**DOKUZ EYLÜL UNIVERSITY**
**GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

# NON-POLYNOMIAL B-SPLINE FUNCTIONS

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**July, 2022**
**İZMİR**

# NON-POLYNOMIAL B-SPLINE FUNCTIONS

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**by**
**Fatma ZÜRNACI YETİŞ**

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**Ph.D. THESIS EXAMINATION RESULT FORM**

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## NON-POLYNOMIAL B-SPLINE FUNCTIONS


## ABSTRACT

This thesis is concerned with the explicit representation of non-polynomial B-spline functions for a wide collection of spline spaces including trigonometric splines, hyperbolic splines, and special Müntz spaces of splines. Divided differences are a basic tool in interpolation and approximation by polynomials and in spline theory. They are directly involved in the definition of B-splines. As in the original definition of B-splines, we represent the non-polynomial B-splines by using non-polynomial divided differences applied to a proper generalization of truncated-power function. Starting with the interpolation problem, we define the non-polynomial divided differences recursively as a generalization of classical divided differences. We also derive the identities and the properties of these non-polynomial divided differences such as symmetry, Leibniz formula, cancellation formula, affine combinations, and ratio of determinants. With the definition of a generalized derivative operator, it is shown that as in the polynomial case, non-polynomial divided differences can be viewed as a discrete analogue of derivatives. A generalization of the Hermite interpolation is also obtained using non-polynomial divided differences with repeated points. Using the link between non-polynomial divided differences and derivatives , we obtain the generalized Taylor series and state Generalized Taylor Theorem. With the definition of definite integral, the non-polynomial divided difference is built by the integration with non-polynomial B-spline functions. Also, we derive a general form of the Peano kernel theorem based on a generalized Taylor expansion. Here, we use the generalized Taylor Theorem with the integral remainder. As in the polynomial case, it is shown that the non-polynomial B-splines are in fact the Peano kernels of non-polynomial divided differences.



**Keywords:** B-spline, Interpolation, Divided difference, Taylor series, Peano kernel theorem.

## POLİNOM OLMAYAN B-SPLİNE FONKSİYONLARI

## ÖZ

Bu tez, trigonometrik spline'lar, hiperbolik spline'lar ve spline'ların özel Müntz uzayları dahil olmak üzere geniş bir spline uzayları koleksiyonu için polinom olmayan B-spline fonksiyonlarının açık gösterimi ile ilgilidir. Bölünmüş farklar, polinomlarla interpolasyon ve yaklaşımda, ve spline teorisinde temel bir araçtır. B-spline'ların tanımında direkt bulunurlar. B-spline'ların orijinal tanımında olduğu gibi, polinom olmayan B-spline'ları, kesik-kuvvet fonksiyonunun uygun bir genelleştirilmesine uygulanan polinom olmayan bölünmüş farkları kullanarak gösteriyoruz. İnterpolasyon probleminden başlayarak, polinom olmayan bölünmüş farkları, klasik bölünmüş farkların bir genellemesi olarak yinelemeli olarak tanımlarız. Simetri, Leibniz formülü, sadeleştirme formülü, afin kombinasyonları ve determinantların oranı gibi bu polinom olmayan bölünmüş farkların özdeşliklerini ve özelliklerini de elde ediyoruz. Genelleştirilmiş bir türev operatörü tanımı ile, polinom durumunda olduğu gibi, polinom olmayan bölünmüş farkların, türevlerin ayrı bir benzeri olarak görülebileceği gösterilmiştir. Tekrarlı noktalarla polinom olmayan bölünmüş farklar kullanılarak Hermite interpolasyonunun bir genellemesi de elde edilir. Polinom olmayan bölünmüş farklar ve türevler arasındaki bağlantıyı kullanarak, genelleştirilmiş Taylor serisini elde ederiz ve Genelleştirilmiş Taylor Teoremini ifade ederiz. Belirli integralin tanımı ile polinom olmayan bölünmüş fark, polinom olmayan B-spline fonksiyonlarının integrasyonuyla oluşturulur. Ayrıca, genelleştirilmiş Taylor serisine dayalı Peano çekirdek teoreminin genel bir biçimini elde ederiz. Burada, integral kalanıyla birlikte genelleştirilmiş Taylor Teoremini kullanıyoruz. Polinom durumunda olduğu gibi, polinom olmayan B-spline'larının aslında polinom olmayan bölünmüş farkların Peano çekirdekleri olduğu gösterilmiştir.

**Anahtar kelimeler:** B-spline, İnterpolasyon, Bölünmüş fark, Taylor serisi, Peano çekirdek teoremi.

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# CHAPTER ONE
# INTRODUCTION

Splines are piecewise polynomials whose derivatives agree up to some order at the joins. Today, splines appear in various areas such as geometric modeling, signal processing, data analysis, visualization, numerical simulation, and probability, etc. To work with splines effectively, one needs a proper basis to represent them. B-splines (where ”B” stands for basis or basic) are the most useful spline basis functions because they possess several features that are essential both theoretically and computationally.

B-splines were first constructed as early as nineteeth century by N. Lobachevsky as convolutions of certain probability distributions. Then, B-splines appeared in papers of Tchakaloff (1934), Popoviciu (1933). But, the modern theory of splines started with the fundamental work of Schoenberg (1946). The original definition of the B-splines is based on the divided differences. There are various approaches to define B-splines. De Boor (1972), Cox (1972) and Mansfield gave the recurrence relation for B-splines. Ramshaw (1987) used the blossoming to define B-splines.

There is always great interest in B-splines. In addition to the classical B-splines, B-splines from non-polynomial spline spaces, such as trigonometric, hyperbolic or Chebychevian ones are studied by Lyche & Winther (1979), Lyche (1979), Schumaker (1983, 1976). The approach to these various types of B-splines is very similar in all cases, that is, it relies on the use of a suitable divided difference operator, applied to a proper generalization of truncated power function. Dişibüyük & Goldman (2016) develop a unified theory of splines for a wide collection of spline spaces $\pi_n(\gamma_1, \gamma_2) = \text{span}\{\gamma_1^{n-k}\gamma_2^k\}_{k=0}^n$, where the functions $\gamma_1$ and $\gamma_2$ are linearly independent. Using the barycentric coordinates of a point on the planar parametric curve $P(t) = (\gamma_1(t), \gamma_2(t))$, $a \leq t \leq b$ satisfying

$$d(x_1, x_2) = \gamma_1(x_1)\gamma_2(x_2) - \gamma_1(x_2)\gamma_2(x_1) \tag{1.1}$$

that never vanishes for distinct $x_1, x_2 \in [a, b]$, Dişibüyük & Goldman (2016) construct compactly supported non-polynomial B-spline basis functions in terms of a recurrance relation and polar form. The space $\pi_n(\gamma_1, \gamma_2)$ includes as special cases standard polynomials, trigonometric polynomials, hyperbolic polynomials and special Müntz spaces, as well as many other spaces. For example,

- if $\gamma_1(x) = 1$ and $\gamma_2(x) = x$, then $d(x_1, x_2) = x_2 - x_1$ never vanishes for distinct values $x_1, x_2 \in [a, b]$ and the space $\pi_n(1, x)$ is the space of polynomials of degree $n$.
- If $\gamma_1(x) = \sin(x)$ and $\gamma_2(x) = \cos(x)$, then $d(x_1, x_2) = \sin(x_1 - x_2)$ and the space $\pi_n(\sin(x), \cos(x))$ is the space of trigonometric polynomials of degree $n$.
- If $\gamma_1(x) = \sinh(x)$ and $\gamma_2(x) = \cosh(x)$, we have the space $\pi_n(\sinh(x), \cosh(x))$ which is the hyperbolic polynomials.
- if $\gamma_1(x) = x^i$ and $\gamma_2(x) = x^j$, $\pi_n(\gamma_1, \gamma_2)$ is a Müntz space generated by span$\left\{ x^{(n-k)i+kj} \right\}_{k=0}^{n}$. Here $i$ and $j$ may be either positive or negative numbers.

It is seen from the examples above that the non-polynomial spline spaces $\pi_n(\gamma_1, \gamma_2)$ include as special cases standard polynomials, trigonometric polynomials, hyperbolic polynomials and special Müntz spaces, as well as many other spaces.

In this thesis, we give an explicit representation of non-polynomial B-spline functions for the space $\pi_n(\gamma_1, \gamma_2)$ by using non-polynomial divided differences applied to a proper generalization of truncated-power function. To define non-polynomial divided differences, we are concerned with the interpolation problem. Then, we give the definition of non-polynomial divided differences as a generalization of classical divided differences. We also derive a number of identities and properties of these non-polynomial divided differences such as symmetry, Leibniz formula, cancellation formula, affine combinations, and ratio of determinants. We define a novel variant of the truncated power function and then we express non-polynomial B-splines explicitly in terms of non-polynomial divided differences of this truncated power function. Defining a certain derivative which is a generalization of the classical derivative, we obtain the connection between non-polynomial divided differences and derivatives. A generalization of the Hermite interpolation is also obtained using non-polynomial divided differences with repeated points. Making use of a certain integration, we build non-polynomial divided differences by integration with non-polynomial B-splines. Also, we obtain a generalization of the Taylor series using non-polynomial divided differences in terms of the generalized derivative operator. Finally, we derive a general form of the Peano kernel theorem based on a generalized Taylor expansion. Here we use the generalized Taylor Theorem with the integral remainder. As in the polynomial case, it is shown that the non-polynomial B-splines are in fact the Peano kernels of non-polynomial divided differences.

To make this thesis fluent and comprehensible, we will briefly summarize polynomial divided differences and B-spline functions together with their properties.

### 1.1 Polynomial Divided Differences

The divided difference is commonly used in numerical analysis and approximation theory, and is related to both Newton interpolation and B-spline approximation. Although divided differences providing the dual functionals for the newton basis are rightly associated with Newton, the term 'divided difference' was, according to Whittaker & Robinson (1946), first used in 1842 by De Morgan (1836).

$f[x_0, \ldots, x_n]$ denotes the divided difference of a function $f : (a, b) \to \mathbb{R}$ at the points $a < x_0 \leq \ldots \leq x_n < b$. This notation is recursively defined by $f[x_0] := f(x_0)$ and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} \quad \text{if} \quad n > 0.$$

Also, the $n$-th order divided difference, $f[x_0, \ldots, x_n]$, of a function $f(x)$ at the (real or complex) points $x_0, \ldots, x_n$, is the coefficient of $x^n$ of the polynomial $p(x)$, degree at most $n$, which agrees with (i.e., interpolates to) $f(x)$ at $x = x_0, \ldots, x_n$ (Lee (1989)). For all $n > 1$ the $n$-th order divided difference of $f$ can be expressed as

$$f[x_0, x_1, \ldots, x_n] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{j=0 \\ j\neq r}}^{n}(x_r - x_j)}$$

(see Corominas (1953)). It is clear from the above expression that $f[x_0, \ldots, x_n]$ is symmetric in its arguments, and that if $f$ and $g$ agree at $x_0, \ldots, x_n$ then

$$f[x_0, \ldots, x_n] = g[x_0, \ldots, x_n].$$

In the literature, many useful identities for divided differences are obtained. The divided difference of a product in terms of divided differences of the factors is defined as

$$(fg)[x_0, x_1, \ldots, x_n] = \sum_{r=0}^{n} f[x_0, \ldots, x_r]g[x_r, \ldots, x_n].$$

This formula is often called the Leibniz rule since it generalizes the Leibniz formula for differentiation, and Leibniz rule has a major role in the development of the spline theory. It is main tool in the derivation of B-spline recurrence relations. Earliest reference for it is Popoviciu (1933) who refers, for the case of uniform spacing, to Jacobsthal (1920). Nonetheless, the formula is generally credited to Steffensen, because of his paper (Stenffensen (1939)). A new proof of Leibniz formula is obtained by Popoviciu as an application of his fundamental formula for divided differences (Popoviciu (1940)). In the case of distinct knots, an interesting proof of Leibniz formula is given by De Boor (1978).

It is convenient to think of the divided difference of a function as a value of the derivative of the function, and Mean value theorem for divided differences (Sahoo & Riedel (1998)) justifies this point of view. The mean value theorem for the $n$-th order divided difference states that if $f : [a, b] \to \mathbb{R}$ is $n$-times continuously differentiable and $x_0, x_1, \ldots, x_n$ are $n + 1$ distinct points in $[a, b]$, then there exists

$\eta \in (\min\{x_0, x_1, \ldots, x_n\}, \max\{x_0, x_1, \ldots, x_n\})$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^n(\eta)}{n!}.$$

Here, $\eta$ is referred to a differential mean value (Isaacson & Keller (1966)). Reduction formula (or cancellation formula) for divided differences (see Micchelli (1980), Lee (1989)) is given as follows

$$f[x_0, x_1, \ldots, x_n] = \{(x - x_{n+1})f\}[x_0, x_1, \ldots, x_{n+1}].$$

The usefulness of this identity is that it allows one to expand or shrink the argument list in a divided difference. Also, divided difference is expressed as an affine combinations of divided differences, i.e.,

$$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} \lambda_k f[x_0, \ldots, x_{k-1}, \tau, x_{k+1}, \ldots, x_{n+1}],$$

where

$$\tau = \sum_k \lambda_k x_k \quad \text{and} \quad \sum_k \lambda_k = 1.$$

This identity is proved by Micchelli (1980) as a preliminary step in obtaining a recurrence formula for multivariate B-splines.

The divided differences possess some representations via integral. An example is Hermite-Genocci formula:

$$f[x_0, x_1, \ldots, x_n] = \int_{\Delta^n} f^{(n)}(x_0 + v_1(x_1 - x_0) + \ldots + v_n(x_n - x_0))dv_1 \ldots dv_n,$$

where $\Delta^n = \{(v_1, \ldots, v_n) | v_i \geq 0$ and $\sum_i v_i \leq 1\}$. It says that a divided difference is an integral average of the derivative of the same order (see e.g. Nörlund (1924), Micchelli (1980) ). Divided difference is also represented by the integration with univariate B-splines (Karlin et al. (1986))

$$f[x_k, x_{k+1}, \ldots, x_{k+n}] = \int_{support} \Big\{\frac{N_{k,n}(x)}{x_{k+n+1} - x_k}\Big\}\Big\{\frac{f^{(n)}}{n!}\Big\}dx.$$

For analytic functions, the divided difference is constructed explicitly using complex contour integration (Davis (1975))

$$f[x_0, x_1, \dots, x_n] = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - x_0) \dots (z - x_n)} dz,$$

where $C$ is any simple closed curve containing $\{x_0, x_1, \dots, x_n\}$ and the function $f$ is analytic inside $C$.

If the knots are not distinct, then the divided difference is defined by a limit process. Namely,

$$f[\underbrace{x_0, \dots, x_0}_{k+1 \text{ terms}}, x_{k+1}, \dots, x_n] = \lim_{x_1, \dots, x_k \to x_0} f[x_0, x_1, \dots, x_n]$$

provided the limit exists. In particular,

$$f[\underbrace{x_0, \dots, x_0}_{k+1 \text{ terms}}] = \lim_{x_1, \dots, x_n \to x_0} f[x_0, x_1, \dots, x_n].$$

Also, the following result is given

$$f[\underbrace{x_0, \dots, x_0}_{n+1 \text{ terms}}] = \lim_{x_1, \dots, x_n \to x_0} f[x_0, x_1, \dots, x_n] = \frac{f^n(x_0)}{n!},$$

where $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and possesses a derivative of order $n$ at $x_0$ (Popoviciu (1933)). In the case of repeated nodes, divided differences are defined as the coefficients of Hermite's interpolation polynomials. Corominas (1953) introduces divided differences over repeated nodes using iterated limits. De Boor (2005) shows that the mean value theorem for divided differences also holds, and Xu et al. (2010) investigates the asymptotic behavior of the differential mean value of divided differences with repetitions.

In addition to polynomial divided differences, generalized divided differences are also studied, which is originally introduced by Popoviciu (1959). In the papers (Mühlbach (1973, 1999)), both classical and generalized divided differences are rewritten as ratios of Vandermonde determinants, and this formula is a recurrence formula. The papers of Mühlbach (1978), Karlin (1968) and Walz (1992) can be referred to readers for more information about generalized divided differences. Using generalized divided differences, authors study B-splines from non-polynomial spline spaces, such as trigonometric, hyperbolic or Chebychevian ones. Schumaker (1983) defines hyperbolic divided differences and, Lyche & Winther (1979) define trigonometric divided differences. Lyche (1979) proposes a scheme for trigonometric divided differences and a Newton-type interpolation formula.

### 1.2 Polynomial B-Spline Functions

B-splines play an important role in computer aided geometric design. They are one of the most powerful and useful tools. There are several approaches to define B-Spline functions. The traditional definition of B-spline uses divided difference of a truncated power function (Schoenberg (1946), Curry & Schoenberg (1947)).
For any fixed value of $n \geq 0$ and for all $i$,

$$N_{i,n}(x) = (x_{i+n+1} - x_i).\,[x_i, \ldots, x_{i+n+1}](t-x)_+^n,$$

where $[x_i, \ldots, x_{i+n+1}]$ denotes the divided difference operator of degree $n+1$ that is applied to the truncated power $(t-x)_+^n$ , regarded as a function of the variable $t$ where

$$(t-x)_+^n = \begin{cases} (t-x)^n, & t \geq x \\ 0, & -\infty < t < x. \end{cases}$$

De Boor (1972), Cox (1972) and Mansfield defined B-spline functions via a recurrence relation. B-splines of degree zero are piecewise constants defined by

$$N_{i,0} = \begin{cases} 1, & x_i < x \leq x_{i+1} \\ 0, & otherwise \end{cases}$$

and those of degree $n > 0$ are defined recursively in terms of those of degree $n-1$ by

$$N_{i,n}(x) = \left(\frac{x - x_i}{x_{i+n} - x_i}\right) N_{i,n-1}(x) + \left(\frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}}\right) N_{i+1,n-1}(x).$$

The interval of support of the B-spline $N_{i,n}$ is $[x_i, x_{i+n+1}]$, and $N_{i,n}$ is positive in the interior of this interval. For $n \geq 1$ the B-spline $N_{i,n}$ is a spline of degree $n$ on $(-\infty, \infty)$, with interval of support $[x_i, x_{i+n+1}]$. For any integer $n \geq 0$, the B-splines of degree $n$ are a basis for splines of degree $n$ defined on the knots $x_i$.

The B-spline basis functions satisfy a two-term differentiation (De Boor (1972)). For $n \geq 2$, it is given that

$$\frac{d}{dx} N_{i,n}(x) = n\left[\frac{N_{i,n-1}(x)}{x_{i+n} - x_i} - \frac{N_{i+1,n-1}(x)}{x_{i+n+1} - x_{i+1}}\right] \tag{1.2}$$

for all real $x$. For $n = 1$, (1.2) holds for all $x$ except at the three knots $x_i$, $x_{i+1}$ and $x_{i+2}$, where the derivative of $N_{i,1}$ is not defined.

Another important identity for B-splines is the Marsden's identity which implies that polynomials are represented as linear combinations of B-splines. For any fixed value of $n \geq 0$,

$$(t - x)^n = \sum_i (t - x_{i+1}) \ldots (t - x_{i+n}) N_{i,n}(x).$$

The B-splines form a partition of unity

$$\sum_i N_{i,n}(x) = 1.$$

The following expression implies that a truncated power function can be written as a sum of multiples of B-splines. For any knot $x_j$ and any integer $n \geq 0$,

$$(x_j - x)_+^n = \sum_{i=-\infty}^{j-n-1} (x_j - x_{i+1}) \ldots (x_j - x_{i+n}) N_{i,n}(x).$$

The outline of the thesis is as follows: In Section 2.1, we investigate the interpolation problem and show that the problem has a unique solution. In Section 2.2, we define non-polynomial divided differences and show that these non-polynomial divided differences satisfy a recurrence relation similar to the one for polynomial divided differences. The error term of the interpolation problem is also found. Some properties and identities of non-polynomial divided differences are investigated in Section 2.2.1. A generalization of the Hermite interpolation is obtained using non-polynomial divided differences with repeated points in Section 2.2.2. In Section 2.3, we define a novel variant of the truncated power function and derive an explicit representation of non-polynomial B-spline functions via non-polynomial divided differences of the truncated power function. In Section 2.4, we define a certain definite integral and show that the non-polynomial divided difference operator is built by the integration with non-polynomial B-splines. In Chapter 3, we obtain a generalization of the Taylor series using the link between non-polynomial divided differences and derivatives. In Chapter 4, a general form of the Peano kernel theorem based on a generalized Taylor expansion is derived. Finally, the findings of this research are summarized and conclusions are given in Chapter 5.

# CHAPTER TWO

# NON-POLYNOMIAL B-SPLINE FUNCTIONS

## 2.1 Interpolation Problem

Let $\Pi_n(\gamma_1, \gamma_2) = \pi_0(\gamma_1, \gamma_2) \oplus \pi_1(\gamma_1, \gamma_2) \oplus \ldots \oplus \pi_n(\gamma_1, \gamma_2)$. Given the values of a function $f(x)$ at $n+1$ distinct values of $x$ say $x_0, x_1, \ldots, x_n$, we can find a function $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$ such that

$$g_n(x_j) = f(x_j) \quad \text{for} \quad 0 \leq j \leq n.$$

For $i = 0, 1, \ldots, n$, define

$$\sigma_i(x) = \begin{cases} 1 & i = 0, \\ d(x_0, x)d(x_1, x) \ldots d(x_{i-1}, x) & i \neq 0. \end{cases}$$

Equation (1.1) guarantees that $d(x_j, x)$ is a linear combination of $\gamma_1(x)$ and $\gamma_2(x)$ for any point $x_j$ which implies that the function $\sigma_i \in \pi_i(\gamma_1, \gamma_2)$ for all $i$. That is, the interpolating function $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$, which assumes the same values as the function $f$ at $x_0, x_1, \ldots, x_n$ can be written as

$$g_n(x) = a_0\sigma_0 + a_1\sigma_1 + \ldots + a_n\sigma_n. \tag{2.1}$$

The coefficients $a_j$ can be determined from the conditions

$$g_n(x_j) = f(x_j) \quad \text{for} \quad 0 \leq j \leq n.$$

This leads to the system of linear equations

$$f(x_j) = a_0\sigma_0(x_j) + a_1\sigma_1(x_j) + \ldots + a_n\sigma_n(x_j), \tag{2.2}$$

for $0 \leq j \leq n$. The system of equations (2.2) has the following matrix form

$$\begin{bmatrix} \sigma_0(x_0) & \sigma_1(x_0) & \dots & \sigma_{n-1}(x_0) & \sigma_n(x_0) \\ \sigma_0(x_1) & \sigma_1(x_1) & \dots & \sigma_{n-1}(x_1) & \sigma_n(x_1) \\ \vdots & & & & \\ \sigma_0(x_n) & \sigma_1(x_n) & \dots & \sigma_{n-1}(x_n) & \sigma_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

Considering $d(x_i, x_j) = 0$ for $i = j$, it is obtained that $\sigma_i(x_j) = 0$ when $i > j$. Then we have matrix form

$$\mathbf{S} \cdot \mathbf{a} = \mathbf{f}, \tag{2.3}$$

where $\mathbf{a} = \begin{bmatrix} a_0 & a_1 & \dots & a_n \end{bmatrix}^T$, $\mathbf{f} = \begin{bmatrix} f(x_0) & f(x_1) & \dots & f(x_n) \end{bmatrix}^T$ and the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & \sigma_1(x_1) & 0 & \dots & 0 \\ \vdots & & & & \\ 1 & \sigma_1(x_n) & \sigma_2(x_n) & \dots & \sigma_n(x_n) \end{bmatrix}.$$

If we evaluate the determinant of **S** via first row, we obtain

$$\det \mathbf{S} = \prod_{j<i}^{n} d(x_j, x_i).$$

The determinant of the matrix **S** is therefore nonzero for distinct nodes $x_0, x_1, \dots, x_n$. Hence, the linear system (2.3) has a unique solution and we have proved the following theorem.

**Theorem 2.1.1.** *Given the values of a function $f(x)$ at $n + 1$ distinct values of $x$ say $x_0, x_1, \dots, x_n$, we can find a unique function $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$ in the form $g_n(x) = a_0\sigma_0(x) + a_1\sigma_1(x) + \dots + a_n\sigma_n(x)$ such that*

$$g_n(x_j) = f(x_j), \quad j = 0, 1, \dots n.$$

Note that the interpolating function can be written in the form

$$g_n(x) = \sum_{i=0}^{n} \sum_{j=0}^{i} c_{i,j} \gamma_1^{i-j}(x) \gamma_2^{j}(x),$$

where $c_{i,j}$ are constant coefficients. We examine existence and uniqueness of the interpolating function $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$ in Lemma 2.1.2 and Lemma 2.1.3.

**Lemma 2.1.2.** *If* $1 \notin span\{\gamma_1, \gamma_2\}$*, then there exists more than one solution satisfying* $g_n(x_j) = f(x_j)$ *for* $0 \le j \le n$.

*Proof.* Since $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$, $g_n(x)$ is written in the form

$$\begin{aligned} g_n(x) = {} & c_{0,0} + c_{1,0}\gamma_1(x) + c_{1,1}\gamma_2(x) + c_{2,0}\gamma_1^2(x) + c_{2,1}\gamma_1(x)\gamma_2(x) + c_{2,2}\gamma_2^2(x) + \\ & \ldots + c_{n,0}\gamma_1^n(x) + c_{n,1}\gamma_1^{n-1}(x)\gamma_2(x) + \ldots + c_{n,n-1}\gamma_1(x)\gamma_2^{n-1}(x) + c_{n,n}\gamma_2^n(x). \end{aligned}$$

Using the interpolation conditions $g_n(x_i) = f(x_i), \quad i = 0, 1, \ldots, n$, we get the system as follows

$$\begin{bmatrix} 1 & \gamma_1(x_0) & \gamma_2(x_0) & \ldots & \gamma_2^n(x_0) \\ 1 & \gamma_1(x_1) & \gamma_2(x_1) & \ldots & \gamma_2^n(x_1) \\ \vdots & & & & \\ 1 & \gamma_1(x_n) & \gamma_2(x_n) & \ldots & \gamma_2^n(x_n) \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ \vdots \\ c_{n,n} \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

There are $\frac{(n+1)(n+2)}{2}$ coefficients. We get a system of linear equations which has fewer equations than unknowns, then the system will have either no solution or infinitely many. However, the consistency is guaranteed by Theorem 2.1.1. So, the linear system has infinitely many solution. □

**Lemma 2.1.3.** *If* $1 \in span\{\gamma_1, \gamma_2\}$*, then there exist a unique solution satisfying* $f(x_j) = g_n(x_j)$ *for* $0 \le j \le n$.

*Proof.* If $1 \in \text{span}\{\gamma_1, \gamma_2\}$, then $\pi_{n-1}(\gamma_1, \gamma_2) \subset \pi_n(\gamma_1, \gamma_2)$ (see Dişibüyük & Goldman (2015)). Since $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$ and $\pi_{n-1}(\gamma_1, \gamma_2) \subset \pi_n(\gamma_1, \gamma_2)$, we get $g_n(x) \in$

$\pi_n(\gamma_1, \gamma_2)$. In this case, $g_n(x)$ can be written as follows

$$g_n(x) = d_0\gamma_1^n + d_1\gamma_1^{n-1}\gamma_2 + \ldots + d_n\gamma_2^n.$$

Using the interpolation conditions $g_n(x_j) = f(x_j), \;\; j = 0, \ldots, n$, we get the following system

$$\mathbf{K} \cdot \mathbf{d} = \mathbf{f},$$

where

$$\mathbf{K} = \begin{bmatrix} \gamma_1^n(x_0) & \gamma_1^{n-1}(x_0)\gamma_2(x_0) & \ldots & \gamma_1^n(x_0) \\ \gamma_1^n(x_1) & \gamma_1^{n-1}(x_1)\gamma_2(x_1) & \ldots & \gamma_1^n(x_1) \\ \vdots & & & \\ \gamma_1^n(x_n) & \gamma_1^{n-1}(x_n)\gamma_2(x_n) & \ldots & \gamma_1^n(x_n) \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix}, \mathbf{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

We know that the functions $\gamma_1^{n-k}\gamma_2^k, k = 0, 1, \ldots, n$ are linearly independent. Thus, there exists a unique solution. $\square$

Note that the uniqueness of the interpolation problem in the space $\pi_n(\gamma_1, \gamma_2)$ is also studied in Dişibüyük (2015).

### 2.2 Non-polynomial Divided Differences

We begin with determining the coefficients of equation (2.1). Since $d(x_i, x_j) = 0$ for $i = j$, the interpolation conditions $g_n(x_0) = f(x_0)$ and $g_n(x_1) = f(x_1)$ gives $a_0 = f(x_0)$ and $a_1 = \dfrac{f(x_1) - f(x_0)}{d(x_0, x_1)}$ respectively.

In general, the coefficient $a_j$ depends on the functions $f, \gamma_1, \gamma_2$ evaluated at the points $x_0, x_1, \ldots, x_j$. To emphasize this dependence, let us write

$$a_j := f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_j], \quad j = 0, 1, \ldots, n.$$

Hence the interpolating function $g_n \in \Pi_n(\gamma_1, \gamma_2)$ that assumes same values as the function $f$ at $x_0, x_1, \ldots, x_n$ is written in the form

$$g_n(x) = \sum_{j=0}^{n} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_j]\sigma_j(x). \tag{2.4}$$

For $x = x_{n-1}$, since $\sigma_n(x_{n-1}) = 0$, equation (2.4) gives

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}]\sigma_{n-1}(x_{n-1}) = f(x_{n-1}) - \sum_{j=0}^{n-2} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_j]\sigma_j(x_{n-1}).$$

On the other hand, let $\tilde{g}_n(x)$ be the interpolating function that interpolate the function $f$ at the points $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n$ where $\tilde{x}_i = x_i$ for $i = 0, 1, \ldots, n-2$, $\tilde{x}_{n-1} = x_n$ and $\tilde{x}_n$ is any number. Then, using similar arguments above, we may write

$$\tilde{g}_n(x) = \sum_{j=0}^{n} f_{\gamma_1,\gamma_2}[\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_j]\tilde{\sigma}_j(x), \tag{2.5}$$

where

$$\tilde{\sigma}_i(x) = \begin{cases} 1 & i = 0, \\ d(\tilde{x}_0, x)d(\tilde{x}_1, x)\ldots d(\tilde{x}_{i-1}, x) & i \neq 0. \end{cases} \tag{2.6}$$

Note that $\tilde{\sigma}_i(x) = \sigma_i(x)$ for $i = 0, 1, \ldots, n-1$. Now substituting $x = \tilde{x}_{n-1}$ in equation (2.5) gives

$$f_{\gamma_1,\gamma_2}[\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{n-1}]\tilde{\sigma}_{n-1}(\tilde{x}_{n-1}) = f(\tilde{x}_{n-1}) - \sum_{j=0}^{n-2} f_{\gamma_1,\gamma_2}[\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_j]\tilde{\sigma}_j(\tilde{x}_{n-1}).$$

Since $\tilde{x}_i = x_i$ for $i = 0, 1, \ldots, n-2$, $\tilde{x}_{n-1} = x_n$ and $\tilde{\sigma}_i(x) = \sigma_i(x)$
for $i = 0, 1, \ldots, n-1$, we may write

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-2}, x_n]\sigma_{n-1}(x_n) = f(x_n) - \sum_{j=0}^{n-2} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_j]\sigma_j(x_n). \tag{2.7}$$

Now substituting $x = x_n$ in equation (2.4) and using equation (2.7) gives

$$\begin{aligned} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n]\sigma_n(x_n) &= f(x_n) - \sum_{j=0}^{n-1} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_j]\sigma_j(x_n) \\ &= f(x_n) - \sum_{j=0}^{n-2} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_j]\sigma_j(x_n) \\ &- f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}]\sigma_{n-1}(x_n) \\ &= f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-2}, x_n]\sigma_{n-1}(x_n) \\ &- f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}]\sigma_{n-1}(x_n). \end{aligned}$$

Finally, dividing both sides by $\sigma_n(x_n)$, we get

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n] = \frac{f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-2}, x_n] - f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}]}{d(x_{n-1}, x_n)}.$$

Hence, we obtain that the $n$th degree divided difference is expressed in terms of the divided differences of degree $n-1$. Thus, we define non-polynomial divided differences recursively as follows.

**Definition 2.2.1.** *Given $n+1$ distinct abscissas $x_j, x_{j+1}, \ldots, x_{j+n}$. The non-polynomial divided difference of order $n$ is defined recursively as*

$$f_{\gamma_1,\gamma_2}[x_j, \ldots, x_{j+n}] = \frac{f_{\gamma_1,\gamma_2}[x_j, \ldots, x_{j+n-2}, x_{j+n}] - f_{\gamma_1,\gamma_2}[x_j, \ldots, x_{j+n-1}]}{d(x_{j+n-1}, x_{j+n})}, \quad (2.8)$$

*where $f_{\gamma_1,\gamma_2}[x_k] = f(x_k)$ for $k = j, j+1, \ldots, j+n$.*

Now using the recurrence relation (2.8), we are going to derive the error between a function $f$ and its interpolating function $g_n$.

**Theorem 2.2.1.** *If $g_n(x) \in \Pi_n(\gamma_1, \gamma_2)$ is the function that interpolates $f$ at $n+1$ distinct points $x_i$, $i = 0, 1, \ldots, n$, then*

$$f(x) - g_n(x) = \sigma_{n+1}(x) f_{\gamma_1,\gamma_2}[x_0, \ldots, x_n, x].$$

*Proof.* From equation (2.8), we may write

$$\begin{aligned} f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n-1},x] =& f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] \\ &+ d(x_n,x) f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n,x]. \end{aligned} \tag{2.9}$$

Similarly, we have

$$f_{\gamma_1,\gamma_2}[x] = f_{\gamma_1,\gamma_2}[x_0] + d(x_0,x) f_{\gamma_1,\gamma_2}[x_0,x].$$

Using equation (2.9) to expand the term $f_{\gamma_1,\gamma_2}[x_0,x]$ in the above equation yields

$$\begin{aligned} f_{\gamma_1,\gamma_2}[x] &= f_{\gamma_1,\gamma_2}[x_0] + d(x_0,x)\left(f_{\gamma_1,\gamma_2}[x_0,x_1] + d(x_1,x) f_{\gamma_1,\gamma_2}[x_0,x_1,x]\right) \\ &= g_1(x) + \sigma_2(x) f_{\gamma_1,\gamma_2}[x_0,x_1,x]. \end{aligned}$$

Repeated application of equation (2.9) gives

$$f(x) = g_n(x) + \sigma_{n+1}(x) f_{\gamma_1,\gamma_2}[x_0,\ldots,x_n,x]$$

which is the desired result. □

### *2.2.1 Properties of Non-Polynomial Divided Differences*

Here we are going to derive properties of the non-polynomial divided difference operator.

Firstly, we show that the divided difference $f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n]$ is symmetric in its arguments $x_0,x_1,\ldots,x_n$ if $1 \in \text{span}\{\gamma_1,\gamma_2\}$ by Lemma 2.2.2.

**Lemma 2.2.2.** *If $1 \in span\{\gamma_1,\gamma_2\}$, then the divided difference $f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n]$ can be expressed as the following symmetric sum of multiples of $f(x_j)$*

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{j=0 \\ j\neq r}}^{n} d(x_j,x_r)}. \tag{2.10}$$

*Proof.* We use induction on $n$. Equation (2.10) holds for $n = 1$, since

$$f_{\gamma_1,\gamma_2}[x_0, x_1] = \frac{f_{\gamma_1,\gamma_2}[x_1] - f_{\gamma_1,\gamma_2}[x_0]}{d(x_0, x_1)} = \frac{f(x_1)}{d(x_0, x_1)} + \frac{f(x_0)}{d(x_1, x_0)}.$$

Assume that (2.10) is true for $n > 1$. Then by equation (2.8), we may write

$$\begin{aligned} d(x_n, x_{n+1}) f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] &= f_{\gamma_1,\gamma_2}[x_0, \ldots, x_{n-1}, x_{n+1}] - f_{\gamma_1,\gamma_2}[x_0, \ldots, x_n] \\ &= f_{\gamma_1,\gamma_2}[\tilde{x}_0, \ldots, \tilde{x}_{n-1}, \tilde{x}_n] - f_{\gamma_1,\gamma_2}[x_0, \ldots, x_n], \end{aligned}$$

where $x_i = \tilde{x}_i$ for $i = 0, 1, \ldots, n-1$ and $\tilde{x}_n = x_{n+1}$. Then the induction hypothesis gives

$$\begin{aligned} &d(x_n, x_{n+1}) f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] \\ &\quad = \sum_{r=0}^{n} \frac{f(\tilde{x}_r)}{\prod_{\substack{j=0\\ j\neq r}}^{n} d(\tilde{x}_j, \tilde{x}_r)} - \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{j=0\\ j\neq r}}^{n} d(x_j, x_r)} \\ &\quad = \sum_{r=0}^{n-1} \frac{f(\tilde{x}_r)}{d(\tilde{x}_n, \tilde{x}_r) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(\tilde{x}_j, \tilde{x}_r)} + \frac{f(\tilde{x}_n)}{\prod_{j=0}^{n-1} d(\tilde{x}_j, \tilde{x}_n)} \\ &\quad\quad - \sum_{r=0}^{n-1} \frac{f(x_r)}{d(x_n, x_r) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} - \frac{f(x_n)}{\prod_{j=0}^{n-1} d(x_j, x_n)}. \end{aligned}$$

Considering again $x_i = \tilde{x}_i$ for $i = 0, 1, \ldots, n-1$ and $\tilde{x}_n = x_{n+1}$, it is obtained that

$$\begin{aligned} &d(x_n, x_{n+1}) f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] \\ &\quad = \sum_{r=0}^{n-1} \frac{f(x_r)}{d(x_{n+1}, x_r) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} + \frac{f(x_{n+1})}{\prod_{j=0}^{n-1} d(x_j, x_{n+1})} \\ &\quad\quad - \sum_{r=0}^{n-1} \frac{f(x_r)}{d(x_n, x_r) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} - \frac{f(x_n)}{\prod_{j=0}^{n-1} d(x_j, x_n)} \\ &\quad = \sum_{r=0}^{n-1} \frac{f(x_r)\left(\frac{1}{d(x_{n+1}, x_r)} - \frac{1}{d(x_n, x_r)}\right)}{\prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} \\ &\quad\quad + \frac{f(x_{n+1})}{\prod_{j=0}^{n-1} d(x_j, x_{n+1})} - \frac{f(x_n)}{\prod_{j=0}^{n-1} d(x_j, x_n)}. \end{aligned}$$

Dividing both sides by $d(x_n, x_{n+1})$, we get

$$\begin{aligned}
&f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] \\
&\qquad = \sum_{r=0}^{n-1} \frac{f(x_r)\left(\frac{1}{d(x_{n+1},x_r)} - \frac{1}{d(x_n,x_r)}\right)}{d(x_n, x_{n+1}) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} + \frac{f(x_{n+1})}{d(x_n, x_{n+1}) \prod_{j=0}^{n-1} d(x_j, x_{n+1})} \\
&\qquad - \frac{f(x_n)}{d(x_n, x_{n+1}) \prod_{j=0}^{n-1} d(x_j, x_n)} \\
&\qquad = \sum_{r=0}^{n-1} \frac{f(x_r)\left(\frac{d(x_n,x_r) - d(x_{n+1},x_r)}{d(x_{n+1},x_r)d(x_n,x_r)}\right)}{d(x_n, x_{n+1}) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} + \frac{f(x_{n+1})}{d(x_n, x_{n+1}) \prod_{j=0}^{n-1} d(x_j, x_{n+1})} \\
&\qquad - \frac{f(x_n)}{d(x_n, x_{n+1}) \prod_{j=0}^{n-1} d(x_j, x_n)}.
\end{aligned}$$

In Dişibüyük & Goldman (2015), it is shown that if $1 \in \text{span}\{\gamma_1, \gamma_2\}$, then the barycentric coordinates sum to one, that is

$$\alpha(x, a, b) + \beta(x, a, b) = \frac{d(x, b)}{d(a, b)} + \frac{d(a, x)}{d(a, b)} = 1. \tag{2.11}$$

Thus, $d(a, x) + d(x, b) = d(a, b)$. Setting $a = x_n$ and $b = x_{n+1}$ in the above equation and using $d(a, b) = -d(b, a)$ gives

$$d(x_n, x_r) - d(x_{n+1}, x_r) = d(x_n, x_r) + d(x_r, x_{n+1}) = d(x_n, x_{n+1}).$$

Hence we obtain

$$\begin{aligned}
f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] &= \sum_{r=0}^{n-1} \frac{f(x_r)}{d(x_{n+1}, x_r)d(x_n, x_r) \prod_{\substack{j=0\\ j\neq r}}^{n-1} d(x_j, x_r)} \\
&\quad + \frac{f(x_{n+1})}{\prod_{\substack{j=0\\ j\neq n+1}}^{n+1} d(x_j, x_{n+1})} + \frac{f(x_n)}{\prod_{\substack{j=0\\ j\neq n}}^{n+1} d(x_j, x_n)} \\
&= \sum_{r=0}^{n-1} \frac{f(x_r)}{\prod_{\substack{j=0\\ j\neq r}}^{n+1} d(x_j, x_r)} + \frac{f(x_{n+1})}{\prod_{\substack{j=0\\ j\neq n+1}}^{n+1} d(x_j, x_{n+1})} \\
&\quad + \frac{f(x_n)}{\prod_{\substack{j=0\\ j\neq n}}^{n+1} d(x_j, x_n)}.
\end{aligned}$$

Therefore,

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] = \sum_{r=0}^{n+1} \frac{f(x_r)}{\prod_{\substack{j=0\\ j\neq r}}^{n+1} d(x_j, x_r)}.$$

This completes the proof. □

**Corollary 2.2.2.1.** *If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$ *, then* $f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n]$ *is symmetric in its arguments, meaning that it is unchanged if we rearrange the argument list in any order.*

*Proof.* It is clear from the symmetric form in Lemma 2.2.2 that each non-polynomial divided difference $f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n]$ is symmetric. □

**Theorem 2.2.3.** *If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$ *and* $f(x) \in \pi_n(\gamma_1, \gamma_2)$*, then*

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] = 0.$$

*Proof.* If $1 \in$ span $\{\gamma_1, \gamma_2\}$ , then there exist constants $c_1$ and $c_2$ such that $c_1\gamma_1(x) + c_2\gamma_2(x) = 1$. Without loss of generality, suppose that $c_2 \neq 0$. Then $\gamma_2(x) = \frac{1-c_1\gamma_1(x)}{c_2}$ which implies $d(x, y) = \frac{1}{c_2}(\gamma_1(x) - \gamma_1(y))$ and span $\{\gamma_1^{n-k}\gamma_2^k\}_{k=0}^n \equiv$ span $\{\gamma_1^k\}_{k=0}^n$. Hence we may write $f(x) = \sum_{i=0}^n a_i\gamma_1^i(x)$ and we have

$$\begin{aligned}
f_{\gamma_1,\gamma_2}[x_0, x] &= \frac{f_{\gamma_1,\gamma_2}[x] - f_{\gamma_1,\gamma_2}[x_0]}{d(x_0, x)} \\
&= \frac{\sum_{i=0}^n a_i\gamma_1^i(x) - \sum_{i=0}^n a_i\gamma_1^i(x_0)}{d(x_0, x)} \\
&= \frac{\sum_{i=0}^n a_i(\gamma_1^i(x) - \gamma_1^i(x_0))}{d(x_0, x)} \\
&= \frac{\sum_{i=0}^n a_i(\gamma_1^i(x) - \gamma_1^i(x_0))}{\gamma_1(x_0)\gamma_2(x) - \gamma_1(x)\gamma_2(x_0)} \\
&= \frac{\sum_{i=0}^n a_i(\gamma_1^i(x) - \gamma_1^i(x_0))}{\gamma_1(x_0)\frac{1-c_1\gamma_1(x)}{c_2} - \gamma_1(x)\frac{1-c_1\gamma_1(x_0)}{c_2}} \\
&= \frac{\sum_{i=0}^n a_i(\gamma_1^i(x) - \gamma_1^i(x_0))}{\frac{1}{c_2}\gamma_1(x_0) - \frac{c_1}{c_2}\gamma_1(x_0)\gamma_1(x) - \frac{1}{c_2}\gamma_1(x) + \frac{c_1}{c_2}\gamma_1(x_0)\gamma_1(x)} \\
&= \sum_{i=0}^n (-a_ic_2)\left(\frac{\gamma_1^i(x) - \gamma_1^i(x_0)}{\gamma_1(x) - \gamma_1(x_0)}\right).
\end{aligned}$$

Using the well known identity $a^m - b^m = (a-b)\sum_{j=0}^{m-1} a^j b^{m-j-1}$, we derive

$$f_{\gamma_1,\gamma_2}[x_0, x] = \sum_{i=0}^{n}\sum_{j=0}^{i-1} (-a_i c_2)\, \gamma_1^j(x)\gamma_1^{i-j-1}(x_0) \in \pi_{n-1}(\gamma_1, \gamma_2).$$

Similarly, we obtain

$$\begin{aligned} f_{\gamma_1,\gamma_2}[x_0, x_1, x] &= \frac{f_{\gamma_1,\gamma_2}[x_0, x] - f_{\gamma_1,\gamma_2}[x_0, x_1]}{d(x_1, x)} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i-2} \left(a_i c_2^2\right) \gamma_1^j(x)\gamma_1^{i-j-1}(x_0), \end{aligned}$$

which implies $f_{\gamma_1,\gamma_2}[x_0, x_1, x] \in \pi_{n-2}(\gamma_1, \gamma_2)$. Continuing this procedure, we have $f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}, x] \in \pi_0(\gamma_1, \gamma_2)$. That is, $f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}, x]$ is a constant function of $x$. Therefore

$$\begin{aligned} f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n, x_{n+1}] &= \frac{f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}, x_{n+1}] - f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n-1}, x_n]}{d(x_n, x_{n+1})} \\ &= 0. \end{aligned}$$

□

The following theorem shows that non-polynomial divided differences satisfy the Leibniz rule as classical divided differences.

**Theorem 2.2.4.** *The non-polynomial divided difference of order $k$ of a product of two function satisfy*

$$(fg)_{\gamma_1,\gamma_2}[x_j, x_{j+1}, \ldots, x_{j+k}] = \sum_{r=0}^{k} f_{\gamma_1,\gamma_2}[x_j, \ldots, x_{j+r}]g_{\gamma_1,\gamma_2}[x_{j+r}, \ldots, x_{j+k}] \quad (2.12)$$

*for $k \geq 0$.*

*Proof.* Without loss of generality, we can set $j = 0$ to simplify the presentation of the proof. Clearly, equation (2.12) is true for $k = 0$. On the other hand,

$$\begin{aligned}
f_{\gamma_1,\gamma_2}[x_0]g_{\gamma_1,\gamma_2}[x_0,x_1] + f_{\gamma_1,\gamma_2}[x_0,x_1]g_{\gamma_1,\gamma_2}[x_1] &= f(x_0)\left\{\frac{g(x_1)-g(x_0)}{d(x_0,x_1)}\right\} \\
&+ g(x_1)\left\{\frac{f(x_1)-f(x_0)}{d(x_0,x_1)}\right\} \\
&= \frac{f(x_1)g(x_1)-f(x_0)g(x_0)}{d(x_0,x_1)} \\
&= \frac{(fg)_{\gamma_1,\gamma_2}[x_1]-(fg)_{\gamma_1,\gamma_2}[x_0]}{d(x_0,x_1)} \\
&= (fg)_{\gamma_1,\gamma_2}[x_0,x_1].
\end{aligned}$$

Hence (2.12) is true for $k = 1$. Now assume that (2.12) is true for some $k \geqslant 0$. We will show that (2.12) is also true for $k + 1$. From equation (2.8), we have

$$(fg)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{k+1}] = \frac{(fg)_{\gamma_1,\gamma_2}[x_0,\ldots,x_{k-1},x_{k+1}]-(fg)_{\gamma_1,\gamma_2}[x_0,\ldots,x_k]}{d(x_k,x_{k+1})}.$$

Setting $x_i = \tilde{x}_i,\ i = 0,\ \ldots, k-1$ and $x_{k+1} = \tilde{x}_k$ in the first term of the numerator yields

$$(fg)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{k+1}] = \frac{(fg)_{\gamma_1,\gamma_2}[\tilde{x}_0,\ldots,\tilde{x}_k]-(fg)_{\gamma_1,\gamma_2}[x_0,\ldots,x_k]}{d(x_k,x_{k+1})}.$$

Applying the induction hypothesis gives

$$\begin{aligned}
(fg)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{k+1}] &= \frac{\sum_{r=0}^{k} f_{\gamma_1,\gamma_2}[\tilde{x}_0,\ldots,\tilde{x}_r]g_{\gamma_1,\gamma_2}[\tilde{x}_r,\ldots,\tilde{x}_k]}{d(x_k,x_{k+1})} \\
&- \frac{\sum_{r=0}^{k} f_{\gamma_1,\gamma_2}[x_0,\ldots,x_r]g_{\gamma_1,\gamma_2}[x_r,\ldots,x_k]}{d(x_k,x_{k+1})} \\
&= \frac{\sum_{r=0}^{k-1} f_{\gamma_1,\gamma_2}[\tilde{x}_0,\ldots,\tilde{x}_r]g_{\gamma_1,\gamma_2}[\tilde{x}_r,\ldots,\tilde{x}_k]}{d(x_k,x_{k+1})} \\
&+ \frac{f_{\gamma_1,\gamma_2}[\tilde{x}_0,\ldots,\tilde{x}_k]g(\tilde{x}_k)}{d(x_k,x_{k+1})} \\
&- \frac{\sum_{r=0}^{k-1} f_{\gamma_1,\gamma_2}[x_0,\ldots,x_r]g_{\gamma_1,\gamma_2}[x_r,\ldots,x_k]}{d(x_k,x_{k+1})} \\
&- \frac{f_{\gamma_1,\gamma_2}[x_0,\ldots,x_k]g(x_k)}{d(x_k,x_{k+1})}.
\end{aligned}$$

Using back substitution $\tilde{x}_i = x_i$ , $i = 0, \dots, k-1$ and $\tilde{x}_k = x_{k+1}$, we get

$$\begin{aligned}(fg)_{\gamma_1,\gamma_2}[x_0, x_1, \dots, x_{k+1}] &= \frac{\sum_{r=0}^{k-1} f_{\gamma_1,\gamma_2}[x_0, \dots, x_r] g_{\gamma_1,\gamma_2}[x_r, \dots, x_{k-1}, x_{k+1}]}{d(x_k, x_{k+1})} \\ &+ \frac{f_{\gamma_1,\gamma_2}[x_0, \dots, x_{k-1}, x_{k+1}] g(x_{k+1})}{d(x_k, x_{k+1})} \\ &- \frac{\sum_{r=0}^{k-1} f_{\gamma_1,\gamma_2}[x_0, \dots, x_r] g_{\gamma_1,\gamma_2}[x_r, \dots, x_k]}{d(x_k, x_{k+1})} \\ &- \frac{f_{\gamma_1,\gamma_2}[x_0, \dots, x_k] g(x_k)}{d(x_k, x_{k+1})}. \end{aligned} \tag{2.13}$$

From equation (2.8), we may write

$$g_{\gamma_1,\gamma_2}[x_r, \dots, x_{k-1}, x_{k+1}] = d(x_k, x_{k+1}) g_{\gamma_1,\gamma_2}[x_r, \dots, x_k, x_{k+1}] + g_{\gamma_1,\gamma_2}[x_r, \dots, x_k]$$

and

$$f_{\gamma_1,\gamma_2}[x_0, \dots, x_{k-1}, x_{k+1}] = d(x_k, x_{k+1}) f_{\gamma_1,\gamma_2}[x_0, \dots, x_k, x_{k+1}] + f_{\gamma_1,\gamma_2}[x_0, \dots, x_k].$$

Substituting these two equations in equation (2.13) and rearranging gives

$$\begin{aligned}(fg)_{\gamma_1,\gamma_2}[x_0, x_1, \dots, x_{k+1}] &= \sum_{r=0}^{k-1} f_{\gamma_1,\gamma_2}[x_0, \dots, x_r] g_{\gamma_1,\gamma_2}[x_r, \dots, x_k, x_{k+1}] \\ &\quad + g(x_{k+1}) f_{\gamma_1,\gamma_2}[x_0, \dots, x_{k+1}] \\ &\quad + f_{\gamma_1,\gamma_2}[x_0, \dots, x_k] \left\{ \frac{g(x_{k+1}) - g(x_k)}{d(x_k, x_{k+1})} \right\} \\ &= \sum_{r=0}^{k} f_{\gamma_1,\gamma_2}[x_0, \dots, x_r] g_{\gamma_1,\gamma_2}[x_r, \dots, x_k, x_{k+1}] \\ &\quad + g(x_{k+1}) f_{\gamma_1,\gamma_2}[x_0, \dots, x_{k+1}] \\ &\quad + f_{\gamma_1,\gamma_2}[x_0, \dots, x_k] g_{\gamma_1,\gamma_2}[x_k, x_{k+1}] \\ &= \sum_{r=0}^{k+1} f_{\gamma_1,\gamma_2}[x_0, \dots, x_r] g_{\gamma_1,\gamma_2}[x_r, \dots, x_{k+1}] \end{aligned}$$

which completes the proof. □

It will be useful to know the non-polynomial divided difference of order $n$ of the functions $\gamma_1^n$ and $\gamma_2^n$ which are given in the following lemma.

**Lemma 2.2.5.** *If* $1 \in$ *span* $\{\gamma_1, \gamma_2\}$*, then*

$$(\gamma_1^n)_{\gamma_1,\gamma_2}[x_j, x_{j+1}, \ldots, x_{j+n}] = \begin{cases} 1, & n = 0 \\ (-c_2)^n, & n \neq 0 \end{cases} \tag{2.14}$$

*and*

$$(\gamma_2^n)_{\gamma_1,\gamma_2}[x_j, x_{j+1}, \ldots, x_{j+n}] = \begin{cases} 1, & n = 0 \\ (c_1)^n, & n \neq 0 \end{cases}, \tag{2.15}$$

*where* $c_1\gamma_1(x) + c_2\gamma_2(x) = 1.$

*Proof.* We will prove only equation (2.14), since the proof of equation (2.15) is similar. Without loss of generality, set $j = 0$.

If $c_2 = 0$, then $\gamma_1(x)$ is a constant function. Hence equation (2.14) is clear for $n = 0$ since $\gamma_1^0 = 1$. For $n \neq 0$, equation (2.14) follows, since $\gamma_1^n(x)$ is also a constant function.

For the case $c_2 \neq 0$, it is clear that equation (2.14) is true for $n = 0$. On the other hand,

$$\gamma_{1\gamma_1,\gamma_2}[x_0, x_1] = \frac{\gamma_1(x_1) - \gamma_1(x_0)}{d(x_0, x_1)} = \frac{\gamma_1(x_1) - \gamma_1(x_0)}{\frac{\gamma_1(x_0)-\gamma_1(x_1)}{c_2}} = (-c_2).$$

That is, equation (2.14) is true for $n = 1$. Now assume that equation (2.14) is true for some $n \geqslant 1$. Since $1 \in$ span $\{\gamma_1, \gamma_2\}$ , the non-polynomial divided differences is symmetric. Thus

$$\begin{aligned} {\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] &= \frac{{\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_1, x_2, \ldots, x_{n+1}] - {\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n]}{d(x_0, x_{n+1})} \\ &= \frac{1}{d(x_0, x_{n+1})}\Big((\gamma_1^n\gamma_1)_{\gamma_1,\gamma_2}[x_1, x_2, \ldots, x_{n+1}] \\ &\qquad - (\gamma_1\gamma_1^n)_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n]\Big), \end{aligned}$$

and using the Leibniz rule, we have

$$\begin{aligned}
&{\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] \\
&\qquad = \frac{1}{d(x_0,x_{n+1})}\Bigg(\sum_{r=0}^{n} {\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_{1+r}]{\gamma_1}_{\gamma_1,\gamma_2}[x_{1+r},x_1,\ldots,x_{n+1}] \\
&\qquad\qquad - \sum_{r=0}^{n} {\gamma_1}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_r]{\gamma_1^n}_{\gamma_1,\gamma_2}[x_r,\ldots,x_n]\Bigg).
\end{aligned}$$

Note that ${\gamma_1}_{\gamma_1,\gamma_2}[x_j,\ldots,x_{j+k}]=0$ for $k>1$ since $\gamma_1\in\pi_1(\gamma_1,\gamma_2)$. Then,

$$\begin{aligned}
{\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] = \frac{1}{d(x_0,x_{n+1})}\Bigg(&{\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_n]{\gamma_1}_{\gamma_1,\gamma_2}[x_n,x_{n+1}] \\
&+ {\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+1}]{\gamma_1}_{\gamma_1,\gamma_2}[x_{n+1}] \\
&- {\gamma_1}_{\gamma_1,\gamma_2}[x_0]{\gamma_1^n}_{\gamma_1,\gamma_2}[x_0,\ldots,x_n] \\
&- {\gamma_1}_{\gamma_1,\gamma_2}[x_0,x_1]{\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_n]\Bigg).
\end{aligned}$$

Since ${\gamma_1}_{\gamma_1,\gamma_2}[x_n,x_{n+1}]={\gamma_1}_{\gamma_1,\gamma_2}[x_0,x_1]=(-c_2)$, we obtain

$$\begin{aligned}
&{\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] \\
&\quad = \frac{1}{d(x_0,x_{n+1})}\Bigg({\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_n](-c_2) + {\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+1}]\gamma_1(x_{n+1}) \\
&\quad - \gamma_1(x_0){\gamma_1^n}_{\gamma_1,\gamma_2}[x_0,\ldots,x_n] - (-c_2){\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_n]\Bigg) \\
&\quad = \frac{1}{d(x_0,x_{n+1})}\Bigg({\gamma_1^n}_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+1}]\gamma_1(x_{n+1}) - \gamma_1(x_0){\gamma_1^n}_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]\Bigg).
\end{aligned}$$

Using induction hypothesis gives

$$\begin{aligned}
{\gamma_1^{n+1}}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] &= \frac{1}{d(x_0,x_{n+1})}\Big((-c_2)^n\gamma_1(x_{n+1}) - \gamma_1(x_0)(-c_2)^n\Big) \\
&= (-c_2)^n{\gamma_1}_{\gamma_1,\gamma_2}[x_0,x_{n+1}] = (-c_2)^n(-c_2) = (-c_2)^{n+1}.
\end{aligned}$$

This completes the proof. $\square$

What follows is the non-polynomial divided difference of order $n$ of any function $f\in\pi_n(\gamma_1,\gamma_2)$.

**Lemma 2.2.6.** *If* $1 \in$ *span* $\{\gamma_1, \gamma_2\}$*, then*

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n}(-1)^{n-i}a_i c_1^i c_2^{n-i}, \tag{2.16}$$

*where* $f(x) = \sum_{i=0}^{n} a_i\gamma_1^{n-i}(x)\gamma_2(x)^i$ *and* $c_1\gamma_1(x) + c_2\gamma_2(x) = 1$.

*Proof.* We consider two cases, $c_2 = 0$ and $c_2 \neq 0$. If $c_2 = 0$, then $\sum_{i=0}^{n}(-1)^{n-i}c_1^i c_2^{n-i}a_i = a_n(c_1)^n$ and $\gamma_1(x) = \frac{1}{c_1}$ which implies

$$f(x) = \sum_{i=0}^{n} a_i\gamma_1^{n-i}(x)\gamma_2(x)^i = \sum_{i=0}^{n} a_i\left(\frac{1}{c_1}\right)^{n-i}\gamma_2^i(x).$$

From Theorem 2.2.3, $\gamma^i_{2\gamma_1,\gamma_2}[x_0, \ldots, x_n] = 0$ for $i < n$. Thus by Lemma 2.2.5

$$\begin{aligned} f(x)_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n] &= \sum_{i=0}^{n} a_i\left(\frac{1}{c_1}\right)^{n-i}\gamma^i_{2\gamma_1,\gamma_2}[x_0, \ldots, x_n] = a_n\gamma^n_{2\;\gamma_1,\gamma_2}[x_0, \ldots, x_n] \\ &= a_n(c_1)^n. \end{aligned}$$

Equation (2.16) is true for the case $c_2 = 0$. If $c_2 \neq 0$, then $\gamma_2(x) = \frac{1-c_1\gamma_1(x)}{c_2}$. So

$$f(x) = \sum_{i=0}^{n} a_i\gamma_1^{n-i}(x)\gamma_2(x)^i = \sum_{i=0}^{n} a_i\gamma_1^{n-i}(x)\big(\frac{1-c_1\gamma_1(x)}{c_2}\big)^i.$$

Subsitituting the binomial expansion of $(1 - c_1\gamma_1(x))^i$ and rearranging, we obtain

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} a_i\gamma_1^{n-i}(x)\sum_{k=0}^{i}\binom{i}{k}\left(\frac{1}{c_2}\right)^{i-k}\left(-\frac{c_1}{c_2}\gamma_1(x)\right)^k \\ &= \sum_{i=0}^{n}(-1)^i a_i c_1^i c_2^{-i}\gamma_1^n(x) + \sum_{j=0}^{n-1} b_j\gamma_1^j(x). \end{aligned}$$

The non-polynomial divided difference of order $n$ of the second summation on the right hand side in the above equation is zero by Theorem 2.2.3. Therefore, from Lemma

2.2.5, we have

$$f(x)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = \sum_{i=0}^{n}(-1)^i a_i c_1^i c_2^{-i} \gamma_1^n{}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n]$$
$$= \sum_{i=0}^{n}(-1)^{n-i} a_i c_1^i c_2^{n-i}.$$

□

**Theorem 2.2.7.** *Let $f(x)$ and $h(x)$ be arbitrary functions, and let $x_0 < x_1 < \ldots < x_n$ be a set of $n+1$ nodes. If $f(x_j) = h(x_j)$ , $j = 0,\ldots,n$, then*

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = h_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n]. \tag{2.17}$$

*Proof.* The proof is by induction on $n$. Let $f(x_j) = h(x_j)$ , $j = 0,\ldots,n$. It is clear that this result holds for $n = 0$. It also holds for $n = 1$,

$$f_{\gamma_1,\gamma_2}[x_0,x_1] = \frac{f(x_1)-f(x_0)}{d(x_0,x_1)} = \frac{h(x_1)-h(x_0)}{d(x_0,x_1)} = h_{\gamma_1,\gamma_2}[x_0,x_1].$$

Let us assume that (2.17) holds for $n \geq 1$. Then let us write

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] = \frac{f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n-1},x_{n+1}] - f_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]}{d(x_n,x_{n+1})}.$$

Let

$$x_i = \tilde{x}_i \ \text{ for } \ i = 0,\ldots,n-1 \ \text{ and } \ x_{n+1} = \tilde{x}_n. \tag{2.18}$$

Hence,

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] = \frac{1}{d(x_n,x_{n+1})}\Big\{f_{\gamma_1,\gamma_2}[\tilde{x}_0,\ldots,\tilde{x}_{n-1},\tilde{x}_n] - f_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]\Big\}.$$

Using induction hypothesis, we obtain

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] = \frac{1}{d(x_n,x_{n+1})}\Big\{h_{\gamma_1,\gamma_2}[\tilde{x}_0,\ldots,\tilde{x}_{n-1},\tilde{x}_n] - h_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]\Big\}.$$

Again using equation (2.18), it yields that

$$\begin{aligned} f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}] &= \frac{1}{d(x_n,x_{n+1})}\Big\{h_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n-1},x_{n+1}] - h_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]\Big\} \\ &= h_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}]. \end{aligned}$$

This completes the proof. $\square$

**Lemma 2.2.8.** *If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$*, then*

$$(\sigma_j)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_k] = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}.$$

*Proof.* We investigate the result in three cases:

**Case 1.** $j > k$:

Using equation (2.10), the divided difference $(\sigma_j)_{\gamma_1,\gamma_2}[x_0,\ldots,x_k]$ can be written as a symmetric sum

$$(\sigma_j)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_k] = \sum_{r=0}^{k} \frac{\sigma_j(x_r)}{\prod_{\substack{i=0\\ i\neq r}}^{k} d(x_i,x_r)}.$$

Since $j > k$, $\sigma_j(x_r) = d(x_0,x_r)d(x_1,x_r)\ldots d(x_{j-1},x_r) = 0$ for $r = 0,\ldots,k$. Therefore, the above summation is equal to zero for $j > k$.

**Case 2.** $j < k$:

We can write $k = j+m,\ m > 0$, and

$$(\sigma_j)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_k] = (\sigma_j)_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{j+m}].$$

Since $\sigma_j \in \pi_j(\gamma_1,\gamma_2)$, the result follows from Theorem 2.2.3. $(j+m)$-th degree divided difference of $\sigma_j$ is equal to zero. Hence $(\sigma_j)_{\gamma_1,\gamma_2}[x_0,\ldots,x_k] = 0$ for $j < k$.

**Case 3.** $j = k$:

$$(\sigma_j)_{\gamma_1,\gamma_2}[x_0,\ldots,x_k] = (\sigma_j)_{\gamma_1,\gamma_2}[x_0,\ldots,x_j] = \sum_{r=0}^{j} \frac{\sigma_j(x_r)}{\prod_{\substack{i=0\\ i\neq r}}^{j} d(x_i,x_r)}.$$

Since $\sigma_j(x_r) = d(x_0, x_r), \ldots, d(x_{j-1}, x_r) = 0$ for $r = 0, \ldots, j-1$, the only nonzero term in the right hand side of the above equation is $\sigma_j(x_j)$. Thus

$$(\sigma_j)_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_k] = \frac{d(x_0, x_j), \ldots, d(x_{j-1}, x_j)}{\prod_{\substack{i=0 \\ i\neq j}}^{j} d(x_i, x_j)} = 1.$$

Hence,

$$(\sigma_j)_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_k] = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}.$$

□

The following lemma gives the cancellation formula of the non-polynomial divided differences. As in the polynomial case, this identity allows one to expand or shrink the argument list in a non-polynomial divided difference.

**Lemma 2.2.9.** *If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$*, then*

$$f(x)_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n] = \{d(x_{n+1}, x)f(x)\}_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}]. \tag{2.19}$$

*Proof.* The proof is by induction on $n$, and equation (2.19) holds for $n = 0$

$$\{d(x_1, x)f(x)\}_{\gamma_1,\gamma_2}[x_0, x_1] = \frac{-d(x_1, x_0)f(x_0)}{d(x_0, x_1)} = f(x_0).$$

Let us assume that equation (2.19) is true for some $n \geq 0$. Then, using the Leibniz rule (Theorem 2.2.4), we can write

$$\{d(x_{n+2}, x)f(x)\}_{\gamma_1,\gamma_2}[x_0, \ldots, x_{n+2}] = \sum_{r=0}^{n+2} d(x_{n+2}, x)_{\gamma_1,\gamma_2}[x_0, \ldots, x_r] f_{\gamma_1,\gamma_2}[x_r, \ldots, x_{n+2}].$$

Since $d(x_{n+2}, x) \in \pi_1(\gamma_1, \gamma_2)$, we have $d(x_{n+2}, x)_{\gamma_1,\gamma_2}[x_0, \ldots, x_r] = 0$ for $r > 1$.

Hence,

$$\begin{aligned}\{d(x_{n+2},x)f(x)\}_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+2}] =& d(x_{n+2},x_0)f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+2}] \\ &+ d(x_{n+2},x)_{\gamma_1,\gamma_2}[x_0,x_1]f_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+2}] \\ =& d(x_{n+2},x_0)f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+2}] \\ &+ f_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+2}]\end{aligned}$$

since $d(x_{n+2},x)_{\gamma_1,\gamma_2}[x_0,x_1] = 1$. Therefore,

$$\begin{aligned}&\{d(x_{n+2},x)f(x)\}_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+2}] \\ &\quad= \frac{d(x_{n+2},x_0)}{d(x_0,x_{n+2})}\{f_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+2}] - f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+1}]\} \\ &\qquad + f_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+2}] \\ &\quad= - f_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+2}] + f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+1}] + f_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+2}] \\ &\quad= f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{n+1}].\end{aligned}$$

This completes the proof. □

**Lemma 2.2.10.** *If $1 \in$ span$\{\gamma_1,\gamma_2\}$, then*

$$(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = \frac{1}{d(x_0,x)d(x_1,x)\ldots d(x_n,x)}. \tag{2.20}$$

*Here $x$ is treated as a constant and the divided difference is taken with respect to $t$.*

*Proof.* We prove this by induction on $n$. Let $1 \in$ span$\{\gamma_1,\gamma_2\}$, then it is clear that (2.20) holds for $n = 1$.

$$(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,x_1] = \frac{\frac{1}{d(x_1,x)} - \frac{1}{d(x_0,x)}}{d(x_0,x_1)} = \frac{1}{d(x_0,x)d(x_1,x)}$$

since $d(x_0,x) - d(x_1,x) = d(x_0,x_1)$. Suppose that (2.20) holds for $n > 1$. Then, we may write

$$
\begin{aligned}
&(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}]\\
&\qquad\qquad =\frac{1}{d(x_0,x_{n+1})}\left\{(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_1,\ldots,x_{n+1}]-(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]\right\}.
\end{aligned}
$$

Using induction hypothesis yields

$$
\begin{aligned}
&(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+1}]\\
&\qquad =\frac{1}{d(x_0,x_{n+1})}\left\{\frac{1}{d(x_1,x)\ldots d(x_{n+1},x)}-\frac{1}{d(x_0,x)\ldots d(x_n,x)}\right\}\\
&\qquad =\frac{1}{d(x_0,x_{n+1})}\cdot\frac{d(x_0,x_{n+1})}{d(x_0,x)\ldots d(x_{n+1},x)}
\end{aligned}
$$

since $d(x_0,x)-d(x_{n+1},x)=d(x_0,x_{n+1})$. Thus, we obtain

$$
(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_{n+2}]=\frac{1}{d(x_0,x)\ldots d(x_{n+1},x)}
$$

which completes the proof. □

Also, equation (2.20) can be written as follows if $t$ is treated as a constant and the non-polynomial divided difference is taken with respect to $x$:

$$
(d(t,x))^{-1}_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n]=\frac{(-1)^n}{d(x_0,t)d(x_1,t)\ldots d(x_n,t)}.
$$

Also, non-polynomial divided difference is expressed as an affine combinations of non-polynomial divided differences.

**Lemma 2.2.11.** *Let $\gamma_1$ and $\gamma_2$ be linear functions and $\tau=\sum\lambda_k x_k$, where $\sum\lambda_k=1$. If $1\in span\{\gamma_1,\gamma_2\}$, then*

$$
f_{\gamma_1,\gamma_2}[x_0,\ldots,x_n]=\sum\lambda_k f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{k-1},\tau,x_{k+1}\ldots,x_n].
$$

*Proof.* If $x_0=x_1=\cdots=x_n$, then $\tau=x_k,\quad k=0,\ldots,n$ and both sides reduce to

$f_{\gamma_1,\gamma_2}[x_0,\ldots,x_0]$. Otherwise, using cancellation formula, we can write

$$f_{\gamma_1,\gamma_2}[x_0,\ldots,x_n] = (d(\tau,x)f(x))_{\gamma_1,\gamma_2}[x_0,\ldots,x_n,\tau].$$

Using the linearity of the functions $\gamma_1$ and $\gamma_2$ yields that $d(\tau,x) = \sum \lambda_k d(x_k,x)$. Therefore, using cancellation formula, linearity and symmetry properties of non-polynomial divided difference, we obtain

$$\begin{aligned}
f_{\gamma_1,\gamma_2}[x_0,\ldots,x_n] =& \left(\sum \lambda_k d(x_k,x)f(x)\right)_{\gamma_1,\gamma_2}[x_0,\ldots,x_n,\tau] \\
=&(\lambda_0 d(x_0,x)f(x))\gamma_1,\gamma_2[x_0,\ldots,x_n,\tau]+ \\
&\cdots + (\lambda_n d(x_n,x)f(x))\gamma_1,\gamma_2[x_0,\ldots,x_n,\tau] \\
=&\lambda_0 f(x)\gamma_1,\gamma_2[x_1,\ldots,x_n,\tau]+ \\
&\cdots + \lambda_k f(x)\gamma_1,\gamma_2[x_0,\ldots,x_{k-1},x_{k-1},\ldots,x_n,\tau] \\
&+ \ldots \lambda_n f(x)\gamma_1,\gamma_2[x_0,\ldots,x_{n-1},\tau] \\
=&\sum \lambda_k f_{\gamma_1,\gamma_2}[x_0,\ldots,x_{k-1},\tau,x_{k+1}\ldots,x_n].
\end{aligned}$$

□

We can express $(d(t,x))^{-1}$ as Marsden's identity for Newton-like basis form (2.6).

**Lemma 2.2.12.** *If* $1 \in$ *span*$\{\gamma_1,\gamma_2\}$*, then*

$$(d(t,x))^{-1} = \sum_{n\geq 0} \frac{(-1)^n \sigma_n(x)}{d(t,x_0)\ldots d(t,x_n)} \tag{2.21}$$

*provided that the nodes* $x_0,x_1,\ldots,x_n$ *are chosen so that the right hand side converges.*

*Proof.* Let

$$\psi_n(t) = \frac{(-1)^n}{d(t,x_0)d(t,x_1)\ldots d(t,x_n)}.$$

We will prove that

$$(d(t,x))^{-1} = \sum_{n\geq 0} \psi_n(t)\sigma_n(x) \quad \text{or equivalently that} \quad 1 = d(t,x)\sum_{n\geq 0}\psi_n(t)\sigma_n(x).$$

Now, we can give the following equalities

$$\sigma_n(x) = d(x_{n-1}, x)\sigma_{n-1}(x), \quad -\psi_{n-1}(x) = d(t, x_n)\psi_n(x),$$
$$d(t, x) = d(x_n, x) + d(t, x_n).$$

Therefore, since the right hand side of equation (2.21) converges,

$$\begin{aligned}
d(t, x)\sum_{n\geq 0}\psi_n(t)\sigma_n(x) &= \sum_{n\geq 0}(d(x_n, x) + d(t, x_n))\psi_n(t)\sigma_n(x) \\
&= \sum_{n\geq 1} d(x_{n-1}, x)\psi_{n-1}(t)\sigma_{n-1}(x) + \sum_{n\geq 0} d(t, x_n)\psi_n(t)\sigma_n(x) \\
&= d(t, x_0)\psi_0(t)\sigma_0(x) + \sum_{n\geq 1}(\psi_{n-1}(t) - \psi_{n-1}(t))\sigma_n(x) \\
&= 1.
\end{aligned}$$

Dividing both sides by $d(t, x)$, we get the result. □

We use the following lemma to define non-polynomial divided difference in terms of ratio of determinants.

**Lemma 2.2.13.** *Let* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$. *Then,*

$$\begin{vmatrix} \gamma_1^n(x_0) & \gamma_1^{n-1}(x_0) & \dots & 1 \\ \gamma_1^n(x_1) & \gamma_1^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ \gamma_1^n(x_n) & \gamma_1^{n-1}(x_n) & \dots & 1 \end{vmatrix} = c_2^{n(n+1)/2}\prod_{j<i}^{n} d(x_j, x_i), \tag{2.22}$$

*and*

$$\begin{vmatrix} \gamma_2^n(x_0) & \gamma_2^{n-1}(x_0) & \dots & 1 \\ \gamma_2^n(x_1) & \gamma_2^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ \gamma_2^n(x_n) & \gamma_2^{n-1}(x_n) & \dots & 1 \end{vmatrix} = (-c_1)^{n(n+1)/2}\prod_{j<i}^{n} d(x_j, x_i), \tag{2.23}$$

*where* $c_1\gamma_1(x) + c_2\gamma_2(x) = 1$ .

*Proof.* We will prove only equation (2.22), since the proof of equation (2.23) is similar.

If $c_2 = 0$, then $\gamma_1(x)$ is a constant function. Since the right-hand side of equation (2.22) has identical rows, the determinant is equal to zero. Therefore equation (2.22) holds.

For the case $c_2 \neq 0$, consider the function

$$V(x_0, \ldots, x_{n-1}, x) = \begin{vmatrix} \gamma_1^n(x_0) & \gamma_1^{n-1}(x_0) & \ldots & 1 \\ \vdots & & & \\ \gamma_1^n(x_{n-1}) & \gamma_1^{n-1}(x_{n-1}) & \ldots & 1 \\ \gamma_1^n(x) & \gamma_1^{n-1}(x) & \ldots & 1 \end{vmatrix}.$$

If we expand $V(x_0, \ldots, x_{n-1}, x)$ with respect to last row we see that it is a function in $\pi_n(\gamma_1, \gamma_2)$. Also, $V(x_0, \ldots, x_{n-1}, x) = 0$ for $x = x_0, \ldots, x_{n-1}$ since the matrix will have identical pairs of rows. Thus, for a constant $C \neq 0$, we may write

$$V(x_0, \ldots, x_{n-1}, x) = C.(\gamma_1(x) - \gamma_1(x_0))(\gamma_1(x) - \gamma_1(x_1)) \ldots (\gamma_1(x) - \gamma_1(x_{n-1}))$$

since it has $n$ zeros at $x = x_0, \ldots, x_{n-1}$. Then, it is rewritten as

$$V(x_0, \ldots, x_{n-1}, x) = C.c_2^n d(x, x_0) d(x, x_1) \ldots d(x, x_{n-1})$$

since $d(x, y) = \frac{1}{c_2}(\gamma_1(x) - \gamma_1(y))$. Notice that $C$ is the coefficient of $\gamma_1^n(x)$ and the coefficient of $\gamma_1^n(x)$ in $|.|$ is minor for the term $\gamma_1^n(x)$, that is

$$(-1)^{n+2} \begin{vmatrix} \gamma_1^{n-1}(x_0) & \ldots & 1 \\ \vdots & & \\ \gamma_1^{n-1}(x_{n-1}) & \ldots & 1 \end{vmatrix} = (-1)^n V(x_0, \ldots, x_{n-1}) = C.$$

Then,

$$V(x_0, \ldots, x_{n-1}, x) = c_2^n V(x_0, \ldots, x_{n-1}) d(x_0, x) d(x_1, x) \ldots d(x_{n-1}, x),$$

and hence $x = x_n$ gives

$$V(x_0, \dots, x_{n-1}, x_n) = c_2^n V(x_0, \dots, x_{n-1}) d(x_0, x_n) d(x_1, x_n) \dots d(x_{n-1}, x_n). \quad (2.24)$$

For $n = 1$, we have

$$V(x_0, x_1) = \begin{vmatrix} \gamma_1(x_0) & 1 \\ \gamma_1(x_1) & 1 \end{vmatrix} = \gamma_1(x_0) - \gamma_1(x_1).$$

Therefore, we get that $V(x_0, x_1) = c_2 d(x_0, x_1)$. For $n = 2$, using equation (2.24), we obtain

$$V(x_0, x_1, x_2) = c_2^2 V(x_0, x_1) d(x_0, x_2) d(x_1, x_2) = c_2^3 d(x_0, x_1) d(x_0, x_2) d(x_1, x_2).$$

Continuing in this way, we get

$$V(x_0, \dots, x_n) = c_2^{n(n+1)/2} \prod_{j<i}^{n} d(x_j, x_i).$$

□

**Lemma 2.2.14.** *Let* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$*. Then,*

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \dots, x_n] = c_2^n \frac{\begin{vmatrix} f(x_0) & \gamma_1^{n-1}(x_0) & \dots & 1 \\ f(x_1) & \gamma_1^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ f(x_n) & \gamma_1^{n-1}(x_n) & \dots & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_1^n(x_0) & \gamma_1^{n-1}(x_0) & \dots & 1 \\ \gamma_1^n(x_1) & \gamma_1^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ \gamma_1^n(x_n) & \gamma_1^{n-1}(x_n) & \dots & 1 \end{vmatrix}},$$

*where* $c_1\gamma_1(x) + c_2\gamma_2(x) = 1$ *and* $c_2 \neq 0$.

*Proof.* Since $1 \in$ span$\{\gamma_1, \gamma_2\}$, $f_{\gamma_1,\gamma_2}[x_0, x_1, \dots, x_n]$ can be written as a symmetric sum

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{i=0\\ i\neq r}}^{n} d(x_i,x_r)}.$$

Therefore,

$$\begin{aligned}f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = &\frac{f(x_0)}{d(x_0,x_1)\ldots d(x_0,x_n)}+\frac{f(x_1)}{d(x_1,x_0)\ldots d(x_1,x_n)}+\ldots\\ &+\frac{f(x_n)}{d(x_n,x_0)\ldots d(x_n,x_{n-1})}.\end{aligned}$$

Equating denominator yields that

$$\begin{aligned}f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n] = \frac{1}{\prod_{j<i}^{n} d(x_j,x_i)}\Bigg\{&f(x_0)\prod_{\substack{j<i\\ i,j\neq 0}} d(x_j,x_i) - f(x_1)\prod_{\substack{j<i\\ i,j\neq 1}} d(x_j,x_i)+\ldots\\ &+(-1)^n f(x_n)\prod_{\substack{j<i\\ i,j\neq n}} d(x_j,x_i)\Bigg\}.\end{aligned}$$

Using equation (2.22), we can write

$$\begin{aligned}&f_{\gamma_1,\gamma_2}[x_0,x_1,\ldots,x_n]\\ &=\frac{f(x_0)c_2^{-n(n-1)/2}\begin{vmatrix}\gamma_1^{n-1}(x_1) & \ldots & 1\\ \\ \gamma_1^{n-1}(x_n) & \ldots & 1\end{vmatrix}+\ldots+(-1)^n f(x_n)c_2^{-n(n-1)/2}\begin{vmatrix}\gamma_1^{n-1}(x_0) & \ldots & 1\\ \\ \gamma_1^{n-1}(x_{n-1}) & \ldots & 1\end{vmatrix}}{c_2^{-n(n+1)/2}\begin{vmatrix}\gamma_1^{n}(x_0) & \gamma_1^{n-1}(x_0) & \ldots & 1\\ \vdots & & & \\ \gamma_1^{n}(x_n) & \gamma_1^{n-1}(x_n) & \ldots & 1\end{vmatrix}}\\ &=c_2^{n}\frac{f(x_0)\begin{vmatrix}\gamma_1^{n-1}(x_1) & \ldots & 1\\ \\ \gamma_1^{n-1}(x_n) & \ldots & 1\end{vmatrix}+\ldots+(-1)^n f(x_n)\begin{vmatrix}\gamma_1^{n-1}(x_0) & \ldots & 1\\ \\ \gamma_1^{n-1}(x_{n-1}) & \ldots & 1\end{vmatrix}}{\begin{vmatrix}\gamma_1^{n}(x_0) & \gamma_1^{n-1}(x_0) & \ldots & 1\\ \vdots & & & \\ \gamma_1^{n}(x_n) & \gamma_1^{n-1}(x_n) & \ldots & 1\end{vmatrix}}.\end{aligned}$$

Hence, we get

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\dots,x_n] = c_2^n \frac{\begin{vmatrix} f(x_0) & \gamma_1^{n-1}(x_0) & \dots & 1 \\ f(x_1) & \gamma_1^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ f(x_n) & \gamma_1^{n-1}(x_n) & \dots & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_1^n(x_0) & \gamma_1^{n-1}(x_0) & \dots & 1 \\ \gamma_1^n(x_1) & \gamma_1^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ \gamma_1^n(x_n) & \gamma_1^{n-1}(x_n) & \dots & 1 \end{vmatrix}}.$$

□

Similarly, if we use equation (2.23) , $n$th order non-polynomial divided difference $f_{\gamma_1,\gamma_2}[x_0,x_1,\dots,x_n]$ is also written in the following determinant formula

$$f_{\gamma_1,\gamma_2}[x_0,x_1,\dots,x_n] = (-c_1)^n \frac{\begin{vmatrix} f(x_0) & \gamma_2^{n-1}(x_0) & \dots & 1 \\ f(x_1) & \gamma_2^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ f(x_n) & \gamma_2^{n-1}(x_n) & \dots & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_2^n(x_0) & \gamma_2^{n-1}(x_0) & \dots & 1 \\ \gamma_2^n(x_1) & \gamma_2^{n-1}(x_1) & \dots & 1 \\ \vdots & & & \\ \gamma_2^n(x_n) & \gamma_2^{n-1}(x_n) & \dots & 1 \end{vmatrix}}, \quad \text{where} \quad c_1 \neq 0.$$

### *2.2.2 Non-Polynomial Divided Differences with Repeated Points*

Consider equation (2.4) and let $x_i \to x_0$ for all $i$, then

$$g_n(x) = \sum_{j=0}^{n} f_{\gamma_1,\gamma_2}[\underbrace{x_0,x_0,\dots,x_0}_{j+1 \text{ terms}}](d(x_0,x))^j. \tag{2.25}$$

For the polynomial case, that is for $\gamma_1(x) = 1$ and $\gamma_2(x) = x$, (2.25) is the Hermite interpolating polynomial that satisfies the conditions

$$g_n^{(k)}(x_0) = f^{(k)}(x_0) \tag{2.26}$$

for $k = 0, 1, \ldots, n$. The conditions (2.26) can be verified by using the following well known property of polynomial divided differences:

$$f[\underbrace{x_0, x_0, \ldots, x_0}_{k+1 \text{ terms}}] = \frac{f^{(k)}(x_0)}{k!}, \tag{2.27}$$

$k = 0, 1, \ldots, n$. However, in general, equation (2.27) is not true for non-polynomial divided difference. For instance,

$$f_{\gamma_1,\gamma_2}[x_0, x_0] = \lim_{x_1 \to x_0} f_{\gamma_1,\gamma_2}[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{d(x_0, x_1)}.$$

Dividing both numerator and denominator of the latter limit by $x_1 - x_0$ and using equation (1.1), one may easily show that

$$f_{\gamma_1,\gamma_2}[x_0, x_0] = \frac{f'(x_0)}{W(x_0)}, \tag{2.28}$$

where $W(x)$ is the Wronskian of the functions $\gamma_1(x)$ and $\gamma_2(x)$. To generalize the classical polynomial Hermite interpolation to non-polynomial case, we need the following definition.

**Definition 2.2.2.** *The derivative of $f$ at a point $x_0$ is*

$$D_{\gamma_1,\gamma_2} f(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{d(x_0, x)}$$

*and the derivative of $f(x)$ with respect to $x$ is the function $D_{\gamma_1,\gamma_2} f(x)$ defined by*

$$D_{\gamma_1,\gamma_2} f(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{d(x, x+h)}$$

*provided that the limits are exist.*

Note that for $\gamma_1(x) = 1$ and $\gamma_2(x) = x$, the above definition is precisely the classical derivative. Hence, the operator $D_{\gamma_1,\gamma_2}$ can be considered as a generalization of the classical derivative operator and can be called ***generalized derivative operator***. From now on, we will use the notation $f^{(k)}(x)$ if the classical derivative applied $k-$times to the function $f(x)$ and the notation $D^k_{\gamma_1,\gamma_2}f(x)$ if the operator $D_{\gamma_1,\gamma_2}$ is applied $k-$times to the function $f(x)$. Using similar technique in obtaining equation (2.28), the relation between the classical derivative and the operator $D_{\gamma_1,\gamma_2}$ is derived as follows:

$$\begin{aligned}
D_{\gamma_1,\gamma_2}f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{d(x,x+h)} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\frac{h}{d(x,x+h)} \\
&= f'(x)\lim_{h\to 0}\frac{h}{d(x,x+h)} \\
&= f'(x)\lim_{h\to 0}\frac{h}{\gamma_1(x)\gamma_2(x+h)-\gamma_1(x+h)\gamma_2(x)\mp\gamma_1(x)\gamma_2(x)} \\
&= f'(x)\lim_{h\to 0}\frac{1}{\gamma_1(x)\frac{\gamma_2(x+h)-\gamma_2(x)}{h}-\gamma_2(x)\frac{\gamma_1(x+h)-\gamma_1(x)}{h}} \\
&= f'(x)\frac{1}{\gamma_1(x)\gamma_2'(x)-\gamma_2(x)\gamma_1'(x)} \\
&= f'(x)\frac{1}{\frac{\partial}{\partial t}d(x,t)|_{t=x}}.
\end{aligned}$$

Hence,

$$D_{\gamma_1,\gamma_2}f(x) = \frac{f'(x)}{W(x)}. \tag{2.29}$$

Throughout this thesis, it is assumed that $W(x) \neq 0$.

$D_{\gamma_1,\gamma_2}$ satisfies the following properties:

1. $D_{\gamma_1,\gamma_2}(f(x) \pm g(x)) = D_{\gamma_1,\gamma_2}f(x) \pm D_{\gamma_1,\gamma_2}g(x)$.
2. $D_{\gamma_1,\gamma_2}(f(x)g(x)) = g(x)D_{\gamma_1,\gamma_2}f(x) + f(x)D_{\gamma_1,\gamma_2}g(x)$.
3. $D_{\gamma_1,\gamma_2}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_{\gamma_1,\gamma_2}f(x) - f(x)D_{\gamma_1,\gamma_2}g(x)}{(g(x))^2}$.
4. $D_{\gamma_1,\gamma_2}(f \circ g)(x) = f'(g(x))D_{\gamma_1,\gamma_2}g(x)$.

provided that $f(x)$ and $g(x)$ are both differentiable. Using the relation (2.29) together with the differentiation rules for classical derivatives, we can easily prove the above properties.

**Lemma 2.2.15.** *If* $1 \in span\{\gamma_1, \gamma_2\}$, *then for any number* $a$

$$D_{\gamma_1,\gamma_2} d(a,x) = 1 \tag{2.30}$$

*and*

$$D^k_{\gamma_1,\gamma_2}(d(a,x))^n = \frac{n!}{(n-k)!}(d(a,x))^{n-k}, \tag{2.31}$$

$k = 1, 2, \ldots, n.$

*Proof.* Proof of equation (2.30):
If $1 \in span\{\gamma_1, \gamma_2\}$, we have $d(a, x+h) - d(a,x) = d(a, x+h) + d(x,a) = d(x, x+h)$. Thus

$$D_{\gamma_1,\gamma_2} d(a,x) = \lim_{h\to 0} \frac{d(a,x+h) - d(a,x)}{d(x,x+h)} = \lim_{h\to 0} \frac{d(x,x+h)}{d(x,x+h)} = 1.$$

Proof of equation (2.31):
$(k)-$th order derivative is written in the following form

$$D^k_{\gamma_1,\gamma_2}(d(a,x))^n = D^{(k-1)}_{\gamma_1,\gamma_2}\big(D_{\gamma_1,\gamma_2}\big((d(a,x))^n\big)\big).$$

Using property $4$ and equation (2.30) yields that

$$\begin{aligned}
D^k_{\gamma_1,\gamma_2}(d(a,x))^n =& D^{(k-1)}_{\gamma_1,\gamma_2}\big(D_{\gamma_1,\gamma_2}\big((d(a,x))^n\big)\big)\\
=& D^{(k-1)}_{\gamma_1,\gamma_2}\big(n(d(a,x))^{n-1}\big)\\
=& nD^{(k-2)}_{\gamma_1,\gamma_2}\big(D_{\gamma_1,\gamma_2}\big((d(a,x))^{n-1}\big)\big)\\
=& nD^{(k-2)}_{\gamma_1,\gamma_2}\big((n-1)(d(a,x))^{n-2}\big).
\end{aligned}$$

Continuing this procedure, we obtain

$$D^k_{\gamma_1,\gamma_2}(d(a,x))^n = n(n-1)\ldots(n-k+1)(d(a,x))^{n-k}$$

which is equal to equation (2.31). □

As classical derivative, generalized derivative satisfies Leibniz formula expressing the derivative of $n$th order of the product of two functions.

**Lemma 2.2.16.** *Let $f(x)$ and $g(x)$ be differentiable functions. Then,*

$$D^{n}_{\gamma_1,\gamma_2}(f.g)(x) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x). \tag{2.32}$$

*Proof.* Suppose that the functions $f$, $g$, $\gamma_1$ and $\gamma_2$ are $(n+1)$ times differantiable. From the product rule we know that

$$D_{\gamma_1,\gamma_2}(f.g) = g.D_{\gamma_1,\gamma_2} f + f.D_{\gamma_1,\gamma_2} g.$$

Thus, Equation (2.32) holds for $n = 1$. Let us assume (2.32) is true for $n \geq 1$. we can write the $(n+1)$th order derivatives in the following form

$$D^{n+1}_{\gamma_1,\gamma_2}(f.g)(x) = D_{\gamma_1,\gamma_2}(D^{n}_{\gamma_1,\gamma_2}(f.g)(x)) = D_{\gamma_1,\gamma_2}\left(\sum_{i=0}^{n} \binom{n}{i} D^{n-i}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x)\right).$$

Then,

$$\begin{aligned}
D^{n+1}_{\gamma_1,\gamma_2}(f.g)(x) &= \sum_{i=0}^{n} \binom{n}{i} D^{n-i+1}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x) + \sum_{i=0}^{n} \binom{n}{i} D^{n-i}_{\gamma_1,\gamma_2} f(x) D^{i+1}_{\gamma_1,\gamma_2} g(x) \\
&= \sum_{i=0}^{n} \binom{n}{i} D^{n-i+1}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x) + \sum_{i=1}^{n+1} \binom{n}{i-1} D^{n+1-i}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x) \\
&= D^{n+1}_{\gamma_1,\gamma_2} f(x) D^{0}_{\gamma_1,\gamma_2} g(x) + \sum_{i=1}^{n} \binom{n}{i} D^{n-i+1}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x) \\
&+ \sum_{i=1}^{n} \binom{n}{i-1} D^{n+1-i}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x) + D^{0}_{\gamma_1,\gamma_2} f(x) D^{n+1}_{\gamma_1,\gamma_2} g(x).
\end{aligned}$$

All sums in the right-hand side can be combined into single sum and we get

$$D^{n+1}_{\gamma_1,\gamma_2}(f.g)(x) = \sum_{i=0}^{n+1} \binom{n+1}{i} D^{n+1-i}_{\gamma_1,\gamma_2} f(x) D^{i}_{\gamma_1,\gamma_2} g(x)$$

which completes the proof. $\square$

Let us compute higher order derivatives of some terms in the space $\pi_n(\gamma_1, \gamma_2)$.

**Lemma 2.2.17.** *If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$ *, then*

$$D^n_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n-1} d(x_k, x)\right) = n! \tag{2.33}$$

*Proof.* The proof is by induction on $n$. From equation (2.30), it is clear that equation (2.33) holds for $n = 1$.

Suppose that (2.33) holds for $n > 1$, then

$$\begin{aligned}
D^{(n+1)}_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n} d(x_k, x)\right) &= D^n_{\gamma_1,\gamma_2}\left(D_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n} d(x_k, x)\right)\right) \\
&= D^n_{\gamma_1,\gamma_2}\left(d(x_0, x)D_{\gamma_1,\gamma_2}\left(\prod_{k=1}^{n} d(x_k, x)\right) + \prod_{k=1}^{n} d(x_k, x)\right) \\
&= D^n_{\gamma_1,\gamma_2}\left(d(x_0, x)D_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n-1} d(x_{k+1}, x)\right)\right) \\
&\quad + D^n_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n-1} d(x_{k+1}, x)\right).
\end{aligned}$$

Using induction hypothesis for the last term, we get

$$D^{(n+1)}_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n} d(x_k, x)\right) = D^n_{\gamma_1,\gamma_2}\left(d(x_0, x)D_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n-1} d(x_{k+1}, x)\right)\right) + n!.$$

Then,

$$\begin{aligned}
D^{(n+1)}_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n} d(x_k,x)\right) &= D^{n}_{\gamma_1,\gamma_2}\Bigg(d(x_0,x)d(x_1,x)D_{\gamma_1,\gamma_2}\left(\prod_{k=1}^{n-1} d(x_{k+1},x)\right)\\
&\quad + d(x_0,x)\prod_{k=1}^{n-1} d(x_{k+1},x)\Bigg) + n!\\
&= D^{n}_{\gamma_1,\gamma_2}\left(d(x_0,x)d(x_1,x)D_{\gamma_1,\gamma_2}\prod_{k=0}^{n-2} d(x_{k+2},x)\right)\\
&\quad + D^{n}_{\gamma_1,\gamma_2}\underbrace{\left(\prod_{k=0,k\neq 1}^{n} d(x_k,x)\right)}_{n \text{ terms}} + n!\\
&= D^{n}_{\gamma_1,\gamma_2}\left(d(x_0,x)d(x_1,x)D_{\gamma_1,\gamma_2}\prod_{k=0}^{n-2} d(x_{k+2},x)\right)\\
&\quad + n! + n!.
\end{aligned}$$

Continuing in this way, we get

$$D^{(n+1)}_{\gamma_1,\gamma_2}\left(\prod_{k=0}^{n} d(x_k,x)\right) = (n+1)n! = (n+1)!.$$

It completes the proof. $\square$

**Lemma 2.2.18.** *Let* $1 \in$ *span*$\{\gamma_1,\gamma_2\}$ *and* $f(x) \in \pi_n(\gamma_1,\gamma_2)$. *Then,*

$$D^{(n+1)}_{\gamma_1,\gamma_2} f(x) = 0.$$

*Proof.* Since $1 \in \text{span}\{\gamma_1,\gamma_2\}$ and $f(x) \in \pi_n(\gamma_1,\gamma_2)$, the function $f(x)$ can be written as $f(x) = \sum_{i=0}^{n} a_i\gamma_1^i$. Then the first order generalized derivative of the function $f(x)$ is as follows

$$\begin{aligned}
D_{\gamma_1,\gamma_2} f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{d(x,x+h)}\\
&= \lim_{h\to 0}\frac{\sum_{i=0}^{n} a_i(\gamma_1^i(x+h)-\gamma_1^i(x))}{\frac{1}{c_2}(\gamma_1(x)-\gamma_1(x+h))}.
\end{aligned}$$

Using the identity $a^m - b^m = (a-b)\sum_{j=0}^{m-1} a^j b^{m-j-1}$ and taking the limit, we obtain

$$D_{\gamma_1,\gamma_2} f(x) = \sum_{i=0}^{n}\sum_{j=0}^{i-1} \left(a_i(-c_2)\right) \gamma_1^j(x)\gamma_1^{i-j-1}(x) = \sum_{i=0}^{n} a_i(-c_2)(i)\gamma_1^{i-1}(x)$$

which implies $D_{\gamma_1,\gamma_2} f(x) \in \pi_{n-1}(\gamma_1,\gamma_2)$. Continuining in this way, we get

$$D^n_{\gamma_1,\gamma_2} f(x) = c \in \pi_0(\gamma_1,\gamma_2),$$

where $c$ is any constant. Thus,

$$D^{(n+1)}_{\gamma_1,\gamma_2} f(x) = D_{\gamma_1,\gamma_2}\left(D^n_{\gamma_1,\gamma_2} f(x)\right) = D_{\gamma_1,\gamma_2}(c) = 0.$$

□

The following theorem gives the value of non-polynomial divided differences with repeated points.

**Theorem 2.2.19.** *If* $1 \in$ *span* $\{\gamma_1,\gamma_2\}$ , *then*

$$f[\underbrace{x_0, x_0, \ldots, x_0}_{n+1 \textit{ terms}}] = \frac{D^n_{\gamma_1,\gamma_2} f(x_0)}{n!}. \tag{2.34}$$

*Proof.* We use induction on $n$. Clearly, from equation (2.28), equation (2.34) is true for $n = 1$. Assume that the induction hypothesis is true for all $k \leqslant n$, i.e.

$$f[\underbrace{x_0, x_0, \ldots, x_0}_{k+1 \text{ terms}}] = \frac{D^k_{\gamma_1,\gamma_2} f(x_0)}{k!}, \quad k \leqslant n. \tag{2.35}$$

We need to show that (2.35) is also true for $k = n+1$. Now consider the error term in Theorem 2.2.1. Letting $x_i \to x_0$ for all $i$ and rearranging the error term by using the induction hypothesis yields

$$f_{\gamma_1,\gamma_2}[\underbrace{x_0, x_0, \ldots, x_0}_{n+1 \text{ terms}}, x] = \frac{f(x) - \sum_{j=0}^{n} f_{\gamma_1,\gamma_2}[\overbrace{x_0, x_0, \ldots, x_0}^{j+1 \text{ terms}}] \, (d(x_0, x))^j}{(d(x_0, x))^{n+1}}$$
$$= \frac{f(x) - \sum_{j=0}^{n} \frac{D^j_{\gamma_1,\gamma_2} f(x_0)}{j!} \, (d(x_0, x))^j}{(d(x_0, x))^{n+1}}.$$

Then,

$$f_{\gamma_1,\gamma_2}[\underbrace{x_0, x_0, \ldots, x_0}_{n+2 \text{ terms}}] = \lim_{x \to x_0} f_{\gamma_1,\gamma_2}[\underbrace{x_0, x_0, \ldots, x_0}_{n+1 \text{ terms}}, x]$$
$$= \lim_{x \to x_0} \frac{f(x) - \sum_{j=0}^{n} \frac{D^j_{\gamma_1,\gamma_2} f(x_0)}{j!} \, (d(x_0, x))^j}{(d(x_0, x))^{n+1}}. \tag{2.36}$$

In the latter limit of equation (2.36), we have an indeterminate form of type $\frac{0}{0}$. Note that equation (2.29) suggests that, instead of the classical derivative operator, we may use the operator $D_{\gamma_1,\gamma_2}$ for L'Hospital rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \lim_{x \to a} \frac{D_{\gamma_1,\gamma_2} f(x) W(\gamma_1, \gamma_2)(x)}{D_{\gamma_1,\gamma_2} g(x) W(\gamma_1, \gamma_2)(x)} = \lim_{x \to a} \frac{D_{\gamma_1,\gamma_2} f(x)}{D_{\gamma_1,\gamma_2} g(x)}.$$

However, from Lemma 2.2.15, we see that $n + 1$ applications of L'Hospital rule is required. Thus

$$f_{\gamma_1,\gamma_2}[\underbrace{x_0, x_0, \ldots, x_0}_{n+2 \text{ terms}}] = \lim_{x \to x_0} \frac{D^{n+1}_{\gamma_1,\gamma_2} f(x) - 0}{(n+1)!} = \frac{D^{n+1}_{\gamma_1,\gamma_2} f(x_0)}{(n+1)!}.$$

This completes the proof. □

**Corollary 2.2.19.1.** *If* $1 \in$ *span* $\{\gamma_1, \gamma_2\}$ *, then the function*

$$g_n(x) = \sum_{j=0}^{n} f_{\gamma_1,\gamma_2}[\overbrace{x_0, x_0, \ldots, x_0}^{j+1 \text{ terms}}] \, (d(x_0, x))^j$$

*satisfies the conditions*

$$D^k_{\gamma_1,\gamma_2} g_n(x_0) = D^k_{\gamma_1,\gamma_2} f(x_0)$$

*for* $k = 0, 1, \ldots, n$.

### 2.3 Explicit representation of non-polynomial B-splines

In this section, our aim is to obtain explicit representation of non-polynomial B-splines. As in the classical B-splines, we will use the following notation for non-polynomial divided difference operator

$$[x_j, x_{j+1}, \ldots, x_{j+n}]_{\gamma_1,\gamma_2} f.$$

We again regard $[x_j, x_{j+1}, \ldots, x_{j+n}]_{\gamma_1,\gamma_2}$ as an operator that acts on $f$.

We begin by the definition of a proper generalization of truncated power function for the space $\pi_n(\gamma_1, \gamma_2)$.

**Definition 2.3.1.** *For any fixed real number $x$ and any non-negative integer $n$, the truncated power function of $t$, $d(x,t)^n_+$ defined for $-\infty < t < \infty$ as follows:*

$$d(x,t)^n_+ = \begin{cases} d(x,t)^n, & x \le t \\ 0, & x > t. \end{cases}$$

When $1 \in$ span $\{\gamma_1, \gamma_2\}$ , the explicit representation of non- polynomial B-splines for the space $\pi_n(\gamma_1, \gamma_2)$ is given in terms of non-polynomial divided differences of the truncated power function in the following theorem.

**Theorem 2.3.1.** *Let $1 \in$ span $\{\gamma_1, \gamma_2\}$. For any $n \ge 0$ and all $i$, non-polynomial B-spline basis function $N_{i,n}(x)$ of degree $n$ (with respect to a given set of knots $\{x_j\}$) is*

$$N_{i,n}(x) = d(x_i, x_{i+n+1})[x_i, x_{i+1}, \ldots, x_{i+n+1}]_{\gamma_1,\gamma_2} d(x,t)^n_+,$$

*where* $[x_i, \ldots, x_{i+n+1}]_{\gamma_1,\gamma_2}$ *denotes a non-polynomial divided difference operator of order* $n+1$ *that is applied to the truncated power function* $d(x,t)_+^n$*, regarded as a function of the variable* $t$*.*

*Proof.* We begin by the Marsden's identity defined in (Dişibüyük & Goldman (2016)):

$$d(x,t)^n = \sum_i \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) N_{i,n}(x). \tag{2.37}$$

Multiplying both sides of the equation (2.37) by $d(x,t)_+^0$, it is obtained that

$$d(x,t)^n d(x,t)_+^0 = \sum_i \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)_+^0 N_{i,n}(x).$$

Applying the divided difference operator with respect to $t$ to both sides with the nodes $x_j, \ldots, x_{j+n+1}$ yields that

$$\begin{aligned} &[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} d(x,t)^n d(x,t)_+^0 \\ &\quad = \sum_i [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)_+^0 N_{i,n}(x). \end{aligned} \tag{2.38}$$

We will show that all the terms in the summation on the right hand side of equation (2.38) are zero except for $i = j$. Therefore, we investigate the non-polynomial divided differences for the terms

$$\prod_{k=1}^{n} d(x_{i+k}, t) d(x,t)_+^0 N_{i,n}(x)$$

in five cases.

**Case 1.** $i < j - n:$

For $x \in (x_i, x_{i+n+1})$, we have two cases. In the first case, $x \in (x_i, x_{j-n}]$, while in the other, $x \in (x_{j-n}, x_{j+1})$. For $x \in (x_i, x_{j-n}]$ and $t \in \{x_j, \ldots, x_{j+n+1}\}$, we get $d(x,t)_+^0 = 1$. For $x \in (x_{j-n}, x_{j+1})$, we obtain either $d(x,t)_+^0 = 0$ or $d(x,t)_+^0 = 1$. If

$d(x,t)^0_+ = 0$, we directly get that

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)^0_+ \right) = 0.$$

Also, for $x$ values where $d(x,t)^0_+ = 1$, we obtain

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)^0_+ \right) = [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right).$$

By Theorem 2.2.3, since $\displaystyle\prod_{k=1}^{n} d(x_{i+k}, t) \in \pi_n(\gamma_1, \gamma_2)$

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) = 0.$$

Hence

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) N_{i,n}(x) = 0 \text{ for } i < j - n.$$

**Case 2.** $j - n \leq i < j$ : Let $i = j - m$ for $1 \leq m \leq n$. Consider the function

$$\begin{aligned} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)^0_+ &= \left( \prod_{k=1}^{n} d(x_{j-m+k}, t) \right) d(x,t)^0_+ \\ &= d(x_{j-m+1}, t) d(x_{j-m+2}, t) \ldots d(x_{j-m+n}, t) d(x,t)^0_+. \end{aligned}$$

The functions $\left( \prod_{k=1}^{n} d(x_{j-m+k}, t) \right) d(x,t)^0_+$ and $\prod_{k=1}^{n} d(x_{j-m+k}, t)$ are then zero for $t \in \{x_j, \ldots, x_{j+n-m}\}$. Moreover, these two functions have the same values for $t \in \{x_{j+n-m+1}, \ldots, x_{j+n+1}\}$ since $d(x,t)^0_+ = 1$ for $x \in (x_i, x_{i+n+1})$. Hence by Lemma 2.2.7,

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)^0_+ = [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right).$$

Since $\prod_{k=1}^{n} d(x_{i+k},t) \in \pi_n(\gamma_1,\gamma_2)$, we get

$$[x_j,\ldots,x_{j+n+1}]_{\gamma_1,\gamma_2}\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)=0.$$

Therefore,

$$[x_j,\ldots,x_{j+n+1}]_{\gamma_1,\gamma_2}\left(\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)d(x,t)_+^0\right)N_{i,n}(x)=0 \text{ for } j-n\leq i<j.$$

**Case 3.** $i>j+n$ :

For $x\in(x_i,x_{i+n+1})$ and $t\in\{x_j,\ldots,x_{j+n+1}\}$, we have $d(x,t)_+^0=0$. Thus,

$$[x_j,\ldots,x_{j+n+1}]_{\gamma_1,\gamma_2}\left(\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)d(x,t)_+^0\right)=0$$

and

$$[x_j,\ldots,x_{j+n+1}]_{\gamma_1,\gamma_2}\left(\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)d(x,t)_+^0\right)N_{i,n}(x)=0 \text{ for } i>j+n.$$

**Case 4.** $j<i\leq j+n$ : Let $i=j+m$ for $1\leq m\leq n$. Then,

$$\prod_{k=1}^{n} d(x_{i+k},t)=\prod_{k=1}^{n} d(x_{j+m+k},t)=0$$

for $t\in\{x_{j+m+1},\ldots,x_{j+n+1}\}$. Moreover, for $t\in\{x_j,\ldots,x_{j+m}\}$ and $x\in(x_i,x_{i+n+1})$, $d(x,t)_+^0=0$. Therefore $\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)d(x,t)_+^0$ agrees with the zero function at the nodes $x_j,\ldots,x_{j+n+1}$. Then, by Lemma 2.2.7

$$[x_j,\ldots,x_{j+n+1}]_{\gamma_1,\gamma_2}\left(\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)d(x,t)_+^0\right)=0.$$

Hence

$$[x_j,\ldots,x_{j+n+1}]_{\gamma_1,\gamma_2}\left(\left(\prod_{k=1}^{n} d(x_{i+k},t)\right)d(x,t)_+^0\right)N_{i,n}(x)=0 \text{ for } j<i\leq j+n.$$

**Case 5.** $i = j$ : For $i = j$ and $x \in (x_j, x_{j+n+1}) = (x_i, x_{i+n+1})$, consider the function of $x$

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \left( \prod_{k=1}^{n} d(x_{j+k}, t) \right) d(x,t)_+^0 \right) N_{i,n}(x).$$

Notice that the following functions of $t$

$$\left( \prod_{k=1}^{n} d(x_{j+k}, t) \right) d(x,t)_+^0 = d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) d(x,t)_+^0 \tag{2.39}$$

and

$$\frac{d(x_j, t)}{d(x_j, x_{j+n+1})} d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) \tag{2.40}$$

coincide at the nodes $x_j, \ldots, x_{j+n+1}$. In fact, for $t = x_{j+1}, \ \ldots, x_{j+n}$,

$$d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) = 0,$$

and for $t = x_j$, $d(x,t)_+^0 = 0$ since $x \in (x_j, x_{j+n+1})$. Hence

$$d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) d(x,t)_+^0 = 0$$

at the nodes $x_j, \ldots, x_{j+n}$. For $t = x_{j+n+1}$,

$$\begin{aligned} &d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) d(x,t)_+^0 \\ &\qquad = d(x_{j+1}, x_{j+n+1}) d(x_{j+2}, x_{j+n+1}) \ldots d(x_{j+n}, x_{j+n+1}) \end{aligned}$$

since $d(x,t)_+^0 = 1$. On the other hand,

$$\frac{d(x_j, t)}{d(x_j, x_{j+n+1})} d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) = 0$$

at the nodes $x_j, \ldots, x_{j+n}$. Finally, for $t = x_{j+n+1}$ the function (2.40) takes the value

$$d(x_{j+1}, x_{j+n+1}) d(x_{j+2}, x_{j+n+1}) \ldots d(x_{j+n}, x_{j+n+1}).$$

Thus, the non-polynomial divided differences of these two functions in (2.39) and

(2.40) are the same at the nodes $x_j, \ldots, x_{j+n+1}$. Hence,

$$\begin{aligned}
&[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \left( \prod_{k=1}^{n} d(x_{j+k}, t) \right) d(x,t)_+^0 \right) N_{i,n}(x) \\
&\quad = [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \frac{d(x_j, t)}{d(x_j, x_{j+n+1})} d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) \right) N_{i,n}(x).
\end{aligned}$$

On the other hand, from Lemma 2.2.8,

$$[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \frac{d(x_j, t)}{d(x_j, x_{j+n+1})} d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) \right) = \frac{1}{d(x_j, x_{j+n+1})}.$$

Thus, from equation (2.38)

$$\begin{aligned}
&[x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( d(x,t)^n d(x,t)_+^0 \right) \\
&\quad = \sum_i [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \left( \prod_{k=1}^{n} d(x_{i+k}, t) \right) d(x,t)_+^0 \right) N_{i,n}(x) \\
&\quad = [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \left( \prod_{k=1}^{n} d(x_{j+k}, t) \right) d(x,t)_+^0 \right) N_{j,n}(x) \\
&\quad = [x_j, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} \left( \frac{d(x_j, t)}{d(x_j, x_{j+n+1})} d(x_{j+1}, t) d(x_{j+2}, t) \ldots d(x_{j+n}, t) \right) N_{j,n}(x) \\
&\quad = \frac{N_{j,n}(x)}{d(x_j, x_{j+n+1})}.
\end{aligned}$$

Then,

$$N_{j,n}(x) = d(x_j, x_{j+n+1}) [x_j, x_{j+1}, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} d(x,t)^n d(x,t)_+^0.$$

Considering $d(x,t)^n d(x,t)_+^0$, we have

$$d(x,t)^{n-1} d(x,t)^1 d(x,t)_+^0 = d(x,t)^{n-1} d(x,t)_+^1,$$

doing this recursively, it yields that

$$d(x,t)^n d(x,t)_+^0 = d(x,t)_+^n.$$

Hence, we obtain

$$N_{j,n}(x) = d(x_j, x_{j+n+1})[x_j, x_{j+1}, \ldots, x_{j+n+1}]_{\gamma_1,\gamma_2} d(x,t)_+^n.$$

□

## 2.4 Non-polynomial B-spline integration

In this section, giving definition of definite integral, we will show that the non-polynomial divided difference operator is built by the integration with non-polynomial B-splines. Our main tools here are the Fundamental Theorem of Calculus, integration by parts, and two term differentiation for non-polynomial B-spline. We begin with the definition of the definite integral.

**Definition 2.4.1.** *Let* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$. *If* $f, \gamma_1$ *and* $\gamma_2$ *are continuous functions defined for* $a \leq x \leq b$ *, we divide the interval into* $n$ *subintervals of width* $d(x_{i-1}, x_i)$ *for* $i = 1, \ldots, n$. *Let* $x_i = x_0 + d(x_0, x_i)$ *for* $i = 1, \ldots, n$ *be the endpoints of these subintervals and let* $x_1^*, x_2^*, \ldots, x_n^*$ *be any sample points in these subintervals, so* $x_i^*$ *lies in the* $i$*th subinterval* $[x_{i-1}, x_i]$ *. Then the definite integral of* $f$ *from* $a$ *to* $b$ *is*

$$\int_a^b f(x) d_{\gamma_1,\gamma_2} x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) d(x_{i-1}, x_i).$$

As Riemann integral, defined integral satisfies the properties of integrals. Also, it is easily shown by using equation (2.11) that if $1 \in$ span$\{\gamma_1, \gamma_2\}$, we can combine integrals of the same function over adjacent intervals:

$$\int_a^c f(x) d_{\gamma_1,\gamma_2} x + \int_c^b f(x) d_{\gamma_1,\gamma_2} x = \int_a^b f(x) d_{\gamma_1,\gamma_2} x.$$

Also, if $1 \in$ span$\{\gamma_1, \gamma_2\}$, we get

$$m.d(a,b) \leq \int_a^b f(t) d_{\gamma_1,\gamma_2} t \leq M.d(a,b),$$

where $m$ and $M$ are absolute minimum and maximum values of $f$ on the interval $[a, b]$.

In the following theorem, we are giving an analogue of the Fundamental Theorem of Calculus for the case $1 \in$ span$\{\gamma_1, \gamma_2\}$.

**Theorem 2.4.1.** *Let* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$. *If* $f, \gamma_1$ *and* $\gamma_2$ *are continuous on* $[a, b]$,

1. *If* $g(x) = \int_a^x f(t) d_{\gamma_1,\gamma_2} t$ *then,* $D_{\gamma_1,\gamma_2} g(x) = f(x)$.
2. $\int_a^b f(t) d_{\gamma_1,\gamma_2} t = F(b) - F(a)$, *where* $F(x)$ *is any antiderivative of* $f$ *such that* $D_{\gamma_1,\gamma_2} F(x) = f(x)$.

*Proof.* The proof is similar with the classical case. $\square$

**Theorem 2.4.2.** *Let* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$. *Assume that* $f(x)$ *and* $g(x)$ *are continuous functions on* $[a, b]$. *If* $g(x)$ *never changes sign on* $[a, b]$, *then there exists* $c \in [a, b]$ *such that*

$$\int_a^b f(x) g(x) d_{\gamma_1,\gamma_2} x = f(c) \int_a^b g(x) d_{\gamma_1,\gamma_2} x.$$

*Proof.* Since $f(x)$ is continuous function on $[a, b]$, by Extreme Value Theorem, there exist $m, M \in [a, b]$ such that

$$m \leq f(x) \leq M.$$

Because $g(x)$ doesn't change sign, either $g(x) \geq 0$ or $g(x) \leq 0$ for $a \leq x \leq b$. For the sake of definiteness, let's assume that $g(x) \geq 0$.

$$\begin{aligned}
&m g(x) \leq f(x) g(x) \leq M g(x) \\
&m \int_a^b g(x) d_{\gamma_1,\gamma_2} x \leq \int_a^b f(x) g(x) d_{\gamma_1,\gamma_2} x \leq M \int_a^b g(x) d_{\gamma_1,\gamma_2} x.
\end{aligned}$$

Assume that $\int_a^b g(x) d_{\gamma_1,\gamma_2} x \neq 0$. Then,

$$m \leq \frac{\int_a^b f(x) g(x) d_{\gamma_1,\gamma_2} x}{\int_a^b g(x) d_{\gamma_1,\gamma_2} x} \leq M.$$

By Intermediate Value Theorem, there exists at least $c \in [a, b]$ such that

$$\frac{\int_a^b f(x)g(x)d_{\gamma_1,\gamma_2}x}{\int_a^b g(x)d_{\gamma_1,\gamma_2}x} = f(c).$$

Hence, we get

$$\int_a^b f(x)g(x)d_{\gamma_1,\gamma_2}x = f(c)\int_a^b g(x)d_{\gamma_1,\gamma_2}x.$$

$\square$

Integration by parts corresponds to the product rule for derivatives, and if $f$ and $g$ are two differentiable functions, then the product rule gives us:

$$D_{\gamma_1,\gamma_2}(f(x)g(x)) = f(x)D_{\gamma_1,\gamma_2}g(x) + g(x)D_{\gamma_1,\gamma_2}f(x).$$

Integrating both sides and then using , we get

$$\int f(x)D_{\gamma_1,\gamma_2}g(x)d_{\gamma_1,\gamma_2}x = f(x)g(x) - \int g(x)D_{\gamma_1,\gamma_2}f(x)d_{\gamma_1,\gamma_2}x.$$

This is the integration by parts formula.
The following theorem implies that non-polynomial divided difference is represented by the integration with non-polynomial B-splines if $1 \in \text{span}\{\gamma_1, \gamma_2\}$.

**Theorem 2.4.3.** *If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$ *, then, for* $f \in C^{n+1}[x_i, x_{i+n+1}]$

$$f_{\gamma_1,\gamma_2}[x_i, \ldots, x_{i+n+1}] = \int_{x_i}^{x_{i+n+1}} \left\{\frac{N_{i,n}(x)}{d(x_i, x_{i+n+1})}\right\}\left\{\frac{D^n_{\gamma_1,\gamma_2}f(x)}{n!}\right\}d_{\gamma_1,\gamma_2}x.$$

*Proof.* The proof is by induction on $n$. For $n = 0$, by Theorem 2.4.1

$$f_{\gamma_1,\gamma_2}[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{d(x_i, x_{i+1})} = \int_{x_i}^{x_{i+1}} \left\{\frac{D_{\gamma_1,\gamma_2}f(x)}{d(x_i, x_{i+1})}\right\}d_{\gamma_1,\gamma_2}x.$$

Assume that the induction hypothesis is true for $n \geq 0$. Consider,

$$\int_{x_i}^{x_{i+n+2}} \Big\{\frac{N_{i,n+1}(x)}{d(x_i, x_{i+n+2})}\Big\}\Big\{\frac{D^{n+1}_{\gamma_1,\gamma_2} f(x)}{(n+1)!}\Big\} d_{\gamma_1,\gamma_2} x.$$

Using integration by parts, we obtain

$$\begin{aligned}
\int_{x_i}^{x_{i+n+2}} &\Big\{\frac{N_{i,n+1}(x)}{d(x_i, x_{i+n+2})}\Big\}\Big\{\frac{D^{n+1}_{\gamma_1,\gamma_2} f(x)}{(n+1)!}\Big\} d_{\gamma_1,\gamma_2} x \\
&= \Big\{\frac{N_{i,n+1}(x)}{d(x_i, x_{i+n+2})} \cdot \frac{D^{n}_{\gamma_1,\gamma_2} f(x)}{(n+1)!}\Big\}\Big|_{x_i}^{x_{i+n+2}} \\
&- \frac{1}{d(x_i, x_{i+n+2})} \int_{x_i}^{x_{i+n+2}} D_{\gamma_1,\gamma_2} N_{i,n+1}(x) \Big\{\frac{D^{n}_{\gamma_1,\gamma_2} f(x)}{(n+1)!}\Big\} d_{\gamma_1,\gamma_2} x.
\end{aligned}$$

Since $N_{i,n+1}(x)$ vanishes on the end points of its support, we get

$$\begin{aligned}
\int_{x_i}^{x_{i+n+2}} &\Big\{\frac{N_{i,n+1}(x)}{d(x_i, x_{i+n+2})}\Big\}\Big\{\frac{D^{n+1}_{\gamma_1,\gamma_2} f(x)}{(n+1)!}\Big\} d_{\gamma_1,\gamma_2} x \\
&= -\frac{1}{d(x_i, x_{i+n+2})} \int_{x_i}^{x_{i+n+2}} D_{\gamma_1,\gamma_2} N_{i,n+1}(x) \Big\{\frac{D^{n}_{\gamma_1,\gamma_2} f(x)}{(n+1)!}\Big\} d_{\gamma_1,\gamma_2} x. \quad (2.41)
\end{aligned}$$

In Dişibüyük & Goldman (2016), two term differentiation formula for the non-polynomial B-spline $N_{i,n}(x)$ are given as follows

$$\frac{d}{dx} N_{i,n}(x) = n\Big[\Big(\frac{d}{dx}\frac{d(x_i, x)}{d(x_i, x_{i+n})}\Big) N_{i,n-1}(x) + \Big(\frac{d}{dx}\frac{d(x, x_{i+n+1})}{d(x_{i+1}, x_{i+n+1})}\Big) N_{i+1,n-1}(x)\Big].$$

By equation (2.29) , we can rewrite the formula,

$$\begin{aligned}
W(\gamma_1,\gamma_2)(x) D_{\gamma_1,\gamma_2} N_{i,n}(x) =& n\Big[\Big(W(\gamma_1,\gamma_2)(x) D_{\gamma_1,\gamma_2} d(x_i, x)\Big) \frac{N_{i,n-1}(x)}{d(x_i, x_{i+n})} \\
&+ \Big(W(\gamma_1,\gamma_2)(x) D_{\gamma_1,\gamma_2} d(x, x_{i+n+1})\Big) \frac{N_{i+1,n-1}(x)}{d(x_{i+1}, x_{i+n+1})}\Big].
\end{aligned}$$

Then, using the facts that $D_{\gamma_1,\gamma_2} d(x_i, x) = 1$ and $D_{\gamma_1,\gamma_2} d(x, x_{i+n+1}) = -1$, we get

$$D_{\gamma_1,\gamma_2} N_{i,n}(x) = n\Big[\frac{N_{i,n-1}(x)}{d(x_i, x_{k+n})} - \frac{N_{i+1,n-1}(x)}{d(x_{i+1}, x_{i+n+1})}\Big].$$

Thus, equation (2.41) is written as

$$
\begin{aligned}
&\int_{x_i}^{x_{i+n+2}} \Big\{\frac{N_{i,n+1}(x)}{d(x_i,x_{i+n+2})}\Big\}\Big\{\frac{D^{n+1}_{\gamma_1,\gamma_2}f(x)}{(n+1)!}\Big\}d_{\gamma_1,\gamma_2}x \\
&= -\frac{1}{d(x_i,x_{i+n+2})}\int_{x_i}^{x_{i+n+2}}(n+1)\Big[\frac{N_{i,n}(x)}{d(x_i,x_{i+n+1})}-\frac{N_{i+1,n}(x)}{d(x_{i+1},x_{i+n+2})}\Big]\Big\{\frac{D^{n}_{\gamma_1,\gamma_2}f(x)}{(n+1)!}\Big\}d_{\gamma_1,\gamma_2}x \\
&= \frac{1}{d(x_i,x_{i+n+2})}\int_{x_{i+1}}^{x_{i+n+2}}\frac{N_{i+1,n}(x)}{d(x_{i+1},x_{i+n+2})}\Big\{\frac{D^{n}_{\gamma_1,\gamma_2}f(x)}{n!}\Big\}d_{\gamma_1,\gamma_2}x \\
&- \frac{1}{d(x_i,x_{i+n+2})}\int_{x_i}^{x_{i+n+1}}\frac{N_{i,n}(x)}{d(x_i,x_{i+n+1})}\Big\{\frac{D^{n}_{\gamma_1,\gamma_2}f(x)}{n!}\Big\}d_{\gamma_1,\gamma_2}x.
\end{aligned}
$$

Hence, by induction hypothesis we deduce that,

$$
\begin{aligned}
\int_{x_i}^{x_{i+n+2}} &\Big\{\frac{N_{i,n+1}(x)}{d(x_i,x_{i+n+2})}\Big\}\Big\{\frac{D^{n+1}_{\gamma_1,\gamma_2}f(x)}{(n+1)!}\Big\}d_{\gamma_1,\gamma_2}x \\
&= \frac{1}{d(x_i,x_{i+n+2})}\Big\{f_{\gamma_1,\gamma_2}[x_{i+1},\ldots,x_{i+n+2}] - f_{\gamma_1,\gamma_2}[x_i,\ldots,x_{i+n+1}]\Big\} \\
&= f_{\gamma_1,\gamma_2}[x_i,\ldots,x_{i+n+2}]
\end{aligned}
$$

which completes the proof.

$\square$

# CHAPTER THREE
# GENERALIZED TAYLOR SERIES

Taylor series have importance for representing an arbitrary function in terms of linear combinations of prescribed functions. The ordinary Taylor series has been generalized by many authors (Riemann (1876), Hardy (1945), Trujillo et al. (1999), Watanabe (1931)). In this section, we obtain a generalization of Taylor series using non-polynomial divided differences in terms of the generalized derivative operator.

Let $f(x)$ has derivatives of all orders on an interval $[a, b]$. We start by assuming that $f(x)$ is any function that can be represented by a power series

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k d(x_0, x)^k \\ &= a_0 + a_1 d(x_0, x) + \ldots + a_n d(x_0, x)^n + \ldots \end{aligned} \tag{3.1}$$

with a positive radius of convergence. Let's determine the coefficients $a_n$. Putting $x = x_0$ in equation (3.1), we get

$$f(x_0) = a_0.$$

Differentiating equation (3.1) term by term gives

$$D_{\gamma_1,\gamma_2} f(x) = a_1 + 2.a_2 d(x_0, x) \ldots + n.a_n d(x_0, x)^{n-1} + \ldots \tag{3.2}$$

Putting $x = x_0$ in equation (3.2), we get

$$D_{\gamma_1,\gamma_2} f(x_0) = a_1.$$

We differentiate both sides of equation (3.2) and obtain

$$D^2_{\gamma_1,\gamma_2} f(x) = 2.a_2 + 3.2.a_3 d(x_0, x) \ldots + n.(n-1).a_n d(x_0, x)^{n-2} + \ldots \tag{3.3}$$

Again we put $x = x_0$ in equation (3.3) . The result is

$$D^2_{\gamma_1,\gamma_2} f(x_0) = 2.a_2.$$

When we continue to differentiate and substitute $x = x_0$, we get

$$D^n_{\gamma_1,\gamma_2} f(x_0) = 2.3 \dots n.a_n = n!a_n.$$

Solving the above equation for $a_n$ gives

$$a_n = \frac{D^n_{\gamma_1,\gamma_2} f(x_0)}{n!}.$$

If $f$ has a power series representation, then the series must be

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{D^k_{\gamma_1,\gamma_2} f(x_0)}{k!} d(x_0, x)^k \\ &= f(x_0) + D^k_{\gamma_1,\gamma_2} f(x_0) d(x_0, x) + \frac{D^2_{\gamma_1,\gamma_2} f(x_0)}{2!} d(x_0, x)^2 + \dots \end{aligned} \tag{3.4}$$

The series in equation (3.4) is called the generalized Taylor series of the function $f$ at $x_0$.
Truncating the series in (3.4) gives the $n$th degree Taylor function of $f(x)$ at $x_0$:

$$\begin{aligned} T_n(x) = f(x_0) + D_{\gamma_1,\gamma_2} f(x_0) d(x_0, x) + &\frac{D^2_{\gamma_1,\gamma_2} f(x_0)}{2!} d(x_0, x)^2 + \\ &\dots + \frac{D^n_{\gamma_1,\gamma_2} f(x_0)}{n!} d(x_0, x)^n. \end{aligned}$$

Note that

$$T_n(x) = \lim_{x_1, x_2, \dots, x_n \to x_0} g_n(x),$$

where $g_n$ interpolates $f$ at $x_0, x_1, \dots, x_n$ . In order to show the generalized Taylor theorem for $f(x)$, we will search the error formula which gives explicit representation for remainder term.

**Lemma 3.0.1.** *Let $x, x_0 \in [a,b]$ and $f, \gamma_1, \gamma_2 \in C^n[a,b]$ and let $f^{n+1}$, $\gamma_1^{n+1}$ and $\gamma_2^{n+1}$ exist in the open interval $(a,b)$ . If $1 \in$ span$\{\gamma_1, \gamma_2\}$, then there exists $\xi_x \in (a,b)$, which depends on $x$, such that*

$$f(x) - T_n(x) = D^{n+1}_{\gamma_1,\gamma_2} f(\xi_x) \frac{(d(x_0,x))^{n+1}}{(n+1)!}.$$

*Proof.* Define a function $h$ by

$$h(x) = f(x) - T_n(x) + K(d(x_0,x))^{n+1}.$$

Clearly $h$ has a zero at $x = x_0$. We will choose $K$ such that $h$ has two zeros including $x = t$. Then

$$h(t) = f(t) - T_n(t) + K(d(x_0,t))^{n+1} = 0$$

which gives

$$K = -\frac{f(t) - T_n(t)}{(d(x_0,t))^{n+1}}.$$

Thus, we have

$$h(x) = f(x) - T_n(x) - \frac{f(t) - T_n(t)}{(d(x_0,t))^{n+1}}(d(x_0,x))^{n+1}.$$

The first $n$ derivatives of $h(x)$ at $x = x_0$ are equal to $0$. Because

$$D^k_{\gamma_1,\gamma_2} T_n(x_0) = D^k_{\gamma_1,\gamma_2} f(x_0)$$

for $k = 0, 1, \ldots, n$, and the first $n$ derivatives of the term $\frac{f(t)-T_n(t)}{(d(x_0,t))^{n+1}}(d(x_0,x))^{n+1}$ at $x = x_0$ are also equal to $0$. It is seen that $h(x_0) = 0$ and $h(t) = 0$. Therefore, we can apply Rolle's Theorem to get that $h'(c_1) = 0$ for some $c_1$ between $x_0$ and $t$. By using the relation $D_{\gamma_1,\gamma_2} f(x) = f'(x)/W(\gamma_1,\gamma_2)(x)$, we obtain that $D_{\gamma_1,\gamma_2} h(c_1) = 0$ for some $c_1$. Now consider $D_{\gamma_1,\gamma_2} h(x)$. We have that $D_{\gamma_1,\gamma_2} h(x_0) = 0$ and $D_{\gamma_1,\gamma_2} h(c_1) = 0$, and then applying Rolle's Theorem again, we obtain that there is some $c_2$ between $x_0$ and $c_1$ such that $D^2_{\gamma_1,\gamma_2} h(c_2) = 0$. This procedure can be repeated $n$ times until we find a $c_n$ between $x_0$ and $c_{n-1}$ such that $D^n_{\gamma_1,\gamma_2} h(c_n) = 0$ . Then, we apply Rolle's Theorem one last time to get that there is some $c_{n+1}$ between $x_0$ and $c_n$ such that $D^{n+1}_{\gamma_1,\gamma_2} h(c_{n+1}) = 0$. Hence, we get

$$D^{n+1}_{\gamma_1,\gamma_2} f(c_{n+1}) - D^{n+1}_{\gamma_1,\gamma_2} T_n(c_{n+1}) - \frac{f(t) - T_n(t)}{(d(x_0,t))^{n+1}} (D^{n+1}_{\gamma_1,\gamma_2}(d(x_0,x))^{n+1})_{x=c_{n+1}} = 0.$$

Here,

$$D^{n+1}_{\gamma_1,\gamma_2} T_n(x) = 0$$

and

$$D^{n+1}_{\gamma_1,\gamma_2}(d(x_0,x))^{n+1} = (n+1)!.$$

Therefore, we obtain

$$D^{n+1}_{\gamma_1,\gamma_2} f(c_{n+1}) - \frac{f(t) - T_n(t)}{(d(x_0,t))^{n+1}} (n+1)! = 0. \tag{3.5}$$

Then, defining $\xi_x = c_{n+1}$ and rearranging equation (3.5), we get

$$f(t) = T_n(t) + D^{n+1}_{\gamma_1,\gamma_2} f(\xi_x) \frac{(d(x_0,t))^{n+1}}{(n+1)!}.$$

Changing $t$ to $x$ and subtracting $T_n(x)$ from both sides completes the proof. $\square$

Now we are going to give a generalization of Taylor Theorem which easily follows from Lemma 3.0.1.

**Theorem 3.0.2** (Generalized Taylor Theorem)**.** *Let* $x, x_0 \in [a,b]$ *and* $f, \gamma_1, \gamma_2 \in C^n[a,b]$ *and let* $f^{n+1}$*,* $\gamma_1^{n+1}$ *and* $\gamma_2^{n+1}$ *exist in the open interval* $(a,b)$ *. If* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$*, then*

$$f(x) = \sum_{k=0}^{n} \frac{D^k_{\gamma_1,\gamma_2} f(x_0)}{k!} d(x_0,x)^k + R_n(x),$$

*where*

$$R_n(x) = D^{n+1}_{\gamma_1,\gamma_2} f(\xi_x) \frac{(d(x_0,x))^{n+1}}{(n+1)!}, \ \ \xi_x \in (a,b).$$

Also, we give the Generalized Taylor Theorem with integral remainder.

**Theorem 3.0.3.** *Let $x, x_0 \in [a,b]$ and $f, \gamma_1, \gamma_2 \in C^{n+1}[a,b]$. If $1 \in$ span$\{\gamma_1, \gamma_2\}$, then $f$ has generalized Taylor series expansion at $x_0$ such that*

$$f(x) = \sum_{k=0}^{n} \frac{D^k_{\gamma_1,\gamma_2} f(x_0)}{k!} d(x_0, x)^k + R_n(x), \tag{3.6}$$

*where*

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} (d(t,x))^n D^{n+1}_{\gamma_1,\gamma_2} f(t) d_{\gamma_1,\gamma_2} t. \tag{3.7}$$

*Proof.* We use induction on $n$. For $n = 1$,

$$R_1(x) = f(x) - T_1(x) = f(x) - f(x_0) - D_{\gamma_1,\gamma_2} f(x_0) d(x_0, x).$$

Consider

$$\int_{x_0}^{x} d(t,x) D^2_{\gamma_1,\gamma_2} f(t) d_{\gamma_1,\gamma_2} t.$$

Applying the integration by parts, we get

$$\begin{aligned} \int_{x_0}^{x} d(t,x) D^2_{\gamma_1,\gamma_2} f(t) d_{\gamma_1,\gamma_2} t =& d(t,x) D_{\gamma_1,\gamma_2} f(t) \Big|_{x_0}^{x} + \int_{x_0}^{x} D_{\gamma_1,\gamma_2} f(t) d_{\gamma_1,\gamma_2} t \\ &= 0 - d(x_0,x) D_{\gamma_1,\gamma_2} f(x_0) + f(x) - f(x_0) \\ &= R_1(x) \end{aligned}$$

Equation (3.7) holds for $n = 1$. Assume that (3.7) is true for $n > 1$. Then, consider

$$\frac{1}{(n+1)!} \int_{x_0}^{x} (d(t,x))^{n+1} D^{n+2}_{\gamma_1,\gamma_2} f(t) d_{\gamma_1,\gamma_2} t.$$

By integration by parts, it is obtained that

$$\begin{aligned}
&\frac{1}{(n+1)!}\int_{x_0}^{x}(d(t,x))^{n+1}D_{\gamma_1,\gamma_2}^{n+2}f(t)d_{\gamma_1,\gamma_2}t\\
&=\frac{1}{(n+1)!}(d(t,x))^{n+1}D_{\gamma_1,\gamma_2}^{n+1}f(t)\Big|_{x_0}^{x}+\frac{n+1}{(n+1)!}\int_{x_0}^{x}(d(t,x))^{n}D_{\gamma_1,\gamma_2}^{n+1}f(t)d_{\gamma_1,\gamma_2}t\\
&=-\frac{1}{(n+1)!}(d(x_0,x))^{n+1}D_{\gamma_1,\gamma_2}^{n+1}f(x_0)+\frac{1}{n!}\int_{x_0}^{x}(d(t,x))^{n}D_{\gamma_1,\gamma_2}^{n+1}f(t)d_{\gamma_1,\gamma_2}t\\
&=-\frac{D_{\gamma_1,\gamma_2}^{n+1}f(x_0)}{(n+1)!}(d(x_0,x))^{n+1}+R_n(x)\\
&=f(x)-T_n(x)-\frac{D_{\gamma_1,\gamma_2}^{n+1}f(x_0)}{(n+1)!}(d(x_0,x))^{n+1}\\
&=f(x)-T_{n+1}(x)\\
&=R_{n+1}(x).
\end{aligned}$$

Therefore, the induction hypothesis is true for $n+1$ and this completes the proof. □

We can give the generalized Taylor formula for functions in the space $\pi_n(\gamma_1,\gamma_2)$. For this generalization, we take our inspiration from the theorem given in (Kac & Cheung, 2002, Theorem 2.1).

**Theorem 3.0.4.** *Let $1 \in span\{\gamma_1,\gamma_2\}$, $x_0$ be a number, $D$ be a linear operator on the space $\pi_n(\gamma_1,\gamma_2)$, and $\{\theta_0,\theta_1,\ldots\}$ be a sequence of functions satifying three conditions:*

*i)* $\theta_0(x_0)=1$ *and* $\theta_k(x_0)=0$ *for any* $k\geq 1$*;*

*ii)* *deg* $\theta_k=k$*;*

*iii)* $D\theta_k(x)=\theta_{k-1}(x)$ *for any* $k\geq 1$*and* $D(1)=0$.

*Then, for any function $f(x)\in\pi_n(\gamma_1,\gamma_2)$, one has the following generalized Taylor formula*

$$f(x)=\sum_{k=0}^{n}(D^kf)(x_0)\theta_k(x).$$

*Proof.* $dim(\pi_n(\gamma_1, \gamma_2)) = n + 1$ (see Dişibüyük & Goldman (2015)). The functions $\{\theta_0, \theta_1, \ldots, \theta_n\}$ are linearly independent by condition (ii). Hence they constitute a basis for the space $\pi_n(\gamma_1, \gamma_2)$. Then, we can express any function $f(x) \in \pi_n(\gamma_1, \gamma_2)$ in the following form

$$f(x) = \sum_{k=0}^{n} c_k \theta_k(x) \tag{3.8}$$

for some unique constants $c_k$. Substituting $x = x_0$ and using condition (i) yields $c_0 = f(x_0)$. $m$ successive application of the linear operator $D$ to both sides of equation (3.8) and using conditions (ii) and (iii) gives

$$(D^m f)(x) = \sum_{k=m}^{n} c_k D^m \theta_k(x) = \sum_{k=m}^{n} c_k \theta_{k-m}(x),$$

where $1 \leq m \leq n$. Again substituting $x = x_0$ and using condition (i) yields that

$$c_m = (D^m f)(x_0), \ 0 \leq m \leq n.$$

□

**Example 1.** *Let* $1 \in$ *span*$\{\gamma_1, \gamma_2\}$. *If* $D = \frac{1}{W(x)} \frac{d}{dx}$ *and* $\theta_k(x) = \frac{(d(x_0,x))^k}{k!}$, *then all the three conditions are satisfied. Because,*

*(i)* $(d(x_0, x))^0 = 1 = \theta_0(x) = \theta_0(x_0)$ *and* $\frac{(d(x_0,x_0))^k}{k!} = 0 = \theta_k(x_0)$ *for* $k \geq 1$

*(ii) deg* $\theta_k$ =*deg*$\left(\frac{(d(x_0,x))^k}{k!}\right) = k$

*By relation* (2.29)*, it is seen that the chosen linear operator* $D$ *is our defined generalized derivative operator* $D_{\gamma_1,\gamma_2}$*. For condition (iii), we get*

$$D\theta_k(x) = D_{\gamma_1,\gamma_2}\left(\frac{(d(x_0, x))^k}{k!}\right) = \frac{(d(x_0, x))^{k-1}}{(k-1)!} = \theta_{k-1}(x)$$

*and* $D(1) = D_{\gamma_1,\gamma_2}(1) = 0$ *by using Lemma 2.2.15 and equation* (2.29).

*Hence all the three conditions are satisfied, and Theorem 3.0.4 gives the generalized Taylor expansion* (3.0.2) *about* $x_0$ *of a function* $f(x) \in \pi_n(\gamma_1, \gamma_2)$ *since* $R_n(x) = 0$.

# CHAPTER FOUR
# PEANO'S THEOREM

The Peano kernel theorem is a well-known method for calculating the errors of approximations including interpolation, quadrature rules, and B-plines (Phillips (2003)). Recently a q-analogue of Peano kernel theorem is given in Budakçı & Oruç (2018). In this section, we derive a new generalization of Peano kernel theorem based on a generalized Taylor expansion. Throughout this section we assume that all conditions stated in Theorem 3.0.3 hold.

Using the truncated power function $d(x,t)_+^n$, the integral remainder for the generalized Taylor expansion of $f$ can be written in the form

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x_n} D_{\gamma_1,\gamma_2}^{n+1} f(t) d(t,x)_+^n d_{\gamma_1,\gamma_2} t. \tag{4.1}$$

**Theorem 4.0.1.** *Let us define*

$$g_t(x) = d(t,x)_+^n,$$

*and let $L$ denote a linear functional that commutes with the operation of integration, and satisfies the further conditions that $L(g_t)$ is defined that $L(f) = 0$ for all $f \in \pi_n$. Then, for all $f, \gamma_1, \gamma_2 \in C^{n+1}[a,b]$,*

$$L(f) = \int_{x_0}^{x_n} D_{\gamma_1,\gamma_2}^{n+1} f(t) K(t) d_{\gamma_1,\gamma_2} t. \tag{4.2}$$

*where*

$$K(t) = K_g(t) = \frac{1}{n!} L(g_t). \tag{4.3}$$

*Proof.* Here, $L(g_t)$ with $g_t(x) = d(t,x)_+^n$ implies that $L$ is applied to the truncated power function $d(t,x)_+^n$ regarded as a function of $x$, with $t$ as a parameter. Hence we get a real number that depends on $t$. Consider $f(x)$ in the form (3.6) with the remainder

$R_n(x)$ given in (4.1). Applying the linear functional $L$ to equation (3.6), we obtain

$$L(f) = \frac{1}{n!} L \int_{x_0}^{x_n} D_{\gamma_1,\gamma_2}^{n+1} f(t)(d(t,x))_+^n d_{\gamma_1,\gamma_2} t$$

since $L$ is linear and annihilates all functions from $\pi_n$. The linear functional $L$ commutes with the operation of integration. Then, we have

$$L(f) = \frac{1}{n!} \int_{x_0}^{x_n} D_{\gamma_1,\gamma_2}^{n+1} f(t) L(d(t,x))_+^n) d_{\gamma_1,\gamma_2} t$$

which completes the proof. □

**Corollary 4.0.1.1.** *If, in addition to the conditions stated in Theorem 4.0.1, the kernel $K(t)$ does not change sign on $[a,b]$, then*

$$L(f) = \frac{D_{\gamma_1,\gamma_2}^{n+1} f(\xi)}{(n+1)!} L(d(x_0,x)^{n+1}), \tag{4.4}$$

*where $x_0 < \xi < x_n$.*

*Proof.* Since $D_{\gamma_1,\gamma_2}^{n+1} f(t)$ is continuous and $K(t)$ does not change on $[a,b]$, we can apply Theorem 2.4.2 to (4.2) to give

$$L(f) = D_{\gamma_1,\gamma_2}^{n+1} f(\xi) \int_{x_0}^{x_n} K(t) d_{\gamma_1,\gamma_2} t, \quad x_0 < \xi < x_n.$$

This holds for all $f, \gamma_1, \gamma_2 \in C^{n+1}[a,b]$. In particular, replacing $f(x)$ with $d(x_0,x)^{n+1}$ in the latter equation yields

$$L(d(x_0,x)^{n+1}) = (n+1)! \int_{x_0}^{x_n} K(t) d_{\gamma_1,\gamma_2} t, \quad x_0 < \xi < x_n,$$

which completes the proof. □

Theorem 2.4.3 shows that non-polynomial B-splines are in fact the Peano kernels of non-polynomial divided differences. In Theorem 2.3.1, non-polynomial B-splines are expressed explicitly in terms of non-polynomial divided differences of truncated power function if $1 \in \text{span}\{\gamma_1, \gamma_2\}$

$$N_{i,n}(x) = d(x_i, x_{i+n+1})[x_i, x_{i+1}, \ldots, x_{i+n+1}]_{\gamma_1,\gamma_2} d(x,t)_+^n.$$

Also, by Lemma 2.2.2, the non-polynomial divided difference $f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_n]$ can be expressed as the symmetric sum of multiples of $f(x_j)$ if $1 \in \text{span}\{\gamma_1, \gamma_2\}$, that is

$$f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] = \sum_{r=0}^{n+1} \frac{f(x_r)}{\prod\limits_{\substack{j=0 \\ j \neq r}}^{n+1} d(x_j, x_r)}.$$

Therefore, we can easily get

$$N_{i,n}(x) = d(x_i, x_{i+n+1}) \sum_{r=i}^{i+n+1} \frac{d(x, x_r)_+^n}{\prod\limits_{\substack{j=i \\ j \neq r}}^{i+n+1} d(x_j, x_r)}. \tag{4.5}$$

Then set

$$L(f) = f_{\gamma_1,\gamma_2}[x_0, x_1, \ldots, x_{n+1}] = \sum_{r=0}^{n+1} \frac{f(x_r)}{\prod\limits_{\substack{j=0 \\ j \neq r}}^{n+1} d(x_j, x_r)}.$$

We know that

$$L(f) = \int_a^b D_{\gamma_1,\gamma_2}^{n+1} f(t) K(t) d_{\gamma_1,\gamma_2} t \tag{4.6}$$

with

$$K(t) = \frac{1}{n!} L(g_t) = \frac{1}{n!} L(d(t,x)_+^n) = \frac{1}{n!} \sum_{r=0}^{n+1} \frac{d(t, x_r)_+^n}{\prod\limits_{\substack{j=0 \\ j \neq r}}^{n+1} d(x_j, x_r)}.$$

Then, by equation (4.5), it is obtained

$$K(t) = \frac{1}{n!}\frac{N_{0,n}(t)}{d(x_0, x_{n+1})}.$$

After combining $K(t)$ with equation (4.6), we get

$$f_{\gamma_1,\gamma_2}[x_0, \ldots, x_{n+1}] = \int_{x_0}^{x_{n+1}} \Big\{\frac{N_{0,n}(t)}{d(x_0, x_{n+1})}\Big\}\Big\{\frac{D^n_{\gamma_1,\gamma_2} f(t)}{n!}\Big\} d_{\gamma_1,\gamma_2} t.$$

We are going to evalute an error bound for a quadrature rule as an application for the general form of Peano kernel theorem. Before the example, we will give the derivation of the quadrature rule.

The most common numerical integration processes are based on interpolation. Let us approximate an integrand $f$ by its interpolating function $g_1$, based on the abscissas $x_0 = a$ and $x_1 = b$ and integrate over $[a, b]$, which gives

$$\int_a^b f(t) d_{\gamma_1,\gamma_2} t \approx \int_a^b g_1(t) d_{\gamma_1,\gamma_2} t.$$

Since

$$\begin{aligned}
\int_a^b g_1(t) d_{\gamma_1,\gamma_2} t =& \int_a^b \left( f(a)\frac{d(b,t)}{d(b,a)} + f(b)\frac{d(a,t)}{d(a,b)} \right) d_{\gamma_1,\gamma_2} t \\
=& \frac{f(a)}{d(b,a)} \int_a^b d(b,t) d_{\gamma_1,\gamma_2} t + \frac{f(b)}{d(a,b)} \int_a^b d(a,t) d_{\gamma_1,\gamma_2} t \\
=& \frac{f(a)}{d(b,a)} \frac{(d(b,t))^2}{2}\Big|_a^b + \frac{f(b)}{d(a,b)} \frac{(d(a,t))^2}{2}\Big|_a^b \\
=& -\frac{f(a)}{d(b,a)} \frac{(d(b,a))^2}{2} + \frac{f(b)}{d(a,b)} \frac{(d(a,b))^2}{2} \\
=& \frac{d(a,b)}{2}(f(a) + f(b)),
\end{aligned}$$

we have

$$\int_a^b f(t) d_{\gamma_1,\gamma_2} t \approx \frac{d(a,b)}{2}(f(a) + f(b)).$$

**Example 2.** *In this example, we will find the error term for the quadrature rule obtained above. We start with*

$$L(f) = \int_a^b f(x) d_{\gamma_1,\gamma_2} x - \frac{d(a,b)}{2}(f(a) + f(b)).$$

*This linear functional* $L$ *annihilates all functions* $f$ *in* $\pi_1$. *If we choose* $g_t(x) = d(t,x)_+$ *in equation (4.3), then*

$$K(t) = L(g_t) = \int_a^b d(t,x)_+ d_{\gamma_1,\gamma_2} x - \frac{d(a,b)}{2}(d(t,a)_+ + d(t,b)_+).$$

*Note that* $L$ *is applied to* $d(t,x)_+$ *regarded as a function of* $x$. *For* $a \leq t \leq b$, *we have* $d(t,a)_+ = 0$ *and* $d(t,b)_+ = d(t,b)$. *Since* $d(t,x)_+ = 0$ *for* $x < t$, *we obtain*

$$\begin{aligned} K(t) = \int_t^b d(t,x) d_{\gamma_1,\gamma_2} x - \frac{d(a,b)}{2} d(t,b) &= \left. \frac{(d(t,x))^2}{2} \right|_t^b - \frac{d(a,b)}{2} d(t,b) \\ &= \frac{1}{2} d(t,b)(d(t,b) + d(b,a)). \end{aligned}$$

*Then,*

$$K(t) = \frac{1}{2} d(t,b) d(t,a).$$

*Since* $\gamma_1$ *and* $\gamma_2$ *are continuous functions, the functions* $d(t,b)$ *and* $d(t,a)$ *are continuous on* $a \leq t \leq b$. *On the other hand,* $d(x_1,x_2)$ *never vanishes for distinct values of* $x_1$ *and* $x_2$. *Hence* $K(t)$ *is continuous on* $a \leq t \leq b$ *and has zeros only at* $t = a$ *and* $t = b$. *Therefore* $K(t)$ *does not change sign on* $[a,b]$, *i.e. we can use Corollary 4.0.1.1, assuming the continuity of* $D^2_{\gamma_1,\gamma_2} f$. *Thus, we evaluate*

$$L(d(x_0,x)^2) = \int_a^b d(x_0,x)^2 d_{\gamma_1,\gamma_2} x - \frac{d(a,b)}{2}(d(x_0,a)^2 + d(x_0,b)^2).$$

*Since $x_0 = a$, we get*

$$\begin{aligned} L(d(x_0,x)^2) &= \int_a^b d(x_0,x)^2 d_{\gamma_1,\gamma_2}x - \frac{d(a,b)}{2}(d(x_0,a)^2 + d(x_0,b)^2) \\ &= \frac{d(a,b)^3}{3} - \frac{d(a,b)^3}{2} \\ &= -\frac{d(a,b)^3}{6}. \end{aligned}$$

*Applying equation (4.4) gives*

$$L(f) = \int_a^b f(x) d_{\gamma_1,\gamma_2}x - \frac{d(a,b)}{2}(f(a) + f(b)) = -\frac{d(a,b)^3}{12} D^2_{\gamma_1,\gamma_2} f(\xi),$$

*where $a < \xi < b$.*

# CHAPTER FIVE
# CONCLUSION

We have dealt with the explicit representation of non-polynomial B-spline functions for a wide collection of spline spaces including trigonometric splines, hyperbolic splines, and special Müntz spaces of splines. Firstly, we have considered the interpolation problem. It has been founded that the problem has a unique solution. Then, we have defined the non-polynomial divided differences recursively. That is, the non-polynomial divided differences satisfy a recurrence relation similar to one for polynomial divided difference, and they are a generalization of classical divided differences. We have derived the identities and the properties of these non-polynomial divided differences such as symmetry, Leibniz formula, cancellation formula, affine combinations, and ratio of determinants. Also, we have searched the non-polynomial divided differences with repeated points. A generalization of the derivative operator has been defined. This generalized derivative operator, as a classical derivative, has satisfied sum, difference, product, and quotient rules. Also, we have searched higher derivatives of some functions as well as the Leibniz rule. With the definition of the generalized derivative operator, it has been shown that as in the polynomial case, non-polynomial divided differences can be viewed as a discrete analogue of derivatives. A generalization of the Hermite interpolation has been also obtained using non-polynomial divided differences with repeated points. We have defined a novel variant of the truncated power function. Then, we have given the explicit representation of non-polynomial B-spline functions using non-polynomial divided differences applied to the generalization of truncated-power function. We have defined a certain definite integral that satisfies the properties of integrals. Then, the non-polynomial divided difference has been built by the integration with non-polynomial B-spline functions.

Repeated points have allowed us to find a generalization of the Taylor series. Using the link between non-polynomial divided differences and derivatives, we have obtained the generalized Taylor series and state Generalized Taylor Theorem.

We have derived a general form of the Peano kernel theorem based on a generalized Taylor expansion. Here, we use the generalized Taylor Theorem with the integral remainder. As in the polynomial case, it has been shown that the non-polynomial B-splines are in fact the Peano kernels of non-polynomial divided difference.

## REFERENCES


Abel, U., & Ivan, M. (2007). The differential mean value of divided differences. *Journal of mathematical analysis and applications*, *325*(1), 560–570.

Abel, U., Ivan, M., & Riedel, T. (2004). The mean value theorem of Flett and divided differences. *Journal of mathematical analysis and applications*, *295*(1), 1–9.

Budakçı, G., & Oruç, H. (2018). A generalization of the Peano kernel and its applications. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, *67*(2), 229–241.

Corominas, E. (1953). Contribution à la théorie de la dérivation d'ordre supérieure. *Bulletin de la Société Mathématique de France*, *81*, 177–222.

Cox, M. G. (1972). The numerical evaluation of B-splines. *IMA Journal of Applied mathematics*, *10*(2), 134–149.

Curry, H., & Schoenberg, I. (1947). On spline distributions and their limits: The Polya distribution functions. *Bulletin of the American Mathematical Society*, *4*, 109.

Davis, P. J. (1975). *Interpolation and approximation*. Courier Corporation.

De Boor, C. (1972). On calculating with B-splines. *Journal of Approximation theory*, *6*(1), 50–62.

De Boor, C. (1978). *A practical guide to splines*, volume 27. Springer-verlag New York.

De Boor, C. (2003). A divided difference expansion of a divided difference. *Journal of Approximation Theory*, *122*(1), 10–12.

De Boor, C. (2005). Divided differences. *arXiv preprint math/0502036*.

De Boor, C., & Höllig, K. (1985). B-splines without divided differences. Technical report, Wisconsin Univ-Madison Mathematics Research Center.

De Morgan, A. (1836). *The differential and integral calculus*. Baldwin and Cradock.

Dişibüyük, Ç. (2015). A functional generalization of the interpolation problem. *Applied Mathematics and Computation*, *256*, 247–251.

Dişibüyük, Ç., & Goldman, R. (2015). A unifying structure for polar forms and for Bernstein Bézier curves. *Journal of Approximation Theory*, *192*, 234–249.

Dişibüyük, Ç., & Goldman, R. (2016). A unified approach to non-polynomial B-spline curves based on a novel variant of the polar form. *Calcolo*, *53*(4), 751–781.

Floater, M., & Lyche, T. (2007). Two chain rules for divided differences and Faa di Bruno's formula. *Mathematics of computation*, *76*(258), 867–877.

Floater, M., & Lyche, T. (2008). Divided differences of inverse functions and partitions of a convex polygon. *Mathematics of computation*, *77*(264), 2295–2308.

Floater, M. S. (2003). Error formulas for divided difference expansions and numerical differentiation. *Journal of Approximation Theory*, *122*(1), 1–9.

Floater, M. S. (2010). A chain rule for multivariate divided differences. *BIT Numerical Mathematics*, *50*(3), 577–586.

Goldman, R. (2002). *Pyramid algorithms: A dynamic programming approach to curves and surfaces for geometric modeling*. Elsevier.

Hardy, G. (1945). Riemann's form of Taylor's series. *Journal of the London Mathematical Society*, *1*(1), 48–57.

Isaacson, E., & Keller, H. B. (1966). Analysis of numerical methods, john willey and sons. *Inc., New York*.

Jacobsthal, E. (1920). Mittelwertbildung und reihentransformation. *Mathematische Zeitschrift*, *6*(1-2), 100–117.

Kac, V. G., & Cheung, P. (2002). *Quantum calculus*, volume 113. Springer.

Karlin, S. (1968). *Total positivity*, volume 1. Stanford University Press.

Karlin, S., Micchelli, C., & Rinott, Y. (1986). Multivariate splines: A probabilistic perspective. *Journal of Multivariate Analysis*, *20*(1), 69–90.

Lee, E. (1989). A remark on divided differences. *The American Mathematical Monthly*, *96*(7), 618–622.

Lyche, T. (1979). A Newton form for trigonometric Hermite interpolation. *BIT Numerical Mathematics*, *19*(2), 229–235.

Lyche, T., & Winther, R. (1979). A stable recurrence relation for trigonometric B-splines. *Journal of Approximation theory*, *25*(3), 266–279.

Micchelli, C. A. (1980). A constructive approach to Kergin interpolation in R k: Multivariate B-splines and Lagrange interpolation. *The Rocky Mountain Journal of Mathematics*, 485–497.

Mühlbach, G. (1973). A recurrence formula for generalized divided differences and some applications. *Journal of Approximation Theory*, *9*(2), 165–172.

Mühlbach, G. (1978). The general Neville-Aitken-algorithm and some applications. *Numerische Mathematik*, *31*(1), 97–110.

Mühlbach, G. (1999). A recurrence relation for generalized divided differences with respect to ect-systems. *Numerical Algorithms*, *22*(3-4), 317–326.

Nörlund, N. E. (1924). *Vorlesungen Über Differenzenrechnung*. Springer Berlin.

Phillips, G. M. (2003). *Interpolation and approximation by polynomials*, volume 14 of *CMS Book in Mathematics/Ouvrages de Math ' ematiques de la SMC*. Springer-Verlag, New York.

Popoviciu, T. (1933). *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*. PhD thesis, Institutul de Arte Grafice" Ardealul.

Popoviciu, T. (1940). Introduction à la théorie des différences divisées. *Bulletin mathématique de la Société Roumaine des Sciences*, *42*(1), 65–78.

Popoviciu, T. (1959). Sur le reste dans certaines formules linéaires d'approximation de l'analyse. *Mathematica (Cluj)*, *1*(24), 95–142.

Powers, R., Riedel, T., & Sahoo, P. (1998). Limit properties of differential mean values. *Journal of mathematical analysis and applications*, *227*(1), 216–226.

Ramshaw, L. (1987). *Blossoming: A connect-the-dots approach to splines*. Digital Equipment Corporation Palo Alto.

Riemann, B. (1876). Versuch einer allgemeinen auffassung der integration und differentiation. *Gesammelte Werke*, *62*(1876).

Sahoo, P., & Riedel, T. (1998). *Mean value theorems and functional equations*. World Scientific.

Schoenberg, I. J. (1946). Contributions to the problem of approximation of equidistant data by analytic functions. part b. on the problem of osculatory interpolation. a second class of analytic approximation formulae. *Quarterly of Applied Mathematics*, *4*(2), 112–141.

Schumaker, L. L. (1976). On Tchebycheffian spline functions. *Journal of Approximation Theory*, *18*(3), 278–303.

Schumaker, L. L. (1983). On hyperbolic splines. *Journal of Approximation Theory*, *38*(2), 144–166.

Stenffensen, J. (1939). Note on divided differences. *Danske Vid. Selsk. Math.-Fys. Medd*, *17*, 1–12.

Tchakaloff, L. (1934). Sur la structure des ensembles linéaires définis par une certaine propriété minimale. *Acta Mathematica*, *63*, 77–97.

Trujillo, J., Rivero, M., & Bonilla, B. (1999). On a Riemann–Liouville generalized Taylor's formula. *Journal of Mathematical Analysis and Applications*, *231*(1), 255–265.

Walz, G. (1992). Generalized divided differences, with applications to generalized B-splines. *Calcolo*, *29*(1-2), 111–123.

Wang, X.-h., & Wang, H.-y. (2006). On the divided difference form of Faà di Bruno's formula. *Journal of Computational Mathematics*, 553–560.

Watanabe, Y. (1931). Notes on the generalized derivatives of Riemann-Liouville and its applications to Leibniz formula i and ii. *Tohoku Math. J*, *34*, 8–14.

Whittaker, E., & Robinson, G. (1946). *The Calculus of Observations: A Treatise on Numerical Mathematics*. Blackie & Sons.

Xu, A., Cui, F., & Hu, Z. (2010). Asymptotic behavior of intermediate points in the differential mean value theorem of divided differences with repetitions. *Journal of mathematical analysis and applications*, *365*(1), 358–362.

Zürnacı, F., & Dişibüyük, Ç. (2019). Non-polynomial divided differences and B-spline functions. *Journal of Computational and Applied Mathematics*, *349*, 579–592.