BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

QI TYPE INTEGRAL INEQUALITIES

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

MELİH TOLUNAY

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ADVISOR: Assoc. Prof. Yüksel SOYKAN

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Advisor: Assoc. Prof. Yüksel SOYKAN ....
Bülent Ecevit University, Faculty of Arts and Science, Department of Mathematics

Member: Assoc. Prof. Yusuf KAYA ....
Bülent Ecevit University, Faculty of Arts and Science, Department of Mathematics

Member: Assist. Prof. Ömer KİŞİ ....
Bartın University, Faculty of Arts and Science, Department of Mathematics

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Assoc. Prof. Ahmet ÖZARSLAN
Director

"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

Melih TOLUNAY

ABSTRACT

Master of Science Thesis

QI TYPE INTEGRAL INEQUALITIES

Melih TOLUNAY

Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Yüksel SOYKAN
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Integral inequalities have been frequently employed in the theory of functional analysis, differential equations and applied sciences such as probability and statistics. Recently, especially an integral inequaliy, which is called Qi Inequality by mathematics community, has been studied by many authors.

In this thesis, we are concerned with Qi type integral inequalities. We present many conjectures on these kind of inequalities. Then, we investigate many papers which gives affirmative answer on these conjectures.

The organization of this thesis is as follows:

In Chapter 1, we give the necessary inequalities and some definitions.

In Chapter 2, we present Qi's open problem and related problems as conjectures.

In Chapter 3, we consider five papers on Qi type integral inequalities from 2001 through 2003.

ABSTRACT (continued)

In Chapter 4, we present four papers on Qi type integral inequalities from 2004 through 2005.

In Chapter 5, we look at five papers on Qi type integral inequalities in 2006.

In Chapter 6, we present four papers on Qi type integral inequalities from 2007 through 2009.

Keywords: Qi type integral inequalities, integral inequalities, inequalities.

Science Code: 403.03.01

ÖZET

Yüksek Lisans Tezi

QI TİPİ İNTEGRAL EŞİTSİZLİKLERİ

Melih TOLUNAY

Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Doc. Dr. Yüksel SOYKAN
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İntegral eşitsizlikleri fonksiyonel analiz teorisinde, diferansiyal denklemlerde ve olasılık istatistik gibi uygulamalı bilimlerde sıkça kullanılır. Son zamanlarda özellikle matematik toplulukları tarafından Qi eşitsizliği olarak adlandırılan bir eşitsizlik birçok yazar tarafından çalışılmıştır.

Bu tezde, Qi tipi integral eşitsizlikleri ile ilgileniyoruz. Bu tip eşitsizlikler üzerine birçok sanı sunuyoruz. Sonrasında bu sanılara olumlu cevap veren birçok makaleyi inceliyoruz.

Bu tezin organizasyonu aşağıdaki gibidir:

Birinci bölümde, gerekli eşitsizlikler ve bazı tanımlar verilmiştir.

İkinci bölümde, Qi'nin açık problemi ve onunla ilişkili problemler sanı olarak verilmiştir.

ÖZET (devam ediyor)

Üçüncü bölümde, 2001 den 2003 boyunca Qi tipi integral eşitsizlikleri üzerine yapılan beş makaleyi ele alıyoruz.

Dördüncü bölümde, 2004 den 2005 boyunca Qi tipi integral eşitsizlikleri üzerine yapılan dört makaleyi sunuyoruz.

Beşinci bölümde, 2006 da Qi tipi integral eşitsizlikleri üzerine yapılan beş makaleyi gözden geçiriyoruz.

Altıncı bölümde, 2007 den 2009 boyunca Qi tipi integral eşitsizlikleri üzerine yapılan dört makaleyi sunuyoruz.

Anahtar Kelimeler: Qi tipi integral eşitsizlikleri, integral eşitsizlikleri, eşitsizlikler.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 INTRODUCTION

In this chapter we present some basic inequalities.

Lemma 1.1 (Jensen's Inequality, [1]) *Suppose that μ is a positive finite measure on a σ-algebra $\mathcal{M}$ in (a,b). If f is a real function in $L^1(a,b)$ and φ is convex in (a,b), then*

$$\varphi\left(\frac{\int_a^b f(x)\,d\mu(x)}{\int_a^b d\mu}\right) \leq \frac{\int_a^b \varphi(f(x))\,d\mu(x)}{\int_a^b d\mu}. \tag{1.1}$$

Lemma 1.2 ([2], p.2) *Suppose that $\mathcal{V}$ is a linear vector space of real valued functions and $f,g \in \mathcal{V}$ with $g \geq 0$. Let F be a positive linear form on $\mathcal{V}$ and φ: $\mathbb{R} \to \mathbb{R}$ be a convex function such that*

(a) $F(g) = 1$,

(b) fg *and* $(\varphi \circ f)g \in \mathcal{V}$.

Then

$$\varphi(F(fg)) \leq F((\varphi \circ f)g).$$

Notice that Lemma 1.2 is in fact a form of the classical Jensen inequality. There is a vast literature on this subject, see, e.g., [3, 4, 5, 6, 7, 8].

Lemma 1.3 ([1]) *Assume that φ is a real differentiable function in (a,b). Then φ is convex in (a,b) if and only if $\varphi'(u) \leq \varphi'(v)$ for all u and v satisfying $a < u < v < b$.*

Theorem 1.4 (Cauchy's Mean Value Theorem (CMVT)) *If the functions h, g differentiable on (a,b) and continuous on $[a,b]$ then there exists a $\xi \in (a,b)$ such that*

$$h'(\xi)(g(b) - g(a)) = g'(\xi)(h(b) - h(a)).$$

Lemma 1.5 (Nehari Inequality, [9, 10]) *Suppose that f, g are nonnegative concave functions on $[a, b]$. Then, for $p, q > 0$ such that $p^{-1} + q^{-1} = 1$, we get*

$$\left(\int_a^b f^p\right)^{\frac{1}{p}} \left(\int_a^b g^q\right)^{\frac{1}{q}} \leq N(p, q) \int_a^b fg, \tag{1.2}$$

where

$$N(p, q) = \frac{6}{(1+p)^{\frac{1}{p}}(1+q)^{\frac{1}{q}}}. \tag{1.3}$$

Lemma 1.6 (Barnes-Godunova-Levin Inequality, [9, 10, 11, 12, 13]) *Suppose that f, g are nonnegative concave functions on $[a, b]$. Then, for $p, q > 1$ we get*

$$\left(\int_a^b f^p\right)^{\frac{1}{p}} \left(\int_a^b g^q\right)^{\frac{1}{q}} \leq B(p, q) \int_a^b fg, \tag{1.4}$$

where

$$B(p, q) = \frac{6(b-a)^{\frac{1}{p}+\frac{1}{q-1}}}{(1+p)^{\frac{1}{p}}(1+q)^{\frac{1}{q}}}. \tag{1.5}$$

For further details about inequalities (1.2) and (1.4), refer to Section 4.3 of the monograph [12].

Lemma 1.7 (Pečarić Inequality, [12, 14]) *Suppose that f, g are nonnegative, concave functions and that fg are μ-integrable on $[a, b]$, where μ is a probability distribution function. Assume that f is non decreasing and g non increasing. Then for $p, q \geq 1$ we get*

$$\left(\int_a^b f^p d\mu\right)^{\frac{1}{p}} \left(\int_a^b g^q d\mu\right)^{\frac{1}{q}} \leq P(p, q; \mu) \int_a^b fg d\mu, \tag{1.6}$$

where

$$P(p, q; \mu) = \frac{\left(\int_a^b (x-a)^p d\mu\right)^{\frac{1}{p}} \left(\int_a^b (b-x)^q d\mu\right)^{\frac{1}{q}}}{\int_a^b (b-x)(x-a) d\mu}. \tag{1.7}$$

1.1.1 Conditions Based On Inequalities Between Means

In this subsection,we will give some notations and inequalities which will be used to Section 4.1.

Definition 1.1 ([15], p.2) *Assume that f is a nonnegative and integrable function on the segment $[a,b]$. The r- mean (or the r-th power mean) of f is defined as*

$$M^{[r]}(f) := \begin{cases} \left(\frac{\int_a^b f(x)^r dx}{b-a}\right)^{\frac{1}{r}} & (r \neq 0,\ +\infty,\ -\infty)\,, \\ \exp\left(\frac{\int_a^b \ln f(x) dx}{b-a}\right) & (r = 0)\,, \\ m & (r = -\infty)\,, \\ M & (r = +\infty)\,. \end{cases}$$

where $m = \inf f(x)$ and $M = \sup f(x)$ for $x \in [a,b]$.

Lemma 1.8 (Power Mean Inequality, [10]) *If f is a nonnegative function on $[a,b]$ and $-\infty \leq r < s \leq +\infty$, then*

$$M^{[r]}(f) \leq M^{[s]}(f)\,.$$

Lemma 1.9 (Berwald Inequality, [13, 14]) *If f is a nonnegative concave function on $[a,b]$, then for $0 < r < s$ we get*

$$M^{[s]}(f) \leq \frac{(r+1)^{\frac{1}{r}}}{(s+1)^{\frac{1}{s}}} M^{[r]}(f)\,.$$

Lemma 1.10 (Thunsdorff Inequality, [13]) *If f is a nonnegative convex function on $[a,b]$ with f(a) = 0, then for $0 < r < s$ we get*

$$M^{[s]}(f) \geq \frac{(r+1)^{\frac{1}{r}}}{(s+1)^{\frac{1}{s}}} M^{[r]}(f)\,.$$

CHAPTER 2

QI INEQUALITY, A REVIEW ON QI TYPE INTEGRAL INEQUALITY

In this chapter, we consider the following Problem 1, Qi's open problem, which appeared at the end of [16]. It was actually posed by Feng Qi in the preprint version [17] and it was formally published in his paper version. Then we present related problems with Problem 1.

In the next chapters, we introduce articles about Qi's open problem and related problems, which are generalizations and special forms of this problem.

For potential availability to interested readers, we list the collection as references of this thesis.

2.1 QI'S OPEN PROBLEM AND RELATED PROBLEMS

In [16], Qi proved the following interesting integral inequality result.

Theorem 2.1 ([16], p.1) *Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=0$.*

(a) *If $0 \leq f'(x) \leq 1$ for $x \in (a,b)$ then*

$$\int_a^b [f(x)]^3 dx \leq \left(\int_a^b f(x) dx \right)^2 . \tag{2.1}$$

(b) *If $f'(x) \geq 1$, then inequality (2.1) reverses.*

(c) *The equality in (2.1) is valid only if $f(x) \equiv 0$ or $f(x) = x - a$.*

As a generalization of inequality 2.1, Qi also obtained the following more general result in [16].

Theorem 2.2 ([16], p.2) *For $n \in \mathbb{N}$, assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$ and $f^{(n)}(x) \geq n!$ then*

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x) dx \right)^{n+1}. \tag{2.2}$$

Qi also proposed an open integral problem at the end of [16] which was actually posed by himself in the preprint version [17].

Remark 2.1 ([16], p.3) *(Problem 1). Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1} \tag{2.3}$$

hold for $t > 1$ (or for $t \geq 1$)?

Qi proposed the Problem 1 for the case $t > 1$ and we include the case $t \geq 1$ in to the Problem 1.
Since then, this Problem 1 (Remark 2.1) has been stimulating much interest of many mathematicians and affirmative answer to it has been established.

Remark 2.2 *In [16], Qi phrased that "the inequality (2.2) is not found in [10,18,19,20] and so maybe it is a new inequality". In fact it is a new result so has attracted many mathematicians research interest and many extensions, generalizations and applications of inequality (2.2) or (2.3) have been investigated in recent years and published articles devoted to answering those problems. For more detailed information, please see the references therein.*

We propose same integral inequality (2.3) *for the case $t < 1$ (or for $t \leq 1$)* as open problem.

Remark 2.3 *(Problem 2). Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1} \tag{2.4}$$

hold for $t < 1$ (or for $t \leq 1$)?

We propose some integral inequality as open problem. The following are the reverses of Problem 1 (Remark 2.1) and Problem 2 (Remark 2.3), respectively.

Remark 2.4 *(Problem 3). Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x)dx\right)^{t-1} \tag{2.5}$$

hold for $t > 1$ (or for $t \geq 1$)?

Remark 2.5 *(Problem 4). Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x)dx\right)^{t-1} \tag{2.6}$$

hold for $t < 1$ (or for $t \leq 1$)?

The following two open problems are special cases of Problems 1 (Remark 2.1) and 2 (Remark 2.4).

Remark 2.6 *(Problem 5). Under what conditions does the inequality*

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x)dx\right)^{n+1} \tag{2.7}$$

hold for $n \in \mathbb{N}$?

Remark 2.7 *(Problem 6). Under what conditions does the inequality*

$$\int_a^b [f(x)]^{n+2} dx \leq \left(\int_a^b f(x)dx\right)^{n+1} \tag{2.8}$$

hold for $n \in \mathbb{N}$?

In [21], L. Bougoffa posed the following problem (see Problem 2 in [21]) which is similar to Problem 1 (Remark 2.1), which is Qi's inequality.

Remark 2.8 *(Problem 7). Under what conditions does the inequality*

$$\int_a^b [f(x)]^t \, dx \leq \left(\int_a^b f(x)\, dx\right)^{1-t} \tag{2.9}$$

hold for $t < 1$?

The following is the reverse of Bougoffa's inequality which is Problem 7 (Remark 2.8).

Remark 2.9 *(Problem 8). Under what conditions does the inequality*

$$\int_a^b [f(x)]^t \, dx \geq \left(\int_a^b f(x)\, dx\right)^{1-t} \tag{2.10}$$

hold for $t < 1$?

The following two open problems are extension of Problems 1, 3, 8 and 2, 4, 7 respectively.

Remark 2.10 *(Problem 9). Under what conditions does the inequality*

$$\int_a^b [f(x)]^\alpha \, dx \geq \left(\int_a^b f(x) \, dx \right)^\beta \tag{2.11}$$

hold for α and β?

Remark 2.11 *(Problem 10). Under what conditions does the inequality*

$$\int_a^b [f(x)]^\alpha \, dx \leq \left(\int_a^b f(x) \, dx \right)^\beta \tag{2.12}$$

hold for α and β?

We can also propose the following generalization of above open problems.

Remark 2.12 *(Problem 11). Under what conditions does the inequality*

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f^\gamma(x) \, dx \right)^\beta \tag{2.13}$$

hold for α, β and γ?

In the case of $\gamma = 1$, Problem 11 is equivalent to Problem 9.

Remark 2.13 *(Problem 12). Under what conditions does the inequality*

$$\int_a^b [f(x)]^\alpha dx \leq \left(\int_a^b f^\gamma(x) \, dx \right)^\beta \tag{2.14}$$

hold for α, β and γ?

In the case of $\gamma = 1$, Problem 12 is equivalent to Problem 10.

Remark 2.14 *(Problem 13). Under what conditions does the inequality*

$$\int_a^b [f(x)]^{\alpha+\gamma} dx \geq \left(\int_a^b f^{\frac{\alpha}{\beta}}(x) \, g^{\frac{\gamma}{\beta}}(x) \, dx \right)^{\beta-1} \tag{2.15}$$

hold for α, β and γ?

Remark 2.15 *(Problem 14). Under what conditions does the inequality*

$$\int_a^b [f(x)]^{\alpha+\gamma} dx \geq \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x) \, g^{\frac{\gamma}{\lambda}}(x) \, dx \right)^{\beta} \tag{2.16}$$

hold for α, β, λ and γ?

Remark 2.16 *(Problem 15). Under what conditions does the inequality*

$$\int_a^b [f(x)]^{\alpha+\gamma} dx \geq \left(\int_a^b (x-a)^\alpha f^\gamma(x)\, dx \right)^\beta \tag{2.17}$$

hold for α, β and γ?

Remark 2.17 *(Problem 16). Under what conditions does the inequality*

$$\frac{\int_a^b f^{\alpha+\gamma}(x)\, dx}{\int_a^b f^{\alpha+\delta}(x)\, dx} \geq \frac{\left(\int_a^b (x-a)^\alpha f^\gamma(x)\, dx \right)^\beta}{\left(\int_a^b (x-a)^\alpha f^\delta(x)\, dx \right)^\lambda} \tag{2.18}$$

hold for α,β,γ,δ and λ?

Remark 2.18 *(Problem 17). Assume that $\varphi(x)$ is a convex function with $\varphi(0) = 0$. Under what conditions does the inequality*

$$\frac{\int_a^b f(x)\, dx}{\int_a^b h(x)\, dx} \geq \frac{\left(\int_a^b \varphi(f(x))\, g(x)\, dx \right)^\beta}{\left(\int_a^b \varphi(h(x))\, g(x)\, dx \right)^\lambda} \tag{2.19}$$

hold for β and λ?

2.2 WORK OF F. QI

In this section, we present the work of F. Qi [16] in detail which gives affirmative answer to Problem 1 firstly for the case $t = 3$ and then for $t = n \in \mathbb{N}$. He used analytic method and mathematical induction to prove integral inequalities.

Theorem 2.3 ([16], p.1) *Suppose that $f(x)$ is a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = 0$.*

(a) *If $0 \leq f'(x) \leq 1$ for $x \in (a, b)$ then*

$$\int_a^b [f(x)]^3 dx \leq \left(\int_a^b f(x) dx \right)^2. \tag{2.20}$$

(b) *If $f'(x) \geq 1$, then inequality (2.20) reverses.*

(c) *The equality in (2.20) is valid only if $f(x) \equiv 0$ or $f(x) = x - a$.*

Proof. For $a \le t \le b$, set

$$F(t) = \left(\int_a^t f(x)dx\right)^2 - \int_a^t [f(x)]^3 dx.$$

Straightforward calculation produces

$$\begin{aligned} F'(t) &= \left\{2\int_a^t f(x)dx - [f(t)]^2\right\} f(t) \triangleq G(t)f(t), \\ G'(t) &= 2\left[1 - f'(t)\right] f(t). \end{aligned}$$

Since $f'(t) \ge 0$ and $f(a) = 0$, hence $f(t)$ is increasing and $f(t) \ge 0$.

(a) When $0 \le f'(t) \le 1$, we get $G'(t) \ge 0$, $G(t)$ increases and $G(t) \ge 0$ because of $G(a) = 0$, thus $F'(t) = G(t)f(t) \ge 0$, $F(t)$ is increasing. Since $F(a) = 0$, we get $F(t) \ge 0$, and $F(b) \ge 0$. Thus the inequality (2.20) is valid.

(b) When $f'(t) \ge 1$, we get $G'(t) \le 0$, $G(t)$ decreases, $G(t) \le 0$, $F'(t) \le 0$, and $F(t)$ is decreasing, then $F(t) \le 0$, the inequality (2.20) reverses.

(c) Since the equality in (2.20) is valid only if $f'(t) = 1$ or $f(t) = 0$, substitution of $f(t) = t + c$ into (2.20) and standard argument leads to $c = -a$.

The proof now is completed. ∎

Corollary 2.1 ([22], p.624) *Suppose that $f(x)$ is a continuous function on the closed interval $[0,1]$ and $f(0) = 0$, its derivative of the first order is bounded by $0 \le f'(x) \le 1$ for $x \in (0,1)$. Then*

$$\int_0^1 [f(x)]^3 dx \le \left(\int_0^1 f(x)dx\right)^2. \tag{2.21}$$

Equality in (2.21) is valid if and only if $f(x) = 0$ or $f(x) = x$.

Theorem 2.4 ([16], p.2) *For $n \in \mathbb{N}$, assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ (i.e. $f \in C^n[a,b]$) such that $f^{(i)}(a) \ge 0$ for $0 \le i \le n-1$ and $f^{(n)}(x) \ge n!$ then*

$$\int_a^b [f(x)]^{n+2} dx \ge \left(\int_a^b f(x)dx\right)^{n+1}. \tag{2.22}$$

Proof . Suppose

$$H(t)=\int_a^t [f(x)]^{n+2}dx-\left[\int_a^t f(x)dx\right]^{n+1}, \quad t\in[a,b]. \tag{2.23}$$

Straightforward calculation produces

$$H'(t)=\left\{[f(x)]^{n+1}-(n+1)\left[\int_a^t f(x)dx\right]^{n}\right\}f(t)\triangleq h_1(t)f(t),$$
$$h_1'(t)=(n+1)\left\{[f(x)]^{n-1}f'(t)-n\left[\int_a^t f(x)dx\right]^{n-1}\right\}f(t)\triangleq(n+1)h_2(t)f(t),$$
$$h_2'(t)=\left\{[f(x)]^{n-2}f''(t)+(n-1)[f(t)]^{n-3}[f'(t)]^2-n(n-1)\left[\int_a^t f(x)dx\right]^{n-2}\right\}f(t)$$
$$\triangleq h_3(t)f(t).$$

By induction, we have

$$h_i'(t)=\left\{f^{(i)}(t)[f(t)]^{n-i}+p_i(t)-\frac{n!}{(n-i)!}\left[\int_a^t f(x)dx\right]^{n-i}\right\}f(t)\triangleq h_{i+1}(t)f(t), \tag{2.24}$$

where $2\le i\le n$ and

$$\begin{aligned} p_2(t) &= (n-1)[f(t)]^{n-3}[f'(t)]^2, \\ p_{i+1}(t)f(t) &= p_i'(t)+(n-i)f^{(i)}(t)[f(t)]^{n-i-1}f'(t). \end{aligned} \tag{2.25}$$

From $f^{(n)}(t)\ge n!$ and $f^{(i)}(a)\ge 0$ for $0\le i\le n-1$, it follows that $f^{(i)}(t)\ge 0$ and are increasing for $0\le i\le n-1$.

Using mathematical induction, it can be easily seen that

$$p_i(t)=\sum_{j_0+\sum_{k=1}^{i-1}k.j_k=n-1} C(j_0,j_1,...,j_{i-1})\prod_{k=0}^{i-1}[f^{(k)}(t)]^{jk},$$

where j_k and $C(j_0,j_1,...,j_{i-1)}$ are non-negative integers, $0\le k\le i-1$.

Thus, we have $p_k'(t)\ge 0$ and $p_{k+1}(t)\ge 0$, then $p_{k-1}'(t)$ and $p_k(t)$ are increasing for $2\le k\le n$. Direct calculation produces

$$h_{n+1}(t)=f^{(n)}(t)+p_n(t)-n!.$$

Considering $f^{(n)}(t)\ge n!$, we obtain $h_{n+1}(t)\ge 0$ and $h_n'(t)\ge 0$, then $h_n(t)$ increases.

By the definitions of $h_i(t)$, we get, for $1\le i\le n-1$,

$$h_{i+1}(a)=f^{(i)}(a)[f(a)]^{n-i}+p_i(a)\ge 0.$$

Hence, using induction on i, we have $h_i'(t) \geq 0$, $h_i(t) \geq 0$, and $h_i(t)$ are increasing for $1 \leq i \leq n$. So $H'(t) \geq 0$ and increases, and $H(t) \geq 0$. The inequality (2.2) follows from $H(b) \geq 0$. Therefore, Theorem is proved. ■

Corollary 2.2 ([16], p.3) *Suppose that $f(x)$ is n-times differentiable on $[a, b]$, $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$ for $0 \leq i \leq n-1$. Then the functions $H(t)$, $h_j(t)$ and $p_k(t)$ defined by the formulae (2.23), (2.24) and (2.25) are increasing and convex, where $1 \leq j \leq n-1$ and $2 \leq k \leq n-2$.*

CHAPTER 3

PAPERS FROM 2001 THROUGH 2003

In this chapter, we present some papers on Qi type inequalities from 2001 through 2003. Papers answering the Problem 1 (Remark 2.1):

- In 2001 (and 2007), work of F. Qi and K. W. Yu (see Section 3.1).
- In 2001, work of N. Towghi (see Section 3.2).
- In 2002, work of T. K. Pogány (see Section 3.3).
- In 2003, work of M. Akkouchi (see Section 3.4).

Paper answering the Problem 5 (Remark 2.6):

- In 2003, work of S. Mazouzi and F. Qi (see Section 3.5).

Papers answering the Problem 9 (Remark 2.10):

- In 2002, work of T. K. Pogány (see Section 3.3).
- In 2003, work of S. Mazouzi and F. Qi (see Section 3.5).

Paper answering the Problem 10 (Remark 2.11):

- In 2002, work of T. K. Pogány (see Section 3.3).

3.1 WORK OF F. QI AND K. W. YU ANSWERING THE PROBLEM 1

In their paper [23] (preprint version) and [24], F. Qi and K. W. Yu gave the following affirmative answer to the Problem 1 (Remark 2.1). In order to prove Theorem 3.1, two lemmas are necessary, namely Lemma 1.1 and Lemma 1.3 .

Theorem 3.1 ([24], p.97) *Assume that f is a continuous function on $[a,b]$. If $\int_a^b f(x)\,dx \geq (b-a)^{t-1}$ for some $t>1$, then*

$$\int_a^b [f(x)]^t\,dx \geq \left[\int_a^b f(x)\,dx\right]^{t-1}. \tag{3.1}$$

Proof. Consider the function $\varphi(x) = x^t$ in (a,b) for $t>1$. It is easy to verify that $\varphi'(u) \leq \varphi'(v)$ is valid for $a<u<v<b$. Then, by Lemma 1.3, the function φ is convex in (a,b). Hence, utilization of Jensen's inequality (1.1) in Lemma 1.1 gives

$$\frac{\int_a^b \varphi(f(x))\,dx}{\int_a^b dx} \geq \varphi\left(\frac{\int_a^b f(x)\,dx}{\int_a^b dx}\right),$$

and then

$$\int_a^b f^t(x)\,dx \geq \frac{\left[\int_a^b f(x)\,dx\right]^t}{(b-a)^{t-1}}. \tag{3.2}$$

Since $\int_a^b f(x)\,dx \geq (b-a)^{t-1}$ for some $t>1$, it follows that

$$\frac{\int_a^b f(x)\,dx}{(b-a)^{t-1}} \geq 1 \tag{3.3}$$

for such $t>1$. Therefore, the desired result follows from combination of (3.2) and (3.3).

■

Corollary 3.1 ([24], p.97) *Assume that f is a continuous function on $[a,b]$ with $b-a \leq 1$. If $\int_a^b f(x)\,dx \geq 1$, then inequality (3.1) is valid for all $t>1$.*

Corollary 3.2 ([24], p.97) *Assume that f is a continuous function on $[a,b]$ with $b-a \leq 1$. If $f(x) \geq 1/(b-a)$ for all $x \in [a,b]$, then inequality (3.1) is valid for all $t>1$.*

3.2 WORK OF N. TOWGHI ANSWERING THE PROBLEM 1

In the paper [25], N. Towghi gave sufficient conditions for the Problem 1 (Remark 2.1) and gave sufficient conditions for inequality (3.4) to hold. His proof is somewhat more simplified than the proof of (2.22) given in Theorem 2.4 and he use slightly weaker assumptions.

Let first set the stage. Assume $f^{(0)}(x) = f(x)$, $f^{(-1)}(x) = \int_a^x f(s)ds$, and $[x]$ denote the greatest integer less than or equal to x. For $t \in (n, n+1]$, where n is a positive integer, suppose $\gamma(t) = t(t-1)(t-2)\cdots[t-(n-1)]$. For $t<1$, let $\gamma(t)=1$.

Theorem 3.2 ([25], p.1) *Suppose $t > 1$, $x \in [a, b]$, and $f^{(i)}(a) \geq 0$ for $i \leq [t-2]$. If $f^{[t-2]}(x) \geq \gamma(t-1)(x-a)^{(t-[t])}$, then $(b-a)^{t-1} \leq \int_a^b f(x)\,dx$, and inequality (2.3) is valid.*

Proof. If $1 < t < 2$ then $f^{[t-2]}(b) \leq \int_a^b f(x)\,dx$, $\gamma(t-1) = 1$, and $(t-[t]) = t-1$. Thus $(b-a)^{t-1} \leq \int_a^b f(x)\,dx$. Assume that $t \in [n, n+1)$, where n is positive integer, and $n \geq 2$. It now follows that

$$\begin{aligned}
\int_a^b f(x)\,dx &\geq \int_a^b \int_a^{x_1} \int_0^{x_2} \cdots \int_a^{x_{n-2}} f^{(n-2)}(x_{n-1})dx_{n-1}dx_{n-1}\cdots dx_1 \qquad (3.4)\\
&\geq \int_a^b \int_a^{x_1} \int_a^{x_2} \cdots \int_a^{x_{n-2}} \gamma(t-1)(x_{n-1}-a)^{(t-n)}dx_{n-1}dx_{n-1}\cdots dx_1\\
&= (b-a)^{(t-1)}.
\end{aligned}$$

To show that inequality (2.3) is valid we define the following quantities,

$$F(t) = \int_a^b [f(x)]^t\,dx,\ G(t) = \left[\int_a^b f(x)dx\right]^t,\ G_I(t) = \left[\frac{G(1)}{b-a}\right]^t.$$

It must be shown that $G(t-1) \leq F(t)$. Note that $G(t-1) = G_I(t)\frac{(b-a)^t}{G(1)}$. By Jensen's inequality $G_I(t) \leq F(t)/(b-a)$. As a result, we have

$$\begin{aligned}
G(t-1) &\leq F(t)\frac{(b-a)^{t-1}}{G(1)}\\
(\text{Inequality } (3.4) &\Rightarrow\ ) \leq F(t).\ \blacksquare
\end{aligned}$$

3.3 WORK OF T. K. POGÁNY ANSWERING THE PROBLEMS 1, 9 AND 10

In [26], T. K. Pogány considered a finite integration domain, i.e. in $\int_a^b f$ we take $-\infty < a < b < \infty$ (possible extensions to an infinite integration domain are left to the interested reader). Denote $\int_a^b f d\mu$ as the weighted (Stieltjes type) integral where μ is nonnegative, monotonic non decreasing, left continuous, finite total variation, Stieltjes measure. In other words μ can be taken to be a probability distribution function.

T. K. Pogány gave an inequality which contains, as a special case, an answer to F. Qi's problem (2.3), see Corollary 3.3.

The following theorem gives an affirmative answer to Problem 9 (Remark 2.10).

Theorem 3.3 ([26], p.3) *Assume that* $\beta > 0$, $\max\{\beta, 1\} < \alpha$ *and suppose that* f^α *is integrable on* $[a, b]$. *For* $f(x) \geq (b-a)^{\frac{\beta-1}{\alpha-\beta}}$, *we get*

$$\int_a^b f^\alpha \geq \left(\int_a^b f\right)^\beta. \tag{3.5}$$

Proof. As $f(x) > 0$ on the integration domain, by the Hölder inequality we have

$$\int_a^b f \leq (b-a)^{1-\frac{1}{\alpha}} \left(\int_a^b f^\alpha\right)^{\frac{1}{\alpha}},$$

i.e.

$$\int_a^b f^\alpha \geq (b-a)^{1-\alpha} \left(\int_a^b f\right)^{\alpha-\beta} \left(\int_a^b f\right)^\beta \geq (b-a)^{1-\beta} (\min f)^{\alpha-\beta} \left(\int_a^b f\right)^\beta.$$

Now, by $f(x) \geq (b-a)^{\frac{\beta-1}{\alpha-\beta}}$ we deduce that $K_1 \geq 1$. Thus, we prove the inequality (3.5). ■

Corollary 3.3 ([26], p.3) *For all* $f(x) \geq (b-a)^{t-2}$, f^t *integrable, the inequality (2.3) is valid for* $t > 1$.

Proof. Put $\alpha = t = \beta + 1$ into (3.5). ■

The following theorem gives an affirmative answer to Problem 10 (Remark 2.11).

Theorem 3.4 ([26], p.3) *Assume that* f *is nonnegative, concave and integrable on* $[a, b]$, $\beta > 0$ *and max*$\{\beta, 1\} < \alpha$. *Assume*

$$f(x) \leq \left(\frac{(1+\alpha)(2\alpha-1)^{\alpha-1}}{6^\alpha(\alpha-1)^{\alpha-1}(b-a)^{1-\beta}}\right)^{\frac{1}{\alpha-\beta}}, \qquad x \in [a, b]. \tag{3.6}$$

Then we obtain

$$\int_a^b f^\alpha \leq \left(\int_a^b f\right)^\beta. \tag{3.7}$$

Proof. Set $g \equiv 1$, $p \equiv \alpha$ into the Nehari inequality (1.2). Then we obviously obtain

$$\begin{aligned}\int_a^b f^\alpha &\leq \frac{N^\alpha(\alpha, \frac{\alpha}{\alpha-1})}{(b-a)^{\alpha-1}} \left(\int_a^b f\right)^{\alpha-\beta} \left(\int_a^b f\right)^\beta \\ &\leq \underbrace{N^\alpha\left(\alpha, \frac{\alpha}{\alpha-1}\right)(b-a)^{1-\beta}(\max f)^{\alpha-\beta}}_{K_2} \left(\int_a^b f\right)^\beta\end{aligned}$$

Here $N(\cdot, \cdot)$ is the Nehari constant given by (1.3). Now, by (3.6) we deduce that $K_2 \leq 1$. Hence, we prove The inequality (3.7). ■

In the next theorem q is a scaling parameter. Therefore, we get an inequality scaled by three parameters.

Theorem 3.5 ([26], p.4) *Suppose that f is nonnegative concave and integrable on $[a,b]$. If*

$$f(x) \leq \left(\frac{(1+\alpha)(1+q)^{\frac{\alpha}{q}}}{6^{\alpha}(b-a)^{1-\beta}} \right)^{\frac{1}{\alpha-\beta}}, \qquad x \in [a,b], \tag{3.8}$$

then (3.7) is valid for all $\alpha > 0$ and $\beta > 0$ such that $\max\{\beta, 1\} < \alpha$ and $q > 1$.

Proof. Substituting $g \equiv 1$, $p \equiv \alpha$ in the Barnes-Godunova-Levin inequality (1.4) and repeating the procedure of previous theorem, we obtain as follows

$$\int_a^b f^{\alpha} \leq B^{\alpha}(\alpha, q)(b-a)^{\alpha-\beta-\frac{\alpha}{q}} (\max f)^{\alpha-\beta} \left(\int_a^b f \right)^{\beta}.$$

Here, $B(\cdot,\cdot)$ is the Barnes-Godunova-Levin constant described by (1.5). Using the assumed upper bound (3.8), we get that $K_3 \leq 1$ and this complete the proof. ■

The following theorem gives an affirmative answer to Problem 9 (Remark 2.10).

Theorem 3.6 ([26], p.4) *Suppose $\beta > 0$, $\max\{\beta, 1\} < \alpha$ and assume that f^{α} is μ-integrable on $[a,b]$. If*

$$f(x) \geq (\mu(b) - \mu(a))^{\frac{\beta-1}{\alpha-\beta}},$$

then

$$\int_a^b f^{\alpha} d\mu \geq \left(\int_a^b f d\mu \right)^{\beta}. \tag{3.9}$$

Proof. The proof is a identical to the proof of Theorem 3.3, hence it is omitted. ■

The following theorem gives an affirmative answer to Problem 10 (Remark 2.11).

Theorem 3.7 ([26], p.4) *Suppose that f is nonnegative, concave non decreasing (non increasing) and μ-integrable on $[a,b]$. If $\beta > 0$, $0 < max\{\beta, 1\} < \alpha$, $q \geq 1$ and*

$$f(x) \leq \frac{(\mu(b) - \mu(a))^{\frac{\alpha}{q(\alpha-\beta)}-1}}{(P(\alpha, q; \mu))^{\frac{\alpha}{(\alpha-\beta)}}}, \qquad x \in [a,b], \tag{3.10}$$

where $P(\alpha, q; \mu)$ is given by (1.7), the following inequality is valid

$$\int_a^b f^{\alpha} d\mu \leq \left(\int_a^b f d\mu \right)^{\beta}. \tag{3.11}$$

Proof. Substituting $p \equiv \alpha$, $g \equiv 1$ into the inequality (1.6) we have

$$\begin{aligned}\int_a^b f^\alpha d\mu &\leq \frac{P^\alpha(\alpha, q; \mu)}{(\mu(b)-\mu(a))^{\frac{\alpha}{q}}} \left(\int_a^b f d\mu\right)^{\alpha-\beta} \left(\int_a^b f d\mu\right)^{\beta} \\ &\leq \frac{P^\alpha(\alpha, q; \mu)(\max f)^{\alpha-\beta}}{(\mu(b)-\mu(a))^{\frac{\alpha}{q}+\beta-\alpha}} \left(\int_a^b f d\mu\right)^{\beta}.\end{aligned}$$

Now, taking into account the upper bound (3.10) of f on $[a, b]$ and the exact value of $P(\cdot, \cdot; \mu)$, we obviously deduce that $K_4 \leq 1$, i.e. the inequality (3.11) is proved. ■

3.4 WORK OF M. AKKOUCHI ANSWERING THE PROBLEM 1

In [27], M. Akkouchi gave some integral inequalities by using analytic and elementary methods. These inequalities get some relationships with certain integral inequalities obtained by Feng Qi in [16], see (2.22) in Theorem 2.4.
His results provided some similar inequalities for certain classes of functions f satisfying weaker conditions than required in the Theorem 2.4. We will present M. Akkouchi's result. Before stating the results,the following Definition 3.1 is needed.

Definition 3.1 ([27], p.122) *Suppose that $[a, b]$ is a finite interval of the real line $\mathbb{R}$. For each real number r, we denote $\mathbb{E}_r(a, b)$ the set of real continuous functions f on $[a, b]$ differentiable on (a, b), such that $f(a) \geq 0$, and $f'(x) \geq r$ for all $x \in (a, b)$.*

Proposition 3.8 ([27], p.122) *Assume $f \in \mathbb{E}_2(a, b)$. Then for each real number $p \in (0, \infty)$, we get*

$$\int_a^b [f(x)]^{2p+1} dx > \left[\int_a^b [f(x)]^p dx\right]^2. \tag{3.12}$$

Proof. We start by proving Proposition 3.8. Suppose that $p > 0$. For every $t \in [a, b]$, we put

$$F(t) = \int_a^t [f(x)]^{2p+1} dx - \left[\int_a^t [f(x)]^p dx\right]^2.$$

Straightforward calculation produces for all $t \in (a, b)$

$$\begin{aligned}F'(t) &= \left[[f(t)]^{p+1} - 2\int_a^t [f(x)]^p dx\right][f(t)]^p := G(t)[f(t)]^p. \\ G'(t) &= [(p+1)f'(t) - 2][f(t)]^p.\end{aligned}$$

Since $p > 0$ and $f \in \mathbb{E}_2(a,b)$ then $f'(t) \geq 2 > \frac{2}{p+1}$, hence f is strictly increasing on $[a,b]$. Then $f(t) > f(a) \geq 0$ for every $t \in (a,b]$. As a result, G is strictly increasing on $[a,b]$. Since $G(a) = [f(a)]^{p+1} \geq 0$, we conclude that $G(t) > 0$ for all $t \in (a,b]$. Therefore, $F'(t) > 0$ for all $t \in (a,b)$. We conclude that F is strictly increasing on $[a,b]$. In particular, we get $F(b) > F(a) = 0$. Thus, the inequality (3.12) is valid. ■

Theorem 3.9 ([27], p.122) *Suppose $\alpha > 0$, and assume $f \in \mathbb{E}_2(a,b)$. Then for each integer $n \geq 1$, we get the following strict inequality:*

$$\int_a^b [f(x)]^{(\alpha+1)2^n-1}\,dx > \left[\int_a^b [f(x)]^{\alpha}\,dx\right]^{2^n}. \tag{3.13}$$

Proof. The proof uses mathematical induction and the above auxiliary proposition, Proposition 3.8. Suppose $\alpha \in (0,\infty)$. For every integer $n \geq 1$, we put $p_n(\alpha) = (\alpha+1)2^n - 1$. So we get $p_n(\alpha) > 0$ and $p_{n+1}(\alpha) = 2p_n(\alpha)+1$, for each integer $n \geq 1$. We point out that $p_1(\alpha) = 2\alpha+1$. Thus a straightforward application of Proposition 3.8 shows that the inequality (3.13) is valid for $n = 1$. Assume that the inequality (3.13) is valid for the integer n and let us prove it for $n+1$. Since $p_{n+1}(\alpha) = 2p_n(\alpha)+1$ and $p_n(\alpha) > 0$ then Proposition 3.8 may be applied and the following strict inequality is obtained:

$$\int_a^b [f(x)]^{(\alpha+1)2^{n+1}-1}\,dx > \left[\int_a^b [f(x)]^{(\alpha+1)2^n-1}dx\right]^2. \tag{3.14}$$

By assumption we get

$$\int_a^b [f(x)]^{(\alpha+1)2^n-1}dx > \left[\int_a^b [f(x)]^{\alpha}dx\right]^{2^n}. \tag{3.15}$$

From (3.14) and (3.15) we see that the inequality (3.13) is valid for $n+1$. Hence M. Akkouchi's result is completely proved. ■

As a consequence of the above theorem, we have the following.

Corollary 3.4 ([27], p.122) *Suppose $f \in \mathbb{E}_2(a,b)$. Then for each integer $n \geq 0$, we get*

$$\left[\int_a^b [f(x)]^{(\alpha+1)2^{n+1}-1}\,dx\right]^{\frac{1}{2^{n+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^n-1}\,dx\right]^{\frac{1}{2^n}}.$$

As a consequence of this result, we have the following.

Corollary 3.5 ([27], p.122) *Suppose $f \in \mathbb{E}_2(a,b)$. Then for each integer $n \geq 2$, we get*

$$\int_a^b [f(x)]^{2^{n+1}-1}\,dx > \left[\int_a^b [f(x)]^3\,dx\right]^{2^{n-1}} > \left[\int_a^b f(x)dx\right]^{2^n}.$$

Remark 3.1 ([27], p.124) *From (3.14) we have all the statements contained in the Corollary 3.4 and Corollary 3.5.*

We end this section by giving a related integral inequality. More precisely, we get the following theorem which gives an affirmative answer to related with Problem 1 (Remark 2.1).

Theorem 3.10 ([27], p.124) *Suppose that* $[a, b]$ *is a closed interval of* $\mathbb{R}$. *Suppose that* $p \geq 1$ *is a real number and assume* $f \in \mathbb{E}_p(a, b)$. *Then we get*

$$\int_a^b [f(x)]^{p+2}dx \geq \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(x)dx\right]^{p+1}. \tag{3.16}$$

Proof. For every $t \in [a, b]$, we put

$$H(t) = \int_a^t [f(x)]^{p+2}dx - \frac{1}{(b-a)^{p-1}} \left[\int_a^t f(x)dx\right]^{p+1}.$$

Straightforward calculation yields for all $t \in (a, b)$

$$\begin{aligned} H'(t) &= \left([f(t)]^{p+1} - \frac{p+1}{(b-a)^{p-1}} \left[\int_a^t f(x)dx\right]^{p}\right) f(x) := h(t)f(t), \\ h'(t) &= (p+1)\left([f(t)]^{p-1}f'(t) - \frac{p}{(b-a)^{p-1}} \left[\int_a^t f(x)dx\right]^{p-1}\right) f(t). \end{aligned}$$

Since f is increasing then $0 \leq \int_a^t f(x)dx \leq (b-a)f(t)$ for all $t \in [a, b]$. Thus, we obtain

$$h'(t) \geq (p+1)(f'(t) - p)[f(t)]^p. \tag{3.17}$$

We conclude from (3.17) that h is increasing on $[a, b]$. Since $h(a) = [f(a)]^{p+1} \geq 0$, this indicates that H is increasing on $[a, b]$. In particular, we get $H(b) \geq H(a) = 0$, which gives the desired inequality. ∎

As a consequence, by replacing $f(x)$ by $pf(x)$ in (3.16), we obtain the following.

Corollary 3.6 ([27], p.124) *Suppose that* $p \geq 1$ *is a real number and assume* $f \in \mathbb{E}_1(0, 1)$. *Then*

$$\int_0^1 [f(x)]^{p+2}dx \geq \frac{1}{p} \left[\int_0^1 f(x)dx\right]^{p+1}.$$

3.5 WORK OF S. MAZOUZI AND F. QI ANSWERING THE PROBLEMS 5 AND 9

In the joint paper [2], by employing a functional inequality introduced in [28], which is an abstract generalization of the classical Jensen's inequality [29], see Lemma 1.2, S. Mazouzi and F. Qi further establish the following functional inequality (3.19) from which inequality (2.3), some integral inequality, and an interesting discrete inequality involving sums can be deduced.

Theorem 3.11 ([2], p.2) *Assume that $\mathcal{V}$ is a linear vector space of real-valued functions, p and q are two real numbers such that $p \geq q \geq 1$. Suppose that f and g are two positive functions in $\mathcal{V}$ and G is a positive linear form on $\mathcal{V}$ such that*

(a) $G(g) > 0$,

(b) fg *and* $gf^p \in \mathcal{V}$.

If

$$[G(g)]^{p-1} \leq [G(gf)]^{p-q}, \tag{3.18}$$

then

$$[G(gf)]^q \leq G(gf^p). \tag{3.19}$$

The new inequality (3.19) has the properties that it is stated for summable functions defined on a finite measure space (X, Σ, μ) whose L^1-norms are bounded from below by some constant involving the measure of the whole space X as well as the exponents p and q.

Proof. If we define a positive linear form $F(u) = \frac{G(u)}{G(g)}$, then we clearly get $F(g) = 1$. From Lemma 1.2, if we take as a convex function $\varphi(x) = x^p$ for $p \geq 1$, then we have

$$[F(gf)]^p \leq F(gf^p),$$

that is,

$$\left[\frac{G(gf)}{G(g)}\right]^p \leq \frac{G(gf^p)}{G(g)}$$

which gives

$$\frac{[G(gf)]^{p-q}}{[G(g)]^{p-1}}\,[G(gf)]^{q} \le G\left(gf^{p}\right).$$

Since inequality (3.18) holds, hence inequality (3.19) follows. ■

As a new positive and concrete answer to F. Qi's problem indicated at the Remark 2.1, we get the following.

Corollary 3.7 ([2], p.3) *Suppose that (X, Σ, μ) is a finite measure space and $\mathcal{V}$ is the space of all integrable functions on X. If p and q are two real numbers such that $p \ge q \ge 1$, and f and g are two positive functions of $\mathcal{V}$ such that*

(a) $\int_X g d\mu > 0$,

(b) *fg and $gf^p \in \mathcal{V}$,*

then

$$\left(\int_X gf d\mu\right)^q \le \int_X gf^p d\mu,$$

provided that $(\int_X gf d\mu)^{p-q} \ge (\int_X g d\mu)^{p-1}$.

Proof. This follows from Theorem 3.11 by taking $G(u) = \int_X u d\mu$ as a positive linear form. ■

Remark 3.2 ([2], p.3) *We notice that if $p = q$ and $G(g) \le 1$, then inequality (3.18) is always fulfilled, and accordingly, we get*

$$[G\left(gf\right)]^p \le G\left(gf^p\right)$$

for all $p \ge 1$.

Remark 3.3 ([2], p.3) *If $\mathcal{V}$ contains the constant functions, then for the function*

$$f = \begin{cases} 0, & p \ge q \ge 1, \\ [G(g)]^{p-q}, & p > q \ge 1, \\ 1, & p = q, G(g) = 1, \end{cases}$$

equality occurs in (3.19).

Remark 3.4 ([2], p.3) *In fact, inequality (3.19) is valid even if inequality (3.18), as only a sufficient condition, is not satisfied. Suppose* $p > q \geq 1$, $m = \frac{q-1}{p-q}$ *and* $c = \left[q\left(\frac{p-q}{p-1}\right)^{q-1}\right]^{\frac{1}{(p-q)}}$. *If* $X = [a,b]$ *is a finite interval of* $\mathbb{R}$ *and* $f(x) = c(x-a)^m$, *then* $\left(\int_a^b f dx\right)^q = \int_a^b f^p dx$. *On the other hand, inequality (3.18) is no longer satisfied if we take* $q\left(\frac{p-q}{p-1}\right)^{p-1} < 1$. *This is because of the fact that* $\left(\int_a^b f dx\right)^{p-q} = q\left(\frac{p-q}{p-1}\right)^{p-1}(b-a)^{p-1}$.

The following corollary gives an affirmative answer to related with Problem 9 (Remark 2.10).

Corollary 3.8 ([2], p.4) *Suppose* $f \in \mathbb{L}^1(a,b)$, *the space of integrable functions on the interval* (a,b) *with respect to the Lebesgue measure, such that* $|f(x)| \geq k(x)$ *a.e. for* $x \in (a,b)$, *where*

$$(b-a)^{\frac{(p-1)}{(p-q)}} \leq \int_a^b k(x)dx < \infty$$

for some $p > q \geq 1$, *thus*

$$\left(\int_a^b |f(x)|\, dx\right)^q \leq \int_a^b |f(x)|^p \, dx.$$

Proof. This follows simply from Lemma 1.2. ■

We now apply Corollary 3.8 to deduce F. Qi's main result, Theorem 2.4, in detail.

The following corollary gives an affirmative answer to Problem 5 (Remark 2.6).

Corollary 3.9 ([2], p.4) *Assume that* $f \in C^n([a,b])$ *satisfies* $f^{(i)}(a) \geq 0$ *and* $f^{(n)}(x) \geq n!$ *for* $x \in [a,b]$, *where* $0 \leq i \leq n-1$ *and* $n \in \mathbb{N}$, *the set of all positive integers, then*

$$\int_a^b [f(x)]^{n+2}\, dx \geq \left(\int_a^b f(x)\, dx\right)^{n+1}. \tag{3.20}$$

Proof. There is an imperfection in this proof and you can see Remark 3.5 for this mistake. Since $f^{(n)}(x) \geq n!$, then successive integrations over $[a,x]$ give

$$f^{(n-k)}(x) \geq \frac{n!}{k!}(x-a)^k, \quad k = 0, 1, \cdots, n-1,$$

thus

$$(x-a)^{n-k} f^{(n-k)}(x) \geq \frac{n!}{k!}(x-a)^n, \quad k = 0, 1, \cdots, n-1.$$

Moreover, Taylor's expansion applied to f with Lagrange remainder states that

$$\begin{aligned} f(x) &= f(a)+\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{f^{(n)}(\xi)}{n!}(x-a)^n \\ &\geq \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(x-a)^n \\ &= 2^n(x-a)^n, \end{aligned}$$

where $\xi\in(a,x)$. However since x is arbitrary and $2^n\geq n+1$ for all $n\in\mathbb{N}$, it follows that

$$f(x)\geq(n+1)(x-a)^n\geq 0$$

for all $x\in(a,b)$. Hence

$$\int_a^b f(x)dx\geq(b-a)^{n+1},$$

and inequality (3.20) follows in consequence of Corollary 3.8. ■

Remark 3.5 *In [15] J. Pečarić and T. Pejković stated that S. Mazouzi and F. Qi gave what appeared to be a basic proof of the inequality under the same conditions (see Corollary 3.9). Unfortunately their proof wasn't correct. Namely, they done a false substitution and arrived at the condition $f(x)\geq(n+1)(x-a)^n$ which is false, e.g. for function $f(x)=x-a$, whereas this function clearly satisfies the conditions of the theorem if $n=1$.*

Remark 3.6 ([2], p.4) *The function*

$$f:[a,b]\to\mathbb{R}^+,\quad x\to f(x)=\frac{(x-a)^{n+1}}{(n+2)^n}$$

for a fixed $n\in\mathbb{N}$ satisfies $f\in C^n([a,b])$ and $f^{(i)}(a)\geq 0$, for $0\leq i\leq n-1$; however, $f^{(n)}(x)=\frac{(n+1)!}{(n+2)^n}(x-a)$ for $x\in[a,b]$. It means that the condition $f^{(n)}\geq n!$ on $[a,b]$ is no longer satisfied. But we get

$$\left(\int_a^b fdx\right)^{n+2}=\int_a^b f^{n+3}dx=\frac{(b-a)^{(n+2)^2}}{(n+2)^{(n+1)(n+2)}}.$$

Finally, we can apply Corollary 3.7 to derive a discrete inequality.

Corollary 3.10 ([2], p.5) *Suppose $X=\{a_1,...,a_N\}$, $f:X\to\mathbb{R}^+$ defined by $f(a_i)=b_i$ for $i=1,...,N$, and let μ be a discrete positive measure given by $\mu(\{a_i\})=\alpha_i>0$ for $i=1,...,N$. If, for $p\geq q\geq 1$*

$$\left(\sum_{i=1}^N\alpha_i\right)^{p-1}\leq\left(\sum_{i=1}^N\alpha_i b_i\right)^{p-q},$$

then we obtain

$$\left(\sum_{i=1}^{N} \alpha_i b_i\right)^q \leq \sum_{i=1}^{N} \alpha_i b_i^p.$$

If, in particular, $\alpha_1 = \cdots = \alpha_N = c > 0$ *satisfies*

$$c^{q-1} \leq \frac{1}{N^{p-1}} \left(\sum_{i=1}^{N} b_i\right)^{p-q},$$

hence

$$\left(\sum_{i=1}^{N} b_i\right)^q \leq \frac{1}{c^{q-1}} \sum_{i=1}^{N} b_i^p. \tag{3.21}$$

Proof. We notice that

$$\begin{aligned}
\left(\int_X f d\mu\right)^{p-q} &= \left(\sum_{i=1}^{N} f(a_i)\mu(\{a_i\})\right)^{p-q} \\
&= \left(\sum_{i=1}^{N} \alpha_i b_i\right)^{p-q} \\
&\geq \left(\sum_{i=1}^{N} \alpha_i\right)^{p-1} \\
&\equiv [\mu(X)]^{p-1}.
\end{aligned}$$

and therefore, the sufficient condition is satisfied. We deduce by Corollary 3.7 that

$$\left(\int_X f d\mu\right)^q = \left(\sum_{i=1}^{N} \alpha_i b_i\right)^q \leq \int_X f^p d\mu = \sum_{i=1}^{N} \alpha_i b_i^p.$$

The proof of inequality (3.21) is a specific case of the above argument, so we are left to the interested reader. ■

CHAPTER 4

PAPERS FROM 2004 THROUGH 2005

In this chapter, we present some papers on Qi type inequalities from 2004 through 2005.

Papers answering the Problem 1 (Remark 2.1):

- In 2004, work of J. Pečarić and T. Pejković (see Section 4.2).
- In 2004, work of J. S. Sun (see Section 4.3).

Paper answering the Problem 3 (Remark 2.4):

- In 2004, work of J. S. Sun (see Section 4.3).

Paper answering the Problem 4 (Remark 2.5):

- In 2004, work of J. S. Sun (see Section 4.3).

Paper answering the Problem 5 (Remark 2.6):

- In 2004, work of J. Pečarić and T. Pejković (see Section 4.2).

Paper answering the Problem 7 (Remark 2.8):

- In 2005, work of L. Bougoffa (see Section 4.4).

Paper answering the Problem 12 (Remark 2.13):

- In 2004, work of J. Pečarić and T. Pejković (see Section 4.1).

4.1 WORK OF J. PEČARIĆ AND T. PEJKOVIĆ ANSWERING THE PROBLEM 12

In the joint paper [15], J. Pečarić and T. Pejković gave sufficient conditions for the following inequality:

$$\left(\int_a^b f(x)^\alpha \, dx\right)^\beta \geq \int_a^b f(x)^\gamma \, dx \tag{4.1}$$

to hold.

This inequality, Problem 12 (Remark 2.13), is a generalization of the inequalities that appear in the papers [14, 16, 23, 25, 26, 30].

In [16], F. Qi considered inequality (4.1) for $\alpha = n+2$, $\beta = 1/(n+1)$, $\gamma = 1$, $n \in \mathbb{N}$ and proved that under conditions

$$f \in C^n([a,b]); \quad f^{(i)}(a) \geq 0, \quad 0 \leq i \leq n-1; \quad f^{(n)}(x) \geq n!, \quad x \in [a,b]$$

the inequality holds.

To obtain some conditions for the inequality (4.1) we will first make progress similarly to T. K. Pogány ([26]) (see also Section 3.3) and in this section we will be using a method from the paper [30].

We want to transform inequality (4.1) to a form more convenient to use. It can readily be seen that if $f(x) \geq 0$, for all $x \in [a,b]$ and $\gamma > 0$, inequality (4.1) is equivalent to

$$\left[\left(\frac{\int_a^b f(x)^\alpha dx}{b-a}\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha\beta}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \left(\frac{\int_a^b f(x)^\gamma \, dx}{b-a}\right)^{\frac{1}{\gamma}}. \tag{4.2}$$

We can write the inequality (4.2) as

$$\left(M^{[\alpha]}(f)\right)^{\frac{\alpha\beta}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq M^{[\gamma]}(f) \tag{4.3}$$

according to the Definition (1.1).

J. Pečarić and T. Pejković state their results in a clear table. Each result is an independent set of conditions that guarantee the inequality (4.1) holds.

Result	Conditions on constants α,β,γ,a,b	Conditions on function f (holds for al l$x \in [a,b]$)	Lemma for the proof
1.	$\alpha \geq \gamma > 0$, $\alpha\beta > \gamma$	$f(x) \geq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}}$	Lemma 1.8
2.	$\alpha \leq \gamma > 0$, $\alpha\beta < 0$	$0 \leq f(x) \leq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}}$	Lemma 1.8
3(i).	$\alpha \geq \gamma > 0$, $\alpha\beta \geq \gamma$, $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$	$f(x) \geq 1$	Lemma 1.8
3(ii).	$\alpha \geq \gamma > 0$, $\alpha\beta \leq \gamma$, $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$	$0 \leq f(x) \leq 1$	Lemma 1.8
4(i).	$0 < \alpha \leq \gamma$, $\alpha\beta \geq \gamma$, $(b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}$	f concave, $f(x) \geq 1$	Lemma 1.9
4(ii).	$0 < \alpha \leq \gamma$, $\alpha\beta \leq \gamma$, $(b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}$	f concave, $0 \leq f(x) \leq 1$	Lemma 1.9
5.	$0 < \alpha \leq \gamma$, $\alpha\beta > \gamma$	f concave, $f(x) \geq$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\alpha\beta-\gamma}}$	Lemma 1.9
6.	$0 < \gamma \leq \alpha$, $\beta < 0$	f concave, $0 \leq f(x) \leq$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}\right)^{\frac{\alpha\beta}{\alpha\beta-\gamma}}$	Lemma 1.9
7.	$0 < \gamma \leq \alpha$, $\alpha\beta > \gamma$	f convex, $f(a) = 0, f(x) \geq$ $\frac{(b-a)^{\frac{\alpha-\gamma}{\alpha(\alpha\beta-\gamma)}}}{(b^{\alpha+1}-a^{\alpha+1})^{\frac{1}{\alpha}}} \cdot \frac{(\alpha+1)^{\frac{\beta}{\alpha\beta-\gamma}}}{(\gamma+1)^{\frac{1}{\alpha\beta-\gamma}}} x$	Lemma 1.10
8.	$0 < \alpha \leq \gamma$, $\beta < 0$	f convex, $f(a) = 0$, $0 \leq f(x) \leq$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}\right)^{\frac{\alpha\beta}{\alpha\beta-\gamma}}$	Lemma 1.10
9.	$0 < \gamma < \alpha$, $\beta < 0$	f concave, $0 \leq f(x) \leq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}}$ $\times \frac{6^{\frac{\alpha\beta}{\gamma(\gamma-\alpha\beta)}}}{\left(\frac{2\alpha-\gamma}{\alpha-\gamma}\right)^{\frac{\beta(\alpha-\gamma)}{\gamma(\gamma-\alpha\beta)}} \left(\frac{\alpha+\gamma}{\gamma}\right)^{\frac{\beta}{\gamma-\alpha\beta}}}$	Lemma 1.5

Remark 4.1 ([15], p.3) *Notice that in the results 4(i). and 5. it is sufficient for the condition on f to hold in endpoints of segment $[a,b]$ (i.e., for $f(a)$ and $f(b)$).*

Remark 4.2 ([15], p.3) *There is just one result in the table got with the help of Lemma 1.5 and none with the Lemma 1.6 because the constants in the conditions are quite complicated.*

Remark 4.3 ([15], p.4) *If we make the substitution* $\gamma \longmapsto 1$, $\beta \longmapsto \frac{1}{\beta}$ *in Result 1, Theorem 3.3, see [26], is acquired.*

We will prove merely a few results after which the method of proving the others will become clear.

Proof:

(a) Result 1. Lemma 1.8 implies that

$$M^{[\alpha]}(f) \geq M^{[\gamma]}(f). \tag{4.4}$$

Moreover

$$M^{[\alpha]}(f) \geq M^{[-\infty]}(f) \geq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}},$$

so by raising this inequality to a power, we have

$$\left(M^{[\alpha]}(f)\right)^{\frac{\alpha\beta-\gamma}{\gamma}} \geq (b-a)^{\frac{-\beta+1}{\gamma}}. \tag{4.5}$$

Multiplying (4.4) and (4.5) we have (4.3). ■

(b) Result 3 (i). $M^{[\alpha]}(f) \geq 1$ because $f(x) \geq 1$, so from $\frac{\alpha\beta}{\gamma} \geq 1$ and Lemma 1.8 it can be easily seen that

$$\left(M^{[\alpha]}(f)\right)^{\frac{\alpha\beta}{\gamma}} \geq M^{[\alpha]}(f) \geq M^{[\gamma]}(f). \tag{4.6}$$

By multiplication of (4.6) and the condition $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$ we have (4.3). ■

(c) Result 5. From

$$M^{[\alpha]}(f) \geq M^{[-\infty]}(f) \geq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\alpha\beta-\gamma}}$$

and $\frac{\alpha\beta-\gamma}{\gamma} > 0$ we get

$$\left(M^{[\alpha]}(f)\right)^{\frac{\alpha\beta-\gamma}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}}. \tag{4.7}$$

Because of Lemma 1.9:

$$M^{[\alpha]}(f) \frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}} \geq M^{[\gamma]}(f). \tag{4.8}$$

From (4.7) and (4.8), by multiplication, we arrive at (4.3). ■

(d) Result 7. Since

$$f(x) \geq \frac{(b-a)^{\frac{\alpha-\gamma}{\alpha(\alpha\beta-\gamma)}}}{(b^{\alpha+1}-a^{\alpha+1})^{\frac{1}{\alpha}}} \cdot \frac{(\alpha+1)^{\frac{\beta}{\alpha\beta-\gamma}}}{(\gamma+1)^{\frac{1}{\alpha\beta-\gamma}}} x$$

by integration it can be seen that we have

$$M^{[\alpha]}(f) \geq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{\frac{1}{\alpha}}}{(\gamma+1)^{\frac{1}{\gamma}}} \right)^{\frac{\gamma}{\alpha\beta-\gamma}}. \tag{4.9}$$

But, Lemma 1.10 implies inequality (4.8). Therefore, from (4.9) and (4.8) it is finally found that inequality (4.3) is valid. ■

In this subsection, inequality (4.1) will be proved under different assumptions including the differentiability of f and boundedness of its derivative.

J. Pečarić and W. Janous proved in [30] the following theorem which gives an affirmative answer to Problem 9 (Remark 2.10) and Problem 10 (Remark 2.11).

Theorem 4.1 ([15], p.5) *Suppose $1 < p \leq 2$ and $r \geq 3$. The differentiable function $f : [0, c] \to \mathbb{R}$ satisfies $f(0) = 0$ and $0 \leq f'(x) \leq M$ for all $0 \leq x \leq c$, c subject to*

$$0 < c \leq \left(\frac{p(p-1)\,2^{2-p} M^{p-r}}{r-1} \right)^{\frac{1}{r-2p+1}}. \tag{4.10}$$

Then we have

$$\left(\int_0^c f(x)dx \right)^p \geq \int_0^c f(x)^r \, dx.$$

(If $f'(x) \geq M$ the reverse inequality is valid under the condition that the second inequality in (4.10) is reversed.)

Remark 4.4 ([15], p.5) *The emphasized words were left out of [30].*

We will prove the following generalization of Theorem 4.1 which gives an affirmative answer to Problem 12 (Remark 2.13).

Theorem 4.2 ([15], p.5) *Suppose $\alpha > 0$, $1 < \beta \leq 2$ and $\gamma \geq 2\alpha + 1$. The differentiable function $f : [0, c] \to \mathbb{R}$ satisfies $f(0) = 0$ and $0 \leq f'(x) \leq M$ for all $0 \leq x \leq c$, c subject to*

$$0 < c \leq \left(\frac{\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma}}{\gamma-\alpha} \right)^{\frac{1}{\gamma-\alpha\beta-\beta+1}}. \tag{4.11}$$

Then we have

$$\left(\int_0^c f(x)^\alpha dx \right)^\beta \geq \int_0^c f(x)^r \, dx.$$

Remark 4.5 ([15], p.5) *For $\alpha = 1$, $\beta = p$, $\gamma = r$, we have Theorem 4.1.*

Proof. From $f(0) = 0$ and $0 \leq f'(x) \leq M$ we get $0 \leq f(x)^\alpha \leq M^\alpha x^\alpha$ and $0 \leq \int_0^x f(t)^\alpha \, dt \leq \frac{M^\alpha x^{\alpha+1}}{\alpha+1}$ for $0 \leq x \leq c$. We now define

$$F(x) := \left(\int_0^x f(x)^\alpha \, dt\right)^\beta - \int_0^x f(t)^\gamma \, dt.$$

Then $F(0) = 0$ and $F(x) = f(x)^\alpha g(x)$, where

$$g(x) := \beta \left(\int_0^x f(t)^\alpha \, dt\right)^{\beta-1} - f(x)^{\gamma-\alpha}.$$

Note that, $g(0) = 0$ and $g'(x) = f(x)^\alpha h(x)$, where

$$h(x) := \beta(\beta-1)\left(\int_0^x f(t)^\alpha \, dt\right)^{\beta-2} - (\gamma-\alpha) f(x)^{\gamma-2\alpha-1} f'(x).$$

From the conditions of the theorem we get

$$\begin{aligned} h(x) &\geq \beta(\beta-1)\left(\frac{M^\alpha x^{\alpha+1}}{\alpha+1}\right)^{\beta-2} - (\gamma-\alpha)(Mx)^{\gamma-2\alpha-1} M \\ &= M^{\gamma-2\alpha} x^{(\alpha+1)(\beta-2)} \left(\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma} - (\gamma-\alpha) x^{\gamma-\alpha\beta-\beta+1}\right) \end{aligned}$$

Therefore, since (4.11) is equivalent to

$$\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma} \geq (\gamma-\alpha) x^{\gamma-\alpha\beta-\beta+1}, \quad x \in [0, c],$$

we get $h(x) \geq 0$, $g'(x) \geq 0$, $g(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$. Thus $F(c) \geq 0$. ■

Substituting $c = a - b$ and translating function f a units to the right $(f(x) \longmapsto f(x-a))$ we have the following theorem.

Theorem 4.3 ([15], p.6) *Suppose $\alpha > 0$, $1 < \beta \leq 2$ and $\gamma \geq 2\alpha + 1$. The differentiable function $f : [a, b] \to \mathbb{R}$ satisfies $f(a) = 0$ and $0 \leq f'(x) \leq M$ for all $a \leq x \leq b$, where*

$$0 < b - a \leq \left(\frac{\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma}}{\gamma-\alpha}\right)^{\frac{1}{\gamma-\alpha\beta-\beta+1}}.$$

Then the inequality (4.1) holds.

4.2 WORK OF J. PEČARIĆ AND T. PEJKOVIĆ ANSWERING THE PROBLEMS 1 AND 5

In the joint paper [31], J. Pečaric and T. Pejković proved the following answer to Problem 1 (Remark 2.1) which will imply a generalization of Problem 5 (Remark 2.6).

Theorem 4.4 ([31], p.2) *Let $f \in C^1([a,b])$ satisfies $f(a) \geq 0$ and $f'(x) \geq (t-2)(x-a)^{t-3}$ for $x \in [a,b]$ and $t \geq 3$. Then*

$$\int_a^b [f(x)]^t \, dx \geq \left(\int_a^b f(x) \, dx \right)^{t-1} \tag{4.12}$$

holds. The equality is valid only if $a = b$ or $f(x) = x - a$ and $t = 3$.

Proof. Suppose that f is a function satisfying the conditions of Theorem 4.4. f is increasing because $f'(x) > 0$ for $x \in (a,b]$, so from $f(\xi) \leq f(x)$ for $\xi \in [a,x]$ we have

$$f(x)(x-a) \geq \int_a^x f(\xi) \, d\xi, \quad \text{for all } x \in [a,b]. \tag{4.13}$$

We now define

$$F(x) \triangleq \int_a^x [f(\xi)]^t \, d\xi - \left(\int_a^x f(\xi) \, d\xi \right)^{t-1}.$$

Then $F(a) = 0$ and $F'(x) = f(x)G(x)$, where

$$G(x) = [f(x)]^{t-1} - (t-1) \left(\int_a^x f(\xi) \, d\xi \right)^{t-2}.$$

Note that, $G(a) = [f(a)]^{t-1} \geq 0$ and

$$G'(x) = (t-1) f(x) \left([f(x)]^{t-3} f'(x) - (t-2) \left(\int_a^x f(\xi) \, d\xi \right)^{t-3} \right).$$

From the conditions of Theorem 4.4 and inequality (4.13) we obtain

$$[f(x)]^{t-3} f'(x) \geq (t-2)(f(x)(x-a))^{t-3} \geq (t-2) \left(\int_a^x f(\xi) \, d\xi \right)^{t-3}. \tag{4.14}$$

Therefore $G'(x) \geq 0$, so with $G(a) \geq 0$ we have $G(x) \geq 0$. From $F(a) = 0$ and $F'(x) = f(x)G(x) \geq 0$ we see that $F(x) \geq 0$ for all $x \in [a,b]$, particularly

$$F(b) = \int_a^b [f(\xi)]^t \, d\xi - \left(\int_a^b f(\xi) \, d\xi \right)^{t-1} \geq 0.$$

The equality in (4.12) holds only if $F'(x) = 0$ for all $x \in [a,b]$ which is equivalent to $f(a) = 0$ and $G'(x) = 0$ and according to (4.14), if $t > 3$, this is valid only for $f(a) = 0$, $f'(x) = (t-2)(x-a)^{t-3}$ and f is constant on $[a,b]$. But the last two conditions cannot hold simultaneously if $b \neq a$. The other possibility for equality to hold is if $f(a) = 0$ and $t = 3$. In that case (4.14) implies that $f'(x) = 1$ on $[a,b]$ thus $f(x) = x - a$. ■

The following corollary gives an affirmative answer to Problem 5 (Remark 2.6).

Corollary 4.1 ([31], p.2) *Let $f \in C^1([a,b])$ satisfies $f(a) \geq 0$ and $f'(x) \geq n(x-a)^{n-1}$ for $x \in [a,b]$ and a positive integer n, then*

$$\int_a^b [f(x)]^{n+2}\,dx \geq \left(\int_a^b f(x)\,dx\right)^{n+1}.$$

Proof. Put $t = n+2$ in Theorem 4.4. ■

Remark 4.6 ([31], p.2) *We now show that Theorem 2.4 follows from Corollary 4.1. Suppose that the function f satisfy the conditions of Theorem 2.4. Since $f^{(n)}(x) \geq n!$, successively integrating $n-1$ times over $[a,x]$ we have $f'(x) \geq n(x-a)n-1$, $x \in [a,b]$. Thus the conditions of Corollary 4.1 are satisfied.*

4.3 WORK OF J. S. SUN ANSWERING THE PROBLEMS 1, 3 AND 4

In the joint paper [23], K. W. Yu and F. Qi had one answer to Problem 1 (Remark 2.1): inequality (2.3) is valid for all $f \in \mathcal{C}[a,b]$ such that $\int_a^b f(x)\,dx \geq (b-a)^{t-1}$ for given $t > 1$ (see Section 3.1 for detail of the proof). Many authors considered different generalizations of open problem cf. [2] (see also Section 3.5), [26] (see also Section 3.3), [31] (see also Section 4.2), [33].

In the paper [32], J. S. Sun gave further extension answer to Problem 1 (Remark 2.1). The following his theorem gives an affirmative answer to Problems 1 (Remark 2.1), Problem 3 (Remark 2.4) and Problem 4 (Remark 2.5) that is :

Theorem 4.5 ([32], p.1) *Suppose that $f(x)$ is differentiable on (a,b) and $f(a) = 0$, we obtain*

(a) *If $f'(x) \geq 0$ and $t \leq 1$, then we get*

$$\int_a^b [f(x)]^t\,dx \leq \left(\int_a^b f(x)\,dx\right)^{t-1} \tag{4.15}$$

(Problem 4)

(b) *If $f'(x) \geq 0$ and $1 \leq t \leq 2$, then we get*

$$\int_a^b [f(x)]^t\,dx \geq \left(\int_a^b f(x)\,dx\right)^{t-1} \tag{4.16}$$

(Problem 1)

(c) *If* $0 \le f'(x) \le (t-2)(x-a)^{t-3}$ *and* $2 \le t \le 3$, *then we get*

$$\int_a^b [f(x)]^t \, dx \le \left(\int_a^b f(x) \, dx \right)^{t-1} \tag{4.17}$$

(Problem 3)

(d) *If* $f'(x) \ge (t-2)(x-a)^{t-3}$ *and* $t \ge 3$, *then we get*

$$\int_a^b [f(x)]^t \, dx \ge \left(\int_a^b f(x) \, dx \right)^{t-1} \tag{4.18}$$

(Problem 1)

Proof. For $x \in [a, b]$, if we set

$$F(x) = \int_a^x [f(\xi)]^t \, d\xi - \left(\int_a^x f(\xi) \, d\xi \right)^{t-1}$$

then $F(a) = 0$ and $F'(x) = f(x)G(x)$, where

$$G(x) = [f(x)]^{t-1} - (t-1) \left(\int_a^x f(\xi) \, d\xi \right)^{t-2}$$

Obviously, $G(a) = 0$ and straightforward computation yields

$$G'(x) = (t-1) f(x) H(x)$$

$$H(x) = [f(x)]^{t-3} f'(x) - (t-2) \left(\int_a^x f(\xi) \, d\xi \right)^{t-3}$$

Since $f'(t) \ge 0$ and $f(a) = 0$, hence $f(t)$ is increasing and $f(x) \ge 0$. Also we get

$$f(x)(x-a) \ge \int_a^x f(\xi) \, d\xi = f(\eta)(x-a) \ge 0 \qquad (a \le \eta \le x) \tag{4.19}$$

(a) When $f'(x) \ge 0$ and $t \le 1$, we get $H(x) \ge 0$ and $G'(x) \le 0$, $G(x)$ decreases and $G(x) \le 0$ because of $G(a) = 0$, so $F'(x) = f(x)G(x) \le 0$, $F(x)$ is decreasing. Since $F(a) = 0$, we get $F(x) \le 0$, and $F(b) \le 0$. Hence, the inequality (4.15) is valid.

(b) When $f'(x) \ge 0$ and $1 \le t \le 2$, we get $H(x) \ge 0$ and $G'(x) \ge 0$, $G(x)$ increases, $G(x) \ge 0$, $F'(x) \ge 0$, and $F(x)$ is increasing, then $F(x) \ge 0$ and $F(b) \ge 0$, the inequality (4.16) follows.

(c) When $0 \le f'(x) \le (t-2)(x-a)^{t-3}$ and $2 \le t \le 3$, from inequality (4.19) we get

$$(f(x)(x-a))^{t-3} - \left(\int_a^x f(\xi) \, d\xi \right)^{t-3} \le 0$$

Hence, we have

$$H(x) \le (t-2)\left((f(x)(x-a))^{t-3} - \left(\int_a^x f(\xi)\, d\xi \right)^{t-3} \right) \le 0$$

Therefore $G'(x) \le 0$, $G(x)$ decreases, $G(x) \le 0$, $F'(x) \le 0$, and $F(x)$ is decreasing, then $F(x) \le 0$ and $F(b) \le 0$, the inequality (4.17) is valid.

(d) When $f'(x) \ge (t-2)(x-a)^{t-3}$ and $t \ge 3$, from inequality (4.19) we obtain

$$(f(x)(x-a))^{t-3} - \left(\int_a^x f(\xi)\, d\xi \right)^{t-3} \ge 0$$

Hence, we have

$$H(x) \ge (t-2)\left((f(x)(x-a))^{t-3} - \left(\int_a^x f(\xi)\, d\xi \right)^{t-3} \right) \ge 0$$

Therefore $G'(x) \ge 0$, $G(x)$ increases, $G(x) \ge 0$, $F'(x) \ge 0$, and $F(x)$ is increasing, then $F(x) \ge 0$ and $F(b) \ge 0$, the inequality (4.18) follows.

Proof is complete. ■

In inequality (4.17), take $t = 3$, $a = 0$ and $b = 1$, then we have

Corollary 4.2 ([22], p.624) *Suppose that $f(x)$ is a continuous function on the closed interval $[0,1]$ and $f(0) = 0$, its derivative of the first order is bounded by $0 \le f'(x) \le 1$, for $x \in [0,1]$, then*

$$\int_0^1 [f(x)]^3\, dx \le \left(\int_0^1 f(x)\, dx \right)^2 \tag{4.20}$$

Equality in (4.20) is valid if and only if $f(x) \equiv 0$ or $f(x) = x$.

In inequality (4.17) and (4.18), take $t = 3$, then we obtain

Corollary 4.3 ([16]) *Suppose that $f(x)$ is differentiable on (a,b) and $f(a) = 0$. If $0 \le f'(x) \le 1$ then*

$$\int_a^b [f(x)]^3\, dx \le \left(\int_a^b f(x)\, dx \right)^2 \tag{4.21}$$

If $f'(x) \ge 1$, then inequality (4.21) reverses. The equality in (4.21) is valid if and only if $f(x) \equiv 0$ or $f(x) = x - a$.

4.4 WORK OF L. BOUGOFFA ANSWERING THE PROBLEM 7

In his paper [21], L. Bougoffa posed Problem 7 (Remark 2.8) which is similar to Problem 1 (Remark 2.1) and he gave an affirmative answer to it.

Before giving main result,which is Proposition 4.7, we need the following.

Proposition 4.6 ([21], p.2) *Suppose that f and g is nonnegative functions with $0 < m \leq \frac{f(x)}{g(x)} \leq M < \infty$ on $[a,b]$. Then for $p>1$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we get*

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx, \tag{4.22}$$

and then we get

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b f(x)\, dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\, dx\right)^{\frac{1}{p}}. \tag{4.23}$$

Proof. From Hölder's inequality, we get

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq \left(\int_a^b f(x)\, dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)\, dx\right)^{\frac{1}{q}},$$

that is,

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq \left(\int_a^b [f(x)]^{\frac{1}{p}} [f(x)]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b [g(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}.$$

Since $[f(x)]^{\frac{1}{p}} \leq M^{\frac{1}{p}} [g(x)]^{\frac{1}{p}}$ and $[g(x)]^{\frac{1}{q}} \leq m^{-\frac{1}{q}} [f(x)]^{\frac{1}{q}}$, from the above inequality it can be seen that

$$\begin{aligned} & \int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \\ \leq \; & M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b [f(x)]^{\frac{1}{p}} [f(x)]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b [g(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}, \end{aligned}$$

that is

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx. \tag{4.24}$$

Thus, the inequality (4.22) is proved.

The inequality (4.23) can be easily seen from substituting the following

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq \left(\int_a^b f(x)\, dx\right)^{\frac{1}{q}} \left(\int_a^b g(x)\, dx\right)^{\frac{1}{p}}$$

into (4.24), which can be obtained using Hölder's inequality. ■

We are ready to give an affirmative answer to Problem 7 (Remark 2.8) is given as follows.

Proposition 4.7 ([21], p.2) *For a given positive integer $p \geq 2$, if $0 < m \leq f(x) \leq M$ on $[a,b]$ with $M \leq \frac{m^{(p-1)^2}}{(b-a)^p}$, then we get*

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \leq \left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}}. \tag{4.25}$$

Proof. Putting $g(x) \equiv 1$ into (4.23) produces

$$\int_a^b [f(x)]^{\frac{1}{p}}\,dx \leq K\left(\int_a^b f(x)\,dx\right)^{1-\frac{1}{p}},$$

where $K = M^{\frac{1}{p^2}}(b-a)^{\frac{1}{p}}/m^{\left(1-\frac{1}{p}\right)^2}$.

From $M \leq m^{(p-1)^2}/(b-a)^p$, we deduce that $K \leq 1$. Therefore, the inequality (4.25) is proved. ■

Remark 4.7 ([21], p.3) *We now debate a basic case of "equality" in Proposition 4.7. If we make the substitution $f(x) = M = m$ and $b - a = 1$ with $p = 2$, so that the equality in (4.25) holds.*

In order to exemplify a possible practical use of Proposition 4.7, we present the following two basic examples in which we can apply inequality (4.25).

Example 4.1 ([21], p.3) *Suppose $f(x) = 8x^2$ on $\left[\frac{1}{2}, 1\right]$ with $M = 8$ and $m = 2$. Taking $p = 2$, we see that the conditions of Proposition 4.7 are satisfied and direct calculation produces*

$$\int_{\frac{1}{2}}^1 \left(8x^2\right)^{\frac{1}{2}}\,dx = \frac{3}{4}\sqrt{2} < \left(\int_{\frac{1}{2}}^1 8x^2 dx\right)^{\frac{1}{2}} = \frac{\sqrt{7}}{\sqrt{3}}.$$

Example 4.2 ([21], p.3) *Suppose $f(x) = e^x$ on $[1,2]$ with $M = e^2$ and $m = e$. Taking $p = 3$, all the conditions of Proposition 4.7 are fulfilled and straightforward computation yields*

$$\int_1^2 (e^x)^{\frac{1}{3}}\,dx = 3\left(e^{\frac{2}{3}} - e^{\frac{1}{3}}\right) \approx 1.65 < \left(\int_1^2 e^x dx\right)^{\frac{2}{3}} = \left(e^2 - e\right)^{\frac{2}{3}} \approx 2.78.$$

CHAPTER 5

PAPERS IN 2006

In this chapter, we present some papers on Qi type inequalities in 2006.

Papers answering the Problem 1 (Remark 2.1):

- In 2006, work of Y. Chen and J. Kimball (see Section 5.1).
- In 2006, work of P. Yan and G. Gyllenberg (see Section 5.5).

Papers answering the Problem 3 (Remark 2.4):

- In 2006, work of Y. Chen and J. Kimball (see Section 5.1).
- In 2006, work of P. Yan and G. Gyllenberg (see Section 5.5).

Papers answering the Problem 5 (Remark 2.6):

- In 2006, work of Y. Chen and J. Kimball (see Section 5.1).
- In 2006, work of P. Yan and G. Gyllenberg (see Section 5.4).

Papers answering the Problem 6 (Remark 2.7):

- In 2006, work of Y. Chen and J. Kimball (see Section 5.1).
- In 2006, work of P. Yan and G. Gyllenberg (see Section 5.4).

Paper answering the Problem 7 (Remark 2.8):

- In 2006, work of W. J. Liu et al. (see Section 5.3).

Paper answering the Problem 8 (Remark 2.9):

- In 2006, work of W. J. Liu et al. (see Section 5.3).

Paper answering the Problem 9 (Remark 2.10):

- In 2006, work of F. Qi et al. (see Section 5.2).

Paper answering the Problem 10 (Remark 2.11):

- In 2006, work of F. Qi et al. (see Section 5.2).

5.1 WORK OF Y. CHEN AND J. KIMBALL ANSWERING THE PROBLEMS 1, 3, 5 AND 6

In the joint paper [34], Y. Chen and J. Kimball gave an answer to Problem 1 (Remark 2.1). The following result is a generalization of Theorem 2.3 and it gives an affirmative answer to Problem 1 (Remark 2.1) and Problem 3 (Remark 2.4).

Theorem 5.1 ([34], p.2) *Suppose that p is a positive number and $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=0$. If $[f^{\frac{1}{p}}]'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then*

$$\int_a^b [f(x)]^{p+2}\,dx \geq \left[\int_a^b f(x)\,dx\right]^{p+1}. \tag{5.1}$$

If $0 \leq [f^{\frac{1}{p}}]'(x) \leq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then the inequality (5.1) reverses.

Proof. Let $[f^{\frac{1}{p}}]'(x) \geq 0$, $x \in (a,b)$. Then $f^{\frac{1}{p}}(x)$ is a non-decreasing function. It can be easily seen that $f(x) \geq 0$ for all $x \in (a,b]$.
If $[f^{\frac{1}{p}}]'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then $f(x) > 0$ for $x \in (a,b]$. Hence both sides of (5.1) are not 0. Now consider the quotient of both sides of (5.1). By using Cauchy's Mean Value Theorem twice, we get

$$\begin{aligned} \frac{\int_a^b [f(x)]^{p+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{p+1}} &= \frac{[f(b_1)]^{p+1}}{(p+1)\left[\int_a^{b_1} f(x)\,dx\right]^{p}} \qquad (a < b_1 < b) \\ &= \left(\frac{[f(b_1)]^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}\int_a^{b_1} f(x)\,dx}\right)^{p} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\left(1+\frac{1}{p}\right) [f(b_2)]^{\frac{1}{p}} f'(b_2)}{(p+1)^{\frac{1}{p}} f(b_2)} \right)^p \qquad (a < b_2 < b_1) \\
&= \left((1+p)^{1-\frac{1}{p}} \left[f^{\frac{1}{p}} \right]' (b_2) \right)^p \qquad\qquad (5.2) \\
&\geq 1.
\end{aligned}
$$

Therefore, the inequality (5.1) holds.

If $f \equiv 0$ on $[a, b]$, then it is obvious that the equation in (5.1) holds. Now, let f be not identically 0 on $[a, b]$. Since $f(x)$ is non-decreasing and non-negative, we may suppose that $f(x) > 0$, $x \in (a, b]$ (otherwise we can find a_1 such that $a_1 < b$, $f(a_1) = 0$ and $f(x) > 0$ for $a_1 < x < b$ and thus we just need to consider f on $(a_1, b]$). This implies that both sides of (5.1) are not 0. Now if $0 \leq [f^{\frac{1}{p}}]'(x) \leq (p+1)^{\frac{1}{p}-1}$, then $(1+p)^{1-\frac{1}{p}}[f^{\frac{1}{p}}]'(b_2) \leq 1$, which, together with (5.2), implies that the inequality (5.1) reverses. ■

Remark 5.1 *In [35], P. Yan and M. Gyllenberg phrased that as a matter of fact, Theorem 5.1 is not true. They explains their claim as follows "To see this, choose* $f(x) = -2\sqrt{x}$, $p = \frac{1}{2}$, $a = 0$, $b = 1$. *It can be easily seen that the conditions of Theorem 5.1 are satisfied, but that inequality (5.1) does not hold. The error in the proof of Theorem 5.1 is the statement that if* $f^{\frac{1}{p}}(x)$ *is a non-decreasing function, then* $f(x) \geq 0$ *for all* $x \in (a, b]$, *which is not true as our example above shows. If one adds* $f(x) \geq 0$ *for all* $x \in (a, b]$ *to the hypotheses, then Theorem 5.1 becomes a valid theorem."*

Note that if $p = 1$, then (5.1) becomes (2.20). Hence Theorem 2.3 is just a special case of Theorem 5.1.

In Theorem 2.3, we see that if $f'(x) = 1$, then $f(x) = x - a$ and the equation in (2.20) holds. A very natural question can be asked as follows: For what polynomial $f(x) = C(x-a)^n$ does the equation in (2.7) hold? It follows that $C = \frac{1}{(n+1)^{(n-1)}}$. The n-th derivative of this polynomial is a constant $\frac{n!}{(n+1)^{(n-1)}}$. This motivates the following theorem. It gives an affirmative answer to Problem 5 (Remark 2.6) and Problem 6 (Remark 2.7).

Theorem 5.2 ([34], p.2) *Assume* $f(x)$ *has derivative of the* n*-th order on the interval* $[a, b]$ *such that* $f^{(i)}(a) = 0$ *for* $i = 0, 1, 2, ..., n-1$. *If* $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ *and* $f^{(n)}(x)$ *is increasing, then the inequality (2.7) holds. If* $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ *and* $f^{(n)}(x)$ *is decreasing, then the inequality (2.7) reverses.*

Proof. Assume that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$. It follows that

$$f(x) \geq \frac{(x-a)^n}{(n+1)^{n-1}}.$$

Using the same argument as in the proof of Theorem 5.1, we get

$$\begin{aligned}
\frac{\int_a^b [f(x)]^{n+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{n+1}} &= \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)\,dx\right]^n} \qquad (a < b_1 < b) \\
&\geq \frac{\frac{(b_1-a)^n}{(n+1)^{n-1}}[f(b_1)]^n}{(n+1)\left[\int_a^{b_1} f(x)\,dx\right]^n} \qquad (5.3) \\
&= \left(\frac{(b_1-a)f(b_1)}{(n+1)\int_a^{b_1} f(x)\,dx}\right)^n. \qquad (5.4)
\end{aligned}$$

Now for the term in (5.4), by using Cauchy's Mean Value Theorem several times, we will get

$$\begin{aligned}
\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,dx} &= 1 + \frac{(b_2-a)f'(b_2)}{f(b_2)} \qquad (a < b_2 < b_1) \\
&= 2 + \frac{(b_3-a)f''(b_3)}{f'(b_3)} \qquad (a < b_3 < b_2) \\
&\cdots \\
&= n + \frac{(b_{n+1}-a)f^{(n)}(b_{n+1})}{f^{(n-1)}(b_{n+1})} \qquad (a < b_{n+1} < b_n). \qquad (5.5)
\end{aligned}$$

But

$$f^{(n-1)}(t) = f^{(n-1)}(t) - f^{(n-1)}(a) = f^{(n)}(t_1)(t-a)$$

for some $t_1 \in (a,t)$. If $f^{(n)}(x)$ is increasing, then $f^{(n)}(t_1) \leq f^{(n)}(t)$. Hence

$$f^{(n-1)}(t) \leq f^{(n)}(t)(t-a). \qquad (5.6)$$

Applying (5.6) to (5.5) produces

$$\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,dx} \geq n+1. \qquad (5.7)$$

(2.22) follows from (5.7) and (5.4).

Let $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ and $f^n(x)$ be decreasing. It is clear that $f^{(n-1)}(t)$ is increasing. If $f^{(n-1)}(t) = 0$ for some $t \in (a,b)$, then $f^{(n-1)}(s) = 0$ for $s \in (a,t)$. Therefore, $f^{(i)}(s) = 0$ for $s \in (a,t)$ and $0 \leq s \leq n-1$. Hence we can suppose that $f^{(n-1)}(x) \neq 0$ for $x \in (a,b)$. By Rolle's Theorem, this means that $f^{(i)}(x) \neq 0$ for $x \in (a,b)$ and for $0 \leq i \leq n-1$. Now that the inequalities (5.3) and (5.6) reverse, it is trivial that the inequality (5.7) reverses, so does (2.7). ■

Remark 5.2 *In [34], Y. Chen and J. Kimball phrased that "Unfortunately there is an additional hypothesis on monotonicity in Theorem 5.2. Our conjecture is that this hypothesis could be dropped. But we are not able to prove it for the moment." However, they proved the following which gives an affirmative answer to Problem 5 (Remark 2.6). Fortunately, in [36], P. Yan and G. Gyllenberg proved Y. Chen and J. Kimball's conjecture by mathematical induction that the conjecture is true (see Theorem 5.13 and its proof in Section 5.4) and in [37], Q. A. Ngô and P. H. Tung proved that this conjecture holds and they used the same technique which was introduced by Qi in [16](see Theorem 6.3 and its proof in Section 6.2).*

Theorem 5.3 ([34], p.3) *Assume $f(x)$ has derivative of the n-th order on the interval $[a,b]$ such that, $f^{(i)}(a) = 0$ for $i = 0, 1, 2, ..., n-1$. If $f^{(n)}(x) \geq \frac{(n+1)!}{n^n}$, then the inequality (2.7) holds.*

Proof. If $f(x) \geq \frac{(n+1)!}{n^n}$, then we have

$$f(x) \geq \frac{n+1}{n^n}(x-a)^n.$$

(5.4) now becomes

$$\frac{\int_a^b [f(x)]^{n+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{n+1}} \geq \left(\frac{(b_1-a)\,f(b_1)}{n\int_a^{b_1} f(x)\,dx}\right)^n. \tag{5.8}$$

Note that all the terms in (5.5) are positive, so we get

$$\frac{(b_1-a)\,f(b_1)}{\int_a^{b_1} f(x)\,dx} \geq n. \tag{5.9}$$

The inequality (2.7) follows from (5.8) and (5.9). ■

The same argument can be used to prove the following result obtained by Pečarić and Pejković [31, Theorem 2], (see also Theorem 4.4 in Section 4.2).

The following theorem gives an affirmative answer to Problems 1 (Remark 2.1).

Theorem 5.4 ([34], p.4) *Suppose that p is a positive number and $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) \geq 0$. If $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$, then the inequality (5.1) is valid.*

Proof. Assume that $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$. Consider the quotient of the two sides of (5.1). By using Cauchy's Mean Value Theorem three times, we get

$$\begin{aligned}
\frac{\int_a^b [f(x)]^{p+2}\, dx}{\left[\int_a^b f(x)\, dx\right]^{p+1}} &= \frac{[f(b_1)]^{p+1}}{(p+1)\left[\int_a^{b_1} f(x)\, dx\right]^{p}} \qquad (a < b_1 < b) \\
&= \frac{[f(b_2)]^{p} f'(b_2)}{(p-1)\left[\int_a^{b_2} f(x)\, dx\right]^{p-1}} \qquad (a < b_2 < b_1) \\
&\geq \left(\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)\, dx}\right)^{p-1} \\
&= \left(1 + \frac{f'(b_3)(b_3-a)}{f(b_3)}\right)^{p-1} \\
&\geq 1.
\end{aligned}$$

The proof is completed. ■

5.2 WORK OF F. QI, A. J. LI, W. Z. ZHAO, D. W. NIU AND J. CAO ANSWERING THE PROBLEMS 9 AND 10

In the joint paper [38], F. Qi et al. established some more extensions and generalizations of inequality (2.7) or (2.3) once again by using [34] extensively (see Section 5.1). Their main results are the following five theorems which give affirmative answers to Problem 9 (Remark 2.10) and Problem 10 (Remark 2.11).

Theorem 5.5 ([38], p.2) *Suppose that $f(x)$ is continuous and not identically zero on $[a,b]$, differentiable in (a,b), with $f(a) = 0$, and suppose that α, β are positive real numbers such that $\alpha > \beta > 1$. If*

$$\left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \gtreqless \frac{(\alpha-\beta)\,\beta^{1/(\beta-1)}}{\alpha-1}$$

for all $x \in (a,b)$, then we get

$$\int_a^b [f(t)]^{\alpha}\, dt \gtreqless \left[\int_a^b f(t)\, dt\right]^{\beta}. \tag{5.10}$$

Proof. If

$$\left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \geq \frac{(\alpha-\beta)\,\beta^{1/(\beta-1)}}{\alpha-1}$$

for $x \in (a,b)$ and $\alpha > \beta > 1$, then $f(x) > 0$ for $x \in (a,b]$. Hence both sides of (5.10) do not equal zero. This allows us to consider the quotient of both sides of (5.10). Using Cauchy's Mean Value Theorem consecutively produces

$$\begin{aligned}
\frac{\left[\int_a^b f(t)\,dt\right]^\beta}{\int_a^b [f(t)]^\alpha\,dt} &= \frac{\beta\left[\int_a^\xi f(t)\,dt\right]^{\beta-1} f(\xi)}{[f(\xi)]^\alpha} && \xi \in (a,b) \\
&= \left\{\frac{\beta^{1/(\beta-1)}\int_a^\xi f(t)\,dt}{[f(\xi)]^{(\alpha-1)/(\beta-1)}}\right\}^{\beta-1} && (5.11) \\
&= \left\{\frac{\beta^{1/(\beta-1)} f(\theta)}{\frac{\alpha-1}{\beta-1}[f(\theta)]^{(\alpha-\beta)/(\beta-1)} f'(\theta)}\right\}^{\beta-1} && \theta \in (a,\xi) \\
&= \left\{\frac{(\alpha-\beta)\beta^{1/(\beta-1)}/(\alpha-1)}{\left[f^{(\alpha-\beta)/(\beta-1)}(\theta)\right]'}\right\}^{\beta-1} && (5.12) \\
&\le 1.
\end{aligned}$$

Therefore, the inequality with direction $\ge$ in (5.10) holds.

If

$$0 \le \left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \le \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1}$$

for $x \in (a,b)$ and $\alpha > \beta > 1$, then the $f^{(\alpha-\beta)/(\beta-1)}(x)$ is non decreasing and $f(x) \ge 0$ for $x \in [a,b]$. Without loss of generality, we may suppose that $f(x) > 0$ for $x \in (a,b]$ (otherwise, we can find a point $a_0 \in (a,b)$ such that $f(a_0) = 0$ and $f(x) > 0$ for $x \in (a_0,b]$ and thus we only need to consider the inequality with direction $\le$ in (5.10) on $[a_0,b]$). This means that both sides of inequality (5.10) are not zero. So, the inequality with direction $\le$ in (5.10) is obtained from (5.12). ■

Theorem 5.6 ([38], p.2) *Suppose that $\alpha \in \mathbb{R}$ and $f(x)$ is continuous on $[a,b]$ and positive in (a,b).*

(a) *For $\beta > 1$, if*

$$\int_a^x f(t)\,dt \lesseqgtr \beta^{1/(1-\beta)}[f(x)]^{(\alpha-1)/(\beta-1)} \tag{5.13}$$

for all $x \in (a,b)$, then inequality (5.10) holds;

(b) *For $0 < \beta < 1$, if inequality (5.13) is reversed, then inequality (5.10) is valid;*

(c) *For $\beta = 1$, if $[f(x)]^{1-\alpha} \lesseqgtr 1$ for all $x \in (a,b)$, then inequality (5.10) is validated.*

Proof. The first and second conclusions can be easily obtained by (5.11) of Theorem 5.5.

For $\beta = 1$, inequality (5.10) is reduced to

$$\int_a^b [f(x)]^\alpha \, dt \gtreqless \int_a^b f(t)\, dt. \tag{5.14}$$

Now consider the quotient of both sides of (5.14). By Cauchy's Mean Value Theorem, we obtain

$$\frac{\int_a^b [f(t)]^\alpha \, dt}{\int_a^b f(t)\, dt} = \frac{[f(\xi)]^\alpha}{f(\xi)} = [f(\xi)]^{\alpha-1}.$$

This completes the third conclusion. ■

Theorem 5.7 ([38], p.2) *Assume $n \in \mathbb{N}$, $1 \le \beta \le n+1$, and $f(x)$ has a derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$ for $0 \le i \le n-1$ and $f^{(n)}(x) \ge 0$.*

(a) *If $f(x) \ge \left[\frac{(x-\alpha)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$ and $f^{(n)}(x)$ is increasing, then the inequality with direction $\ge$ in (5.10) is valid.*

(b) *If $0 \le f(x) \le \left[\frac{(x-\alpha)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$ and $f^{(n)}(x)$ is decreasing, then the inequality with direction $\le$ in (5.10) holds.*

Proof. By using of the condition that $f(x) \ge \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$ and Cauchy's Mean Value Theorem, we have

$$\begin{aligned}
\frac{\int_a^b [f(x)]^\alpha \, dx}{\left[\int_a^b f(x)\, dx\right]^\beta} &= \frac{[f(b_1)]^{\alpha-1}}{\beta\left[\int_a^{b_1} f(x)\, dx\right]^{\beta-1}} \qquad a < b_1 < b & (5.15)\\
&\ge \frac{(b_1-a)^{\beta-1}[f(b_1)]^{\beta-1}/\beta^{\beta-2}}{\beta\left[\int_a^{b_1} f(x)\, dx\right]^{\beta-1}} & (5.16)\\
&= \left[\frac{(b_1-a) f(b_1)}{\beta\int_a^{b_1} f(x)\, dx}\right]^{\beta-1}. & (5.17)
\end{aligned}$$

Now for the term in (5.17), by using Cauchy's Mean Value Theorem several times, we get

$$\begin{aligned}
\frac{(b_1-a) f(b_1)}{\int_a^{b_1} f(x)\, dx} &= 1 + \frac{(b_2-a) f'(b_2)}{f(b_2)} \qquad a < b_2 < b_1\\
&= 2 + \frac{(b_3-a) f''(b_3)}{f'(b_3)} \qquad a < b_3 < b_2\\
&\cdots\\
&= n + \frac{(b_{n+1}-a) f^{(n)}(b_{n+1})}{f^{(n+1)}(b_{n+1})} \qquad a < b_{n+1} < b_{n.} & (5.18)
\end{aligned}$$

However $f^{(n-1)}(t) = f^{(n-1)}(t) - f^{(n-1)}(a) = (t-a) f^{(n)}(t_1)$ for some $t_1 \in (a,t)$. If $f^{(n)}(x)$ is increasing, then $f^{(n)}(t_1) \le f^{(n)}(t)$. So we get

$$0 < f^{(n-1)}(t) \le f^{(n)}(t)(t-a). \tag{5.19}$$

Applying (5.19) to (5.18) produces

$$\frac{(b_1 - a) f(b_1)}{\int_a^{b_1} f(x)\,dx} \ge n+1. \tag{5.20}$$

Thus,

$$\frac{\int_a^b [f(x)]^\alpha\,dx}{\left[\int_a^b f(x)\,dx\right]^\beta} \ge \left(\frac{n+1}{\beta}\right)^{\beta-1} \tag{5.21}$$

for $1 \le \beta \le n+1$. Then the inequality with direction $\ge$ in (5.10) follows.

Assume that

$$0 \le f(x) \le \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$$

and $f^{(n)}(x)$ is decreasing. The statement of the theorem implies that the inequalities (5.16) and (5.19) reverse, this means that the inequalities (5.20) and (5.21) reverse also, thus the inequality with direction $\le$ in (5.10) follows. ■

Theorem 5.8 ([38], p.2) *Assume $n \in \mathbb{N}$, $1 < \beta \le n+1$, and $f(x)$ has a derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$ for $0 \le i \le n-1$ and $f^{(n)}(x) \ge 0$.*

(a) *If $f(x) \ge \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)}$, then the inequality with direction $\ge$ in (5.10) is valid.*

(b) *If $0 \le f(x) \le \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)}$, then the inequality with direction $\le$ in (5.10) holds.*

Proof. If

$$f(x) \ge \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)},$$

(5.15) becomes

$$\frac{\int_a^b [f(x)]^\alpha\,dx}{\left[\int_a^b f(x)\,dx\right]^\beta} \ge \left[\frac{(b_1-a) f(b_1)}{(\beta-1)\int_a^{b_1} f(x)\,dx}\right]^{\beta-1}.$$

Note that if all the terms in (5.18) are positive, then $\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)dx} \geq n$. Thus for $1 < \beta \leq n+1$, the inequality with direction $\geq$ in (5.10) follows.

If

$$0 \leq f(x) \leq \left[\frac{\beta (x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}} \right]^{1/(\alpha-\beta)},$$

the inequality with direction $\leq$ in (5.10) is obtained from a similar argument as above. ■

Theorem 5.9 ([38], p.3) *Assume that α, β are positive numbers, $\alpha > \beta \geq 2$ and $f(x)$ are continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) \geq 0$. If*

$$\left[f^{(\alpha-\beta)}(x)\right]' \geq \frac{\beta(\beta-1)(\alpha-\beta)(x-a)^{\beta-2}}{\alpha-1}$$

for $x \in (a,b)$, then the inequality with direction $\geq$ in (5.10) is valid.

Proof. Let

$$\left[f^{(\alpha-\beta)}(x)\right]' \geq \frac{\beta(\beta-1)(\alpha-\beta)(x-a)^{\beta-2}}{\alpha-1}.$$

Now consider the quotient of the two sides of (5.10). Applying Cauchy's Mean Value Theorem three times gives

$$\begin{aligned}
\frac{\int_a^b [f(x)]^\alpha dx}{\left[\int_a^b f(x)\,dx\right]^\beta} &= \frac{[f(b_1)]^{\alpha-1}}{\beta\left[\int_a^{b_1} f(x)\,dx\right]^{\beta-1}} \\
&\geq \frac{(\alpha-1)[f(b_2)]^{\alpha-3} f'(b_2)}{\beta(\beta-1)\left[\int_a^{b_2} f(x)\,dx\right]^{\beta-2}} \qquad a < b_2 < b_1 \\
&\geq \left[\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)\,dx}\right]^{\beta-2} \qquad a < b_3 < b_2 \\
&= \left[1 + \frac{f'(b_3)(b_3-a)}{f(b_3)}\right]^{\beta-2} \\
&\geq 1.
\end{aligned}$$

The proof is completed. ■

Remark 5.3 ([38], p.3) *Theorem 5.9 generalizes a result obtained in [31, Theorem 2] by Pečarić and Pejković (see also Theorem 4.4 in Section 4.2).*

5.3 WORK OF W. J. LIU, C. C. LI AND J. W. DONG ANSWERING THE PROBLEMS 7 AND 8

Similar to Problem 1 (Remark 2.1), in the paper [21] L. Bougoffa proposed Problem 7 (Remark 2.8). By using Hölder's inequality, L. Bougoffa obtained an answer to Problem 7 (see Proposition 2 in [21]). (see also Proposition 4.7 in Section 4.4). W. J. Liu et al. noted that the condition, which is given Proposition 2 in [21],

$$0 < m \leq f(x) \leq M \text{ on } [a, b] \text{ with } M \leq m^{(p-1)^2} / (b-a)^p$$

is not fulfilled when $\min\limits_{[a,b]} f(x) = 0$.

In the joint paper [39], W. J. Liu et al. firstly gave an answer to Problem 7 (Remark 2.8) and Problem 8 (Remark 2.9), in which they allow $\min\limits_{[a,b]} f(x) = 0$ and p unnecessarily to be an integer. Secondly, they obtained a consolidation of Qi's inequality which is Problem 1 (Remark 2.1) and Bougoffa's inequality which is Problem 7 (Remark 2.8).

Theorem 5.10 ([39], p.5) *Suppose that $p > 2$ is a positive number and $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = 0$.*
If $[f^{p-2}]'(x) \geq p^p (p-2) / (p-1)^{p+1}$ for $x \in (a, b)$, then

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b f(x) dx \right)^{1-\frac{1}{p}}. \tag{5.22}$$

If $0 \leq [f^{p-2}]'(x) \leq p^p (p-2) / (p-1)^{p+1}$ for $x \in (a, b)$, then the inequality (5.22) reverses.

Proof. If $f \equiv 0$ on $[a, b]$, then it is clear that the equation in (5.22) is valid. Assume now that f is not identically 0 on $[a, b]$ and $[f^{p-2}]'(x) \geq 0$ for $x \in (a, b)$, we may suppose $f(x) > 0$, $x \in (a, b]$. This implies that both sides of (5.22) are not 0.
If $[f^{p-2}]'(x) \geq p^p (p-2) / (p-1)^{p+1}$ for $x \in (a, b)$, then $f(x) > 0$ for $x \in (a, b]$. Thus both sides of (5.22) are not 0. By using Cauchy's Mean Value Theorem twice, we get

$$\begin{aligned} \frac{\int_a^b [f(x)]^{\frac{1}{p}} dx}{\left(\int_a^b f(x) dx \right)^{1-\frac{1}{p}}} &= \frac{[f(b_1)]^{\frac{1}{p}-1}}{\left(1-\frac{1}{p}\right)\left(\int_a^{b_1} f(x) dx \right)^{-\frac{1}{p}}} \quad (a < b_1 < b) \\ &= \left(\frac{\int_a^{b_1} f(x) dx}{\left(1-\frac{1}{p}\right)^p [f(b_1)]^{p-1}} \right)^{\frac{1}{p}} \end{aligned} \tag{5.23}$$

$$
\begin{aligned}
&= \left(\frac{1}{\left(1-\frac{1}{p}\right)^{p}(p-1)\left[f\left(b_{2}\right)\right]^{p-3} f^{\prime}\left(b_{2}\right)}\right)^{\frac{1}{p}} \quad\left(a<b_{2}<b_{1}\right) \\
&= \left(\frac{1}{\frac{(p-1)^{p+1}}{p^{p}(p-2)}\left[f^{p-2}\right]^{\prime}\left(b_{2}\right)}\right)^{\frac{1}{p}} \\
&\leq 1 .
\end{aligned}
$$

Therefore, the inequality (5.22) is valid.

If $0 \leq \left[f^{p-2}\right]^{\prime}(x) \leq p^{p}(p-2) /(p-1)^{p+1}$, then $\frac{(p-1)^{p+1}}{p^{p}(p-2)}\left[f^{p-2}\right]^{\prime}\left(b_{2}\right) \leq 1$, which, together with (5.23), implies that the inequality (5.22) reverses. ■

In the paper [34], Y. Chen and J. Kimball gave an answer to Problem 1 (we also give it and its proof as Theorem 5.1 in Section 5.1).

Thus, combining Theorem 5.10 and Theorem 5.1, the authors obtained another result of their paper, which gives a consolidation of Qi's inequality which is Problem 1 (Remark 2.1) and Bougoffa's inequality which is Problem 7 (Remark 2.8). They said "To our best knowledge, this result is not found in the literature."

Theorem 5.11 ([39], p.7) *Suppose that $p>2$ is a positive number and $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a)=0$.*

(a) *If $\left[f^{p-2}\right]^{\prime}(x) \geq p^{p}(p-2) /(p-1)^{p+1}$ and $\left[f^{\frac{1}{p-2}}\right]^{\prime}(x) \geq(p-1)^{\frac{1}{p-2}-1}$ for $x \in(a, b)$, then*

$$
\left(\int_{a}^{b}[f(x)]^{\frac{1}{p}} d x\right)^{p} \leq\left(\int_{a}^{b} f(x) d x\right)^{p-1} \leq \int_{a}^{b}[f(x)]^{p} d x . \tag{5.24}
$$

(b) *If $0 \leq \left[f^{p-2}\right]^{\prime}(x) \leq p^{p}(p-2) /(p-1)^{p+1}$ and $0 \leq \left[f^{\frac{1}{p-2}}\right]^{\prime}(x) \leq(p-1)^{\frac{1}{p-2}-1}$ for $x \in(a, b)$, then the inequality (5.24) reverses.*

Corollary 5.1 ([39], p.7) *Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a)=0$.*

(a) *If $f^{\prime}(x) \geq \frac{27}{16}$ for $x \in(a, b)$, then*

$$
\left(\int_{a}^{b}[f(x)]^{\frac{1}{3}} d x\right)^{3} \leq\left(\int_{a}^{b} f(x) d x\right)^{2}<\int_{a}^{b}[f(x)]^{3} d x . \tag{5.25}
$$

(b) *If* $0 \le f'(x) \le 1$ *for* $x \in (a,b)$*, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{3}}\,dx\right)^3 > \left(\int_a^b f(x)\,dx\right)^2 \ge \int_a^b [f(x)]^3\,dx. \tag{5.26}$$

Proof. Set $p = 3$ in Theorem 5.11. ■

In order to illustrate a possible practical use of Corollary 5.1, we will give two simple examples in which we can apply inequality (5.25) and inequality (5.26).

Example 5.1 ([39], p.8) *Assume that* $f(x) = e^x - e$ *on* $[1,2]$*, we see that* $f'(x) > e > \frac{27}{16}$ *for* $x \in (1,2)$*, other conditions of Corollary 5.1 are satisfied and direct calculation produces*

$$\begin{aligned} \left(\int_1^2 (e^x - e)^{\frac{1}{3}}\,dx\right)^3 &\approx 1.56 < \left(\int_1^2 (e^x - e)\,dx\right)^2 \\ &\approx 3.81 < \int_1^2 (e^x - e)^3\,dx \\ &\approx 18.74. \end{aligned}$$

Example 5.2 ([39], p.8) *Suppose that* $f(x) = \frac{e^x - e}{10}$ *on* $[1,2]$*, then* $\frac{e}{10} \le f'(x) \le \frac{e^2}{10}$*, other conditions of Corollary 5.1 are satisfied and straightforward computation yields that*

$$\begin{aligned} \left[\int_1^2 \left(\frac{e^x - e}{10}\right)^{\frac{1}{3}} dx\right]^3 &\approx 0.156 > \left(\int_1^2 \frac{e^x - e}{10} dx\right)^2 \\ &\approx 0.038 > \int_1^2 \left(\frac{e^x - e}{10}\right)^3 dx \\ &\approx 0.019. \end{aligned}$$

5.4 WORK OF P. YAN AND M. GYLLENBERG ANSWERING THE PROBLEMS 5 AND 6

In [34], Y. Chen and J. Kimball studied a very interesting Qi-type integral inequality and proved (we also give it and its proof as Theorem 5.2 in Section 5.1). Then they conjectured that the additional hypothesis on monotonicity in Theorem 5.2 could be dropped (see Remark 5.2).

In the joint paper [35], P. Yan and G. Gyllenberg proved Chen and Kimball's conjecture, which is Theorem 5.13, by mathematical induction that the conjecture is true. As a matter of fact, Theorem 5.13 holds under slightly weaker assumptions (existence of $f^{(n)}(x)$ at the endpoints $x = a$, $x = b$ is not needed) and it gives an affirmative answer to Problem 5 (Remark 2.6) and Problem 6 (Remark 2.7) which we present here.

Before giving the proof of conjecture, we need a lemma (Lemma 5.12) and we will use the Cauchy's mean value theorem (CMVT), which is given as Theorem 1.4 in Section 1.1, to prove the following lemma.

Lemma 5.12 ([35], p.2) *Suppose $n \in \mathbb{Z}^+$. Assume $f(x)$ has a derivative of the n-th order on the interval (a, b) and $f^{(n-1)}(x)$ is continuous on $[a, b]$ such that $f^{(i)}(a) = 0$ for $i = 0, 1, 2, ..., n-1$.*

(a) *If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$, then*

$$(f(x))^{n+1} \geq (n+1)\left(\int_a^x f(s)\, ds\right)^n \qquad \text{for} \quad x \in [a, b].$$

(b) *If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$, then*

$$(f(x))^{n+1} \leq (n+1)\left(\int_a^x f(s)\, ds\right)^n \qquad \text{for} \quad x \in [a, b].$$

Proof. First notice that if f is identically 0, then the statement is clearly true. Let f be not identically 0 on $[a, b]$. Then the assumption implies that $f(x) \geq 0$ for $x \in [a, b]$. If $\int_a^x f(s)\, ds = 0$ for some $x \in (a, b]$ then $f(s) = 0$ for all $s \in [a, x]$. Thus we can suppose that $\int_a^x f(s)\, ds > 0$ for all $x \in (a, b]$. Otherwise, we can find $a_1 \in (a, b)$ such that $\int_a^x f(s)\, ds = 0$ for $x \in [a, a_1]$ and $\int_a^x f(s)\, ds > 0$ for $x \in (a_1, b)$ and thus we only need to consider f on $[a_1, b]$.

(a) Assume that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$.

(1) $n = 1$. By CMVT, for every $x \in (a, b]$, there exists a $b_1 \in (a, x)$ such that

$$\frac{(f(x))^2}{2\int_a^x f(s)\, ds} = \frac{2f(b_1) f'(b_1)}{2f(b_1)} = f'(b_1) \geq 1.$$

Thus (a) is true for $n = 1$.

(2) Let (a) be true for $n = k > 1$. We prove that (a) is true for $n = k + 1$. It then follows by mathematical induction that (a) is true for $n = 1, 2, ...$ By CMVT, for every $x \in (a, b]$, there exists a $b_1 \in (a, x)$ such that

$$\begin{aligned}
\frac{(f(x))^{k+2}}{(k+2)\left(\int_a^x f(s)\,ds\right)^{k+1}} &= \frac{1}{(k+2)}\left(\frac{(f(x))^{\frac{k+2}{k+1}}}{\int_a^x f(s)\,ds}\right)^{k+1} \\
&= \frac{1}{(k+2)}\left(\frac{\frac{k+2}{k+1}(f(b_1))^{\frac{1}{k+1}} f'(b_1)}{f(b_1)}\right)^{k+1} \\
&= \frac{(k+2)^k}{(k+1)^{k+1}} \frac{(f'(b_1))^{k+1}}{(f(b_1))^k} \\
&= \frac{\left(\left(\frac{k+2}{k+1}\right)^k f'(b_1)\right)^{k+1}}{(k+1)\left(\int_a^{b_1}\left(\frac{k+2}{k+1}\right)^k f'(s)\,ds\right)^k} \geq 1.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{d^k}{dx^k}\left[\left(\frac{k+2}{k+1}\right)^k f'(x)\right] &= \left(\frac{k+2}{k+1}\right)^k f^{(k+1)}(x) \\
&\geq \left(\frac{k+2}{k+1}\right)^k \frac{(k+1)!}{(k+2)^k} \\
&= \frac{k!}{(k+1)^{k-1}}
\end{aligned}$$

for $x \in (a, b)$, by the induction assumption that (a) is true for $n = k$.

Thus (a) is true for $n = 1, 2, ...$

(b) The proof of the second part is similar so we leave out the details.

The proof of the lemma is completed. ■

We now ready to prove the conjecture as a theorem.

Theorem 5.13 (Conjecture in [34]) *Suppose $n \in \mathbb{Z}^+$. Assume $f(x)$ has derivative of the n-th order on the interval $[a, b]$ such that $f^{(i)}(a) = 0$ for $i = 0, 1, 2, ..., n-1$.*

(a) *If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$, then the inequality (2.7) is valid.*

(b) *If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$, then the inequality (2.7) is reversed.*

Proof. As in the proof of Lemma 5.12, we can suppose that $\int_a^x f(s)\,ds > 0$ for any $x \in (a, b]$.

(a) Assume that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$. By CMVT and Lemma 5.12, in case (a), there exists a $b_1 \in (a,x)$ such that

$$\frac{\int_a^b [f(x)]^{n+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)\,dx\right]^{n}} \geq 1.$$

So (a) is proved.

(b) Assume that $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$. By CMVT and Lemma 5.12, in case (b), there exists a $b_1 \in (a,x)$ such that

$$\frac{\int_a^b [f(x)]^{n+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)\,dx\right]^{n}} \leq 1.$$

The proof of the conjecture is completed. ■

As the proofs show, we actually have the following slightly stronger result which is a generalization of Proposition 1.1 in [16] (which we present as Theorem 2.3 in Section 2.2), Theorem 4 and Theorem 5 in [34] (which we present as Theorem 5.2 and Theorem 5.3 in Section 5.1).

Theorem 5.14 ([35], p.4) *Suppose $n \in \mathbb{Z}^+$. Assume $f(x)$ has derivative of the n-th order on the interval (a,b) and $f^{(n-1)}(x)$ is continuous on $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,...,n-1$.*

(a) *If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then the inequality (2.7) is valid.*

(b) *If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then the inequality (2.7) is reversed.*

Remark 5.4 *For a different proof for the Theorem 5.13 (Y. Chen and J. Kimball's Conjecture in [34]) see Theorem 6.3 and its proof in Section 6.2*

5.5 WORK OF P. YAN AND M. GYLLENBERG ANSWERING THE PROBLEMS 1 AND 3

In the joint paper [36], P. Yan and M. Gyllenberg gave new answers to Qi's problem, which is Problem 1 (Remark 2.1), and some new results concerning the integral inequality (5.27), that is

$$\int_a^b [f(x)]^{p+2}\,dx \geq \left[\int_a^b f(x)\,dx\right]^{p+1}, \tag{5.27}$$

and its reversed form, which are modified forms of Problem 1 (Remark 2.1) and Problem 3 (Remark 2.4) as $p = t - 2$, which extend related results in the references such as [16, 31, 34, 35] which we have already investigated in detail above. The following result is a generalization of Theorems 3, 4 and 5 in [34] (which we present as Theorem 5.1, Theorem 5.2 and Theorem 5.3 in Section 5.1), Proposition 1.1 in [16] (which we present as Theorem 2.3 in Section 2.2) and Theorem 2 in [35] (which we present as Theorem 5.13 in Section 5.4).

The following theorem gives an affirmative answer to Problem 1 (Remark 2.1) and Problem 3 (Remark 2.4).

Theorem 5.15 ([36], p.2) *Suppose that k is a non-negative integer and p is a positive number such that $p > k$. Assume that $f(x)$ has a derivative of the $(k+1)$-th order on the interval (a, b) such that $f^{(k)}(x)$ is continuous on $[a, b]$, $f(x)$ is non-negative on $[a, b]$ and $f^{(i)}(a) = 0$ for $i = 0, 1, 2, ..., k$.*

(a) *If*

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \qquad x \in (a, b)\,,$$

then inequality (5.27) is valid.

(b) *If*

$$0 \leq \left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \leq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \qquad x \in (a, b)\,,$$

then inequality (5.27) is reversed.

Proof. First notice that if $f \equiv 0$ on $[a, b]$, then Theorem 5.15 is clear. Assume f is not identically 0 on $[a, b]$. If $\int_a^x f(s)\,ds = 0$ for some $x \in (a, b]$ then $f(s) = 0$ for all $s \in [a, x]$, because $f(x)$ is non-negative on $[a, b]$. Hence we can suppose that $\int_a^x f(s)\,ds > 0$ for all $x \in (a, b]$. (Otherwise, we can find $a_1 \in (a, b)$ such that $\int_a^x f(s)\,ds = 0$ for $x \in [a, a_1]$ and $\int_a^x f(s)\,ds > 0$ for $x \in (a_1, b)$ and thus we only need to consider f on $[a_1, b]$).

(a) Assume that

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \qquad x \in (a, b)\,.$$

(1) $k = 0 < p$. By Cauchy's mean value theorem (CMVT), which is given as Theorem 1.4 in Section 1.1, by using CMVT twice, there exist $a < b_2 < b_1 < b$ such that

$$\begin{aligned}
\frac{\int_a^b (f(x))^{p+2}\,dx}{\left(\int_a^b f(x)\,dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)\,dx\right)^p} \\
&= \frac{1}{p+1}\left(\frac{(f(b_1))^{\frac{p+1}{p}}}{\int_a^{b_1} f(x)\,dx}\right)^p \\
&= \frac{(p+1)^{p-1}}{p^p}\frac{(f'(b_2))^p}{(f(b_2))^{p-1}} \geq 1,
\end{aligned}$$

since $\left(f^{\frac{1}{p}}\right)'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$.

Thus (a) is true for $k = 0$.

(2) $k = 1 < p$. By using CMVT three times, there exist $a < b_3 < b_2 < b_1 < b$ such that

$$\begin{aligned}
\frac{\int_a^b (f(x))^{p+2}\,dx}{\left(\int_a^b f(x)\,dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)\,dx\right)^p} \\
&= \frac{1}{p+1}\left(\frac{(f(b_1))^{\frac{p+1}{p}}}{\int_a^{b_1} f(x)\,dx}\right)^p \\
&= \frac{(p+1)^{p-1}}{p^p}\left(\frac{(f'(b_2))^{\frac{p}{p-1}}}{f(b_2)}\right)^{p-1} \\
&= \frac{(p+1)^{p-1}}{p^p}\left(\frac{p}{p-1}\right)^{p-1}\frac{(f''(b_3))^{p-1}}{(f'(b_3))^{p-2}} \\
&\geq 1,
\end{aligned}$$

since

$$\left((f')^{\frac{1}{p-1}}\right)'(x) \geq \left(\frac{p}{(p+1)^{p-1}}\right)^{\frac{1}{p-1}}, \qquad x \in (a,b).$$

Thus (a) is true for $k = 1 < p$.

(3) $1 < k < p$. By using CMVT $(k+2)$ times and mathematical induction, there exist $a < b_{k+2} < \cdots < b_1 < b$ such that

$$\begin{aligned}
\frac{\int_a^b (f(x))^{p+2}\,dx}{\left(\int_a^b f(x)\,dx\right)^{p+1}} &= \frac{1}{p+1}\frac{(f(b_1))^{p+1}}{\left(\int_a^{b_1} f(x)\,dx\right)^{p}} \\
&= \frac{(p+1)^{p-1}}{p^p}\frac{(f'(b_2))^{p}}{(f(b_2))^{p-1}} \\
&= \cdots \\
&= \frac{(p+1)^{p-1}}{p(p-1)\cdots(p-k+1)(p-k)^{p-k}}\frac{\left(f^{(k+1)}(b_{k+2})\right)^{p-k}}{\left(f^{(k)}(b_{k+2})\right)^{p-k-1}} \\
&\geq 1,
\end{aligned}$$

since

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b).$$

Thus (a) is true for $0 \leq k < p$.

(b) The proof of the second part can be done similarly.

The proof of Theorem 5.15 is completed. ■

Remark 5.5 ([36], p.4) *If $p = 1$ and $k = 0$, then Theorem 5.15 is reduced to Proposition 1.1 in [16]. (See also Theorem 2.3 in Section 2.2). If $p = k + 1$, then*

$$\left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}} = \frac{(k+1)!}{(k+2)^k}.$$

In [35] Y. Chen and J. Kimball proved the conjecture in [34] (i.e., Theorem 2 in [35]). (See also Theorem 5.13 in Section 5.4). It is clear that Theorem 5.15 is a generalization of Proposition 1.1 in [16] (which we present as Theorem 2.3 in Section 2.2), Theorem 3, Theorem 4 and Theorem 5 in [34] (which we present as Theorem 5.1, Theorem 5.2 and Theorem 5.3 in Section 5.1), and Theorem 2 in [35] (which we present as Theorem 5.13 in Section 5.4).

The following result is a generalization of Theorem 2 in [31] (which we present as Theorem 4.4 in Section 4.2) and Proposition 1.3 in [16] (which we present as Theorem 2.4 in Section 2.2). It gives an affirmative answer to Problem 1. (Remark 2.1).

Theorem 5.16 ([36], p.4) *Suppose that k is a non-negative integer and suppose that p is a positive number such that $p > k$. Assume that $f(x)$ has a derivative of the $(k+1)$-th order on the interval (a,b) such that $f^{(k)}(x)$ is continuous on $[a,b]$, $f^{(i)}(a) = 0$ for $i = 0,1,2,\ldots,k-1$, and $f^{(k)}(a) \geq 0$. If*

$$f^{(k+1)}(x) \geq \frac{(k+1)!\binom{p}{k+1}(p-k+1)^{p-k-1}}{(p+1)^{p-1}}(x-a)^{p-k-1}$$

for $x \in (a,b)$, then inequality (5.27) is valid.

Proof. As in the proof of Theorem 5.15 we can suppose that $\int_a^x f(s)\,ds > 0$ for all $x \in (a,b]$.

(a) $k = 0$. By using CMVT three times, there exist $a < b_3 < b_2 < b_1 < b$ such that

$$\begin{aligned}
\frac{\int_a^b (f(x))^{p+2}\,dx}{\left(\int_a^b f(x)\,dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)\,dx\right)^p} \\
&\geq \frac{(f(b_2))^{p-1} f'(b_2)}{p\left(\int_a^{b_2} f(x)\,dx\right)^{p-1}} \\
&\geq \frac{(f(b_2))^{p-1}(b_2-a)^{p-1}}{\left(\int_a^{b_2} f(x)\,dx\right)^{p-1}},
\end{aligned}$$

since $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$. Then

$$\begin{aligned}
\frac{(f(b_2))^{p-1}(b_2-a)^{p-1}}{\left(\int_a^{b_2} f(x)\,dx\right)^{p-1}} &= \left(\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)\,dx}\right)^{p-1} \\
&= \left(1 + \frac{f'(b_3)(b_3-a)}{f(b_3)}\right)^{p-1} \geq 1.
\end{aligned}$$

Thus Theorem 5.16 is true for $k = 0$.

(b) $k = 1 < p$. By using CMVT four times, there exist $a < b_4 < b_3 < b_2 < b_1 < b$ such that

$$\begin{aligned}
\frac{\int_a^b (f(x))^{p+2}\,dx}{\left(\int_a^b f(x)\,dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)\,dx\right)^p} \\
&= \frac{1}{(p+1)}\left(\frac{(f(b_1))^{\frac{p+1}{p}}}{\int_a^{b_1} f(x)\,dx}\right)^p
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(p+1)^{p-1}}{p^p} \frac{(f'(b_2))^p}{\left(\int_a^{b_2} f'(x)\,dx\right)^{p-1}} \\
&\geq \frac{(p+1)^{p-1}}{p^{p-1}} \frac{1}{p-1} \frac{(f'(b_3))^{p-2} f''(b_3)}{\left(\int_a^{b_3} f'(x)\,dx\right)^{p-2}} \\
&\geq \left(\frac{f'(b_3)(b_3-a)}{\int_a^{b_3} f'(x)\,dx}\right)^{p-2},
\end{aligned}
$$

since

$$
f''(x) \geq (p-1)\left(\frac{p}{p+1}\right)^{p-1} (x-a)^{p-2}, \qquad x \in (a,b).
$$

Then

$$
\left(\frac{f'(b_3)(b_3-a)}{\int_a^{b_3} f'(x)\,dx}\right)^{p-2} = \left(\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)\,dx}\right)^{p-1} = \left(1 + \frac{f''(b_4)(b_4-a)}{f'(b_4)}\right)^{p-2} \geq 1.
$$

Thus Theorem 5.16 is true for $k=1$.

(c) $1 < k < p$. By using CMVT $(k+3)$ times, there exist $a < b_{k+3} < \cdots < b_1 < b$ such that

$$
\begin{aligned}
\frac{\int_a^b (f(x))^{p+2}\,dx}{\left(\int_a^b f(x)\,dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)\,dx\right)^p} \\
&= \frac{(p+1)^{p-1}}{p^p} \frac{(f'(b_2))^p}{\left(\int_a^{b_2} f'(x)\,dx\right)^{p-1}} \\
&= \cdots \\
&= \frac{(p+1)^{p-1}}{p(p-1)\cdots(p-k+2)(p-k+1)^{p-k+1}} \\
&\quad \times \frac{\left(f^{(k)}(b_{k+1})\right)^{p-k+1}}{\left(\int_a^{b_{k+1}} f^{(k)}(x)\,dx\right)^{p-k}} \\
&\geq \frac{(p+1)^{p-1}}{p(p-1)\cdots(p-k+2)(p-k+1)^{p-k}(p-k)} \\
&\quad \times \frac{\left(f^{(k)}(b_{k+2})\right)^{p-k-1} f^{(k+1)}(b_{k+2})}{\left(\int_a^{b_{k+2}} f^{(k)}(x)\,dx\right)^{p-k-1}} \\
&\geq \left(\frac{f^{(k)}(b_{k+2})(b_{k+2}-a)}{\int_a^{b_{k+2}} f^k(x)\,dx}\right)^{p-k-1},
\end{aligned}
$$

since

$$f^{(k+1)}(x) \geq \frac{(k+1)!\binom{p}{k+1}(p-k+1)^{p-k-1}}{(p+1)^{p-1}}(x-a)^{p-k-1}, \qquad x \in (a,b).$$

Then

$$\left(\frac{f^{(k)}(b_{k+2})(b_{k+2}-a)}{\int_a^{b_{k+2}} f^k(x)\,dx}\right)^{p-k-1} = \left(1+\frac{f^{(k+1)}(b_{k+3})(b_{k+3}-a)}{f^{(k)}(b_{k+3})}\right)^{p-k-1} \geq 1.$$

The proof of Theorem 5.16 is completed. ∎

Remark 5.6 ([36], p.6) *If $k=0$, Theorem 5.16 is reduced to Theorem 2 in [31] (which we present as Theorem 4.4 in Section 4.2). If $k=0$ and $p=n$, then*

$$\frac{(k+1)!\binom{p}{k+1}(p-k+1)^{p-k-1}}{(p+1)^{p-1}}(x-a)^{p-k-1} = n(x-a)^{n-1}.$$

It follows that Proposition 1.3 in [16] (which we present as Theorem 2.4 in Section 2.2) is a corollary of Theorem 5.16. In fact, suppose f satisfy the conditions of Proposition 1.3 in [16]. Since $f^{(n)}(x) \geq n!$, successively integrating $n-1$ times over $[a,x]$, we obtain $f'(x) \geq n(x-a)^{n-1}$ for $x \in (a,b)$. Thus, Theorem 5.16 is a generalization of Proposition 1.3 in [16] and Theorem 2 in [31].

CHAPTER 6

PAPERS FROM 2007 THROUGH 2009

In this chapter, we present some papers on Qi type inequalities from 2007 through 2009.
Papers answering the Problem 5 (Remark 2.6):

- In 2007, work of Q. A. Ngô and P. H. Tung (see Section 6.2).

- In 2008, work of W. T. Sulaiman (see Section 6.3).

Paper answering the Problem 9 (Remark 2.10):

- In 2007, work of Y. Hong (see Section 6.1).

Paper answering the Problem 11 (Remark 2.12):

- In 2008, work of W. T. Sulaiman (see Section 6.3).

Paper answering the Problem 12 (Remark 2.13):

- In 2008, work of W. T. Sulaiman (see Section 6.3).

Paper answering the Problem 13 (Remark 2.14):

- In 2009, work of W. Liu et al. (see Section 6.4).

Paper answering the Problem 14 (Remark 2.15):

- In 2009, work of W. Liu et al. (see Section 6.4).

6.1 WORK OF Y. HONG ANSWERING THE PROBLEM 9

In the paper [40], Y. Hong proposed his main result. It gives an affirmative answer to Problem 9 (Remark 2.10) which we present here.

Theorem 6.1 ([40], p.1244) *Assume* $\alpha > \beta \geq 2$, $m = [\beta]$, $f(x) \in C^1[a,b]$, $f'(x) \geq f(x) \geq 0$ *and* $\left[f^{\alpha-\beta}(x)\right]' \geq (\alpha-\beta)\frac{\beta(\beta-1)\cdots(\beta-m+1)}{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}(x-a)^{\beta-m}$. *Then*

$$\int_a^b [f(x)]^\alpha \, dx \geq \left(\int_a^b f(x)\, dx\right)^\beta . \tag{6.1}$$

Where $[\beta]$ *denote the integer part of* β.

Proof. Since $f'(x) \geq 0$, $f(x)$ is a increasing function on $[a,b]$. Suppose

$$F(x) = \int_a^x [f(u)]^\alpha \, du - \left(\int_a^x f(u)\, du\right)^\beta, \quad x \in [a,b].$$

Then $F(a) = 0$, and we get

$$\begin{aligned} F'(x) &= [f(x)]^\alpha - \beta\left(\int_a^x f(u)\, du\right)^{\beta-1} f(x) \\ &= f(x)\left\{[f(x)]^{\alpha-1} - \beta\left(\int_a^x f(u)\, du\right)^{\beta-1}\right\} \triangleq f(x)\, F_1(x), \end{aligned}$$

where

$$F_1(x) = [f(x)]^{\alpha-1} - \beta\left(\int_a^x f(u)\, du\right)^{\beta-1}.$$

Clearly, $F_1(a) = [f(a)]^{\alpha-1} \geq 0$ and

$$\begin{aligned} F_1'(x) &= (\alpha-1)[f(x)]^{\alpha-2} f'(x) - \beta(\beta-1)\left(\int_a^x f(u)\, du\right)^{\beta-2} f(x) \\ &\geq (\alpha-1)[f(x)]^{\alpha-1} - \beta(\beta-1)\left(\int_a^x f(u)\, du\right)^{\beta-2} f(x) \\ &= f(x)\left\{(\alpha-1)[f(x)]^{\alpha-2} - \beta(\beta-1)\left(\int_a^x f(u)\, du\right)^{\beta-2}\right\} \\ &\triangleq f(x)\, F_2(x), \end{aligned}$$

where

$$F_2(x) = (\alpha-1)[f(x)]^{\alpha-2} - \beta(\beta-1)\left(\int_a^x f(u)\, du\right)^{\beta-2}.$$

Clearly, $F_2(a) = (\alpha-1)[f(a)]^{\alpha-2} \geq 0$ and

$$\begin{aligned}
F_2'(x) &= (\alpha-1)(\alpha-2)[f(x)]^{\alpha-3} f'(x) - \beta(\beta-1)(\beta-2)\left(\int_a^x f(u)\,du\right)^{\beta-3} f(x) \\
&\geq (\alpha-1)(\alpha-2)[f(x)]^{\alpha-2} - \beta(\beta-1)(\beta-2)\left(\int_a^x f(u)\,du\right)^{\beta-3} f(x) \\
&= f(x)\left\{(\alpha-1)(\alpha-2)[f(x)]^{\alpha-3} - \beta(\beta-1)(\beta-2)\left(\int_a^x f(u)\,du\right)^{\beta-3}\right\} \\
&\triangleq f(x)\,F_3(x).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
F_{m-1}(x) &= (\alpha-1)(\alpha-2)\cdots(\alpha-m+2)[f(x)]^{\alpha-m+1} \\
&\quad -\beta(\beta-1)(\beta-2)\cdots(\beta-m+2)\left(\int_a^x f(u)\,du\right)^{\beta-m+1}.
\end{aligned}$$

Clearly, $F_{m-1}(x) = (\alpha-1)(\alpha-2)\cdots(\alpha-m+2)[f(x)]^{\alpha-m+1} \geq 0$ and

$$\begin{aligned}
F_{m-1}'(x) &= (\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f(x)]^{\alpha-m} f'(x) \\
&\quad -\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)\left(\int_a^x f(u)\,du\right)^{\beta-m} f(x) \\
&\geq (\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f(x)]^{\alpha-m} f'(x) \\
&\quad -\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)[f(x)(x-a)]^{\beta-m} f(x) \\
&= [f(x)]^{\beta-m+1}\left\{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f(x)]^{\alpha-\beta-1} f'(x)\right\} \\
&\quad -[f(x)]^{\beta-m+1}\left\{\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)(x-a)^{\beta-m}\right\} \\
&= [f(x)]^{\beta-m+1}\left\{\frac{1}{\alpha-\beta}(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)\left[f^{\alpha-\beta}(x)\right]'\right\} \\
&\quad -[f(x)]^{\beta-m+1}\left\{\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)(x-a)^{\beta-m}\right\}.
\end{aligned}$$

Since

$$\left[f^{\alpha-\beta}(x)\right]' \geq (\alpha-\beta)\frac{\beta(\beta-1)\cdots(\beta-m+1)}{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}(x-a)^{\beta-m},$$

we have $F_{m-1}'(x) \geq 0$ for $x \in [a,b]$, so $F_{m-1}(x)$ is increasing on $[a,b]$, thus $F_{m-1}(x) \geq F_{m-1}(a) \geq 0$. It follows that $F_{m-2}'(x) \geq 0$, hence $F_{m-2}(x)$ is increasing on $[a,+\infty)$, i.e. $F_{m-2}(x) \geq F_{m-2}(a) \geq 0$. We obtain $F_1(x) \geq 0$, so $F'(x) \geq 0$. Therefore $F(x)$ is increasing on $[a,b]$, i.e. $F(b) \geq F(a) \geq 0$, That is

$$\int_a^b [f(x)]^\alpha\,dx \geq \left(\int_a^b f(x)\,dx\right)^\beta. \blacksquare$$

Corollary 6.1 ([40], p.1245) *Assume* $t \geq 3$, $m = [t-1]$, $f(x) \in C^1[a,b]$ *on the interval* $[a,b]$, $f'(x) \geq f(x) \geq 0$ *and* $f'(x) \geq (t-m)(x-a)^{t-m-1}$. *Then*

$$\int_a^b [f(x)]^t \, dx \geq \left(\int_a^b f(x) \, dx \right)^{t-1}.$$

Proof. Put $\alpha = t$, $\beta = t-1$ in (6.1). ■

Corollary 6.2 ([40], p.1246) *Assume* $n, k \in \mathbb{Z}_+$, $k \geq 2$, $f(x) \in C^1[a,b]$, $f'(x) \geq f(x) \geq 0$ *and* $\left[f^k(x)\right]' \geq k\frac{k!n!}{(k+n-1)!}$. *Then*

$$\int_a^b [f(x)]^{n+k} \, dx \geq \left(\int_a^b f(x) \, dx \right)^{n}.$$

Proof. Put $\alpha = n+k$, $\beta = k$ in (6.1). ■

Corollary 6.3 ([40], p.1246) *Assume* $n, k \in \mathbb{Z}_+$, $f(x) \in C^1[a,b]$ *on the interval* $[a,b]$, $f'(x) \geq f(x) \geq 0$ *and* $f'(x) \geq 1$. *Then*

$$\int_a^b [f(x)]^{n+2} \, dx \geq \left(\int_a^b f(x) \, dx \right)^{n+1}.$$

Proof. Put $\alpha = n+2$, $\beta = n+1$ in (6.1). ■

Example 6.1 ([40], p.1246) *Assume* $n \in \mathbb{Z}_+$, $f(x) = e^x$. *Then* $f'(x) = e^x = f(x) > 0$ *and* $f'(x) = e^x \geq 1$ *for* $x \in [0,1]$. *By Corollary 6.1, we have*

$$\int_0^1 (e^x)^{n+2} \, dx \geq \left(\int_0^1 e^x dx \right)^{n+1}. \tag{6.2}$$

Hence

$$e^{n+2} - 1 \geq (n+2)(e-1)^{n+1}, \quad n \in \mathbb{Z}_+.$$

Remark 6.1 ([40], p.1246) *If* $n \geq 3$, $t = n+2$, $f(x) = e^x \, (x \in [0,1])$, *then* $f(x)$ *doesn't satisfy the condition in [31] (see Section 4.2):* $f'(x) \geq (t-2)(x-a)^{t-3}$ *i.e.* $e^x \geq nx^{n-1}$; $f(x)$ *doesn't also satisfy the condition in [16] (see Section 2.2):* $f^{(n)}(x) \geq n!$ *i.e.* $e^x \geq n!$. *Hence (6.2) can't be obtained by the theorems in [16, 31].*

6.2 WORK OF Q. A. NGÔ AND P. H. TUNG ANSWERING THE PROBLEM 5

In [34], Chen and Kimball proposed a theorem which is given as Theorem 5.2 in Section 5.1. After proving the Theorem 5.2, Chen and Kimball proposed a conjecture (see Remark 5.2). The conjecture is that the monotony assumption of Theorem 5.2 could be dropped. In the joint paper [37], Q. A. Ngô and P. H. Tung proved that this conjecture holds which we present in this section. They used the same technique which was introduced by Qi in [16]. See also for another proof, In [36], P. Yan and G. Gyllenberg proved Y. Chen and J. Kimball's conjecture by mathematical induction that the conjecture is true (see Theorem 5.13 in Section 5.4 which we presented its proof there).
Q. A. Ngô and P. H. Tung considered the case $n = 2$ for inequality 2.7 in Problem 5 (Remark 2.6) as the first step in the process.

Lemma 6.2 ([37], p.2) *Assume $f(x)$ has continuous derivative of the 2-nd order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$, where $i = 0, 1$, and $f^{(2)}(x) \geq \frac{2}{3}$, then*

$$\int_a^b f^4(x)dx \geq \left(\int_a^b f(x)dx\right)^3 . \tag{6.3}$$

Proof. Since $f^{(2)}(x) \geq \frac{2}{3} > 0$ then f' is (strictly) increasing in $[a,b]$. It follows from $f'(a) = 0$ that $f'(x) > f'(a) = 0$ for every $a < x \leq b$. Thus f is also increasing in $[a,b]$. Let

$$H(x) = \int_a^x f^4(x)dx - \left(\int_a^x f(x)dx\right)^3 , \; x \in [a,b] .$$

Straightforward computation yields

$$H'(x) = \left(f^3(x) - 3\left(\int_a^x f(x)dx\right)^2\right) f(x) =: h_1(x)f(x),$$

which produces

$$h_1'(x) = 3\left(f(x)f'(x) - 2\int_a^x f(x)dx\right) f(x) =: 3h_2(x)f(x).$$

It follows that

$$h_2'(x) = (f'(x))^2 + f(x)f''(x) - 2f(x)$$

and

$$h_2'(x) = (f'(x))^2 + f(x)f''(x) - 2f(x) \geq (f'(x))^2 + \left(\frac{2}{3} - 2\right) f(x) =: h_3(x).$$

Hence

$$h_3'(x) = 2f'(x)f''(x) - \frac{4}{3}f'(x) \geq 2f'(x)\left(f''(x) - \frac{2}{3}\right) \geq 0.$$

Thus $h_3(x)$, $h_2(x)$ and $h_1(x)$ are increasing and then $H(x)$ is also increasing. Therefore $H(b) \geq H(a) = 0$ which completes this proof. ■

Now we state Q. A. Ngô and P. H. Tung's main result which gives an affirmative answer to Problem 5 (Remark 2.6).

Theorem 6.3 ([37], p.3) *Suppose that n is a positive integer. Assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$, where $0 \leq i \leq n-1$, and $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$, then*

$$\int_a^b f^{n+2}(x)dx \geq \left(\int_a^b f(x)dx\right)^{n+1}. \tag{6.4}$$

Proof. Letting

$$g(x) = \frac{(n+1)^{n-1}}{n!} f(x),$$

it can be easily seen that $g^{(n)}(x) \geq 1$ for all x.

The problem now is to prove that the inequality below is true

$$\int_a^b g^{n+2}(x)dx \geq \frac{(n+1)^{n-1}}{n!}\left(\int_a^b g(x)dx\right)^{n+1}.$$

Let

$$G(x) = \int_a^x g^{n+2}(t)dt - \frac{(n+1)^{n-1}}{n!}\left(\int_a^x g(t)dt\right)^{n+1}.$$

Direct calculation produces that

$$G'(x) = g(x)\left(g^{n+1}(x) - \frac{(n+1)^n}{n!}\left(\int_a^x g(t)dt\right)^n\right) = g(x)g_1(x).$$

We will show $g_1(x) \geq 0$ by induction. According to Lemma 6.2, the case $n = 2$ is proved. Denote

$$g_2(x) = g^{\frac{n+1}{n}}(x) - \frac{(n+1)}{\sqrt[n]{n!}}\int_a^x g(t)dt.$$

It can be easily seen that the function $h(x) := g'(x)$ satisfies the following conditions

(a) $h^{(k)}(a) = 0$ for all $k \leq n-2$, and

(b) $h^{(n-1)}(x) \geq 1$.

So, by induction

$$h^n(x) \geq \frac{n^{n-1}}{(n-1)!}\left(\int_a^x h(t)dt\right)^{n-1}$$

or equivalently

$$g'(x) \geq \sqrt[n]{\frac{n^{n-1}}{(n-1)!}} g^{\frac{n-1}{n}}(x).$$

Therefore,

$$\frac{n+1}{n} g^{\frac{1}{n}}(x) g'(x) \geq \frac{n+1}{n}\sqrt[n]{\frac{n^{n-1}}{(n-1)!}} g(x).$$

Hence,

$$g^{\frac{n+1}{n}}(x) \geq \frac{n+1}{n}\sqrt[n]{\frac{n^{n-1}}{(n-1)!}} \int_a^x g(x)dx.$$

Then, the conclusion $g_2(x) \geq 0$ follows from the fact that

$$\frac{n+1}{n}\sqrt[n]{\frac{n^{n-1}}{(n-1)!}} = \frac{n+1}{\sqrt[n]{n!}},$$

which produces $g_1(x) \geq 0$. Then $G(x) \geq 0$. The proof of Theorem 6.3 is completed. ■

6.3 WORK OF W. T. SULAIMAN ANSWERING THE PROBLEMS 5, 11 AND 12

In the paper [41], W. T. Sulaiman gave an affirmative answer to Problem 5 (Remark 2.6), Problem 11 (Remark 2.12) and Problem 12 (Remark 2.13) which we present here.

Theorem 6.4 ([41], p.891) *Let f be positive and has continuous 2nd derivative on the interval $[a.b]$ such that $f(a) = 0$, $f'(a) = 0$, and suppose $\gamma > \alpha > 0$, $\beta > 1$, $\beta(\alpha+1) > (\gamma+1)$. If*

$$\frac{f(t) f''(t)}{(f'(t))^2} \geq \beta(\alpha+1) - (\gamma+1), \tag{6.5}$$

then

$$\int_a^b f^\gamma(x)\,dx \geq \left(\int_a^b f^\alpha(x)\,dx\right)^\beta. \tag{6.6}$$

If the inequality (6.5) reverses, then the inequality (6.6) reverses as well.

Proof. By the hypothesis, $f''(t) > 0$, then $f'(t)$ is increasing and thus $f'(t) > f'(a) = 0$. Set

$$F(t) = \int_a^t f^{\gamma}(x)\,dx - \left(\int_a^t f^{\alpha}(x)\,dx\right)^{\beta}.$$

We get

$$F'(t) = f^{\gamma}(t) - \beta f^{\alpha}(t)\left(\int_a^t f^{\alpha}(x)\,dx\right)^{\beta-1} = 0,$$

if

$$\beta\left(\int_a^t f^{\alpha}(x)\,dx\right)^{\beta-1} = f^{\gamma-\alpha}(t). \tag{6.7}$$

$$\begin{aligned} F''(t) &= \gamma f^{\gamma-1}(t)\,f'(t) - \beta(\beta-1)\,f^{2\alpha}(t)\left(\int_a^t f^{\alpha}(x)\,dx\right)^{\beta-2} \\ &\quad -\alpha\beta f^{2\alpha-1}(t)\,f'(t)\left(\int_a^t f(x)\,dx\right)^{\beta-1}. \end{aligned}$$

Now, for t satisfying equality (6.7), we get

$$\begin{aligned} F''(t) &= (\gamma-\alpha)\,f^{\gamma-1}(t)\,f'(t) - (\beta-1)\,f^{\alpha+\gamma}(t)\left(\int_a^t f^{\alpha}(x)\,dx\right)^{-1} \\ &= \frac{(\gamma-\alpha)\,f^{\gamma-1}(t)\,f'(t)}{\int_a^t f^{\alpha}(x)\,dx}\left(\int_a^t f^{\alpha}(x)\,dx - \frac{(\beta-1)\,f^{\alpha+1}(t)}{(\gamma-\alpha)\,f'(t)}\right). \end{aligned}$$

If we set $G(t)$ for the quantity in the brackets above, we get

$$\begin{aligned} G'(t) &= f^{\alpha}(t) - \frac{(\beta-1)}{(\gamma-\alpha)}\left((\alpha+1)\,f^{\alpha}(t) - \frac{f^{\alpha+1}(t)\,f''(t)}{(f'(t))^2}\right) \\ &= f^{\alpha}(t)\left(1 - \frac{(\beta-1)}{(\gamma-\alpha)}\left(\alpha+1-\frac{f(t)\,f''(t)}{(f'(t))^2}\right)\right) \geq 0, \end{aligned}$$

by the hypothesis.

Hence g is nondecreasing. But $G(a) = 0$, then $G(t) \geq 0$. It follows now that $F''(t) \geq 0$. That is F attains its minimum when $t = a$. But $F(a) \geq 0$, then $F(t) \geq 0$. The proof is now completed. ■

Remark 6.2 ([41], p.892) *In Theorem 6.4, if f is convex, the theorem holds as $f''(t) \geq 0$. But if f is concave, that is $f''(t) \leq 0$, and nondecreasing, then the theorem holds if $\gamma + 1 > \beta(\alpha + 1)$.*

Theorem 6.5 ([41], p.893) *Suppose that n is a positive integer. Assume $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a)=0$, where $0 \leq i \leq n-1$. Suppose that α,β,γ are positive numbers such that $\alpha\beta > \gamma$. If*

$$\left(f^{(n)}(x)\right)^{\gamma-\alpha\beta} \geq \frac{(n\gamma+1)\,n!}{(n\alpha+1)\,\beta}(b-a)^{\beta(n\alpha+1)-(n\gamma+1)},$$

then inequality (6.6) holds true. In particular for $\gamma = n+2$, $\beta = n+1$, $\alpha = 1$, the following inequality,

$$\int_a^b f^{n+2}(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)^{n+1},$$

which is given as inequality (2.7) in Problem 5 (Remark 2.6), is valid.

Proof. Taylor's expansion applied to f with Lagrange remainder states that

$$\begin{aligned} f(x) &= f(a) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f(\xi)}{n!}(x-a)^n \qquad \text{for some } \xi \in (a,x) \\ &= \frac{f(\xi)}{n!}(x-a)^n. \end{aligned}$$

On substituting for $f(x)$ above in inequality (6.6), we have:

$$\int_a^b \left(\frac{f^{(n)}(\xi)}{n!}(x-a)^n\right)^{\gamma} dx \geq \left(\int_a^b \left(\frac{f^{(n)}(\xi)}{n!}(x-a)^n\right)^{\alpha} dx\right)^{\beta}.$$

Simplifying gives:

$$\left(f^{(n)}(\xi)\right)^{\gamma-\alpha\beta} \geq \frac{(n\gamma+1)\,n!}{(n\alpha+1)\,\beta}(b-a)^{\beta(n\alpha+1)-(n\gamma+1)}.$$

Hence in order to have inequality (6.6) satisfied, we obtain

$$\left(f^{(n)}(x)\right)^{\gamma-\alpha\beta} \geq \frac{(n\gamma+1)\,n!}{(n\alpha+1)\,\beta}(b-a)^{\beta(n\alpha+1)-(n\gamma+1)}. \blacksquare$$

6.4 WORK OF W. LIU, Q. A. NGÔ AND V. N. HUY ANSWERING THE PROBLEMS 13 AND 14

In the joint paper [42], W. Liu et al. studied the similar combination between inequality (2.3) as Qi's open problem in Problem 1 (Remark 2.1) and inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx, \tag{6.8}$$

where f, g are positive functions on $[a, b]$. Then, they studied some inequalities which is similar to inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b (x-a)^\alpha g^\beta(x)\,dx \tag{6.9}$$

or the inequality 6.8. In this section, we present some theorems coresponding the inequalities (6.8) and (6.9),after then we present their work. They also proposed some open problems in the end of their paper [42], which are given as Problem 15 (Remark 2.16), Problem 16 (Remark 2.17) and Problem 17 (Remark 2.18).

Now we present some theorems coresponding to the inequalities (6.8) and (6.9).

In [43], Q. A. Ngô et al. studied very interesting integral inequality and proved the following result

Theorem 6.6 ([42], p.202) *Suppose that $f(x)$ is a continous function on $[0, 1]$ satisfying*

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt, \qquad \forall x \in [0,1].$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x^\alpha f(x)\,dx \geq \frac{1}{\alpha+2}$$

and

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x f^\alpha(x)\,dx$$

are valid for every positive real number $\alpha > 0$.

Then, They, Q. A. Ngô et al., proposed the following open problem

Problem 6.1 ([42], p.202) *Under what conditions does the inequality*

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^\alpha f^\beta(x)\,dx \tag{6.10}$$

hold for α and β?

Theorem 6.6 is a generalization of Problem 2 (when $\alpha = 1$) of the 2nd International Mathematical Competition for University Students, Plovdiv, Bulgaria, 2–7 August, 1995. Shortly after the paper [43] was published, in [44], W. J. Liu gave an affirmative answer to the above Problem 6.1 and obtained the following result (see also Bougoffa [45], Boukerrioua and Guezane-Lakoud [46]).

Theorem 6.7 ([42], p.203) *Suppose that $f(x) \geq 0$ is a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f^{\beta}(t)\,dt \geq \int_x^1 t^{\beta} dt, \qquad \forall x \in [0,1].$$

Then the inequality (6.10) is valid for every positive real number $\alpha > 0$ and $\beta > 0$.

In [47], W. J. Liu obtained an extension of the Problem 6.1.

Theorem 6.8 ([42], p.203) *Suppose that $f(x) \geq 0$ is a continuous function on $[a,b]$ satisfying*

$$\int_x^b f^{\min\{1,\beta\}}(t)\,dt \geq \int_x^b (t-a)^{\min\{1,\beta\}}\,dt, \qquad \forall x \in [a,b].$$

Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b (x-a)^{\alpha} f^{\beta}(x)\,dx \tag{6.11}$$

is valid for every positive real number $\alpha > 0$ and $\beta > 0$.

It is obvious that inequality (6.11) is a special case of the following inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx$$

where f,g are positive functions on $[a,b]$.

In [48], N. S. Hoang obtained the following result. It is needed for the proof of Theorem 6.10.

Theorem 6.9 ([42], p.203) *Suppose $f \in L^1[a,b]$, $g \in C([a,b])$ such that $f(x)$, $g(x) \geq 0$ and g is non decreasing on $[a,b]$. If*

$$\int_x^b f(t)\,dt \geq \int_x^b g(t)\,dt, \qquad \forall x \in [a,b],$$

then inequality (6.8) is valid for every positive real number $\alpha > 0$ and $\beta > 0$ satisfying $\alpha + \beta \geq 1$.

The following theorem gives an affirmative answer to Problem 13 (Remark 2.14).

Theorem 6.10 ([42], p.203) *Suppose that $f(x)$, $g(x) \geq 0$ are continuous functions on $[a,b]$, g is non decreasing. Suppose that λ, α, β are positive constants with $\alpha+\beta \geq \lambda > 1$. If*

$$\int_x^b f(t)\,dt \geq \int_x^b g(t)\,dt, \qquad \forall x \in [a,b]$$

and

$$\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(t)\,dt \geq (b-a)^{\lambda-1},$$

then the following inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x)\,g^{\frac{\beta}{\lambda}}(x)\,dx\right)^{\lambda-1}$$

is valid.

Proof. By utilizing Theorem 1.1 in [24] which is given as Theorem 3.1 in Section 3.1, we conclude that

$$\int_a^b f^{\alpha+\beta}(x)\,dx = \int_a^b \left(f^{\frac{\alpha+\beta}{\lambda}}(x)\right)^{\lambda} dx \geq \left(\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(x)\,dx\right)^{\lambda-1}.$$

On the other hand, using Theorem 6.9, we obtain that

$$\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(x)\,dx \geq \int_a^b f^{\frac{\alpha}{\lambda}}(x)\,g^{\frac{\beta}{\lambda}}(x)\,dx.$$

Summing up the above two inequalities, we have the result. ■

The following theorem gives an affirmative answer to Problem 14 (Remark 2.15).

Theorem 6.11 ([42], p.204) *Suppose that $f(x)$, $g(x) \geq 0$ are continuous functions on $[a,b]$, g is non decreasing. Suppose that λ, γ, α, β are positive constant with $\alpha+\beta \geq \lambda > \gamma > 1$. If*

$$\int_x^b f(t)\,dt \geq \int_x^b g(t)\,dt, \qquad \forall x \in [a,b]$$

and

$$\left(f^{\frac{\alpha+\beta}{\lambda}\cdot\frac{\lambda-\gamma}{\gamma-1}}(x)\right)' \geq \frac{(\lambda-\gamma)\,\gamma^{\frac{1}{\gamma-1}}}{\lambda-1}, \qquad \forall x \in (a,b),$$

then the following inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x)\,g^{\frac{\beta}{\lambda}}(x)\,dx\right)^{\gamma}$$

is valid.

Proof. By utilizing Theorem 1.1 in [38] which is given as Theorem 5.5 in Section 5.2, we conclude that

$$\int_a^b f^{\alpha+\beta}(x)\,dx = \int_a^b \left(f^{\frac{\alpha+\beta}{\lambda}}(x)\right)^{\lambda} dx \geq \left(\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(x)\,dx\right)^{\gamma}.$$

Then we use the same proof as Theorem 6.10 to have the result. ■

Theorem 6.12 ([42], p.205) *Suppose that $f(x) > 0$ is a continuous and decreasing function on $[a,b]$. Then the inequality*

$$\frac{\int_a^b f^{\beta}(x)\,dx}{\int_a^b f^{\gamma}(x)\,dx} \geq \frac{\int_a^b (x-a)^{\alpha} f^{\beta}(x)\,dx}{\int_a^b (x-a)^{\alpha} f^{\gamma}(x)\,dx} \tag{6.12}$$

is valid for every positive real number $\alpha > 0$ and $\beta \geq \gamma > 0$. If $f(x)$ is increasing, then the inequality (6.12) reverses.

Proof. We just prove the case "$\geq$". For this purpose, we merely need to prove

$$\int_a^b f^{\beta}(x)\,dx \int_a^b (x-a)^{\alpha} f^{\gamma}(x)\,dx \geq \int_a^b f^{\gamma}(x)\,dx \int_a^b (x-a)^{\alpha} f^{\beta}(x)\,dx,$$

which is equivalent to

$$\int_a^b f^{\beta}(x)\,dx \int_a^b (y-a)^{\alpha} f^{\gamma}(y)\,dy \geq \int_a^b f^{\gamma}(x)\,dx \int_a^b (y-a)^{\alpha} f^{\beta}(y)\,dy.$$

That is to prove

$$\int_a^b \int_a^b f^{\gamma}(x) f^{\gamma}(y) (y-a)^{\alpha} \left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right) dxdy \geq 0.$$

Put

$$G = \int_a^b \int_a^b f^{\gamma}(x) f^{\gamma}(y) (y-a)^{\alpha} \left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right) dxdy.$$

So

$$G = \int_a^b \int_a^b f^{\gamma}(x) f^{\gamma}(y) (x-a)^{\alpha} \left(f^{\beta-\gamma}(y) - f^{\beta-\gamma}(x)\right) dxdy.$$

Thus

$$2G = \int_a^b \int_a^b f^{\gamma}(x) f^{\gamma}(y) \left((y-a)^{\alpha} - (x-a)^{\alpha}\right) \left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right) dxdy.$$

Since $f(x) \geq 0$ is a continuous and decreasing function on $[a,b]$, we get

$$\left((y-a)^{\alpha} - (x-a)^{\alpha}\right) \left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right) > 0$$

for all $x, y \in [a,b]$. Hence $2G \geq 0$. The proof is completed. ■

Remark 6.3 ([42], p.205) *The following special case of the Theorem 6.12 (for $f(x) \geq 0$ being a continuous and decreasing function on $[0,1]$)*

$$\frac{\int_0^1 f^2(x)\,dx}{\int_0^1 f(x)\,dx} \geq \frac{\int_0^1 xf^2(x)\,dx}{\int_0^1 xf(x)\,dx}$$

gives an answer to the test question 5 in the Mathematics Contest of Tsinghua University, China in 1985.

Similar to the method used in the proof of the foregoing theorem, the following three theorems are obtained.

Theorem 6.13 ([42], p.206) *Suppose that $f(x), g(x) \geq 0$ are continuous functions on $[a,b]$ such that $f(x)$ is decreasing and $g(x)$ is increasing. Then the inequality*

$$\frac{\int_a^b f^\beta(x)\,dx}{\int_a^b f^\gamma(x)\,dx} \geq \frac{\int_a^b g^\alpha(x)\,f^\beta(x)\,dx}{\int_a^b g^\alpha(x)\,f^\gamma(x)\,dx} \tag{6.13}$$

is valid for every positive real number $\alpha > 0$ and $\beta \geq \gamma > 0$. If $f(x)$ is increasing, then the inequality (6.13) reverses.

Proof. We just prove the case "$\geq$". For this purpose, we merely need to prove

$$\int_a^b f^\beta(x)\,dx \int_a^b g^\alpha(x)\,f^\gamma(x)\,dx \geq \int_a^b f^\gamma(x)\,dx \int_a^b g^\alpha(x)\,f^\beta(x)\,dx,$$

which is equivalent to

$$\int_a^b f^\beta(x)\,dx \int_a^b g^\alpha(y)\,f^\gamma(y)\,dy \geq \int_a^b f^\gamma(x)\,dx \int_a^b g^\alpha(y)\,f^\beta(y)\,dy.$$

That is to prove

$$\int_a^b \int_a^b f^\gamma(x)\,f^\gamma(y)\,g^\alpha(y)\left[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right]dxdy \geq 0.$$

Set

$$G = \int_a^b \int_a^b f^\gamma(x)\,f^\gamma(y)\,g^\alpha(y)\left[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right]dxdy.$$

It follows now that

$$G = \int_a^b \int_a^b f^\gamma(x)\,f^\gamma(y)\,g^\alpha(x)\left[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right]dxdy.$$

Thus

$$2G=\int_a^b\int_a^b f^{\gamma}(x)\,f^{\gamma}(y)\,(g^{\alpha}(y)-g^{\alpha}(x))\left(f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)\right)dxdy.$$

Since $f(x)\geq 0$ is a continuous and decreasing function on $[a,b]$, we get

$$(g^{\alpha}(y)-g^{\alpha}(x))\left(f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)\right)>0$$

for all $x,y\in[a,b]$. Hence $2G\geq 0$. This complete the proof. ∎

Theorem 6.14 ([42], p.207) *Suppose that $f(x)\geq 0$ is a continuous function on $[a,b]$ and satisfies*

$$\left[(y-a)^{\alpha}f^{\alpha}(x)-(x-a)^{\alpha}f^{\alpha}(y)\right]\left[f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)\right]\geq 0,\quad \forall x,y\in[a,b]. \tag{6.14}$$

Then the inequality

$$\frac{\int_a^b f^{\alpha+\beta}(x)\,dx}{\int_a^b f^{\alpha+\gamma}(x)\,dx}\geq\frac{\int_a^b (x-a)^{\alpha}f^{\beta}(x)\,dx}{\int_a^b (x-a)^{\alpha}f^{\gamma}(x)\,dx} \tag{6.15}$$

is valid for every positive real number $\alpha>0$ and $\beta\geq\gamma>0$. If (6.14) reverses, then the inequality (6.15) reverses.

Proof. We just prove the case "$\geq$". For this purpose, we merely need to prove

$$H:=\int_a^b\int_a^b f^{\alpha+\gamma}(x)\,f^{\gamma}(y)\,(y-a)^{\alpha}\left[f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)\right]dxdy\geq 0.$$

Note that since

$$H=\int_a^b\int_a^b f^{\gamma}(x)\,f^{\alpha+\gamma}(y)\,(x-a)^{\alpha}\left[f^{\beta-\gamma}(y)-f^{\beta-\gamma}(x)\right]dxdy,$$

we get

$$2H=\int_a^b\int_a^b f^{\gamma}(x)\,f^{\gamma}(y)\left[(y-a)^{\alpha}f^{\alpha}(x)-(x-a)^{\alpha}f^{\alpha}(y)\right]\left[f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)\right]dxdy.$$

From the condition (6.14) we have that $H\geq 0$. ∎

Theorem 6.15 ([42], p.207) *Suppose that $f(x)$,$g(x)\geq 0$ is continuous functions on $[a,b]$ and satisfy*

$$\left[g^{\alpha}(y)\,f^{\alpha}(x)-g^{\alpha}(x)\,f^{\alpha}(y)\right]\left[f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)\right]\geq 0,\quad \forall x,y\in[a,b]. \tag{6.16}$$

Then the inequality

$$\frac{\int_a^b f^{\alpha+\beta}(x)\,dx}{\int_a^b f^{\alpha+\gamma}(x)\,dx} \geq \frac{\int_a^b g^{\alpha}(x)\,f^{\beta}(x)\,dx}{\int_a^b g^{\alpha}(x)\,f^{\gamma}(x)\,dx} \tag{6.17}$$

is valid for every positive real number $\alpha > 0$ and $\beta \geq \gamma > 0$. If the inequality (6.16) reverses, then the inequality (6.17) reverses.

Proof. We just prove the case "$\geq$". For this purpose, we merely need to prove

$$H := \int_a^b \int_a^b f^{\alpha+\gamma}(x)\,f^{\gamma}(y)\,g^{\alpha}(y)\left[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right]dxdy \geq 0.$$

Note that since

$$H = \int_a^b \int_a^b f^{\gamma}(x)\,f^{\alpha+\gamma}(y)\,g^{\alpha}(x)\left[f^{\beta-\gamma}(y) - f^{\beta-\gamma}(x)\right]dxdy,$$

we get

$$2H = \int_a^b \int_a^b f^{\gamma}(x)\,f^{\gamma}(y)\left[g^{\alpha}(y)\,f^{\alpha}(x) - g^{\alpha}(x)\,f^{\alpha}(y)\right]\left[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right]dxdy.$$

From the condition (6.16) we obtain $H \geq 0$. ■

Their next result is the following theorem which will be used frequently in the rest of this section.

Theorem 6.16 ([42], p.208) *Suppose that $f(x), g(x), h(x) \geq 0$ are continuous functions on $[a, b]$ such that*

$$(g(x) - g(y))\left(\frac{f(y)}{h(y)} - \frac{f(x)}{h(x)}\right) \geq 0. \tag{6.18}$$

Then the inequality

$$\frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geq \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b h(x)\,g(x)\,dx} \tag{6.19}$$

is valid. If the inequality(6.18) reverses, then the inequality (6.19) reverses.

Proof. We need to prove that

$$\int_a^b f(x)\,dx \int_a^b h(x)\,g(x)\,dx \geq \int_a^b h(x)\,dx \int_a^b f(x)\,g(x)\,dx,$$

which is equivalent to

$$\int_a^b f(x)\,dx \int_a^b h(y)\,g(y)\,dy \geq \int_a^b h(x)\,dx \int_a^b f(y)\,g(y)\,dy.$$

That is to prove

$$\int_a^b \int_a^b g(y)\,(f(x)\,h(y) - f(y)\,h(x))\,dxdy \geq 0.$$

Denote

$$G = \int_a^b \int_a^b g(y)\,(f(x)\,h(y) - f(y)\,h(x))\,dxdy.$$

It follows that

$$G = \int_a^b \int_a^b g(x)\,(f(y)\,h(x) - f(x)\,h(y))\,dxdy.$$

Thus

$$\begin{aligned} 2G &= \int_a^b \int_a^b (g(x) - g(y))\,(f(y)\,h(x) - f(x)\,h(y))\,dxdy \\ &= \int_a^b \int_a^b h(x)\,h(y)\,(g(x) - g(y)) \left(\frac{f(y)}{h(y)} - \frac{f(x)}{h(x)} \right) dxdy. \end{aligned}$$

From the hypotheses, it can be easily seen that $2G \geq 0$. The proof is completed. ■

Remark 6.4 ([42], p.209) *It is trivially to see that the inequality (6.18) is valid for* f,g,h *such that either*

(a) g *is increasing and* $\frac{f}{h}$ *is decreasing or*

(b) g *is decreasing and* $\frac{f}{h}$ *is increasing.*

By applying Theorem 6.16, we get

Theorem 6.17 ([42], p.209) *Suppose that* $f(x)$,$h(x) > 0$ *are continuous functions on* $[a, b]$ *with* $f(x) \leq h(x)$ *for all* x *and such that* $\frac{f(x)}{h(x)}$ *is decreasing and* $f(x)$ *is increasing. Then the inequality*

$$\frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geq \frac{\int_a^b f^p(x)\,dx}{\int_a^b h^p(x)\,dx} \tag{6.20}$$

is valid for all $p \geq 1$. *If* $\frac{f(x)}{h(x)}$ *is increasing, then the inequality (6.20) reverses.*

Proof. Since $p \geq 1$ and f is increasing, then $g(x) := f^{p-1}(x)$ is also increasing. By applying the Theorem 6.16, we conclude that

$$\frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geq \frac{\int_a^b f(x)\,f^{p-1}(x)\,dx}{\int_a^b h(x)\,f^{p-1}(x)\,dx}. \tag{6.21}$$

Also, since $f(x) \geq h(x)$ for all x, then

$$\frac{\int_a^b f(x) f^{p-1}(x)\,dx}{\int_a^b h(x) f^{p-1}(x)\,dx} \geq \frac{\int_a^b f^p(x)\,dx}{\int_a^b h^p(x)\,dx}. \tag{6.22}$$

Summing up (6.21) and (6.22), we have the result. ∎

Involving convex functions, we get

Theorem 6.18 ([42], p.209) *Suppose that $f(x)$,$h(x) > 0$ are continuous functions on $[a, b]$ with $f(x) \leq h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x)$ is increasing. Let $\varphi(x)$ be a convex function with $\varphi(0) = 0$. Then the inequality*

$$\frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geq \frac{\int_a^b \varphi(f(x))\,dx}{\int_a^b \varphi(h(x))\,dx}$$

is valid.

Proof. Since φ is convex with $\varphi(0) = 0$, then $\frac{\varphi(x)}{x}$ is increasing. This and the fact that $f(x) \leq h(x)$ produce

$$\frac{\varphi(f(x))}{f(x)} \leq \frac{\varphi(h(x))}{h(x)}$$

Also, f and $\frac{\varphi(x)}{x}$ are increasing, then the following function

$$g(x) = \frac{\varphi(f(x))}{f(x)}$$

is increasing, too. This helps us to conclude that

$$\frac{\int_a^b \varphi(f(x))\,dx}{\int_a^b \varphi(h(x))\,dx} = \frac{\int_a^b f(x)\frac{\varphi(f(x))}{f(x)}dx}{\int_a^b h(x)\frac{\varphi(h(x))}{h(x)}dx} \leq \frac{\int_a^b f(x)\frac{\varphi(f(x))}{f(x)}dx}{\int_a^b h(x)\frac{\varphi(f(x))}{f(x)}dx} \leq \frac{\int_a^b f(x)}{\int_a^b h(x)}.$$

We complete the proof. ∎

The following theorem slightly improves Theorem 6.18.

Theorem 6.19 ([42], p.210) *Assume that $f(x)$,$g(x)$,$h(x) > 0$ are continuous functions on $[a, b]$ with $f(x) \leq h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x)$,$g(x)$ are increasing. Let $\varphi(x)$ be a convex function with $\varphi(0) = 0$. Then the inequality*

$$\frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geq \frac{\int_a^b \varphi(f(x))\,g(x)\,dx}{\int_a^b \varphi(h(x))\,g(x)\,dx}$$

is valid.

Proof. Similar to the proof of the Theorem 6.18, we get

$$\begin{aligned}\frac{\int_a^b \varphi(f(x))\,g(x)\,dx}{\int_a^b \varphi(h(x))\,g(x)\,dx} &= \frac{\int_a^b \frac{\varphi(f(x))}{f(x)} g(x)\,f(x)\,dx}{\int_a^b \frac{\varphi(h(x))}{h(x)} g(x)\,h(x)\,dx} \\ &\le \frac{\int_a^b \frac{\varphi(f(x))}{f(x)} g(x)\,f(x)\,dx}{\int_a^b \frac{\varphi(f(x))}{f(x)} g(x)\,h(x)\,dx} \le \frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx}.\end{aligned}$$

We complete the proof. ■

Lastly, W. Liu et al. proposed the following open problems.

Problem 6.2 ([42], p.210) *Can the condition (6.14) in the Theorem 6.14 (or (6.16) in the Theorem 6.15) be improved?*

Also they proposed Problem 15 (Remark 2.16), Problem 16 (Remark 2.17) and Problem 17 (Remark 2.18), which are given in Chapter 2, in the end of their paper [42].

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CURRICULUM VITAE

Melih Tolunay was born in Zonguldak in 1989. He completed his primary, secondary and high school education in Zonguldak. He graduated in Zonguldak İMKB Anatolian Teacher Training High School in 2006. He attended in mathematics teacher education programme at Ondokuz Mayıs University in 2006 and graduated in 2015. He completed his post graduate education in department of mathematics at Bülent Ecevit University in 2017.

ADDRESS:

Address: Yeşil Mah. Akşemsettin Cad. Tersane Sitesi D/1 Blok Daire: 6
Merkez / ZONGULDAK

Phone: 05432353667
E-mail: melihtolunay@hotmail.com