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Q-RUNG ORTHOPAIR FUZZY SETS AND THEIR SIMILARITY MEASURES

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WALEED SADEQ JAAYWEL JAAYWEL

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By Waleed Sadeq Jaaywel JAAYWEL

February 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Advisor **:** Assoc. Prof. Dr. Faruk KARAASLAN

Examining Committee Members:

Chairman **:** Prof. Dr. Aslıhan SEZGİN
Mathematics Education Departments
Amasya University

Member **:** Assoc. Prof. Dr. Faruk KARAASLAN
Mathematics
Çankırı Karatekin University

Member **:** Asst. Prof. Dr. Şerifenur CEBESOY ERDAL
Mathematics
Çankırı Karatekin University

Approved for the Graduate School of Natural and Applied Sciences

Prof. Dr. İbrahim ÇİFTÇİ
Director of Graduate School

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Waleed Sadeq Jaaywel JAAYWEL

ABSTRACT

Q-RUNG ORTHOPAIR FUZZY SETS AND THEIR SIMILARITY MEASURES

Waleed Sadeq Jaaywel JAAYWEL
Master of Science in Mathematics
Advisor: Assoc. Prof. Dr. Faruk KARAASLAN
February 2022

This thesis consists of five sections. In Section 1, a detailed literature review and motivation of the thesis are presented. In Section 2, some fundamental definitions are given to be used in the next part of the thesis. In Section 3, Cosine, weighted Cosine, cotangent and weighted cotangent similarity measure methods between intuitionistic fuzzy sets existing in literature are presented and they are supported by examples. In Section 4, some similarity and weighted similarity measure methods of q-rung orthopair fuzzy sets based on distance measure are presented in the light of existing studies in the literature. Also, Dice and generalized Dice similarity measure methods for q-rung orthopair fuzzy sets in literature and examples of them are presented. In Section 5, some explanations about methods are given.

2022, 97 pages

Keywords: Intutitionistic fuzzy set, q-rung othopair fuzzy set, Distance measure, Similarity measure

ÖZET

Q-RUNG ORTHOPAIR BULANIK KÜMELER VE BENZERLİK ÖLÇÜMLERİ

Waleed Sadeq Jaaywel JAAYWEL
Matematik, Yüksek Lisans
Tez Danışmanı: Doç. Dr. Faruk KARAASLAN
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Bu tez beş bölümden oluşmaktadır. Bölüm 1'de ayrıntılı bir literatür taraması ve tezin motivasyonu sunulmaktadır. Bölüm 2'de, tezin bir sonraki bölümünde kullanılmak üzere bazı temel tanımlar verilmiştir. Bölüm 3'te, literatürde var olan sezgisel bulanık kümeler arasında Kosinüs, ağırlıklı Kosinüs, kotanjant ve ağırlıklı kotanjant benzerlik ölçüm yöntemleri sunulmuş ve örneklerle desteklenmiştir. Bölüm 4'te, literatürde mevcut çalışmalar ışığında, uzaklık ölçümüne dayalı q-rung ortopair bulanık kümelerinin bazı benzerlik ve ağırlıklı benzerlik ölçüm yöntemleri sunulmuştur. Ayrıca, literatürdeki q-rung ortopair bulanık kümeler için Dice ve genelleştirilmiş Dice benzerlik ölçüm yöntemleri ve bunlara ilişkin örnekler sunulmuştur. Bölüm 5'te yöntemlerle ilgili bazı açıklamalara yer verilmiştir.

2021, 97 sayfa

Anahtar Kelimeler: Sezgisel bulanık küme, q-rung othopair bulanık küme, Mesafe ölçümü, Benzerlik ölçümü

PREFACE AND ACKNOWLEDGEMENTS

In this thesis, we presented a compilation of studies on similarity measurements of q-ROFS. We especially benefited from the references Ye (2011) Shi and Ye (2013), Tian (2013), Rajarajeswari and Uma (2013), Liu *et al.* (2019), Peng and Liu (2019), and Jan *et al.* (2020). You can find these resources in the references.

At the beginning of my speech, I must first thank God Almighty, who enabled me to reach this high scientific stage, and paved the way for me to be among you today to discuss my master's thesis.

My beloved father, my dear mother, my brothers and friends, I cannot forget your support for me and what you gave for me, so you have all my love, and whatever words of thanks I say to you, I will not give you what you deserve.

I also extend my highest thanks and gratitude to Assoc. Prof. Dr. Faruk KARAASLAN, who kindly accepted the supervision of the master's thesis, and who gave me his precious time and his vast knowledge and experience, which constituted a great addition to the research work, as his guidance and advice were the beacon that I used in my entire research work. May God Almighty reward him well.

Waleed Sadeq Jaaywel JAAYWEL
Çankırı-2022

CONTENTS

LIST OF SYMBOLS

$\mathcal{X}$	A non-empty Set
x	Element of $\mathcal{X}$
$\mathcal{A}$	Fuzzy Set
$\mathcal{N}(x)$	Fuzzy Number
ρ	Pythagorean Fuzzy Set
$\hat{\mathcal{A}}$	Subset of q-Rung Orthopair Fuzzy Set
$\hat{\mathcal{C}}$	Subset of q-Rung Orthopair Fuzzy Set
$\hat{\mathcal{S}}$	Subset of q-Rung Orthopair Fuzzy Set
$\hat{\mathcal{A}}^*$	Crisp Function
$\mathcal{S}(\mathcal{Q})$	Score Function
$\mathcal{A}(\mathcal{Q})$	Accurancy Function
$\mathcal{S}$	Similarity Measure
$\mathcal{Q}$	q-Rung Orthopair
$\mathcal{F}$	Distance Measure
η	Membership function
γ	Non-membership function
π	Hesitancy Degree
λ	Interval Between 0 and 1 q-Rung Orthopair Fuzzy Set
τ	Interval Between 0 and 1 q-Rung Orthopair Fuzzy Set
$\mathscr{s}$	Interval Between 0 and 1 to q-Rung Orthopair Fuzzy Set
ω	Interval Weighted for q-Rung Orthopair Fuzzy Set
κ	Power of element in q-ROFS
Cos	Fuzzy Cosine Similarity
Cot	Fuzzy Cotangent Similarity
$\mathcal{G}$	Generalized Dice Similarity Measures
$\mathcal{W}$	Weighted Dice Similarity Measures
$\forall$	For all

LIST OF ABBREVIATIONS

FS	Fuzzy Set
PFS	Pythagorean Fuzzy Set
PFN	Pythagorean Fuzzy Number
q-ROFS	q-Rung Orthopair Fuzzy Set
q-ROFN	q-Rung Orthopair Fuzzy Number
IFS	Intuitionistic Fuzzy Set
IFN	Intuitionistic Fuzzy Number
IFCS	Intuitionistic Fuzzy Cosine Similarity
WIFCS	Weighted Intuitionistic Fuzzy Cosine Similarity
IFCotS	Intuitionistic Fuzzy Cotangent Similarity
WIFCotS	Weighted Intuitionistic Fuzzy Cotangent Similarity
$d_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Euclidean distance measure
$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Euclidean Distance Measure
$\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Cosine Similarity Measure
$\mathcal{C}_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Cosine Similarity Measure
$\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Similarity Measure
$\mathcal{S}_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Similarity Measure
$\mathcal{D}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Distance Measure
$\mathcal{D}_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Distance Measure
$\mathcal{F}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Distance Measure
$\mathcal{S}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Similarity Measure
$\mathcal{D}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Dice Similarity Measures
$\mathcal{D}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Dice Similarity Measures
$\mathcal{D}_{\mathcal{G}_{q\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Generalized Dice Similarity Measures
$\mathcal{D}_{\mathcal{G}_{Wq\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	Weighted Generalized Dice Similarity Measures

LIST OF TABLES

1. INTRODUCTION

In real life, human beings face many problems involving uncertainty. For years, problems involving uncertainty have been an important issue to deal with. Although classical set theory and modeling methods are useful tools from time to time, they have not been sufficient in modeling many problems. Therefore, about such problems, researchers have worked on modeling and decision-making methods that enable more precise and precise decision-making. One of the most important of these studies is the fuzzy set approach put forward by Zadeh (1965). Zadeh (1965) The membership function, which is a larger extension of the characteristic function in classical sets, was used to create the fuzzy set. A fuzzy set is a function that maps a non-empty having started universe to the closed interval [0,1]. The membership function specifies just what an element of the initial universe has a place in the fluffy set. In general, a fluffy set A's participation potential is known as μ_A. Although fuzzy set is a useful tool, some problems are encountered regarding the determination of the membership function during modeling of real problems. In some cases the membership function can be ambiguous. To deal with such situations, the concept of type 2 fuzzy set was developed., which allows the use and grading of linguistic variables, was also defined by Zadeh in 1975. If an element in a fuzzy set has a membership degree of 0.6 and a non-membership degree of 0.4, the hesitation situation is not taken into account. For example, if an element has a membership value of 0.6 and a non-membership of 0.2, the fuzzy set is not sufficient for modeling. Thus, Atanassov introduced the concept of the heuristic fuzzy set in 1986 as a useful tool for modeling hesitant situations. Two functions characterize an intuitionistic fuzzy set, called membership (μ) and non-membership (ν) functions, defined from the initial universe to the closed interval [0,1]. The sum of the images of any element of the initial universe under the functions μ and ν is in the closed span [0,1]. For example, if a component has a membership degree of 0.7 and a non-membership degree of 0.2, there is a 0.1 degree hesitation for that component. Although intuitionistic fuzzy sets are well-known for their ability to model hesitancy, they do have significant drawbacks. For example, if an element's membership and non-membership degrees are 0.6 and 0.7, respectively, the sum of these values cannot be stated with intuitionistic fuzzy values because the sum is more than 1.

Because the sum of membership and non-membership degrees corresponding to each element in an intuitive fuzzy set cannot exceed 1, the sum of membership and non-membership degrees corresponding to each element in an intuitive fuzzy set cannot exceed 1. Yager proposed the Pythagorean Fuzzy Set as a solution to this limitation (2013). The sum of the squares of an element's degrees of membership and non-membership is in the closed interval [0,1] in this set. If the degrees of participation and non-membership in the above model are 0.8 and 0.9, respectively, it is insufficient for modeling in Pythagorean Fuzzy Sets since $0.8^2 + 0.9^2 > 1$. Therefore, Yager et al. (2017, 2018) introduced the q-rung orthopair fuzzy set concept as a generalization of fuzzy sets, intuitive fuzzy sets, and Pythagorean Fuzzy sets. In a q-rung orthopair fuzzy set

$\mathcal{Q} = \{(x_i, (\eta_{\mathcal{Q}}(x_i), \gamma_{\mathcal{Q}}(x_i)) | x_i \in \mathcal{X}\}$, $\eta_{\mathcal{Q}}^{\kappa}(x_i) + \gamma_{\mathcal{Q}}^{\kappa}(x_i) \leq 1$ such that $\kappa \geq 3$. When the degrees of membership and nonmembership are 0.8 and 0.9, respectively,, then $0.8^5 + 0.9^5 \leq 1$.

Information similarity is a significant concept in decision-making processes, (Abdel-Basset et al. 2019, Chiclana et al. 2013, Farhadinia 2013; 2014; 2016, Farhadinia and Herrara 2018, Farhadinia and Xu 2017, Li and Cheng 2002, Kamis et al. 2018a, 2018b, 2019; Mitchell 2003, Tang et al. 2020; Tian 2013, Xia and Xu 2010, Ye 2011). In particular, general agreement, grouping creation, image processing, pattern recognition, machine learning, and scheme selection are all important. Indeed, a variety of similarity measures have recently emerged, such as the statistical similarity measure proposed to quantify the similarity between two random vectors (Huang et al. 2017); the similarity measure used in subtractive clustering algorithm for endmember extraction (Yadav et al. 2020); the Dice similarity measure proposed for face recognition problem (Singh and Kumar 2020); and the similarity measure suggested to combine the traditional fuzziness with the traditional fuzziness with the traditional (Seal et al. 2020).

Recently, there has been a lot of work on similarity measurement. One of the most important of these is the similarity measurement between q-ROFSs. Some of the recent studies are: Similarity measures between q-ROFSs were defined by Peng and Liu

(2019) as additive inverses of the q-ROFSs have the same number of distance measurements. Liu et al. (2019) use the mean of cosine and Euclidean-based similarity measures for q-ROFS-based TOPSIS approaches, but Peng and Dai (2019) use Xia and Xu's (2010) methodology for q-ROFS-based TOPSIS techniques. By substituting square powers with qpowers, Wang et al. (2020) introduced the cosine and cotangent similarity measurements for PFSs to q-powers.

In this thesis, we presented a compilation of studies on similarity measurements of q-ROFS. First, we aimed to make the formulas more understandable in terms of application by examining the similarity measures presented by Ye (2011) Shi and Ye (2013), Tian (2013) and Rajarajeswari and Uma (2013) and presenting examples. Later, Liu *et al.* (2019), Peng and Liu (2019), and Jan *et al.* (2020) similarity measures between q-ROPF clusters were examined and examples were presented.

2. PRELIMINARIES

We'll go over some basic concepts including IFS, PFS, and q-ROFS, as well as their related features, in this part, $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ denotes a finite, discrete, and nonempty discourse set throughout the work.

Definition 2.1 (Zadeh 1965) Let $\mathcal{X}$ be a set that isn't empty. Then, by mapping as follows, a fuzzy set (FS) $\mathcal{A}$ over $\mathcal{X}$ is determined:

$$\eta_{\mathcal{A}}: \mathcal{X} \to [0,1].$$

Here, $\eta_{\mathcal{A}}$ is referred to as the membership function of the FS $\mathcal{A}$. $\eta_{\mathcal{A}}(x)$ is called membership grade of x in $\mathcal{X}$ and $\eta_{\mathcal{A}}(x) \in [0,1]$.
An FS can be expressed as a set of pairs as follows:

$$\mathcal{A} = \{(x, \eta_{\mathcal{A}}(x)): x \in \mathcal{X},\ \eta_{\mathcal{A}}(x) \in [0,1]\}.$$

Also, the readers can encounter different notations of the fuzzy sets.

Some of them are below:

- $\{\frac{\eta_{\mathcal{A}}(x)}{x}: x \in \mathcal{X},\ \eta_{\mathcal{A}}(x) \in [0,1]\}$
- If $\mathcal{X}$ is finite and crisp, then $\mathcal{A} = \sum_{x\in\mathcal{X}} \eta_{\mathcal{A}}(x)/x$

If $\mathcal{X}$ is infinite, then $\mathcal{A} = \int_{x\in\mathcal{X}} \eta_{\mathcal{A}}(x)/x$.
Note that "/" doesn't denote the division and $\int$ doesn't denote the integral operator.

Remark 1: Elements of $\mathcal{X}$ the membership grade of which are 0 don't appear in the fuzzy sets. If $\eta_{\mathcal{A}}(x) = 1$, then element x is a full member of fuzzy set . If $0 < \eta_{\mathcal{A}}(x) < 1$, then element x is the partially member of fuzzy set $\mathcal{A}$.
In a fuzzy set, if an element has 0.6 membership degree, we can say that its non-membership degree is 0.4. However, this is not sufficient to express the hesitancy. To overcome such a situation, the concept of the intuitionistic fuzzy set was put forward by Atanassov (1986) as a generalization of the fuzzy sets.

Definition 2.2 (Atanassov 1986) Let $\mathcal{X}$ be a fixed set. An intuitionistic fuzzy set (IFS) $\hat{\mathcal{A}}$ over $\mathcal{X}$ is defined as follows:

$$\hat{\mathcal{A}} = \{(x, (\eta_{\hat{\mathcal{A}}}(x), \gamma_{\hat{\mathcal{A}}}(x))|x \in \mathcal{X}\},$$

where $\eta_{\hat{\mathcal{A}}}: \mathcal{X} \to [0,1]$ and $\gamma_{\hat{\mathcal{A}}}: \mathcal{X} \to [0,1]$ represent the degree of membership and non-membership degree of $x \in \mathcal{X}$, respectively with condition $0 \leq \eta_{\hat{\mathcal{A}}}(x) + \gamma_{\hat{\mathcal{A}}}(x) \leq 1$ for all $x \in \mathcal{X}$. Here, $\pi_{\mathcal{A}}(x) = 1 - \eta_{\hat{\mathcal{A}}}(x) - \gamma_{\hat{\mathcal{A}}}(x)$ is hesitancy degree of $x \in \mathcal{X}$.

Based on the definition of IFS, it can be said that each ordinary FS is an IFS defined as follows:

$$\hat{\mathcal{A}} = \{(x_i, (\eta_{\hat{\mathcal{A}}}(x), 1 - \eta_{\hat{\mathcal{A}}}(x)))|x \in \mathcal{X}\}.$$

If the set $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ has only one element, that is $\mathcal{X} = \{x\}$, then the IFS is denoted by $\hat{\mathcal{A}} = (\eta_{\hat{\mathcal{A}}}(x), \gamma_{\hat{\mathcal{A}}}(x)) = (\eta_{\hat{\mathcal{A}}}, \gamma_{\hat{\mathcal{A}}})$.

For convenience, we define an intuitionistic fuzzy number (IFS_N) as space $\hat{\mathcal{A}} = (\eta_{\hat{\mathcal{A}}}, \gamma_{\hat{\mathcal{A}}})$.

Example 2.1. Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ be the set of parties attending an election in a subprovince where are 2000 voter in which. The results obtained at the end of the selection are given in (Table 2.1).

Table 2.1 Result of the election

Parties	x_1	x_2	x_3	x_4
Results	990	155	400	145

If we consider $\eta_{\hat{\mathcal{A}}}(x) = \frac{\mathcal{N}(x)}{2000}$ as a membership function where $\mathcal{N}(x)$ means the numbers of results by x , then we can express these results as an FS by

$$\mathcal{A} = \{(x_1, 0.495), (x_2, 0.775), (x_3, 0.2), (x_4, 0.0725)\}$$

According to fuzzy set $\mathcal{A}$ non-membership degree of x_1 is $1 - 0.495 = 0.505$. But this may be not true because 300 voters don't attend the election. As a result, it can be said that every FS is an IFS, but the converse is not true.

In an IFS, the summation of a membership degree and non-membership degree of an element is a value in the interval [0, 1]. If the summation of a membership degree and non-membership degree of an element is greater than 1, IFS is not a useful tool for displaying the issue. Yager (2010) defined the Pythagorean fuzzy set as a conjecture of FSs and IFSs in order to address this issue.

Definition 2.3 (Yager 2010) A Pythagorean fuzzy set (PFS) ρ defined on a non-empty set $\mathcal{X}$ is an object having the form $\rho = \left\{\left(x, (\eta_\rho(x), \gamma_\rho(x)\right) : x \in \mathcal{X}\right\}$, where $\eta_\rho : \mathcal{X} \to [0,1]$ and $\gamma_\rho : \mathcal{X} \to [0,1]$ are mappings, defining the degrees of membership and non-membership of $x \in \mathcal{X}$, respectively. Here, $0 \leq \eta^2{}_\rho(x) + \gamma^2{}_\rho(x) \leq 1$, $\forall x \in \mathcal{X}$, and the hesitancy degree it is $\pi_\rho(x) = \sqrt{1 - \eta^2{}_\rho(x) - \gamma^2{}_\rho(x)}$. Obviously, $0 \leq \pi_\rho(x_i) \leq 1$ for any $x \in \mathcal{X}$.

If the set $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ has only element one, that means $\mathcal{X} = \{x\}$, then the PFS $\rho = \left(\eta_\rho(x), \gamma_\rho(x)\right) = (\eta_\rho, \gamma_\rho)$ is called a Pythagorean fuzzy number(PFN).

Example 2.2 Let $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$, be a non-empty set. Then, Pythagorean fuzzy set (PFS) ρ over $\mathcal{X}$ can be written as follows:

$\rho = \{(x_1, (0.9{,}0.4), (x_2, (0.7{,}0.5)), (x_3, (0.4{,}0.7)), (x_4, (0.6{,}0.5)), (x_5, (0.3{,}0.5))\}$.

Here, for x_1, x_2, x_3 and $x_4 \in \mathcal{X}$ although $\eta_\rho(x) + \gamma_\rho(x) \geq 1$, sum of square of membership and non-membership degrees corresponding these elements less or equal then 1.

Also, for $x_1 \in \mathcal{X}$, hesitancy degree $\pi_\rho(x_1) = \sqrt{1 - (\eta_\rho(x_1))^2 - (\gamma_\rho(x_1))^2} = 0.17$.

By the similar way, hesitancy degrees of other elements can be computed.

Definition 2.4 (Yager 2018) Let $\mathcal{X}$=$\{x_1, x_2, \dots, x_n\}$ be a fixed set. A q-rung orthopair fuzzy set (q-ROFS) $\mathcal{Q}$ in $\mathcal{X}$ is defined as follows:

$$\mathcal{Q} = \{(x_i, (\eta_{\mathcal{Q}}(x_i), \gamma_{\mathcal{Q}}(x_i)) | x_i \in \mathcal{X}\},$$

where $\eta_{\mathcal{Q}} : \mathcal{X} \to [0,1]$ means the degree of membership and $\gamma_{\mathcal{Q}} : \mathcal{X} \to [0,1]$ means the degree of non-membership such that $0 \leq \eta_{\mathcal{Q}}{}^\kappa(x_i) + \gamma_{\mathcal{Q}}{}^\kappa(x_i) \leq 1$, $\forall x_i \in \mathcal{X}$. Here, the degree of indeterminacy membership of $x_i \in \mathcal{X}$ is defined by

$$\pi_{Q}(x_i) = \sqrt[\kappa]{1 - {\eta_{Q}}^{\kappa}(x_i) - {\gamma_{Q}}^{\kappa}(x_i)}\,, \forall x_i \in \mathcal{X}.$$

Example 2.3 Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ be a fixed set and let us consider the following set $Q = \{\langle x_1, (0.7,0.8)_{\kappa}\rangle, \langle x_2, (0.6,0.9)_{\kappa}\rangle, \langle x_3, (0.9,0.5)_{\kappa}\rangle, \langle x_4, (0.65,0.85)_{\kappa}\rangle\}$. For $\kappa = 3$ and $i = 1,2,3,4$, Q is a q-ROFS. We can show that they satisfy the condition $0 \leq {\eta^3}_{Q}(x_i) + {\gamma^3}_{Q}(x_i) \leq 1\,, \forall x_i \in \mathcal{X}$.

For i=1, $\eta_{Q}(x_1) = 0.7$ and $\gamma_{Q}(x_1) = 0.8$, then $0 \leq (0.7)^3 + (0.8)^3 \leq 1$ and

$$0 \leq 0.855 \leq 1.$$

For i=2, $\eta_{Q}(x_2) = 0.6$ and $\gamma_{Q}(x_2) = 0.9$, then $0 \leq (0.6)^3 + (0.9)^3 \leq 1$ and

$$0 \leq 0.945 \leq 1.$$

For i=3, $\eta_{Q}(x_i) = 0.9$ and $\gamma_{Q}(x_i) = 0.5$, then $0 \leq (0.9)^3 + (0.5)^3 \leq 1$ and

$$0 \leq 0.854 \leq 1.$$

For i=4, $\eta_{Q}(x_4) = 0.75$ and $\gamma_{Q}(x_4) = 0.85$, then, $0 \leq (0.85)^3 + (0.85)^3 \leq 1$ and

$$0 \leq 0.888 \leq 1.$$

Then, the degree of membership and the degree of non-membership satisfy the condition of q-rung orthpair fuzzy set. Also, we find the degree of indeterminacy membership by using following formula:

$$\pi_{Q}(x_i) = \sqrt[\kappa]{1 - {\eta_{Q}}^{\kappa}(x_i) - {\gamma_{Q}}^{\kappa}(x_i)}\,, \forall x_i \in \mathcal{X}.$$

For $\eta_{Q}(x_1) = 0.7$ and $\gamma_{Q}(x_1) = 0.8$, $\pi_{Q}(x_1) = \sqrt[3]{1 - (0.7)^3 - (0.8)^3} = 0.5253$

For $\eta_{Q}(x_2) = 0.6$ and $\gamma_{Q}(x_2) = 0.9$, $\pi_{Q}(x_2) = \sqrt[3]{1 - (0.6)^3 - (0.9)^3} = 0.3802$

For $\eta_{Q}(x_3) = 0.9$ and $\gamma_{Q}(x_3) = 0.5$ $\pi_{Q}(x_3) = \sqrt[3]{1 - (0.9)^3 - (0.5)^3} = 0.8175$

For $\eta_{Q}(x_4) = 0.65$ and $\gamma_{Q}(x_i) = 0.85$ $\pi_{Q}(x_4) = \sqrt[3]{1 - (0.65)^3 - (0.85)^3} = 0.4809$

If the set $\mathcal{X}$ has only one element, which means that $\mathcal{X} = \{x\}$, then the q-ROFS $\mathcal{Q}$ is denoted as $\mathcal{A} = (\eta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x))_{\kappa}$ and this is called as q-rung othopair fuzzy number (q-ROFN).

Definition 2.5 (Yager 2018) Let $\mathcal{Q}_1 = (\eta_{\mathcal{Q}_1}(x), \gamma_{\mathcal{Q}_1}(x))_{\kappa}$ and $\mathcal{Q}_2 = (\eta_{\mathcal{Q}_2}(x), \gamma_{\mathcal{Q}_2}(x))_{\kappa}$ be two q-ROFNs over $\mathcal{X}$. Then, operations between two q-ROFNs $\mathcal{Q}_1$and $\mathcal{Q}_2$ can be defined as:

1. $\mathcal{Q}_1 \cup \mathcal{Q}_2 = \left(\max\left\{\eta_{\mathcal{Q}_1}(x), \eta_{\mathcal{Q}_2}(x)\right\}, min\left\{\gamma_{\mathcal{Q}_1}(x), \gamma_{\mathcal{Q}_2}(x)\right\}\right)_{\kappa}$
2. $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \left(\min\left\{\eta_{\mathcal{Q}_1}(x), \eta_{\mathcal{Q}_2}(x)\right\}, max\left\{\gamma_{\mathcal{Q}_1}(x), \gamma_{\mathcal{Q}_2}(x)\right\}\right)_{\kappa}$
    - If $\mathcal{Q}_1 = (\eta_{\mathcal{Q}_1}(x), \gamma_{\mathcal{Q}_1}(x))_{\kappa}$, then ${\mathcal{Q}_1}^{\mathcal{C}} = (\gamma_{\mathcal{Q}_1}(x), \eta_{\mathcal{Q}_1}(x))_{\kappa}$, where ${\mathcal{Q}_1}^{\mathcal{C}}$ means the converse of $\mathcal{Q}_1$.
    - If $\eta_{\mathcal{Q}_1}(x) \leq \eta_{\mathcal{Q}_2}(x)$ and $\gamma_{\mathcal{Q}_1}(x) \geq \gamma_{\mathcal{Q}_2}$, then $\mathcal{Q}_1 \leq \mathcal{Q}_2$.

Defintion 2.6 (Liu and Wang 2018) Let $\mathcal{Q} = (\eta_{\mathcal{Q}}(x), \gamma_{\mathcal{Q}}(x))_{\kappa}$ be a q-ROFN. The score function $\mathcal{S}(\mathcal{Q})$ and the accuracy function $\mathcal{A}(\mathcal{Q})$ are defined as follows:

- $\mathcal{S}(\mathcal{Q}) = {\eta_{\mathcal{Q}}}^{\kappa} - {\gamma_{\mathcal{Q}}}^{\kappa}$;
- $\mathcal{A}(\mathcal{Q}) = {\eta_{\mathcal{Q}}}^{\kappa} + {\gamma_{\mathcal{Q}}}^{\kappa}$,

respectively, $-1 \leq \mathcal{S}(\mathcal{Q}) \leq 1$, and $0 \leq \mathcal{A}(\mathcal{Q}) \leq 1,\ \kappa \geq 1$.

Lemma 2.1 (Liu and Wang 2018) Let $\mathcal{Q}_1 = (\eta_{\mathcal{Q}_1}(x), \gamma_{\mathcal{Q}_1}(x))_{\kappa}$, and $\mathcal{Q}_2 = (\eta_{\mathcal{Q}_2}(x), \gamma_{\mathcal{Q}_2}(x))_{\kappa}$ be any two q-ROFNs. Then the score function $\mathcal{S}(\mathcal{Q}_1)$ and $\mathcal{S}(\mathcal{Q}_2)$ of $\mathcal{Q}_1$and $\mathcal{Q}_2$ and the accuracy functions $\mathcal{A}(\mathcal{Q}_1)$ and $\mathcal{A}(\mathcal{Q}_2)$ of $\mathcal{Q}_1$and $\mathcal{Q}_2$ are defined as follows:

1. If $\mathcal{S}(\mathcal{Q}_1) > \mathcal{S}(\mathcal{Q}_2)$, then $\mathcal{Q}_1 > \mathcal{Q}_2$.
2. If $\mathcal{S}(\mathcal{Q}_1) = \mathcal{S}(\mathcal{Q}_2)$, then
    - If $\mathcal{A}(\mathcal{Q}_1) > \mathcal{A}(\mathcal{Q}_2)$, then $\mathcal{Q}_1 > \mathcal{Q}_2$;
    - If $\mathcal{A}(\mathcal{Q}_1) = \mathcal{A}(\mathcal{Q}_2)$, then $\mathcal{Q}_1 = \mathcal{Q}_2$.

3. SOME SIMILARITY MEASURES BETWEEN INTUITIONISTIC FUZZY SETS

In this section, we present some similarity measure methods between two IFSs which are introduced by Ye (2011), and their examples which are presented by Shi and Ye (2013).

Let $\hat{\mathcal{A}}$={$(x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j)) | x_j \in \mathcal{X}$} and $\hat{\mathcal{C}}$={$(x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j)) | x_j \in \mathcal{X}$} be two IFSs over $\mathcal{X}$. Then, the intuitionistic fuzzy cosine similarity (IFCS) measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ proposed by Ye (2011) is defined as follows:

1. $\mathrm{IFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)}\,\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)}}.$

In the above formula, Ye (2011) doesn't consider the hesitancy degree of the IF elements. Shi and Ye (2013) defined the IFCS measure between $\hat{\mathcal{A}}$and $\hat{\mathcal{C}}$ by considering hesitancy degrees of the IF elements as follows:

2. $\mathrm{IFCS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)+\pi_{\hat{\mathcal{A}}}(x_j)\pi_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)+\pi^2_{\hat{\mathcal{A}}}(x_j)}\,\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)+\pi^2_{\hat{\mathcal{C}}}(x_j)}}.$

Example 3.1 Let $\hat{\mathcal{A}} = \{(x_1, 0.4,0.3), (x_2, 0.2,0.5), (x_3, 0.6,0.2), (x_4, 0.3,0.1)\}$ and $\hat{\mathcal{C}}$=$\{(x_1, 0.1,0.2), (x_2, 0.3,0.5), (x_3, 0.5,0.4), (x_4, 0.4,0.6)\}$ be two IFSs over $\mathcal{X}$. Then, the according to the IFC^1 measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

- $\mathrm{IFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)}\,\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)}}$

$$\mathrm{IFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{4}\left(\begin{array}{c}\frac{0.4\times0.1+0.3\times0.2}{\sqrt{0.4^2+0.3^2}\sqrt{0.1^2+0.2^2}} + \frac{0.2\times0.3+0.5\times0.5}{\sqrt{0.2^2+0.5^2}\sqrt{0.3^2+0.5^2}} + \\ \frac{0.6\times0.5+0.2\times0.4}{\sqrt{0.6^2+0.2^2}\sqrt{0.5^2+0.4^2}} + \frac{0.3\times0.4+0.1\times0.6}{\sqrt{0.3^2+0.1^2}\sqrt{0.4^2+0.6^2}}\end{array}\right)$$

$$= \frac{1}{4}(0.8944 + 0.9872 + 0.9383 + 0.7893)$$

$$= 0.9051.$$

Example 3.2 Let $\hat{\mathcal{A}} = \{(x_1, 0.4,0.3,0.3), (x_2, 0.2,0.5,0.3), (x_3, 0.6,0.2,0.2),$ $(x_4, 0.3, 0.1,0.6)\}$ and $\hat{\mathcal{C}}$=$\{(x_1, 0.1,0.2,0.7), (x_2, 0.3,0.5,0.2), (x_3, 0.5,0.4,0.1),$ $(x_4, 0.4,0.6,0)\}$ be two IFSs over $\mathcal{X}$, then the according to the intuitionistic fuzzy cosin (IFC^1) measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

- $\text{IFCS}^2(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\eta_{\hat{\mathcal{A}}}(x_j)\, \eta_{\hat{\mathcal{C}}}(x_j) + \gamma_{\hat{\mathcal{A}}}(x_j)\, \gamma_{\hat{\mathcal{C}}}(x_j) + \pi_{\hat{\mathcal{A}}}(x_j)\pi_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j) + \gamma^2_{\hat{\mathcal{A}}}(x_j) + \pi^2_{\hat{\mathcal{A}}}(x_j)}\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j) + \gamma^2_{\hat{\mathcal{C}}}(x_j) + \pi^2_{\hat{\mathcal{C}}}(x_j)}}$

$$= \frac{1}{4}\left(\begin{array}{c} \frac{0.4 \times 0.1 + 0.3 \times 0.2 + 0.3 \times 0.7}{\sqrt{0.4^2 + 0.3^2 + 0.3^2}\sqrt{0.1^2 + 0.2^2 + 0.7^2}} + \frac{0.2 \times 0.3 + 0.5 \times 0.5 + 0.3 \times 0.2}{\sqrt{0.2^2 + 0.5^2 + 0.3^2}\sqrt{0.3^2 + 0.5^2 + 0.2^2}} \\ + \\ \frac{0.6 \times 0.5 + 0.2 \times 0.4 + 0.2 \times 0.1}{\sqrt{0.6^2 + 0.2^2 + 0.2^2}\sqrt{0.5^2 + 0.4^2 + 0.1^2}} + \frac{0.3 \times 0.4 + 0.1 \times 0.6 + 0.6 \times 0}{\sqrt{0.3^2 + 0.1^2 + 0.6^2}\sqrt{0.4^2 + 0.6^2 + 0^2}} \end{array}\right)$$

$$= \frac{1}{4}(0.7324 + 0.9736 + 0.9304 + 0.3680)$$

$$= 0.7511.$$

Ye (2016) developed two IFCS measures between two (IFSs) $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ based on cosine function.

3. $\text{IFCS}^3(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x) - \eta_{\hat{\mathcal{C}}}(x)|, \\ |\gamma_{\hat{\mathcal{A}}}(x) - \gamma_{\hat{\mathcal{C}}}(x)|, \\ |\pi_{\hat{\mathcal{A}}}(x) - \pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

4. $\text{IFCS}^4(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{4}\left(\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x) - \eta_{\hat{\mathcal{C}}}(x)| + \\ |\gamma_{\hat{\mathcal{A}}}(x) - \gamma_{\hat{\mathcal{C}}}(x)| + \\ |\pi_{\hat{\mathcal{A}}}(x) - \pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

Example 3.3 Let $\hat{\mathcal{A}} = \{(x_1, 0.4,0.3,0.3), (x_2, 0.2,0.5,0.3), (x_3, 0.6,0.2,0.2), (x_4, 0.3,0.1,0.6)\}$ and $\hat{\mathcal{C}}$=$\{(x_1, 0.1,0.2,0.7), (x_2, 0.3,0.5,0.2), (x_3, 0.5,0.4,0.1), (x_4, 0.4,0.6,0)\}$ be two IFSs over $\mathcal{X}$, then the according to the intuitionistic fuzzy cosine similarity (IFCS^1) and (IFCS^2) measures between two intuitionistic fuzzy sets $\hat{\mathcal{A}}$and $\hat{\mathcal{C}}$ can be calculated as follows:

- $\mathrm{IFCS}^3(\mathcal{A},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x) - \eta_{\hat{\mathcal{C}}}(x)|,\\ |\gamma_{\hat{\mathcal{A}}}(x) - \gamma_{\hat{\mathcal{C}}}(x)|,\\ |\pi_{\hat{\mathcal{A}}}(x) - \pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

$$= \frac{1}{4}\left(cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.4 - 0.1|,\\ |0.3 - 0.2|,\\ |0.3 - 0.7|\end{pmatrix}\right)\right] + cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.2 - 0.3|,\\ |0.5 - 0.5|,\\ |0.3 - 0.2|\end{pmatrix}\right)\right] + cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.6 - 0.5|,\\ |0.2 - 0.4|,\\ |0.2 - 0.1|\end{pmatrix}\right)\right] + cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.3 - 0.4|,\\ |0.1 - 0.6|,\\ |0.6 - 0|\end{pmatrix}\right)\right]\right)$$

$$= \frac{1}{4}\left(cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.3|,\\ |0.1|,\\ |-0.4|\end{pmatrix}\right)\right] + cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|-0.1|,\\ |0|,\\ |0.1|\end{pmatrix}\right)\right] + cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.1|,\\ |-0.2|,\\ |0.1|\end{pmatrix}\right)\right] + cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|- 0.1|,\\ |-0.5|,\\ |0.6|\end{pmatrix}\right)\right]\right)$$

$$= \frac{1}{4}\left(cos\left[\frac{\pi}{2}(0.4)\right] + cos\left[\frac{\pi}{2}(0.1)\right] + cos\left[\frac{\pi}{2}(0.2)\right] + cos\left[\frac{\pi}{2}(0.6)\right]\right)$$

$$= \frac{1}{4}(0.8090 + 0.9876 + 0.9510 + 0.5877)$$

$$= 0.8338.$$

- $\mathrm{IFCS}^4(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cos\left[\frac{\pi}{4}\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x) - \eta_{\hat{\mathcal{C}}}(x)| +\\ |\gamma_{\hat{\mathcal{A}}}(x) - \gamma_{\hat{\mathcal{C}}}(x)| +\\ |\pi_{\hat{\mathcal{A}}}(x) - \pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right]$

$$= \frac{1}{4}\left(cos\left[\frac{\pi}{4}\begin{pmatrix}|0.4 - 0.1| +\\ |0.3 - 0.2| +\\ |0.3 - 0.7|\end{pmatrix}\right] + cos\left[\frac{\pi}{4}\begin{pmatrix}|0.2 - 0.3| +\\ |0.5 - 0.5| +\\ |0.3 - 0.2|\end{pmatrix}\right] + cos\left[\frac{\pi}{4}\begin{pmatrix}|0.6 - 0.5| +\\ |0.2 - 0.4| +\\ |0.2 - 0.1|\end{pmatrix}\right] + cos\left[\frac{\pi}{4}\begin{pmatrix}|0.3 - 0.4| +\\ |0.1 - 0.6| +\\ |0.6 - 0|\end{pmatrix}\right]\right)$$

$$= \frac{1}{4}\left(cos\left[\frac{\pi}{4}\begin{pmatrix}|0.3| +\\ |0.1| +\\ |-0.4|\end{pmatrix}\right] + cos\left[\frac{\pi}{4}\begin{pmatrix}|-0.1| +\\ |0| +\\ |0.1|\end{pmatrix}\right] + cos\left[\frac{\pi}{4}\begin{pmatrix}|0.1| +\\ |-0.2| +\\ |0.1|\end{pmatrix}\right] + cos\left[\frac{\pi}{4}\begin{pmatrix}|- 0.1| +\\ |-0.5| +\\ |0.6|\end{pmatrix}\right]\right)$$

$$= \frac{1}{4}\left(cos\left[\frac{\pi}{4}(0)\right] + cos\left[\frac{\pi}{4}(0)\right] + cos\left[\frac{\pi}{4}(0)\right] + cos\left[\frac{\pi}{4}(0)\right]\right)$$

$$= \frac{1}{4}(1 + 1 + 1 + 1)$$

$$= 1.0000.$$

The intuitionistic fuzzy cotangent similarity (IFCotS) measure between any two IFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ was proposed by Tian (2013) as follows:

5. $\quad \text{IFCot}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x)-\eta_{\hat{\mathcal{C}}}(x)|,\\|\gamma_{\hat{\mathcal{A}}}(x)-\gamma_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

Consider the degree of membership, non-membership, and hesitancy, and then the IFCotS measure between two IFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ proposed by Rajarajeswari and Uma (2013) as follows:

6. $\quad \text{IFCot}^2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x)-\eta_{\hat{\mathcal{C}}}(x)|,\\|\gamma_{\hat{\mathcal{A}}}(x)-\gamma_{\hat{\mathcal{C}}}(x)|,\\|\pi_{\hat{\mathcal{A}}}(x)-\pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

Example 3.4 Let $\hat{\mathcal{A}} = \{(x_1, 0.4,0.3,0.3), (x_2, 0.2,0.5,0.3), (x_3, 0.6,0.2,0.2),$ $(x_4, 0.3,0.1,0.6)\}$ and $\hat{\mathcal{C}}$=$\{(x_1, 0.1,0.2,0.7), (x_2, 0.3,0.5,0.2), (x_3, 0.5,0.4,0.1),$ $(x_4, 0.4,0.6,0)\}$ be two IFSs over $\mathcal{X}$. Then, their IFCotS^1 and IFCotS^2 measures between two IFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

- $IFCot^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x)-\eta_{\hat{\mathcal{C}}}(x)|,\\|\gamma_{\hat{\mathcal{A}}}(x)-\gamma_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

$$= \frac{1}{4}\begin{pmatrix} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.4-\ 0.1|,\\|0.3-0.2|\end{pmatrix}\right)\right]\\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.2-\ 0.3|,\\|0.5-0.5|\end{pmatrix}\right)\right]\\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.6-\ 0.5|,\\|0.2-0.4|\end{pmatrix}\right)\right]\\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3-\ 0.4|,\\|0.1-0.6|\end{pmatrix}\right)\right]\end{pmatrix}$$

$$= \frac{1}{4}\left(cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3|,\\|0.1|\end{pmatrix}\right)\right] + cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\|0|\end{pmatrix}\right)\right]\right.$$
$$\left. + cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.1|,\\|-0.2|\end{pmatrix}\right)\right] + cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\|-0.5|\end{pmatrix}\right)\right]\right)$$

$$= \frac{1}{4}\begin{pmatrix} cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.3)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.1)\right] \\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.2)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.5)\right] \end{pmatrix}$$

$$= \frac{1}{4}(0.6128+0.8540+0.7265+0.4142)$$

$$= 0.6518.$$

- $\text{IFCotS}^2\left(\hat{\mathcal{A}},\hat{\mathcal{C}}\right)=\frac{1}{n}\sum_{j=1}^{n} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x)-\eta_{\hat{\mathcal{C}}}(x)|,\\ |\gamma_{\hat{\mathcal{A}}}(x)-\gamma_{\hat{\mathcal{C}}}(x)|,\\ |\pi_{\hat{\mathcal{A}}}(x)-\pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$

$$= \frac{1}{4}\begin{pmatrix} cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.4-0.1|,\\|0.3-0.2|,\\|0.3-0.7|\end{pmatrix}\right)\right] \\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.2-0.3|,\\|0.5-0.5|,\\|0.3-0.2|\end{pmatrix}\right)\right] \\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.6-0.5|,\\|0.2-0.4|,\\|0.2-0.1|\end{pmatrix}\right)\right] \\ +cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3-0.4|,\\|0.1-0.6|,\\|0.6-0|\end{pmatrix}\right)\right] \end{pmatrix}$$

$$= \frac{1}{4}\left(cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3|,\\|0.1|,\\|-0.4|\end{pmatrix}\right)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\|0|,\\|0.1|\end{pmatrix}\right)\right]\right.$$
$$\left.+\,cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.1|,\\|-0.2|,\\|0.1|\end{pmatrix}\right)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\|-0.5|,\\|0.6|\end{pmatrix}\right)\right]\right)$$

$$= \frac{1}{4}\left(cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.4)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.1)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.2)\right]+cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.6)\right]\right)$$

$$= \frac{1}{4}(0.5090+0.8540+0.7265+0.3249)$$

$$= 0.6036.$$

In some practical applications, weights of the elements may be different. Consider the weighting vector of the elements in an IFS, the weighted intuitionistic fuzzy cosine similairty (WIFCS) measure, and weighted intuitionistic fuzzy cotangent similarity (WIFCotS) measure between any two IFSs, $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be shown as follows (Ye 2011, Shi and Ye Tian 2013, Rajarajeswari and Uma 2013, Ye 2016)

7. $$\mathrm{WIFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\left[\frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)}\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)}}\right]$$

8. $$\mathrm{WIFCS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\left[\frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)+\pi_{\hat{\mathcal{A}}}(x_j)\pi_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)+\pi^2_{\hat{\mathcal{A}}}(x_j)}\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)+\pi^2_{\hat{\mathcal{C}}}(x_j)}}\right]$$

9. $$\mathrm{WIFCS}^3(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\, cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x_j)-\eta_{\hat{\mathcal{C}}}(x_j)|,\\ |\gamma_{\hat{\mathcal{A}}}(x_j)-\gamma_{\hat{\mathcal{C}}}(x_j)|,\\ |\pi_{\hat{\mathcal{A}}}(x_j)-\pi_{\hat{\mathcal{C}}}(x_j)|\end{pmatrix}\right)\right]$$

10. $$\mathrm{WIFCS}^4(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j cos\left[\frac{\pi}{4}\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x_j)-\eta_{\hat{\mathcal{C}}}(x_j)|+\\ |\gamma_{\hat{\mathcal{A}}}(x_j)-\gamma_{\hat{\mathcal{C}}}(x_j)|+\\ |\pi_{\hat{\mathcal{A}}}(x_j)-\pi_{\hat{\mathcal{C}}}(x_j)|\end{pmatrix}\right]$$

11. $$\mathrm{WIFCotS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\, cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x_j)-\eta_{\hat{\mathcal{C}}}(x_j)|,\\ |\gamma_{\hat{\mathcal{A}}}(x_j)-\gamma_{\hat{\mathcal{C}}}(x_j)|\end{pmatrix}\right)\right]$$

12. $$\mathrm{WIFCotS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\, cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x)-\eta_{\hat{\mathcal{C}}}(x)|,\\ |\gamma_{\hat{\mathcal{A}}}(x)-\gamma_{\hat{\mathcal{C}}}(x)|,\\ |\pi_{\hat{\mathcal{A}}}(x)-\pi_{\hat{\mathcal{C}}}(x)|\end{pmatrix}\right)\right]$$

where, $\omega_j (j = 1,2,\dots,n)$ denotes the weight of elements x_j, $j = (1,2,\dots,n)$ which satisfies the condition of $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

Example 3.5 Let

$\hat{\mathcal{A}} = \{(x_1, 0.4,0.3,0.3), (x_2, 0.2,0.5,0.3), (x_3, 0.6,0.2,0.2), (x_4, 0.3,0.1,0.6)\}$ and

$\hat{\mathcal{C}} = \{(x_1, 0.1,0.2,0.7), (x_2, 0.3,0.5,0.2), (x_3, 0.5,0.4,0.1), (x_4, 0.4,0.6,0)\}$ be two IFSs over $\mathcal{X}$. Assume that $\omega_j = 0.1,0.2,0.3,0.4$. Then, the according to the weighted intuitionistic fuzzy cosine similarity (WIFC) measure, and weighted intuitionistic fuzzy cotangent (WIFCot) similarity measure between any two intuitionistic fuzzy sets (IFSs), $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as:

- $\mathrm{WIFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\left[\frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)}\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)}}\right]$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times(\frac{0.4\times 0.1+0.3\times 0.2}{\sqrt{0.4^2+0.3^2}\sqrt{0.1^2+0.2^2}})+0.2\times(\frac{0.2\times 0.3+0.5\times 0.5}{\sqrt{0.2^2+0.5^2}\sqrt{0.3^2+0.5^2}})+ \\ 0.3\times(\frac{0.6\times 0.5+0.2\times 0.4}{\sqrt{0.6^2+0.2^2}\sqrt{0.5^2+0.4^2}})+0.4\times(\frac{0.3\times 0.4+0.1\times 0.6}{\sqrt{0.3^2+0.1^2}\sqrt{0.4^2+0.6^2}}) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times 0.8944+0.2\times 0.9872+ \\ 0.3\times 0.9383+0.4\times 0.7893 \end{pmatrix}$$

$$= 0.2210.$$

- $\mathrm{WIFCS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\left[\frac{\eta_{\hat{\mathcal{A}}}(x_j)\,\eta_{\hat{\mathcal{C}}}(x_j)+\gamma_{\hat{\mathcal{A}}}(x_j)\,\gamma_{\hat{\mathcal{C}}}(x_j)+\pi_{\hat{\mathcal{A}}}(x_j)\pi_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\eta^2_{\hat{\mathcal{A}}}(x_j)+\gamma^2_{\hat{\mathcal{A}}}(x_j)+\pi^2_{\hat{\mathcal{A}}}(x_j)}\sqrt{\eta^2_{\hat{\mathcal{C}}}(x_j)+\gamma^2_{\hat{\mathcal{C}}}(x_j)+\pi^2_{\hat{\mathcal{C}}}(x_j)}}\right]$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times\frac{0.4\times 0.1+0.3\times 0.2+0.3\times 0.7}{\sqrt{0.4^2+0.3^2+0.3^2}\sqrt{0.1^2+0.2^2+0.7^2}} \\ +0.2\times\frac{0.2\times 0.3+0.5\times 0.5+0.3\times 0.2}{\sqrt{0.2^2+0.5^2+0.3^2}\sqrt{0.3^2+0.5^2+0.2^2}} \\ +0.3\times\frac{0.6\times 0.5+0.2\times 0.4+0.2\times 0.1}{\sqrt{0.6^2+0.2^2+0.2^2}\sqrt{0.5^2+0.4^2+0.1^2}} \\ +0.4\times\frac{0.3\times 0.4+0.1\times 0.6+0.6\times 0}{\sqrt{0.3^2+0.1^2+0.6^2}\sqrt{0.4^2+0.6^2+0^2}} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times 0.7324+0.2\times 0.9736 \\ + \\ 0.3\times 0.9304+0.4\times 0.3680 \end{pmatrix}$$

$$= 0.1735.$$

- $\mathrm{WIFCS}^3(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j\,cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x_j)-\eta_{\hat{\mathcal{C}}}(x_j)|, \\ |\gamma_{\hat{\mathcal{A}}}(x_j)-\gamma_{\hat{\mathcal{C}}}(x_j)|, \\ |\pi_{\hat{\mathcal{A}}}(x_j)-\pi_{\hat{\mathcal{C}}}(x_j)|\end{pmatrix}\right)\right]$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.4-\ 0.1|,\\ |0.3-0.2|,\\ |0.3-0.7|\end{pmatrix}\right)\right] \\ +0.2\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.2-\ 0.3|,\\ |0.5-0.5|,\\ |0.3-0.2|\end{pmatrix}\right)\right] \\ +0.3\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.6-\ 0.5|,\\ |0.2-0.4|,\\ |0.3-0.1|\end{pmatrix}\right)\right] \\ +0.4\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.3-\ 0.4|,\\ |0.1-0.6|,\\ |0.6-0|\end{pmatrix}\right)\right]\end{pmatrix}$$

$$= \frac{1}{4}\left(0.1\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.3|,\\ |0.1|,\\ |-0.4|\end{pmatrix}\right)\right] + 0.2\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|-0.1|,\\ |0|,\\ |0.1|\end{pmatrix}\right)\right] + 0.3 \times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|0.1|,\\ |0.2|,\\ |0.1|\end{pmatrix}\right)\right] + 0.4\times cos\left[\frac{\pi}{2}\left(max\begin{pmatrix}|-\ 0.1|,\\ |-0.5|,\\ |0.6|\end{pmatrix}\right)\right]\right)$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cos\left[\frac{\pi}{2}(0.4)\right] + 0.2\times cos\left[\frac{\pi}{2}(0.1)\right] \\ +0.3\times cos\left[\frac{\pi}{2}(0.2)\right] + 0.4\times cos\left[\frac{\pi}{2}(0.6)\right]\end{pmatrix}$$

$$= \frac{1}{4}(0.1\times 0.9999 + 0.2\times 0.9999 + 0.3\times 0.9999 + 0.4\times 0.9998)$$

$$= 0.2499.$$

- $\text{WIFCS}^4(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j cos\left[\frac{\pi}{4}\begin{pmatrix}\left|\eta_{\hat{\mathcal{A}}}(x_j) -\ \eta_{\hat{\mathcal{C}}}(x_j)\right| + \\ \left|\gamma_{\hat{\mathcal{A}}}(x_j) -\ \gamma_{\hat{\mathcal{C}}}(x_j)\right| + \\ \left|\pi_{\hat{\mathcal{A}}}(x_j) -\ \pi_{\hat{\mathcal{C}}}(x_j)\right|\end{pmatrix}\right]$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cos\left[\frac{\pi}{4}\begin{pmatrix}|0.4-\ 0.1| + \\ |0.3-0.2| + \\ |0.3-0.7|\end{pmatrix}\right] + 0.2\times cos\left[\frac{\pi}{4}\begin{pmatrix}|0.2-\ 0.3| + \\ |0.5-0.5| + \\ |0.3-0.2|\end{pmatrix}\right] + \\ 0.3\times cos\left[\frac{\pi}{4}\begin{pmatrix}|0.6-\ 0.5| + \\ |0.2-0.4| + \\ |0.2-0.1|\end{pmatrix}\right] + 0.4\times cos\left[\frac{\pi}{4}\begin{pmatrix}|0.3-\ 0.4| + \\ |0.1-0.6| + \\ |0.6-0|\end{pmatrix}\right]\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cos\left[\frac{\pi}{4}\begin{pmatrix}|0.3|+\\|0.1|+\\|-0.4|\end{pmatrix}\right]+0.2\times cos\left[\frac{\pi}{4}\begin{pmatrix}|-0.1|+\\|0|+\\|0.1|\end{pmatrix}\right] \\ +0.3\times cos\left[\frac{\pi}{4}\begin{pmatrix}|0.1|+\\|-0.2|+\\|0.1|\end{pmatrix}\right]+0.4\times cos\left[\frac{\pi}{4}\begin{pmatrix}|-0.1|+\\|-0.5|+\\|0.6|\end{pmatrix}\right]\end{pmatrix}$$

$$= \frac{1}{4}\left(0.1\times cos\left[\frac{\pi}{4}(0)\right]+0.2\times cos\left[\frac{\pi}{4}(0)\right]+0.3cos\left[\frac{\pi}{4}(0)\right]+0.4\times cos\left[\frac{\pi}{4}(0)\right]\right)$$

$$= \frac{1}{4}(0.1\times 1+0.2\times 1+0.3\times 1+0.4\times 1)$$

$$= 0.2500.$$

- $\text{WIFCotS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x_j)-\eta_{\hat{\mathcal{C}}}(x_j)|,\\|\gamma_{\hat{\mathcal{A}}}(x_j)-\gamma_{\hat{\mathcal{C}}}(x_j)|\end{pmatrix}\right)\right]$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.4-0.1|,\\|0.3-0.2|\end{pmatrix}\right)\right] \\ +0.2\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.2-0.3|,\\|0.5-0.5|\end{pmatrix}\right)\right] \\ +0.3\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.6-0.5|,\\|0.2-0.4|\end{pmatrix}\right)\right] \\ +0.4\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3-0.4|,\\|0.1-0.6|\end{pmatrix}\right)\right]\end{pmatrix}$$

$$= \frac{1}{4}\left(0.1\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3|,\\|0.1|\end{pmatrix}\right)\right]+0.2\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\|0|\end{pmatrix}\right)\right]+0.3\right.$$
$$\left.\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.1|,\\|-0.2|\end{pmatrix}\right)\right]+0.4\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\|-0.5|\end{pmatrix}\right)\right]\right)$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.3)\right]+0.2\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.1)\right] \\ +0.3\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.2)\right]+0.4\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.5)\right]\end{pmatrix}$$

$$= \frac{1}{4}(0.1\times 0.6128+0.2\times 0.8540+0.3\times 0.7265+0.4\times 0.4142)$$

$$= 0.1539.$$

- $\mathrm{WIFCotS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\omega_j \, cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|\eta_{\hat{\mathcal{A}}}(x_j)-\eta_{\hat{\mathcal{C}}}(x_j)|,\\ |\gamma_{\hat{\mathcal{A}}}(x_j)-\gamma_{\hat{\mathcal{C}}}(x_j)|,\\ |\pi_{\hat{\mathcal{A}}}(x_j)-\pi_{\hat{\mathcal{C}}}(x_j)|\end{pmatrix}\right)\right]$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.4-0.1|,\\ |0.3-0.2|,\\ |0.3-0.7|\end{pmatrix}\right)\right] \\ +0.2\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.2-0.3|,\\ |0.5-0.5|,\\ |0.3-0.2|\end{pmatrix}\right)\right] \\ +0.3\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.6-0.5|,\\ |0.2-0.4|,\\ |0.2-0.1|\end{pmatrix}\right)\right] \\ +0.4\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3-0.4|,\\ |0.1-0.6|,\\ |0.6-0|\end{pmatrix}\right)\right]\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.3|,\\ |0.1|,\\ |-0.4|\end{pmatrix}\right)\right] \\ +0.2\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\ |0|,\\ |0.1|\end{pmatrix}\right)\right] \\ +0.3\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|0.1|,\\ |-0.2|,\\ |0.1|\end{pmatrix}\right)\right] \\ +0.4\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(max\begin{pmatrix}|-0.1|,\\ |-0.5|,\\ |0.6|\end{pmatrix}\right)\right]\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0.1\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.4)\right]+0.2\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.1)\right] \\ +0.3\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.2)\right]+0.4\times cot\left[\frac{\pi}{4}+\frac{\pi}{4}(0.6)\right]\end{pmatrix}$$

$$= \frac{1}{4}(0.1\times 0.5090+0.2\times 0.8540+0.3\times 0.7265+0.4\times 0.3249)$$

$= 0.1424.$

Results obtained from the above similarity measures methods are given in (Table 3.1).

Table 3.1 Similarity Measures

Similarity Measures	**Results**
$\mathrm{IFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.9051
$\mathrm{IFCS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7511
$\mathrm{IFCS}^3(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.8338
$\mathrm{IFCS}^4(\hat{\mathcal{A}},\hat{\mathcal{C}})$	1.0000
$\mathrm{IFCotS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.6518
$\mathrm{IFCotS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.6036
$\mathrm{WIFCS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2210
$\mathrm{WIFCS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.1735
$\mathrm{WIFCS}^3(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2499
$\mathrm{WIFCS}^4(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2500
$\mathrm{WIFCotS}^1(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.1539
$\mathrm{WIFCotS}^2(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.1424

4. SIMILARITY MEASURES OF Q-RUNG ORTHOPAIR FUZZY SETS

In this section, we present some similarity measures of q-ROFSs existing in literature (Liu et al. 2019, Peng and Liu 2019, Jan et al. 2020) with their examples.

4.1 Similarity Measures Based on Distance Measure

Definition 4.1.1 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$, the Euclidean distance measure, denoted by $d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$, is defined as follows:

$$d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sqrt{\frac{1}{2n}\sum_{x_j \in \mathcal{X}} (|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2 + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2)}.$$

The weighted Euclidean distance measure that corresponds, referred to $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$, is defined as follows:

$$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sqrt{\tfrac{1}{2}\textstyle\sum_{x_j \in \mathcal{X}} \omega_j (|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2 + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2)},$$

where $\omega_j (j = 1,2, \ldots, n)$ denotes the weight of elementx_j, $j = (1,2, \ldots, n)$ which satisfies the condition of $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 4.1.1 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the weighted Euclidean distance measure $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ satisfies the following requirements:

(1) $0 \leq d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = d_{wq\mathcal{ROF}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0$ iff $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$,

$\gamma_{\mathcal{A}}(x_j) = \gamma_{\hat{C}}(x_j)$.

Proof.

(1) Since $0 \leq \eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\mathcal{A}}(x_j), \eta_{\hat{C}}(x_j), \gamma_{\hat{C}}(x_j) \leq 1$, then $\kappa \geq 1$, $0 \leq |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{C}}{}^{\kappa}(x_j)|^2 \leq 1, 0 \leq |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{C}}{}^{\kappa}(x_j)|^2 \leq 1$.Thus, we have

$$0 \leq d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{C}) \leq \sqrt{\frac{1}{2}} \cdot \sqrt{2 \sum_{j=1}^{n} \omega_j} = 1.$$

(2) Since

$$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{C}) = \sqrt{\frac{1}{2} \sum_{x_j \in \mathcal{X}} \omega_j (|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{C}}{}^{\kappa}(x_j)|^2 + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{C}}{}^{\kappa}(x_j)|^2)} = \sqrt{\frac{1}{2} \sum_{x_j \in \mathcal{X}} \omega_j (|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{C}}{}^{\kappa}(x_j)|^2 + |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{C}}{}^{\kappa}(x_j)|^2)} = d_{wq\mathcal{ROF}}(\hat{C}, \hat{\mathcal{A}}),$$

(3) Since $\hat{\mathcal{A}} = \hat{C} \Leftrightarrow \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{C}}{}^{\kappa}(x_j)\right| = 0$ and $\left|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{C}}{}^{\kappa}(x_j)\right| = 0 \Leftrightarrow d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{C}) = 0$.

Example 4.1.1 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8,0.7)_{\kappa}\rangle, \langle x_2, (0.5,0.9)_{\kappa}\rangle, \langle x_3, (0.6,0.5)_{\kappa}\rangle\}$ and $\hat{C}$=$\{\langle x_1, (0.6,0.9)_{\kappa}\rangle, \langle x_2, (0.8,0.7)_{\kappa}\rangle, \langle x_3, (0.7,0.6)_{\kappa}\rangle\}$, be any two q-ROFSs over $\mathcal{X}$, assume that $\kappa = 3$. Then, using the Euclidean distance measurement system $d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{C})$ between $\hat{\mathcal{A}}$ and $\hat{C}$ can be obtained as follows:

- $d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{C}) = \sqrt{\frac{1}{2n} \sum_{x_j \in \mathcal{X}} (|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{C}}{}^{\kappa}(x_j)|^2 + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{C}}{}^{\kappa}(x_j)|^2)}$

$$d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{C}) = \sqrt{\frac{1}{2(3)} \begin{pmatrix} (|0.8^3 - 0.6^3|^2 + |0.7^3 - 0.9^3|^2) + \\ (|0.5^3 - 0.8^3|^2 + |0.9^3 - 0.7^3|^2) + \\ (|0.6^3 - 0.7^3|^2 + |0.5^3 - 0.6^3|^2) \end{pmatrix}}$$

$$= \sqrt{\frac{1}{6} \begin{pmatrix} (0.0876 + 0.1489) + \\ (0.1497 + 0.1489) + \\ (0.0161 + 0.0082) \end{pmatrix}}$$

$$= \sqrt{0.0932}$$

$$= 0.3052.$$

Let $\hat{\mathcal{A}}$={$\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9)_\kappa\rangle,\langle x_3,(0.6,0.5)_\kappa\rangle$} and $\hat{\mathcal{C}}$={$\langle x_1,(0.6,0.9)_\kappa\rangle,\langle x_2,(0.8,0.7)_\kappa\rangle,\langle x_3,(0.7,0.6)_\kappa\rangle$}, be any two q-ROFSs over $\mathcal{X}$, if $\kappa=3$ and $\omega_j=(0.25,0.4,0.35)$. Then, the following can be obtained by using the weighted Euclidean distance measure between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$:

- $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sqrt{\frac{1}{2}\sum_{x_j\in X}\omega_j(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2+|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2)}$

$$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sqrt{\frac{1}{2}\begin{pmatrix}0.25\times(|0.8^3-0.6^3|^2+|0.7^3-0.9^3|^2)+\\0.4\times(|0.5^3-0.8^3|^2+|0.9^3-0.7^3|^2)+\\0.35\times(|0.6^3-0.7^3|^2+|0.5^3-0.6^3|^2)\end{pmatrix}}$$

$$=\sqrt{\frac{1}{2}\begin{pmatrix}0.25\times(0.0876+0.1489)+\\0.4\times(0.1497+0.1489)+\\0.35\times(0.0161+0.0082)\end{pmatrix}}$$

$$=\sqrt{0.0935}$$

$$=0.3057.$$

Definition 4.1.2 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$={$\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}$} and $\hat{\mathcal{C}}$={$\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}$} be any two q-ROFSs over $\mathcal{X}$. Then, a cosine similarity measure denoted by $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ is defined as follows:

$$\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}.$$

Theorem 4.1.2 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$={$\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}$} and $\hat{\mathcal{C}}$={$\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}$} be any two q-ROFSs over $\mathcal{X}$. Then, a cosine similarity measure $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0\le\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\le 1$,

(2) $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1$ which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

1. For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have

$0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then,

$\sqrt{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)} \leq$
$\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$, and so

$$0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)$$
$$\leq \sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Thus, $0 \leq \frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}} \leq 1$. It is concluded that

$$0 \leq \frac{1}{n}\sum_{j=1}^{n} \frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}} \leq 1.$$

2. Since

$$\begin{aligned}
\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) &= \frac{1}{n}\sum_{j=1}^{n} \frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}} \\
&= \frac{1}{n}\sum_{j=1}^{n} \frac{\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}} \\
&= \mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{C}},\hat{\mathcal{A}}),
\end{aligned}$$

3. Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$, then $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=1$. Then, we complete that process of proofs. If we consider the weight of $x_j \in \mathcal{X}$, then the weighted cosine similarity measure $\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between q-ROFSs can be defined as follows.

Example 4.1.2 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9)_\kappa\rangle,\langle x_3,(0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9)_\kappa\rangle,\langle x_2,(0.8,0.7)_\kappa\rangle,\langle x_3,(0.7,0.6)_\kappa\rangle$, be any two q-ROFSs over $\mathcal{X}$, and assume that $\kappa=3$. Then, the cosine similarity measure $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\;\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}$$

$$=\frac{1}{3}\left[\begin{array}{c}\dfrac{0.8^3\times\ 0.6^3+0.7^3\times 0.9^3}{\sqrt{(0.8^3)^2+(0.7^3)^2}\ \sqrt{(0.6^3)^2+(0.9^3)^2}}+\\ \dfrac{0.5^3\times\ 0.8^3+0.9^3\times 0.7^3}{\sqrt{(0.5^3)^2+(0.9^3)^2}\ \sqrt{(0.8^3)^2+(0.7^3)^2}}+\\ \dfrac{0.6^3\times\ 0.7^3+0.5^3\times 0.6^3}{\sqrt{(0.6^3)^2+(0.5^3)^2}\ \sqrt{(0.7^3)^2+(0.6^3)^2}}\end{array}\right]$$

$$=\frac{1}{3}[0.7696+0.6889+0.9993]$$

$$=0.8192.$$

Definition 4.1.3 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$, assume ω_j is the weight of $x_j\in\mathcal{X}$ and $\sum_{j=1}^{n}\omega_j=1,(0\le\omega_j\le 1)$. Then, the weighted cosine similarity measure $\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ is defined as follows:

$$\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sum_{j=1}^{n}\omega_j\frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}.$$

Theorem 4.1.3 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}=\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j\in\mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, $\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ satisfies the following properties:

(1) $0\le\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\le 1$,

(2) $\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) $\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=1$ if $\hat{\mathcal{A}}=\hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j)=\eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j)=\gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

1. For $\kappa\ge 1$, $0\le\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\le 1$. Then, from Cauchy-Schwarz inequality, we have

$$0\le\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)$$
$$\le\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right).$$

Then, $\sqrt{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}$

$\le\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$, and so $0\le$ $\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\le$ $\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$.

Thus, $0\le\frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}\le 1$. It is concluded that

$$0\le\sum_{j=1}^{n}\omega_j\frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}\le 1.$$

2. Since

$$\mathcal{C}_{W_{q\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n}\omega_j \frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}$$

$$= \sum_{j=1}^{n}\omega_j \frac{\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}}$$

$$= \mathcal{C}_{W_{q\mathcal{ROF}}}(\hat{\mathcal{C}},\hat{\mathcal{A}}),$$

3. Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$, then $\mathcal{C}_{W_{q\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1$. Then, we complete this process of proofs.

Example 4.1.3 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9)_\kappa\rangle,\langle x_3,(0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9)_\kappa\rangle,\langle x_2,(0.8,0.7)_\kappa\rangle,\langle x_3,(0.7,0.6)_\kappa\rangle\}$, be any two q-ROFSs over $\mathcal{X}$, if $\kappa = 3$ and $\omega_j = (0.25,0.4,0.35)$. The weighted cosine similarity measure $\mathcal{C}_{W_{q\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can then be calculated as follows:

$$\mathcal{C}_{W_{q\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n}\omega_j \frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}$$

$$\mathcal{C}_{W_{q\mathcal{ROF}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \begin{bmatrix} 0.25 \times \dfrac{0.8^3 \times\ 0.6^3 + 0.7^3 \times 0.9^3}{\sqrt{(0.8^3)^2 + (0.7^3)^2}\ \sqrt{(0.6^3)^2 + (0.9^3)^2}} + \\ 0.4 \times \dfrac{0.5^3 \times\ 0.8^3 + 0.9^3 \times 0.7^3}{\sqrt{(0.5^3)^2 + (0.9^3)^2}\ \sqrt{(0.8^3)^2 + (0.7^3)^2}} + \\ 0.35 \times \dfrac{0.6^3 \times\ 0.7^3 + 0.5^3 \times 0.6^3}{\sqrt{(0.6^3)^2 + (0.5^3)^2}\ \sqrt{(0.7^3)^2 + (0.6^3)^2}} \end{bmatrix}$$

$$= [0.1924 + 0.2755 + 0.3497]$$

$$= 0.8176.$$

A similarity measure is a function satisfying the condition given in the following Lemma.

Lemma 4.1.1 (Liu *et al.* 2019) Assume that there are any two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$. Then, a similarity measure defined by $\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ satisfies the following properties:

(1) $0 \leq \hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,
(2) $\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \hat{\mathcal{S}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,
(3) $\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, if and only if $\hat{\mathcal{A}} = \hat{\mathcal{C}}$,

Then, the relationship between the distance measure $d(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ and similarity measure $\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ can be expressed as $d(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$.
According to Definitions 4.1.2 and 4.1.3, cosine similarity measures are not certified similarity measures. Because they do not satisfy property (3) of Lemma 4.1.1 in certain arrangements.

Definition 4.1.4 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$, assume ω_j is the weight of $x_j \in \mathcal{X}$ and $\sum_{j=1}^{n} \omega_j = 1 (0 \leq \omega_j \leq 1; j = 1,2, \ldots, n)$. Then, the weighted similarity measure $\mathcal{S}_{\mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ is defined as follows:

$$\mathcal{S}_{\mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + 1 - d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})}{2}$$

where,

$$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sqrt{\frac{1}{2} \sum_{x_j \in \mathcal{X}} \omega_j (|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2 + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)|^2)},$$

$$\mathcal{C}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sum_{j=1}^{n} \omega_j \frac{\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\, \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\, \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)}{\sqrt{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j))^2 + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j))^2}\, \sqrt{(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j))^2 + (\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j))^2}}.$$

The weighted similarity measure $\mathcal{S}_{\mathcal{W}q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ is reduced to the following similarity measure $\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$, if $\omega = (\omega_1, \omega_2, \ldots, \omega_n) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$.

Definition 4.1.5 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the similarity measure $\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ is defined as follows:

$$\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + 1 - d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})}{2}$$

where,

$$d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sqrt{\frac{1}{2n}\sum_{x_j \in \mathcal{X}}(|\eta^{\kappa}_{\hat{\mathcal{A}}}(x_j) - \eta^{\kappa}{}_{\hat{\mathcal{C}}}(x_j)|^2 + |\gamma^{\kappa}{}_{\hat{\mathcal{A}}}(x_j) - \gamma^{\kappa}{}_{\hat{\mathcal{C}}}(x_j)|^2)}\ ,$$

$$\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\eta^{\kappa}_{\hat{\mathcal{A}}}(x_j)\, \eta^{\kappa}_{\hat{\mathcal{C}}}(x_j) + \gamma^{\kappa}_{\hat{\mathcal{A}}}(x_j)\, \gamma^{\kappa}_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\ \sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}.$$

Theorem 4.1.4 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the weighted cosine similarity measure $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ satisfies the following properties:

(1) $0 \leq \mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, iff $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) Since $0 \leq \mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$, according to Theorem 4.1.3, since $0 \leq d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$, it's a distance measure based on Lemma 2.1 (Liu and Wang 2018), then $0 \leq \frac{\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + 1 - d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})}{2} \leq 1$, that is, $0 \leq \mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$ obtained.

(2) Since $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$ and $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = d_{wq\mathcal{ROF}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$, according to Theorem 4.1.1 and 4.1.3, then $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$.

(3) If $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, then $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + 1 - d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 2$, that is, $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 + d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$. For all $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$, $0 \leq \mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$ and

$0 \leq d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$ it must simultaneously $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$ and $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0$. According to Theorem 4.1.1, when $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0$, $\hat{\mathcal{A}} = \hat{\mathcal{C}}$. In contrast, if $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, we have $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$ and $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0$, then $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$. Thus, we have $\mathcal{S}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$ iff $\hat{\mathcal{A}} = \hat{\mathcal{C}}$.

Example 4.1.4 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8{,}0.7)_\kappa\rangle, \langle x_2, (0.5{,}0.9)_\kappa\rangle, \langle x_3, (0.6{,}0.5)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6{,}0.9)_\kappa\rangle, \langle x_2, (0.8{,}0.7)_\kappa\rangle, \langle x_3, (0.7{,}0.6)_\kappa\rangle\}$be any two q-ROFSs over $\mathcal{X}$, assume that $\kappa = 3$ and $\omega_j = \left(\frac{1}{4}, \frac{2}{5}, \frac{7}{20}\right)$. Then, the weighted similarity measure $\mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ and the similarity measure $\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be obtained as follows:

- The corresponding weighted similarity measure $\mathcal{S}_{WqROF}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ is obtained as follows:

$$\mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + 1 - d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})}{2}$$

Since $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0.3057$ and $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0.8176$, then

$$\mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{0.8176 + 1 - 0.3057}{2}$$

$$\mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0.7559.$$

- The similarity measure $\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ is obtained as follows:

$$\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + 1 - d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})}{2}$$

Since $d_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0.3052$ and $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0.8192$, then

$$\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{0.8192 + 1 - 0.3052}{2}$$

$$\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0.757.$$

Definition 4.1.6 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the distance measure $\mathcal{D}_{q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ is characterized as follows:

$$\mathcal{D}_{q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1-\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})+d_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})}{2}$$

where,

$$d_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sqrt{\frac{1}{2n}\sum_{x_j\in\mathcal{X}}(|\eta^{\kappa}_{\hat{\mathcal{A}}}(x_j)-\eta^{\kappa}{}_{\hat{\mathcal{C}}}(x_j)|^2+|\gamma^{\kappa}{}_{\hat{\mathcal{A}}}(x_j)-\gamma^{\kappa}{}_{\hat{\mathcal{C}}}(x_j)|^2)}\,,$$

$$\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{\eta^{\kappa}_{\hat{\mathcal{A}}}(x_j)\,\eta^{\kappa}_{\hat{\mathcal{C}}}(x_j)+\gamma^{\kappa}_{\hat{\mathcal{A}}}(x_j)\,\gamma^{\kappa}_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\ \sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}}.$$

Definition 4.1.7 (Liu *et al.* 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa}\rangle|x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa}\rangle|x_j\in\mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$, assume ω_j is the weight of $x_j\in\mathcal{X}$ and $\sum_{j=1}^{n}\omega_j=1(0\le\omega_j\le1;j=1,2,\dots,n)$. Then, a weighted distance measure $\mathcal{D}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ can be defined as follows:

$$\mathcal{D}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=1-\mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1-\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})-d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})}{2}$$

where,

$$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sqrt{\frac{1}{2}\sum_{x_j\in\mathcal{X}}\omega_j(|\eta^{\kappa}_{\hat{\mathcal{A}}}(x_j)-\eta^{\kappa}{}_{\hat{\mathcal{C}}}(x_j)|^2+|\gamma^{\kappa}{}_{\hat{\mathcal{A}}}(x_j)-\gamma^{\kappa}{}_{\hat{\mathcal{C}}}(x_j)|^2)}$$

$$\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sum_{j=1}^{n}\omega_j\frac{\eta^{\kappa}_{\hat{\mathcal{A}}}(x_j)\,\eta^{\kappa}_{\hat{\mathcal{C}}}(x_j)+\gamma^{\kappa}_{\hat{\mathcal{A}}}(x_j)\,\gamma^{\kappa}_{\hat{\mathcal{C}}}(x_j)}{\sqrt{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}\ \sqrt{\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}},$$

The weighted distance measure $\mathcal{D}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ is reduced to the following distance measure $\mathcal{D}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$, if there are $\omega=(\omega_1,\omega_2,\dots,\omega_n)=(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})$.

Example 4.1.5 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7)_{\kappa}\rangle,\langle x_2,(0.5,0.9)_{\kappa}\rangle,\langle x_3,(0.6,0.5)_{\kappa}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9)_{\kappa}\rangle,\langle x_2,(0.8,0.7)_{\kappa}\rangle,\langle x_3,(0.7,0.6)_{\kappa}\rangle\}$ be any two q-ROFSs over $\mathcal{X}$, assume that $\kappa=3$ and $\omega_j=(\frac{1}{4},\frac{2}{5},\frac{7}{20})$. Then, the distance measure $\mathcal{D}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$, and the weighted distance measure $\mathcal{D}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ can be calculated as follows:

- The corresponding weighted similarity measure $\mathcal{S}_{WqROF}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1 - \mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) + d_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})}{2}$$

Since $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 0.3057$ and $\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 0.8192$, then

$$\mathcal{D}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1 - 0.8192 + 0.3057}{2}$$

$$\mathcal{D}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 0.2432.$$

- $\mathcal{D}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1 - \mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) - d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})}{2}$

Since $d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 0.3057$ and $\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 0.8176$, then

$$\mathcal{D}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1 - 0.8176 - 0.3057}{2}$$

$$\mathcal{D}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 0.2441.$$

Results obtained in examples are given in (Table 4.1).

Table 4.1 Similarity Measures Based on Distance Measure

Methods	Results
$d_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3052
$d_{wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3057
$\mathcal{C}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.8192
$\mathcal{C}_{Wq\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.8176
$\mathcal{S}_{q\mathcal{ROF}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.757
$\mathcal{S}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7559
$\mathcal{D}_{q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2441
$\mathcal{D}_{Wq\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2432

Definition 4.1.8 (Peng and Liu 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{S}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{S}}}(x_j), \gamma_{\hat{\mathcal{S}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any three q-ROFSs over $\mathcal{X}$. Then, the distance measure $\mathcal{F}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ is a mapping

$\mathcal{F}$: q-ROFS($\mathcal{X}$) ×q-ROFS($\mathcal{X}$) ⟶ [0,1], given as follows:

1. $0 \leq \mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$;
2. $\mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{F}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$;
3. $\mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0$ iff $\hat{\mathcal{A}} = \hat{\mathcal{C}}$;
4. $\mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{A}}^{*})= 1$ iff $\hat{\mathcal{A}}^{*}$ is a crisp set ;
5. $\mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq \mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{S}})$ and $\mathcal{F}(\hat{\mathcal{C}}, \hat{\mathcal{S}}) \leq \mathcal{F}(\hat{\mathcal{A}}, \hat{\mathcal{S}})$, Iff $\hat{\mathcal{A}} \subseteq \hat{\mathcal{C}} \subseteq \hat{\mathcal{S}}$.

Theorem 4.1.5 (Peng and Liu 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_i, (\eta_{\hat{\mathcal{A}}}(x_i), \gamma_{\hat{\mathcal{A}}}(x_i))_\kappa\rangle | x_i \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_i, (\eta_{\hat{\mathcal{C}}}(x_i), \gamma_{\hat{\mathcal{C}}}(x_i))_\kappa\rangle | x_i \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, $\mathcal{F}_i(\hat{\mathcal{A}}, \hat{\mathcal{C}})$, $(i = 1,2,3 \ldots,13)$ given as follows are distance measures:

(1) $\mathcal{F}_1(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$;

(2) $\mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$;

(3) $\mathcal{F}_3(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{4|\mathcal{X}|}(\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|) + \sum_{x_i \in \mathcal{X}}|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)| + |\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$;

(4) $\mathcal{F}_4(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$;

(5) $\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}{1+|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}$;

(6) $\mathcal{F}_6(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2\sum_{x_i \in \mathcal{X}}|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}{\sum_{x_i \in \mathcal{X}}(1+|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)}$;

(7) $\mathcal{F}_7(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \lambda\frac{\sum_{x_i \in \mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i \in \mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)} - \tau\frac{\sum_{x_i \in \mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i \in \mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$, where $(\lambda + \tau = 1)$ and $\lambda, \tau \in [0,1]$;

(8) $\mathcal{F}_8(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{\lambda}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)} - \frac{\tau}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$, where $(\lambda + \tau = 1)$ and $\lambda, \tau \in [0,1]$;

(9) $\mathcal{F}_9(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}$;

(10) $\mathcal{F}_{10}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{\sum_{x_i \in \mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}{\sum_{x_i \in \mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}$;

(11) $\mathcal{F}_{11}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right) + \left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) \wedge \left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right) + \left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) \vee \left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$;

(12) $\mathcal{F}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{\sum_{x_i\in\mathcal{X}}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\wedge\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}{\sum_{x_i\in\mathcal{X}}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\vee\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}$;

(13) $\mathcal{F}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sqrt[\kappa]{\begin{array}{l}\frac{1}{2|\mathcal{X}|\,(s_1+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|s_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}+\\ \frac{1}{2|\mathcal{X}|\,(s_2+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|s_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}\end{array}}$.

Proof .

For q-ROFSs to qualify as a reasonable measure of distance in order to $\mathcal{F}_i(\hat{\mathcal{A}},\hat{\mathcal{C}})$, where $i = (1,2,3 \dots,13)$, should satisfy $(\mathcal{F}_1)-(\mathcal{F}_5)$ of axiomatic requirements. Since it is easy to prove $(\mathcal{F}_1)-(\mathcal{F}_4)$, we only prove $(\mathcal{F}_5)$ of each of distance measures $\mathcal{F}_i(\hat{\mathcal{A}},\hat{\mathcal{C}})$. We have $\eta_{\hat{\mathcal{A}}}(x_i) \le \eta_{\hat{\mathcal{C}}}(x_i) \le \eta_{\hat{\mathcal{S}}}(x_i)$ and $\gamma_{\hat{\mathcal{A}}}(x_i) \ge \gamma_{\hat{\mathcal{C}}}(x_i) \ge \gamma_{\hat{\mathcal{S}}}(x_i)$, for $\forall x_i \in \mathcal{X}$, if $\hat{\mathcal{A}} \subseteq \hat{\mathcal{C}} \subseteq \hat{\mathcal{S}}$.

Thus, $|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \le |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)|$, $|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \le |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)|$ and $|\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| = \left|1-\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(1-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right| = \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|$, $|\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\pi_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)| = \left|1-\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(1-\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right| = \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|$.

1. By the formula of the distance measure, we have

$$\mathcal{F}_1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\Big(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|+|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|\Big),$$

$$\mathcal{F}_1(\hat{\mathcal{A}},\hat{\mathcal{S}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\Big(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)|+|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)| + \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|\Big).$$

Let $\mathcal{F}_1 = \mathcal{F}_1(\hat{\mathcal{A}},\hat{\mathcal{S}}) - \mathcal{F}_1(\hat{\mathcal{A}},\hat{\mathcal{C}})$; then we have

$$\mathcal{F}_1 = \frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\Big(\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right| - \left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)+\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|\Big).$$

Suppose we have four cases to consider for $\mathcal{F}_1$:

Case 1. $\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \geq max\{\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i), \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\}$, then

$$\mathcal{F}_1 = \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right) \geq 0\,;$$

Case 2. $\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \leq min\{\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i), \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\}$, then

$$\mathcal{F}_1 = \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\left(\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right) \geq 0\,;$$

Case 3. $\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \leq \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)$, then

$$\mathcal{F}_1 = \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)$$

$$\geq \tfrac{1}{|\mathcal{X}|}\textstyle\sum_{x_i\in\mathcal{X}}\Big(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \big(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\big) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\Big) = \tfrac{1}{|\mathcal{X}|}\textstyle\sum_{x_i\in\mathcal{X}}\big(\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\big) \geq 0;$$

Case 4. $\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \leq \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)$, then

$$\mathcal{F}_1 = \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\left(\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)$$

$$\geq \tfrac{1}{|\mathcal{X}|}\textstyle\sum_{x_i\in\mathcal{X}}\Big(\big(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\big) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\Big) = \tfrac{1}{|\mathcal{X}|}\textstyle\sum_{x_i\in\mathcal{X}}\big(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\big) \geq 0\,.$$

From the four cases mentioned above, we have it $\mathcal{F}_1 \geq 0$, which means that, $\mathcal{F}_1(\hat{\mathcal{A}}, \hat{\mathcal{S}}) \geq \mathcal{F}_1(\hat{\mathcal{A}}, \hat{\mathcal{C}})$. Similarly $\mathcal{F}_1(\hat{\mathcal{A}}, \hat{\mathcal{S}}) \geq \mathcal{F}_1(\hat{\mathcal{C}}, \hat{\mathcal{S}})$. We complete the process of proofs.

2. By the formula of the distance measure, we have

$$\mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\left(\left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|\right),$$

$$= \tfrac{1}{2|\mathcal{X}|}\textstyle\sum_{x_i\in\mathcal{X}}\Big(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \big(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\big)\Big),$$

$$\mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{S}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\left(\left|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|\right),$$

$$= \frac{1}{2|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) \right) \right).$$

Let $\mathcal{F}_1 = \mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{S}}) - \mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}})$; then we have

$$\mathcal{F}_1 = \frac{1}{2|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \left(\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) + \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) \right) \geq 0,$$

Thus, $\mathcal{F}_1 = \mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{S}}) \geq \mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}})$. In other words, $\mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{S}}) \geq \mathcal{F}_2(\hat{\mathcal{C}}, \hat{\mathcal{S}})$. We complete the process of proofs.

3. It can be shown in the same way with (1).

4. By the formula of the distance measure, we have

$$\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}{1 + |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}$$

$$\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{1}{\frac{1}{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|} + 1}$$

$$\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{S}}) = \frac{2}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)|}{1 + |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}$$

$$\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{S}}) = \frac{2}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{1}{\frac{1}{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)|} + 1}.$$

So, $\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq \mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{S}})$. Because $|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)| \geq |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|$. Similarly, $\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq \mathcal{F}_5(\hat{\mathcal{C}}, \hat{\mathcal{S}})$. We complete the process of proofs.

It can be shown in the same way from (5).

5. By the formula of the distance measure, we have

$$\mathcal{F}_7(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \lambda \frac{\sum_{x_i \in \mathcal{X}} \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \right)}{\sum_{x_i \in \mathcal{X}} \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \right)} - \tau \frac{\sum_{x_i \in \mathcal{X}} \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \right)}{\sum_{x_i \in \mathcal{X}} \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \right)}$$

$$\mathcal{F}_7(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \lambda \frac{\sum_{x_i \in \mathcal{X}} \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)}{\sum_{x_i \in \mathcal{X}} \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)} - \tau \frac{\sum_{x_i \in \mathcal{X}} \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)}{\sum_{x_i \in \mathcal{X}} \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)},$$

$$\mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{S}}) = 1-\lambda\frac{\sum_{x_i\in\mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i\in\mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)}-\tau\frac{\sum_{x_i\in\mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i\in\mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)}$$

$$\mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{S}}) = 1-\lambda\frac{\sum_{x_i\in\mathcal{X}}\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)}{\sum_{x_i\in\mathcal{X}}\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)}-\tau\frac{\sum_{x_i\in\mathcal{X}}\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)}{\sum_{x_i\in\mathcal{X}}\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)}.$$

So, $\mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{C}}) \le \mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{S}})$. Because $\frac{\sum_{x_i\in\mathcal{X}}\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)}{\sum_{x_i\in\mathcal{X}}\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)} \ge \frac{\sum_{x_i\in\mathcal{X}}\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)}{\sum_{x_i\in\mathcal{X}}\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)}, \frac{\sum_{x_i\in\mathcal{X}}\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)}{\sum_{x_i\in\mathcal{X}}\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)} \ge \frac{\sum_{x_i\in\mathcal{X}}\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)}{\sum_{x_i\in\mathcal{X}}\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)}$. Similarly, $\mathcal{F}_5(\hat{\mathcal{A}},\hat{\mathcal{S}}) \le \mathcal{F}_5(\hat{\mathcal{C}},\hat{\mathcal{S}})$.We complete the process of proofs.

6. It can be shown in the same way with (7).

7. By the formula of the distance measure, we have

$$\mathcal{F}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sqrt[\kappa]{\begin{gathered}\frac{1}{2|\mathcal{X}|\,(\mathcal{s}_1+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|\mathcal{s}_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}+\\ \frac{1}{2|\mathcal{X}|\,(\mathcal{s}_2+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|\mathcal{s}_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}\end{gathered}},$$

$$\mathcal{F}_{13}(\hat{\mathcal{A}},\hat{\mathcal{S}}) = \sqrt[\kappa]{\begin{gathered}\frac{1}{2|\mathcal{X}|\,(\mathcal{s}_1+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|\mathcal{s}_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}+\\ \frac{1}{2|\mathcal{X}|\,(\mathcal{s}_2+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|\mathcal{s}_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}\end{gathered}}$$

Since $\left|\mathcal{s}_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|$

$= \left(\mathcal{s}_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\mathcal{s}_1\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right),$

$= \left(\mathcal{s}_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)-\left(\mathcal{s}_1\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right),$

$\left|\mathcal{s}_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|$

$= \left(\mathcal{s}_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\mathcal{s}_1\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right),$

$= \left(\mathcal{s}_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)-\left(\mathcal{s}_1\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)$

$\left|\mathcal{s}_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|$

$= \left(s_2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - s_2\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right) - \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right),$

$= \left(s_2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) - \left(s_2\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)$

$\left|s_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right) - \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|$

$= \left(s_2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - s_1\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right) - \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right),$

$= \left(s_2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) - \left(s_2\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right).$

If $\hat{\mathcal{A}} \subseteq \hat{\mathcal{C}} \subseteq \hat{\mathcal{S}}$, then we have $\eta_{\hat{\mathcal{S}}} \geq \eta_{\hat{\mathcal{C}}} \geq \eta_{\hat{\mathcal{A}}}, \gamma_{\hat{\mathcal{S}}} \leq \gamma_{\hat{\mathcal{C}}} \leq \gamma_{\hat{\mathcal{A}}}$. Therefore,

$s_1\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) \geq s_1\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \geq s_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i),$

$s_2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \geq s_2\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) \geq s_2\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i).$

Thus,

$\left|\left(s_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) - \left(s_1\eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|$

$\geq \left|\left(s_1\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) - \left(s_1\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|,$

$\left|\left(s_2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) - \left(s_2\gamma_{\hat{\mathcal{S}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{S}}}{}^{\kappa}(x_i)\right)\right|$

$\geq \left|\left(s_1\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right) - \left(s_2\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|.$

Therefore, $\mathcal{F}_{13}(\hat{\mathcal{A}}, \hat{\mathcal{S}}) \geq \mathcal{F}_{13}(\hat{\mathcal{C}}, \hat{\mathcal{S}})$ and $\mathcal{F}_{13}(\hat{\mathcal{A}}, \hat{\mathcal{S}}) \geq \mathcal{F}_{13}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$. We complete this process of proofs.

Example 4.1.6 Let $\hat{\mathcal{A}}$={$\langle x_1, (0.8{,}0.7)_\kappa\rangle, \langle x_2, (0.5{,}0.9)_\kappa\rangle, \langle x_3, (0.6{,}0.5)_\kappa$} and $\hat{\mathcal{C}}$={$\langle x_1, (0.6{,}0.9)_\kappa\rangle, \langle x_2, (0.8{,}0.7)_\kappa\rangle, \langle x_3, (0.7{,}0.6)_\kappa\rangle$}, assume that $\kappa = 3$ and $\omega_j = \left(\frac{1}{4}, \frac{2}{5}, \frac{7}{20}\right)$, $(s_1 = 0.6, s_2 = 0.4)$ be any two q-ROFSs over $\mathcal{X}$. Then, the distance measure $\mathcal{F}_i(\hat{\mathcal{A}}, \hat{\mathcal{C}})$, such that $(i = 1{,}2{,}3 \ldots{,}13)$, and $(\lambda = 0.4$ and $\tau = 0.6)$. When λ , $\tau \in [0,1]$ can be calculated as :

- $\mathcal{F}_1(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|X|}\sum_{x_i \in X}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$

$$\mathcal{F}_1(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|3|}\begin{pmatrix} (|0.8^3 - 0.6^3| + |0.7^3 - 0.9^3| + |0.525^3 - 0.380^3|) + \\ (|0.5^3 - 0.8^3| + |0.9^3 - 0.7^3| + |0.526^3 - 0.525^3|) + \\ (|0.6^3 - 0.7^3| + |0.5^3 - 0.6^3| + |0.870^3 - 0.761^3|) \end{pmatrix}$$

$$= \frac{1}{2|3|}\begin{pmatrix} (|0.296| + |-0.386| + |0.089|) + \\ (|-0.387| + |0.386| + |0.0008|) + \\ (|-0.127| + |-0.091| + |0.207|) \end{pmatrix}$$

$$= \frac{1}{2|3|}(1.9698)$$

$$= 0.3283.$$

- $\mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))|)$

$$\mathcal{F}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{2|3|}\begin{pmatrix} (|0.8^3 - 0.6^3 - (0.7^3 - 0.9^3)|) + \\ (|0.5^3 - 0.8^3 - (0.9^3 - 0.7^3)|) + \\ (|0.6^3 - 0.7^3 - (0.5^3 - 0.6^3)|) \end{pmatrix}$$

$$= \frac{1}{2|3|}\begin{pmatrix} (|0.682|) + \\ (|-0.773|) + \\ (-0.036) \end{pmatrix}$$

$$= \frac{1}{2|3|}(1.491)$$

$$= 0.2485.$$

- $\mathcal{F}_3(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{4|\mathcal{X}|}(\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| +$
$|\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|) + \sum_{x_j \in \mathcal{X}}|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)| + |\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$

$$\mathcal{F}_3(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{4|3|}\begin{pmatrix} (|0.8^3 - 0.6^3| + |0.7^3 - 0.9^3| + |0.525^3 - 0.380^3| + (|0.8^3 - 0.7^3| + |0.6^3 - 0.9^3|)) + \\ (|0.5^3 - 0.8^3| + |0.9^3 - 0.7^3| + |0.526^3 - 0.525^3| + (|0.5^3 - 0.9^3| + |0.8^3 - 0.7^3|)) + \\ (|0.6^3 - 0.7^3| + |0.5^3 - 0.6^3| + |0.870^3 - 0.761^3| + (|0.6^3 - 0.5^3| + |0.7^3 - 0.6^3|)) \end{pmatrix}$$

$$= \frac{1}{4|3|}\begin{pmatrix} (|0.296| + |-0.386| + |0.089| + (|0.169| + |-0.513|)) + \\ (|-0.387| + |0.386| + |0.0008| + (|-0.604| + |-0.169|)) + \\ (|-0.127| + |-0.091| + |0.217| + (|0.091| + |0.127|)) \end{pmatrix}$$

$$= \frac{1}{4|3|}(3.652)$$

$$= 0.3043.$$

- $\mathcal{F}_4(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$

$$\mathcal{F}_4(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{3}\begin{pmatrix} (|0.8^3 - 0.6^3| \vee |0.7^3 - 0.9^3|) + \\ (|0.5^3 - 0.8^3| \vee |0.9^3 - 0.7^3|) + \\ (|0.6^3 - 0.7^3| \vee |0.5^3 - 0.6^3|) \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} (|0.296| \vee |-0.386|) + \\ (|-0.387| \vee |0.386|) + \\ (|-0.127| \vee |-0.091|) \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} |-0.386| + \\ |-0.387| + \\ |-0.127| \end{pmatrix}$$

$$= 0.3.$$

- $\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}} \frac{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}{1 + |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}$

$$\mathcal{F}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2}{|3|}\begin{pmatrix} \dfrac{|0.8^3 - 0.6^3| \vee |0.7^3 - 0.9^3|}{1 + (|0.8^3 - 0.6^3| \vee |0.7^3 - 0.9^3|)} + \\ \dfrac{|0.5^3 - 0.8^3| \vee |0.9^3 - 0.7^3|}{1 + (|0.5^3 - 0.8^3| \vee |0.9^3 - 0.7^3|)} + \\ \dfrac{|0.6^3 - 0.7^3| \vee |0.5^3 - 0.6^3|}{1 + (|0.6^3 - 0.7^3| \vee |0.5^3 - 0.6^3|)} \end{pmatrix}$$

$$= \frac{2}{|3|}\begin{pmatrix} \dfrac{|0.296| \vee |-0.386|}{1 + (|0.296| \vee |-0.386|)} + \\ \dfrac{|-0.387| \vee |0.386|}{1 + (|-0.387| \vee |0.386|)} + \\ \dfrac{|-0.127| \vee |-0.091|}{1 + (|-0.127| \vee |-0.091|)} \end{pmatrix}$$

$$= \frac{2}{|3|}\begin{pmatrix} \dfrac{|-0.386|}{1 + |-0.386|} + \\ \dfrac{|-0.387|}{1 + |-0.387|} + \\ \dfrac{|-0.127|}{1 + |-0.127|} \end{pmatrix}$$

$$= \frac{2}{|3|}(0.6702)$$

$$= 0.4468.$$

- $\mathcal{F}_6(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)}{\sum_{x_i \in \mathcal{X}}(1 + |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)}$

$$\mathcal{F}_6(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 2\left(\frac{(|0.8^3 - 0.6^3| \vee |0.7^3 - 0.9^3|) + (|0.5^3 - 0.8^3| \vee |0.9^3 - 0.7^3|) + (|0.6^3 - 0.7^3| \vee |0.5^3 - 0.6^3|)}{(1 + |0.8^3 - 0.6^3| \vee |0.7^3 - 0.9^3|) + (1 + |0.5^3 - 0.8^3| \vee |0.9^3 - 0.7^3|) + (1 + |0.6^3 - 0.7^3| \vee |0.5^3 - 0.6^3|)}\right)$$

$$= 2\left(\frac{(|0.296| \vee |-0.386|) + (|-0.387| \vee |0.386|) + (|-0.127| \vee |-0.091|)}{(1 + |0.296| \vee |-0.386|) + (1 + |-0.387| \vee |0.386|) + (1 + |-0.127| \vee |-0.091|)}\right)$$

$$= 2\left(\frac{(|-0.386|) + (|-0.387|) + (|-0.127|)}{(1 + |-0.386|) + (1 + |-0.387|) + (1 + |-0.127|)}\right)$$

$$= 0.4615.$$

- $\mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \lambda \frac{\sum_{x_i\in\mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i\in\mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)} - \tau \frac{\sum_{x_i\in\mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i\in\mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$, where $\lambda = 0.4$ and $\tau = 0.6$, $(\lambda + \tau = 1)$ and $\lambda, \tau \in [0,1]$

$$\mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \begin{pmatrix} 1 - 0.4\left(\frac{(0.8^3\wedge0.6^3)+(0.5^3\wedge0.8^3)+(0.6^3\wedge0.7^3)}{(0.8^3\vee0.6^3)+(0.5^3\vee0.8^3)+(0.6^3\vee0.7^3)}\right) \\ -0.6\left(\frac{(0.7^3\wedge0.9^3)+(0.9^3\wedge0.7^3)+(0.5^3\wedge0.6^3)}{(0.7^3\vee0.9^3)+(0.9^3\vee0.7^3)+(0.5^3\vee0.6^3)}\right) \end{pmatrix}$$

$$= \left(1 - 0.4\left(\frac{0.6^3 + 0.5^3 + 0.6^3}{0.8^3 + 0.8^3 + 0.7^3}\right) - 0.6\left(\frac{0.7^3 + 0.7^3 + 0.5^3}{0.9^3 + 0.9^3 + 0.6^3}\right)\right)$$

$$= (0.8370 - 0.2906)$$

$$= 0.5464.$$

- $\mathcal{F}_8(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{\lambda}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}} \frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)} - \frac{\tau}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}} \frac{\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$, where $\lambda = 0.4$ and $\tau = 0.6$, $(\lambda + \tau = 1)$ and $\lambda, \tau \in [0,1]$

$$\mathcal{F}_8(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \begin{pmatrix} 1 - \frac{0.4}{3}\left(\frac{(0.8^3\wedge0.6^3)+(0.5^3\wedge0.8^3)+(0.6^3\wedge0.7^3)}{(0.8^3\vee0.6^3)+(0.5^3\vee0.8^3)+(0.6^3\vee0.7^3)}\right) \\ -\frac{0.6}{3}\left(\frac{(0.7^3\wedge0.9^3)+(0.9^3\wedge0.7^3)+(0.5^3\wedge0.6^3)}{(0.7^3\vee0.9^3)+(0.9^3\vee0.7^3)+(0.5^3\vee0.6^3)}\right) \end{pmatrix}$$

$$= \left(1 - \frac{0.4}{3}\left(\frac{0.6^3 + 0.5^3 + 0.6^3}{0.8^3 + 0.8^3 + 0.7^3}\right) - \frac{0.6}{3}\left(\frac{0.7^3 + 0.7^3 + 0.5^3}{0.9^3 + 0.9^3 + 0.6^3}\right)\right)$$

$$= (0.9456 - 0.0968)$$

$$= 0.8488.$$

- $\mathcal{F}_9(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}} \frac{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}$

$$\mathcal{F}_9(\hat{\mathcal{A}},\hat{\mathcal{C}})$$

$$= 1 - \frac{1}{3}\left(\frac{(0.8^3\wedge0.6^3) + (0.7^3\wedge0.9^3) + (0.5^3\wedge0.8^3) + (0.9^3\wedge0.7^3) + (0.6^3\wedge0.7^3) + (0.5^3\wedge0.6^3)}{(0.8^3\vee0.6^3) + (0.7^3\vee0.9^3) + (0.5^3\vee0.8^3) + (0.9^3\vee0.7^3) + (0.6^3\vee0.7^3) + (0.5^3\vee0.6^3)}\right)$$

$$= 1 - \frac{1}{3}\left(\frac{(0.6^3) + (0.7^3) + (0.5^3) + (0.7^3) + (0.6^3) + (0.5^3)}{(0.8^3) + (0.9^3) + (0.8^3) + (0.9^3) + (0.7^3) + (0.6^3)}\right)$$

$$= 1 - \frac{1}{3}(0.4498)$$

$$= 0.8500.$$

- $\mathcal{F}_{10}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{\sum_{x_i\in\mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}{\sum_{x_i\in\mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))+(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}$

$$\mathcal{F}_{10}(\hat{\mathcal{A}},\hat{\mathcal{C}})$$
$$= 1 - \left(\frac{(0.8^3\wedge 0.6^3) + (0.7^3\wedge 0.9^3) + (0.5^3\wedge 0.8^3) + (0.9^3\wedge 0.7^3) + (0.6^3\wedge 0.7^3) + (0.5^3\wedge 0.6^3)}{(0.8^3\vee 0.6^3) + (0.7^3\vee 0.9^3) + (0.5^3\vee 0.8^3) + (0.9^3\vee 0.7^3) + (0.6^3\vee 0.7^3) + (0.5^3\vee 0.6^3)}\right)$$
$$= 1 - \left(\frac{(0.6^3) + (0.7^3) + (0.5^3) + (0.7^3) + (0.6^3) + (0.5^3)}{(0.8^3) + (0.9^3) + (0.8^3) + (0.9^3) + (0.7^3) + (0.6^3)}\right)$$
$$= 1 - (0.4498)$$
$$= 0.5502.$$

- $\mathcal{F}_{11}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\wedge\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\vee\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$

$$\mathcal{F}_{11}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{3}\left(\frac{\begin{pmatrix}(0.8^3\wedge 0.6^3) + \\ (1-0.7^3)\wedge(1-0.9^3)\end{pmatrix} + \begin{pmatrix}(0.5^3\wedge 0.8^3) + \\ (1-0.9^3)\wedge(1-0.7^3)\end{pmatrix} + \begin{pmatrix}(0.6^3\wedge 0.7^3) + \\ (1-0.5^3)\wedge(1-0.6^3)\end{pmatrix}}{\begin{pmatrix}(0.8^3\vee 0.6^3) + \\ (1-0.7^3)\vee(1-0.9^3)\end{pmatrix} + \begin{pmatrix}(0.5^3\vee 0.8^3) + \\ (1-0.9^3)\vee(1-0.7^3)\end{pmatrix} + \begin{pmatrix}(0.6^3\vee 0.7^3) + \\ (1-0.5^3)\vee(1-0.6^3)\end{pmatrix}}\right)$$
$$= 1 - \frac{1}{3}\left(\frac{\left((0.6^3) + (1-0.9^3)\right) + \left((0.5^3) + (1-0.9^3)\right) + \left((0.6^3) + (1-0.6^3)\right)}{\left((0.8^3) + (1-0.7^3)\right) + \left((0.8^3) + (1-0.7^3)\right) + \left((0.7^3) + (1-0.5^3)\right)}\right)$$
$$= 1 - \frac{1}{3}\left(\frac{(0.216 + 0.271) + (0.125 + 0.271) + (0.216 + 0.784)}{(0.512 + 0.675) + (0.512 + 0.657) + (0.343 + 0.875)}\right)$$
$$= 0.8252.$$

- $\mathcal{F}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{\sum_{x_i\in\mathcal{X}}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\wedge\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}{\sum_{x_j\in\mathcal{X}}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\vee\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}$

$$\mathcal{F}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \left(\frac{\begin{pmatrix}(0.8^3\wedge 0.6^3) + \\ (1-0.7^3)\wedge(1-0.9^3)\end{pmatrix} + \begin{pmatrix}(0.5^3\wedge 0.8^3) + \\ (1-0.9^3)\wedge(1-0.7^3)\end{pmatrix} + \begin{pmatrix}(0.6^3\wedge 0.7^3) + \\ (1-0.5^3)\wedge(1-0.6^3)\end{pmatrix}}{\begin{pmatrix}(0.8^3\vee 0.6^3) + \\ (1-0.7^3)\vee(1-0.9^3)\end{pmatrix} + \begin{pmatrix}(0.5^3\vee 0.8^3) + \\ (1-0.9^3)\vee(1-0.7^3)\end{pmatrix} + \begin{pmatrix}(0.6^3\vee 0.7^3) + \\ (1-0.5^3)\vee(1-0.6^3)\end{pmatrix}}\right)$$
$$= 1 - \left(\frac{\left((0.6^3) + (1-0.9^3)\right) + \left((0.5^3) + (1-0.9^3)\right) + \left((0.6^3) + (1-0.6^3)\right)}{\left((0.8^3) + (1-0.7^3)\right) + \left((0.8^3) + (1-0.7^3)\right) + \left((0.7^3) + (1-0.5^3)\right)}\right)$$
$$= 1 - \left(\frac{(0.216 + 0.271) + (0.125 + 0.271) + (0.216 + 0.784)}{(0.512 + 0.675) + (0.512 + 0.657) + (0.343 + 0.875)}\right)$$
$$= 0.4757.$$

$$\mathcal{F}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sqrt[\kappa]{\begin{array}{l}\frac{1}{2|\mathcal{X}|\,(s_1+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|s_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}+\\ \frac{1}{2|\mathcal{X}|\,(s_2+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|s_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}\end{array}}$$

$$\mathcal{F}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sqrt[3]{\left(\begin{array}{l}\frac{1}{2|3|\,(0.6+1)^3}\left\{\begin{array}{l}|0.6(0.8^3-0.6^3)-(0.7^3-0.9^3)|^3\\+|0.6(0.5^3-0.8^3)-(0.9^3-0.7^3)|^3\\+|0.6(0.6^3-0.7^3)-(0.5^3-0.6^3)|^3\end{array}\right\}+\\ \frac{1}{2|3|\,(0.4+1)^3}\left\{\begin{array}{l}|0.4(0.7^3-0.9^3)-(0.8^3-0.6^3)|^3\\+|0.4(0.9^3-0.7^3)-(0.5^3-0.8^3)|^3\\+|0.4(0.5^3-0.6^3)-(0.6^3-0.7^3)|^3\end{array}\right\}\end{array}\right)}$$

$$= \sqrt[3]{\left(\begin{array}{l}\frac{1}{2|3|\,(0.6+1)^3}\left\{\begin{array}{c}0.2351\\+0.2897\\+0.0754\end{array}\right\}+\\ \frac{1}{2|3|\,(0.4+1)^3}\left\{\begin{array}{c}0.1803\\+0.2123\\+0.0343\end{array}\right\}\end{array}\right)}$$

$$= \sqrt[3]{0.02442+0.0259}$$

$$= 0.3691.$$

Theorem 4.1.6 (Peng and Liu 2019) If $\lambda = \tau = 0.5$ $(\imath = 1,2,\ldots,13)$. Then, we have,

(1) $\mathcal{F}_{\imath}(\hat{\mathcal{A}}^{*},\hat{\mathcal{C}}) = (\hat{\mathcal{A}},\hat{\mathcal{C}}^{*}),\ \imath \neq 11, 12;$

(2) $\mathcal{F}_{\imath}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{F}_{\imath}(\hat{\mathcal{A}}\cap\hat{\mathcal{C}},\hat{\mathcal{A}}\cup\hat{\mathcal{C}});$

(3) $\mathcal{F}_{\imath}(\hat{\mathcal{A}},\hat{\mathcal{A}}\cup\hat{\mathcal{C}}) = \mathcal{F}_{\imath}(\hat{\mathcal{C}},\hat{\mathcal{A}}\cap\hat{\mathcal{C}},);$

(4) $\mathcal{F}_{\imath}(\hat{\mathcal{A}},\hat{\mathcal{A}}\cap\hat{\mathcal{C}}) = \mathcal{F}_{\imath}(\hat{\mathcal{C}},\hat{\mathcal{A}}\cup\hat{\mathcal{C}},).$

Definition 4.1.9 (Peng and Liu 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa}\rangle | x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa}\rangle | x_j\in\mathcal{X}\}$ and $\hat{\mathcal{S}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{S}}}(x_j),\gamma_{\hat{\mathcal{S}}}(x_j))_{\kappa}\rangle | x_j\in\mathcal{X}\}$ be any three q-ROFSs over $\mathcal{X}$. Then, a similarity measure $\mathcal{S}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ is mapping
$\boldsymbol{\mathcal{S}}$: q-ROFS$(\mathcal{X})$ ×q-ROFS$(\mathcal{X}) \longrightarrow [0,1]$, given as follows:

1. $\mathcal{S}_1 = 0 \leq \mathcal{S}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \leq 1;$

2. $\mathcal{S}_2 = \mathcal{S}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{S}(\hat{\mathcal{C}},\hat{\mathcal{A}});$

3. $\mathcal{S}_3 = \mathcal{S}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 0$ iff $\hat{\mathcal{A}} = \hat{\mathcal{C}}$;

4. $\mathcal{S}_4 = \mathcal{S}(\hat{\mathcal{A}}, \hat{\mathcal{A}}^*)= 1$ iff $\hat{\mathcal{A}}^*$ is a crisp set;

5. Iff $\hat{\mathcal{A}} \subseteq \hat{\mathcal{C}} \subseteq \hat{\mathcal{S}}$, then $\mathcal{S}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \le \mathcal{S}(\hat{\mathcal{A}}, \hat{\mathcal{S}})$ and $\mathcal{S}(\hat{\mathcal{C}}, \hat{\mathcal{S}}) \le \mathcal{S}(\hat{\mathcal{A}}, \hat{\mathcal{S}})$.

Theorem 4.1.7 (Peng and Liu 2019) Let $\hat{\mathcal{A}}$=$\{\langle x_i, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_i))_\kappa\rangle | x_i \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_i, (\eta_{\hat{\mathcal{C}}}(x_i), \gamma_{\hat{\mathcal{C}}}(x_i))_\kappa\rangle | x_i \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the distance measure $\mathcal{S}_i(\hat{\mathcal{A}}, \hat{\mathcal{C}})$, such that $(i = 1,2,3 \ldots,13)$ given as follows:

(1) $\mathcal{S}_1(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{1}{2|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)| + |\pi_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \pi_{\hat{\mathcal{C}}}{}^\kappa(x_i)|)$;

(2) $\mathcal{S}_2(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{1}{2|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|)$;

(3) $\mathcal{S}_3(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{1}{4|\mathcal{X}|}(\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)| + |\pi_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \pi_{\hat{\mathcal{C}}}{}^\kappa(x_i)|) + \sum_{x_i \in \mathcal{X}}|\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i)| + |\eta_{\hat{\mathcal{C}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|)$;

(4) $\mathcal{S}_4(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1 - \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|)$;

(5) $\mathcal{S}_5(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{|\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|}{1 + |\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|}$;

(6) $\mathcal{S}_6(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{x_i \in \mathcal{X}}|1 - \eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|}{\sum_{x_i \in \mathcal{X}}(1 + |\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i)|)}$;

(7) $\mathcal{S}_7(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \lambda\frac{\sum_{x_i \in \mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{\sum_{x_i \in \mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i))} + \tau\frac{\sum_{x_i \in \mathcal{X}}(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{\sum_{x_i \in \mathcal{X}}(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}$, where $(\lambda + \tau = 1)$, and $\lambda, \tau \in [0,1]$;

(8) $\mathcal{S}_8(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\lambda}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i))} + \frac{\tau}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}$, where $\lambda + \tau = 1$ and $\lambda, \tau \in [0,1]$;

(9) $\mathcal{S}_9(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}$;

(10) $\mathcal{S}_{10}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{x_i \in \mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{\sum_{x_i \in \mathcal{X}}(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}$;

(11) $\mathcal{S}_{11}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}}\frac{(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)) + (1 - \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i)) \wedge (1 - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}{(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^\kappa(x_i)) + (1 - \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_i)) \vee (1 - \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_i))}$;

(12) $\mathcal{S}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{\sum_{x_i\in\mathcal{X}}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\wedge\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}{\sum_{x_i\in\mathcal{X}}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\vee\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}$;

(13) $\mathcal{S}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) =$

$$1-\sqrt[\kappa]{\begin{matrix}\frac{1}{2|\mathcal{X}|\,(\mathscr{s}_1+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|\mathscr{s}_1\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}+\\ \frac{1}{2|\mathcal{X}|\,(\mathscr{s}_2+1)^{\kappa}}\sum_{x_i\in\mathcal{X}}\left\{\left|\mathscr{s}_2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}\end{matrix}}.$$

Example 4.1.7 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7)_{\kappa}\rangle,\langle x_2,(0.5,0.9)_{\kappa}\rangle,\langle x_3,(0.6,0.5)_{\kappa}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9)_{\kappa}\rangle,\langle x_2,(0.8,0.7)_{\kappa}\rangle,\langle x_3,(0.7,0.6)_{\kappa}\rangle\}$, assume that $\kappa=3$ and $\omega_{j}=(\frac{1}{4},\frac{2}{5},\frac{7}{20})$, $(\mathscr{s}_1=0.6,\mathscr{s}_2=0.4)$ be any two q-ROFSs over $\mathcal{X}$. Then, the similarity measure $\mathcal{S}_i(\hat{\mathcal{A}},\hat{\mathcal{C}})$, such that $(i=1,2,3\ldots,13)$ and $(\lambda=0.4$ and $\tau=0.6)$.
$\lambda,\tau\in[0,1]$ can be calculated as :

- $\mathcal{S}_1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1-\frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|+|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|+ |\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$

$$\mathcal{S}_1(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1-\frac{1}{2|3|}\begin{pmatrix}(|0.8^3-0.6^3|+|0.7^3-0.9^3|+|0.525^3-0.380^3|)+\\(|0.5^3-0.8^3|+|0.9^3-0.7^3|+|0.526^3-0.525^3|)+\\(|0.6^3-0.7^3|+|0.5^3-0.6^3|+|0.870^3-0.761^3|)\end{pmatrix}$$

$$= 1-\frac{1}{2|3|}\begin{pmatrix}(|0.296|+|-0.386|+|0.089|)+\\(|-0.387|+|0.386|+|0.0008|)+\\(|-0.127|+|-0.091|+|0.207|)\end{pmatrix}$$

$$= 1-\frac{1}{2|3|}(1.9698)$$

$$= 0.6717.$$

- $\mathcal{S}_2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1-\frac{1}{2|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)|)$

$$\mathcal{S}_2(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1-\frac{1}{2|3|}\begin{pmatrix}(|0.8^3-0.6^3-(0.7^3-0.9^3)|)+\\(|0.5^3-0.8^3-(0.9^3-0.7^3)|)+\\(|0.6^3-0.7^3-(0.5^3-0.6^3)|)\end{pmatrix}$$

$$= 1-\frac{1}{2|3|}\begin{pmatrix}(|0.682|)+\\(|-0.773|)+\\(-0.036)\end{pmatrix}$$

$$= 1 - \frac{1}{2|3|}(1.491)$$
$$= 0.7515.$$

- $\mathcal{S}_3(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{4|\mathcal{X}|}(\sum_{x_i\in\mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| + |\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|) + \sum_{x_j\in\mathcal{X}}|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)| + |\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$

$$\mathcal{S}_3(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{4|3|}\begin{pmatrix}\left(|0.8^3-0.6^3| + |0.7^3-0.9^3| + |0.525^3-0.380^3| + (|0.8^3-0.7^3| + |0.6^3-0.9^3|)\right) + \\ \left(|0.5^3-0.8^3| + |0.9^3-0.7^3| + |0.526^3-0.525^3| + (|0.5^3-0.9^3| + |0.8^3-0.7^3|)\right) + \\ \left(|0.6^3-0.7^3| + |0.5^3-0.6^3| + |0.870^3-0.761^3| + (|0.6^3-0.5^3| + |0.7^3-0.6^3|)\right)\end{pmatrix}$$

$$= 1 - \frac{1}{4|3|}\begin{pmatrix}\left(|0.296| + |-0.386| + |0.089| + (|0.169| + |-0.513|)\right) + \\ \left(|-0.387| + |0.386| + |0.0008| + (|-0.604| + |-0.169|)\right) + \\ \left(|-0.127| + |-0.091| + |0.217| + (|0.091| + |0.127|)\right)\end{pmatrix}$$

$$= 1 - \frac{1}{4|3|}(3.652)$$
$$= 0.6959.$$

- $\mathcal{S}_4(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}(|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)$

$$\mathcal{S}_4(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \frac{1}{3}\begin{pmatrix}(|0.8^3-0.6^3| \vee |0.7^3-0.9^3|) + \\ (|0.5^3-0.8^3| \vee |0.9^3-0.7^3|) + \\ (|0.6^3-0.7^3| \vee |0.5^3-0.6^3|)\end{pmatrix}$$

$$= 1 - \frac{1}{3}\begin{pmatrix}(|0.296| \vee |-0.386|) + \\ (|-0.387| \vee |0.386|) + \\ (|-0.127| \vee |-0.091|)\end{pmatrix}$$

$$= 1 - \frac{1}{3}\begin{pmatrix}|-0.386| + \\ |-0.387| + \\ |-0.127|\end{pmatrix}$$
$$= 0.7.$$

- $\mathcal{S}_5(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i\in\mathcal{X}}\frac{|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}{1 + |\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)| \vee |\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) - \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|}$

$$\mathcal{S}_5(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{|3|}\begin{pmatrix}\frac{|0.8^3-0.6^3| \vee |0.7^3-0.9^3|}{1 + (|0.8^3-0.6^3| \vee |0.7^3-0.9^3|)} + \\ \frac{|0.5^3-0.8^3| \vee |0.9^3-0.7^3|}{1 + (|0.5^3-0.8^3| \vee |0.9^3-0.7^3|)} + \\ \frac{|0.6^3-0.7^3| \vee |0.5^3-0.6^3|}{1 + (|0.6^3-0.7^3| \vee |0.5^3-0.6^3|)}\end{pmatrix}$$

$$= \frac{1}{|3|}\begin{pmatrix} \frac{|0.296|\vee|-0.386|}{1+(|0.296|\vee|-0.386|)} + \\ \frac{|-0.387|\vee|0.386|}{1+(|-0.387|\vee|0.386|)} + \\ \frac{|-0.127|\vee|-0.091|}{1+(|-0.127|\vee|-0.091|)} \end{pmatrix}$$

$$= \frac{1}{|3|}\begin{pmatrix} \frac{|-0.386|}{1+|-0.386|} + \\ \frac{|-0.387|}{1+|-0.387|} + \\ \frac{|-0.127|}{1+|-0.127|} \end{pmatrix}$$

$$= \frac{1}{|3|}(0.6702)$$

$$= 0.2234.$$

- $\mathcal{S}_6(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{x_i \in \mathcal{X}}(1-|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|\vee|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)}{\sum_{x_i \in \mathcal{X}}(1+|\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|\vee|\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)|)}$

$$\mathcal{S}_6(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \left(\frac{\left(\begin{matrix}1-\\|0.8^3-0.6^3|\vee|0.7^3-0.9^3|\end{matrix}\right)+\left(\begin{matrix}1-\\|0.5^3-0.8^3|\vee|0.9^3-0.7^3|\end{matrix}\right)+\left(\begin{matrix}1-\\|0.6^3-0.7^3|\vee|0.5^3-0.6^3|\end{matrix}\right)}{\left(\begin{matrix}1+\\|0.8^3-0.6^3|\vee|0.7^3-0.9^3|\end{matrix}\right)+\left(\begin{matrix}1+\\|0.5^3-0.8^3|\vee|0.9^3-0.7^3|\end{matrix}\right)+\left(\begin{matrix}1+\\|0.6^3-0.7^3|\vee|0.5^3-0.6^3|\end{matrix}\right)}\right)$$

$$= \left(\frac{\left(\begin{matrix}1-\\|0.296|\vee|-0.386|\end{matrix}\right)+\left(\begin{matrix}1-\\|-0.387|\vee|0.386|\end{matrix}\right)+\left(\begin{matrix}1-\\|-0.127|\vee|-0.091|\end{matrix}\right)}{\left(\begin{matrix}1+\\|0.296|\vee|-0.386|\end{matrix}\right)+\left(\begin{matrix}1+\\|-0.387|\vee|0.386|\end{matrix}\right)+\left(\begin{matrix}1+\\|-0.127|\vee|-0.091|\end{matrix}\right)}\right)$$

$$= \left(\frac{(|0.704|)+(|0.613|)+(|0.873|)}{(1+|-0.386|)+(1+|-0.387|)+(1+|-0.127|)}\right)$$

$$= 0.5615.$$

- $\mathcal{S}_7(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \lambda\frac{\sum_{x_i \in \mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i \in \mathcal{X}}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)} + \tau\frac{\sum_{x_i \in \mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\wedge\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\sum_{x_i \in \mathcal{X}}\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\vee\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$, where $\lambda = 0.4$ and

$\tau = 0.6$, $(\lambda + \tau = 1)$ and $\lambda, \tau \in [0,1]$

$$\mathcal{S}_7(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \begin{pmatrix} 0.4\left(\frac{(0.8^3\wedge0.6^3)+(0.5^3\wedge0.8^3)+(0.6^3\wedge0.7^3)}{(0.8^3\vee0.6^3)+(0.5^3\vee0.8^3)+(0.6^3\vee0.7^3)}\right) \\ +0.6\left(\frac{(0.7^3\wedge0.9^3)+(0.9^3\wedge0.7^3)+(0.5^3\wedge0.6^3)}{(0.7^3\vee0.9^3)+(0.9^3\vee0.7^3)+(0.5^3\vee0.6^3)}\right) \end{pmatrix}$$

$$= \left(0.4\left(\frac{0.6^3 + 0.5^3 + 0.6^3}{0.8^3 + 0.8^3 + 0.7^3}\right) + 0.6\left(\frac{0.7^3 + 0.7^3 + 0.5^3}{0.9^3 + 0.9^3 + 0.6^3}\right)\right)$$

$$= (0.1629 + 0.2906)$$

$$= 0.4535.$$

- $\mathcal{S}_8(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\lambda}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)} + \frac{\tau}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$, where $\lambda = 0.4$

and $\tau = 0.6$, $(\lambda + \tau = 1)$ and $\lambda, \tau \in [0,1]$

-

$$\mathcal{S}_8(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \left(\begin{array}{c} \frac{0.4}{3}\left(\frac{(0.8^3 \wedge 0.6^3) + (0.5^3 \wedge 0.8^3) + (0.6^3 \wedge 0.7^3)}{(0.8^3 \vee 0.6^3) + (0.5^3 \vee 0.8^3) + (0.6^3 \vee 0.7^3)}\right) \\ + \frac{0.6}{3}\left(\frac{(0.7^3 \wedge 0.9^3) + (0.9^3 \wedge 0.7^3) + (0.5^3 \wedge 0.6^3)}{(0.7^3 \vee 0.9^3) + (0.9^3 \vee 0.7^3) + (0.5^3 \vee 0.6^3)}\right) \end{array}\right)$$

$$= \left(\frac{0.4}{3}\left(\frac{0.6^3 + 0.5^3 + 0.6^3}{0.8^3 + 0.8^3 + 0.7^3}\right) + \frac{0.6}{3}\left(\frac{0.7^3 + 0.7^3 + 0.5^3}{0.9^3 + 0.9^3 + 0.6^3}\right)\right)$$

$$= (0.0543 + 0.0968)$$

$$= 0.1511.$$

- $\mathcal{S}_9(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|} \sum_{x_i \in \mathcal{X}} \frac{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}{(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}$

$$\mathcal{S}_9(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{3}\left(\frac{\begin{pmatrix}(0.8^3 \wedge 0.6^3) + \\ (0.7^3 \wedge 0.9^3)\end{pmatrix} + \begin{pmatrix}(0.5^3 \wedge 0.8^3) + \\ (0.9^3 \wedge 0.7^3)\end{pmatrix} + \begin{pmatrix}(0.6^3 \wedge 0.7^3) + \\ (0.5^3 \wedge 0.6^3)\end{pmatrix}}{\begin{pmatrix}(0.8^3 \vee 0.6^3) + \\ (0.7^3 \vee 0.9^3)\end{pmatrix} + \begin{pmatrix}(0.5^3 \vee 0.8^3) + \\ (0.9^3 \vee 0.7^3)\end{pmatrix} + \begin{pmatrix}(0.6^3 \vee 0.7^3) + \\ (0.5^3 \vee 0.6^3)\end{pmatrix}}\right)$$

$$= \frac{1}{3}\left(\frac{(0.6^3) + (0.7^3) + (0.5^3) + (0.7^3) + (0.6^3) + (0.5^3)}{(0.8^3) + (0.9^3) + (0.8^3) + (0.9^3) + (0.7^3) + (0.6^3)}\right)$$

$$= \frac{1}{3}(0.4498)$$

$$= 0.1499.$$

- $\mathcal{S}_{10}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{x_i \in \mathcal{X}} (\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}{\sum_{x_i \in \mathcal{X}} (\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)) + (\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))}$

$$\mathcal{S}_{10}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \left(\frac{\begin{pmatrix}(0.8^3 \wedge 0.6^3)+\\(0.7^3 \wedge 0.9^3)\end{pmatrix}+\begin{pmatrix}(0.5^3 \wedge 0.8^3)+\\(0.9^3 \wedge 0.7^3)\end{pmatrix}+\begin{pmatrix}(0.6^3 \wedge 0.7^3)+\\(0.5^3 \wedge 0.6^3)\end{pmatrix}}{\begin{pmatrix}(0.8^3 \vee 0.6^3)+\\(0.7^3 \vee 0.9^3)\end{pmatrix}+\begin{pmatrix}(0.5^3 \vee 0.8^3)+\\(0.9^3 \vee 0.7^3)\end{pmatrix}+\begin{pmatrix}(0.6^3 \vee 0.7^3)+\\(0.5^3 \vee 0.6^3)\end{pmatrix}}\right)$$

$$= \left(\frac{(0.6^3)+(0.7^3)+(0.5^3)+(0.7^3)+(0.6^3)+(0.5^3)}{(0.8^3)+(0.9^3)+(0.8^3)+(0.9^3)+(0.7^3)+(0.6^3)}\right)$$

$$= 0.4498.$$

- $\mathcal{S}_{11}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{|\mathcal{X}|}\sum_{x_i \in \mathcal{X}} \frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\wedge\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)\right)\vee\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)}$

$$\mathcal{S}_{11}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{3}\left(\frac{\begin{pmatrix}(0.8^3 \wedge 0.6^3)+\\(1-0.7^3)\wedge(1-0.9^3)\end{pmatrix}+\begin{pmatrix}(0.5^3 \wedge 0.8^3)+\\(1-0.9^3)\wedge(1-0.7^3)\end{pmatrix}+\begin{pmatrix}(0.6^3 \wedge 0.7^3)+\\(1-0.5^3)\wedge(1-0.6^3)\end{pmatrix}}{\begin{pmatrix}(0.8^3 \vee 0.6^3)+\\(1-0.7^3)\vee(1-0.9^3)\end{pmatrix}+\begin{pmatrix}(0.5^3 \vee 0.8^3)+\\(1-0.9^3)\vee(1-0.7^3)\end{pmatrix}+\begin{pmatrix}(0.6^3 \vee 0.7^3)+\\(1-0.5^3)\vee(1-0.6^3)\end{pmatrix}}\right)$$

$$= \frac{1}{3}\left(\frac{\left((0.6^3)+(1-0.9^3)\right)+\left((0.5^3)+(1-0.9^3)\right)+\left((0.6^3)+(1-0.6^3)\right)}{\left((0.8^3)+(1-0.7^3)\right)+\left((0.8^3)+(1-0.7^3)\right)+\left((0.7^3)+(1-0.5^3)\right)}\right)$$

$$= \frac{1}{3}\left(\frac{(0.216+0.271)+(0.125+0.271)+(0.216+0.784)}{(0.512+0.675)+(0.512+0.657)+(0.343+0.875)}\right)$$

$$= 0.1722.$$

- $\mathcal{S}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{\sum_{x_i \in \mathcal{X}}\left(\left(\eta_{\mathcal{A}}{}^{\kappa}(x_i) \wedge \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\mathcal{A}}{}^{\kappa}(x_i)\right)\wedge\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}{\sum_{x_j \in \mathcal{X}}\left(\left(\eta_{\mathcal{A}}{}^{\kappa}(x_i) \vee \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)+\left(1-\gamma_{\mathcal{A}}{}^{\kappa}(x_i)\right)\vee\left(1-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right)}$

$$\mathcal{S}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \left(\frac{\begin{pmatrix}(0.8^3 \wedge 0.6^3)+\\(1-0.7^3)\wedge(1-0.9^3)\end{pmatrix}+\begin{pmatrix}(0.5^3 \wedge 0.8^3)+\\(1-0.9^3)\wedge(1-0.7^3)\end{pmatrix}+\begin{pmatrix}(0.6^3 \wedge 0.7^3)+\\(1-0.5^3)\wedge(1-0.6^3)\end{pmatrix}}{\begin{pmatrix}(0.8^3 \vee 0.6^3)+\\(1-0.7^3)\vee(1-0.9^3)\end{pmatrix}+\begin{pmatrix}(0.5^3 \vee 0.8^3)+\\(1-0.9^3)\vee(1-0.7^3)\end{pmatrix}+\begin{pmatrix}(0.6^3 \vee 0.7^3)+\\(1-0.5^3)\vee(1-0.6^3)\end{pmatrix}}\right)$$

$$= \left(\frac{\left((0.6^3)+(1-0.9^3)\right)+\left((0.5^3)+(1-0.9^3)\right)+\left((0.6^3)+(1-0.6^3)\right)}{\left((0.8^3)+(1-0.7^3)\right)+\left((0.8^3)+(1-0.7^3)\right)+\left((0.7^3)+(1-0.5^3)\right)}\right)$$

$$= \left(\frac{(0.216+0.271)+(0.125+0.271)+(0.216+0.784)}{(0.512+0.675)+(0.512+0.657)+(0.343+0.875)}\right)$$

$$= 0.5167.$$

- $\mathcal{S}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \sqrt[\kappa]{\begin{array}{l}\frac{1}{2|\mathcal{X}|\,(s_1+1)^{\kappa}}\sum_{x_i \in \mathcal{X}}\left\{\left|s_1\left(\eta_{\mathcal{A}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}+\\ \frac{1}{2|\mathcal{X}|\,(s_2+1)^{\kappa}}\sum_{x_i \in \mathcal{X}}\left\{\left|s_2\left(\gamma_{\mathcal{A}}{}^{\kappa}(x_i)-\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)-\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i)-\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i)\right)\right|^{\kappa}\right\}\end{array}}$

$$\mathcal{S}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1 - \sqrt[3]{\left(\begin{array}{l}\dfrac{1}{2|3|\,(0.6+1)^3}\left\{\begin{array}{c}|0.6(0.8^3-0.6^3)-(0.7^3-0.9^3)|^3\\+|0.6(0.5^3-0.8^3)-(0.9^3-0.7^3)|^3\\+|0.6(0.6^3-0.7^3)-(0.5^3-0.6^3)|^3\end{array}\right\}+\\ \dfrac{1}{2|3|\,(0.4+1)^3}\left\{\begin{array}{c}|0.4(0.7^3-0.9^3)-(0.8^3-0.6^3)|^3\\+|0.4(0.9^3-0.7^3)-(0.5^3-0.8^3)|^3\\+|0.4(0.5^3-0.6^3)-(0.6^3-0.7^3)|^3\end{array}\right\}\end{array}\right)}$$

$$= 1 - \sqrt[3]{\left(\begin{array}{l}\dfrac{1}{2|3|\,(0.6+1)^3}\left\{\begin{array}{c}0.2351\\+0.2897\\+0.0754\end{array}\right\}+\\ \dfrac{1}{2|3|\,(0.4+1)^3}\left\{\begin{array}{c}0.1803\\+0.2123\\+0.0343\end{array}\right\}\end{array}\right)}$$

$$= 1 - \sqrt[3]{0.02442 + 0.0259}$$

$$= 0.6309.$$

Theorem 4.1.8 (Peng and Liu 2019) If $(\lambda = \tau = 0.5)$, $(\imath = 1,2,\ldots,13)$. Then, we have,

(1) $\mathcal{S}_\imath(\hat{\mathcal{A}}^*,\hat{\mathcal{C}}) = (\hat{\mathcal{A}},\hat{\mathcal{C}}^*)$, $\imath \neq 11,12$;

(2) $\mathcal{S}_\imath(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{S}_\imath(\hat{\mathcal{A}} \cap \hat{\mathcal{C}}, \hat{\mathcal{A}} \cup \hat{\mathcal{C}})$;

(3) $\mathcal{S}_\imath(\hat{\mathcal{A}},\hat{\mathcal{A}} \cup \hat{\mathcal{C}}) = \mathcal{S}_\imath(\hat{\mathcal{C}},\hat{\mathcal{A}} \cap \hat{\mathcal{C}},)$;

(4) $\mathcal{S}_\imath(\hat{\mathcal{A}},\hat{\mathcal{A}} \cap \hat{\mathcal{C}}) = \mathcal{S}_\imath(\hat{\mathcal{C}},\hat{\mathcal{A}} \cup \hat{\mathcal{C}},)$.

For each distance measure, results obtained from examples are given in (Table 4.2).

Table 4.2 Distance Measures and Similarity Measures

Distance Measures	Results	Similarity Measures	Results
$\mathcal{F}_1(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3283	$\mathcal{S}_1(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.6717
$\mathcal{F}_2(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2485	$\mathcal{S}_2(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7515
$\mathcal{F}_3(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3043	$\mathcal{S}_3(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.6959
$\mathcal{F}_4(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3000	$\mathcal{S}_4(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7000
$\mathcal{F}_5(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.4468	$\mathcal{S}_5(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.2234
$\mathcal{F}_6(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.6415	$\mathcal{S}_6(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.5615
$\mathcal{F}_7(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.5464	$\mathcal{S}_7(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.4535
$\mathcal{F}_8(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.8488	$\mathcal{S}_8(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.1511
$\mathcal{F}_9(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.8500	$\mathcal{S}_9(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.1499
$\mathcal{F}_{10}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.5502	$\mathcal{S}_{10}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.4498
$\mathcal{F}_{11}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.8202	$\mathcal{S}_{11}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.1722
$\mathcal{F}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.4757	$\mathcal{S}_{12}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.5167
$\mathcal{F}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3691	$\mathcal{S}_{13}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.6309

4.2 Dice Similarity Measure for q-Rung Orthopair Fuzzy Set

The purpose of this subject is to develop a new game for ideas of Dice similarity measure (DSM) and weighted Dice similarity measure (WDSM) for q-rung orthopair (q-ROFSs). It has four types, and to solve the conditions of (DSM) and (WDSM) to (q-ROFSs) proof throughout this part $\hat{\mathcal{A}} = (\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_i), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_i))$ and $\hat{\mathcal{C}} = (\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_i), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_i))$, we will notice two (q-ROFSs) on the finite general set $\mathcal{X}$.

Definition 4.2.1 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then a dice similarity measures $\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be characterized as follows:

$$\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}.$$

Theorem 4.2.1 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, a dice similarity measures $\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the accompanying properties:

(1) $0 \le \mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \le 1$,

(2) $\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \ge 1$, $0 \le \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \le 1$. Then, from Cauchy-Schwarz inequality, we have,

$0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then, $2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$.

Thus, $0 \leq \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1$. It is concluded that

$$0 \leq \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1.$$

(2) Since

$$\begin{aligned}
\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) &= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}}).
\end{aligned}$$

(3) Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$, then using Definition 4.2.1 we have,

$$\begin{aligned}
\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) &= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2} \\
&= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)^2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)^2\right)}{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)} = 1.
\end{aligned}$$

Then, we complete this process of proofs.

Example 4.2.1 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9)_\kappa\rangle,\langle x_3,(0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9)_\kappa\rangle,\langle x_2,(0.8,0.7)_\kappa\rangle,\langle x_3,(0.7,0.6)_\kappa\rangle\}$ be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa=3$ and. Then, a dice similarity measures $\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{3}\begin{pmatrix}\frac{2\left((0.8^3\times 0.6^3)+(0.7^3\times 0.9^3)\right)}{(0.8^3)^2+(0.7^3)^2+(0.6^3)^2+(0.9^3)^2}+\\ \frac{2\left((0.5^3\times 0.8^3)+(0.9^3\times 0.7^3)\right)}{(0.5^3)^2+(0.9^3)^2+(0.8^3)^2+(0.7^3)^2}+\\ \frac{2\left((0.6^3\times 0.7^3)+(0.5^3\times 0.6^3)\right)}{(0.6^3)^2+(0.5^3)^2+(0.7^3)^2+(0.6^3)^2}\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}0.4919+\\ 0.4078+\\ 0.7731\end{pmatrix}$$

$$=0.5576.$$

Definition 4.2.2 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the weighted dice similarity measures $\mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be defined as follows:

$$\mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\sum_{j=1}^{n}\omega_j\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}.$$

Theorem 4.2.2 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the weighted dice similarity measures $\mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0 \leq \mathcal{D}_{1_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{1_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{1_{WqROFS}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{1_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, which means that, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have $0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then, $2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$.

Thus, $0 \leq \omega_j \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1$. It is concluded that

$$0 \leq \sum_{j=1}^{n} \omega_j \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1.$$

(2) Since

$$\begin{aligned}
\mathcal{D}_{1_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) &= \sum_{j=1}^{n} \omega_j \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \mathcal{D}_{WqROF}(\hat{\mathcal{C}}, \hat{\mathcal{A}}),
\end{aligned}$$

(3) Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Then, by using Definition 4.2.2 we have

$$\mathcal{D}_{1_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \omega_j \frac{\left(2\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)2\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + 2\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\, 2\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \sum_{j=1}^{n} \omega_j \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2} = 1.$$

Then, we complete this process of proofs.

Example 4.2.2 Let $\hat{\mathcal{A}}$={$\langle x_1, (0.8,0.7)_\kappa\rangle, \langle x_2, (0.5,0.9)_\kappa\rangle, \langle x_3, (0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}$={$\langle x_1, (0.6,0.9)_\kappa\rangle, \langle x_2, (0.8,0.7)_\kappa\rangle, \langle x_3, (0.7,0.6)_\kappa\rangle\}$ be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and $\omega_j = (\frac{1}{4}, \frac{2}{5}, \frac{7}{20})$. Then, the weighted dice similarity measures $\mathcal{D}_{1_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be obtained as follows:

$$\mathcal{D}_{1_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \omega_j \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \begin{pmatrix} \left(\frac{1}{4}\right)\left(\frac{2\left((0.8^3\times 0.6^3)+(0.7^3\times 0.9^3)\right)}{(0.8^3)^2+(0.7^3)^2+(0.6^3)^2+(0.9^3)^2}\right) + \\ \left(\frac{2}{5}\right)\left(\frac{2\left((0.5^3\times 0.8^3)+(0.9^3\times 0.7^3)\right)}{(0.5^3)^2+(0.9^3)^2+(0.8^3)^2+(0.7^3)^2}\right) + \\ \left(\frac{7}{20}\right)\left(\frac{2\left((0.6^3\times 0.7^3)+(0.5^3\times 0.6^3)\right)}{(0.6^3)^2+(0.5^3)^2+(0.7^3)^2+(0.6^3)^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} (0.25)(0.4919) + \\ (0.4)(0.4078) + \\ (0.35)(0.7731) \end{pmatrix}$$

$$= 0.5566.$$

Definition 4.2.3 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$={$\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$={$\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then a dice similarity measures $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}.$$

Theorem 4.2.3 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, a dice similarity measures $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0\le\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\le 1$,

(2) $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}}=\hat{\mathcal{C}}$, then $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=1$, which means that, $\eta_{\hat{\mathcal{A}}}(x_j)=\eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j)=\gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa\ge 1$, $0\le\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j),\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\le 1$.

Then, from Cauchy-Schwarz inequality, we have

$0\le\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\le\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then,

$2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\le\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so

$0\le\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j),\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j),\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j),\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\le\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$.

Thus, $0\le\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}\le 1$. It is concluded that

$$0\le\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}\le 1.$$

(2) Since

$$\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$$

$$=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$=\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}).$$

(3) Since $\hat{\mathcal{A}}=\hat{\mathcal{C}}$, $\eta_{\hat{\mathcal{A}}}(x_j)=\eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j)=\gamma_{\hat{\mathcal{C}}}(x_j)$, $\pi_{\hat{\mathcal{A}}}(x_j)=\pi_{\hat{\mathcal{C}}}(x_j)$.
Then, by using Definition 4.2.3 we have,

$$\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$$

$$=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}$$

$=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}{2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)}=1$. Then, we complete this process of proofs.

Example 4.2.3 Let $\hat{\mathcal{A}}$={$\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9)_\kappa\rangle,\langle x_3,(0.6,0.5)_\kappa$} and $\hat{\mathcal{C}}$={$\langle x_1,(0.6,0.9)_\kappa\rangle,\langle x_2,(0.8,0.7)_\kappa\rangle,\langle x_3,(0.7,0.6)_\kappa\rangle$}, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa=3$. Then, a dice similarity measures $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \left(\frac{1}{3}\right)\left(\begin{array}{c}\left(\frac{2\left((0.8^3\times\ 0.6^3)+(0.7^3\times\ 0.9^3)+(0.525^3\times\ 0.380^3)\right)}{(0.8^3)^2+(0.7^3)^2+(0.525^3)^2+(0.6^3)^2+(0.9^3)^2+(080.3^3)^2}\right)+\\ \left(\frac{2\left((0.5^3\times\ 0.8^3)+(0.9^3\times\ 0.7^3)+(0.526^3\times\ 0.525^3)\right)}{(0.5^3)^2+(0.9^3)^2+(0.526^3)^2+(0.8^3)^2+(0.7^3)^2+(0.525^3)^2}\right)+\\ \left(\frac{2\left((0.6^3\times\ 0.7^3)+(0.5^3\times\ 0.6^3)+(0.870^3\times\ 0.761^3)\right)}{(0.6^3)^2+(0.5^3)^2+(0.870^3)^2+(0.7^3)^2+(0.6^3)^2+(0.761^3)^2}\right)\end{array}\right)$$

$$= \left(\frac{1}{3}\right)\left(\begin{array}{c}(0.7647)+\\(0.7048)+\\(0.9159)\end{array}\right)$$

$$= 0.7951.$$

Definition 4.2.4 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$={$\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}$} and $\hat{\mathcal{C}}$={$\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}$} be any two q-ROFNs over $\mathcal{X}$, then the weighted dice similarity measures $\mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$$
$$= \sum_{j=1}^{n} \omega_j \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Theorem 4.2.4 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$={$\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}$} and $\hat{\mathcal{C}}$={$\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}$} be any two q-ROFNs over $\mathcal{X}$. Then, the weighted dice similarity measures $\mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0 \leq \mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{2_{WqROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have

$0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then,

$2(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)) \leq \sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so

$0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$.

Thus, $0 \leq \omega_j \frac{2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \leq 1$. It is concluded that

$$0 \leq \sum_{j=1}^{n}\omega_j \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \leq 1.$$

(2) Since

$$\begin{aligned}
&\mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \\
&= \sum_{j=1}^{n}\omega_j \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \\
&= \sum_{j=1}^{n}\omega_j \frac{2\sum_{j=1}^{n}\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{1}\left(\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \\
&= \mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}}).
\end{aligned}$$

(3) Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j), \pi_{\hat{\mathcal{A}}}(x_j) = \pi_{\hat{\mathcal{C}}}(x_j)$. Then, by using Definition 4.2.4 we have,

$\mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$

$$=\sum_{j=1}^{n}\omega_j\frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$=\sum_{j=1}^{n}\omega_j\frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$=\sum_{j=1}^{n}\omega_j\frac{2\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)}=1$. Then, we complete the process of proofs.

Example 4.2.4 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9,0.5)_\kappa\rangle,\langle x_3,(0.6,0.5,0.8)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9,0.3)_\kappa\rangle,\langle x_2,(0.8,0.7,0.5)_\kappa\rangle,\langle x_3,(0.7,0.6,0.7)_\kappa\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa=3$ and $\omega_j=(\frac{1}{4},\frac{2}{5},\frac{7}{20})$. Then, the weighted dice similarity measures $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$\mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$

$$=\sum_{j=1}^{n}\omega_j\frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$=\begin{pmatrix}\left(\frac{1}{4}\right)\left(\frac{2\left((0.8^3\times\ 0.6^3)+(0.7^3\times\ 0.9^3)+(0.525^3\times\ 0.380^3)\right)}{(0.8^3)^2+(0.7^3)^2+(0.525^3)^2+(0.6^3)^2+(0.9^3)^2+(0.380^3)^2}\right)+\\ \left(\frac{2}{5}\right)\left(\frac{2\left((0.5^3\times\ 0.8^3)+(0.9^3\times\ 0.7^3)+(0.526^3\times\ 0.525^3)\right)}{(0.5^3)^2+(0.9^3)^2+(0.526^3)^2+(0.8^3)^2+(0.7^3)^2+(0.525^3)^2}\right)+\\ \left(\frac{7}{20}\right)\left(\frac{2\left((0.6^3\times\ 0.7^3)+(0.5^3\times\ 0.6^3)+(0.870^3\times\ 0.761^3)\right)}{(0.6^3)^2+(0.5^3)^2+(0.870^3)^2+(0.7^3)^2+(0.6^3)^2+(0.761^3)^2}\right)\end{pmatrix}$$

$$=\begin{pmatrix}(0.25)(0.7507)+\\(0.4)(0.6916)+\\(0.35)(0.9159)\end{pmatrix}$$

$=0.7848.$

Definition 4.2.5 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then a dice similarity measures $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can characterized as follows:

$$\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Theorem 4.2.5 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, a dice similarity measures $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0 \leq \mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have

$0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then, $\sum_{j=1}^{n} 2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq$ $\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so

$0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 +$
$\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right).$

Thus, $0 \leq \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1$. İt is concluded that

$$0 \leq \sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1.$$

(2) Since

$$\begin{aligned}
\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) &= \sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \sum_{j=1}^{n} \frac{2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{C}},\hat{\mathcal{A}}).
\end{aligned}$$

(3) Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, that is to say, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$,
Then, by using Definition 4.2.5 we have

$$\begin{aligned}
\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) &= \sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2} \\
&= \sum_{j=1}^{n} \frac{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)}{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)} = 1.
\end{aligned}$$

Then, we complete this process of proofs.

Example 4.2.5 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8,0.7)_\kappa\rangle, \langle x_2, (0.5,0.9)_\kappa\rangle, \langle x_3, (0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6,0.9)_\kappa\rangle, \langle x_2, (0.8,0.7)_\kappa\rangle, \langle x_3, (0.7,0.6)_\kappa\rangle\}$ be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and. Then, a dice similarity measures $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \left(\frac{2\begin{pmatrix}(0.8^3\times 0.6^3+0.7^3\times 0.9^3)+\\(0.5^3\times 0.8^3+0.9^3\times 0.7^3)+(0.6^3\times 0.7^3+0.5^3\times 0.6^3)\end{pmatrix}}{\begin{matrix}((0.8^3)^2+(0.7^3)^2+(0.6^3)^2+(0.9^3)^2)+\\((0.5^3)^2+(0.9^3)^2+(0.8^3)^2+(0.7^3)^2)+((0.6^3)^2+(0.5^3)^2+(0.7^3)^2+(0.6^3)^2)\end{matrix}}\right)$$

$$= \left(\frac{2(0.3606+0.3140+0.1010)}{1.1369+1.2392+1.1466}\right)$$

$$= \left(\frac{1.5512}{3.5227}\right)$$

$$= 0.4403.$$

Definition 4.2.6 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the weighted dice similarity measures $\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{2\sum_{j=1}^{n} \omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n} \omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \sum_{j=1}^{n} \omega_j{}^2\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}.$$

Theorem 4.2.6 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, a dice similarity measures $\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0 \le \mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \le 1$,

(2) $\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have

$0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then,

$\sum_{j=1}^{n} 2\omega_j{}^2 \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \sum_{j=1}^{n} \omega_j{}^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so

$0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$.

Thus, $0 \leq \frac{2\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \omega_j{}^2\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \leq 1$. İt is concluded that

$$0 \leq \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1.$$

(2) Since

$$\begin{aligned}
\mathcal{D}_{3_{W q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) &= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \\
&= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \\
&= \mathcal{D}_{3_{W q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}}).
\end{aligned}$$

(3) Since $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$,

Then, by using Definition 4.2.6 we have

$$\begin{aligned}
\mathcal{D}_{3_{W q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) &= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \\
&= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)} \\
&= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)}{2\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)} = 1.
\end{aligned}$$

Then, we complete this process of proofs.

Example 4.2.6 Let $\hat{\mathcal{A}}=\{\langle x_1,(0.8,0.7)_\kappa\rangle,\langle x_2,(0.5,0.9)_\kappa\rangle,\langle x_3,(0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}=\{\langle x_1,(0.6,0.9)_\kappa\rangle,\langle x_2,(0.8,0.7)_\kappa\rangle,\langle x_3,(0.7,0.6)_\kappa\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa=3$ and $\omega_j=(\frac{1}{4},\frac{2}{5},\frac{7}{20})$. Then, the weighted dice similarity measures $\mathcal{D}_{3_{\mathcal{WqROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{3_{\mathcal{WqROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{2\sum_{j=1}^{n}\omega_j^{\,2}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j^{\,2}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)+\sum_{j=1}^{n}\omega_j^{\,2}\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$=\left(\frac{2\left(\begin{array}{c}\left(\frac{1}{4}\right)^2(0.8^3\times 0.6^3+0.7^3\times 0.9^3)+\\ \left(\frac{2}{5}\right)^2(0.5^3\times 0.8^3+0.9^3\times 0.7^3)+\left(\frac{7}{20}\right)^2(0.6^3\times 0.7^3+0.5^3\times 0.6^3)\end{array}\right)}{\left(\frac{1}{4}\right)^2\left(\begin{array}{c}(0.8^3)^2+(0.7^3)^2+\\(0.6^3)^2+(0.9^3)^2\end{array}\right)+\left(\frac{2}{5}\right)^2\left(\begin{array}{c}(0.5^3)^2+(0.9^3)^2+\\(0.8^3)^2+(0.7^3)^2\end{array}\right)+\left(\frac{7}{20}\right)^2\left(\begin{array}{c}(0.6^3)^2+(0.5^3)^2+\\(0.7^3)^2+(0.6^3)^2\end{array}\right)}\right)$$

$$=\left(\frac{2(0.0225+0.0502+0.0123)}{0.1813+0.2971+0.0667}\right)$$

$$=\left(\frac{0.17}{0.5451}\right)$$

$$=0.3118.$$

Definition 4.2.7 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle|x_j\in\mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then, at that point, the dice similarity measures $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$$
$$=\frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Theorem 4.2.7 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, a dice similarity measures $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0 \leq \mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^\kappa(x_j), \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j), \pi_{\hat{\mathcal{A}}}{}^\kappa(x_j), \eta_{\hat{\mathcal{C}}}{}^\kappa(x_j), \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j), \pi_{\hat{\mathcal{C}}}{}^\kappa(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have

$0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j) + \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j) + \pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2\right) + \left(\pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2\right)$. Then, $2\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j) + \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j) + \pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\, \pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2\right) + \left(\pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2\right)$, and so $0 \leq \eta_{\hat{\mathcal{A}}}{}^\kappa(x_j), \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j), \pi_{\hat{\mathcal{A}}}{}^\kappa(x_j), \eta_{\hat{\mathcal{C}}}{}^\kappa(x_j), \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j), \pi_{\hat{\mathcal{C}}}{}^\kappa(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2\right)$.

Thus, $0 \leq \frac{2\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\, \eta_{\hat{\mathcal{C}}}{}^\kappa(x_j) + \gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\, \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j) + \pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)}{\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2} \leq 1$. It is concluded that

$$0 \leq \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\overline{\mathcal{A}}}{}^\kappa(x_j)\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right) + \left(\gamma_{\overline{\mathcal{A}}}{}^\kappa(x_j)\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right) + \left(\pi_{\overline{\mathcal{A}}}{}^\kappa(x_j)\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)\right)}{\left(\sum_{j=1}^{n}\left(\eta_{\overline{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\overline{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\overline{\mathcal{A}}}{}^\kappa(x_j)\right)^2 + \sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^\kappa(x_j)\right)^2\right)} \leq 1.$$

(2) Since

$\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$

$$= \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$= \frac{2\sum_{j=1}^{n}\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\sum_{j=1}^{n}\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{C}},\hat{\mathcal{A}}).$$

(3) Since $\hat{\mathcal{A}}=\hat{\mathcal{C}}$, that is to say $\eta_{\hat{\mathcal{A}}}(x_j)=\eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j)=\gamma_{\hat{\mathcal{C}}}(x_j)$, $\pi_{\hat{\mathcal{A}}}(x_j)=\pi_{\hat{\mathcal{C}}}(x_j)$. Then, by using Definition 4.2.7 we have,

$\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$

$$= \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \frac{2\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \frac{2\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)+\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}{2\left(\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)} = 1.$$

Then, we complete the process of proofs.

Example 4.2.7 Let $\hat{\mathcal{A}}$={$\langle x_1,(0.8,0.7,0.5)_\kappa\rangle,\langle x_2,(0.5,0.9,0.5)_\kappa\rangle,\langle x_3,(0.6,0.5,0.8)_\kappa$} and $\hat{\mathcal{C}}$={$\langle x_1,(0.6,0.9,0.3)_\kappa\rangle,\langle x_2,(0.8,0.7,0.5)_\kappa\rangle,\langle x_3,(0.7,0.6,0.7)_\kappa\rangle$}, be any two q-ROFNs over $\mathcal{X}$. Assume that $\kappa=3$. Then, a dice similarity measures $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{2\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\begin{array}{c}\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\\ \sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\end{array}}$$

$$= \left(\frac{2\left(\left(\begin{matrix} 0.8^3 \times 0.6^3 + \\ 0.7^3 \times 0.9^3 + 0.525^3 \times 0.380^3 \end{matrix} \right) + \left(\begin{matrix} 0.5^3 \times 0.8^3 + \\ 0.9^3 \times 0.7^3 + 0.526^3 \times 0.525^3 \end{matrix} \right) + \left(\begin{matrix} 0.6^3 \times 0.7^3 + \\ 0.5^3 \times 0.6^3 + 0.870^3 \times 0.761^3 \end{matrix} \right) \right)}{\left(\begin{matrix} (0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2 \\ +(0.6^3)^2 + (0.9^3)^2 + (0.380^3)^2 \end{matrix} \right) + \left(\begin{matrix} (0.5^3)^2 + (0.9^3)^2 + (0.526^3)^2 \\ +(0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2 \end{matrix} \right) + \left(\begin{matrix} (0.6^3)^2 + (0.5^3)^2 + (0.870^3)^2 \\ +(0.7^3)^2 + (0.6^3)^2 + (0.761^3)^2 \end{matrix} \right)} \right)$$

$$= \left(\frac{2(0.3685 + 0.3351 + 0.3912)}{0.9818 + 0.9689 + 0.8544} \right)$$

$$= \left(\frac{2.1896}{2.8051} \right)$$

$$= 0.7805.$$

Definition 4.2.8 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the weighted dice similarity measures $\mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) \right)}{\sum_{j=1}^{n} \omega_j^2 \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 + \sum_{j=1}^{n} \omega_j^2 \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2}.$$

Theorem 4.2.8 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, a weighted dice similarity measures $\mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ satisfies the following properties:

(1) $0 \leq \mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{4_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Proof.

(1) For $\kappa \geq 1$, $0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq 1$. Then, from Cauchy-Schwarz inequality, we have

$0 \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$. Then, $2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) \leq \sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$, and so

$0 \leq \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j), \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j), \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \leq \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)$.

Thus, $0 \leq \frac{2\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)} \leq 1$. It is concluded that

$$0 \leq \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \leq 1.$$

(2) Since

$$\begin{aligned}
&\mathcal{D}_{4\mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \\
&= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \sum_{j=1}^{n}\omega_j{}^2\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2} \\
&= \mathcal{D}_{4\mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{C}}, \hat{\mathcal{A}},).
\end{aligned}$$

(3) Since $\widehat{\mathcal{A}} = \hat{\mathcal{C}}$, which means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$, $\pi_{\hat{\mathcal{A}}}(x_j) = \pi_{\hat{\mathcal{C}}}(x_j)$.

Then, by using Definition 4.2.8 we have,

$$\mathcal{D}_{4\mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2\sum_{j=1}^{n}\omega_j{}^2\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \sum_{j=1}^{n}\omega_j{}^2\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2}$$

$$= \frac{2\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$= \frac{2\sum_{j=1}^{n} \omega_j^2 \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) + \pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2}{2\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)} = 1.$$

Then, we complete the process of proofs.

Example 4.2.8 Let $\hat{\mathcal{A}}=\{\langle x_1, (0.8,0.7,0.5)_{\kappa}\rangle, \langle x_2, (0.5,0.9,0.5)_{\kappa}\rangle, \langle x_3, (0.6,0.5,0.8)_{\kappa}\}$ and $\hat{\mathcal{C}}=\{\langle x_1, (0.6,0.9,0.3)_{\kappa}\rangle, \langle x_2, (0.8,0.7,0.5)_{\kappa}\rangle, \langle x_3, (0.7,0.6,0.7)_{\kappa}\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and $\omega_j = (\frac{1}{4}, \frac{2}{5}, \frac{7}{20})$. Then, the weighted dice similarity measures $\mathcal{D}_{4_{W q \mathcal{R}\mathcal{O}\mathcal{F}\mathcal{S}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{4_{W q \mathcal{R}\mathcal{O}\mathcal{F}\mathcal{S}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{2\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\begin{gathered}\sum_{j=1}^{n} \omega_j^2 \left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \\ \sum_{j=1}^{n} \omega_j^2 \left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\end{gathered}}$$

$$= \left(\frac{2\left(\left(\frac{1}{4}\right)^2 \begin{pmatrix} (0.8^3 \times 0.6^3) + \\ (0.7^3 \times 0.9^3) + \\ (0.525^3 \times 0.380^3) \end{pmatrix} + \left(\frac{2}{5}\right)^2 \begin{pmatrix} (0.5^3 \times 0.8^3) + \\ (0.9^3 \times 0.7^3) + \\ (0.526^3 \times 0.525^3) \end{pmatrix} + \left(\frac{7}{20}\right)^2 \begin{pmatrix} (0.6^3 \times 0.7^3) + \\ (0.5^3 \times 0.6^3) + \\ (0.870^3 \times 0.761^3) \end{pmatrix} \right)}{\begin{pmatrix} \left(\frac{1}{4}\right)^2 \begin{pmatrix} (0.8^3)^2 + (0.7^3)^2 \\ +(0.525^3)^2 \end{pmatrix} + \\ \left(\frac{1}{4}\right)^2 \begin{pmatrix} (0.6^3)^2 + (0.9^3)^2 \\ +(0.380^3)^2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \left(\frac{2}{5}\right)^2 \begin{pmatrix} (0.5^3)^2 + (0.9^3)^2 \\ +(0.526^3)^2 \end{pmatrix} + \\ \left(\frac{2}{5}\right)^2 \begin{pmatrix} (0.8^3)^2 + (0.7^3)^2 \\ +(0.525^3)^2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \left(\frac{7}{20}\right)^2 (0.6^3)^2 + (0.5^3)^2 \\ +(0.870^3)^2 + \\ \left(\frac{7}{20}\right)^2 \begin{pmatrix} (0.7^3)^2 + (0.6^3)^2 \\ +(0.761^3)^2 \end{pmatrix} \end{pmatrix}} \right)$$

$$= \left(\frac{2\left(\left(\frac{1}{4}\right)^2 (0.3685) + \left(\frac{2}{5}\right)^2 (0.3351) + \left(\frac{7}{20}\right)^2 (0.3912) \right)}{\begin{gathered}\left(\left(\frac{1}{4}\right)^2 (0.2610) + \left(\frac{1}{4}\right)^2 (0.4310) \right) + \left(\left(\frac{2}{5}\right)^2 (0.4899) + \left(\frac{2}{5}\right)^2 (0.2610) \right) \\ + \left(\left(\frac{7}{20}\right)^2 (0.4733) + \left(\frac{7}{20}\right)^2 (0.3047) \right)\end{gathered}} \right)$$

$$= \left(\frac{2(0.0934)}{0.2585} \right)$$

$$= 0.7223.$$

Results in Examples related to Dice similarity measures are given in (Table 4.3)

Table 4.3 Dice Similarity Measures

Dice Similarity Measures	Results
$\mathcal{D}_{1_{q\mathcal{ROFS}}}(\mathcal{A},\hat{\mathcal{C}})$	0.5576
$\mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.5566
$\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7951
$\mathcal{D}_{\mathcal{G}_{2}{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7848
$\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.4403
$\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.3118
$\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7805
$\mathcal{D}_{4_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$	0.7223

4.3 Generalized Dice Similarity Measures for q-rung Orthopair Fuzzy Sets

We have introduced new measures of similarity in the above section to Dice Similarity Measures for q-rung Orthopair Fuzzy Sets, which we will expand on in this section $\mathcal{A}=\left(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j)\right)$ and $\hat{\mathcal{C}}=\left((\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j)\right)$, for the general constraint set that refers for any two of q-ROFNs over X. We add this section the innovative concepts similarity conditions.

Definition 4.3.1 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j\in\mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be defined as follows:

$$\mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})=\frac{1}{n}\sum_{j=1}^{n}\frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)+\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)+(1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Theorem 4.3.1 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j\in\mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j\in\mathcal{X}\}$ to be any two q-ROFSs over $\mathcal{X}$, then the generalized dice similarity measures $\mathcal{D}_{q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$. possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1$. This means that, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$,

$\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.1 Let $\hat{\mathcal{A}}$={$\langle x_1,(0.8{,}0.7)_\kappa\rangle,\langle x_2,(0.5{,}0.9)_\kappa\rangle,\langle x_3,(0.6{,}0.5)_\kappa$} and $\hat{\mathcal{C}}$={$\langle x_1,(0.6{,}0.9)_\kappa\rangle,\langle x_2,(0.8{,}0.7)_\kappa\rangle,\langle x_3,(0.7{,}0.6)_\kappa\rangle$} be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and $\sigma = 0.5$. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_{1}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n}\frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$= \frac{1}{3}\begin{pmatrix}\left(\dfrac{((0.8^3 \times 0.6^3) + (0.7^3 \times 0.9^3))}{0.5 \times ((0.8^3)^2 + (0.7^3)^2) + (1-0.5) \times ((0.6^3)^2 + (0.9^3)^2)}\right) + \\ \left(\dfrac{((0.5^3 \times 0.8^3) + (0.9^3 \times 0.7^3))}{0.5 \times ((0.5^3)^2 + (0.9^3)^2) + (1-0.5) \times ((0.8^3)^2 + (0.7^3)^2)}\right) + \\ \left(\dfrac{((0.6^3 \times 0.7^3) + (0.5^3 \times 0.6^3))}{0.5 \times ((0.6^3)^2 + (0.5^3)^2) + (1-0.5) \times ((0.7^3)^2 + (0.6^3)^2)}\right)\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}0.7529 + \\ 0.6776 + \\ 0.8922\end{pmatrix}$$

$$= 0.7741.$$

Definition 4.3.2 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$={$\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}$} and $\hat{\mathcal{C}}$={$\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}$} be any two q-ROFNs over $\mathcal{X}$, then the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{1}{}_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ is defined as follows:

$$\mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \omega_j \frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Theorem 4.3.2 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ to be any two q-ROFSs over $\mathcal{X}$, then the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1$. This means that, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.2 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8,0.7)_{\kappa}\rangle, \langle x_2, (0.5,0.9)_{\kappa}\rangle, \langle x_3, (0.6,0.5)_{\kappa}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6,0.9)_{\kappa}\rangle, \langle x_2, (0.8,0.7)_{\kappa}\rangle, \langle x_3, (0.7,0.6)_{\kappa}\rangle\}$ be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and $\sigma = 0.5$, such that $\omega_j = (\frac{1}{4}, \frac{2}{5}, \frac{7}{20})$. Then, the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_1 q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_1 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \omega_j \frac{\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\,\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}$$

$$= \begin{pmatrix} \left(\frac{1}{4}\right)\left(\frac{((0.8^3 \times 0.6^3) + (0.7^3 \times 0.9^3))}{0.5 \times ((0.8^3)^2 + (0.7^3)^2) + (1-0.5) \times ((0.6^3)^2 + (0.9^3)^2)}\right) + \\ \left(\frac{2}{5}\right)\left(\frac{((0.5^3 \times 0.8^3) + (0.9^3 \times 0.7^3))}{0.5 \times ((0.5^3)^2 + (0.9^3)^2) + (1-0.5) \times ((0.8^3)^2 + (0.7^3)^2)}\right) + \\ \left(\frac{7}{20}\right)\left(\frac{((0.6^3 \times 0.7^3) + (0.5^3 \times 0.6^3))}{0.5 \times ((0.6^3)^2 + (0.5^3)^2) + (1-0.5) \times ((0.7^3)^2 + (0.6^3)^2)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0.1882 + \\ 0.1694 + \\ 0.3122 \end{pmatrix}$$

$= 0.6698.$

Definition 4.3.3 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sum_{j=1}^{n} \frac{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

Theorem 4.3.3 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}=\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}=\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1,$

(2) $\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}}),$

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$. This means that, $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.3 Let $\hat{\mathcal{A}}=\{\langle x_1, (0.8,0.7)_\kappa\rangle, \langle x_2, (0.5,0.9)_\kappa\rangle, \langle x_3, (0.6,0.5)_\kappa\}$ and $\hat{\mathcal{C}}=\{\langle x_1, (0.6,0.9)_\kappa\rangle, \langle x_2, (0.8,0.7)_\kappa\rangle, \langle x_3, (0.7,0.6)_\kappa\rangle\}$ be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and $\sigma = 0.5$. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_{2\,q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{j=1}^{n}\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) + \gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)}{\sigma\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + (1-\sigma)\sum_{j=1}^{n}\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)}.$$

$$= \left(\frac{\left((0.8^3 \times 0.6^3 + 0.7^3 \times 0.9^3) + (0.5^3 \times 0.8^3 + 0.9^3 \times 0.7^3) + (0.6^3 \times 0.7^3 + 0.5^3 \times 0.6^3)\right)}{\begin{pmatrix} (0.5)((0.8^3)^2 + (0.7^3)^2) + \\ (1-0.5)((0.6^3)^2 + (0.9^3)^2) \end{pmatrix} + \begin{pmatrix} (0.5)((0.5^3)^2 + (0.9^3)^2) + \\ (1-0.5)((0.8^3)^2 + (0.7^3)^2) \end{pmatrix} + \begin{pmatrix} (0.5)((0.6^3)^2 + (0.5^3)^2) + \\ (1-0.5)((0.7^3)^2 + (0.6^3)^2) \end{pmatrix}} \right)$$

$$= \left(\frac{(0.3606 + 0.3140 + 0.1010)}{0.9037 + 0.3861 + 0.0792} \right)$$

$$= \left(\frac{0.7756}{1.369} \right)$$

$$= 0.5665.$$

Definition 4.3.4 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{j=1}^n \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j) \eta_{\hat{\mathcal{C}}}{}^\kappa(x_j) \right) + \left(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j) \gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j) \right) \right)}{\sigma \sum_{j=1}^n \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^\kappa(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^\kappa(x_j) \right)^2 \right) + (1-\sigma) \sum_{j=1}^n \omega_j^2 \left(\left(\eta_{\hat{\mathcal{C}}}{}^\kappa(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^\kappa(x_j) \right)^2 \right)}.$$

Theorem 4.3.4 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, a weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_2 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$. This means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.4 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8{,}0.7)_\kappa \rangle, \langle x_2, (0.5{,}0.9)_\kappa \rangle, \langle x_3, (0.6{,}0.5)_\kappa \}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6{,}0.9)_\kappa \rangle, \langle x_2, (0.8{,}0.7)_\kappa \rangle, \langle x_3, (0.7{,}0.6)_\kappa \rangle\}$ be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and $\sigma = 0.5$, such that $\omega_j = (\frac{1}{4}, \frac{2}{5}, \frac{7}{20})$. Then, the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_2 W q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_2 W q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) \right)}{\sigma \sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 \right) + (1-\sigma) \sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 \right)}$$

$$= \left(\frac{\left(\left(\frac{1}{4}\right)^2 \begin{pmatrix} 0.8^3 \times 0.6^3 + \\ 0.7^3 \times 0.9^3 \end{pmatrix} + \left(\frac{2}{5}\right)^2 \begin{pmatrix} 0.5^3 \times 0.8^3 + \\ 0.9^3 \times 0.7^3 \end{pmatrix} + \left(\frac{7}{20}\right)^2 \begin{pmatrix} 0.6^3 \times 0.7^3 + \\ 0.5^3 \times 0.6^3 \end{pmatrix} \right)}{\begin{pmatrix} (0.5)\left(\frac{1}{4}\right)^2 ((0.8^3)^2 + (0.7^3)^2) + \\ (1-0.5)\left(\frac{1}{4}\right)^2 ((0.6^3)^2 + (0.9^3)^2) \end{pmatrix} + \begin{pmatrix} (0.5)\left(\frac{2}{5}\right)^2 ((0.5^3)^2 + (0.9^3)^2) + \\ (1-0.5)\left(\frac{2}{5}\right)^2 ((0.8^3)^2 + (0.7^3)^2) \end{pmatrix} + \begin{pmatrix} (0.5)\left(\frac{7}{20}\right)^2 ((0.6^3)^2 + (0.5^3)^2) + \\ (1-0.5)\left(\frac{7}{20}\right)^2 ((0.7^3)^2 + (0.6^3)^2) \end{pmatrix}} \right)$$

$$= \left(\frac{(0.0225 + 0.0502 + 0.0123)}{0.0295 + 0.0747 + 0.0118} \right)$$

$$= \left(\frac{0.085}{0.116} \right)$$
$$= 0.7327.$$

Definition 4.3.5 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa} \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa} \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_3 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{\mathcal{G}_3 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n} \sum_{j=1}^{n} \frac{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right) \right)}{\sum_{j=1}^{n} \begin{pmatrix} \sigma \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j) \right)^2 \right) + \\ (1-\sigma) \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j) \right)^2 \right) \end{pmatrix}}.$$

Theorem 4.3.5 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa} \rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa} \rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_3 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_{3}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{\mathcal{G}_{3}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_{3}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_{3}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$. This means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.5 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8,0.7,0.5)_\kappa\rangle, \langle x_2, (0.5,0.9,0.5)_\kappa\rangle, \langle x_3, (0.6,0.5,0.8)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6,0.9,0.3)_\kappa\rangle, \langle x_2, (0.8,0.7,0.5)_\kappa\rangle, \langle x_3, (0.7,0.6,0.7)_\kappa\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{3}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_{3}{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\sum_{j=1}^{n}\begin{pmatrix} \sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \\ (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)\end{pmatrix}}$$

$$= \left(\frac{1}{3}\right)\left(\frac{\begin{pmatrix}(0.8^3 \times 0.6^3) + \\ (0.7^3 \times 0.9^3) + (0.525^3 \times 0.380^3)\end{pmatrix} + \\ \begin{pmatrix}(0.5^3 \times 0.8^3) + \\ (0.9^3 \times 0.7^3) + (0.526^3 \times 0.525^3)\end{pmatrix} + \begin{pmatrix}(0.6^3 \times 0.7^3) + \\ (0.5^3 \times 0.6^3) + (0.870^3 \times 0.761^3)\end{pmatrix}}{\begin{pmatrix}0.5 \times ((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2) + \\ (1-0.5) \times ((0.6^3)^2 + (0.9^3)^2 + (0.380^3)^2)\end{pmatrix} + \\ \begin{pmatrix}0.5 \times ((0.5^3)^2 + (0.9^3)^2 + (0.526^3)^2) + \\ (1-0.5) \times ((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2)\end{pmatrix} + \begin{pmatrix}0.5 \times ((0.6^3)^2 + (0.5^3)^2 + (0.870^3)^2) + \\ (1-0.5) \times ((0.7^3)^2 + (0.6^3)^2 + (0.761^3)^2)\end{pmatrix}}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{(0.3685) + (0.3351) + (0.3912)}{(0.4281) + (0.4402) + (0.4390)}\right)$$

$$= 0.2791.$$

Definition 4.3.6 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the wieghted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_{3}{}_{\mathcal{W}q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \frac{\omega_j\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\sigma\sum_{j=1}^{n}\begin{pmatrix}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)+\\(1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)\end{pmatrix}\right)}.$$

Theorem 4.2.6 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{A}}}(x_j),\gamma_{\hat{\mathcal{A}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j,(\eta_{\hat{\mathcal{C}}}(x_j),\gamma_{\hat{\mathcal{C}}}(x_j))_{\kappa}\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFSs over $\mathcal{X}$, the wieghted generalized dice similarity measures are used $\mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ between two q-ROFSs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \le \mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \le 1$,

(2) $\mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \mathcal{D}_{3 W q \mathcal{ROFS}}(\hat{\mathcal{C}},\hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = 1$. This means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.6 Let $\hat{\mathcal{A}}$=$\{\langle x_1,(0.8,0.7,0.5)_{\kappa}\rangle,\langle x_2,(0.5,0.9,0.5)_{\kappa}\rangle,\langle x_3,(0.6,0.5,0.8)_{\kappa}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1,(0.6,0.9,0.3)_{\kappa}\rangle,\langle x_2,(0.8,0.7,0.5)_{\kappa}\rangle,\langle x_3,(0.7,0.6,0.7)_{\kappa}\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and σ=0.5, such that $\omega_j = (\frac{1}{4},\frac{2}{5},\frac{7}{20})$. Then, the wieghted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_3 W q \mathcal{ROFS}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \sum_{j=1}^{n} \frac{\omega_j\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\sigma\sum_{j=1}^{n}\begin{pmatrix}\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right)+\\(1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2+\left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)\end{pmatrix}\right)}$$

$$= \left(\frac{\left(\frac{1}{4}\right)\left(\begin{matrix}(0.8^3 \times 0.6^3) + \\ (0.7^3 \times 0.9^3) + (0.525^3 \times 0.380^3)\end{matrix}\right) + \left(\frac{2}{5}\right)\left(\begin{matrix}(0.5^3 \times 0.8^3) + \\ (0.9^3 \times 0.7^3) + (0.526^3 \times 0.525^3)\end{matrix}\right) + \left(\frac{7}{20}\right)\left(\begin{matrix}(0.6^3 \times 0.7^3) + \\ (0.5^3 \times 0.6^3) + (0.870^3 \times 0.761^3)\end{matrix}\right)}{0.5\left(\left(\begin{matrix}((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2) + \\ (1-0.5)((0.6^3)^2 + (0.9^3)^2 + (0.380^3)^2)\end{matrix}\right) + \left(\begin{matrix}((0.5^3)^2 + (0.9^3)^2 + (0.526^3)^2) + \\ (1-0.5)((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2)\end{matrix}\right) + \left(\begin{matrix}((0.6^3)^2 + (0.5^3)^2 + (0.870^3)^2) + \\ (1-0.5)((0.7^3)^2 + (0.6^3)^2 + (0.761^3)^2)\end{matrix}\right)\right)} \right)$$

$$= \left(\frac{(0.0921) + (0.1340) + (0.1369)}{0.5(0.6912 + 0.7686 + 0.6751)}\right)$$

$$= \left(\frac{0.363}{1.0674}\right)$$

$= 0.3400.$

Definition 4.2.7 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sum_{j=1}^{n} \frac{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\begin{matrix}\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \\ (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)\end{matrix}}.$$

Theorem 4.3.7 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$. This means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$,

$\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.7 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8,0.7,0.5)_\kappa\rangle, \langle x_2, (0.5,0.9,0.5)_\kappa\rangle, \langle x_3, (0.6,0.5,0.8)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6,0.9,0.3)_\kappa\rangle, \langle x_2, (0.8,0.7,0.5)_\kappa\rangle, \langle x_3, (0.7,0.6,0.7)_\kappa\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and σ=0.5. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_4 q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \sum_{j=1}^{n} \frac{\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\begin{array}{c}\sigma\left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \\ (1-\sigma)\left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)\end{array}}$$

$$= \left(\frac{\begin{array}{c}\left(\begin{array}{c}(0.8^3 \times 0.6^3) + \\ (0.7^3 \times 0.9^3) + (0.525^3 \times 0.380^3)\end{array}\right) + \\ \left(\begin{array}{c}(0.5^3 \times 0.8^3) + \\ (0.9^3 \times 0.7^3) + (0.526^3 \times 0.525^3)\end{array}\right) + \left(\begin{array}{c}(0.6^3 \times 0.7^3) + \\ (0.5^3 \times 0.6^3) + (0.870^3 \times 0.761^3)\end{array}\right)\end{array}}{\begin{array}{c}\left(\begin{array}{c}0.5 \times ((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2) + \\ (1-0.5) \times ((0.6^3)^2 + (0.9^3)^2 + (0.380^3)^2)\end{array}\right) + \\ \left(\begin{array}{c}0.5 \times ((0.5^3)^2 + (0.9^3)^2 + (0.526^3)^2) + \\ (1-0.5) \times ((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2)\end{array}\right) + \left(\begin{array}{c}0.5 \times ((0.6^3)^2 + (0.5^3)^2 + (0.870^3)^2) + \\ (1-0.5) \times ((0.7^3)^2 + (0.6^3)^2 + (0.761^3)^2)\end{array}\right)\end{array}}\right)$$

$$= \left(\frac{(0.3685) + (0.3351) + (0.3912)}{(0.4281) + (0.4402) + (0.4390)}\right)$$

$= 0.8374$.

Definition 4.3.8 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$, then the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_4 Wq\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can defined as follows:

$$\mathcal{D}_{\mathcal{G}_4 Wq\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{j=1}^{n} \omega_j{}^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)\right)}{\left(\begin{array}{c}\sigma \sum_{j=1}^{n} \omega_j{}^2 \left(\left(\eta_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}{}^{\kappa}(x_j)\right)^2\right) + \\ (1-\sigma) \sum_{j=1}^{n} \omega_j{}^2 \left(\left(\eta_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}{}^{\kappa}(x_j)\right)^2\right)\end{array}\right)}.$$

Theorem 4.3.8 (Jan *et al.* 2020) Let $\hat{\mathcal{A}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{A}}}(x_j), \gamma_{\hat{\mathcal{A}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ and $\hat{\mathcal{C}}$=$\{\langle x_j, (\eta_{\hat{\mathcal{C}}}(x_j), \gamma_{\hat{\mathcal{C}}}(x_j))_\kappa\rangle | x_j \in \mathcal{X}\}$ be any two q-ROFNs over $\mathcal{X}$. Then, the weighted generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ possesses the following properties:

(1) $0 \leq \mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \leq 1$,

(2) $\mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{C}}, \hat{\mathcal{A}})$,

(3) If $\hat{\mathcal{A}} = \hat{\mathcal{C}}$, then $\mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = 1$. This means that $\eta_{\hat{\mathcal{A}}}(x_j) = \eta_{\hat{\mathcal{C}}}(x_j)$, $\gamma_{\hat{\mathcal{A}}}(x_j) = \gamma_{\hat{\mathcal{C}}}(x_j)$.

Example 4.3.8 Let $\hat{\mathcal{A}}$=$\{\langle x_1, (0.8,0.7,0.5)_\kappa\rangle, \langle x_2, (0.5,0.9,0.5)_\kappa\rangle, \langle x_3, (0.6,0.5,0.8)_\kappa\}$ and $\hat{\mathcal{C}}$=$\{\langle x_1, (0.6,0.9,0.3)_\kappa\rangle, \langle x_2, (0.8,0.7,0.5)_\kappa\rangle, \langle x_3, (0.7,0.6,0.7)_\kappa\rangle\}$, be any two q-ROFNs over $\mathcal{X}$, assume that $\kappa = 3$ and σ=0.5, such that $\omega_j = \left(\frac{1}{4}, \frac{2}{5}, \frac{7}{20}\right)$. Then, the generalized dice similarity measures $\mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ between two q-ROFNs $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ can be calculated as follows:

$$\mathcal{D}_{\mathcal{G}_4 \mathcal{W}q\mathcal{ROFS}}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) = \frac{\sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}^{\kappa}(x_j)\eta_{\hat{\mathcal{C}}}^{\kappa}(x_j)\right) + \left(\gamma_{\hat{\mathcal{A}}}^{\kappa}(x_j)\gamma_{\hat{\mathcal{C}}}^{\kappa}(x_j)\right) + \left(\pi_{\hat{\mathcal{A}}}^{\kappa}(x_j)\pi_{\hat{\mathcal{C}}}^{\kappa}(x_j)\right)\right)}{\begin{array}{c}\sigma \sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{A}}}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{A}}}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{A}}}^{\kappa}(x_j)\right)^2\right) + \\ (1-\sigma) \sum_{j=1}^{n} \omega_j^2 \left(\left(\eta_{\hat{\mathcal{C}}}^{\kappa}(x_j)\right)^2 + \left(\gamma_{\hat{\mathcal{C}}}^{\kappa}(x_j)\right)^2 + \left(\pi_{\hat{\mathcal{C}}}^{\kappa}(x_j)\right)^2\right)\end{array}}$$

$$= \left(\frac{\begin{array}{c}\left(\frac{1}{4}\right)^2 \left(\begin{array}{c}(0.8^3 \times 0.6^3) + \\ (0.7^3 \times 0.9^3) + (0.525^3 \times 0.380^3)\end{array}\right) + \\ \left(\frac{2}{5}\right)^2 \left(\begin{array}{c}(0.5^3 \times 0.8^3) + \\ (0.9^3 \times 0.7^3) + (0.526^3 \times 0.525^3)\end{array}\right) + \left(\frac{7}{20}\right)^2 \left(\begin{array}{c}(0.6^3 \times 0.7^3) + \\ (0.5^3 \times 0.6^3) + (0.870^3 \times 0.761^3)\end{array}\right)\end{array}}{\begin{array}{c}\left(\begin{array}{c}0.5 \times \left(\frac{1}{4}\right)^2 ((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2) + \\ (1-0.5) \times \left(\frac{1}{4}\right)^2 ((0.6^3)^2 + (0.9^3)^2 + (0.380^3)^2)\end{array}\right) + \\ \left(\begin{array}{c}0.5 \times \left(\frac{2}{5}\right)^2 ((0.5^3)^2 + (0.9^3)^2 + (0.526^3)^2) + \\ (1-0.5) \times \left(\frac{2}{5}\right)^2 ((0.8^3)^2 + (0.7^3)^2 + (0.525^3)^2)\end{array}\right) + \left(\begin{array}{c}0.5 \times \left(\frac{7}{20}\right)^2 ((0.6^3)^2 + (0.5^3)^2 + (0.870^3)^2) + \\ (1-0.5) \times \left(\frac{7}{20}\right)^2 ((0.7^3)^2 + (0.6^3)^2 + (0.761^3)^2)\end{array}\right)\end{array}} \right)$$

$$= \left(\frac{(0.0230) + (0.0536) + (0.0479)}{(0.0306) + (0.0775) + (0.0523)}\right)$$

$$= 0.7761.$$

Results are given in (Table 4.4) for all generalized Dice similarity measures.

Table 4.4 Generalized Dice Similarity Measures

Generalized Dice Similarity Measures	Results
$\mathcal{D}_{\mathcal{G}_1{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.7741
$\mathcal{D}_{\mathcal{G}_2{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.6698
$\mathcal{D}_{\mathcal{G}_2{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.5665
$\mathcal{D}_{\mathcal{G}_2{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.7327
$\mathcal{D}_{\mathcal{G}_3{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.2791
$\mathcal{D}_{\mathcal{G}_3{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.3400
$\mathcal{D}_{\mathcal{G}_4{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.8374
$\mathcal{D}_{\mathcal{G}_4{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$	0.7761

Through the above examples for the values of DSM and GDSM, we noticed that our highest values in GDSM to be available it is $\mathcal{D}_{\mathcal{G}_4{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$. We notice that our highest values in DSM to be available it is $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$, we are notice from examples above in DSM and GDSM, when comparing if the value increases in $\mathcal{D}_{\mathcal{G}_1{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ the value decreasing $\mathcal{D}_{\mathcal{G}_1{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ according to increases $\mathcal{D}_{\mathcal{G}_1{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ and the same is true for DSM, when $\mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ increases the value $\mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ it is decreasing, also DSM the value in $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ be equal to value in $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ increases, then the value $\mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ are decreasing according to increases $\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ and the same is true for GDSM when $\mathcal{D}_{\mathcal{G}_2{}_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ increases the value $\mathcal{D}_{\mathcal{G}_2{}_{q\mathcal{ROFS}}}(\hat{\mathcal{A}}, \hat{\mathcal{C}})$ it is decreasing, when the value in $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ decreasing then the value $\mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ increase according to the increases $\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$, and the same is true for GDSM, when $\mathcal{D}_{\mathcal{G}_{3_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ decrease, then the value in $\mathcal{D}_{\mathcal{G}_{3_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ increase, finally we notice, when the value in $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ increases, then the value $\mathcal{D}_{4_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ decrease according to the increase $\mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$, and the same is true for GDSM, when $\mathcal{D}_{\mathcal{G}_{4_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ increases, then the value in $\mathcal{D}_{\mathcal{G}_{4_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})$ are decreasing.

The idea can be summarized in the following order :

- $\left(\mathcal{D}_{\mathcal{G}_{1_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{1_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \text{ and } \mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)$, such that:

  $\left(\mathcal{D}_{\mathcal{G}_{1_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{1_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{1_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{1_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)$,

  and we notice

- $\left(\mathcal{D}_{\mathcal{G}_{2_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{2_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \text{ and } \mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)$,

such that:

$$\left(\mathcal{D}_{2_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{2_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{2_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{2_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right),$$

and we notice

- $\left(\mathcal{D}_{\mathcal{G}_{3_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) < \mathcal{D}_{\mathcal{G}_{3_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \text{ and } \mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)$,

such that

$\left(\mathcal{D}_{3_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{3_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{3_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{3_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)$, and we notice

- $\left(\mathcal{D}_{\mathcal{G}_{4_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{4_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \text{ and } \mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{4_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)$,

such that:

$$\left(\mathcal{D}_{\mathcal{G}_{4_{q\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{4_{q\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{\mathcal{G}_{4_{Wq\mathcal{ROFS}}}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) > \mathcal{D}_{4_{Wq\mathcal{ROFS}}}(\hat{\mathcal{A}},\hat{\mathcal{C}})\right).$$

5. CONCLUSIONS AND RECOMMENDATION

In this thesis, a compilation of some similarity measures available in the literature for q-ROFSs has been made. Similarity measures and associated features were taken from cited articles. It is aimed to make the existing measurements more understandable by producing examples related to these definitions. It is aimed that this study will be a regular and collective resource for researchers who intend to apply the similarity measurements of q-ROFSs to real problems.

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CURRICULUM VITAE

Personal Information

Name and Surname : Waleed Sadeq Jaaywel JAAYWEL

Education

MSc	Çankırı Karatekin University Graduate School of Natural and Applied Sciences Mathematics	2022-Present
Undergraduate	Thi Qar University College of Mathematics and computer scienes, Department of Mathematics	2013-2017