability analysis. In the derivation of the laminated beam theory we assume that

$$M_{yy} = M_{zy} = 0 \quad (5.14)$$

everywhere in the beam. The classical laminated plate theory constitutive equation for symmetric laminates in the absence of in-plane forces, is given by (Reddy, 1997):

$$\begin{Bmatrix} M_{zz} \\ M_{yy} \\ M_{zy} \end{Bmatrix} = - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial z^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial z \partial y} \end{Bmatrix} \quad (5.15)$$

in inverse form of Eq. (5.15)

$$\begin{Bmatrix} \frac{\partial^2 w}{\partial z^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial z \partial y} \end{Bmatrix} = - \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{Bmatrix} M_{zz} \\ M_{yy} \\ M_{zy} \end{Bmatrix} \quad (5.16)$$

where (D_{ij}^*) denote the elements of the inverse matrix of (D_{ij}) . In view of the assumption (5.14), we have from Eq. (5.16),

$$\frac{\partial^2 w}{\partial z^2} = -D_{11}^* M_{zz}, \quad \frac{\partial^2 w}{\partial y^2} = -D_{12}^* M_{zz}, \quad 2 \frac{\partial^2 w}{\partial z \partial y} = -D_{16}^* M_{zz} \quad (5.17)$$

Eq. (5.17) indicates that the transverse deflection w may be independent on the coordinate y due to the Poisson effect (D_{ij}^*) and anisotropic shear coupling (D_{16}^*). These effects can be neglected only for long beams. The length-to-width ratio for which the transverse deflection can be assumed to be independent of y is a function of the lamination scheme.

In the following derivations, the laminated beam under consideration is long enough to make the effects of the Poisson ratio and shear coupling on the deflection negligible. The transverse deflection can be treated only as a function of coordinate z and time t ,

$$w = w(z, t) \quad (5.18)$$

and can be written as;

$$\frac{\partial^2 w}{\partial z^2} = -D_{11}^* M_{zz} \quad (5.19)$$

In order to cast Eq. (5.19) in the familiar form used in the classical Euler–Bernoulli beam theory, the effective flexural modulus can be obtained as,

$$M = bM_{zz} \rightarrow E_s^b = \frac{12}{t^3 D_{11}^*} \quad (5.20)$$

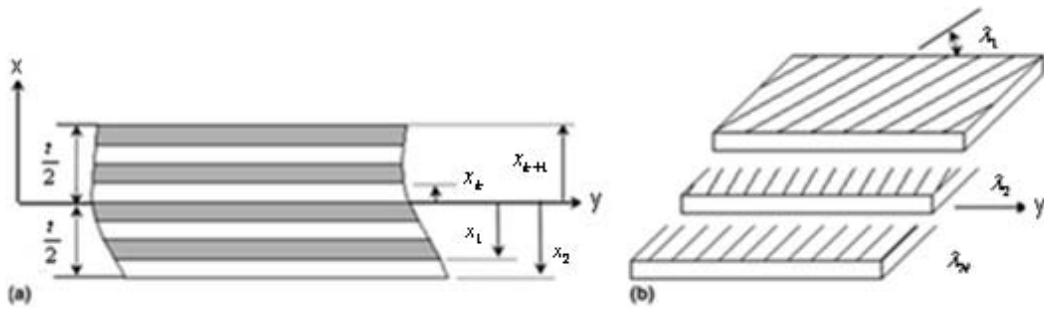


Fig. 5.3. Layer numbering used for a typical laminated beam and orientation angles.

As shown in Fig. 5.3.(a), the laminates have the same thickness and x is axis of symmetry. D_{ij} is called the bending stiffness, and is calculated using Eq. (5.21) for each laminate in Fig. 5.3.(b),

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N \bar{Q}_{ij}^{(k)} (x_{k+1}^3 - x_k^3) \quad (5.21)$$

where

$$c = \cos \lambda, \quad s = \sin \lambda$$

$$\bar{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{33})s^2c^2 + Q_{22}s^4$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{33})s^2c^2 + Q_{12}(s^4 + c^4)$$

$$\bar{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{33})s^2c^2 + Q_{22}c^4$$

$$\bar{Q}_{13} = (Q_{11} - Q_{12} - 2Q_{33})sc^3 + (Q_{12} - Q_{22} + 2Q_{33})s^3c$$

$$\bar{Q}_{23} = (Q_{11} - Q_{12} - 2Q_{33})s^3c + (Q_{12} - Q_{22} + 2Q_{33})sc^3$$

$$\bar{Q}_{33} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{33})s^2c^2 + Q_{33}(s^4 + c^4);$$

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad \nu_{21} = \nu_{12} \frac{E_2}{E_1}$$

CHAPTER SIX

DISCUSSIONS OF RESULTS

In this chapter, some of the results obtained for the frequencies and buckling loads of composite curved beams in-plane directions are shown and compared with Ansys results in order to indicate the degree of accuracy and adaptability that can be expected in the application of the finite element method. Moreover, the effects of subtended angle, variations of orientation angle, static and dynamic load parameters on the stability regions of laminated composite curved beam are shown in graphics.

In this study, C1, C2, C3, C4, C5 and C6, represent the different type orientation angles of curved beams. The explanation of these orientation angles is as follows:

C1: [0 0 0 0],

C2: [90 90 90 90],

C3: [0 90 90 0],

C4: [90 0 0 90],

C5: [0 45 -45 0],

C6: [0 60 -60 0],

The static stability and free vibrations of laminated curved beams have been analyzed with Matlab. The variation of the first natural frequency and critical buckling load depending on number of elements is given in Table 6.1 and the table is formed for the orientation angle [0 0 0 0] of a curved beam. As shown in Table 6.1, if the number of element increases, the accuracy of the results increases. The number of elements used the throughout the analyses is 30. The first four natural frequencies and the first critical buckling load obtained with the present element were compared with the results of Ansys and given in Table 6.2 and Table 6.3, respectively. As seen from the tables, close agreements are found.

Analysis of natural frequency with Ansys of a laminated curved beam for [0 0 0 0] orientation angle has been investigated and the first four mode shapes of the beam

are shown in Figure 6.1. Similarly, in Figure 6.2, the buckling load analysis of a laminated curved beam with Ansys for $[0\ 0\ 0\ 0]$ orientation angle has been shown for the first two modes.

As shown in Figure 6.3, if the subtended angle increases, the fundamental frequencies of the curved beams decrease. Effect of subtended angle on the critical buckling load is shown in Figure 6.4. In Figure 6.4, the variation in graphic is similar to the one shown in Figure 6.3. When the subtended angle of an arch increases, as a result the curved beam becomes more flexible. Thus, as shown in the figure, the critical buckling load decreases. It can be noticed from the figure that critical buckling load values of beams with C1, C3, C5 and C6 orientations are close to each other. The beam having C1 orientation is more stable in comparison to the other beams due to the greatest critical buckling load.

As shown in Figure 6.5, when the variation of fundamental frequencies with respect to the radius of curvature of laminated curved beams are studied, if the radius increases, the fundamental frequencies of curved beams decrease. There is a similar phenomenon shown in Figure 6.6. As the radius of curvature increases, the critical buckling load decreases.

From Figure 6.7, it can be noticed that the first and second dynamic instability regions narrow because of the increase in the subtended angle of the arch and the dynamic stability and the unstable regions move towards origin and are close to each other. C1 laminated curved beam is less stable compared to other beams. In addition, when the dynamic load parameter increases, the unstable region widens. From the figure, it can be noticed that the sequences of these stability regions move downwards as C1, C3, C5, C6, C4 and C2 at $\theta=60^\circ$ and $\theta=120^\circ$.

As shown in Figure 6.8 (a) and (b), if the static load parameter increases, disturbing frequency moves towards origin. Thus, the curved composite beam under periodic loading becomes unstable at a small disturbing frequency and small

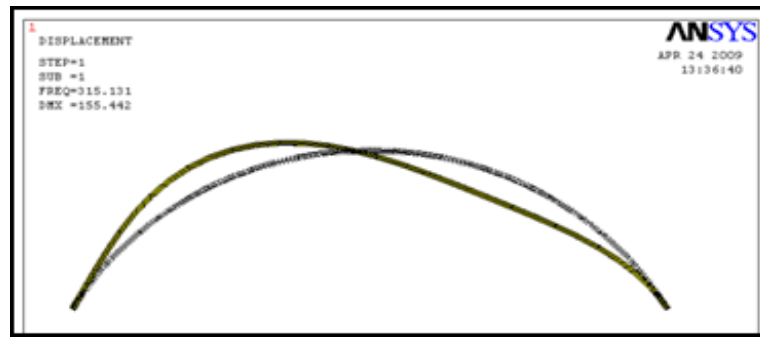
dynamic load parameter. In addition, in Figure 6.9 and Figure 6.10, similar phenomenon takes place.

Table 6.1 The variation of critical buckling load and fundamental frequency of laminated curved beam for $[0\ 0\ 0]$ orientation angle with respect to the number of elements $E_1=129.207\text{ GPa}$, $E_2=E_3=9.42512\text{ GPa}$, $G_{12}=5.15658\text{ GPa}$, $G_{13}=4.3053\text{ GPa}$, $G_{23}=2.5414\text{ GPa}$, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254\text{ m}$, $t=0.004\text{ m}$, $\rho=1550.0666\text{ kg/m}^3$, $\theta=120^\circ$

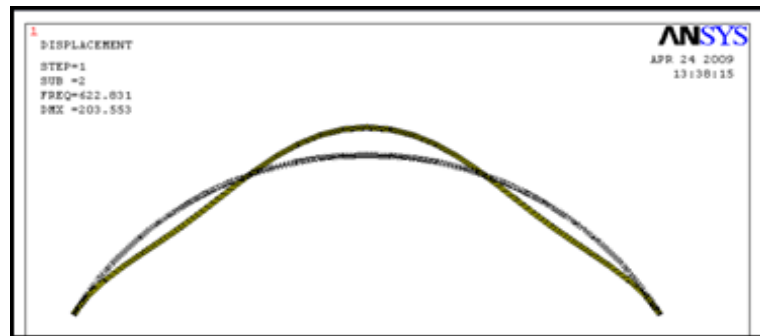
Number of elements	f_1 (rad/sn)	p_1 (N/m)
2	427,8223	32112,9207
4	321,0542	22606,403
6	318,6456	22094,8115
8	318,2143	21992,287
10	318,0944	21962,5695
12	318,051	21951,5789
14	318,0322	21946,7881
16	318,023	21944,4385
18	318,0182	21943,1792
20	318,0153	21942,4561
30	318,01	21941,341

Table 6.2 The comparison between the values obtained by using Ansys and the values of present Finite Element Model of the first four natural frequency of a laminated curved beam
 $E_1=129.207$ GPa, $E_2=E_3=9.42512$ GPa, $G_{12}=5.15658$ GPa, $G_{13}=4.3053$ GPa, $G_{23}=2.5414$ GPa, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254$ m, $t=0.004$ m, $\rho=1550.0666$ kg/m³, $\theta=120^\circ$

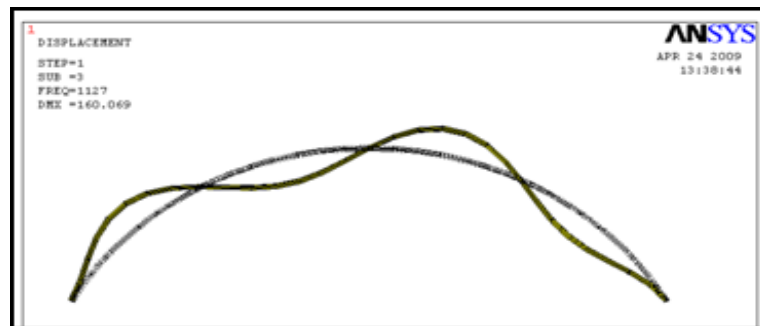
Frequency / fiber orientation	f_1 (rad/sn) (ansys)	f_1 (rad/sn) (matlab)	% error f_1	f_2 (rad/sn) (ansys)	f_2 (rad/sn) (matlab)	% error f_2	f_3 (rad/sn) (ansys)	f_3 (rad/sn) (matlab)	% error f_3	f_4 (rad/sn) (ansys)	f_4 (rad/sn) (matlab)	% error f_4
[0 0 0 0]	315.13	318.01	-0.905	622.83	632.851	-1.583	1126.5	1152.306	-2.239	1663.4	1714.957	-3.006
[90 90 90 90]	85.816	85.889	-0.084	170.70	170.923	-0.13	310.91	311.22	-0.099	463.16	463.184	-0.005
[0 90 90 0]	296.32	297.514	-0.401	584.64	592.062	-1.253	1057.3	1078.037	-1.923	1555.7	1604.424	-3.036
[90 0 0 90]	138.17	133.931	3.165	274.87	266.528	3.129	499.78	485.299	2.983	745.33	722.262	3.193
[0 45 -45 0]	298.17	295.120	1.033	588.89	587.297	0.271	1065.8	1069.362	-0.333	1571.6	1591.512	-1.251
[0 60 -60 0]	296.51	294.172	0.794	585.32	585.412	-0.015	1059.0	1065.929	-0.65	1559.7	1586.404	-1.683



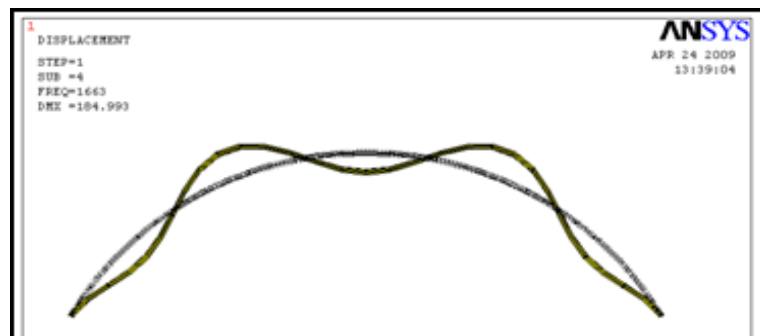
(a)



(b)



(c)

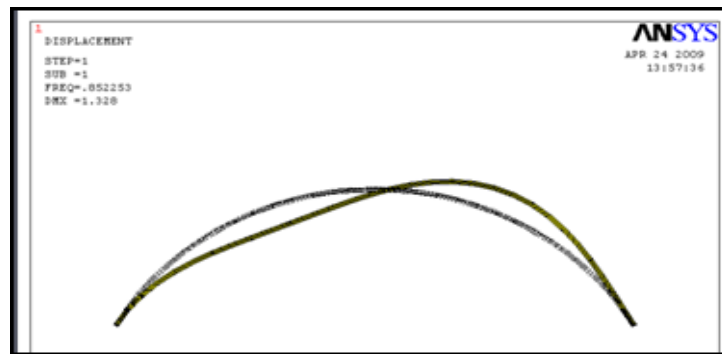


(d)

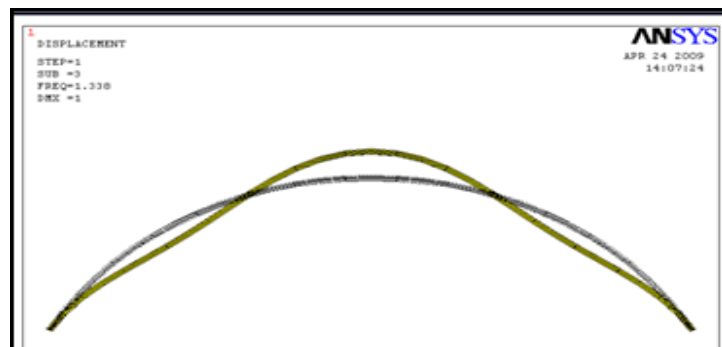
Figure 6.1 The first four vibration mode shapes of laminated curved beam for $[0\ 0\ 0\ 0]$ orientation angle $E_1= 129.207\ \text{GPa}$, $E_2=E_3=9.42512\ \text{GPa}$, $G_{12}=5.15658\ \text{GPa}$, $G_{13}=4.3053\ \text{GPa}$, $G_{23}=2.5414\ \text{GPa}$, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254\ \text{m}$, $t=0.004\ \text{m}$, $\rho=1550.0666\ \text{kg/m}^3$, $\theta=120^\circ$, $R=0.25\text{m}$ (a) first mode shape, (b) second mode shape, (c) third mode shape, (d) fourth mode shape

Table 6.3 The comparison between the values obtained by using Ansys and the values of present Finite Element Model of the buckling load of a laminated curved beam $E_1=129.207$ GPa, $E_2=E_3=9.42512$ GPa, $G_{12}=5.15658$ GPa, $G_{13}=4.3053$ GPa, $G_{23}=2.5414$ GPa, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254$ m, $t=0.004$ m, $\rho=1550.0666$ kg/m³, $\theta=120^\circ$

Buckling / fiber orientation	p_1 (N/m) ansys	p_1 (N/m) matlab	% error
[0 0 0 0]	21647,15	21941,341	-1.34
[90 90 90 90]	1599,3618	1600,53	-0.072
[0 90 90 0]	19166,586	19204,142	-0.195
[90 0 0 90]	4146,296	3891,768	6.54
[0 45 -45 0]	19329,4	18896,296	2.292
[0 60 -60 0]	19164,554	18775,190	2.073



(a)



(b)

Figure 6.2 The first two buckling load mode shapes of laminated curved beam for [0 0 0 0] orientation angle $E_1=129.207$ GPa, $E_2=E_3=9.42512$ GPa, $G_{12}=5.15658$ GPa, $G_{13}=4.3053$ GPa, $G_{23}=2.5414$ GPa, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254$ m, $t=0.004$ m, $\rho=1550.0666$ kg/m³, $\theta=120^\circ$, $R=0.25$ m (a) first mode shape, (b) second mode shape

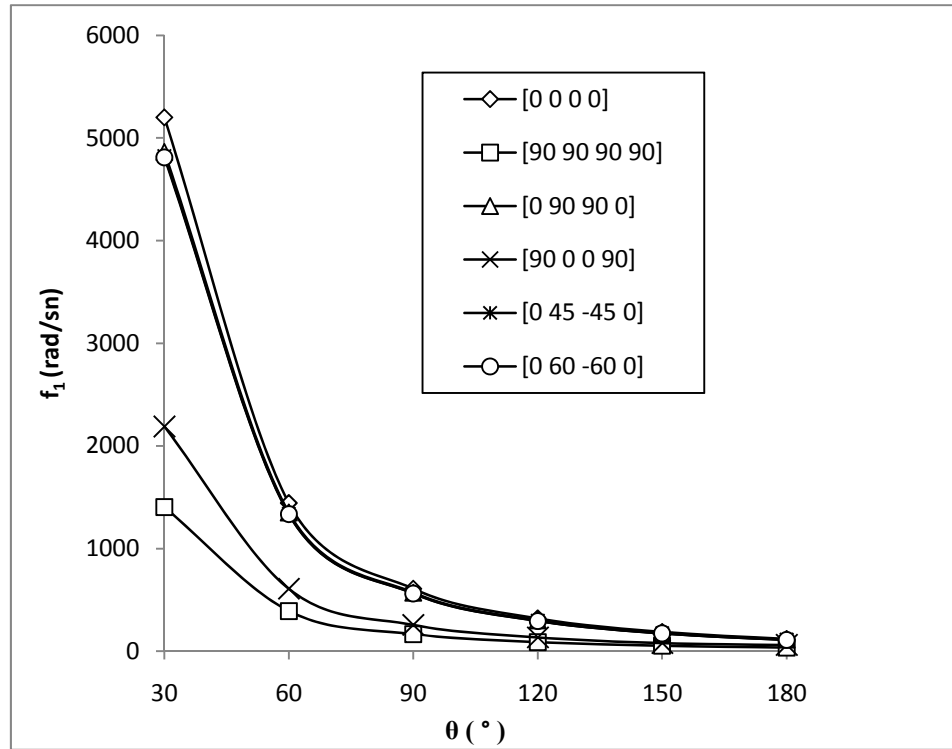


Figure 6.3 The effect of subtended angle on the fundamental frequency variation of a laminated curved beam $E_1 = 129.207$ GPa, $E_2 = E_3 = 9.42512$ GPa, $G_{12} = 5.15658$ GPa, $G_{13} = 4.3053$ GPa, $G_{23} = 2.5414$ GPa, $\nu_{12} = 0.3$, $\nu_{21} = \nu_{12} \times E_2/E_1$, $b = 0.0254$ m, $t = 0.004$ m, $\rho = 1550.0666$ kg/m³, $R = 0.25$ m

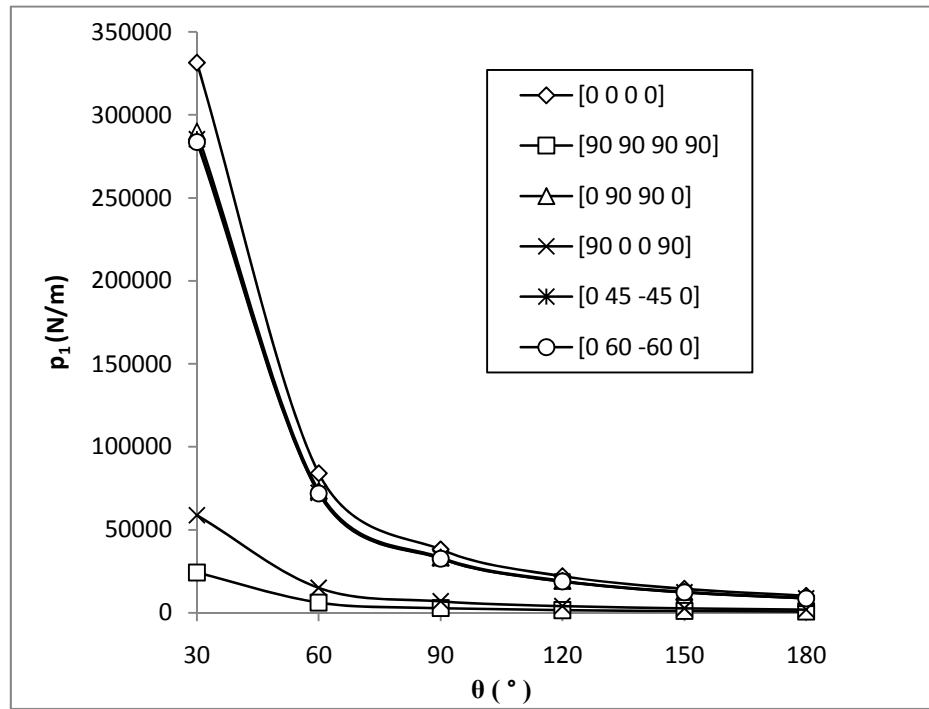


Figure 6.4 The effect of subtended angle on the critical buckling load variation of a laminated curved beam $E_1 = 129.207$ GPa, $E_2 = E_3 = 9.42512$ GPa, $G_{12} = 5.15658$ GPa, $G_{13} = 4.3053$ GPa, $G_{23} = 2.5414$ GPa, $\nu_{12} = 0.3$, $\nu_{21} = \nu_{12} \times E_2/E_1$, $b = 0.0254$ m, $t = 0.004$ m, $\rho = 1550.0666$ kg/m³, $R = 0.25$ m

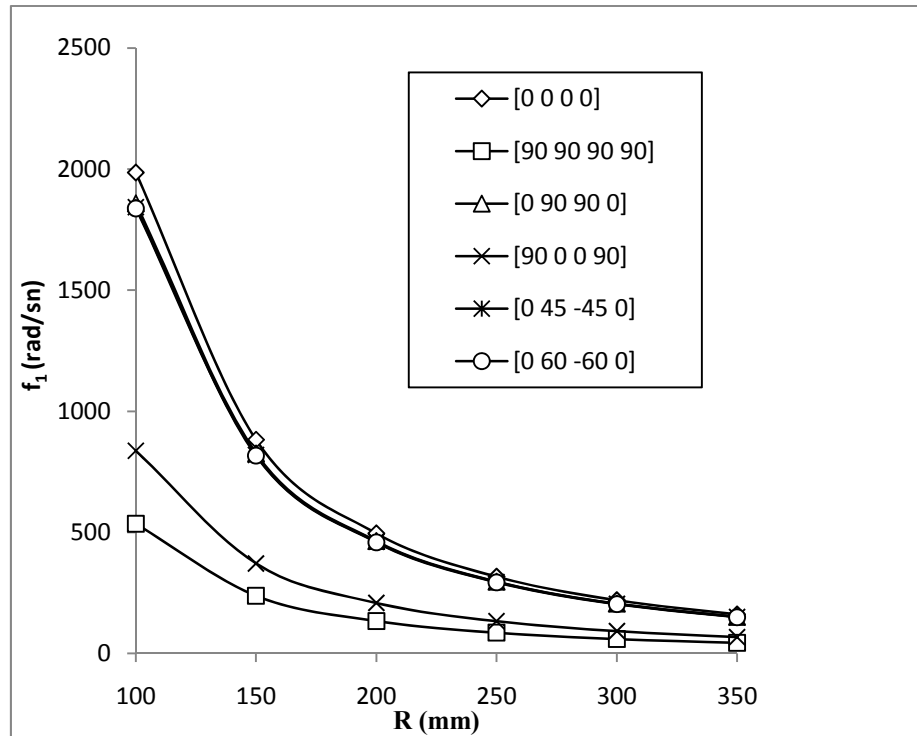


Figure 6.5 The effect of radius of curvature on the fundamental frequency variation of a laminated curved beam $E_1 = 129.207$ GPa, $E_2 = E_3 = 9.42512$ GPa, $G_{12} = 5.15658$ GPa, $G_{13} = 4.3053$ GPa, $G_{23} = 2.5414$ GPa, $\nu_{12} = 0.3$, $\nu_{21} = \nu_{12} \times E_2/E_1$, $b = 0.0254$ m, $t = 0.004$ m, $\rho = 1550.0666$ kg/m³, $\theta = 120^\circ$

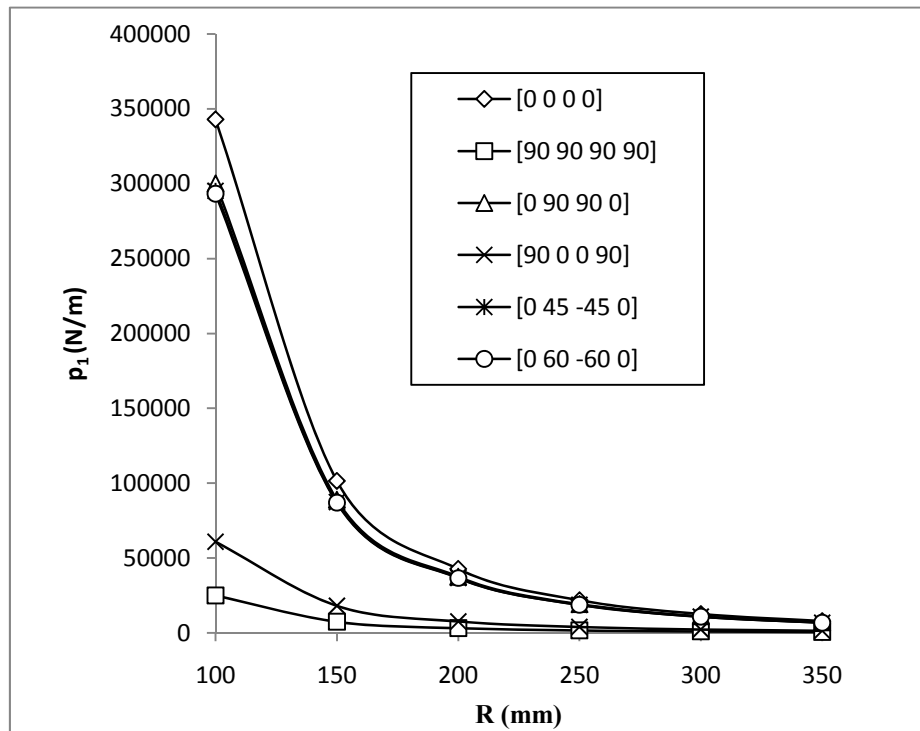
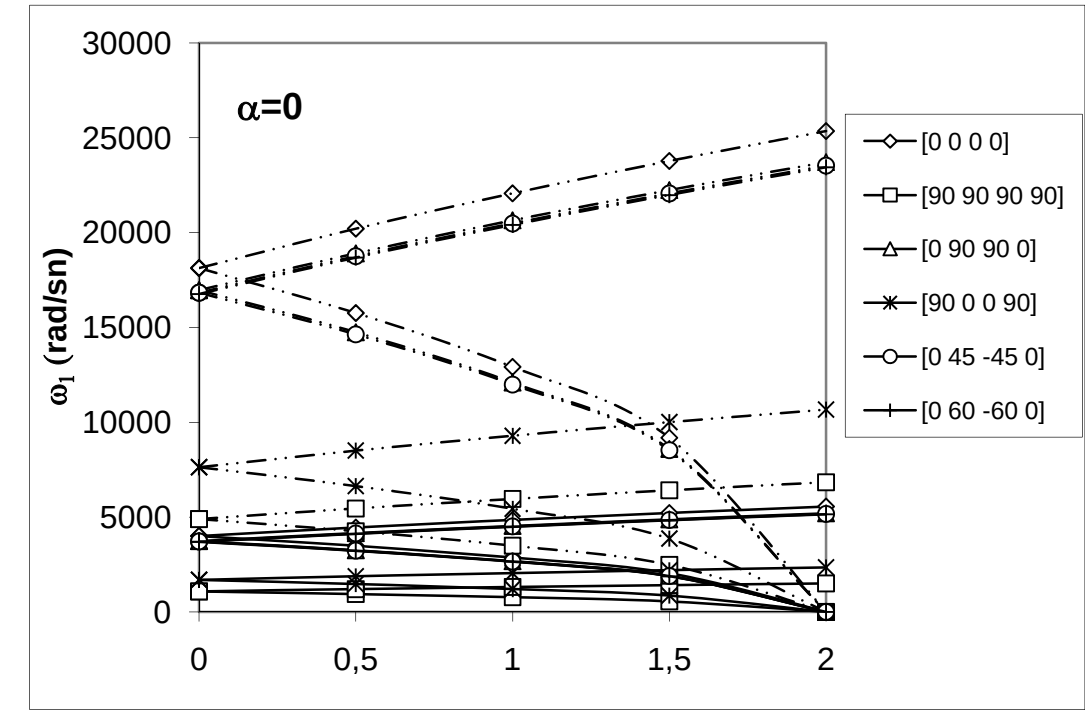
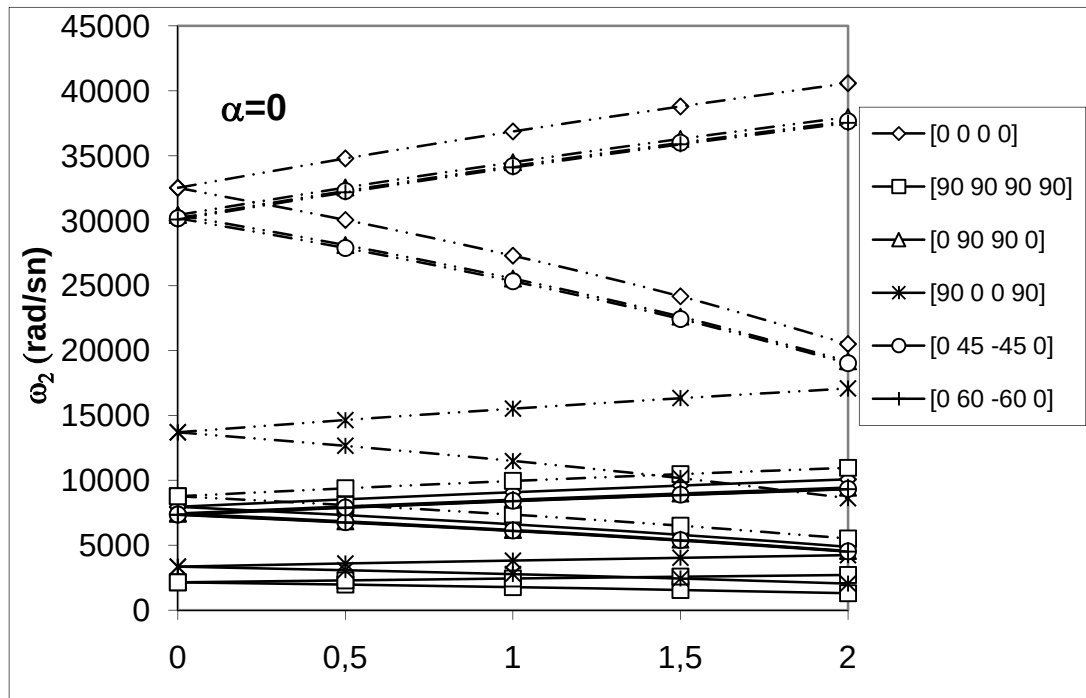


Figure 6.6 The effect of radius of curvature on the critical buckling load variation of a laminated curved beam $E_1 = 129.207$ GPa, $E_2 = E_3 = 9.42512$ GPa, $G_{12} = 5.15658$ GPa, $G_{13} = 4.3053$ GPa, $G_{23} = 2.5414$ GPa, $\nu_{12} = 0.3$, $\nu_{21} = \nu_{12} \times E_2/E_1$, $b = 0.0254$ m, $t = 0.004$ m, $\rho = 1550.0666$ kg/m³, $\theta = 120^\circ$

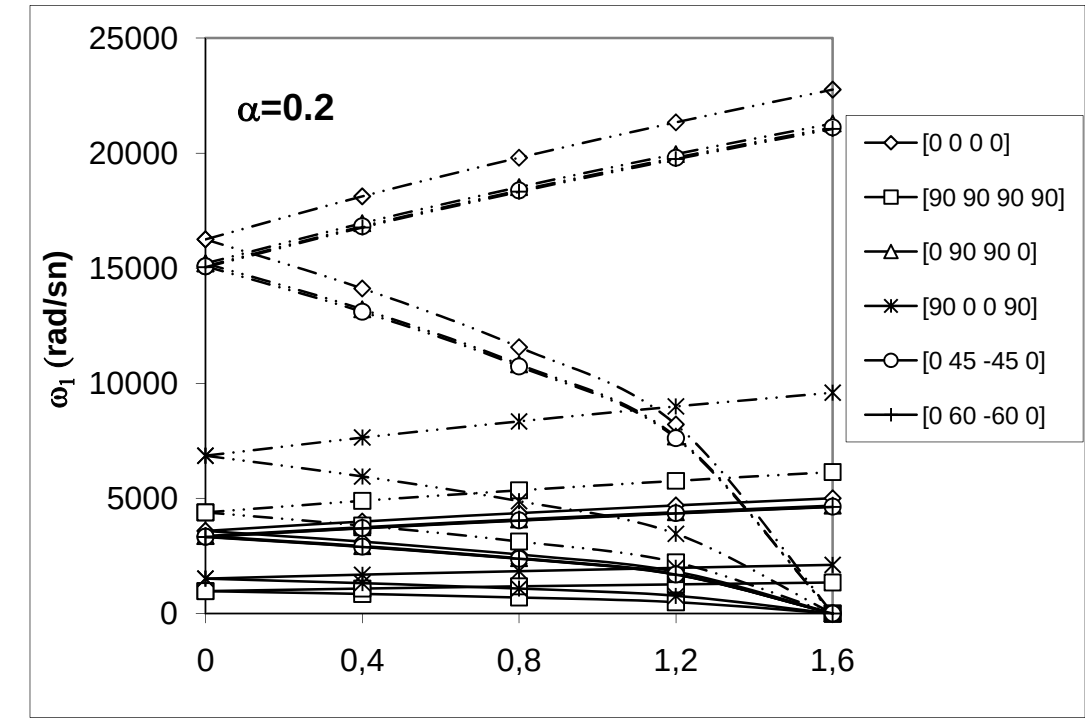


(a)

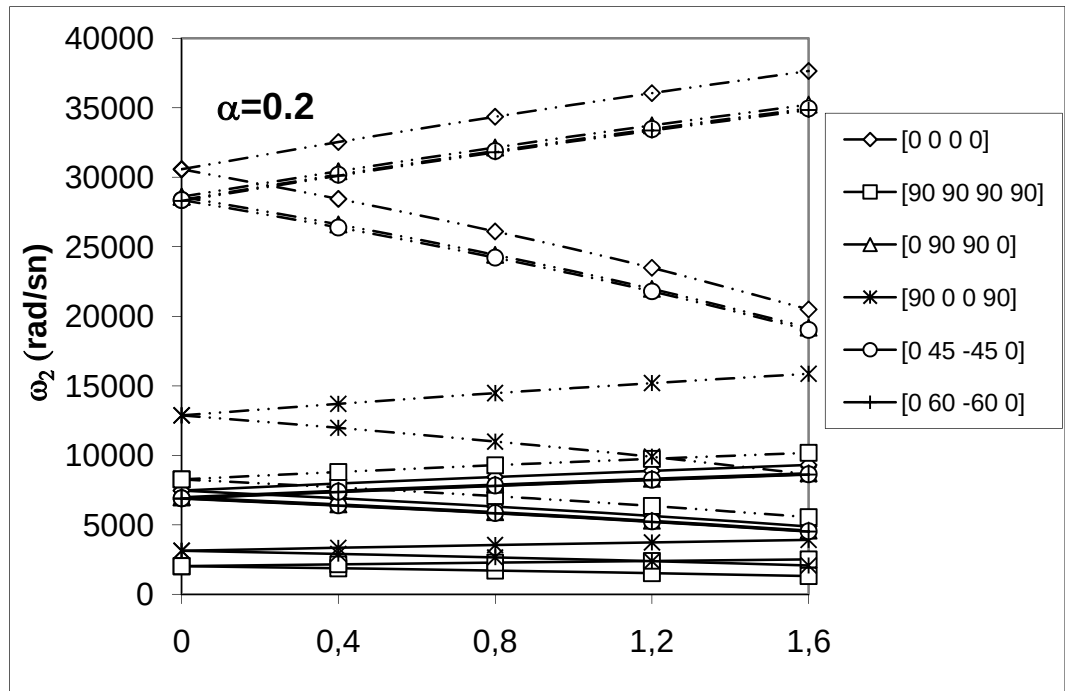


(b)

Figure 6.7 The effect of dynamic load parameter on the first and second dynamic stabilities regions of laminated curved beams for two different subtended angle. (a) First unstable region. (b) Second unstable region. $E_1=129.207$ GPa, $E_2=E_3=9.42512$ GPa, $G_{12}=5.15658$ GPa, $G_{13}=4.3053$ GPa, $G_{23}=2.5414$ GPa, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254$ m, $t=0.004$ m, $\rho=1550.0666$ kg/m³, $R=0.25$ m, $\theta=60^\circ$, _____ $\theta=120^\circ$

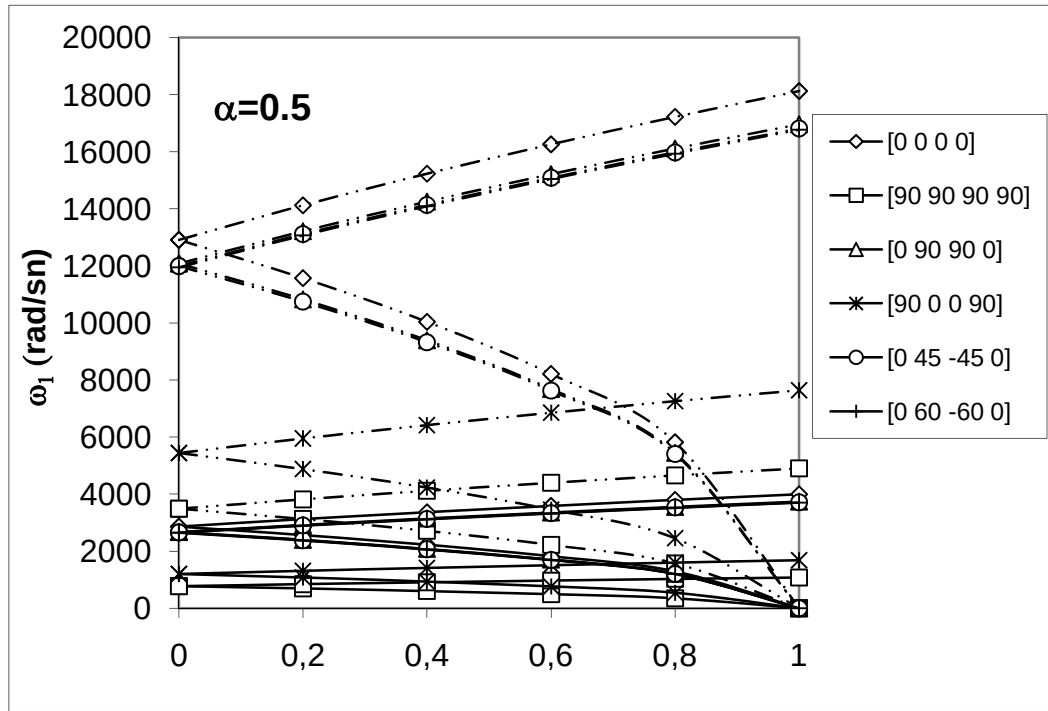


(a)

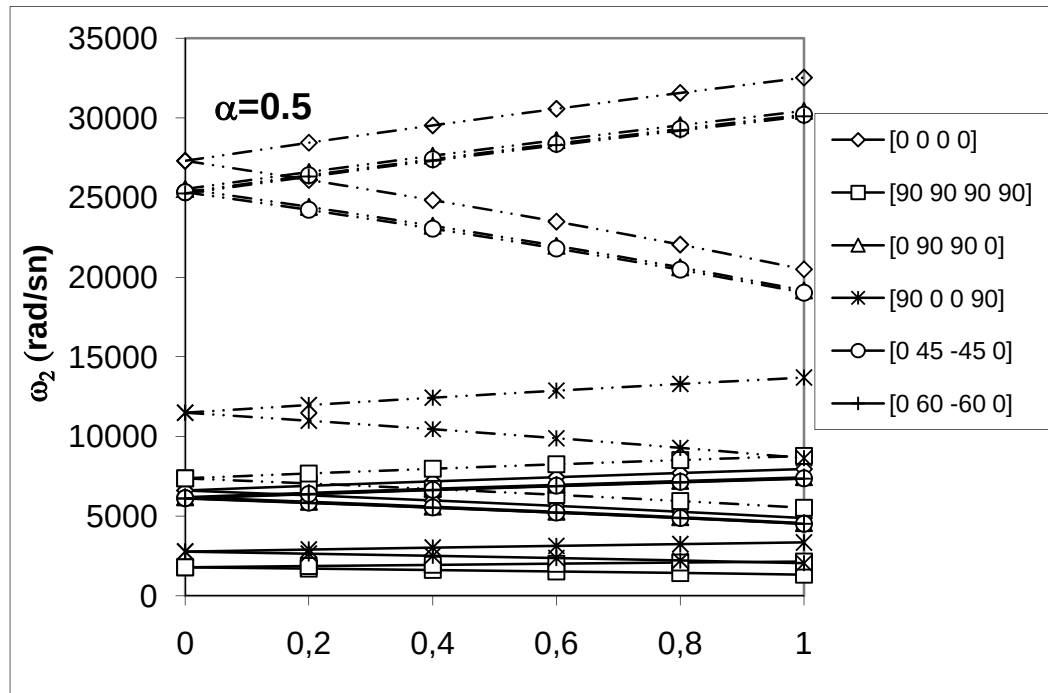


(b)

Figure 6.8 The effect of dynamic load parameter on the first and second dynamic stability regions of laminated curved beams for two different subtended angle. (a) First unstable region. (b) Second unstable region. $E_1=129.207$ GPa, $E_2=E_3=9.42512$ GPa, $G_{12}=5.15658$ GPa, $G_{13}=4.3053$ GPa, $G_{23}=2.5414$ GPa, $\nu_{12}=0.3$, $\nu_{21}=\nu_{12} \times E_2/E_1$, $b=0.0254$ m, $t=0.004$ m, $\rho=1550.0666$ kg/m³, $R=0.25$ m, $\theta=60^\circ$, _____ $\theta=120^\circ$

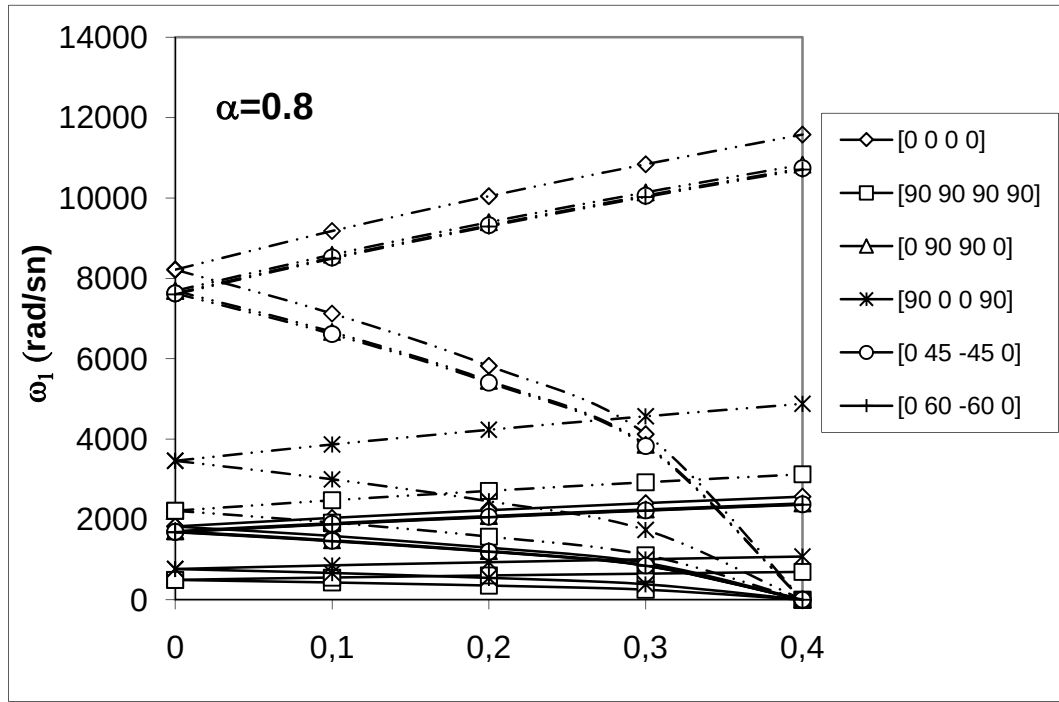


(a)

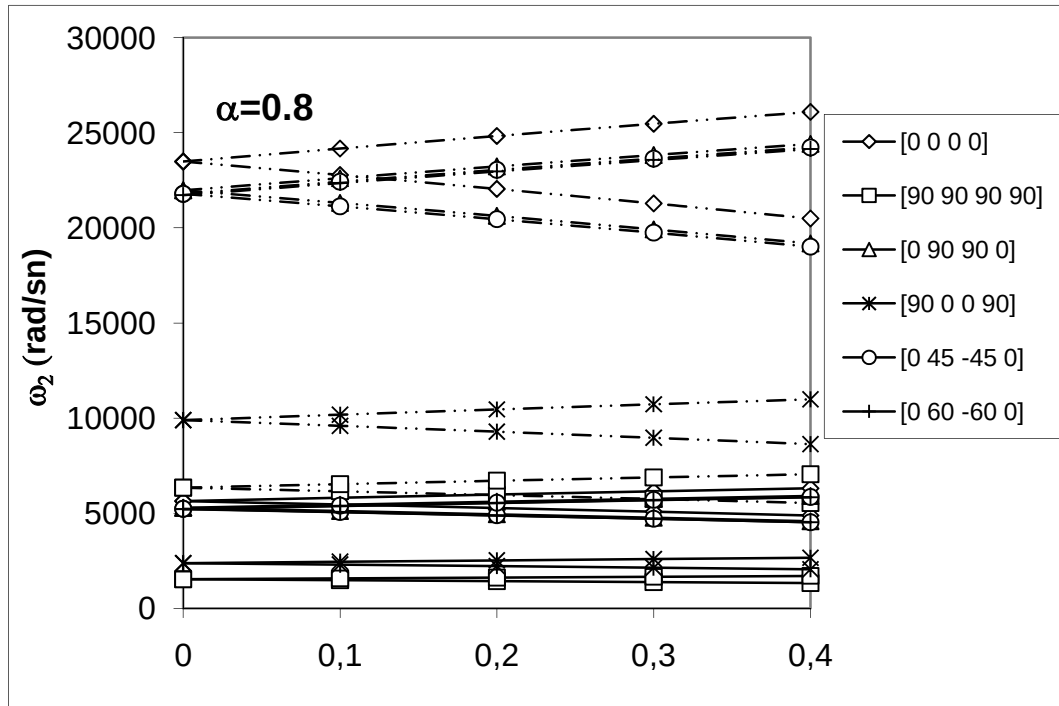


(b)

Figure 6.9 The effect of dynamic load parameter on the first and second dynamic stabilities regions of laminated curved beams for two different subtended angle. (a) First unstable region. (b) Second unstable region. $E_1 = 129.207$ GPa, $E_2 = E_3 = 9.42512$ GPa, $G_{12} = 5.15658$ GPa, $G_{13} = 4.3053$ GPa, $G_{23} = 2.5414$ GPa, $\nu_{12} = 0.3$, $\nu_{21} = \nu_{12} \times E_2/E_1$, $b = 0.0254$ m, $t = 0.004$ m, $\rho = 1550.0666$ kg/m³, $R = 0.25$ m, $\theta = 60^\circ$, _____ $\theta = 120^\circ$



(a)



(b)

Figure 6.10 The effect of dynamic load parameter on the first and second dynamic stabilities regions of laminated curved beams for two different subtended angle. (a) First unstable region. (b) Second unstable region. $E_1 = 129.207$ GPa, $E_2 = E_3 = 9.42512$ GPa, $G_{12} = 5.15658$ GPa, $G_{13} = 4.3053$ GPa, $G_{23} = 2.5414$ GPa, $\nu_{12} = 0.3$, $\nu_{21} = \nu_{12} \times E_2/E_1$, $b = 0.0254$ m, $t = 0.004$ m, $\rho = 1550.0666$ kg/m³, $R = 0.25$ m, $\theta = 60^\circ$, _____ $\theta = 120^\circ$

CHAPTER SEVEN

CONCLUSIONS

In this thesis in plane free vibration, static and dynamic stability analysis of laminated curved beams for various orientation angle and subtended angles are studied and the following conclusions are drawn.

The finite element method used in this investigation is ideal for the vibration, static and dynamic stability analysis of laminated curved beams. The results show that the finite element method gives very good accuracy, when the comparison made with the results obtained from Ansys.

When the subtended angle increases, first and second disturbing frequency of laminated curved beam decreases, this phenomenon can also be explained as follows; when the subtended angle of a laminated curved beam increases, the length of laminated curved beams also increases, consequently beam becomes very flexible.

When the subtended angle of a laminated curved beam increases, the dynamic instability regions narrow.

If the static load parameter increases, disturbing frequency moves towards origin.

If the static load parameter is equal to 0.2, 0.5 and 0.8, the initial disturbing frequency moves towards origin. The dynamic load parameter is bounded between the values of zero and 1.6, 1 and 0.4 respectively.

Finally, by changing the static and dynamic load parameters, dynamic stability of laminated curved beam may be conserved.

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APPENDIX

LIST OF SYMBOLS

A_s	Area of curved beam cross section
G_s	Shear modulus of curved beam
E_s	Modulus of elasticity of curved beam
f	Natural frequency
ρ	Density of curved beam
I_{xxs}	Second moment of area of cross section about axis X
d	Length of beam element
θ	Subtended angle
β_d	Dynamic component of load
α	Static component of load
R	Radius of curved beam
$[M]$	Global mass (inertia) matrix
$[K]$	Global stiffness matrix
$[K_{go}]$	Global geometric stiffness matrix
b	Width of the curved beam
t	Thickness of the curved beam
$[]^T$	Denotes transpose of a matrix
$[]^{-1}$	Denotes inverse of a matrix
$\{ \}$	Denotes column matrix
N	Number of layers
ω	Disturbing frequency
λ	Orientation angle