q-HYPERGEOMETRIC SERIES

by

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ABSTRACT

q-HYPERGEOMETRIC SERIES

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Master, Department of Mathematics

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This thesis is a survey for some properties of q-hypergeometric series which has many applications in number theory, physics, statistics and computer science.

The first chapter deals with the basic definitions for q-calculus and necessary lemmas and theorems that will be used later.

In the second chapter, some summation and transformation formulas for q-hypergeometric series were introduced.

Keywords: q-calculus, q-binomial theorem, q-hypergeometric series.

ÖZET

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Bu tezde sayılar teorisi, fizik, istatistik ve bilgisayar alanlarında birçok uygulamaya sahip olan q-hipergeometrik serilerin bazı özellikleri verilmiştir.

Birinci bölüm, tez için gerekli olan temel tanımlar ve ikinci bölümde kullanacağımız bazı temel teoremler içermektedir.

İkinci bölüm ise bazı toplama ve dönüşüm formülleriyle birlikte iki q-hipergeometrik seri arasındaki bağıntılar sunulmuştur.

Anahtar Kelimeler : q-analiz, q-binom teoremi, q-hipergeometrik seri.

To My Family

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TABLE OF CONTENTS

ABSTRACT iii

ÖZET iv

ACKNOWLEDGEMENTS vi

1 PRELIMINARIES 1
- 1.1 Useful identities . . . 3
- 1.2 The q-binomial theorem . . . 15
- 1.3 The triple product identity . . . 22
- 1.4 The Carlitz inversions . . . 24

2 Q-HYPERGEOMETRIC SERIES 27
- 2.1 Heine's transformation formula . . . 35
- 2.2 Iteration of Heine's transformation formula . . . 37
- 2.3 Euler's transformation formula . . . 38
- 2.4 Gauss summation formula . . . 39
- 2.5 Chu-Vandermonde's formula . . . 42
- 2.6 Jackson's transformation formula . . . 44
- 2.7 The Bailey-Daum summation formula . . . 46
- 2.8 The q-Pfaff-Saalschütz summation formula . . . 49
- 2.9 The terminating q-Dougall-Dixon formula . . . 51
- 2.10 The Sears balanced transformation formulas . . . 54
- 2.11 The Watson transformation formula . . . 62
- 2.12 Jackson's q-Dougall-Dixon formula . . . 66

REFERENCES 67

CHAPTER 1

PRELIMINARIES

The main purpose of this chapter is to present some basic definitions and notations for q-calculus [1-8]. We also derive q-binomial theorem and triple product identity [4, 9-11]. Finally, we give Carlitz inversion relations [11].

Let us introduce some basic definitions and notations.

Definition 1.1 *Let q be a fixed nonzero real or complex number with $|q| < 1$. The q-shifted factorial $(a;q)_n$ is defined for any complex parameter a by*

$$(a;q)_n = \begin{cases} 1 & ; n = 0 \\ \prod_{j=0}^{n-1}(1-aq^j) & ; n \in \mathbb{N} := \{1, 2, ...\}. \end{cases}$$

for negative subscripts,

$$(a;q)_{-n} = \frac{1}{(1-aq^{-1})(1-aq^{-2})...(1-aq^{-n})} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}.$$

Furthermore,

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

Throughout the thesis we assume that q is a real or complex number satisfying $|q| < 1$.

Definition 1.2 *The q-analogue of a complex number a is denoted by*

$$[a]_q = \frac{1-q^a}{1-q}.$$

Definition 1.3 *Let n be a natural number. The q-factorial function is given by*

$$[n]_q! = \begin{cases} 1 & ; n = 0 \\ \dfrac{(q;q)_n}{(1-q)^n} & ; n \geq 1. \end{cases}$$

Definition 1.4 *Given a natural number n and a non-negative integer k, the q-binomial coefficient is defined as follows:*

$$\begin{aligned} \binom{n}{k}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!} \\ &= \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \qquad (n \geq k) \end{aligned}$$

.

Definition 1.5 *The q-difference operator D_q of a differentiable function $f(x)$ at x is given by*

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Corollary 1.6

$$\lim_{q \to 1} [a]_q = a$$

$$\lim_{q \to 1} [n]_q! = n!$$

$$\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k}$$

and

$$\lim_{q \to 1} D_q f(x) = \frac{df}{dx}$$

where $n!$, $\binom{n}{k}$ *and* $\dfrac{df}{dx}$ *classical factorial, binomial coefficients and ordinary derivative, respectively.*

1.1 Useful identities

In this section, we shall give some lemmas [4, 9] which are used throughout this thesis.

Lemma 1.7 *Let n be a natural number. Then the identities are valid:*

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.1}$$

$$(a;q)_n = (a^{-1}q^{1-n};q)_n(-a)^n q^{\binom{n}{2}}. \tag{1.2}$$

$$(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-\binom{n}{2}}. \tag{1.3}$$

$$(aq^{-n};q)_n = (a^{-1}q;q)_n \left(\frac{-a}{q}\right)^n q^{-\binom{n}{2}}. \tag{1.4}$$

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(a^{-1}q;q)_n}{(b^{-1}q;q)_n}\left(\frac{a}{b}\right)^n. \tag{1.5}$$

$$(a^{-1}q^{1-n};q)_n = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}. \tag{1.6}$$

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n. \tag{1.7}$$

$$(a^2;q^2)_n = (a;q)_n(-a;q)_n. \tag{1.8}$$

$$\frac{(\sqrt{a}q;q)_n(-\sqrt{a}q;q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n} = \frac{1-aq^{2n}}{1-a}. \tag{1.9}$$

Lemma 1.8 *Let n and k be natural numbers. Then the following identities hold:*

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-qa^{-1})^k q^{\binom{k}{2}-nk}. \tag{1.10}$$

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \frac{(a;q)_n(b^{-1}q^{1-n};q)_k}{(b;q)_n(a^{-1}q^{1-n};q)_k}\left(\frac{b}{a}\right)^k. \tag{1.11}$$

$$(aq^{-n};q)_{n-k} = \frac{(a^{-1}q;q)_n}{(a^{-1}q;q)_k}\left(\frac{-a}{q}\right)^{n-k} q^{\binom{k}{2}-\binom{n}{2}}. \tag{1.12}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}. \tag{1.13}$$

$$(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}. \tag{1.14}$$

Lemma 1.9 *Let n and k be natural numbers. Then the following identities are valid:*

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{1.15}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k(a^{-1}q;q)_n}{(a^{-1}q^{1-k};q)_n}q^{-nk}. \tag{1.16}$$

$$(aq^{-kn};q)_n = \frac{(a^{-1}q;q)_{kn}}{(a^{-1}q;q)_{(k-1)n}}(-a)^n q^{\binom{n}{2}-kn^2}. \tag{1.17}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k. \tag{1.18}$$

$$(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}. \tag{1.19}$$

$$(aq^{kn};q)_n = \frac{(a;q)_{(k+1)n}}{(a;q)_{kn}}. \tag{1.20}$$

$$(aq^{-k-n};q)_n = \frac{(a^{-1}q;q)_{n+k}}{(a^{-1}q;q)_k}\left(\frac{-a}{q}\right)^n q^{-nk-\binom{n}{2}}. \tag{1.21}$$

Proof. We prove (1.1) – (1.21) by means of the q-shifted factorial.

Proof of (1.1):

$$\begin{aligned}(a;q)_n &= (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})\\ &= \frac{(1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})\,[(1-aq^n)(1-aq^{n+1})\ldots]}{[(1-aq^n)(1-aq^{n+1})\ldots]}\\ &= \frac{(a;q)_\infty}{(aq^n;q)_\infty}.\end{aligned}$$

Proof of (1.2):

$$\begin{aligned}(a;q)_n &= (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})\\ &= (1-a^{-1})(1-a^{-1}q^{-1})(1-a^{-1}q^{-2})\ldots(1-a^{-1}q^{1-n})(-a)^n q^{[1+2+\ldots+(n-1)]}.\end{aligned}$$

By using the fact that

$$1+2+3+...+(n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$$

we can write

$$\begin{aligned}(a;q)_n &= (1-a^{-1}q^{1-n})\ldots(1-a^{-1}q^{-2})(1-a^{-1}q^{-1})(1-a^{-1})(-a)^n q^{\binom{n}{2}}\\ &= (a^{-1}q^{1-n};q)_n(-a)^n q^{\binom{n}{2}}.\end{aligned}$$

Proof of (1.3):

$$\begin{aligned}(a;q^{-1})_n &= (1-a)(1-aq^{-1})(1-aq^{-2})\ldots(1-aq^{-n+1})\\ &= (1-a^{-1})(1-a^{-1}q)(1-a^{-1}q^2)\ldots(1-a^{-1}q^{n-1})(-a)^n q^{-[1+2+\ldots+(n-1)]}\\ &= (a^{-1};q)_n(-a)^n q^{-\binom{n}{2}}.\end{aligned}$$

Proof of (1.4):

$$(aq^{-n};q)_n = (1-aq^{-n})(1-aq^{-n+1})\ldots(1-aq^{-2})(1-aq^{-1})$$

$$
\begin{aligned}
(aq^{-n};q)_n &= (1-a^{-1}q^n)(1-a^{-1}q^{n-1})\dots(1-a^{-1}q^2)(1-a^{-1}q)(-a)^n q^{-[n+(n-1)+\dots+1]} \\
&= (1-a^{-1}q)(1-a^{-1}q^2)\dots(1-a^{-1}q^{n-1})(1-a^{-1}q^n)(-a)^n q^{-n-\binom{n}{2}} \\
&= (a^{-1}q;q)_n\left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}.
\end{aligned}
$$

Proof of (1.5):

By equation (1.4)

$$
\begin{aligned}
\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} &= \frac{(a^{-1}q;q)_n\left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}}{(b^{-1}q;q)_n\left(-\frac{b}{q}\right)^n q^{-\binom{n}{2}}} \\
&= \frac{(a^{-1}q;q)_n}{(b^{-1}q;q)_n}\left(\frac{a}{b}\right)^n.
\end{aligned}
$$

Proof of (1.6):

$$
\begin{aligned}
(a^{-1}q^{1-n};q)_n &= (1-a^{-1}q^{1-n})(1-a^{-1}q^{2-n})\dots(1-a^{-1}q^{-1})(1-a^{-1}) \\
&= (1-aq^{n-1})(1-aq^{n-2})\dots(1-aq)(1-a)(-a^{-1})^n q^{-[(n-1)+(n-2)+\dots+1]} \\
&= (1-a)(1-aq)\dots(1-aq^{n-2})(1-aq^{n-1})(-a^{-1})^n q^{-\binom{n}{2}} \\
&= (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}.
\end{aligned}
$$

Proof of (1.7):

$$
\begin{aligned}
(a;q)_{2n} &= (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots(1-aq^{2n-2})(1-aq^{2n-1}) \\
&= [(1-a)(1-aq^2)\dots(1-aq^{2n-2})][(1-aq)(1-aq^3)\dots(1-aq^{2n-1})] \\
&= (a;q^2)_n(aq;q^2)_n.
\end{aligned}
$$

Proof of (1.8):

$$
\begin{aligned}
(a^2;q^2)_n &= (1-a^2)(1-a^2q^2)\dots(1-a^2q^{2n-2}) \\
&= (1-a)(1+a)(1-aq)(1+aq)\dots(1-aq^{n-1})(1+aq^{n-1})
\end{aligned}
$$

$$(a^2;q^2)_n = [(1-a)(1-aq)\dots(1-aq^{n-1})][(1+a)(1+aq)\dots(1+aq^{n+1})]$$
$$= (a;q)_n(-a;q)_n.$$

Proof of (1.9):

$$\begin{aligned}\frac{(\sqrt{a}q;q)_n(-\sqrt{a}q;q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n} &= \frac{[(1-\sqrt{a}q)(1-\sqrt{a}q^2)\dots(1-\sqrt{a}q^{n-1})(1-\sqrt{a}q^n)]}{[(1-\sqrt{a})(1-\sqrt{a}q)(1-\sqrt{a}q^2)\dots(1-\sqrt{a}q^{n-1})]}\\ &\times \frac{[(1+\sqrt{a}q)(1+\sqrt{a}q^2)\dots(1+\sqrt{a}q^{n-1})(1+\sqrt{a}q^n)]}{[(1+\sqrt{a})(1+\sqrt{a}q)(1+\sqrt{a}q^2)\dots(1+\sqrt{a}q^{n-1})]}\\ &= \frac{(1-aq^2)(1-aq^4)\dots(1-aq^{2n-2})(1-aq^{2n})}{(1-a)(1-aq^2)(1-aq^4)\dots(1-aq^{2n-2})}\\ &= \frac{1-aq^{2n}}{1-a}.\end{aligned}$$

Proof of (1.10):

Case 1. $n > k$

$$\begin{aligned}(a;q)_{n-k} &= (1-a)(1-aq)\dots(1-aq^{n-k-1})\\ &= \frac{(1-a)(1-aq)\dots(1-aq^{n-k-1})\left[(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{n-1})\right]}{[(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{n-1})]}\\ &= \frac{(a;q)_n}{(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{n-1})}\\ &= \frac{(a;q)_n}{(1-a^{-1}q^{k-n})(1-a^{-1}q^{k-n-1})\dots(1-a^{-1}q^{1-n})(-a)^k q^{[(n-k)+(n-k+1)+\dots(n-1)]}}\end{aligned}$$

Since

$$\begin{aligned}(n-k)+(n-k+1)+\dots+(n-1) &= nk-[k+(k-1)+\dots+1]\\ &= nk-k-\binom{k}{2},\end{aligned}$$

we have

$$\begin{aligned}(a;q)_{n-k} &= \frac{(a;q)_n}{(1-a^{-1}q^{1-n})\dots(1-a^{-1}q^{k-n-1})(1-a^{-1}q^{k-n})(-a)^k q^{nk-k-\binom{k}{2}}}\\ &= \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-qa^{-1})^k q^{\binom{k}{2}-nk}.\end{aligned}$$

Case 2. $n < k$

$$
\begin{aligned}
(a;q)_{n-k} = (a;q)_{-(k-n)} &= \frac{1}{(aq^{n-k};q)_{k-n}} \\
&= \frac{1}{(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{-1})} \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{n-1})]}{(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{-1})\,[(1-a)(1-aq)\dots(1-aq^{n-1})]} \\
&= \frac{(a;q)_n}{(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{n-1})} \\
&= \frac{(a;q)_n}{(1-a^{-1}q^{k-n})(1-a^{-1}q^{k-n-1})\dots(1-a^{-1}q^{1-n})(-a)^k q^{[(n-k)+(n-k+1)+\dots(n-1)]}} \\
&= \frac{(a;q)_n}{(1-a^{-1}q^{1-n})\dots(1-a^{-1}q^{k-n-1})(1-a^{-1}q^{k-n})(-a)^k q^{nk-k-\binom{k}{2}}} \\
&= \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-qa^{-1})^k q^{\binom{k}{2}-nk}.
\end{aligned}
$$

Proof of (1.11):

By using (1.10), we get

$$
\begin{aligned}
\frac{(a;q)_{n-k}}{(b;q)_{n-k}} &= \frac{\dfrac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-qa^{-1})^k q^{\binom{k}{2}-nk}}{\dfrac{(b;q)_n}{(b^{-1}q^{1-n};q)_k}(-qb^{-1})^k q^{\binom{k}{2}-nk}} \\
&= \frac{(a;q)_n}{(b;q)_n}\frac{(b^{-1}q^{1-n};q)_k}{(a^{-1}q^{1-n};q)_k}\left(\frac{b}{a}\right)^k.
\end{aligned}
$$

Proof of (1.12):

Case 1. $n > k$

$$
\begin{aligned}
(aq^{-n};q)_{n-k} &= (1-aq^{-n})(1-aq^{-n+1})\dots(1-aq^{-k-1}) \\
&= \frac{(1-aq^{-n})\dots(1-aq^{-k-1})\left[(1-aq^{-k})(1-aq^{-k+1})\dots(1-aq^{-1})\right]}{[(1-aq^{-k})(1-aq^{-k+1})\dots(1-aq^{-1})]} \\
&= \frac{(1-a^{-1}q^{n})(1-a^{-1}q^{n-1})\dots(1-a^{-1}q)(-a)^n q^{-[n+(n-1)+\dots+1]}}{(1-a^{-1}q^{k})(1-a^{-1}q^{k-1})\dots(1-a^{-1}q)(-a)^k q^{-[k+(k-1)+\dots+1]}} \\
&= \frac{(1-a^{-1}q)\dots(1-a^{-1}q^{n-1})(1-a^{-1}q^{n})(-a)^n q^{-n-\binom{n}{2}}}{(1-a^{-1}q)\dots(1-a^{-1}q^{k-1})(1-a^{-1}q^{k})(-a)^k q^{-k-\binom{k}{2}}} \\
&= \frac{(a^{-1}q;q)_n}{(a^{-1}q;q)_k}\left(\frac{-a}{q}\right)^{n-k} q^{\binom{k}{2}-\binom{n}{2}}.
\end{aligned}
$$

Case 2. $n < k$

$$
\begin{aligned}
(aq^{-n};q)_{n-k} &= (aq^{-n};q)_{-(k-n)} = \frac{1}{(aq^{-k};q)_{k-n}} \\
&= \frac{1}{(1-aq^{-k})(1-aq^{-k+1})\dots(1-aq^{-n-1})} \\
&= \frac{1}{(1-a^{-1}q^{k})(1-a^{-1}q^{k-1})\dots(1-a^{-1}q^{n+1})(-a)^{k-n}q^{-[k+(k-1)+\dots+(n+1)]}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&k+(k-1)+\dots+(n+1) \\
&= k+(k-1)\dots+(n+1)+[n+(n-1)+\dots+1]-[n+(n-1)+\dots+1] \\
&= k+[(k-1)+(k-2)+\dots+1]-n-[(n-1)+(n-2)+\dots+1] \\
&= k-n+\binom{k}{2}-\binom{n}{2},
\end{aligned}
$$

we get

$$
\begin{aligned}
(aq^{-n};q)_{n-k} &= \frac{(-a)^{n-k}q^{k-n+\binom{k}{2}-\binom{n}{2}}}{(1-a^{-1}q^{n+1})\dots(1-a^{-1}q^{k-1})(1-a^{-1}q^{k})} \\
&= \frac{[(1-a^{-1}q)(1-a^{-1}q^{2})\dots(1-a^{-1}q^{n})](-a)^{n-k}q^{k-n+\binom{k}{2}-\binom{n}{2}}}{[(1-a^{-1}q)(1-a^{-1}q^{2})\dots(1-a^{-1}q^{n})](1-a^{-1}q^{n+1})\dots(1-a^{-1}q^{k})} \\
&= \frac{(a^{-1}q;q)_n}{(a^{-1}q;q)_k}\left(\frac{-a}{q}\right)^{n-k}q^{\binom{k}{2}-\binom{n}{2}}.
\end{aligned}
$$

Proof of (1.13):

Case 1. $n > k$

$$
\begin{aligned}
(aq^{k};q)_{n-k} &= (1-aq^{k})(1-aq^{k+1})\dots(1-aq^{n-1}) \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{k-1})](1-aq^{k})(1-aq^{k+1})\dots(1-aq^{n-1})}{[(1-a)(1-aq)\dots(1-aq^{k-1})]} \\
&= \frac{(a;q)_n}{(a;q)_k}.
\end{aligned}
$$

Case 2. $n < k$

$$
\begin{aligned}
(aq^k;q)_{-(k-n)} &= \frac{1}{(aq^n;q)_{k-n}} = \frac{1}{(1-aq^n)(1-aq^{n+1})\dots(1-aq^{k-1})} \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{n-1})]}{[(1-a)(1-aq)\dots(1-aq^{n-1})](1-aq^n)(1-aq^{n+1})\dots(1-aq^{k-1})} \\
&= \frac{(a;q)_n}{(a;q)_k}.
\end{aligned}
$$

Proof of (1.14):

Case 1. $n > k$

$$
\begin{aligned}
(aq^{2k};q)_{n-k} &= (1-aq^{2k})(1-aq^{2k+1})\dots(1-aq^{n+k-1}) \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{2k-1})](1-aq^{2k})(1-aq^{2k+1})\dots(1-aq^{n+k-1})}{[(1-a)(1-aq)\dots(1-aq^{2k-1})]} \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{n-1})][(1-aq^n)(1-aq^{n+1})\dots(1-aq^{n+k-1})]}{(1-a)(1-aq)\dots(1-aq^{2k-1})} \\
&= \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}.
\end{aligned}
$$

Case 2. $n < k$

$$
\begin{aligned}
(aq^{2k};q)_{n-k} &= (aq^{2k};q)_{-(k-n)} = \frac{1}{(aq^{n+k};q)_{k-n}} \\
&= \frac{1}{(1-aq^{n+k})(1-aq^{n+k+1})\dots(1-aq^{2k-1})} \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{n+k-1})]}{[(1-a)(1-aq)\dots(1-aq^{n+k-1})](1-aq^{n+k})(1-aq^{n+k+1})\dots(1-aq^{2k-1})} \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{n-1})](1-aq^n)\dots(1-aq^{n+k-1})}{[(1-a)(1-aq)\dots(1-aq^{2k-1})} \\
&= \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}.
\end{aligned}
$$

Proof of (1.15):

$$
\begin{aligned}
(q^{-n};q)_k &= (1-q^{-n})(1-q^{-n+1})\dots(1-q^{-n+k-1}) \\
&= (1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})(-1)^k q^{-[n+(n-1)+\dots+(n-k+1)]} \\
&= (1-q^{n-k+1})\dots(1-q^{n-1})(1-q^n)(-1)^k q^{\binom{k}{2}-nk}
\end{aligned}
$$

$$
\begin{aligned}
(q^{-n};q)_k &= \frac{[(1-q)(1-q^2)\dots(1-q^{n-k})](1-q^{n-k+1})\dots(1-q^n)}{[(1-q)(1-q^2)\dots(1-q^{n-k})]}(-1)^k q^{\binom{k}{2}-nk}\\
&= \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}.
\end{aligned}
$$

Proof of (1.16):

$$
\begin{aligned}
(aq^{-n};q)_k &= (1-aq^{-n})(1-aq^{-n+1})\dots(1-aq^{-n+k-1})\\
&= (1-a^{-1}q^n)(1-a^{-1}q^{n-1})\dots(1-a^{-1}q^{n-k+1})(-a)^k q^{-[n+(n-1)+\dots+(n-k+1)]}\\
&= (1-a^{-1}q^n)(1-a^{-1}q^{n-1})\dots(1-a^{-1}q^{n-k+1})(-a)^k q^{-nk+\binom{k}{2}}\\
&= (1-a^{-1}q^n)(1-a^{-1}q^{n-1})\dots(1-a^{-1}q^{n-k+1})\\
&\times \frac{\left[(1-a^{-1}q^{n-k})(1-a^{-1}q^{n-k-1})\dots(1-a^{-1}q)\right]}{[(1-a^{-1}q^{n-k})(1-a^{-1}q^{n-k-1})\dots(1-a^{-1}q)]}(-aq^{-n})^k q^{\binom{k}{2}}\\
&= \frac{(a^{-1}q;q)_n(-aq^{-n})^k q^{\binom{k}{2}}}{(1-a^{-1}q^{n-k})(1-a^{-1}q^{n-k-1})\dots(1-a^{-1}q)}\\
&\times \frac{\left[(1-a^{-1})(1-a^{-1}q^{-1})\dots(1-a^{-1}q^{1-k})\right]}{[(1-a^{-1})(1-a^{-1}q^{-1})\dots(1-a^{-1}q^{1-k})]}\\
&= \frac{(a^{-1}q;q)_n(-aq^{-n})^k q^{\binom{k}{2}}(1-a^{-1})(1-a^{-1}q^{-1})\dots(1-a^{-1}q^{1-k})}{(a^{-1}q^{1-k};q)_n}\\
&= \frac{(a^{-1}q;q)_n q^{-nk}(1-a)(1-aq)\dots(1-aq^{k-1})}{(a^{-1}q^{1-k};q)_n}\\
&= \frac{(a;q)_k(a^{-1}q;q)_n}{(a^{-1}q^{1-k};q)_n}q^{-nk}.
\end{aligned}
$$

Proof of (1.17):

$$
\begin{aligned}
(aq^{-kn};q)_n &= (1-aq^{-kn})(1-aq^{-kn+1})\dots(1-aq^{-kn+n-1})\\
&= (1-a^{-1}q^{kn})(1-a^{-1}q^{kn-1})\dots(1-a^{-1}q^{kn-n+1})(-a)^n\\
&\times q^{-[kn+(kn-1)+\dots+(kn-n+1)]}
\end{aligned}
$$

Since

$$
\begin{aligned}
kn+(kn-1)+\dots+(kn-n+1) &= (kn)n-[1+2+\dots+(n-1)]\\
&= kn^2-\binom{n}{2},
\end{aligned}
$$

we get

$$
\begin{aligned}
(aq^{-kn};q)_n &= (-a)^n q^{\binom{n}{2}-kn^2}(1-a^{-1}q^{kn})(1-a^{-1}q^{kn-1})\dots(1-a^{-1}q^{kn-n+1}) \\
&\quad\times \frac{[(1-a^{-1}q^{kn-n})(1-a^{-1}q^{kn-n-1})\dots(1-a^{-1}q)]}{[(1-a^{-1}q^{kn-n})(1-a^{-1}q^{kn-n-1})\dots(1-a^{-1}q)]} \\
&= \frac{(a^{-1}q;q)_{kn}}{(a^{-1}q;q)_{(k-1)n}}(-a)^n q^{\binom{n}{2}-kn^2}.
\end{aligned}
$$

Proof of (1.18):

$$
\begin{aligned}
(a;q)_{n+k} &= (1-a)(1-aq)\dots(1-aq^{n-1})(1-aq^n)\dots(1-aq^{n+k-1}) \\
&= (a;q)_n(aq^n;q)_k.
\end{aligned}
$$

Proof of (1.19):

$$
\begin{aligned}
(aq^n;q)_k &= (1-aq^n)(1-aq^{n+1})\dots(1-aq^{n+k-1}) \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{n-1})](1-aq^n)(1-aq^{n+1})\dots(1-aq^{n+k-1})}{[(1-a)(1-aq)\dots(1-aq^{n-1})]} \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{k-1})][(1-aq^k)(1-aq^{k+1})\dots(1-aq^{n+k-1})]}{[(1-a)(1-aq)\dots(1-aq^{n-1})]} \\
&= \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}.
\end{aligned}
$$

Proof of (1.20):

$$
\begin{aligned}
(aq^{kn};q)_n &= (1-aq^{kn})(1-aq^{kn+1})\dots(1-aq^{kn+n-1}) \\
&= \frac{[(1-a)(1-aq)\dots(1-aq^{kn-1})](1-aq^{kn})\dots(1-aq^{kn+n-1})}{[(1-a)(1-aq)\dots(1-aq^{kn-1})]} \\
&= \frac{(a;q)_{(k+1)n}}{(a;q)_{kn}}.
\end{aligned}
$$

Proof of (1.21):

$$
\begin{aligned}
(aq^{-k-n};q)_n &= (1-aq^{-k-n})(1-aq^{-k-n+1})\dots(1-aq^{-k-1}) \\
&= (-a)^n q^{-[(k+n)+\dots+(k+1)]}(1-a^{-1}q^{k+n})(1-a^{-1}q^{k+n-1})\dots(1-a^{-1}q^{k+1})
\end{aligned}
$$

$$
\begin{aligned}
(aq^{-k-n};q)_n &= (-a)^n q^{-nk-n-\binom{n}{2}}(1-a^{-1}q^{k+n})(1-a^{-1}q^{k+n-1})\dots(1-a^{-1}q^{k+1}) \\
&\quad\times \frac{[(1-a^{-1}q^k)(1-a^{-1}q^{k-1})\dots(1-a^{-1}q)]}{[(1-a^{-1}q^k)(1-a^{-1}q^{k-1})\dots(1-a^{-1}q)]} \\
&= \frac{(a^{-1}q;q)_{n+k}}{(a^{-1}q;q)_k}\left(\frac{-a}{q}\right)^n q^{-nk-\binom{n}{2}}.
\end{aligned}
$$

□

Lemma 1.10 *For* $\quad n \geq k \geq i$ *and* $q^{-1} = \dfrac{1}{q}$,

$$
\binom{n+1}{k}_q = \binom{n}{k}_q q^k + \binom{n}{k-1}_q. \tag{1.22}
$$

$$
\binom{n+1}{k}_q = \binom{n}{k}_q + q^{n-k+1}\binom{n}{k-1}_q. \tag{1.23}
$$

$$
\binom{n}{k}_{q^{-1}} = \binom{n}{k}_q q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}}. \tag{1.24}
$$

$$
\binom{n}{k}_q \times \binom{k}{i}_q = \binom{n}{i}_q \times \binom{n-i}{k-i}_q. \tag{1.25}
$$

Proof. We prove (1.22) - (1.25) with the help of the q-binomial coefficients.

Proof of (1.22):

$$
\begin{aligned}
\binom{n+1}{k}_q &= \frac{(q;q)_{n+1}}{(q;q)_k(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n(1-q^{n+1})}{(q;q)_k(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k+1}}[(1-q^{n-k+1})q^k + (1-q^k)] \\
&= \frac{(q;q)_n(1-q^{n-k+1})}{(q;q)_k(q;q)_{n-k}(1-q^{n-k+1})}q^k + \frac{(q;q)_n(1-q^k)}{(q;q)_{k-1}(1-q^k)(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}q^k + \frac{(q;q)_n}{(q;q)_{k-1}(q;q)_{n-k+1}} \\
&= \binom{n}{k}_q q^k + \binom{n}{k-1}_q.
\end{aligned}
$$

Proof of (1.23):

$$\begin{aligned}
\binom{n+1}{k}_q &= \frac{(q;q)_{n+1}}{(q;q)_k(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n(1-q^{n+1})}{(q;q)_k(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k+1}}[(1-q^{n-k+1}) + q^{n-k+1}(1-q^k)] \\
&= \frac{(q;q)_n(1-q^{n-k+1})}{(q;q)_k(q;q)_{n-k}(1-q^{n-k+1})} + q^{n-k+1}\frac{(q;q)_n(1-q^k)}{(q;q)_{k-1}(1-q^k)(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} + q^{n-k+1}\frac{(q;q)_n}{(q;q)_{k-1}(q;q)_{n-k+1}} \\
&= \binom{n}{k}_q + q^{n-k+1}\binom{n}{k-1}_q.
\end{aligned}$$

Proof of (1.24):

$$\binom{n}{k}_{q^{-1}} = \frac{(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_k(q^{-1};q^{-1})_{n-k}}.$$

In (1.3), replacing a by q^{-1}, we find

$$(q^{-1};q^{-1})_n = (-1)^n(q;q)_n q^{-n-\binom{n}{2}}$$

and so we can write

$$\begin{aligned}
\binom{n}{k}_{q^{-1}} &= \frac{(-1)^n(q;q)_n q^{-n-\binom{n}{2}}}{(-1)^k(q;q)_k q^{-k-\binom{k}{2}}(-1)^{n-k}(q;q)_{n-k}q^{-(n-k)-\binom{n-k}{2}}} \\
&= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}} \\
&= \binom{n}{k}_q q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}}.
\end{aligned}$$

Proof of (1.25):

$$\begin{aligned}
\binom{n}{k}_q \times \binom{k}{i}_q &= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \times \frac{(q;q)_k}{(q;q)_i(q;q)_{k-i}} \\
&= \frac{(q;q)_n}{(q;q)_{n-k}} \times \frac{1}{(q;q)_i(q;q)_{k-i}}\left[\frac{(q;q)_{n-i}}{(q;q)_{n-i}}\right]
\end{aligned}$$

$$\begin{aligned}\binom{n}{k}_q \times \binom{k}{i}_q &= \frac{(q;q)_n}{(q;q)_i(q;q)_{n-i}} \times \frac{(q;q)_{n-i}}{(q;q)_{k-i}(q;q)_{n-k}} \\ &= \binom{n}{i}_q \times \binom{n-i}{k-i}_q .\end{aligned}$$

□

1.2 The q-binomial theorem

The classical Binomial theorem [10] reads as follows:

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} z^n = \frac{1}{(1-z)^c} \quad ; \quad (|z| < 1)$$

where $(c)_n = c(c+1)(c+2)\cdots(c+n-1)$. Now, we give its q-analogue by the following theorem.

Theorem 1.11 *Let $|z| < 1$. Then we have*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.26}$$

Proof. Let

$$h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \ |z| < 1.$$

Applying the q-difference operator D_q to the both sides of this equation, we obtain

$$D_q h_a(z) = \frac{h_a(z) - h_a(qz)}{(1-q)z} = \frac{1}{(1-q)z}\left[\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n - \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(qz)^n\right]$$

and so

$$\begin{aligned}h_a(z) - h_a(qz) &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n - \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(qz)^n \\ &= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n}(1-q^n) z^n \\ &= \sum_{n=1}^{\infty} \frac{(a;q)_n(1-q^n)}{(q;q)_{n-1}(1-q^n)} z^n.\end{aligned}$$

$$h_a(z) - h_a(qz) = \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n.$$

Replacing n by $n+1$, we have

$$h_a(z) - h_a(qz) = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1}.$$

Since

$$(a;q)_{n+1} = (1-a)(aq;q)_n,$$

one gets

$$\begin{aligned} h_a(z) - h_a(qz) &= (1-a)z \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n} z^n \\ &= (1-a)z h_{aq}(z). \end{aligned} \tag{1.27}$$

Now consider the difference

$$\begin{aligned} h_a(z) - h_{aq}(z) &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n - \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n \\ &= \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n. \end{aligned}$$

By using the relation

$$(a;q)_n - (aq;q)_n = (aq;q)_{n-1}(aq^n - a)$$

we can write

$$\begin{aligned} h_a(z) - h_{aq}(z) &= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} (aq^n - a) z^n \\ &= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}(1-q^n)}{(q;q)_{n-1}(1-q^n)} z^n. \end{aligned}$$

$$h_a(z) - h_{aq}(z) = -az \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^{n-1}.$$

Replacing n by $n+1$, we have

$$\begin{aligned} h_a(z) - h_{aq}(z) &= -az \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n} z^n \\ &= -az h_{aq}(z). \end{aligned} \tag{1.28}$$

Thus from equations (1.27) and (1.28), we obtain

$$h_{aq}(z) = \frac{h_a(z)}{(1-az)}$$

and

$$\begin{aligned} h_a(z) - h_a(qz) &= (1-a)z \frac{h_a(z)}{(1-az)} \\ h_a(z)\left[1 - \frac{(1-a)z}{(1-az)}\right] &= h_a(qz) \\ h_a(z) &= \frac{(1-az)}{(1-z)} h_a(qz). \end{aligned} \tag{1.29}$$

In (1.29), if we write qz in place of z, we find

$$h_a(qz) = \frac{(1-aqz)}{(1-qz)} h_a(q^2 z).$$

Substitution of this result into (1.29) gives

$$h_a(z) = \frac{(1-az)(1-aqz)}{(1-z)(1-qz)} h_a(q^2 z).$$

Applying this process $n-2$ times, we arrive at

$$\begin{aligned} h_a(z) &= \frac{(1-az)(1-aqz)(1-aq^2z)\cdots(1-aq^{n-1}z)}{(1-z)(1-zq)(1-zq^2)\cdots(1-zq^{n-1})} h_a(q^n z) \\ &= \frac{(az;q)_n}{(z;q)_n} h_a(q^n z). \end{aligned}$$

since $q^n \to 0$ as $n \to \infty$ and $h_a(0) = 1$, we find

$$\begin{aligned} h_a(z) &= \frac{(az;q)_\infty}{(z;q)_\infty} h_a(0) \\ &= \frac{(az;q)_\infty}{(z;q)_\infty}, \end{aligned}$$

which is the desired result. □

As a result of q-binomial theorem we can give the following corollary.

Corollary 1.12

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{1.30}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} z^n = (z;q)_\infty. \tag{1.31}$$

$$\sum_{n=0}^{N} \binom{N}{n}_q (-1)^n q^{\binom{n}{2}} z^n = (z;q)_N. \tag{1.32}$$

$$\sum_{n=0}^{\infty} \binom{N+n-1}{n}_q z^n = \frac{1}{(z;q)_N}. \tag{1.33}$$

Proof.

Proof of (1.30): Setting $a = 0$ in equation (1.26), we have

$$\sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} z^n = \frac{(0;q)_\infty}{(z;q)_\infty}.$$

Since $(0;q)_n = 1$ and $(0;q)_\infty = 1$ we find

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},$$

which is (1.30).

Proof of (1.31): If we take $\dfrac{1}{a}$ and az in places of a and z in equation (1.26), respectively, then we get

$$\sum_{n=0}^{\infty} \frac{(1/a;q)_n}{(q;q)_n}(az)^n = \frac{(z;q)_\infty}{(az;q)_\infty}\cdot \tag{1.34}$$

By means of the following relation

$$\begin{aligned}\left(\frac{1}{a};q\right)_n &= \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a}q\right)\dots\left(1-\frac{1}{a}q^{n-1}\right)\\ &= \left(\frac{-1}{a}\right)^n \left[(1-a)(q-a)\cdots(q^{n-1}-a)\right],\end{aligned}$$

we can rewrite equation (1.34) as

$$\sum_{n=0}^{\infty} \frac{(-1)^n(1-a)(q-a)\cdots(q^{n-1}-a)}{(q;q)_n} z^n = \frac{(z;q)_\infty}{(az;q)_\infty}\cdot$$

In this equation taking $a=0$, we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n q q^2 q^3 \dots q^{n-1}}{(q;q)_n} z^n = \frac{(z;q)_\infty}{(0;q)_\infty}$$

or

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} z^n = (z;q)_\infty,$$

which is the desired result.

Proof of (1.32): In a similar manner, substitution of $a=q^{-N}$ in equation (1.26) yields

$$\sum_{n=0}^{\infty} \frac{(q^{-N};q)_n}{(q;q)_n} z^n = \frac{(q^{-N}z;q)_\infty}{(z;q)_\infty}\cdot \tag{1.35}$$

By the definition of q-shifted factorial, the right hand side of (1.35) can be written as

$$\begin{aligned}\frac{(q^{-N}z;q)_\infty}{(z;q)_\infty} &= \frac{(1-zq^{-N})(1-zq^{-N+1})\dots(1-zq^{-1})(1-z)(1-zq)\dots}{(1-z)(1-zq)\dots} \\ &= (q^{-N}z;q)_N\end{aligned}$$

and by using (1.15) and the relation

$$\frac{(q;q)_N}{(q;q)_n(q;q)_{N-n}} = \binom{N}{n}_q$$

the left hand side of (1.35) takes the form

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q^{-N};q)_n}{(q;q)_n}z^n &= \sum_{n=0}^{\infty}\frac{(q;q)_N}{(q;q)_n(q;q)_{N-n}}(-1)^n q^{\binom{n}{2}-Nn}z^n \\ &= \sum_{n=0}^{\infty}\binom{N}{n}_q(-1)^n q^{\binom{n}{2}-Nn}z^n.\end{aligned}$$

Thus, (1.35) becomes

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{N}{n}_q(-1)^n q^{\binom{n}{2}-Nn}z^n &= \sum_{n=0}^{N}\binom{N}{n}_q(-1)^n q^{\binom{n}{2}-Nn}z^n \\ &\quad+\sum_{n=N+1}^{\infty}\binom{N}{n}_q(-1)^n q^{\binom{n}{2}-Nn}z^n \\ &= (q^{-N}z;q)_N.\end{aligned}$$

Since $\binom{N}{n}_q = 0$ for $n \geq N+1$, the second summation is zero. So we have

$$\sum_{n=0}^{N}\binom{N}{n}_q(-1)^n q^{\binom{n}{2}-Nn}z^n = (q^{-N}z;q)_N$$

or

$$\sum_{n=0}^{N}\binom{N}{n}_q(-1)^n q^{\binom{n}{2}}\left(\frac{z}{q^N}\right)^n = (q^{-N}z;q)_N.$$

In this equation, we replace $\frac{z}{q^N}$ by z to obtain

$$\sum_{n=0}^{\mathbb{N}} \binom{N}{n}_q (-1)^n q^{\binom{n}{2}} z^n = (z;q)_N.$$

This is the required result.

Proof of (1.33): Finally, setting $a = q^N$ in q-binomial theorem one gets

$$\sum_{n=0}^{\infty} \frac{(q^N;q)_n}{(q;q)_n} z^n = \frac{(q^N z;q)_\infty}{(z;q)_\infty}. \tag{1.36}$$

By using the q-shifted factorial the left hand side of (1.36) is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(q^N;q)_n}{(q;q)_n} z^n &= \sum_{n=0}^{\infty} \frac{[(1-q)(1-q^2)\dots(1-q^{N-1})](q^N;q)_n}{[(1-q)(1-q^2)\dots(1-q^{N-1})](q;q)_n} z^n \\
&= \sum_{n=0}^{\infty} \frac{(q;q)_{N+n-1}}{(q;q)_n (q;q)_{N-1}} z^n \\
&= \sum_{n=0}^{\infty} \binom{N+n-1}{n}_q z^n
\end{aligned}$$

and the right hand side of (1.36) is

$$\begin{aligned}
\frac{(q^N z;q)_\infty}{(z;q)_\infty} &= \frac{(1-zq^N)(1-zq^{N+1})\dots}{[(1-z)(1-zq)\dots(1-zq^{N-1})]\,(1-zq^N)(1-zq^{N+1})\dots} \\
&= \frac{1}{(z;q)_N}.
\end{aligned}$$

Setting these in (1.36), we find

$$\sum_{n=0}^{\infty} \binom{N+n-1}{n}_q z^n = \frac{1}{(z;q)_N}.$$

Thus the proof is completed. □

Now by using the result (1.32) of q-binomial theorem we give the following theorem.

1.3 The triple product identity

Theorem 1.13 *For* $z \in \mathbb{C} \setminus \{0\}$

$$(z;q)_\infty (q/z;q)_\infty (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k.$$

Proof. We take $N = 2n$ in (1.32) to obtain

$$(z;q)_{2n} = \sum_{k=0}^{2n} \binom{2n}{k}_q (-1)^k q^{\binom{k}{2}} z^k.$$

Replacing k by $n+k$, we have

$$\begin{aligned}(z;q)_{2n} &= \sum_{k=-n}^{n} \binom{2n}{n+k}_q (-1)^{n+k} q^{\binom{n+k}{2}} z^{n+k} \\ &= \sum_{k=-n}^{n} \binom{2n}{n+k}_q (-1)^{n+k} q^{\frac{(n+k)(n+k-1)}{2}} z^{n+k}.\end{aligned}$$

Again replacing z by zq^{-n} we can write the last equation as follows:

$$(zq^{-n};q)_{2n} = \sum_{k=-n}^{n} \binom{2n}{n+k}_q (-1)^{n+k} q^{\frac{(n+k)(n+k-1)}{2}} (zq^{-n})^{n+k}. \tag{1.37}$$

Now, using equation (1.4) and (1.18), one gets

$$\begin{aligned}(zq^{-n};q)_n &= (q/z;q)_n (-z/q)^n q^{-\binom{n}{2}} \\ &= (-1)^n z^n q^{\frac{-n^2-n}{2}} (q/z;q)_n.\end{aligned} \tag{1.38}$$

and

$$\begin{aligned}(zq^{-n};q)_{2n} &= (zq^{-n};q)_n (z;q)_n \\ &= (-1)^n q^{\frac{-n^2-n}{2}} (q/z;q)_n (z;q)_n z^n.\end{aligned} \tag{1.39}$$

Noting that

$$\binom{2n}{n+k}_q = \frac{(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}}$$

and setting these two equations (1.38) and (1.39) into (1.37) we find

$$(-1)^n z^n q^{\frac{-n^2-n}{2}}(q/z;q)_n(z;q)_n = \sum_{k=-n}^{n} \frac{(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}} \times (-1)^{n+k} q^{\frac{(n+k)(n+k-1)}{2}} q^{-n^2-nk} z^{n+k}. \quad (1.40)$$

Since

$$\frac{(n+k)(n+k-1)}{2} - n^2 - nk = \frac{-n^2-n}{2} + \binom{k}{2},$$

equation (1.40) becomes

$$(-1)^n z^n q^{\frac{-n^2-n}{2}}(q/z;q)_n(z;q)_n = (-1)^n z^n q^{\frac{-n^2-n}{2}} \sum_{k=-n}^{n} \frac{(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}} \times (-1)^k q^{\binom{k}{2}} z^k$$

or

$$(q/z;q)_n(z;q)_n = \sum_{k=-n}^{n} \frac{(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} z^k.$$

When $n \to \infty$, this gives

$$\begin{aligned}(q/z;q)_\infty(z;q)_\infty &= \sum_{k=-\infty}^{\infty} \frac{(q;q)_\infty}{(q;q)_\infty(q;q)_\infty} (-1)^k q^{\binom{k}{2}} z^k \\ &= \frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k\end{aligned}$$

or

$$(z;q)_\infty(q/z;q)_\infty(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k.$$

Thus the proof is completed. □

1.4 The Carlitz inversions

In this part, we shall introduce Carlitz inverse series relations which will be used in the next chapter.

Theorem 1.14 *Let $\{a_i\}$ and $\{b_j\}$ be two complex sequences such that the polynomials defined by*

$$\phi(x;0) = 1 \tag{1.41}$$

and

$$\phi(x;n) = \prod_{k=0}^{n-1}(a_k + xb_k),\ \textit{for}\ n = 1, 2, ... \tag{1.42}$$

differ from zero for $x = q^n$ with n being non-negative integers. Then we have the following inverse series relations

$$F(n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} \phi(q^k;n)G(k),\ n = 0, 1, 2, ... \tag{1.43}$$

$$G(n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q \frac{a_k + q^k b_k}{\phi(q^n;k+1)} F(k),\ n = 0, 1, 2, ... \tag{1.44}$$

Proof. In system (1.43)-(1.44), F and G are unique. Thus, in order to prove this theorem it is sufficient to verify (1.43).

We first prove (1.43) by assuming (1.44) holds. By this assumption, setting

$$G(k) = \sum_{i=0}^{k}(-1)^i \binom{k}{i}_q \frac{a_i + q^i b_i}{\phi(q^k;i+1)} F(i)$$

into (1.43) and using (1.25) we can get

$$\begin{aligned}
F(n) &= \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} \phi(q^k;n)G(k) \\
&= \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} \phi(q^k;n) \sum_{i=0}^{k}(-1)^i \binom{k}{i}_q \frac{a_i + q^i b_i}{\phi(q^k;i+1)} F(i) \\
&= \sum_{i=0}^{n}(a_i + q^i b_i) \binom{n}{i}_q F(i) \sum_{k=i}^{n}(-1)^{k+i} \binom{n-i}{k-i}_q \frac{\phi(q^k;n)}{\phi(q^k;i+1)} q^{\binom{n-k}{2}}.
\end{aligned}$$

Let $\ell = k - i$, then

$$F(n) = \sum_{i=0}^{n} (a_i + q^i b_i) \binom{n}{i}_q F(i) \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell}_q \frac{\phi(q^{i+\ell}; n)}{\phi(q^{i+\ell}; i+1)} q^{\binom{n-i-\ell}{2}}. \quad (1.45)$$

Now, let $S(i, n)$ denote the inner sum with respect to ℓ:

$$S(i, n) := \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell}_q \frac{\phi(q^{i+\ell}; n)}{\phi(q^{i+\ell}; i+1)} q^{\binom{n-i-\ell}{2}}.$$

For $i = n$, this gives

$$S(n, n) = \frac{\phi(q^n; n)}{\phi(q^n; n+1)} = \frac{\prod_{k=0}^{n-1} (a_k + q^n b_k)}{\prod_{k=0}^{n} (a_k + q^n b_k)} = \frac{1}{a_n + q^n b_n}.$$

This shows that when $i = n$ (1.45) reduces to F(n). Thus, to complete the proof we prove that (1.45) is valid for $0 \le i \le n - 1$.
In (1.42), replacing x by $q^{i+\ell}$ and n by $i + 1$, we have

$$\phi(q^{i+\ell}; n) = \prod_{k=0}^{n-1} (a_k + q^{i+\ell} b_k)$$

and

$$\phi(q^{i+\ell}; i+1) = \prod_{k=0}^{i} (a_k + q^{i+\ell} b_k).$$

Since $\phi(q^{i+\ell}; n)$ and $\phi(q^{i+\ell}; i+1)$ are polynomials, $\dfrac{\phi(q^{i+\ell}; n)}{\phi(q^{i+\ell}; i+1)}$ is also a polynomial in q^ℓ of degree $n - i - 1$, namely

$$\frac{\phi(q^{i+\ell}; n)}{\phi(q^{i+\ell}; i+1)} = \sum_{j=0}^{n-i-1} C_j q^{\ell(n-i-j-1)},$$

where $\{C_j\}_{j=0}^{n-i-1}$ are constants. Now by using this equation we can rewrite the sum $S(i, n)$ as

$$S(i,n) = \sum_{\ell=0}^{n-i} (-1)^\ell \binom{n-i}{\ell}_q q^{\binom{n-i-\ell}{2}} \sum_{j=0}^{n-i-1} C_j q^{\ell(n-i-j-1)}. \tag{1.46}$$

By means of the relation

$$\binom{n-i-\ell}{2} = \binom{n-i}{2} + \binom{\ell}{2} - \ell(n-i-1),$$

one gets

$$q^{\binom{n-i-\ell}{2}} = q^{\binom{n-i}{2}+\binom{\ell}{2}-\ell(n-i-1)}.$$

Substitution of this in (1.46) gives

$$S(i,n) = \sum_{j=0}^{n-i-1} C_j q^{\binom{n-i}{2}} \sum_{\ell=0}^{n-i} (-1)^\ell q^{\binom{\ell}{2}} \binom{n-i}{\ell}_q q^{-\ell j}. \tag{1.47}$$

In equation (1.32) replacing N, n and z by $n-i$, ℓ and q^{-j} we find

$$\sum_{\ell=0}^{n-i} (-1)^\ell q^{\binom{\ell}{2}} \binom{n-i}{\ell}_q q^{-\ell j} = (q^{-j};q)_{n-i}.$$

Setting this into (1.47), we have

$$S(i,n) = \sum_{j=0}^{n-i-1} C_j q^{\binom{n-i}{2}} (q^{-j};q)_{n-i},$$

which vanishes for $0 \leq j < n-i$. $\square$

CHAPTER 2

Q-HYPERGEOMETRIC SERIES

As we mentioned before, q-hypergeometric series (or basic hypergeometric series) has many applications in number theory, physics, statistics, computer science etc. In this chapter, we introduce the definition of basic hypergeometric series and give their convergence properties. We also present transformation and summation formulas for these series. Here the notations and some results were given in [3, 4] and [7, 9-13].

Definition 2.1 *Let* $\{a_i\}$ *and* $\{b_j\}$ $(i = 0, 1, 2, ..., r; j = 1, 2, ..., s)$ *be complex numbers such that* $b_j \neq q^{-n}$ *with* $n \in \mathbb{N}_0 := \{0, 1, ...\}$ *for all* $j = 1, 2, ..., s$. *Then the basic hypergeometric series (or q-hypergeometric series) with variable* z *is defined by*

$$
\begin{aligned}
{}_{1+r}\phi_s(a_0, a_1, ..., a_r; b_1, b_2, ..., b_s; q, z) = \sum_{n=0}^{\infty} & \frac{(a_0; q)_n (a_1; q)_n ... (a_r; q)_n}{(b_1; q)_n (b_2; q)_n ... (b_s; q)_n (q; q)_n} \\
& \times \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n. \qquad (2.1)
\end{aligned}
$$

Since $(q^{-k}, q)_n = 0$ for $n = k + 1, \ldots$ it is easily seen that when $a_i = q^{-k}$ with $k \in \mathbb{N}_0$, then the q-hypergeometric series is a polynomial of z.

Theorem 2.2 ***(Convergence properties)*** *For the q-hypergeometric series the convergence properties as follows :*

(i) the series is convergent for all $z \in \mathbb{C}$, *if* $s > r$

(ii) the series is convergent only when $z = 0$, *if* $s < r$

(iii) the series is convergent for $|z| < 1$, *if* $s = r$

Proof. Let A_n denoted by

$$
A_n = \frac{(a_0; q)_n (a_1; q)_n \ldots (a_r; q)_n}{(b_1; q)_n (b_2; q)_n \ldots (b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n
$$

then

$$\frac{A_{n+1}}{A_n} = \frac{\dfrac{(a_0;q)_{n+1}(a_1;q)_{n+1}\dots(a_r;q)_{n+1}}{(b_1;q)_{n+1}(b_2;q)_{n+1}\dots(b_s;q)_{n+1}(q;q)_{n+1}}\left[(-1)^{n+1}q^{\binom{n+1}{2}}\right]^{s-r} z^{n+1}}{\dfrac{(a_0;q)_n(a_1;q)_n\dots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\dots(b_s;q)_n(q;q)_n}\left[(-1)^{n}q^{\binom{n}{2}}\right]^{s-r} z^{n}}$$
$$= \frac{(1-a_0q^n)(1-a_1q^n)\dots(1-a_rq^n)}{(1-b_1q^n)(1-b_2q^n)\dots(1-b_sq^n)(1-q^{n+1})}(-q^n)^{s-r}z.$$

Since $0<|q|<1$, we get $|q^n|\to 0$ as $n\to\infty$. So we have

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \begin{cases} 0 & \text{if } r<s \\ \infty & \text{if } r>s \text{ and } z\neq 0 \\ |z| & \text{if } r=s. \end{cases}$$

Thus, by the ratio test we can say that

If $r<s$, the ${}_{1+r}\phi_s$ series converges for all $z\in\mathbb{C}$.

If $r>s$, the ${}_{1+r}\phi_s$ series diverges for all $z\in\mathbb{C}\backslash\{0\}$.

If $r=s$, the ${}_{1+r}\phi_s$ series converges for $|z|<1$. $\square$

We classify the q-hypergeometric series into two main types. We now assume that $r=s$. If

$$qa_0a_1\dots a_r = b_1b_2\dots b_r$$

then, the ${}_{1+r}\phi_r$ series is called balanced. If

$$qa_0 = a_1b_1 = \dots = a_rb_r$$

then, the ${}_{1+r}\phi_r$ series is called well-poised. In particular, if also $a_1=-a_2=q\sqrt{a_0}$, it is said to be very-well-poised.

Corollary 2.3 *The limit relation between ordinary hypergeometric series and q-hypergeometric series is given by*

$$\lim_{q\to 1} {}_{1+r}\phi_s\left(q^{a_0}, q^{a_1}, \ldots, q^{a_r}; q^{b_1}, q^{b_2}, \ldots, q^{b_s}; q, \frac{(-1)^{r-s}z}{(1-q)^{r-s}}\right)$$
$$= {}_{1+r}F_s(a_0, a_1, \ldots, a_r; b_1, b_2, \ldots, b_s; z),$$

where

$$ {}_{1+r}F_s(a_0, a_1, \ldots, a_r; b_1, b_2, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n}{n!}$$

is the ordinary hypergeometric series with the classical pochhammer symbol $(c)_n = c(c+1)\ldots(c+n-1)$.

Proof. By the definition of q-number and q-shifted factorial, we have

$$\frac{(q^c;q)_n}{(1-q)^n} = \frac{(1-q^c)}{(1-q)} \frac{(1-q^{c+1})}{(1-q)} \cdots \frac{(1-q^{c+n-1})}{(1-q)} = [c]_q [c+1]_q \ldots [c+n-1]_q$$

and now taking the limit of both sides of this equation when q approaches 1, we find

$$\lim_{q\to 1} \frac{(q^c;q)_n}{(1-q)^n} = c(c+1)\ldots(c+n-1) = (c)_n. \tag{2.2}$$

On the other hand, by the definition of basic hypergeometric series we can write

$$ {}_{1+r}\phi_s(q^{a_0}, q^{a_1}, \ldots, q^{a_r}; q^{b_1}, q^{b_2}, \ldots, q^{b_s}; q, \frac{(-1)^{r-s}z}{(1-q)^{r-s}})$$
$$= \sum_{n=0}^{\infty} \frac{(q^{a_0};q)_n (q^{a_1};q)_n \ldots (q^{a_r};q)_n}{(q^{b_1};q)_n (q^{b_2};q)_n \ldots (q^{b_s};q)_n (q;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} \left[\frac{(-1)^{r-s}z}{(1-q)^{r-s}}\right]^n.$$
$$= \sum_{n=0}^{\infty} \frac{[(q^{a_0};q)_n/(1-q)^n][(q^{a_1};q)_n/(1-q)^n] \ldots [(q^{a_r};q)_n/(1-q)^n]}{[(q^{b_1};q)_n/(1-q)^n][(q^{b_2};q)_n/(1-q)^n] \ldots [(q^{b_s};q)_n/(1-q)^n][(q;q)_n/(1-q)^n]}$$
$$\times [(1-q)^{r-s}]^n \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} \left[\frac{(-1)^{r-s}z}{(1-q)^{r-s}}\right]^n.$$

Hence, by using the relation (2.2), we obtain

$$\lim_{q\to 1} {}_{1+r}\phi_s\left(q^{a_0}, q^{a_1}, \ldots, q^{a_r}; q^{b_1}, q^{b_2}, \ldots, q^{b_s}; q, \frac{(-1)^{r-s}z}{(1-q)^{r-s}}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n}{n!}$$
$$= {}_{1+r}F_s(a_0, a_1, \ldots, a_r; b_1, b_2, \ldots, b_s; z).$$

This is the desired result. □

Now we give some summation and transformation formulas for basic hypergeometric series.

Theorem 2.4 *Let* $|z| < 1$*. Then we have*

$${}_2\phi_1(a, b; cq^{-1}; q, z) - {}_2\phi_1(a, b; c; q, z) = cz\frac{(1-a)(1-b)}{(q-c)(1-c)}{}_2\phi_1(aq, bq; cq; q, z). \quad (2.3)$$

$${}_2\phi_1(aq, b; c; q, z) - {}_2\phi_1(a, b; c; q, z) = az\frac{(1-b)}{(1-c)}{}_2\phi_1(aq, bq; cq; q, z). \quad (2.4)$$

$${}_2\phi_1(aq, b; cq; q, z) - {}_2\phi_1(a, b; c; q, z) = az\frac{(1-b)(1-c/a)}{(1-c)(1-cq)}{}_2\phi_1(aq, bq; cq^2; q, z). \quad (2.5)$$

$${}_2\phi_1(aq, bq^{-1}; c; q, z) - {}_2\phi_1(a, b; c; q, z) = az\frac{(1-b/aq)}{(1-c)}{}_2\phi_1(aq, b; cq; q, z). \quad (2.6)$$

$$b(1-a){}_2\phi_1(aq, b; c; q, z) - a(1-b){}_2\phi_1(a, bq; c; q, z) = (b-a){}_2\phi_1(a, b; c; q, z). \quad (2.7)$$

Proof.

Proof of (2.3):

By the definition of basic hypergeometric series we get

$$
{}_2\phi_1(a,b;c;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n. \tag{2.8}
$$

$$
{}_2\phi_1(a,b;cq^{-1};q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(cq^{-1};q)_n(q;q)_n}z^n.
$$

Then, we can write

$$
\begin{aligned}
{}_2\phi_1(a,b;cq^{-1};q,z)-{}_2\phi_1(a,b;c;q,z)&=\sum_{n=0}^{\infty}\left[\frac{(a;q)_n(b;q)_n}{(cq^{-1};q)_n(q;q)_n}-\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}\right]z^n\\
&=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n\left[(c;q)_n-(cq^{-1};q)_n\right]}{(c;q)_n(cq^{-1};q)_n(q;q)_n}z^n. \qquad (2.9)
\end{aligned}
$$

Since

$$
(c;q)_n=(c;q)_{n-1}(1-cq^{n-1}) \tag{2.10}
$$

and

$$
(cq^{-1};q)_n=(1-cq^{-1})(c;q)_{n-1}, \tag{2.11}
$$

we can rewrite (2.9) as,

$$
\begin{aligned}
{}_2\phi_1(a,b;cq^{-1};q,z)-{}_2\phi_1(a,b;c;q,z)&=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n(c;q)_{n-1}(1-q^n)cq^{-1}}{(c;q)_n(1-cq^{-1})(c;q)_{n-1}(q;q)_n}z^n\\
&=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n(1-q^n)c}{(c;q)_n(q;q)_n(q-c)}z^n\\
&=\sum_{n=0}^{\infty}\frac{(1-a)(aq;q)_{n-1}(1-b)(bq;q)_{n-1}}{(1-c)(cq;q)_{n-1}(q;q)_{n-1}(1-q^n)}\\
&\quad\times\frac{(1-q^n)c}{(q-c)}z^n\\
&=\frac{c(1-a)(1-b)}{(1-c)(q-c)}\sum_{n=0}^{\infty}\frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(cq;q)_{n-1}(q;q)_{n-1}}z^n.
\end{aligned}
$$

We replace n by $n+1$ to obtain

$$
{}_2\phi_1(a,b;cq^{-1};q,z)-{}_2\phi_1(a,b;c;q,z)=\frac{c(1-a)(1-b)}{(1-c)(q-c)}\sum_{n=-1}^{\infty}\frac{(aq;q)_n(bq;q)_n}{(cq;q)_n(q;q)_n}z^{n+1}
$$

$$
\begin{aligned}
{}_2\phi_1(a,b;cq^{-1};q,z) - {}_2\phi_1(a,b;c;q,z) &= \frac{cz(1-a)(1-b)}{(1-c)(q-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_n(bq;q)_n}{(cq;q)_n(q;q)_n} z^n \\
&= \frac{cz(1-a)(1-b)}{(1-c)(q-c)} {}_2\phi_1(aq,bq;cq;q,z),
\end{aligned}
$$

which is the desired result.

Proof of (2.4):

In (2.8), replacing a by aq, one gets

$$
{}_2\phi_1(aq,b;c;q,z) - {}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(c;q)_n(q;q)_n} \left[(aq;q)_n - (a;q)_n\right] z^n.
$$

By means of the relations,

$$
(a;q)_n = (1-a)(aq;q)_{n-1} \tag{2.12}
$$

and

$$
(aq;q)_n = (aq;q)_{n-1}(1-aq^n) \tag{2.13}
$$

we have

$$
\begin{aligned}
{}_2\phi_1(aq,b;c;q,z) - {}_2\phi_1(a,b;c;q,z) = &= \sum_{n=0}^{\infty} \frac{(b;q)_n(aq;q)_{n-1}a(1-q^n)}{(c;q)_n(q;q)_n} z^n \\
&= \sum_{n=0}^{\infty} \frac{a(1-b)(bq;q)_{n-1}(aq;q)_{n-1}(1-q^n)}{(1-c)(cq;q)_{n-1}(q;q)_{n-1}(1-q^n)} z^n \\
&= \frac{a(1-b)}{(1-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(cq;q)_{n-1}(q;q)_{n-1}} z^n.
\end{aligned}
$$

Now, writing n in place of $n+1$, we find

$$
\begin{aligned}
{}_2\phi_1(aq,b;c;q,z) - {}_2\phi_1(a,b;c;q,z) = &= az\frac{(1-b)}{(1-c)} \sum_{n=-1}^{\infty} \frac{(aq;q)_n(bq;q)_n}{(cq;q)_n(q;q)_n} z^n \\
&= az\frac{(1-b)}{(1-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_n(bq;q)_n}{(cq;q)_n(q;q)_n} z^n \\
&= az\frac{(1-b)}{(1-c)} {}_2\phi_1(aq,bq;cq;q,z),
\end{aligned}
$$

which is the required result.

Proof of (2.5):

We prove in a similar way that of (2.3) and (2.4). In equation (2.8), replacing a and c by aq and cq, respectively, and by using relation (2.12), we have

$$
\begin{aligned}
&{}_2\phi_1(aq,b;cq;q,z) - {}_2\phi_1(a,b;c;q,z) \\
&\quad = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n}\left[\frac{(aq;q)_n}{(cq;q)_n} - \frac{(a;q)_n}{(c;q)_n}\right] z^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n}\left[\frac{(aq;q)_n(c;q)_n - (a;q)_n(cq;q)_n}{(cq;q)_n(c;q)_n}\right] z^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(b;q)_n(aq;q)_{n-1}(cq;q)_{n-1}[(1-aq^n)(1-c)-(1-a)(1-cq^n)]}{(q;q)_n(cq;q)_n(c;q)_n} z^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(b;q)_n(aq;q)_{n-1}(cq;q)_{n-1}[(cq^n-c)+(a-aq^n)]}{(q;q)_n(cq;q)_{n-1}(1-cq^n)(c;q)_n} z^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_n(1-q^n)(a-c)}{(1-cq^n)(c;q)_n(q;q)_n} z^n \\
&\quad = \sum_{n=0}^{\infty} \frac{a(1-c/a)(1-b)(aq;q)_{n-1}(bq;q)_{n-1}(1-q^n)}{(1-cq^n)(1-c)(cq;q)_{n-1}(q;q)_{n-1}(1-q^n)} z^n \\
&\quad = \frac{a(1-c/a)(1-b)}{(1-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(cq;q)_n(q;q)_{n-1}} z^n \\
&\quad = \frac{a(1-c/a)(1-b)}{(1-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(1-cq)(cq^2;q)_{n-1}(q;q)_{n-1}} z^n \\
&\quad = \frac{az(1-c/a)(1-b)}{(1-c)(1-cq)} \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(cq^2;q)_{n-1}(q;q)_{n-1}} z^{n-1}.
\end{aligned}
$$

If write $n+1$ in place of n, it follows that

$$
\begin{aligned}
{}_2\phi_1(aq,b;cq;q,z) - {}_2\phi_1(a,b;c;q,z) &= \frac{az(1-c/a)(1-b)}{(1-c)(1-cq)} \sum_{n=-1}^{\infty} \frac{(aq;q)_n(bq;q)_n}{(cq^2;q)_n(q;q)_n} z^n \\
&= \frac{az(1-c/a)(1-b)}{(1-c)(1-cq)} \sum_{n=0}^{\infty} \frac{(aq;q)_n(bq;q)_n}{(cq^2;q)_n(q;q)_n} z^n \\
&= \frac{az(1-c/a)(1-b)}{(1-c)(1-cq)} {}_2\phi_1(aq,bq;cq^2;q,z).
\end{aligned}
$$

Proof of (2.6)

From equations (2.8) and (2.10)-(2.13), we can write

$$
\begin{aligned}
&{}_2\phi_1(aq, bq^{-1}; c; q, z) - {}_2\phi_1(a, b; c; q, z)\\
&\quad= \sum_{n=0}^{\infty} \frac{[(aq;q)_n(bq^{-1};q)_n - (a;q)_n(b;q)_n]}{(c;q)_n(q;q)_n} z^n\\
&\quad= \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_{n-1}[(1-aq^n)(1-bq^{-1}) - (1-a)(1-bq^{n-1})]}{(c;q)_n(q;q)_n} z^n\\
&\quad= \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_{n-1}[bq^{n-1} - bq^{-1} + a - aq^n]}{(c;q)_n(q;q)_n} z^n\\
&\quad= \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_{n-1}[(1-q^n)(-b/q) + a(1-q^n)]}{(c;q)_n(q;q)_n} z^n\\
&\quad= \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_{n-1}(1-q^n)(a-b/q)}{(c;q)_n(q;q)_n} z^n\\
&\quad= \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_{n-1}(1-q^n)(a-b/q)}{(1-c)(cq;q)_{n-1}(q;q)_{n-1}(1-q^n)} z^{n-1}z.\\
&\quad= az\frac{(1-b/aq)}{(1-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_{n-1}(b;q)_{n-1}}{(cq;q)_{n-1}(q;q)_{n-1}} z^{n-1}.
\end{aligned}
$$

Now, replacing n by $n+1$, we find

$$
\begin{aligned}
{}_2\phi_1(aq, bq^{-1}; c; q, z) - {}_2\phi_1(a, b; c; q, z) &= az\frac{(1-b/aq)}{(1-c)} \sum_{n=-1}^{\infty} \frac{(aq;q)_n(b;q)_n}{(cq;q)_n(q;q)_n} z^n\\
&= az\frac{(1-b/aq)}{(1-c)} \sum_{n=0}^{\infty} \frac{(aq;q)_n(b;q)_n}{(cq;q)_n(q;q)_n} z^n\\
&= az\frac{(1-b/aq)}{(1-c)} {}_2\phi_1(aq, b; cq; q, z).
\end{aligned}
$$

Proof of (2.7):

Again by equation (2.8), we have

$$
\begin{aligned}
&b(1-a){}_2\phi_1(aq, b; c; q, z) - a(1-b){}_2\phi_1(a, bq; c; q, z)\\
&\qquad= b(1-a)\sum_{n=0}^{\infty} \frac{(aq;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n - a(1-b)\sum_{n=0}^{\infty} \frac{(a;q)_n(bq;q)_n}{(c;q)_n(q;q)_n} z^n\\
&\qquad= \sum_{n=0}^{\infty} \frac{[b(1-a)(aq;q)_n(b;q)_n - a(1-b)(a;q)_n(bq;q)_n]}{(c;q)_n(q;q)_n} z^n.
\end{aligned}
$$

Thus, by using the two equations

$$(1-a)(aq;q)_n = (a;q)_n(1-aq^n),$$

and

$$(1-b)(bq;q)_n = (b;q)_n(1-bq^n),$$

we get

$$\begin{aligned}
&b(1-a){}_2\phi_1(aq,b;c;q,z) - a(1-b){}_2\phi_1(a,bq;c;q,z)\\
&= \sum_{n=0}^{\infty} \frac{[b(a;q)_n(1-aq^n)(b;q)_n - a(b;q)_n(1-bq^n)(a;q)_n]}{(c;q)_n(q;q)_n} z^n\\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} [b(1-aq^n) - a(1-bq^n)]\, z^n\\
&= (b-a)\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n\\
&= (b-a){}_2\phi_1(a,b;c;q,z).
\end{aligned}$$

Thus, the proof is completed. □

2.1 Heine's transformation formula

Theorem 2.5 *For $|z| < 1$ and $|b| < 1$, we have*

$${}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty} {}_2\phi_1(c/b,z;az;q,b). \tag{2.14}$$

Proof. By the definition of basic hypergeometric series ${}_2\phi_1$, we write

$$\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n\\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n[(1-bq^n)(1-bq^{n+1})\ldots]}{(q;q)_n(c;q)_n[(1-bq^n)(1-bq^{n+1})\ldots]}\\
&\times \frac{[(1-cq^n)(1-cq^{n+1})\ldots]}{[(1-cq^n)(1-cq^{n+1})\ldots]} z^n.
\end{aligned}$$

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_\infty(cq^n;q)_\infty}{(q;q)_n(c;q)_\infty(bq^n;q)_\infty} z^n \\
&= \frac{(b;q)_\infty}{(c,q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty} z^n. \qquad (2.15)
\end{aligned}
$$

In Theorem 1.11, replacing a and z by c/b and bq^n, respectively, we have

$$
\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} = \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m.
$$

Setting this result in the left hand side of (2.15), we can get

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c,q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m \\
&= \frac{(b;q)_\infty}{(c,q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(zq^m)^n. \qquad (2.16)
\end{aligned}
$$

On the other hand, again by Theorem 1.11, we can write the inner sum

$$
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(zq^m)^n = \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}.
$$

Substitution of this gives

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty}{(c,q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}. \qquad (2.17)
$$

Multiplying and dividing equation (2.17) by $(az;q)_m(z;q)_m$, we find

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty}{(c,q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} \frac{(az;q)_m(z;q)_m}{(z;q)_m(az;q)_m}.
$$

Now, by using the relations

$$
(azq^m;q)_\infty(az;q)_m = (az;q)_\infty,
$$

and

$$
(zq^m;q)_\infty(z;q)_m = (z;q)_\infty,
$$

we can rewrite the last equation as

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c,q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m(z;q)_m(az;q)_\infty}{(q;q)_m(az;q)_m(z;q)_\infty}b^m \\
&= \frac{(b;q)_\infty(az;q)_\infty}{(c,q)_\infty(z;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m(z;q)_m}{(q;q)_m(az;q)_m}b^m \\
&= \frac{(b;q)_\infty(az;q)_\infty}{(c,q)_\infty(z;q)_\infty}{}_2\phi_1(c/b,z;az;q,b).
\end{aligned}
$$

Thus the proof is completed. □

2.2 Iteration of Heine's transformation formula

Theorem 2.6 *For $|z|<1$ and $|c/b|<1$, we have*

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}{}_2\phi_1(abz/c,b;bz;q,c/b). \tag{2.18}
$$

Proof. By using the symmetry property,

$$
{}_2\phi_1(c/b,z;az;q,b) = {}_2\phi_1(z,c/b;az;q,b)
$$

In this case, the Heine's transformation formula can be rewritten as

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}{}_2\phi_1(z,c/b;az;q,b). \tag{2.19}
$$

In (2.14), replacing the parameters a, b, c and z by z, c/b, az and b, respectively, we have

$$
{}_2\phi_1(z,c/b;az;q,b) = \frac{(c/b;q)_\infty(bz;q)_\infty}{(az;q)_\infty(b;q)_\infty}{}_2\phi_1(abz/c,b;bz;q,c/b).
$$

Putting this in the left side of (2.19), one gets

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\left[\frac{(c/b;q)_\infty(bz;q)_\infty}{(az;q)_\infty(b;q)_\infty}{}_2\phi_1(abz/c,b;bz;q,c/b)\right].
$$

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(c/b;q)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty} {}_2\phi_1(abz/c,b;bz;q,c/b). $$

This is the desired result. □

2.3 Euler's transformation formula

Theorem 2.7 *For $|z| < 1$ and $|abz/c| < 1$, we have*

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} {}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{2.20} $$

Proof. By means of the symmetry property of ${}_2\phi_1$

$$ {}_2\phi_1(abz/c,b;bz;q,c/b) = {}_2\phi_1(b,abz/c;bz;q,c/b), $$

equation (2.18) can be rewritten as

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(c/b;q)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty} {}_2\phi_1(b,abz/c;bz;q,c/b). \tag{2.21} $$

Now, in (2.14), we take b, abz/c, bz and c/b in place of a, b, c and z, respectively, to obtain

$$ {}_2\phi_1(b,abz/c;bz;q,c/b) = \frac{(abz/c;q)_\infty (c;q)_\infty}{(bz;q)_\infty (c/b;q)_\infty} {}_2\phi_1(c/a,c/b;c;q,abz/c). $$

Setting this result into equation (2.21), we find

$$ \begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \frac{(c/b;q)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty} \left[\frac{(abz/c;q)_\infty (c;q)_\infty}{(bz;q)_\infty (c/b;q)_\infty} {}_2\phi_1(c/a,c/b;c;q,abz/c) \right] \\ &= \frac{(abz/c;q)_\infty}{(z;q)_\infty} {}_2\phi_1(c/a,c/b;c;q,abz/c), \end{aligned} $$

which is the desired result. □

2.4 Gauss summation formula

Theorem 2.8 *For $|z| < 1$ and $|c/ab| < 1$, we have*

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}. \tag{2.22} $$

Proof. Putting $z = c/ab$ in Heine's transformation formula and by using the definition of the basic hypergeometric series ${}_2\phi_1$, we find

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, c/ab) &= \frac{(b; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} {}_2\phi_1(c/b, c/ab; c/b; q, b) \\ &= \frac{(b; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b; q)_n (c/ab; q)_n}{(c/b; q)_n (q; q)_n} b^n \\ &= \frac{(b; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/ab; q)_n}{(q; q)_n} b^n. \end{aligned} \tag{2.23} $$

In equation (1.26), replacing a and z by c/ab and b, respectively, we get

$$ \sum_{n=0}^{\infty} \frac{(c/ab; q)_n}{(q; q)_n} b^n = \frac{(c/a; q)_\infty}{(b; q)_\infty}. $$

Putting this in (2.23), we obtain

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, c/ab) &= \frac{(b; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} \frac{(c/a; q)_\infty}{(b; q)_\infty} \\ &= \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}. \end{aligned} $$

Therefore, the proof is completed.

Second Proof: We now prove the Gauss summation formula by using only the q-binomial theorem. By the definition of ${}_2\phi_1$, we get

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, c/ab) &= \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} \Big(\frac{c}{ab}\Big)^n \\ &= \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} \Big(\frac{c}{ab}\Big)^n \left[\frac{(bq^n; q)_\infty (cq^n; q)_\infty}{(bq^n; q)_\infty (cq^n; q)_\infty}\right]. \end{aligned} $$

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,c/ab) &= \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_\infty(cq^n;q)_\infty}{(q;q)_n(c;q)_\infty(bq^n;q)_\infty}\Big(\frac{c}{ab}\Big)^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\Big(\frac{c}{ab}\Big)^n\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}. \qquad (2.24)
\end{aligned}
$$

In (1.26), setting $a = c/b$ and $z = bq^n$ we find

$$
\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}(bq^n)^k = \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}.
$$

Substituting this in (2.24) we have

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,c/ab) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\Big(\frac{c}{ab}\Big)^n\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}(bq^n)^k \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}b^k\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\left(\frac{cq^k}{ab}\right)^n. \qquad (2.25)
\end{aligned}
$$

Again by using q-binomial theorem for the sum on n, one gets

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,c/ab) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_k}{(q;q)_k}b^k\frac{(cq^k/b;q)_\infty}{(cq^k/ab;q)_\infty} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/b;q)_\infty}{(q;q)_k(cq^k/ab;q)_\infty}b^k\left[\frac{(c/ab;q)_k}{(c/ab;q)_k}\right] \\
&= \frac{(b;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/ab;q)_k}{(q;q)_k}b^k \\
&= \frac{(b;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}\frac{(c/a;q)_\infty}{(b;q)_\infty} \\
&= \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}.
\end{aligned}
$$

Therefore, the proof is completed. □

Corollary 2.9 *Let $|z| < 1$. Then we have*

$$
\sum_{n=0}^{\infty}\frac{q^{n(n-1)}z^n}{(q;q)_n(z;q)_n} = \frac{1}{(z;q)_\infty} \qquad (2.26)
$$

Proof.

Setting $a = A^{-1}$, $b = B^{-1}$ and $c = z$ in the Gauss summation formula and by the definition of q-hypergeometric series, we find

$$
\begin{aligned}
{}_2\phi_1(A^{-1}, B^{-1}; z; q, ABz) &= \sum_{n=0}^{\infty} \frac{(A^{-1}; q)_n (B^{-1}; q)_n}{(z; q)_n (q; q)_n} (ABz)^n \\
&= \frac{(Az; q)_\infty (Bz; q)_\infty}{(z; q)_\infty (ABz; q)_\infty}. \qquad (2.27)
\end{aligned}
$$

From (1.3), we have

$$
(a^{-1}; q)_n = (-a^{-1})^n q^{\binom{n}{2}} (a; q^{-1})_n. \qquad (2.28)
$$

By means of this equation, we can rewrite (2.27) as follows :

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\left[(-A^{-1})^n q^{\binom{n}{2}} (A; q^{-1})_n\right] \left[(-B^{-1})^n q^{\binom{n}{2}} (B; q^{-1})_n\right]}{(z; q)_n (q; q)_n} A^n B^n z^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(A; q^{-1})_n (B; q^{-1})_n q^{n(n-1)}}{(z; q)_n (q; q)_n} z^n \\
&\quad = \frac{(Az; q)_\infty (Bz; q)_\infty}{(z; q)_\infty (ABz; q)_\infty}.
\end{aligned}
$$

In this equation taking, $A \to 0$ and $B \to 0$ and using the equations

$$
\begin{aligned}
(0; q^{-1})_n &= 1, \\
(0; q)_n &= 1, \qquad (2.29)
\end{aligned}
$$

we find

$$
\sum_{n=0}^{\infty} \frac{(0; q^{-1})_n (0; q^{-1})_n q^{n(n-1)}}{(z; q)_n (q; q)_n} z^n = \frac{(0; q)_\infty (0; q)_\infty}{(z; q)_\infty (0; q)_\infty}
$$

and so

$$
\sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n}{(q; q)_n (z; q)_n} = \frac{1}{(z; q)_\infty}.
$$

This proves the corollary. □

2.5 Chu-Vandermonde's formula

Theorem 2.10 *Let* $|cq^n/b| < 1$. *Then we have*

$$ {}_2\phi_1(q^{-n}, b; c; q, cq^n/b) = \frac{(c/b;q)_n}{(c;q)_n}. \tag{2.30} $$

and

$$ {}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b;q)_n}{(c;q)_n} b^n. \tag{2.31} $$

Proof.

Proof of (2.30) : In the Gauss summation formula setting $a = q^{-n}$ and using the equations

$$ (c/b;q)_\infty = (c/b;q)_n (cq^n/b;q)_\infty, $$

and

$$ (c;q)_\infty = (c;q)_n (cq^n;q)_\infty, $$

we can write

$$ \begin{aligned} {}_2\phi_1(q^{-n}, b; c; q, cq^n/b) &= \frac{(cq^n,q)_\infty (c/b;q)_\infty}{(c,q)_\infty (cq^n/b;q)_\infty} \\ &= \frac{(cq^n,q)_\infty (c/b;q)_n (cq^n/b;q)_\infty}{(c,q)_n (cq^n,q)_\infty (cq^n/b;q)_\infty} \\ &= \frac{(c/b;q)_n}{(c;q)_n}. \end{aligned} $$

Thus the proof is completed.

Proof of (2.31): By definition of q-hypergeometric series, we rewrite (2.30) explictly as

$$ {}_2\phi_1(q^{-n}, b; c; q, cq^n/b) = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (b;q)_k}{(c;q)_k (q;q)_k} \left(\frac{cq^n}{b}\right)^k = \frac{(c/b;q)_n}{(c;q)_n}. $$

Since $(q^{-n};q)_k = 0$ for $k = n+1, n+2, \ldots$ the last equation takes the form

$$ \sum_{k=0}^{n} \frac{(q^{-n};q)_k (b;q)_k}{(c;q)_k (q;q)_k} \left(\frac{cq^n}{b}\right)^k = \frac{(c/b;q)_n}{(c;q)_n}. $$

or

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_{n-k}(b;q)_{n-k}}{(c;q)_{n-k}(q;q)_{n-k}} \left(\frac{cq^n}{b}\right)^{n-k} = \frac{(c/b;q)_n}{(c;q)_n}. \tag{2.32}$$

Considering that

$$(a;q)_{n-k} = (-1)^k a^{-k} q^{\binom{k+1}{2}-nk} \frac{(a;q)_n}{(q^{1-n}/a;q)_k},$$

we can write (2.32) as follows:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{(q^{-n};q)_{n-k}(b;q)_{n-k}}{(c;q)_{n-k}(q;q)_{n-k}} \left(\frac{cq^n}{b}\right)^{n-k} &= \sum_{k=0}^{n} \frac{\left[q^{nk}(q^{-n};q)_n/(q;q)_k\right]}{\left[c^{-k}(c;q)_n/(q^{1-n}/c;q)_k\right]} \\
&\quad \times \frac{\left[b^{-k}(b;q)_n/(q^{1-n}/b;q)_k\right]}{\left[q^{-k}(q;q)_n/(q^{-n};q)_k\right]} \left(\frac{cq^n}{b}\right)^{n-k} \\
&= \frac{(q^{-n};q)_n(b;q)_n}{(q;q)_n(c;q)_n} \left(\frac{cq^n}{b}\right)^{n} \\
&\quad \times \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{1-n}/c;q)_k}{(q^{1-n}/b;q)_k(q;q)_k} q^k
\end{aligned}$$

and thus by the definition of ${}_2\phi_1$ we have

$$\frac{(q^{-n};q)_n(b;q)_n}{(q;q)_n(c;q)_n} \left(\frac{cq^n}{b}\right)^{n} {}_2\phi_1(q^{-n}, q^{1-n}/c; q^{1-n}/b; q, q) = \frac{(c/b;q)_n}{(c;q)_n},$$

which gives

$$\begin{aligned}
{}_2\phi_1(q^{-n}, q^{1-n}/c; q^{1-n}/b; q, q) &= \frac{(c/b;q)_n}{(c;q)_n} \frac{(q;q)_n(c;q)_n}{(q^{-n};q)_n(b;q)_n} \left(\frac{b}{cq^n}\right)^{n} \\
&= \frac{(c/b;q)_n(q;q)_n}{(b;q)_n(q^{-n};q)_n} \left(\frac{b}{cq^n}\right)^{n}.
\end{aligned}$$

In this equation, if we replace q^{1-n}/c and q^{1-n}/b by B and C, respectively, and use the relation

$$(q^{-n};q)_n = (-1)^n q^{-\binom{n+1}{2}}(q;q)_n,$$

we obtain

$${}_2\phi_1(q^{-n}, B; C; q, q) = \frac{(C/B;q)_n(q;q)_n}{(q^{1-n}/C;q)_n(-1)^n q^{-\binom{n+1}{2}}(q;q)_n} \left(\frac{B}{Cq^n}\right)^{n}.$$

$$
{}_2\phi_1(q^{-n}, B; C; q, q) = \frac{(C/B;q)_n}{(q^{1-n}/C;q)_n(-1)^n q^{-\binom{n+1}{2}+n^2}}\left(\frac{B}{C}\right)^n.
$$

Since

$$
(q^{1-n}/C;q)_n = (-1)^n q^{-\binom{n}{2}} C^{-n}(C;q)_n
$$

and

$$
q^{-\binom{n}{2}-\binom{n+1}{2}+n^2} = 1,
$$

we find

$$
\begin{aligned}
{}_2\phi_1(q^{-n}, B; C; q, q) &= \frac{(C/B;q)_n}{(-1)^n q^{-\binom{n}{2}} C^{-n}(C;q)_n(-1)^n q^{-\binom{n+1}{2}+n^2}}\left(\frac{B}{C}\right)^n \\
&= \frac{(C/B;q)_n}{(C;q)_n q^{-\binom{n}{2}-\binom{n+1}{2}+n^2}} B^n \\
&= \frac{(C/B;q)_n}{(C;q)_n} B^n.
\end{aligned}
$$

Finally, replacing the parameters B and C by b and c, respectively, this gives the desired result. $\square$

2.6 Jackson's transformation formula

Theorem 2.11 *For* $|z| < 1$, *we have*

$$
{}_2\phi_1(a, b; c; q, z) = \frac{(az;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(c/b;q)_n}{(q;q)_n(c;q)_n(az;q)_n}(-bz)^n q^{\binom{n}{2}}.
$$

Proof. In (2.30), if we write c/b instead of b, one gets

$$
{}_2\phi_1(q^{-k}, c/b; c; q, bq^k) = \frac{(b;q)_k}{(c;q)_k}. \tag{2.33}
$$

By using the definition of ${}_2\phi_1$, the left hand side of (2.33) becomes

$$
{}_2\phi_1(q^{-k}, c/b; c; q, bq^k) = \sum_{n=0}^{\infty}\frac{(q^{-k};q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(bq^k)^n
$$

and thus we have

$$\sum_{n=0}^{\infty}\frac{(q^{-k};q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(bq^k)^n=\sum_{n=0}^{k}\frac{(q^{-k};q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(bq^k)^n$$
$$+\sum_{n=k+1}^{\infty}\frac{(q^{-k};q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(bq^k)^n.$$

Since $(q^{-k};q)_n=0$ for $n=k+1,k+2\ldots$, the second sum zero , so we have

$$ {}_2\phi_1(q^{-k},c/b;c;q,bq^k)=\sum_{n=0}^{k}\frac{(q^{-k};q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(bq^k)^n=\frac{(b;q)_k}{(c;q)_k}\cdot \tag{2.34}$$

Setting this result in the series form of ${}_2\phi_1$, we find

$$\begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k}z^k \\ &= \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k\sum_{n=0}^{k}\frac{(q^{-k};q)_n(c/b;q)_n}{(q;q)_n(c;q)_n}(bq^k)^n. \end{aligned}$$

Now replacing k by $n+k$ and using the series relation

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k}A(n,k)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A(n,n+k)$$

we can write the final equation as

$$ {}_2\phi_1(a,b;c;q,z)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a;q)_{n+k}}{(q;q)_{n+k}}\frac{(q^{-(n+k)};q)_n(c/b;q)_n}{(q;q)_n(c;q)_n}(bq^{n+k})^n z^{n+k}.$$

Since

$$(q^{-(n+k)};q)_n=\frac{(q;q)_{n+k}}{(q;q)_k}(-1)^n q^{\binom{n}{2}-n(n+k)}$$

or

$$\frac{(q^{-(n+k)};q)_n}{(q;q)_{n+k}}=\frac{(-1)^n q^{\binom{n}{2}-n(n+k)}}{(q;q)_k}$$

and

$$(a;q)_{n+k}=(a;q)_n(aq^n;q)_k,$$

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a;q)_{n+k}(c/b;q)_n}{(q;q)_n(c;q)_n}\frac{(-1)^n q^{\binom{n}{2}-n(n+k)}}{(q;q)_k}(bq^{n+k})^n z^{n+k}\\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a;q)_{n+k}(c/b;q)_n}{(q;q)_k(c;q)_n(q;q)_n}(-bz)^n z^k q^{\binom{n}{2}}\\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(aq^n;q)_k(c/b;q)_n}{(q;q)_k(c;q)_n(q;q)_n}(-bz)^n z^k q^{\binom{n}{2}}\\
&= \sum_{n=0}^{\infty}\frac{(a;q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(-bz)^n q^{\binom{n}{2}}\sum_{k=0}^{\infty}\frac{(aq^n;q)_k}{(q;q)_k}z^k.
\end{aligned}
$$

By the q-binomial theorem, the inner sum is equal to $\frac{(aq^n z;q)_\infty}{(z;q)_\infty}\cdot$ Thus we have

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \sum_{n=0}^{\infty}\frac{(a;q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(-bz)^n q^{\binom{n}{2}}\frac{(aq^n z;q)_\infty}{(z;q)_\infty}\\
&= \sum_{n=0}^{\infty}\frac{(a;q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(-bz)^n q^{\binom{n}{2}}\frac{(aq^n z;q)_\infty}{(z;q)_\infty}\left[\frac{(az;q)_n}{(az;q)_n}\right]\\
&= \sum_{n=0}^{\infty}\frac{(a;q)_n(c/b;q)_n}{(q;q)_n(c;q)_n}(-bz)^n q^{\binom{n}{2}}\frac{(az;q)_\infty}{(az;q)_n(z;q)_\infty}\\
&= \frac{(az;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(c/b;q)_n}{(q;q)_n(c;q)_n(az;q)_n}(-bz)^n q^{\binom{n}{2}}.
\end{aligned}
$$

This is the required result. □

2.7 The Bailey-Daum summation formula

Theorem 2.12 *For $|q/b|<1$, we have*

$$
{}_2\phi_1(a,b;aq/b;q,-q/b) = \frac{(-q;q)_\infty(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}.
$$

Proof. Replacing a, b, c and z by b, a, aq/b and $-q/b$ in Heine's transformation formula and using the series representation of ${}_2\phi_1$, we have

$$
\begin{aligned}
{}_2\phi_1(b,a;aq/b;q,-q/b) &= \frac{(a;q)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}{}_2\phi_1(q/b,-q/b;-q;q,a)\\
&= \frac{(a;q)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/b;q)_n(-q/b;q)_n}{(-q;q)_n(q;q)_n}a^n. \quad (2.35)
\end{aligned}
$$

Now, we consider the sum

$$\sum_{n=0}^{\infty} \frac{(q/b;q)_n(-q/b;q)_n}{(-q;q)_n(q;q)_n} a^n.$$

By means of the following relations

$$(q^2/b^2;q^2)_n = (q/b;q)_n(-q/b;q)_n,$$

and

$$(q^2;q^2)_n = (q;q)_n(-q;q)_n,$$

we get

$$\sum_{n=0}^{\infty} \frac{(q/b;q)_n(-q/b;q)_n}{(-q;q)_n(q;q)_n} a^n = \sum_{n=0}^{\infty} \frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n} a^n. \tag{2.36}$$

In theorem (1.11), if we write q^2/b^2, q^2, a in place of a, q and z, respectively, then we obtain the left hand side of (2.36) as follows:

$$\sum_{n=0}^{\infty} \frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n} a^n = \frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty}. \tag{2.37}$$

Hence from (2.36) and (2.37)

$$\sum_{n=0}^{\infty} \frac{(q/b;q)_n(-q/b;q)_n}{(-q;q)_n(q;q)_n} a^n = \frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty}.$$

Substitution of this in (2.35) gives

$${}_2\phi_1(b,a;aq/b;q,-q/b) = \frac{(a;q)_\infty(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} \frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty}.$$

Considering

$$(a;q)_\infty = (aq;q^2)_\infty(a;q^2)_\infty,$$

we find

$$
\begin{aligned}
{}_2\phi_1(b, a; aq/b; q, -q/b) &= \frac{(a, q^2)_\infty (aq; q^2)_\infty (-q; q)_\infty}{(aq/b; q)_\infty (-q/b; q)_\infty} \frac{(aq^2/b^2; q^2)_\infty}{(a; q^2)_\infty} \\
&= \frac{(-q, q)_\infty (aq; q^2)_\infty (aq^2/b^2; q^2)_\infty}{(aq/b; q)_\infty (-q/b; q)_\infty}.
\end{aligned}
$$

Therefore the proof is completed. □

As a result of the Bailey-Daum summation formula we can give the following corollary.

Corollary 2.13

$$
\sum_{n=0}^{\infty} \frac{(a; q)_n q^{n(n+1)/2}}{(q; q)_n} = (aq; q^2)_\infty (-q; q)_\infty. \tag{2.38}
$$

Proof. Setting $b = B^{-1}$ in Bailey's formula and by using the definition q-hypergeometric series ${}_2\phi_1$, we obtain

$$
{}_2\phi_1(a, B^{-1}; aBq; q, -Bq) = \sum_{n=0}^{\infty} \frac{(a; q)_n (B^{-1}; q)_n}{(aBq; q)_n (q; q)_n} (-Bq)^n = \frac{(-q, q)_\infty (aq; q^2)_\infty (aB^2q^2; q^2)_\infty}{(aBq; q)_\infty (-Bq; q)_\infty}.
$$

By (2.28), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a; q)_n (B^{-1}; q)_n}{(aBq; q)_n (q; q)_n} (-Bq)^n &= \sum_{n=0}^{\infty} \frac{(a; q)_n [(-B^{-1})^n q^{\binom{n}{2}} (B; q^{-1})_n]}{(aBq; q)_n (q; q)_n} (-B)^n q^n \\
&= \sum_{n=0}^{\infty} \frac{(a; q)_n q^{n(n+1)/2} (B; q^{-1})_n}{(aBq; q)_n (q; q)_n}.
\end{aligned}
$$

Now, let $B \to 0$. So, one gets

$$
\sum_{n=0}^{\infty} \frac{(a; q)_n q^{n(n+1)/2} (0; q^{-1})_n}{(0; q)_n (q; q)_n} = \frac{(-q, q)_\infty (aq; q^2)_\infty (0; q^2)_\infty}{(0; q)_\infty (0; q)_\infty}.
$$

by using (2.29) this gives

$$
\sum_{n=0}^{\infty} \frac{(a; q)_n q^{n(n+1)/2}}{(q; q)_n} = (aq; q^2)_\infty (-q; q)_\infty.
$$

This proves the corollary. $\square$

2.8 The q-Pfaff-Saalschütz summation formula

Theorem 2.14 *For $|q| < 1$, we have*

$$
{}_3\phi_2(q^{-n}, a, b; c, abq^{1-n}/c; q, q) = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n}.
$$

Proof. Recall the q-Euler transformation formula,

$$
{}_2\phi_1(a, b; c; q, z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} {}_2\phi_1(c/a, c/b; c; q, abz/c)
$$

or

$$
\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/a;q)_k(c/b;q)_k}{(c;q)_k(q;q)_k} \left(\frac{abz}{c}\right)^k. \tag{2.39}
$$

In q-binomial theorem writing ab/c instead of a we have

$$
\sum_{n=0}^{\infty} \frac{(ab/c;q)_n}{(q;q)_n} z^n = \frac{(abz/c;q)_\infty}{(z;q)_\infty}.
$$

Setting this into the right side of (2.39) and by using the series manipulation

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n-k) \tag{2.40}
$$

one gets

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n &= \sum_{n=0}^{\infty} \frac{(ab/c;q)_n}{(q;q)_n} z^n \sum_{k=0}^{\infty} \frac{(c/a;q)_k(c/b;q)_k}{(c;q)_k(q;q)_k} \left(\frac{abz}{c}\right)^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(ab/c;q)_{n-k}}{(q;q)_{n-k}} z^{n-k} \frac{(c/a;q)_k(c/b;q)_k}{(c;q)_k(q;q)_k} \left(\frac{abz}{c}\right)^k \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{(c/a;q)_k(c/b;q)_k(ab/c;q)_{n-k}}{(c;q)_k(q;q)_k(q;q)_{n-k}} \left(\frac{ab}{c}\right)^k\right] z^n.
\end{aligned}
$$

Equating the coefficients of z^n from both members, we have

$$\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} = \sum_{k=0}^{n} \frac{(c/a;q)_k(c/b;q)_k(ab/c;q)_{n-k}}{(c;q)_k(q;q)_k(q;q)_{n-k}} \left(\frac{ab}{c}\right)^k. \tag{2.41}$$

Noting that

$$\begin{aligned}\frac{(ab/c;q)_{n-k}}{(q;q)_{n-k}}\left(\frac{ab}{c}\right)^k &= \frac{(ab/c;q)_n(q^{-n};q)_k q^k}{(q;q)_n(cq^{1-n}/ab;q)_k\left(\frac{ab}{c}\right)^k}\left(\frac{ab}{c}\right)^k \\ &= \frac{(ab/c;q)_n(q^{-n};q)_k}{(q;q)_n(cq^{1-n}/ab;q)_k}q^k,\end{aligned}$$

we can write (2.41) as

$$\begin{aligned}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} &= \sum_{k=0}^{n} \frac{(c/a;q)_k(c/b;q)_k}{(c;q)_k(q;q)_k}\frac{(ab/c;q)_n(q^{-n};q)_k}{(q;q)_n(cq^{1-n}/ab;q)_k}q^k \\ &= \frac{(ab/c;q)_n}{(q;q)_n}\sum_{k=0}^{n} \frac{(c/a;q)_k(c/b;q)_k(q^{-n};q)_k}{(c;q)_k(q;q)_k(cq^{1-n}/ab;q)_k}q^k\end{aligned}$$

or

$$\frac{(a;q)_n(b;q)_n}{(c;q)_n(ab/c;q)_n} = \sum_{k=0}^{n} \frac{(c/a;q)_k(c/b;q)_k(q^{-n};q)_k}{(c;q)_k(q;q)_k(cq^{1-n}/ab;q)_k}q^k.$$

In this equation replacing a and b by c/a and c/b, respectively, we obtain

$$\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} = \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k(q^{-n};q)_k}{(c;q)_k(q;q)_k(abq^{1-n}/c;q)_k}q^k$$

or by the definition of q-hypergeometric series

$$\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} = {}_3\phi_2(q^{-n},a,b;c,abq^{1-n}/c;q,q).$$

Thus completes the proof. $\square$

2.9 The terminating q-Dougall-Dixon formula

Theorem 2.15 *For* $|aq^{1+n}/bc| < 1$, *we have*

$$ {}_6\phi_5\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+n}; q, \frac{aq^{1+n}}{bc}\right) = \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}. $$

Proof. In the q-Pfaff-Saalschütz theorem, replacing a, b and c by aq^n, aq/bc and aq/b, respectively, we can write

$$ {}_3\phi_2(q^{-n}, aq^n, aq/bc; aq/b, aq/c; q, q) = \frac{(q^{1-n}/b;q)_n(c;q)_n}{(aq/b;q)_n(cq^{-n}/a;q)_n} \tag{2.42} $$

By using equation (1.6), we obtain

$$ (b^{-1}q^{1-n};q)_n = (b;q)_n(-1)^n(b)^{-n}q^{-\binom{n}{2}}, $$

$$ (cq^{-n}/a;q)_n = (aq/c;q)_n(-1)^n\left(\frac{aq}{c}\right)^{-n}q^{-\binom{n}{2}} $$

and so

$$ \frac{(b^{-1}q^{1-n};q)_n}{(cq^{-n}/a;q)_n} = \frac{(b;q)_n}{(aq/c;q)_n}\left(\frac{aq}{bc}\right)^n. \tag{2.43} $$

Putting these in (2.42), we have

$$ {}_3\phi_2(q^{-n}, aq^n, aq/bc; aq/b, aq/c; q, q) = \frac{(b;q)_n(c;q)_n}{(aq/b;q)_n(aq/c;q)_n}\left(\frac{aq}{bc}\right)^n. \tag{2.44} $$

By the definition of q-hypergeometric series the left hand side of (2.44) can be written as follows:

$$ {}_3\phi_2(q^{-n}, aq^n, aq/bc; aq/b, aq/c; q, q) = \sum_{k=0}^{\infty}\frac{(q^{-n};q)_k(aq^n;q)_k(aq/bc;q)_k}{(aq/b;q)_k(aq/c;q)_k(q;q)_k}q^k. \tag{2.45} $$

Using the fact that $(q^{-n};q)_k = 0$ for $k = n+1, n+2, ...$ equation (2.45) becomes

$$ {}_3\phi_2(q^{-n}, aq^n, aq/bc; aq/b, aq/c; q, q) = \sum_{k=0}^{n}\frac{(q^{-n};q)_k(aq^n;q)_k(aq/bc;q)_k}{(aq/b;q)_k(aq/c;q)_k(q;q)_k}q^k. \tag{2.46} $$

On the other hand, from equations (1.15) and (1.19), we have

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} q^{-\binom{n}{2}-k},$$

and

$$(aq^n;q)_k = \frac{(a;q)_k (aq^k;q)_n}{(a;q)_n}.$$

Substitution of these in the right side of (2.46) yields

$$\begin{aligned} {}_3\phi_2(q^{-n}, aq^n, aq/bc; aq/b, aq/c; q, q) = &\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} q^{-\binom{n}{2}} \frac{(a;q)_k (aq^k;q)_n}{(a;q)_n} \\ &\times \frac{(aq/bc;q)_k}{(aq/b;q)_k (aq/c;q)_k}. \end{aligned} \tag{2.47}$$

Thus from (2.44) and (2.47) we have

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} (aq^k;q)_n \frac{(a;q)_k (aq/bc;q)_k}{(aq/b;q)_k (aq/c;q)_k} \\ = q^{\binom{n}{2}} \frac{(a;q)_n (b;q)_n (c;q)_n}{(aq/b;q)_n (aq/c;q)_n} \left(\frac{aq}{bc}\right)^n. \end{aligned} \tag{2.48}$$

We now consider the Carlitz inversions (1.43) and (1.44). If we take $a_k = 1$ and $b_k = -aq^k$, then

$$\begin{aligned} \phi(x;n) &= \prod_{k=0}^{n-1} (a_k + xb_k) = \prod_{k=0}^{n-1} (1 - axq^k) = (ax;q)_n, \\ \phi(q^k;n) &= \prod_{k=0}^{n-1} (1 - aq^{2k}) = (aq^k;q)_n. \end{aligned}$$

So, (1.43) and (1.44) are given by

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} (aq^k;q)_n G(k) \tag{2.49}$$

and

$$G(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q \frac{1 - aq^{2k}}{(aq^n;q)_{k+1}} F(k), \tag{2.50}$$

respectively. When we choose

$$F(n) = q^{\binom{n}{2}} \frac{(a;q)_n (b;q)_n (c;q)_n}{(aq/b;q)_n (aq/c;q)_n} \left(\frac{aq}{bc}\right)^n$$

and

$$G(k) = \frac{(a;q)_k (aq/bc;q)_k}{(aq/b;q)_k (aq/c;q)_k}$$

equation (2.48) is in the form (2.49). So by (2.50), we can write

$$\frac{(a;q)_n (aq/bc;q)_n}{(aq/b;q)_n (aq/c;q)_n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q \frac{1-aq^{2k}}{(aq^n;q)_{k+1}} q^{\binom{k}{2}} \frac{(a;q)_k (b;q)_k (c;q)_k}{(aq/b;q)_k (aq/c;q)_k} \left(\frac{aq}{bc}\right)^k . \tag{2.51}$$

By using the fact that

$$\binom{n}{k}_q = (-1)^k q^{nk-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_k}$$

and

$$\frac{1-aq^{2k}}{1-a} = \frac{(\sqrt{a}q;q)_k (-\sqrt{a}q;q)_k}{(\sqrt{a};q)_k (-\sqrt{a};q)_k},$$

we can rewrite (2.51) as follows :

$$\begin{aligned} \frac{(a;q)_n (aq/bc;q)_n}{(aq/b;q)_n (aq/c;q)_n} = \sum_{k=0}^{n} & \frac{(1-a)}{(aq^n;q)_{k+1}} \frac{(\sqrt{a}q;q)_k (-\sqrt{a}q;q)_k}{(\sqrt{a};q)_k (-\sqrt{a};q)_k} \\ & \times \frac{(q^{-n};q)_k (a;q)_k (b;q)_k (c;q)_k}{(q;q)_k (aq/b;q)_k (aq/c;q)_k} \left(\frac{aq^{n+1}}{bc}\right)^k . \end{aligned} \tag{2.52}$$

Since

$$(aq^n;q)_{k+1} = (1-aq^n)(aq^{n+1};q)_k,$$

(2.52) gives

$$\begin{aligned} \frac{(1-aq^n)}{(1-a)} \frac{(a;q)_n (aq/bc;q)_n}{(aq/b;q)_n (aq/c;q)_n} = \sum_{k=0}^{n} & \frac{(a;q)_k (\sqrt{a}q;q)_k (-\sqrt{a}q;q)_k (b;q)_k}{(\sqrt{a};q)_k (-\sqrt{a};q)_k (aq/b;q)_k (aq/c;q)_k} \\ & \times \frac{(c;q)_k (q^{-n};q)_k}{(aq^{n+1};q)_k (q;q)_k} \left(\frac{aq^{n+1}}{bc}\right)^k . \end{aligned}$$

or

$$\frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n} = {}_6\phi_5\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1}; q, \frac{aq^{n+1}}{bc}\right),$$

which is the desired result. □

2.10 The Sears balanced transformation formulas

Theorem 2.16 *Let $|q| < 1$. Then, we have*

$$\begin{aligned}
&{}_4\phi_3(q^{-n}, a, c, e; b, d, aceq^{1-n}/bd; q, q) \\
&= {}_4\phi_3(q^{-n}, a, b/c, b/e; b, bd/ce, aq^{1-n}/d; q, q) \times \frac{(d/a;q)_n(bd/ce;q)_n}{(d;q)_n(bd/ace;q)_n}
\end{aligned} \tag{2.53}$$

and

$$\begin{aligned}
&{}_4\phi_3(q^{-n}, a, c, e; b, d, aceq^{1-n}/bd; q, q) \\
&= {}_4\phi_3(q^{-n}, b/c, d/c, bd/ace; bd/ac, bd/ce, q^{1-n}/c; q, q) \times \frac{(c;q)_n(bd/ac;q)_n(bd/ce;q)_n}{(b;q)_n(d;q)_n(bd/ace;q)_n}.
\end{aligned} \tag{2.54}$$

Proof.

Equation (2.54) is a result of (2.53). So we assume that (2.53) holds. Replacing the parameters c, e, b, d by b/c, b/e, b, bd/ce, respectively, in (2.53) and then using symmetric property, we have

$$\begin{aligned}
&{}_4\phi_3(q^{-n}, a, b/c, b/e; b, bd/ce, aq^{1-n}/d; q, q) \\
&\quad = {}_4\phi_3(q^{-n}, b/c, a, b/e; bd/ce, b, aq^{1-n}/d; q, q) \\
&\quad = {}_4\phi_3(q^{-n}, b/c, bd/ace, d/c; bd/ce, bd/ac, q^{1-n}/c; q, q) \times \frac{(c;q)_n(bd/ac;q)_n}{(b;q)_n(d/a;q)_n}.
\end{aligned}$$

or

$$
\begin{aligned}
&{}_4\phi_3(q^{-n}, a, b/c, b/e; b, bd/ce, aq^{1-n}/d; q, q) \\
&= {}_4\phi_3(q^{-n}, b/c, d/c, bd/ace; bd/ce, bd/ac, q^{1-n}/c; q, q) \times \frac{(c;q)_n(bd/ac;q)_n}{(b;q)_n(d/a;q)_n}.
\end{aligned}
$$

Setting this result in the right side of (2.53), we find

$$
\begin{aligned}
&{}_4\phi_3(q^{-n}, a, c, e; b, d, aceq^{1-n}/bd; q, q) \\
&= {}_4\phi_3(q^{-n}, b/c, d/c, bd/ace; bd/ce, bd/ac, q^{1-n}/c; q, q) \\
&\quad \times \frac{(c;q)_n(bd/ac;q)_n}{(b;q)_n(d/a;q)_n}\frac{(d/a;q)_n(bd/ce;q)_n}{(d;q)_n(bd/ace;q)_n} \\
&= {}_4\phi_3(q^{-n}, b/c, d/c, bd/ace; bd/ce, bd/ac, q^{1-n}/c; q, q) \\
&\quad \times \frac{(c;q)_n(bd/ac;q)_n(bd/ce;q)_n}{(b;q)_n(d;q)_n(bd/ace;q)_n},
\end{aligned}
$$

which is the desired result. □

Replacing the base q by its inverse q^{-1} and then using the fact that

$$
\binom{n}{k}_{q^{-1}} = \binom{n}{k}_q q^{\binom{k}{2}-\binom{n}{2}+\binom{n-k}{2}}
$$

we can write the Carlitz inversions (1.43) and (1.44) as

$$
F(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}_q \phi(q^{-k}; n)G(k) \qquad n = 0, 1, 2, \ldots \tag{2.55}
$$

and

$$
G(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{n-k}{2}}\frac{a_k + q^{-k}b_k}{\phi(q^{-n}; k+1)}F(k) \qquad n = 0, 1, 2, \ldots \quad , \tag{2.56}
$$

respectively, with

$$
\phi(x; 0) = 1 \quad , \quad \phi(x, n) = \prod_{k=0}^{n-1}(a_k + xb_k) \qquad n = 1, 2, \ldots \quad .
$$

If we choose $a_k = 1$, $b_k = -aceq^k/bd$, then we have

$$\phi(x,n) = \prod_{k=0}^{n-1}\left(1 - xaceq^k/bd\right) = (acex/bd;q)_n$$

and so

$$\phi(q^{-k},n) = (aceq^{-k}/bd;q)_n,$$

$$\phi(q^{-n},k+1) = (aceq^{-n}/bd;q)_{k+1},$$

$$\frac{a_k + q^{-k}b_k}{\phi(q^{-n};k+1)} = \frac{1 - ace/bd}{(aceq^{-n}/bd;q)_{k+1}}.$$

In this case, the corresponding inversions (2.55) and (2.56) take the form

$$F(n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q (aceq^{-k}/bd;q)_n G(k) \tag{2.57}$$

and

$$G(n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}} \frac{1 - ace/bd}{(aceq^{-n}/bd;q)_{k+1}} F(k), \tag{2.58}$$

respectively.

Now we can prove (2.53). By the definition of q-hypergeometric series and by means of the relations

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k \binom{n}{k}_q q^{\binom{n-k}{2}-\binom{n}{2}-k}$$

and

$$(aceq^{1-n}/bd;q)_k = \frac{(aceq^{-n}/bd;q)_{k+1}}{1 - aceq^{-n}/bd}$$

we can write

$$\begin{aligned}
&{}_4\phi_3(q^{-n},a,c,e;b,d,aceq^{1-n}/bd;q,q)\left[\frac{1-bd/ace}{1-bdq^n/ace}q^{\binom{n+1}{2}}\right] \\
&= \sum_{k=0}^{n}\frac{(q^{-n};q)_k(a;q)_k(c;q)_k(e;q)_k}{(b;q)_k(d;q)_k(aceq^{1-n}/bd;q)_k(q;q)_k}q^k\left[\frac{1-bd/ace}{1-bdq^n/ace}q^{\binom{n+1}{2}}\right].
\end{aligned}$$

$$
\begin{aligned}
&{}_4\phi_3(q^{-n},a,c,e;b,d,aceq^{1-n}/bd;q,q)\left[\frac{1-bd/ace}{1-bdq^n/ace}q^{\binom{n+1}{2}}\right]\\
&\quad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{n-k}{2}-\binom{n}{2}+\binom{n+1}{2}}\frac{(a;q)_k(c;q)_k(e;q)_k}{(b;q)_k(d;q)_k(aceq^{-n}/bd;q)_{k+1}}\\
&\quad\times(1-aceq^{-n}/bd)\left(\frac{1-bd/ace}{1-bdq^n/ace}\right). \qquad (2.59)
\end{aligned}
$$

Noting that

$$\binom{n+1}{2}-\binom{n}{2}=n$$

and

$$(1-aceq^{-n}/bd)\left(\frac{1-bd/ace}{1-bdq^n/ace}\right)q^n=1-ace/bd$$

we can write equation (2.59) as follows:

$$
\begin{aligned}
&{}_4\phi_3(q^{-n},a,c,e;b,d,aceq^{1-n}/bd;q,q)\left[\frac{1-bd/ace}{1-bdq^n/ace}q^{\binom{n+1}{2}}\right]\\
&\quad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{n-k}{2}}\frac{(a;q)_k(c;q)_k(e;q)_k}{(b;q)_k(d;q)_k(aceq^{-n}/bd;q)_{k+1}}(1-ace/bd).
\end{aligned}
$$

Thus, (2.53) can be stated equivalently as

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{n-k}{2}}\frac{(a;q)_k(c;q)_k(e;q)_k}{(b;q)_k(d;q)_k(aceq^{-n}ace/bd;q)_{k+1}}(1-ace/bd)\\
&\quad={}_4\phi_3(q^{-n},a,b/c,b/e;b,bd/ce,aq^{1-n}/d;q,q)\frac{(d/a;q)_n(bd/ce;q)_n}{(d;q)_n(bd/ace;q)_n}\\
&\quad\times\left[\frac{1-bd/ace}{1-bdq^n/ace}q^{\binom{n+1}{2}}\right]. \qquad (2.60)
\end{aligned}
$$

Since

$$\frac{1}{(bd/ace;q)_n}\left[\frac{1-bd/ace}{1-bdq^n/ace}\right]=\frac{1}{(bdq/ace;q)_n},$$

we can rewrite (2.60) in the form

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{n-k}{2}}\frac{(a;q)_k(c;q)_k(e;q)_k}{(b;q)_k(d;q)_k(aceq^{-n}/bd;q)_{k+1}}(1-ace/bd)\\
&\quad={}_4\phi_3(q^{-n},a,b/c,b/e;b,bd/ce,aq^{1-n}/d;q,q)\frac{(d/a;q)_n(bd/ce;q)_n}{(d;q)_n(bdq/ace;q)_n}q^{\binom{n+1}{2}}. \qquad (2.61)
\end{aligned}
$$

This expression is in the form of (2.58) with

$$F(k) = \frac{(a;q)_k(c;q)_k(e;q)_k}{(b;q)_k(d;q)_k}$$

and

$$G(n) = {}_4\phi_3(q^{-n}, a, b/c, b/e; b, bd/ce, aq^{1-n}/d; q, q)\frac{(d/a;q)_n(bd/ce;q)_n}{(d;q)_n(bdq/ace;q)_n}q^{\binom{n+1}{2}}.$$

Thus, to prove (2.53) it is sufficient to show that the following relation, corresponding to the relation (2.57),

$$\begin{aligned}\frac{(a;q)_n(c;q)_n(e;q)_n}{(b;q)_n(d;q)_n} &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}_q (aceq^{-k}/bd;q)_n\frac{(d/a;q)_k(bd/ce;q)_k}{(d;q)_k(bdq/ace;q)_k}q^{\binom{k+1}{2}}\\ &\times {}_4\phi_3(q^{-k}, a, b/c, b/e; b, bd/ce, aq^{1-k}/d; q, q). \qquad (2.62)\end{aligned}$$

holds.

Recalling the definition of q-hypergeometric series and noting that $(q^{-k};q)_i = 0$ for $i = k+1, k+2, ...$, we can write the series ${}_4\phi_3$ in the right side of (2.62),

$${}_4\phi_3(q^{-k}, a, b/c, b/e; b, bd/ce, aq^{1-k}/d; q, q) = \sum_{i=0}^{k}\frac{(q^{-k};q)_i(a;q)_i(b/c;q)_i(b/e;q)_i}{(b;q)_i(bd/ce;q)_i(aq^{1-k}/d;q)_i(q;q)_i}q^i.$$

Putting this in (2.62) we have

$$\begin{aligned}\frac{(a;q)_n(c;q)_n(e;q)_n}{(b;q)_n(d;q)_n} &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}_q (aceq^{-k}/bd;q)_n\frac{(d/a;q)_k(bd/ce;q)_k}{(d;q)_k(bdq/ace;q)_k}q^{\binom{k+1}{2}}\\ &\times \sum_{i=0}^{k}\frac{(q^{-k};q)_i(a;q)_i(b/c;q)_i(b/e;q)_i}{(b;q)_i(bd/ce;q)_i(aq^{1-k}/d;q)_i(q;q)_i}q^i. \qquad (2.63)\end{aligned}$$

Now, let T denote the right side of (2.63). Since

$$\binom{n}{k}_q = (-1)^k\frac{(q^{-n};q)_k}{(q;q)_k}q^{nk-\binom{k}{2}}$$

and

$$q^{nk-\binom{k}{2}+\binom{k+1}{2}} = q^{k(n+1)},$$

we get

$$\begin{aligned}
T &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} q^{k(n+1)} (aceq^{-k}/bd;q)_n \frac{(d/a;q)_k (bd/ce;q)_k}{(d;q)_k (bdq/ace;q)_k} \\
&\times \sum_{i=0}^{k} \frac{(q^{-k};q)_i (a;q)_i (b/c;q)_i (b/e;q)_i}{(b;q)_i (bd/ce;q)_i (aq^{1-k}/d;q)_i (q;q)_i} q^i \\
&= \sum_{i=0}^{n} \frac{(a;q)_i (b/c;q)_i (b/e;q)_i}{(b;q)_i (bd/ce;q)_i (q;q)_i} q^i \sum_{k=i}^{n} \frac{(q^{-n};q)_k (d/a;q)_k (bd/ce;q)_k}{(d;q)_k (bdq/ace;q)_k (q;q)_k} \\
&\times (aceq^{-k}/bd;q)_n \frac{(q^{-k};q)_i}{(aq^{1-k}/d;q)_i} q^{k(n+1)},
\end{aligned}$$

We now consider the inner sum. Replacing k by $i+j$, one gets

$$\begin{aligned}
&\sum_{k=i}^{n} \frac{(q^{-n};q)_k (d/a;q)_k (bd/ce;q)_k}{(d;q)_k (bdq/ace;q)_k (q;q)_k} \frac{(aceq^{-k}/bd;q)_n (q^{-k};q)_i}{(aq^{1-k}/d;q)_i} q^{k(n+1)} \\
&= \sum_{j=0}^{n-i} \frac{(q^{-n};q)_{i+j} (d/a;q)_{i+j} (bd/ce;q)_{i+j}}{(d;q)_{i+j} (bdq/ace;q)_{i+j} (q;q)_{i+j}} \frac{(aceq^{-i-j}/bd;q)_n (q^{-i-j};q)_i}{(aq^{1-i-j}/d;q)_i} q^{(i+j)(n+1)}.
\end{aligned}$$

By using the relations

$$\begin{aligned}
\frac{(q^{-i-j};q)_i}{(aq^{1-i-j}/d;q)_i} &= \left(\frac{d}{aq}\right)^i \frac{(q^{1+j};q)_i}{(dq^j/a;q)_i} = \left(\frac{d}{aq}\right)^i \frac{(q;q)_{i+j}}{(d/a;q)_{i+j}} \frac{(d/a;q)_j}{(q;q)_j}, \\
(aceq^{-i-j}/bd;q)_n &= \frac{(aceq^{-i-j}/bd;q)_{i+j}}{(aceq^{n-i-j}/bd;q)_{i+j}} (ace/bd;q)_n \\
&= \frac{(bdq/ace;q)_{i+j}}{(bdq^{1-n}/ace;q)_i} \frac{(ace/bd;q)_n}{(bdq^{1+i-n}/ace;q)_j} q^{-n(i+j)},
\end{aligned}$$

and

$$(q^{-n};q)_{i+j} = (q^{-n};q)_i (q^{i-n};q)_j,$$

$$(bd/ce;q)_{i+j} = (bd/ce;q)_i (bdq^i/ce;q)_j,$$

$$(d;q)_{i+j} = (d;q)_i (dq^i;q)_j.$$

the inner sum can be written as

$$
\begin{aligned}
&\sum_{k=i}^{n}\frac{(q^{-n};q)_k(d/a;q)_k(bd/ce;q)_k}{(d;q)_k(bdq/ace;q)_k(q;q)_k}\frac{(aceq^{-k}/bd;q)_n(q^{-k};q)_i}{(aq^{1-k}/d)_i}q^{k(n+1)}\\
&=\sum_{j=0}^{n-i}\frac{(q^{-n};q)_{i+j}(d/a;q)_{i+j}(bd/ce;q)_{i+j}}{(d;q)_{i+j}(bdq/ace;q)_{i+j}(q;q)_{i+j}}\frac{(bdq/ace;q)_{i+j}}{(bdq^{1-n}/ace;q)_i}\frac{(ace/bd;q)_n}{(bdq^{1+i-n}/ace;q)_j}\\
&\times q^{-n(i+j)}q^{(i+j)(n+1)}\left(\frac{d}{aq}\right)^i\frac{(q;q)_{i+j}}{(d/a;q)_{i+j}}\frac{(d/a;q)_j}{(q;q)_j}\\
&=(ace/bd;q)_n\frac{(q^{-n};q)_i(bd/ce;q)_i}{(d;q)_i(bdq^{1-n}/ace;q)_i}\left(\frac{d}{a}\right)^i\sum_{j=0}^{n-i}\frac{(q^{i-n};q)_j(d/a;q)_j(bdq^i/ce;q)_j}{(q;q)_j(dq^i;q)_j(bdq^{1+i-n}/ace;q)_j}q^j\\
&=(ace/bd;q)_n\frac{(q^{-n};q)_i(bd/ce;q)_i}{(d;q)_i(bdq^{1-n}/ace;q)_i}\left(\frac{d}{a}\right)^i\\
&\times\quad {}_3\phi_2(q^{i-n},d/a,bdq^i/ce;dq^i,bdq^{1+i-n}/ace;q,q).
\end{aligned}
\tag{2.64}
$$

Now, in the q-Pfaff-Saalschütz formula, writing $n-i$, d/a, bdq^i/ce and dq^i instead of n, a, b and c, respectively, we can get

$$
{}_3\phi_2(q^{i-n},d/a,bdq^i/ce;dq^i,bdq^{1+i-n}/ace;q,q)=\frac{(aq^i;q)_{n-i}(ce/b;q)_{n-i}}{(dq^i;q)_{n-i}(ace/bd;q)_{n-i}}\cdot \tag{2.65}
$$

Since

$$
\frac{(ce/b;q)_{n-i}}{(ace/bd;q)_{n-i}}=\frac{(ce/b;q)_n(bdq^{1-n}/ace;q)_i}{(ace/bd;q)_n(bq^{1-n}/ce;q)_i}\left(\frac{a}{d}\right)^i,
$$

(2.65) becomes

$$
{}_3\phi_2(q^{i-n},d/a,bdq^i/ce;dq^i,bdq^{1+i-n}/ace;q,q)=\frac{(aq^i;q)_{n-i}}{(dq^i;q)_{n-i}}\frac{(ce/b;q)_n(bdq^{1-n}/ace;q)_i}{(ace/bd;q)_n(bq^{1-n}/ce;q)_i}\left(\frac{a}{d}\right)^i.
$$

and thus setting this result into (2.64) we have

$$
\begin{aligned}
&\sum_{k=i}^{n}\frac{(q^{-n};q)_k(d/a;q)_k(bd/ce;q)_k}{(d;q)_k(bdq/ace;q)_k(q;q)_k}\frac{(aceq^{-k}/bd;q)_n(q^{-k};q)_i}{(aq^{1-k}/d)_i}q^{k(n+1)}\\
&=(ace/bd;q)_n\frac{(q^{-n};q)_i(bd/ce;q)_i}{(d;q)_i(bdq^{1-n}/ace;q)_i}\left(\frac{d}{a}\right)^i\\
&\times\frac{(aq^i;q)_{n-i}(ce/b;q)_n(bdq^{1-n}/ace;q)_i}{(dq^i;q)_{n-i}(ace/bd;q)_n(bq^{1-n}/ce;q)_i}\left(\frac{a}{d}\right)^i.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
T &= \sum_{i=0}^{n} \frac{(a;q)_i(b/c;q)_i(b/e;q)_i}{(b;q)_i(bd/ce;q)_i(q;q)_i}\left(\frac{dq}{a}\right)^i \frac{(q^{-n};q)_i(bd/ce;q)_i}{(d;q)_i(bdq^{1-n}/ace;q)_i} \\
&\quad \times (ace/bd;q)_n \frac{(aq^i;q)_{n-i}(ce/b;q)_{n-i}}{(dq^i;q)_{n-i}(ace/bd;q)_{n-i}} \\
&= \sum_{i=0}^{n} \frac{(a;q)_i(b/c;q)_i(b/e;q)_i}{(b;q)_i(bd/ce;q)_i(q;q)_i}\left(\frac{dq}{a}\right)^i \frac{(q^{-n};q)_i(bd/ce;q)_i}{(d;q)_i(bdq^{1-n}/ace;q)_i} \\
&\quad \times (ace/bd;q)_n \frac{(aq^i;q)_{n-i}}{(dq^i;q)_{n-i}} \frac{(ce/b;q)_n(bdq^{1-n}/ace;q)_i}{(ace/bd;q)_n(bq^{1-n}/ce;q)_i}\left(\frac{a}{d}\right)^i.
\end{aligned}$$

Considering that

$$(d;q)_n = (d;q)_i(dq^i;q)_{n-i}$$

and

$$(a;q)_n = (a;q)_i(aq^i;q)_{n-i}$$

we can write

$$\begin{aligned}
T &= (ce/b;q)_n \frac{(a;q)_n}{(d;q)_n} \sum_{i=0}^{n} \frac{(q^{-n};q)_i(b/c;q)_i(b/e;q)_i}{(b;q)_i(bq^{1-n}/ce;q)_i(q;q)_i} q^i \\
&= (ce/b;q)_n \frac{(a;q)_n}{(d;q)_n} {}_3\phi_2(q^{-n}, b/c, b/e; b, bq^{1-n}/ce; q, q). \qquad (2.66)
\end{aligned}$$

In the q- Pfaff-Saalschütz formula again replacing a, b and c by b/c, b/e and b, respectively we find

$${}_3\phi_2(q^{-n}, b/c, b/e; b, bq^{1-n}/ce; q, q) = \frac{(c;q)_n(e;q)_n}{(b;q)_n(ce/b;q)_n}.$$

Finally, substituting this result in (2.66), we obtain

$$T = (ce/b;q)_n \frac{(a;q)_n}{(d;q)_n} \frac{(c;q)_n(e;q)_n}{(b;q)_n(ce/b;q)_n} = \frac{(a;q)_n(c;q)_n(e;q)_n}{(d;q)_n(b;q)_n}$$

This completes the proof. □

2.11 The Watson transformation formula

Theorem 2.17 *Let $|a^2q^{2+n}/bcde| < 1$. Then we have*

$$
{}_8\phi_7\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, \frac{a^2q^{2+n}}{bcde}\right)
$$
$$
= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}{}_4\phi_3(q^{-n}, b, c, aq/de; aq/d, aq/e, bcq^{-n}/a; q, q). \tag{2.67}
$$

Proof. By using the definition of basic hypergeometric series and noting that $(q^{-n};q)_k = 0$ for $k = n+1, \ldots$ the left side of (2.67) can be written in the form

$$
{}_8\phi_7\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, \frac{a^2q^{2+n}}{bcde}\right)
$$
$$
= \sum_{k=0}^{n} \frac{(a;q)_k(\sqrt{a}q;q)_k(-\sqrt{a}q;q)_k(b;q)_k(c;q)_k(q^{-n};q)_k}{(\sqrt{a};q)_k(-\sqrt{a};q)_k(aq/b;q)_k(aq/c;q)_k(aq^{n+1};q)_k(q;q)_k}\left(\frac{aq^{1+n}}{bc}\right)^k
$$
$$
\times \frac{(d;q)_k(e;q)_k}{(aq/d;q)_k(aq/e;q)_k}\left(\frac{aq}{de}\right)^k. \tag{2.68}
$$

In (2.44), setting $n = k$, $b = d$ and $c = e$, we have

$$
\begin{aligned}
\frac{(d;q)_k(e;q)_k}{(aq/d;q)_k(aq/e;q)_k}\left(\frac{aq}{de}\right)^k &= {}_3\phi_2(q^{-k}, aq^k, aq/de; aq/d, aq/e; q, q) \\
&= \sum_{i=0}^{k} \frac{(q^{-k};q)_i(aq^k;q)_i(aq/de;q)_i}{(q;q)_i(aq/d;q)_i(aq/e;q)_i} q^i.
\end{aligned}
$$

Substituting this result in (2.68), we get

$$
{}_8\phi_7\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, \frac{a^2q^{2+n}}{bcde}\right)
$$
$$
= \sum_{k=0}^{n} \frac{(\sqrt{a}q;q)_k(-\sqrt{a}q;q)_k}{(\sqrt{a};q)_k(-\sqrt{a};q)_k} \frac{(b;q)_k(c;q)_k(q^{-n};q)_k(a;q)_k}{(aq/b;q)_k(aq/c;q)_k(aq^{n+1};q)_k(q;q)_k}\left(\frac{aq^{1+n}}{bc}\right)^k
$$
$$
\times \sum_{i=0}^{k} \frac{(q^{-k};q)_i(aq^k;q)_i(aq/de;q)_i}{(q;q)_i(aq/d;q)_i(aq/e;q)_i} q^i. \tag{2.69}
$$

Now using the relations

$$
\frac{(\sqrt{a}q;q)_k(-\sqrt{a}q;q)_k}{(\sqrt{a};q)_k(-\sqrt{a};q)_k} = \frac{1-aq^{2k}}{1-a},
$$

$$(a;q)_{k+i} = (a;q)_k(aq^k;q)_i \tag{2.70}$$

the final equation takes the form

$$\begin{aligned}
&{}_8\phi_7\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, \frac{a^2q^{2+n}}{bcde}\right)\\
&= \sum_{i=0}^{n} \frac{(aq/de;q)_i}{(q;q)_i(aq/d;q)_i(aq/e;q)_i} q^i \sum_{k=i}^{n} \frac{1-aq^{2k}}{1-a} \frac{(b;q)_k(c;q)_k(q^{-n};q)_k}{(aq/b;q)_k(aq/c;q)_k(aq^{n+1};q)_k}\\
&\times \frac{(a;q)_{k+i}(q^{-k};q)_i}{(q;q)_k}\left(\frac{aq^{1+n}}{bc}\right)^k. \qquad (2.71)
\end{aligned}$$

Let Ω denote the inner sum with respect to k. Putting k-i=j and noting that

$$\frac{(q^{-i-j};q)_i}{(q;q)_{i+j}} = \frac{(-1)^i q^{-\binom{i+1}{2}-ij}}{(q;q)_j}$$

and again using (2.70), we have

$$\begin{aligned}
\Omega &= \sum_{j=0}^{n-i} \frac{1-aq^{2i+2j}}{1-a} \frac{(b;q)_{i+j}(c;q)_{i+j}(q^{-n};q)_{i+j}}{(aq/b;q)_{i+j}(aq/c;q)_{i+j}(aq^{n+1};q)_{i+j}} \frac{(a;q)_{2i+j}(q^{-i-j};q)_i}{(q;q)_{i+j}}\left(\frac{aq^{1+n}}{bc}\right)^{i+j}\\
&= \sum_{j=0}^{n-i} \frac{1-aq^{2i+2j}}{1-a} \frac{(b;q)_i(bq^i;q)_j(c;q)_i(cq^i;q)_j(q^{-n};q)_i(q^{-n+i};q)_j}{(aq/b;q)_i(aq^{1+i}/b;q)_j(aq/c;q)_i(aq^{1+i}/c;q)_j(aq^{n+1};q)_i(aq^{n+1+i};q)_j}\\
&\times (a;q)_{2i}(aq^{2i};q)_j \frac{(-1)^i q^{-\binom{i+1}{2}-ij}}{(q;q)_j}\left(\frac{aq^{1+n}}{bc}\right)^i\left(\frac{aq^{1+n}}{bc}\right)^j.
\end{aligned}$$

Multiplying and dividing the left side of the last equation by $(1-aq^{2i})$

$$\begin{aligned}
\Omega &= (-1)^i q^{-\binom{i+1}{2}}(a;q)_{2i}\frac{1-aq^{2i}}{1-a} \frac{(b;q)_i(c;q)_i(q^{-n};q)_i}{(aq/b;q)_i(aq/c;q)_i(aq^{n+1};q)_i}\left(\frac{aq^{1+n}}{bc}\right)^i\\
&\times \sum_{j=0}^{n-i} \frac{1-aq^{2i+2j}}{1-aq^{2i}} \frac{(aq^{2i};q)_j(bq^i;q)_j(cq^i;q)_j(q^{-n+i};q)_j}{(q;q)_j(aq^{1+i}/b;q)_j(aq^{1+i}/c;q)_j(aq^{n+1+i};q)_j}\left(\frac{aq^{1+n-i}}{bc}\right)^j.
\end{aligned}$$

In the view of the relations

$$(a;q)_{2i}\frac{1-aq^{2i}}{1-a} = (aq;q)_{2i},$$

$$\frac{(\sqrt{a}q^{i+1};q)_j(-\sqrt{a}q^{i+1};q)_j}{(\sqrt{a}q^{i};q)_j(-\sqrt{a}q^{i};q)_j}=\frac{1-aq^{2i+2j}}{1-aq^{2i}}$$

and the definition of basic hypergeometric series we can write

$$\begin{aligned}
\Omega &= (-1)^i q^{-\binom{i+1}{2}}(aq;q)_{2i}\frac{(b;q)_i(c;q)_i(q^{-n};q)_i}{(aq/b;q)_i(aq/c;q)_i(aq^{n+1};q)_i}\left(\frac{aq^{1+n}}{bc}\right)^i\\
&\times\sum_{j=0}^{n-i}\frac{(\sqrt{a}q^{i+1};q)_j(-\sqrt{a}q^{i+1};q)_j}{(\sqrt{a}q^{i};q)_j(-\sqrt{a}q^{i};q)_j}\frac{(aq^{2i};q)_j(bq^i;q)_j(cq^i;q)_j(q^{-n+i};q)_j}{(q;q)_j(aq^{1+i}/b;q)_j(aq^{1+i}/c;q)_j(aq^{n+1+i};q)_j}\\
&\times\left(\frac{aq^{1+n-i}}{bc}\right)^j\\
&= (-1)^i q^{-\binom{i+1}{2}}(aq;q)_{2i}\frac{(b;q)_i(c;q)_i(q^{-n};q)_i}{(aq/b;q)_i(aq/c;q)_i(aq^{n+1};q)_i}\left(\frac{aq^{1+n}}{bc}\right)^i\\
&\times {}_6\phi_5(aq^{2i},\sqrt{a}q^{1+i},-\sqrt{a}q^{1+i},bq^i,cq^i,q^{-n+i}\\
&\qquad ;\sqrt{a}q^i,-\sqrt{a}q^i,aq^{1+i}/b,aq^{1+i}/c,aq^{1+n+i};q,aq^{1+n-i}/bc) \quad (2.72)
\end{aligned}$$

Now, in q-Dougall-Dixon formula, replacing a, b, c and n by aq^{2i}, bq^i, cq^i and $n-i$, respectively, one gets

$$\begin{aligned}
&{}_6\phi_5(aq^{2i},\sqrt{a}q^{1+i},-\sqrt{a}q^{1+i},bq^i,cq^i,q^{-n+i}\\
&\qquad ;\sqrt{a}q^i,-\sqrt{a}q^i,aq^{1+i}/b,aq^{1+i}/c,aq^{1+n+i};q,aq^{1+n-i}/bc)\\
&= \frac{(aq^{1+2i};q)_{n-i}(aq/bc;q)_{n-i}}{(aq^{1+i}/b;q)_{n-i}(aq^{1+i}/c;q)_{n-i}}.
\end{aligned}$$

Substitution of this (2.72) gives

$$\begin{aligned}
\Omega &= (-1)^i q^{-\binom{i+1}{2}}(aq;q)_{2i}\frac{(b;q)_i(c;q)_i(q^{-n};q)_i}{(aq/b;q)_i(aq/c;q)_i(aq^{n+1};q)_i}\left(\frac{aq^{1+n}}{bc}\right)^i\\
&\times\frac{(aq^{1+2i};q)_{n-i}(aq/bc;q)_{n-i}}{(aq^{1+i}/b;q)_{n-i}(aq^{1+i}/c;q)_{n-i}}.
\end{aligned}$$

Thus, if we set this result into (2.71), we obtain

$$
\begin{aligned}
&{}_8\phi_7\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, \frac{a^2q^{2+n}}{bcde}\right) \\
&= \sum_{i=0}^{n} \frac{(aq/de;q)_i}{(q;q)_i(aq/d;q)_i(aq/e;q)_i} q^i \times \frac{(aq^{1+2i};q)_{n-i}(aq/bc;q)_{n-i}}{(aq^{1+i}/b;q)_{n-i}(aq^{1+i}/c;q)_{n-i}} \\
&\times (-1)^i q^{-\binom{i+1}{2}} (aq;q)_{2i} \frac{(b;q)_i(c;q)_i(q^{-n};q)_i}{(aq/b;q)_i(aq/c;q)_i(aq^{1+n};q)_i} \left(\frac{aq^{1+n}}{bc}\right)^i.
\end{aligned}
$$

Finally by means of the following equations

$$
\begin{aligned}
(aq/bc;q)_{n-i} &= (-1)^i q^{\binom{i}{2}-ni} \left(\frac{bc}{a}\right)^i \frac{(aq/bc;q)_n}{(bcq^{-n}/a;q)_i} \\
(aq;q)_{2i} \frac{(aq^{1+2i};q)_{n-i}}{(aq^{1+n};q)_i} &= (aq;q)_n \\
(aq/b;q)_i(aq^{1+i}/b;q)_{n-i} &= (aq/b;q)_n \\
(aq/c;q)_i(aq^{1+i}/c;q)_{n-i} &= (aq/c;q)_n
\end{aligned}
$$

and the definition of ${}_4\phi_3$, we have

$$
\begin{aligned}
&{}_8\phi_7\left(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, \frac{a^2q^{2+n}}{bcde}\right) \\
&= \sum_{i=0}^{n} \frac{(aq/de;q)_i}{(q;q)_i(aq/d;q)_i(aq/e;q)_i} q^i (aq;q)_n (-1)^i q^{\binom{i}{2}-ni} \left(\frac{bc}{a}\right)^i \frac{(aq/bc;q)_n}{(bcq^{-n}/a;q)_i} \\
&\times (-1)^i q^{-\binom{i+1}{2}} \frac{(b;q)_i(c;q)_i(q^{-n};q)_i}{(aq/b;q)_i(aq/c;q)_i(aq^{1+i}/b;q)_{n-i}(aq^{1+i}/c;q)_{n-i}} \left(\frac{aq^{1+n}}{bc}\right)^i \\
&= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n} \sum_{i=0}^{n} \frac{(q^{-n};q)_i(aq/de;q)_i(b;q)_i(c;q)_i}{(q;q)_i(aq/d;q)_i(aq/e;q)_i(bcq^{-n}/a;q)_i} q^i \\
&= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n} {}_4\phi_3(q^{-n}, b, c, aq/de; aq/d, aq/e, bcq^{-n}/a; q, q),
\end{aligned}
$$

which is the required result. □

2.12 Jackson's q-Dougall-Dixon formula

Theorem 2.18 *Let $a^2q^{1+n} = bcde$. Then we have*

$$
\begin{aligned}
&{}_8\phi_7(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, q) \\
&= \frac{(aq;q)_n(aq/bc;q)_n(aq/bd;q)_n(aq/cd;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/bcd;q)_n}. \qquad (2.73)
\end{aligned}
$$

Proof. For $a^2q^{1+n} = bcde$ or $aq/de = bcq^{-n}/a$, the Watson transformation formula becomes

$$
\begin{aligned}
&{}_8\phi_7(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, q) \\
&= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n} {}_4\phi_3(q^{-n}, b, c, bcq^{-n}/a; aq/d, aq/e, bcq^{-n}/a; q, q) \\
&= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(b;q)_k(c;q)_k(bcq^{-n}/a;q)_k}{(aq/d;q)_k(aq/e;q)_k(bcq^{-n}/a;q)_k} q^k
\end{aligned}
$$

So, by the definition of ${}_3\phi_2$, we can write

$$
\begin{aligned}
&{}_8\phi_7(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, q) \\
&= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n} {}_3\phi_2(q^{-n}, b, c; aq/d, aq/e; q, q).
\end{aligned}
$$

Finally, in the q-Pfaff-Saalshütz theorem, writing b, c and aq/d in place of the parameters a, b and c, respectively and noting that $bcdq^{-n}/a = aq/e$, we have

$$
\begin{aligned}
&{}_8\phi_7(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, q) \\
&= \frac{(aq;q)_n(aq/bc;q)_n(aq/bd;q)_n(aq/dc;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/bcd;q)_n}.
\end{aligned}
$$

Thus the proof is completed. □

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INDEX

pochhammer symbol, 29

q-binomial theorem, 15

q-factorial function, 2

Bailey-Daum summation formula, 46

Carlitz inversions, 24

Chu-Vandermonde's formula, 42

Euler's transformation formula, 38

Gauss Summation formula, 39

Heine's transformation formula, 35

Iteration of Heine's transformation formula, 37

Jackson's q-Dougall-Dixon formula, 66

Jackson's transformation formula, 44

q-analogue of a number , 1

q-binomial coefficients, 2

q-difference operator, 2

q-Dougall-Dixon formula, 51

q-Hypergeometric Series, 27

q-Pfaff-Saalschütz summation formula, 49

q-shifted factorial, 1

Sears balanced transformation formulas, 54

The triple product identity, 22

Watson transformation formula, 62