QUADRATIC CURVATURE GRAVITY THEORIES IN VARIOUS DIMENSIONS WITH AND WITHOUT TORSION

Doctoral Dissertation

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DOCTORAL DISSERTATION

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Supervisor: Prof. Dr. Hakan CEBECİ

Eskişehir
Eskişehir Technical University
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ABSTRACT

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In this thesis work, generalized gravity models are examined in n-dimensional spacetimes. Generalized gravity theories are described by a Lagrangian that involves a gravitational function expressed in terms of the scalar curvature, Ricci-squared and Riemann-squared terms in general. The models are investigated for two seperate cases, in which the connection of the spacetime is torsion-free and the connection of the spacetime involves spacetime torsion. By using exterior algebra formalism, the field equations of the theory are obtained by making independent variations of the gravitational action with respect to metric co-frame and connection fields. 3-dimensional gravity models such as Topologically Massive Gravity, New Massive Gravity, Minimal Massive Gravity and Generalized Massive Gravity are examined within the same framework. For certain generalized gravity models, some new exact solutions are achieved. Among them are the constant curvature solutions of quadratic curvature gravity model in a spacetime involving torsion. In addition, Lifshitz-type and Bianchi-type static and stationary solutions and Schrödinger-type solutions of R^2-corrected gravity model are presented. Also, pp-wave solutions of Minimal Massive 3D Gravity theory are examined for both vacuum case and for the gravity model that is minimally coupled to Maxwell-Chern-Simons electromagnetic theory.

Keywords : Connection, metric compatibility, field equations, modified gravity theories.

ÖZET

ÇEŞİTLİ BOYUTLARDA BURULMA İÇEREN VE İÇERMEYEN KUADRATİK EĞRİLİKLİ KÜTLEÇEKİM KURAMLARI

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Fizik Anabilim Dalı
Yüksek Enerji ve Plazma Fiziği Bilim Dalı
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Danışman: Prof. Dr. Hakan CEBECİ

Bu tez çalışmasında, n-boyutlu uzay-zamanlarda genelleştirilmiş kütleçekim modelleri incelenmiştir. Genelleştirilmiş kütleçekim kuramları, genel olarak skaler eğrilik, Ricci kare ve Riemann kare terimleri cinsinden ifade edilen bir kütleçekim fonksiyonu içeren Lagrangian ile ifade edilir. Modeller, uzay-zaman bağlantısı burulma içeren ve içermeyen durumlar için incelenmiştir. Dış cebir formalizmini kullanarak, kuramın alan denklemleri kütleçekim eyleminin metrik ve bağlantı alanlarına göre bağımsız varyasyonundan elde edilmiştir. Topolojik Kütleli Kütleçekim, Yeni Kütleli Kütleçekim, Minimal Kütleli Kütleçekim ve Genelleştirilmiş Kütleli Kütleçekim gibi 3-boyutlu kütleçekim modelleri aynı çerçeve içinde incelenmiştir. Belli genelleştirilmiş kütleçekim modelleri için yeni çözümler elde edilmiştir. Bunların arasında, burulma içeren bir uzay-zamanda kuadratik (ikinci dereceden) eğrilikli kütleçekim modelinin sabit eğrilikli çözümleri bulunmaktadır. Ek olarak, R^2-düzeltilmiş kütleçekim modelinin Lifshitz tipi ve Bianchi tipi statik ve durağan çözümleri ve Schrödinger tipi çözümler sunulmuştur. Ayrıca, 3-boyutlu Minimal Kütleli Kütleçekim kuramının vakum ve minimal olarak Maxwell-Chern-Simons elektromanyetik kuramına bağlanmış durumları için pp-dalga çözümleri incelenmiştir.

Anahtar Sözcükler : Bağlantı, metrik uyumluluk, alan denklemleri, modifiye gravitasyon teorileri.

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20/01/2021

STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with “scientific plagiarism detection program”used by Eskiehir Technical University, and that “it does not have any plagiarism”whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

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GLOSSARY OF ABBREVIATIONS

Throughout this thesis work, the following abbreviations are used. The notation is always defined at its first occurrence in the text and in all other places that ambiguities may arise.

GR	:	General Relativity
IR	:	Infrared
UV	:	Ultraviolet
TMG	:	Topological Massive Gravity
NMG	:	New Massive Gravity
MMG	:	Minimal Massive Gravity
GMG	:	Generalized Massive Gravity
GGM	:	Generalized Gravity Model
QCG	:	Quadratic Curvature Gravity

1. INTRODUCTION

General Relativity (GR), which is considered as a theory that describes the macroscopic geometrical properties of spacetime, has been accepted as the most successful and elegant theory of gravity in the last century. Despite GR's enormous influence and impact on the science community, it wasn't long before that the first proposals appeared for its possible modifications and extensions since it has been shown that GR is not renormalizable and, therefore cannot be properly quantized [1, 2]. In addition to this significant deficiency of the theory, GR cannot explain the accelerated expansion of the Universe as well as the existence of galaxy rotation curves at galactic scales. Because of such shortcomings of the theory, some possible extensions have been proposed. Such extensions can be seen in many different forms. The most promising one is that the gravitational Lagrangian involves a function of the curvature scalar R such that the theory can be identified as $f(R)$ gravity model. In addition, there exist some other gravity models such as $f(T)$ (with T being the torsion scalar), $f(G)$ (with G being the Gauss-Bonnet invariant) or $f(R, U, V)$ (with U being the Ricci-squared scalar and V being the Kretchmann scalar).

The simplest of them, namely $f(R)$ gravity theory, belongs to the general class of higher order gravity theories admitting higher-order metric derivatives in field equations. The theory is either the addition of specific higher-order curvature invariants to the Einstein-Hilbert action, or more straightforwardly, the Lagrangian as a general function of the scalar curvature R. The motivation for $f(R)$ gravity theories come from cosmology and astrophysics for seeking generalizations of GR (reviewed in depth in [3, 4]). These theories are of a specific concern due to the fact that higher curvature modifications of GR leads to a gravity theory that exhibits good renormalization properties at one loop [5]. The results are also confirmed in [6].

To obtain the field equations of $f(R)$ gravity theory or the generalized gravity model involving curvature, Ricci and Kretchmann scalars in the Lagrangian, two different formalisms can be realized, namely the standard metric formalism and the Palatini formalism [7]. In the standard metric formalism (rigorously studied in [8]), the field equations can be obtained from the variations with respect to the metric tensor taking the connection to be Levi-Civita (torsion-free) such that the connection depends on the spacetime metric. On the other hand, in the Palatini formalism, metric and connection fields are considered to be independent field variables if one considers the variation of the action. These two methods can be shown to be identical for the GR action but they differ for the higher curvature gravity models. Actually, there is even another effectively available formalism of $f(R)$ gravity: metric-affine $f(R)$ gravity [9]. This happens if one uses the Palatini variation but leaves the assumption that the action of the matter fields is independent of the connection. In fact, metric-affine $f(R)$ gravity is the most general one of these theories. If additional assumptions are made, it also reduces to metric or Palatini $f(R)$ gravity theory. Although the metric and Palatini formulations of $f(R)$ gravity theory and its physical consequences have been widely discussed in the literature, it is useful to express its different formulation in exterior algebraic notation since it proves to be practical and powerful in calculations. Even though the exterior algebraic notation is not a common formulation, it is proven to be useful to reveal the existence of propagating torsion produced when one makes independent variations of metric and connections.

In general, the generalized gravity models are examined together with the priori assumption that the connections are metric compatible. It is also possible to drop this assumption and investigate such models within the non-metric compatible spacetime framework. However, in non-metric compatible gravity theories, one should be careful in raising and lowering the indices of the tensor-valued forms since when covariant exterior derivative operator acts on these forms one can encounter extra terms coming from non-metricity fields. As far as the higher curvature gravity models such as $f(R)$ and its extensions are concerned, obtaining the field equations from the Lagrangians involving higher curvature terms with non-metric compatible connection requires elaborate work. Throughout the thesis work, the metric compatible connections are assumed. This assumption leads to anti-symmetric connection and curvature forms unlike the non-metric compatible theories where in this case the connection and curvature forms involve both anti-symmetric and symmetric terms. To mention the physical importance of the modifications of the Einstein-Hilbert action to include the higher order curvature terms, such generalized models are mostly investigated by the motivation that the theory exhibits improved divergences in both infrared (IR) and ultraviolet (UV) scales. From the field theoretical point of view, the generalized gravity models can offer renormalizable theories and as a consequence they are considered to be good candidates for a quantum gravity model. In most of the studies in the literature, generalized gravity theories are concentrated on quadratic models [10], since quadratic theories are assumed to be general prototypes of higher-order theories. The quadratic terms in the action density are considered as restricted to a strong gravity regimes and they are strongly suppressed by small couplings as well as in the simple effective field theory considerations.

Furthermore, from the cosmological point of view, the corrections to GR are crucial when one attempts to investigate the physics in the early universe or near black hole singularities. Moreover, these corrections are expected to be consistent for large scales, i.e., in the late universe.

In this thesis work, by employing exterior algebra formalism, generalized gravity models are examined in n-dimensional spacetimes considering that extended gravity consists of a Lagrangian that is a gravitational function of the curvature scalar R, Ricci-squared scalar U, Kretchmann scalar V in general. The field equations are obtained by first constraining that the connection is the Levi-Civita (i.e., torsion-free) and then making independent variations of the Lagrangian density with respect to metric co-frame and connection fields. Next, quadratic curvature gravity theories are studied including three dimensional massive gravity theories. Some new exact analytical solutions of quadratic curvature gravity theories in various dimensions are presented. Among them are the constant curvature solutions with and without torsion, Lifshitz [11], Bianchi [12] and Schrödinger [13] type solutions. Specifically, the matter coupling in Minimal massive 3D Gravity theory is considered as well taking the matter Lagrangian to be Maxwell-Chern-Simons electromagnetic theory.

The work is organized as follows: In Section 2, the field equations of n-dimensional Generalized Gravity Model are obtained by first considering that the connections of the spacetime are constrained to be torsion-free (i.e., the connections are considered to be Levi-Civita). Then by making independent variations of the corresponding Lagrangian with respect to co-frames and connection fields, the field equations that involve propagating torsion are presented. In the same section, n-dimensional quadratic curvature gravity models including 3-dimensional massive gravity theories such as Topological Massive Gravity (TMG), New Massive Gravity (NMG), Minimal Massive Gravity (MMG) and Generalized Massive Gravity (GMG) are examined as well. In Section 3, some new exact solutions of Quadratic Curvature Gravity are presented. Among them are the constant curvature solutions of quadratic curvature gravity theory with and without torsion. In addition, Lifshitz and Bianchi type static/stationary solutions and Schrödinger type solutions of R^2-corrected gravity theory are obtained. Some thermodynamic properties are also examined for these types of solutions. In Section 4, vacuum/matter coupled pp-wave solutions of Minimal Massive Gravity (MMG) are found. Section 5 is devoted to conclusions of this thesis work. The exterior algebra formalism, miscellaneous identities and detailed derivations of the field equations presented throughout this work are explicitly illustrated in Appendices.

2. GENERALIZED GRAVITY MODELS IN HIGHER DIMENSIONS

In this section, n-dimensional Generalized Gravity Model (GGM) is discussed. This model is described by Lagrangian n-forms such that the Lagrangian is considered to be a gravitational function of curvature invariants including scalar curvature, Ricci-squared and Riemann-squared terms (also known as Kretschmann invariant) in general. Being a subset of such models, $f(R)$ gravity models play an important role especially in explaining the accelerated expansion of the Universe. GGM can be examined in two different frameworks. Either the spacetime is constrained to be a Levi-Civita spacetime where torsion-free connection is imposed as a constraint, or the spacetime is considered to include torsion (or even non-metricity) where in that case independent variations of the action with respect to the co-frame and connection fields are considered. Then, GGM (with the matter coupling in general) can be described by the following action

$$I[e,\omega] = \int \mathcal{L}[e,\omega]\,, \tag{2.1}$$

where the Lagrangian density is

$$\mathcal{L}[e,\omega] = f(R,U,V) * 1 + \mathcal{L}(matter)\,. \tag{2.2}$$

Here, $f(R,U,V)$ is the gravitational function expressed in terms of the scalar curvature R, the Ricci-squared scalar $U = - * (P_a \wedge *P^a)$ and the Kretschmann scalar (sometimes it is also called Riemann-squared scalar) $V = -2 * (R_{ab} \wedge *R^{ab})$ which are also identified as the curvature invariants and $\mathcal{L}(matter)$ represents the matter Lagrangian. Hodge $*$ map is defined with respect to the metric g so that the oriented volume form is given as $*1 = e^0 \wedge e^1 \wedge \ldots \wedge e^n$ where $\wedge$ denotes the wedge product. The basic field variables are the co-frame 1-forms $\{e^a\}$, in terms of which the spacetime metric can be given by $g = \eta_{ab} e^a \otimes e^b$, where $\eta_{ab} = diag(-+...+)$. The connection 1-forms, denoted by $\{\omega^a{}_b\}$, can be physically interpreted as the gravitational gauge potentials. The connection structure of the spacetime is assumed to be metric-compatible throughout this thesis work. Metric compatibility (i.e., $Dg = 0$ or equivalently $D\eta_{ab} = 0$, where D denotes the covariant exterior derivative operator acting on form fields) implies that $\omega^{ab} = -\omega^{ba}$ and the first Cartan structure equation is given by

$$T^a = De^a = de^a + \omega^a{}_b \wedge e^b\,, \tag{2.3}$$

where T^a represents the torsion 2-forms and d is exterior derivative operator acting on differential forms. The curvature 2-forms R^{ab} can be obtained from the second Cartan structure equation

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\,. \tag{2.4}$$

By expressing the curvature 2-forms in terms of coframes in the form

$$R^{ab} = \frac{1}{2} R^{ab}{}_{cd} e^c \wedge e^d \,, \tag{2.5}$$

one can read the components of Riemann curvature tensor $R^{ab}{}_{cd}$. The corresponding Ricci 1-forms P^a and curvature scalar R are defined by the contractions as

$$P^a = \iota_b R^{ba} = R^{ba}{}_{bc}\, e^c \equiv P^a{}_c\, e^c \tag{2.6}$$

and

$$R = \iota_a P^a \,, \tag{2.7}$$

respectively. Here, ι_a denotes the interior (inner) derivative with respect to the frame vector field X_a and $P^a{}_c$ signifies component of the Ricci 1-forms. Throughout this work, the gravitational coupling constant is taken to be equal to unity and one can see Appendix A for further details of the exterior algebra notation and operators on differential forms.

First, one can examine GGM under the constraint that the connections are Levi-Civita (i.e., $T^a = 0$). Therefore, one should consider the variations of the action whose Lagrangian density can be expressed by

$$\bar{\mathcal{L}}[e, \omega, \lambda] = \mathcal{L}[e, \omega] + T^a \wedge \lambda_a \,, \tag{2.8}$$

where $\{\lambda_a\}$ denotes the Lagrange multiplier $(n-2)$-forms that ensure the torsion-free connection structure. Then, variations of the action density (2.8) with respect to co-frame 1-forms e^a, connection 1-forms $\omega^a{}_b$ and the Lagrange multiplier $(n-2)$-forms λ^a yields

$$\begin{aligned}
\delta\bar{\mathcal{L}} \;\; = \;\; & \delta e^c \wedge \Bigg[f * e_c - \tau_c + D\lambda_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\
& + \left(\frac{\partial f}{\partial U}\right)\Big(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) \\
& + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *\,(R_{ab} \wedge e_c) - V * e_c\Big)\Bigg] \\
& + \delta\omega^{ab} \wedge \Bigg[D\left(\left(\frac{\partial f}{\partial R}\right) *\,(e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) *\,(P_a \wedge e_b - P_b \wedge e_a)\right) \\
& + 4D\left(\left(\frac{\partial f}{\partial V}\right) *\,R_{ab}\right) + \Omega_{ab} + \frac{1}{2}\,(e_b \wedge \lambda_a - e_a \wedge \lambda_b)\Bigg] + \delta\lambda_a \wedge T^a \\
& + d\Bigg[\left(\frac{\partial f}{\partial R}\right)(\delta\omega^{ab}) \wedge *(e_a \wedge e_b) + \left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) \\
& + 4\left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge *R_{ab} + (\delta e^a) \wedge \lambda_a\Bigg] \,,
\end{aligned} \tag{2.9}$$

where δ denotes variation of the fields. Here, $\tau_c \equiv -\frac{\delta\bar{\mathcal{L}}(matter)}{\delta e^c}$ is the variational term obtained from the co-frame variation, defined as the stress-energy $(n-1)$-form, while $\Omega_{ab} = \frac{\delta\bar{\mathcal{L}}(matter)}{\delta\omega^{ab}}$ is the variational term obtained from the connection variation of the matter Lagrangian, also defined as the hypermomentum current $(n-1)$-form. Note that the combination of the material and the higher derivative curvature terms defines the energy and momentum exchange. Further details of the geometry and algebra used in this work are presented in Appendix A. Additionally, required identities are explicitly illustrated in Appendix B and detailed calculations of the variational procedure for GGM are given in Appendix C.

In the variation process of the gravitational action, there are also some surface terms as in GR. If one aims to examine the thermodynamical properties of the theory and attempts to calculate physical quantities such as energy and field momentum, surface terms deserve a detailed investigation. Now, if one demands that the variations of the gravitational fields vanish on the boundary, the closed forms (or surface terms) under the integral do not give any contribution to the field equations:

$$\int_M d\Psi = \int_{\partial M} \Psi = 0 \,. \tag{2.10}$$

Hence, the variational principle

$$\int_M \delta \bar{\mathcal{L}} = 0 \tag{2.11}$$

gives the following field equations:

$$\begin{aligned} & f * e_c - \tau_c + D\lambda_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\ & + \left(\frac{\partial f}{\partial U}\right)\Big({P^a}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) \\ & + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big) = 0 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} & D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) + 4D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) \\ & + \Omega_{ab} + \frac{1}{2}\left(e_b \wedge \lambda_a - e_a \wedge \lambda_b\right) = 0 \end{aligned} \tag{2.13}$$

together with the constraint relation

$$T^a = de^a + {\omega^a}_b \wedge e^b = 0 \,. \tag{2.14}$$

which ensures that the connections are Levi-Civita. Here, one should begin with the connection equation (2.13) to solve the Lagrange multiplier $(n-1)$-forms. For this purpose, if one defines

$$\begin{aligned} \Lambda_{ab} \;=\; & 2D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + 2D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) \\ & + 8D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + 2\Omega_{ab} \,, \end{aligned} \tag{2.15}$$

(2.13) takes the following form

$$\Lambda_{ab} = e_a \wedge \lambda_b - e_b \wedge \lambda_a \,. \tag{2.16}$$

Application of the interior product operator ι^a onto both sides with the identity (B.2) gives

$$\begin{aligned}
\iota^a \Lambda_{ab} &= \iota^a \left(e_a \wedge \lambda_b - e_b \wedge \lambda_a \right) \\
&= (\iota^a e_a) \wedge \lambda_b - e_a \wedge \iota^a \lambda_b - (\iota^a e_b) \wedge \lambda_a + e_b \wedge \iota^a \lambda_a \\
&= \lambda_b + e_b \wedge \iota^a \lambda_a \,,
\end{aligned} \tag{2.17}$$

since $\iota^a e_b = \delta^a{}_b$ and $\iota^a e_a = n$. Further application of the inner product operator ι^b on (2.17) reads

$$\begin{aligned}
\iota^b \iota^a \Lambda_{ab} &= \iota^b (\lambda_b + e_b \wedge \iota^a \lambda_a) \\
&= \iota^b \lambda_b + (\iota^b e_b) \wedge \iota^a \lambda_a - e_b \wedge \iota^b \iota^a \lambda_a \\
&= 4 \iota^a \lambda_a \,.
\end{aligned} \tag{2.18}$$

Back substitution of (2.18) into (2.17) yields

$$\begin{aligned}
\lambda_b &= \iota^a \Lambda_{ab} - e_b \wedge \iota^a \lambda_a \\
&= \iota^a \Lambda_{ab} - \frac{1}{4} e_b \wedge \iota^c \iota^a \Lambda_{ac} \,.
\end{aligned} \tag{2.19}$$

After some long and cumbersome calculations (the details can be found in Appendix C), the corresponding Lagrange multiplier $(n-2)$-forms can be exactly calculated as

$$\begin{aligned}
\lambda_b &= 2 \left(\iota^a D \left(\frac{\partial f}{\partial R} \right) \right) * (e_a \wedge e_b) + 2 \left(\iota^a D \left(\frac{\partial f}{\partial U} \right) \right) * (P_a \wedge e_b - P_b \wedge e_a) \\
&\quad + \left(\frac{\partial f}{\partial U} \right) (\iota_b * dR + 2 * DP_b) + 8 \left(\iota^a D \left(\frac{\partial f}{\partial V} \right) \right) * R_{ab} + 2 \iota^a \Omega_{ab} \\
&\quad + 8 \left(\frac{\partial f}{\partial V} \right) * DP_b - \frac{1}{2} e_b \wedge \iota^c \iota^a \Omega_{ac} \,.
\end{aligned} \tag{2.20}$$

Therefore, the Einstein field equation of GGM is obtained as

$$
\begin{aligned}
& f * e_c - \tau_c + \left(\frac{\partial f}{\partial R}\right)\left(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\right) \\
& + \left(\frac{\partial f}{\partial U}\right)\left(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\right) \\
& + \left(\frac{\partial f}{\partial V}\right)\left(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\right) \\
& + D\left[2\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_c) + 2\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_c - P_c \wedge e_a)\right. \\
& + \left(\frac{\partial f}{\partial U}\right)(\iota_c * dR + 2 * DP_c) + 8\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right) * R_{ac} + 2\iota^a \Omega_{ac} \\
& \left. + 8\left(\frac{\partial f}{\partial V}\right) * DP_c - \frac{1}{2} e_c \wedge \iota^b \iota^a \Omega_{ab}\right] = 0\,. \qquad (2.21)
\end{aligned}
$$

It is also interesting to note that, if one multiplies (2.21) by e^c, one gets the trace of GGM Einstein field equation as

$$
\begin{aligned}
& \left[nf - 2R\left(\frac{\partial f}{\partial R}\right) - 4U\left(\frac{\partial f}{\partial U}\right) - 4V\left(\frac{\partial f}{\partial V}\right) - \tau\right] * 1 \\
& -2\left[(n-1)D\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) + \left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) dR + R\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right)\right] \wedge *e_a \\
& -2\left[(n-2)D\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 2D\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right] \wedge *P_a \\
& -2\left[(n-2)\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 2\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right] D * P_a \\
& -\left[nD\left(\frac{\partial f}{\partial U}\right) + 2D\left(\frac{\partial f}{\partial V}\right)\right] \wedge *dR \\
& -\left[n\left(\frac{\partial f}{\partial U}\right) + 2\left(\frac{\partial f}{\partial V}\right)\right] d * dR + 2e^c \wedge D(\iota^a \Omega_{ac}) = 0\,. \qquad (2.22)
\end{aligned}
$$

One can easily see that, when $\frac{\partial f}{\partial U} = 1$ or $\frac{\partial f}{\partial U} = 0$ and $\frac{\partial f}{\partial V} = 1$ or $\frac{\partial f}{\partial V} = 0$, the trace equation is a fourth order equation for Levi-Civita connection structure.

On the other hand, one can also consider that the connection 1-forms $\omega^a{}_b$ and the co-frame 1-forms e^a are varied independently without any constraint (i.e., $T^a \neq 0$). Independent variations of the Lagrangian density (2.2) with respect to e^a and $\omega^a{}_b$ yields

$$\begin{aligned}
\delta\mathcal{L} \;=\; & \delta e^c \wedge \Bigg[f * e_c - \tau_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\
& + \left(\frac{\partial f}{\partial U}\right)\Big(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) \\
& + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge * (R_{ab} \wedge e_c) - V * e_c\Big)\Bigg] \\
& + \delta\omega^{ab} \wedge \Bigg[D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) \\
& + 4D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + \Omega_{ab}\Bigg] \\
& + d\Bigg[\left(\frac{\partial f}{\partial R}\right)(\delta\omega^{ab}) \wedge *(e_a \wedge e_b) + \left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) \\
& + 4\left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge *R_{ab}\Bigg] \,. \qquad (2.23)
\end{aligned}$$

Then the variational principle (2.11) implies the following field equations:

$$\begin{aligned}
& f * e_c - \tau_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\
& + \left(\frac{\partial f}{\partial U}\right)\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) \\
& + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big) = 0 \qquad (2.24)
\end{aligned}$$

and

$$D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) + 4D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + \Omega_{ab} = 0\,. \qquad (2.25)$$

Now, upon multiplication of Einstein field equation by e^c, the trace equation becomes

$$\left(nf - 2R\left(\frac{\partial f}{\partial R}\right) - 4U\left(\frac{\partial f}{\partial U}\right) - 4V\left(\frac{\partial f}{\partial V}\right) - \tau\right) * 1 = 0\,, \qquad (2.26)$$

where the details are again given in Appendix C. It should be noted that if there is no matter coupling in the action density (2.8) (i.e., $\mathcal{L}(matter) = 0$), field equations and trace expressions can be simply obtained by dropping the related matter terms. The basic feature of GGM considered in this work is that one can clearly see the effect of all terms in the field equations. It should also be noted that for unconstrained variations of GGM, one can inevitably encounter a propagating torsion.

In the same vein, if one considers $f = f(R)$ as a specific higher curvature gravity model where gravitational action is a function of curvature scalar only, independent variation of the action with respect to co-frame and connection fields yields the field equations

$$f(R) * e_c - \tau_c + \left(\frac{df(R)}{dR}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) = 0 \qquad (2.27)$$

and

$$D\left(\left(\frac{df(R)}{dR}\right)*(e_a \wedge e_b)\right) + \Omega_{ab} = 0\,. \tag{2.28}$$

Here, one can notice that, when the action does not involve any matter fields, the connection equation (2.28) can equivalently be expressed in the following form by using the definition of the exterior covariant derivative operator D:

$$\left(\frac{d^2 f(R)}{dR^2}\right) dR \wedge *(e_a \wedge e_b) + \left(\frac{df(R)}{dR}\right) T^c \wedge *(e_a \wedge e_b \wedge e_c) = 0\,. \tag{2.29}$$

It is seen that this equation is not an algebraic equation for torsion T^a, i.e., T^a cannot be determined algebraically. To see this, one can decompose the connection 1-forms in a spacetime that involves torsion as

$$\omega^a{}_b = \Gamma^a{}_b + K^a{}_b\,, \tag{2.30}$$

where $\Gamma^a{}_b$ denotes the connection 1-forms of the vanishing torsion (i.e., the Levi-Civita connection) while $K^a{}_b$ denotes the contorsion 1-forms which determine the torsion through the relation

$$T^a(\omega) = K^a{}_b \wedge e^b\,. \tag{2.31}$$

Now, curvature 2-forms R^{ab} can also be expressed in terms of connection-free curvature two-forms and contorsions (and therefore in terms of torsion field) in the form

$$R^{ab}(\omega) = R^{ab}(\Gamma) + D(\Gamma)K^{ab} + K^a{}_c \wedge K^{cb} \tag{2.32}$$

such that curvature scalar $R = \iota_a \iota_b R^{ba}$ of the theory involves torsion intrinsically. Therefore, equation (2.29) becomes a non-algebraic equation in torsion field. In this sense, the gravity model inevitably produces propagating torsion. On the contrary, constant scalar curvature solution imply that the theory becomes equivalent to torsion-free model.
Interestingly, one can also discuss the solutions of $f(R)$ model with torsion in the absence of matter. Indeed, calculating the trace of the Einstein field equation for $f = f(R)$ yields

$$nf(R) - 2R\left(\frac{df(R)}{dR}\right) = 0\,. \tag{2.33}$$

Equation (2.33) can automatically be solved when the gravitational function $f(R)$ has the following form:

$$\begin{aligned} 2R\left(\frac{df(R)}{dR}\right) &= nf(R), \\ \int \frac{df(R)}{f(R)} &= \frac{n}{2}\int \frac{dR}{R}, \\ f(R) &= C_0 R^{n/2}\,, \end{aligned} \tag{2.34}$$

with C_0 is a constant where in 4-dimensional spacetime the solution gives R^2 gravity. The

4-dimensional uncharged and charged black hole solutions of this model have recently been examined in [14]. The model can be shown to be conformally scale invariant in generic dimensions, i.e., the gravitational action is invariant under the conformal scaling

$$e^a \to e^{\sigma(x)} e^a \,, \tag{2.35}$$

where $\sigma(x)$ is any dimensionless scalar field.
Before closing this section, one can also examine the field equations of $f(R)$ gravity this time demanding that connections are the Levi-Civita. Then, the Einstein field equation (in the absence of matter) can be given by

$$f(R) * e_c + \left(\frac{df(R)}{dR}\right)\left(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\right) + 2D\left(\left(\frac{d^2 f(R)}{dR^2}\right) \iota_c * dR\right) = 0 \,, \tag{2.36}$$

whose trace is

$$\left(nf(R) - 2R\left(\frac{df(R)}{dR}\right)\right) * 1 - 2(n-1)D\left(\left(\frac{d^2 f(R)}{dR^2}\right) * dR\right) = 0 \,. \tag{2.37}$$

Now, it should be noted that the trace equation (2.37) can admit constant scalar curvature solutions hence in this sense, AdS solutions with constant curvature can be the solutions of $f(R)$ gravity models with and without torsion. On the other hand, if $f(R)$ theory has (minimal or non-minimal) couplings to matter, then such theories with and without torsion admit spacetime solutions whose scalar curvatures are not constant. When matter coupling exists, the solutions obtained in both approaches (i.e., without torsion and with torsion) are not equivalent.

2.1. Quadratic Curvature Gravity Theory In n-Dimensional Spacetime

The quadratic curvature gravity (QCG) model in n-dimensional spacetime is discussed in this subsection. QCG model can be considered as a specific model of generalized gravity in the sense that the model is described by the Lagrangian density that involves R^2, U and the celebrated Gauss-Bonnet term. In the absence of matter fields, one can consider the most general QCG model in n-dimensions ($n \geq 4$) in the form

$$\mathcal{L}_{QCG} = \frac{1}{2} R^{ab} \wedge *(e_a \wedge e_b) + \frac{\alpha}{2} R^2 * 1 + \frac{\beta}{2} P_a \wedge *P^a + \frac{\gamma}{4} R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d) \,, \tag{2.38}$$

where α, β, γ denote coupling constants. Here, the first term on the right hand side of (2.38) is the celebrated Einstein-Hilbert term and the last one is known as the Gauss-Bonnet term which can also be expressed in terms of curvature scalar in the form

$$R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d) = \left(R^2 - 4U + V\right) * 1 \,. \tag{2.39}$$

With the modifications of the coupling constants and using the identity (B.36), the action density of QCG model can simply be expressed as

$$\mathcal{L}_{QCG} = f(R, U, V) * 1 \,, \tag{2.40}$$

with

$$f(R, U, V) = \frac{1}{2}\left(\left(\alpha + \frac{\gamma}{2}\right) R^2 + R\right) + \left(\frac{\beta}{2} - \gamma\right) U + \frac{\gamma}{4} V \,. \tag{2.41}$$

Here, expressing the gravitational function in terms of the curvature invariants, one can get the field equations of QCG model with the aid of GGM field equations presented previously. First, one can consider that the connections of spacetime are constrained to be Levi-Civita. Then, by introducing the Lagrange multiplier $(n-2)$-form to ensure that the connections to be torsion-free, one can make the variations of the following Lagrangian with respect to co-frames e^a, connection 1-forms $\omega^a{}_b$ and auxiliary field (Lagrange multiplier) λ_a:

$$\bar{\mathcal{L}}_{QCG} \;=\; f(R, U, V) * 1 + \lambda_a \wedge T^a \,. \tag{2.42}$$

The variations of the action density with respect to co-frame fields e^a, connection fields $\omega^a{}_b$ and the Lagrange multiplier $(n-2)$-forms λ^a take the form

$$\begin{aligned}
\delta\bar{\mathcal{L}}_{QCG} \;=\; & \delta e^c \wedge \Big[f * e_c + D\lambda_c + \left(\left(\alpha + \frac{\gamma}{2}\right) R + \frac{1}{2}\right)\left(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\right) \\
& + \left(\frac{\beta}{2} - \gamma\right)\left(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\right) \\
& + \frac{\gamma}{4}\left(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\right)\Big] \\
& + \delta\omega^{ab} \wedge \Big[D\left(\left(\left(\alpha + \frac{\gamma}{2}\right) R + \frac{1}{2}\right) * (e_a \wedge e_b)\right) \\
& + \left(\frac{\beta}{2} - \gamma\right) D * (P_a \wedge e_b - P_b \wedge e_a) + \gamma D * R_{ab} + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b)\Big] \\
& + \delta\lambda_a \wedge \left[de^a + \omega^a{}_b \wedge e^b\right] + d\Big[\left(\left(\alpha + \frac{\gamma}{2}\right) R + \frac{1}{2}\right)(\delta\omega^{ab}) \wedge *(e_a \wedge e_b) \\
& + \left(\frac{\beta}{2} - \gamma\right)\left(\delta\omega^{ab}\right) \wedge *(P_a \wedge e_b - P_b \wedge e_a) + \gamma(\delta\omega^{ab}) \wedge *R_{ab} + (\delta e^a) \wedge \lambda_a\Big] .
\end{aligned} \tag{2.43}$$

Since the last term is a total derivative and does not contribute to the equations of motion, the variational principle (2.11) yields:

$$\begin{aligned}
& f * e_c + D\lambda_c + \left(\left(\alpha + \frac{\gamma}{2}\right) R + \frac{1}{2}\right)\left(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\right) \\
& + \left(\frac{\beta}{2} - \gamma\right)\left(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\right) \\
& + \frac{\gamma}{4}\left(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\right) = 0
\end{aligned} \tag{2.44}$$

and

$$D\left(\left(\left(\alpha+\frac{\gamma}{2}\right)R+\frac{1}{2}\right)*(e_a\wedge e_b)\right)+\left(\frac{\beta}{2}-\gamma\right)D*(P_a\wedge e_b-P_b\wedge e_a)$$
$$+\gamma D*R_{ab}+\frac{1}{2}(e_b\wedge\lambda_a-e_a\wedge\lambda_b)=0\,. \tag{2.45}$$

together with $T^a=0$. The Lagrange-multiplier $(n-2)$-forms can easily be calculated by using (2.20) and properly identifying the term Λ_{ab} :

$$\lambda_b=\left(2\alpha+\frac{\beta}{2}\right)\iota_b*dR+\beta*DP_b\,, \tag{2.46}$$

owing to the fact that QCG model considered here is a special case of GGM (i.e., details of the calculations for Lagrange multiplier $(n-2)$-form of GGM are given in (C.50)-(C.74) and these are also compatible with QCG). Here, one should notice that the general expression of the Lagrange multiplier $(n-2)$-form is additionally simplified by using (B.2). Then, the Einstein field equation of QCG model is given by

$$f*e_c+\left(\left(\alpha+\frac{\gamma}{2}\right)R+\frac{1}{2}\right)\left(R^{ab}\wedge*(e_a\wedge e_b\wedge e_c)-R*e_c\right)$$
$$+\left(\frac{\beta}{2}-\gamma\right)\left(P^a{}_c*P_a+2(\iota_cR^{ba})\wedge*(P_a\wedge e_b)-P^a\wedge*(P_a\wedge e_c)-U*e_c\right)$$
$$+\frac{\gamma}{4}\left(-2(\iota_cR^{ab})\wedge*R_{ab}+2R^{ab}\wedge*(R_{ab}\wedge e_c)-V*e_c\right)$$
$$+D\left(\left(2\alpha+\frac{\beta}{2}\right)\iota_c*dR+\beta*DP_c\right)=0\,. \tag{2.47}$$

It is also of interest to obtain the trace of Einstein equation by considering exterior multiplication of Einstein equation by co-frame e^a. Such a calculation yields

$$((n-4)f+R)*1-\left(2(n-1)\alpha+\frac{n\beta}{2}\right)d*dR=0\,. \tag{2.48}$$

Next, one can study the same action by considering independent variations of co-frames e^a and connections $\omega^a{}_b$. In that case, the resulting field equations are

$$f*e_c+\left(\left(\alpha+\frac{\gamma}{2}\right)R+\frac{1}{2}\right)\left(R^{ab}\wedge*(e_a\wedge e_b\wedge e_c)-R*e_c\right)$$
$$+\left(\frac{\beta}{2}-\gamma\right)\left(P^a{}_c*P_a+(\iota_cR^{ba})\wedge*(P_a\wedge e_b-P_b\wedge e_a)-P^a\wedge*(P_a\wedge e_c)-U*e_c\right)$$
$$+\frac{\gamma}{4}\left(-2(\iota_cR^{ab})\wedge*R_{ab}+2R^{ab}\wedge*(R_{ab}\wedge e_c)-V*e_c\right)=0 \tag{2.49}$$

and

$$\left(\alpha+\frac{\gamma}{2}\right)dR\wedge*(e_a\wedge e_b)+\left(\left(\alpha+\frac{\gamma}{2}\right)R+\frac{1}{2}\right)T^c\wedge*(e_a\wedge e_b\wedge e_c)$$
$$+\left(\frac{\beta}{2}-\gamma\right)D*(P_a\wedge e_b-P_b\wedge e_a)+\gamma D*R_{ab}=0\,. \tag{2.50}$$

Here, all the field equations ($T^a = 0$ or $T^a \neq 0$) in higher dimensions are easily obtained by substituting gravitational functional (2.41) of QCG into the field equations of GGM. On the other hand, the field equations (again $T^a = 0$ or $T^a \neq 0$) of QCG model are clearly derived without using the identity for the Gauss-Bonnet term (2.39) in Appendix D.

2.2. Quadratic Curvature Gravity Models In 3-Dimensional Spacetimes

Three-dimensional gravity theories are considered as toy models in constructing a consistent theory of quantum gravity in higher dimensions. Therefore, they attracted too much attention in recent years. It is known that Einstein theory of gravity in $(2+1)$ dimensions has no propagating degrees of freedom such that the theory in its own is non-dynamical. By adding gravitational Chern-Simons term and a cosmological constant to Einstein-Hilbert term, one can obtain Topologically Massive Gravity (TMG). Some modifications of the theory can be made possible by introducing additional quadratic curvature terms resulting in New Massive Gravity (NMG), Minimal Massive Gravity (MMG) and Generalized Massive Gravity (GMG). In what follows, these 3-dimensional gravity models are investigated within two different formalisms, namely by first considering that the connections are constrained to be Levi-Civita and then considering that the co-frame and connection 1-form fields are independent.

2.2.1. Topological Massive Gravity

The three-dimensional gravity theory known as Topologically Massive Gravity (TMG) [15] includes the Einstein-Hilbert term, a cosmological constant and a gravitational Chern-Simons term as a deformation. The theory is also known as Chern-Simons gravity and renowned Lagrangian is described by

$$\mathcal{L}_{TMG} = -\frac{\sigma}{2} R_{ab} \wedge *(e^a \wedge e^b) - \Lambda * 1 + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right) , \quad (2.51)$$

where σ is a dimensionless parameter which defines the sign of Einstein-Hilbert term ($\sigma = \pm 1$), Λ is a cosmological constant and μ is a mass parameter (or the Chern-Simons coupling constant). This dynamical theory that describes a massive graviton admits both static and stationary (rotating) classes of exact black hole solutions with a negative cosmological constant. In the limit that $|\mu| \to \infty$, the TMG action reduces to the 3-dimensional GR action with a cosmological constant (for $\sigma = -1$).

Under the constraint that the connection is Levi-Civita, one can consider the variation of the following action density with respect to orthonormal 1-form field e^a, connection 1-form field $\omega^a{}_b$ and Lagrange multiplier $(n-1)$-form field λ_a:

$$\delta\bar{\mathcal{L}}_{TMG} = -\frac{\sigma}{2} R_{ab} \wedge *(e^a \wedge e^b) - \Lambda * 1 + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right) + \lambda_a \wedge T^a . \quad (2.52)$$

Then, corresponding variations lead to

$$
\begin{aligned}
\bar{\mathcal{L}}_{TMG} \;\; = \;\; & \delta e_a \wedge (\sigma G^a - \Lambda * e^a + D\lambda^a) + (\delta\lambda_a) \wedge T^a \\
& + \delta\omega_{ab} \wedge \left(-\frac{\sigma}{2} D * (e^a \wedge e^b) + \frac{1}{2\mu} R^{ba} - \frac{1}{2}(e^a \wedge \lambda^b - e^b \wedge \lambda^a) \right) \\
& + d\left(-\frac{\sigma}{2} \left(\delta\omega^{ab}\right) \wedge *(e_a \wedge e_b) + \frac{1}{4\mu} \left(\delta\omega^{ab}\right) \wedge \omega_{ba} + (\delta e^a) \wedge \lambda_a \right) \; , \quad (2.53)
\end{aligned}
$$

where

$$
G_a \equiv -\frac{1}{2} R^{bc} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{1}{2}\epsilon_{abc} R^{bc} \quad (2.54)
$$

denotes the Einstein 2-forms with the identification $*(e_a \wedge e_b \wedge e_c) = \epsilon_{abc}$ in three-dimensions. It can easily be seen that, the co-frame variations give

$$
\sigma G^a - \Lambda * e^a + D\lambda^a = 0 \, , \quad (2.55)
$$

while variations of the connection yield

$$
-\frac{\sigma}{2} D * (e^a \wedge e^b) + \frac{1}{2\mu} R^{ba} - \frac{1}{2}(e^a \wedge \lambda^b - e^b \wedge \lambda^a) = 0 \, . \quad (2.56)
$$

With some algebra (the compatible details can be reached for $\alpha = 0$ in (E.30)-(E.34)), one can see that, equation (2.56) admits a unique solution for the Lagrange multiplier $(n-2)$-forms as

$$
\lambda^a = -\frac{1}{2\mu} \left(2P^a - \frac{1}{2} R e^a \right) = -\frac{1}{2\mu} Y^a \, , \quad (2.57)
$$

where

$$
Y^a = P^a - \frac{1}{4} R e^a \quad (2.58)
$$

denotes the Schouten 1-form field. Then, the celebrated Einstein field equation of TMG can be obtained as

$$
\sigma G^a - \Lambda * e^a - \frac{1}{\mu} C^a = 0 \, , \quad (2.59)
$$

where

$$
C^a \equiv DY^a = dY^a + \omega^a{}_b \wedge Y^b \quad (2.60)
$$

symbolize a symmetric and traceless Cotton 2-form in terms of the Schouten 1-form field Y^a. It has been shown that TMG admits a variety of solutions. To mention some of them are classes of solutions that are locally equivalent to biaxially squashed AdS_3 or to AdS pp-waves where an almost complete classification has been recently made in [19]. In addition, the supersymmetric version of the solutions of TMG has been obtained in [16, 17].

2.2.2. New Massive Gravity

To avoid the bulk versus boundary clash problem which is present in TMG, some other massive gravity models have been proposed to obtain holographically consistent theories of massive gravity. One of them is New Massive Gravity (NMG) introduced in [20]. NMG is a

three-dimensional theory of gravity that involves R^2 and U terms specifically. In this sense, NMG can be considered to be a quadratic curvature gravity model in 3-dimensions. In the literature, it has been shown that the theory admits some interesting physical solutions like AdS waves [21], warped AdS black holes [22], type D and type N solutions [23]. It has also been proven that the theory also admits rotating asymptotically AdS black hole solution [20, 24] as well as a class of asymptotically Lifshitz solutions (together with Lifshitz black hole solution with a dynamical exponent $z = 3$ where $z = 1$ corresponds to AdS) [25].
The general NMG model can be described by three-dimensional QCG action density as

$$\mathcal{L} = \frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b) + \Lambda * 1 + \frac{\alpha}{2}R^2 * 1 + \frac{\beta}{2}P_a \wedge *P^a \,. \tag{2.61}$$

One should also recall that the action density (2.61) can be written in the form

$$\mathcal{L} = (f(R,U) + \Lambda) * 1 \,, \tag{2.62}$$

where

$$f(R,U) = \frac{1}{2}(\alpha R^2 + R) + \frac{\beta}{2}U \,. \tag{2.63}$$

As in the previous section, the field equations can be obtained by considering either the connections to be Levi-Civita (i.e., with constraint $T^a = 0$) or by making independent variations of action with respect to co-frame e^a and connection $\omega^a{}_b$. First, one can examine the variation of action density by constraining the connections to be Levi-Civita. Then, the constrained Lagrangian reads

$$\bar{\mathcal{L}} = \left(\frac{1}{2}\left(\alpha R^2 + R\right) + \frac{\beta}{2}U + \Lambda\right) * 1 + \lambda_a \wedge T^a \,. \tag{2.64}$$

Then, the variation of (2.64) with respect to the independent variables e^a, $\omega^a{}_b$ and λ^a gives

$$\begin{aligned}
\delta\bar{\mathcal{L}} \;=\; & \delta e^c \wedge \Bigg[\left(\alpha R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \left(\Lambda - \frac{\alpha}{2}R^2\right) * e_c + D\lambda_c \\
& + \frac{\beta}{2}\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c)\Big)\Bigg] \\
& + \delta\omega^{ab} \wedge \Bigg[\frac{1}{2}D\left[*\left(e_a \wedge e_b\right)\right] + \alpha D\left[R * \left(e_a \wedge e_b\right)\right] + \frac{\beta}{2}D\left[*\left(P_a \wedge e_b - P_b \wedge e_a\right)\right] \\
& + \frac{1}{2}\left(e_b \wedge \lambda_a - e_a \wedge \lambda_b\right)\Bigg] + \delta\lambda_a \wedge T^a \\
& + d\left[(\delta\omega^{ab}) \wedge \left(\left(\frac{1}{2} + \alpha R\right) * \left(e_a \wedge e_b\right) + \frac{\beta}{2} * \left(P_a \wedge e_b - P_b \wedge e_a\right)\right) + (\delta e^a) \wedge \lambda_a\right] .
\end{aligned} \tag{2.65}$$

By minimal action principle, the field equations of model can be obtained as

$$\begin{aligned}
& \left(\alpha R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \left(\Lambda - \frac{\alpha}{2}R^2\right) * e_c + D\lambda_c \\
& + \frac{\beta}{2}\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c)\Big) = 0
\end{aligned} \tag{2.66}$$

and

$$D\left((\frac{1}{2}+\alpha R)*(e_a\wedge e_b)\right)+\frac{\beta}{2}D\left(*(P_a\wedge e_b-P_b\wedge e_a)\right)+\frac{1}{2}(e_b\wedge\lambda_a-e_a\wedge\lambda_b)=0\,. \quad (2.67)$$

Lagrange multiplier $(n-2)$-forms can be easily written from (2.20) by using gravitation function (2.63) in the form

$$\lambda_a=\left(2\alpha+\frac{\beta}{2}\right)\iota_a*dR+\beta*DP_a\,. \quad (2.68)$$

Since the details of the procedure is outlined in (C.50)-(C.74), the Einstein field equation of the theory becomes

$$\begin{aligned}&\left(\alpha R+\frac{1}{2}\right)R^{ab}\wedge*(e_a\wedge e_b\wedge e_c)+\left(\Lambda-\frac{\alpha}{2}R^2\right)*e_c\\&+\frac{\beta}{2}\left(P^a{}_c*P_a+(\iota_cR^{ba})\wedge*(P_a\wedge e_b-P_b\wedge e_a)-P^a\wedge*(P_a\wedge e_c)\right)\\&+D\left(\left(2\alpha+\frac{\beta}{2}\right)\iota_c*dR+\beta*DP_c\right)=0\,. \end{aligned}\quad (2.69)$$

To be complete, one can also obtain the trace of the Einstein field equation (2.69) as:

$$\left(\frac{1}{2}\left(\alpha R^2-R\right)+\frac{\beta}{2}U-3\Lambda\right)*1+\left(4\alpha+\frac{3\beta}{2}\right)d*dR=0\,. \quad (2.70)$$

One can also consider independent variations of $\omega^a{}_b$ and e^a. In that case, the field equations become

$$\begin{aligned}&\left(\alpha R+\frac{1}{2}\right)R^{ab}\wedge*(e_a\wedge e_b\wedge e_c)+\left(\Lambda-\frac{\alpha}{2}R^2\right)*e_c\\&+\frac{\beta}{2}\left(P^a{}_c*P_a+(\iota_cR^{ba})\wedge*(P_a\wedge e_b-P_b\wedge e_a)-P^a\wedge*(P_a\wedge e_c)\right)=0\end{aligned}\quad (2.71)$$

and

$$D\left((\frac{1}{2}+\alpha R)*(e_a\wedge e_b)+\frac{\beta}{2}*(P_a\wedge e_b-P_b\wedge e_a)\right)=0\,, \quad (2.72)$$

which can also be expressed in alternative form

$$\alpha\,dR\wedge*(e_a\wedge e_b)+(\frac{1}{2}+\alpha R)T^c\,\epsilon_{abc}+\frac{\beta}{2}D\left(*(P_a\wedge e_b-P_b\wedge e_a)\right)=0\,. \quad (2.73)$$

At this stage, it should be noted that by taking the coupling constants

$$\frac{\alpha}{2}=\frac{3}{8m^2},\qquad\frac{\beta}{2}=-\frac{1}{m^2} \quad (2.74)$$

with correct mass dimensions, one can obtain the familiar form of NMG Lagrangian density n-form from (2.61) as given in [20]:

$$\mathcal{L}_{NMG}=\left(\frac{1}{2}R+\Lambda-\frac{1}{m^2}\left(P_a\wedge*P^a-\frac{3}{8}R^2\right)\right)*1\,. \quad (2.75)$$

The Einstein field equation of NMG with $T^a = 0$ can be written as

$$
\begin{aligned}
&\left(\frac{3}{4m^2}R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \left(\Lambda - \frac{3}{8m^2}R^2\right) * e_c \\
&-\frac{1}{m^2}\left(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c)\right) \\
&+D\left(\iota_c * dR - 4 * DP_c\right) = 0 \qquad (2.76)
\end{aligned}
$$

while the trace of Einstein field equation becomes

$$
\frac{1}{2}\left(\frac{3}{4m^2}R^2 - R\right) - \frac{1}{m^2}U - 3\Lambda = 0\,. \qquad (2.77)
$$

One can easily see from the trace expression that there is no contribution from the second term of (2.70) for NMG theory. Before closing this section, one can also obtain NMG theory field equations for independent variations of the connection 1-forms and the co-frame 1-forms by taking (2.74) as

$$
\begin{aligned}
&\left(\frac{3}{4m^2}R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \left(\Lambda - \frac{3}{8m^2}R^2\right) * e_c \\
&-\frac{1}{m^2}\left(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c)\right) = 0 \qquad (2.78)
\end{aligned}
$$

and

$$
D\left((\frac{1}{2} + \frac{3}{4m^2}R) * (e_a \wedge e_b) - \frac{1}{m^2} * (P_a \wedge e_b - P_b \wedge e_a)\right) = 0\,, \qquad (2.79)
$$

which can be alternatively written as

$$
\frac{3}{4m^2}\, dR \wedge *(e_a \wedge e_b) + (\frac{1}{2} + \frac{3}{4m^2}R)T^c\, \epsilon_{abc} - \frac{1}{m^2} D * (P_a \wedge e_b - P_b \wedge e_a) = 0\,. \qquad (2.80)
$$

Finally, the equations (2.73) and (2.75) suggest that vacuum field equations of NMG (or 3 dimensional QCG theory in general) can admit constant curvature solutions with constant torsion where the torsion can take the specific form

$$
T_a = \frac{1}{2}\,\tau\, \epsilon_{abc}\, e^b \wedge e^c \qquad (2.81)
$$

where τ denotes constant torsion. This solution is to be discussed in Section 3.

2.2.3. Generalized Massive Gravity

It is also possible to construct a 3-dimensional massive gravity theory with quadratic curvature terms by adding gravitational Chern-Simons term to NMG Lagrangian. The resulting theory can be identified as Generalized Massive Gravity (GMG) in 3-dimensions. Then the Lagrangian of GMG can be expressed in the form

$$
\mathcal{L}_{GMG} \;=\; \mathcal{L}_{NMG} + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right)\,. \qquad (2.82)
$$

This can be explicitly given by

$$\begin{aligned}\mathcal{L}_{GMG} &= \frac{1}{2}R^{ab}\wedge *(e_a\wedge e_b)+\Lambda *1+\frac{3}{8m^2}R^2*1-\frac{1}{m^2}P_a\wedge *P^a\\ &+\frac{1}{4\mu}\left(\omega^a{}_b\wedge d\omega^b{}_a+\frac{2}{3}\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a\right),\end{aligned} \tag{2.83}$$

where α and β denotes coupling constants of NMG and μ is the mass parameter. Introducing the Lagrange multiplier $(n-2)$-form to ensure that the connections to be torsion-free, variation of GMG Lagrangian with the constraint term reads

$$\begin{aligned}\delta\bar{\mathcal{L}}_{GMG} &= \frac{1}{2}R^{ab}\wedge *(e_a\wedge e_b)+\Lambda *1+\frac{3}{8m^2}R^2*1-\frac{1}{m^2}P_a\wedge *P^a\\ &+\frac{1}{4\mu}\left(\omega^a{}_b\wedge d\omega^b{}_a+\frac{2}{3}\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a\right)+\lambda_a\wedge T^a.\end{aligned} \tag{2.84}$$

Then, the variation of GMG Lagrangian under torsion-free constraint yields

$$\begin{aligned}\delta\bar{\mathcal{L}}_{GMG} &= \delta e^c\wedge\Bigg[\left(\frac{3}{4m^2}R+\frac{1}{2}\right)R^{ab}\wedge *(e_a\wedge e_b\wedge e_c)+\left(\Lambda-\frac{3}{8m^2}R^2\right)*e_c+D\lambda_c\\ &-\left(\frac{1}{m^2}\right)\Big(P^a{}_c*P_a+(\iota_cR^{ba})\wedge *(P_a\wedge e_b-P_b\wedge e_a)-P^a\wedge *(P_a\wedge e_c)\Big)\Bigg]\\ &+\delta\omega^{ab}\wedge\Bigg[D*(e_a\wedge e_b)+\frac{3}{4m^2}D\left(R*(e_a\wedge e_b)\right)-\frac{1}{m^2}D*(P_a\wedge e_b-P_b\wedge e_a)\\ &+\frac{1}{2\mu}R_{ba}+\frac{1}{2}(e_b\wedge\lambda_a-e_a\wedge\lambda_b)\Bigg]+\delta\lambda_a\wedge T^a\\ &+d\Bigg[(\delta\omega^{ab})\wedge\left(\left(\frac{1}{2}+\frac{3}{4m^2}R\right)*(e_a\wedge e_b)-\frac{1}{m^2}*(P_a\wedge e_b-P_b\wedge e_a)+\frac{1}{4\mu}\omega_{ba}\right)\\ &+(\delta e^a)\wedge\lambda_a\Bigg].\end{aligned} \tag{2.85}$$

Since the variational contribution of the Chern-Simons term is given through (E.6)-(E.11), the field equations of GMG can be easily written by using minimal action principle as

$$\begin{aligned}&\left(\frac{3}{4m^2}R+\frac{1}{2}\right)R^{ab}\wedge *(e_a\wedge e_b\wedge e_c)+\left(\Lambda-\frac{3}{8m^2}R^2\right)*e_c+D\lambda_c\\ &-\left(\frac{1}{m^2}\right)\Big(P^a{}_c*P_a+(\iota_cR^{ba})\wedge *(P_a\wedge e_b-P_b\wedge e_a)-P^a\wedge *(P_a\wedge e_c)\Big)=0\end{aligned} \tag{2.86}$$

and

$$\begin{aligned}&D*(e_a\wedge e_b)+\frac{3}{4m^2}D[R*(e_a\wedge e_b)]-\frac{1}{m^2}D*(P_a\wedge e_b-P_b\wedge e_a)+\frac{1}{2\mu}R_{ba}\\ &+\frac{1}{2}(e_b\wedge\lambda_a-e_a\wedge\lambda_b)=0.\end{aligned} \tag{2.87}$$

Using the definition (2.16) in connection equation (2.87) gives the Lagrange multiplier of GGG model as

$$\lambda_a = \frac{1}{2m^2}\iota_a * dR + \beta * DP_a - \frac{1}{\mu}Y_a\,. \tag{2.88}$$

Therefore the Einstein field equation of GMG becomes

$$\begin{aligned}
&\left(\frac{3}{4m^2}R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \left(\Lambda - \frac{3}{8m^2}R^2\right) * e_c \\
&-\left(\frac{1}{m^2}\right)\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c)\Big) \\
&+D\left(\frac{1}{2m^2}\iota_c * dR + \beta * DP_c - \frac{1}{\mu}Y_c\right) = 0\,.
\end{aligned} \tag{2.89}$$

On the other hand, it is also possible to consider independent variations of GMG Lagrangian with respect to co-frmaes e^a and connections $\omega^a{}_b$. Such a variation yields following field equations:

$$\begin{aligned}
&\left(\frac{3}{4m^2}R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \left(\Lambda - \frac{3}{8m^2}R^2\right) * e_c \\
&-\left(\frac{1}{m^2}\right)\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c)\Big) = 0
\end{aligned} \tag{2.90}$$

and

$$D * (e_a \wedge e_b) + \frac{3}{4m^2}D\left(R * (e_a \wedge e_b)\right) - \frac{1}{m^2}D * (P_a \wedge e_b - P_b \wedge e_a) + \frac{1}{2\mu}R_{ba} = 0\,. \tag{2.91}$$

The equation coming from the connection variation can also be expressed in the form

$$\begin{aligned}
&\left(1 + \frac{3}{4m^2}R\right)\epsilon_{abc}T^c + \frac{3}{4m^2}dR \wedge *(e_a \wedge e_b) \\
&-\frac{1}{m^2}D * (P_a \wedge e_b - P_b \wedge e_a) + \frac{1}{2\mu}R_{ba} = 0
\end{aligned} \tag{2.92}$$

Looking at the last expression, one can immediately conclude that independent connection variations leads to propagating torsion in the theory. Therefore, the field equations involving torsion considerably differ from the field equations obtained under torsion-free constraint.

2.2.4. Minimal Massive Gravity

Minimal Massive Gravity (MMG) is a three-dimensional gravity model which has attracted much attention in recent years. MMG is proposed as the extension of TMG with an auxiliary field in the action such that at the very end the extension leads to a particular form of curvature squared terms in the Einstein field equation. MMG is proposed to elude the bulk/boundary clash that exists in TMG. It has been shown that [26], although the bulk properties of MMG and TMG are identical, the boundary properties are not same if one calculates central charges of corresponding conformal field theory. The geometrical analysis of the unitarity conditions of MMG are investigated in [27]. Matter-coupled MMG theory has been first investigated in [28]. In [28, 29], it has also been shown that for a specific value of the cosmological constant, MMG has a unique maximally-symmetric vacuum solution which is also identified as 'merger point'. In addition, circularly symmetric static and stationary solutions of MMG at its merger points are given in [45]. An alternative derivation of the minimal massive 3D gravity [31] and an alternative approach to matter coupling in Minimal Massive 3D gravity [32] are presented by using the algebra of exterior forms. Yet there is another work [33] in which MMG theory is examined within exterior algebra formalism. In this section, the field equations of the 3-dimensional MMG theory with and without matter coupling are examined. The details of the calculations for vacuum and matter coupled cases are presented in Appendix D.

i. MMG field equations without matter (Vacuum field equations)

Lagrangian density of MMG, introduced in [26], is given in terms of exterior forms as

$$\begin{aligned}\mathcal{L}_{MMG} &= -\frac{\sigma}{2}R_{ab}(\omega)\wedge *(e^a\wedge e^b)+\Lambda*1+\frac{1}{4\mu}\left(\omega^a{}_b\wedge d\omega^b{}_a+\frac{2}{3}\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a\right)\\ &\quad+\lambda_a\wedge T^a(\omega)+\frac{\alpha}{2}\lambda_a\wedge\lambda_b\wedge *(e^a\wedge e^b)\,. \qquad (2.93)\end{aligned}$$

As usual, μ describes the mass parameter inherited from TMG while λ_a is introduced as the auxiliary 1-form field where for the coupling constant $\alpha=0$, it enforces the vanishing torsion constraint for the TMG model. Furthermore, σ is taken to be a dimensionless parameter which defines the sign of Einstein-Hilbert term (usually taken as $\sigma=\pm1$) and Λ again denotes the cosmological constant.
The variation of the Lagrangian density $\mathcal{L}_{MMG}$ with respect to e^a, $\omega^a{}_b$ and λ^a leads

$$\begin{aligned}\delta\mathcal{L}_{MMG} &= \delta e_a\wedge\left(\sigma G^a(\omega)+\Lambda*e^a+D(\omega)\lambda^a+\frac{\alpha}{2}\epsilon^{abc}\lambda_b\wedge\lambda_c\right)\\ &\quad+\delta\omega_{ab}\wedge\left(-\frac{\sigma}{2}D(\omega)*(e^a\wedge e^b)+\frac{1}{2\mu}R^{ba}(\omega)-\frac{1}{2}(e^a\wedge\lambda^b-e^b\wedge\lambda^a)\right)\\ &\quad+\delta\lambda_a\wedge\left(T^a(\omega)+\alpha\epsilon^{abc}e_b\wedge\lambda_c\right)\\ &\quad+d\left(-\frac{\sigma}{2}\left(\left(\delta\omega^{ab}\right)\wedge*(e_a\wedge e_b)\right)+\frac{1}{4\mu}\left(\delta\omega^{ab}\right)\wedge\omega_{ba}+(\delta e^a)\wedge\lambda_a\right) \quad (2.94)\end{aligned}$$

Assuming that the variation of connection and co-frame fields vanish on the boundary of spacetime manifold, one can conclude that the closed forms do not contribute to the field equations (i.e., they satisfy (2.10)). Therefore, the variational principle (2.11) implies

$$\sigma G^a(\omega)+\Lambda*e^a+D(\omega)\lambda^a+\frac{\alpha}{2}\epsilon^{abc}\lambda_b\wedge\lambda_c=0\,, \qquad (2.95)$$

$$-\frac{\sigma}{2}D(\omega)*(e^a\wedge e^b)+\frac{1}{2\mu}R^{ba}(\omega)-\frac{1}{2}(e^a\wedge\lambda^b-e^b\wedge\lambda^a)=0\,, \qquad (2.96)$$

$$T^a(\omega)+\alpha\epsilon^{abc}e_b\wedge\lambda_c=0\,. \qquad (2.97)$$

From equation (2.97), torsion T^a can be solved as

$$T^a = -\alpha\epsilon^{abc} e_b \wedge \lambda_c \,. \tag{2.98}$$

Then, by using Cartan structure equation

$$K^a{}_b \wedge e^b = T^a \,, \tag{2.99}$$

one can obtain contorsion 1-forms $K^a{}_b$ in the form

$$K^a{}_b = \alpha\,\epsilon^a{}_{bc}\lambda^c \,. \tag{2.100}$$

This suggests that one can decompose the connection 1-form $\omega^a{}_b$ into the Riemannian part, denoted by $\Gamma^a{}_b$, and non-Riemannian part in the form

$$\omega^{ab} = \Gamma^{ab} + \alpha\epsilon^{abc}\lambda_c \,. \tag{2.101}$$

In addition, the curvature 2-form can be decomposed by using (2.101) in the Cartan's second structure equation as

$$R^{ab}(\omega) = R^{ab}(\Gamma) + \alpha\epsilon^{abc} D(\Gamma)\lambda_c + \alpha^2\lambda^a \wedge \lambda^b \,. \tag{2.102}$$

Then, the field equations (2.95) and (2.96) can be rewritten in terms of Riemannian quantities by using (2.98) and (2.102). They take the forms

$$-\frac{\sigma}{2}\epsilon^{abc} R_{bc}(\Gamma) + \Lambda * e^a + (1+\alpha\sigma) D(\Gamma)\lambda^a - \frac{\alpha}{2}(1+\alpha\sigma)\epsilon^{abc}\lambda_b \wedge \lambda_c = 0 \,, \tag{2.103}$$

and

$$-\frac{1}{2\mu} R^{ab}(\Gamma) - \frac{\alpha}{2\mu}\epsilon^{ab}{}_c D(\Gamma)\lambda^c - \frac{\alpha^2}{2\mu}\lambda^a \wedge \lambda^b - \frac{1}{2}(1+\alpha\sigma)(e^a \wedge \lambda^b - e^b \wedge \lambda^a) = 0 \,, \tag{2.104}$$

respectively. To solve equation (2.104) for λ_a, one can multiply Einstein field equation by $\epsilon^{df}{}_a$ and substitute the term $\epsilon^{df}{}_a D(\Gamma)\lambda^a$ in (2.104). After some straightforward calculations, one finally ends up with the field equation which is fully expressed in terms of Levi-Civita connection Γ^{ab} of the form

$$-\frac{1}{2\mu(1+\alpha\sigma)} R^{ab}(\Gamma) - \frac{\alpha\Lambda}{2\mu(1+\alpha\sigma)} e^a \wedge e^b - \frac{1}{2}(1+\alpha\sigma)\left(e^a \wedge \lambda^b - e^b \wedge \lambda^a\right) = 0 \,. \tag{2.105}$$

Equation (2.105) can be exactly solved for the auxiliary field λ_a as (see appendix D for detailed calculations)

$$\lambda_a = -\frac{1}{2\mu(1+\alpha\sigma)^2}\left(2Y_a + \alpha\Lambda e_a\right) \,, \tag{2.106}$$

where Y^a denotes the Schouten 1-form, identified in (2.58). Then, final form of the Einstein field equation in terms of the Levi-Civita connection becomes

$$
\begin{aligned}
&-\frac{\sigma}{2}\epsilon^{abc}R_{bc}(\Gamma)+\Lambda*e^{a}-\frac{1}{2\mu(1+\alpha\sigma)}D(\Gamma)\left(2Y^{a}+\alpha\Lambda e^{a}\right)\\
&-\frac{\alpha}{8\mu^{2}(1+\alpha\sigma)^{3}}\epsilon^{abc}\left(2Y_{b}+\alpha\Lambda e_{b}\right)\wedge\left(2Y_{c}+\alpha\Lambda e_{c}\right)=0
\end{aligned}
\tag{2.107}
$$

or equivalently

$$
\begin{aligned}
&-\left(\frac{\sigma}{2}+\frac{\alpha^{2}\Lambda}{4\mu^{2}(1+\alpha\sigma)^{3}}\right)\epsilon^{abc}R_{bc}(\Gamma)+\left(\Lambda-\frac{\alpha^{3}\Lambda^{2}}{4\mu^{2}(1+\alpha\sigma)^{3}}\right)*e^{a}\\
&-\frac{1}{\mu(1+\alpha\sigma)}C^{a}-\frac{\alpha}{2\mu^{2}(1+\alpha\sigma)^{3}}\epsilon^{abc}Y_{b}\wedge Y_{c}=0\,,
\end{aligned}
\tag{2.108}
$$

where C^a denotes the Cotton 2-forms. It can be clearly seen that the last term in (2.108) involves the quadratic-curvature terms which is a consequence of the MMG field equations.

ii. MMG field equations with Maxwell-Chern-Simons coupling

In this subsection, MMG model is coupled to the electromagnetic Maxwell-Chern-Simons term where the Lagrangian three-form reads

$$
\begin{aligned}
\bar{\mathcal{L}}_{MMG} &= \mathcal{L}_{MMG}+\mathcal{L}(matter)\\
&= \mathcal{L}_{MMG}-\frac{1}{2}F\wedge *F-\frac{1}{2}mA\wedge F\,.
\end{aligned}
\tag{2.109}
$$

Here, $\mathcal{L}_{MMG}$ denotes renowned MMG Lagrangian (2.93) and the Maxwell field is defined as $F = dA$, where A denotes the vector potential. The last term is identified as Chern-Simons term with the coupling constant m. The variation of $\bar{\mathcal{L}}_{MMG}$ with respect to e^a, $\omega^a{}_b$, λ^a and A yields

$$
\begin{aligned}
\delta\bar{\mathcal{L}}_{MMG} &= \delta e^{a}\wedge\Big(\sigma G_{a}(\omega)+\Lambda*e_{a}+D(\omega)\lambda_{a}+\frac{\alpha}{2}\epsilon_{abc}\lambda^{b}\wedge\lambda^{c}+\tau_{a}[F]\Big)\\
&+\delta\omega^{ab}\wedge\left(-\frac{\sigma}{2}D(\omega)*(e_{a}\wedge e_{b})+\frac{1}{2\mu}R_{ba}(\omega)-\frac{1}{2}(e_{a}\wedge\lambda_{b}-e_{b}\wedge\lambda_{a})\right)\\
&+\delta\lambda^{a}\wedge\Big(T_{a}(\omega)+\alpha\epsilon_{abc}e^{b}\wedge\lambda^{c}\Big)+(\delta A)\wedge(d*F+mF)\\
&+d\left[-\frac{\sigma}{2}\Big(\delta\omega^{ab}\wedge*(e_{a}\wedge e_{b})\Big)+\frac{1}{4\mu}\Big(\delta\omega^{ab}\Big)\wedge\omega_{ba}+(\delta e^{a})\wedge\lambda_{a}+\frac{1}{2}(\delta A)\wedge A\right],
\end{aligned}
\tag{2.110}
$$

where

$$
\tau^{a}[F]=\frac{1}{2}\left(\iota^{a}F\wedge *F-F\wedge\iota^{a}*F\right)
\tag{2.111}
$$

is the electromagnetic field energy momentum 2-form. The variation with respect to the related fields gives the field equations of the model as

$$
\sigma G_{a}(\omega)+\Lambda*e_{a}+D(\omega)\lambda_{a}+\frac{\alpha}{2}\epsilon_{abc}\lambda^{b}\wedge\lambda^{c}+\tau_{a}[F]=0\,,
\tag{2.112}
$$

$$-\frac{\sigma}{2}D(\omega)*(e_a\wedge e_b)+\frac{1}{2\mu}R_{ba}(\omega)-\frac{1}{2}(e_a\wedge\lambda_b-e_b\wedge\lambda_a)=0\,, \tag{2.113}$$

$$T_a(\omega)+\alpha\epsilon_{abc}e^b\wedge\lambda^c=0 \tag{2.114}$$

and

$$d*F+mF=0\,. \tag{2.115}$$

As can easily be seen, (2.114) and (2.97) are identical. Therefore, torsion can be solved as $T^a=-\alpha\epsilon_{abc}e^b\wedge\lambda^c$ and Cartan structure equation leads writing contorsion 1-forms as $K^a{}_b=\alpha\,\epsilon^a{}_{bc}\lambda^c$. This yields the decomposition of the connection 1-form and curvature 2-form into Riemannian and non-Riemannian parts as given in (2.101) and (2.102). Then, the field equations according to orthonormal 1-forms and connection 1-forms can be fully expressed in terms of Riemannian quantities as

$$-\frac{\sigma}{2}\epsilon_{abc}R^{bc}(\Gamma)+\Lambda*e_a+(1+\alpha\sigma)D(\Gamma)\lambda_a-\frac{\alpha}{2}(1+\alpha\sigma)\epsilon_{abc}\lambda^b\wedge\lambda^c+\tau_a[F]=0\,, \tag{2.116}$$

$$-\frac{1}{2\mu}R_{ab}(\Gamma)-\frac{\alpha}{2\mu}\epsilon_{abc}D(\Gamma)\lambda^c-\frac{\alpha^2}{2\mu}\lambda_a\wedge\lambda_b-\frac{1}{2}(1+\alpha\sigma)(e_a\wedge\lambda_b-e_b\wedge\lambda_a)=0 \tag{2.117}$$

respectively. Here, multiplying the Einstein field equation ϵ^{dfa} and then substituting the term $\epsilon^{dfa}D(\Gamma)\lambda_a$ in (2.117) gives

$$-R^{ab}(\Gamma)-\alpha\Lambda e^a\wedge e^b-\mu(1+\alpha\sigma)^2\left(e^a\wedge\lambda^b-e^b\wedge\lambda^a\right)+\alpha\epsilon^{abc}\tau_c[F]=0\,. \tag{2.118}$$

The auxiliary field 1-form λ_a can be exactly solved from (2.118) as

$$\lambda^b=-\frac{1}{\mu(1+\alpha\sigma)^2}\left(Y^b(\Gamma)+\frac{1}{2}\alpha\Lambda e^b+\alpha\left(-\tau^{cb}[F]e_c+\frac{1}{2}\tau[F]e^b\right)\right)\,. \tag{2.119}$$

Here, redefining a stress-energy tensor as

$$-\tau^{cb}[F]e_c+\frac{1}{2}\tau[F]e^b=-*\hat{\tau}^b[F] \tag{2.120}$$

yields the following

$$\hat{\tau}^b[F]=-\tau^b[F]+\frac{1}{2}\tau[F]*e^b\,, \tag{2.121}$$

so that the auxiliary field 1-form reads

$$\lambda^b=-\frac{1}{\mu(1+\alpha\sigma)^2}\left(Y^b(\Gamma)+\frac{1}{2}\alpha\Lambda e^b+\alpha*\hat{\tau}^b[F]\right)\,. \tag{2.122}$$

Then, in terms of Riemannian quantities, the Einstein field equation becomes

$$
\begin{aligned}
&\left(-\frac{\sigma}{2}-\frac{\Lambda\alpha^2}{4\mu^2(1+\alpha\sigma)^3}\right)\epsilon_{abc}R^{bc}-\frac{1}{\mu(1+\alpha\sigma)}C_a+\tau_a[F] \\
&+\left(\Lambda-\frac{\Lambda^2\alpha^3}{4\mu^2(1+\alpha\sigma)^3}\right)*e_a-\frac{\alpha}{2\mu(1+\alpha\sigma)^3}\epsilon_{abc}Y^b(\Gamma)\wedge Y^c(\Gamma) \\
&-\frac{\alpha}{\mu(1+\alpha\sigma)^3}D(\Gamma)*\hat{\tau}_a[F]-\frac{\alpha^3}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}*\hat{\tau}^b[F]\wedge *\hat{\tau}^c[F] \\
&-\frac{\alpha^2}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\left(Y^b(\Gamma)\wedge *\hat{\tau}^c[F]-Y^c(\Gamma)\wedge *\hat{\tau}^b[F]\right) \\
&-\frac{\alpha^3\Lambda}{4\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\left(e^b\wedge *\hat{\tau}^c[F]-e^c\wedge *\hat{\tau}^b[F]\right)=0\,.
\end{aligned}
\tag{2.123}
$$

Although the field equations with/without matter coupling with respect to connection 1-forms seem equivalent, their auxiliary field 1-forms are different since there is an explicit contribution coming from matter field. If one drops the matter field, the related field equations are compatible with the ones presented in vacuum case.
One can also consider the field equation in the following form

$$
\mathcal{E}_a+\tau_a[F]+\mathcal{O}_a[\hat{\tau}]=0\,, \tag{2.124}
$$

where

$$
\begin{aligned}
\mathcal{E}_a \;=\; &\left(-\frac{\sigma}{2}-\frac{\Lambda\alpha^2}{4\mu^2(1+\alpha\sigma)^3}\right)\epsilon_{abc}R^{bc}-\frac{1}{\mu(1+\alpha\sigma)}C_a+\left(\Lambda-\frac{\Lambda^2\alpha^3}{4\mu^2(1+\alpha\sigma)^3}\right)*e_a \\
&-\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\,\epsilon_{abc}Y^b(\Gamma)\wedge Y^c(\Gamma)
\end{aligned}
\tag{2.125}
$$

and

$$
\begin{aligned}
\mathcal{O}_a[\hat{\tau}] \;=\; &-\frac{\alpha}{\mu(1+\alpha\sigma)^3}D(\Gamma)*\hat{\tau}_a[F]-\frac{\alpha^2}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\Big(\alpha*\hat{\tau}^b[F]\wedge *\hat{\tau}^c[F] \\
&+\left(Y^b(\Gamma)\wedge *\hat{\tau}^c[F]-Y^c(\Gamma)\wedge *\hat{\tau}^b[F]\right)+\frac{\alpha\Lambda}{2}\left(e^b\wedge *\hat{\tau}^c[F]-e^c\wedge *\hat{\tau}^b[F]\right)\Big)\,.
\end{aligned}
\tag{2.126}
$$

Here, $\mathcal{E}_a$ is the stress tensor of pure MMG theory and $\mathcal{O}_a[\tau]$ is the stress tensor contribution coming from the electromagnetic field. The usual conservation condition implies

$$
-\frac{\alpha}{\mu^2(1+\alpha\sigma)^3}\,\epsilon_{abc}C^b\wedge Y^c+D\mathcal{O}_a[\hat{\tau}]=0\,. \tag{2.127}
$$

Eliminating the Cotton 2-form C_a from the Einstein field equation (2.123) reads

$$
D\mathcal{O}_a[\hat{\tau}]=\frac{\alpha}{\mu(1+\alpha\sigma)^2}\epsilon_{abc}\left(\tau^b[F]+\mathcal{O}^b[\hat{\tau}]\right)\wedge Y^c(\Gamma)\,, \tag{2.128}
$$

which is known as the consistency relation in the literature [28, 32]. One can see that electromagnetic field contribution ensures that the field equation of matter-coupled MMG theory is consistent on shell. As a final remark, one can find details of the derivation of field equations in Appendix E.

3. SOME NEW EXACT SOLUTIONS OF GENERALIZED GRAVITY MODELS

In this section, exact constant curvature solutions of higher curvature models are derived. Firstly, QCG model is examined for constant curvature with/without torsion in higher dimensions. The solutions presented at first have a cross product structure of two constant curvature manifolds and the equations of motion yield relations for the coupling constants of the theory in order to have solution containing nontrivial torsion. Then, constant curvature solutions are obtained assuming that QCG Lagrangian independent from co-frame and connection fields. On the other hand, under the constraint that the connection is Levi-Civita, Lifshitz type static/stationary solutions of R^2-corrected gravity are investigated with some thermodynamical properties. Moreover, Bianchi type static/stationary solutions and Schrödinger type solutions of R^2-corrected gravity are obtained in higher dimensions as well.

Quadratic curvature gravity model can admit different classes of solutions. Among them are cosmological solutions of model with and without torsion in 4-dimensional spacetime. For $n = 5$, one can investigate Lifshitz (Li_5) solutions (where $z = 1$ corresponds to AdS_5 solutions). In addition, one can examine solutions of the form $Li_3 \times \Omega_2$ (for $z = 1$ corresponding to $AdS_3 \times \Omega_2$) with and without torsion where Ω corresponds to constant curvature spacetime. Bañados-Teitelboim-Zanelli (BTZ) [35] type solutions of the form $BTZ \times \Omega_2$ are of particular importance as well (where BTZ solutions correspond to 3-dimensional static and stationary solutions). On the other hand, for $n = 6$, the model can admit a variety of solutions. Among them are Li_6 ($z = 1$ corresponding to AdS_6) solutions, $Li_3 \times \Omega_3$ ($z = 1$ corresponding to $AdS_3 \times \Omega_3$), $Li_4 \times \Omega_2$ ($z = 1$ corresponding to $AdS_4 \times \Omega_2$). In addition solutions of the form $Li_3 \times Bianchi_3$ ($z = 1$ case corresponding to AdS-type), $BTZ \times \Omega_3$ with and without torsion deserve a particular investigation.

3.1. Constant Curvature Solutions Of Quadratic Curvature Gravity With And Without Torsion

In what follows, constant curvature vacuum field equations with and without torsion in spacetimes with $n \geq 6$ are discussed with the motivation of [36]. Moreover, in [37], general forms of symmetries are discussed in the presence of torsion. In the same work, how torsion affects the maximal symmetries are also studied. To this end, higher dimensional QCG Lagrangian density is considered as

$$\begin{aligned}\mathcal{L}_{QCG} &= \frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b) + \Lambda * 1 + \frac{\alpha}{2}R^2 * 1 + \frac{\beta}{2}R^{ab} \wedge *R_{ab} \\ &\quad + \frac{\eta}{4}R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)\,, \qquad (3.1)\end{aligned}$$

where Λ is a cosmological constant, α, β and η denote constants. Expressing the Gauss-Bonnet term in terms of R^2, $(Ricci)^2$ and $(Riemann)^2$ by using (2.39) yields

$$\mathcal{L}_{QCG} = \left(\frac{1}{2}R + \frac{1}{2}\left(\alpha + \frac{\eta}{2}\right)R^2 + \Lambda - U + \frac{1}{4}(\beta+\eta)V\right) * 1\,, \tag{3.2}$$

where in this case the gravitational function of the theory is

$$f = \frac{1}{2}\left(\left(\alpha + \frac{\eta}{2}\right)R^2 + R + 2\Lambda - U + \frac{1}{4}(\beta+\eta)V\right). \tag{3.3}$$

Introducing the Lagrange multiplier $(n-2)$-form to ensure that the connections to be the Levi-Civita gives

$$\bar{\mathcal{L}}_{QCG} = f * 1 + \lambda_a \wedge T^a\,. \tag{3.4}$$

The variations with respect to e^a, ${\omega^a}_b$ and λ^a implies the following field equations

$$\begin{aligned}&\left(\alpha R + \frac{1}{2}\right) D * (e_a \wedge e_b) + \beta D * R_{ab} + \frac{\eta}{2} D\left(R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)\right)\\ &+\frac{1}{2}\left(e_b \wedge \lambda_a - e_a \wedge \lambda_b\right) = 0\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\left(\alpha R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_f) + \left(\Lambda - \frac{\alpha}{2}R^2\right) * e_f\\ &-\frac{\beta}{2}\left((\iota_f R^{ab}) \wedge *R_{ab} - R^{ab} \wedge \iota_f * R_{ab}\right)\\ &+\frac{\eta}{4} R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d \wedge e_f) + D\lambda_f = 0\,.\end{aligned} \tag{3.6}$$

naturally with $T^a = 0$. Here, the Lagrange multiplier $(n-2)$-form of the model can be given by

$$\lambda_a = 2\alpha \iota_a * dR + 2\beta * DP_a\,. \tag{3.7}$$

Thus, the final form of the Einstein field equation is

$$\begin{aligned}&\left(\alpha R + \frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_f) + \left(\Lambda - \frac{\alpha}{2}R^2\right) * e_f\\ &-\frac{\beta}{2}\left((\iota_f R^{ab}) \wedge *R_{ab} - R^{ab} \wedge \iota_f * R_{ab}\right)\\ &+\frac{\eta}{4} R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d \wedge e_f) + 2D\left(\alpha \iota_f * dR + \beta * DP_f\right) = 0\,.\end{aligned} \tag{3.8}$$

When the connection of spacetime involves torsion, the field equations of the model via the variational principle can be written as

$$\left(\alpha R + \frac{1}{2}\right) D * (e_a \wedge e_b) + \beta D * R_{ab} + \frac{\eta}{2} D\left(R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)\right) = 0 \tag{3.9}$$

and

$$\left(\alpha R+\frac{1}{2}\right)R^{ab}\wedge *(e_a\wedge e_b\wedge e_f)+\left(\Lambda-\frac{\alpha}{2}R^2\right)*e_f$$
$$-\frac{\beta}{2}\left((\iota_f R^{ab})\wedge *R_{ab}-R^{ab}\wedge\iota_f *R_{ab}\right)+\frac{\eta}{4}R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d\wedge e_f)=0\,. \tag{3.10}$$

Recall that, one can immediately obtain associated field equations with appropriate coefficients with the aid of the calculations presented in Appendix C and Appendix D.

3.1.1. Constant curvature solutions with torsion

In this subsection, one can consider the n-dimensional metric

$$g=\eta_{ab}e^a\otimes e^b \tag{3.11}$$

where in the splitted form, this metric can be written as

$$g=\eta_{\alpha\beta}e^\alpha\otimes e^\beta+\eta_{ij}e^i\otimes e^j\,. \tag{3.12}$$

Here $\eta_{\alpha\beta}e^\alpha\otimes e^\beta$ corresponds to metric of $(n-3)$-dimensional subspace while $\eta_{ij}e^i\otimes e^j$ corresponds to 3-dimensional spacetime metric. Indices range as $i=\{0,1,2\}$ and $\alpha=\{3,4,\ldots,(n-1)\}$. Now, for the product spaces with maximally symmetric submanifolds, one can deal with the following ansatz for curvature 2-forms with respect to torsion-free connections:

$$\bar{R}^{\alpha\beta}=\bar{\Lambda}e^\alpha\wedge e^\beta\,,\qquad \bar{R}^{ij}=\tilde{\Lambda}e^i\wedge e^j\qquad (n\geq 6) \tag{3.13}$$

with $\bar{\Lambda}$ and $\tilde{\Lambda}$ being related to the curvature constants of the submanifolds. In addition, one can consider

$$T^\alpha=0\,,\qquad T^i=\frac{1}{2}\tau\epsilon^i{}_{jk}e^j\wedge e^k \tag{3.14}$$

for torsion ansatz. By using

$$2\omega_{ab}=\iota_a(T_b-de_b)-\iota_b(T_a-de_a)-(\iota_a\,\iota_b(T_c-de_c))e^c\,, \tag{3.15}$$

and assuming that $e^\alpha=e^\alpha(x^\alpha)$ and $e^i=e^i(x^i)$, one can obtain

$$\omega_{\alpha\beta}=\bar{\omega}_{\alpha\beta}\,,\qquad \omega_{\alpha i}=0\,,\qquad \omega_{ij}=\bar{\omega}_{ij}+K_{ij} \tag{3.16}$$

with

$$K_{ij}=\frac{1}{2}\tau\epsilon_{jik}\,e^k \tag{3.17}$$

determined from

$$K^i{}_j\wedge e^j=T^i\,. \tag{3.18}$$

Curvature 2-forms are obtained from (2.4) (i.e., the second Cartan structure equation). By using this relation, one can calculate

$$R^{\alpha\beta} = \bar{R}^{\alpha\beta}\,, \qquad R^{\alpha i} = 0\,, \qquad R^{ij} = \bar{R}^{ij} + \bar{D}K^{ij} + K^{i}{}_{k} \wedge K^{kj}\,. \tag{3.19}$$

Then, the field equations (3.9) obtained from the variation with respect to connection field ω^{ab} yields the following equations:

$$\left[\bar{\Lambda}\left(\alpha(n-3)(n-4) + \beta + \frac{\eta}{2}(n-5)(n-6)\right) + (6\alpha+1)\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right) + \frac{1}{2}\right]\tau = 0 \tag{3.20}$$

and

$$\left[(n-3)(n-4)\bar{\Lambda}\left(\alpha + \frac{\eta}{2}\right) + (6\alpha+\beta)\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right) + \frac{1}{2}\right]\tau = 0\,. \tag{3.21}$$

On the other hand, from the Einstein field equation (3.10), one obtains the following equations:

$$\begin{aligned}
&\left(6\alpha\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right) + \alpha(n-3)(n-4)\bar{\Lambda} + \frac{1}{2}\right)\left((n-4)(n-5)\bar{\Lambda} + \left(\tilde{\Lambda} + \frac{\tau^2}{4}\right)\right) \\
&+\Lambda - \frac{\alpha}{2}\left(6\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right) + (n-3)(n-4)\bar{\Lambda}\right) - \frac{\beta}{2}\left(-(n-4)(n-7)\bar{\Lambda}^2 + \left(\tilde{\Lambda} + \frac{\tau^2}{4}\right)^2\right) \\
&+\frac{\eta}{4}(n-4)(n-5)\bar{\Lambda}\left((n-6)(n-7)\bar{\Lambda} + 2\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right)\right) = 0
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
&\left(6\alpha\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right) + \alpha(n-3)(n-4)\bar{\Lambda} + \frac{1}{2}\right)\left((n-3)(n-4)\bar{\Lambda} + \left(\tilde{\Lambda} + \frac{\tau^2}{4}\right)\right) \\
&+\Lambda - \frac{\alpha}{2}\left(6\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right) + (n-3)(n-4)\bar{\Lambda}\right) - \frac{\beta}{2}\left(-(n-3)(n-4)\bar{\Lambda}^2 + \left(\tilde{\Lambda} + \frac{\tau^2}{4}\right)^2\right) \\
&+\frac{\eta}{4}(n-3)(n-4)\bar{\Lambda}\left((n-5)(n-6)\bar{\Lambda} + 2\left(\tilde{\Lambda} + \frac{\tau^2}{4}\right)\right) = 0
\end{aligned} \tag{3.23}$$

respectively. As can easily be seen that $\tau = 0$ solves the equations of motion but it is out of the subject of this subsection (i.e., $\tau = 0$ yields torsion-free constant curvature solutions). Here, the field equations imply two classes of solutions. In the first class, equations (3.20) and (3.21) require that

$$\tilde{\Lambda} = -\frac{\tau^2}{4} \tag{3.24}$$

and

$$\bar{\Lambda} = -\frac{1}{2}\left((n-3)(n-4)\left(\alpha + \frac{\eta}{2}\right)\right)\,. \tag{3.25}$$

In addition, the equations (3.22) and (3.23) enforces the condition

$$\eta = \frac{\beta}{(2n-9)} \tag{3.26}$$

for the coupling constant η and also the relation

$$\Lambda = \frac{(2n-9)}{4(2\alpha(2n-9)+\beta)} \,. \tag{3.27}$$

It means that quadratic curvature gravity model admits the first class of constant curvature solutions with torsion presented above it the relations (3.24), (3.25), (3.26) and (3.27) hold simultaneously.
On the other hand, the second class of solution requires

$$\tilde{\Lambda} = -\frac{\tau^2}{4} + \frac{(\beta+(9-2n)\eta)}{(\beta-1)}\,\bar{\Lambda}\,, \tag{3.28}$$

$$\bar{\Lambda} = -\frac{1}{2}\left((n-3)(n-4)\left(\alpha+\frac{\eta}{2}\right)+(6\alpha+\beta)\frac{(\beta+(9-2n)\eta)}{(\beta-1)}\right) \tag{3.29}$$

and

$$\eta = 2\,, \tag{3.30}$$

$$\begin{aligned}\Lambda \;=\; & \left(6\alpha\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)+\alpha(n-3)(n-4)\bar{\Lambda}+\frac{1}{2}\right)\left(-(n-3)(n-4)\bar{\Lambda}-\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)\right)\\ & +\frac{\alpha}{2}\left(6\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)+(n-3)(n-4)\bar{\Lambda}\right)+\frac{\beta}{2}\left(-(n-3)(n-4)\bar{\Lambda}^2+\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)^2\right)\\ & -\frac{\eta}{4}(n-3)(n-4)\bar{\Lambda}\left((n-5)(n-6)\bar{\Lambda}+2\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)\right)\,.\end{aligned} \tag{3.31}$$

It is seen that the second class of constant curvature solution is valid if the constant curvature $\bar{\Lambda}$ of the $(n-3)$-dimensional subspace is to be expressed in terms of the coupling constants of the gravity model and the dimension of the spacetime. Also, this class fixes the coupling constant η as $\eta = 2$. The study as presented here also discussed for five dimensional Einstein-Gauss-Bonnet gravity [38].
Before closing this subsection, one can also consider 3-dimensional QCG model in which the Lagrangian density is given as follows:

$$\mathcal{L} = \frac{1}{2}R^{ab}\wedge *(e_a\wedge e_b)+\Lambda * 1+\frac{\alpha}{2}R^2 * 1+\frac{\beta}{2}R^{ab}\wedge *R_{ab}\,, \tag{3.32}$$

where α and β denote coupling constants. In this case, indices are $i = 0, 1, 2$. Choosing torsion and curvature ansatz as

$$T^i = \frac{1}{2}\tau\epsilon^i{}_{jk}e^j\wedge e^k\,, \qquad \bar{R}^{ij} = \tilde{\Lambda}e^i\wedge e^j\,. \tag{3.33}$$

gives the following field equations

$$\left((6\alpha+\beta)\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)+\frac{1}{2}\right)\tau = 0 \tag{3.34}$$

and

$$\left(6\alpha\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)+\frac{1}{2}\right)\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)+\Lambda-18\alpha\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)^2-\frac{\beta}{2}\left(\tilde{\Lambda}+\frac{\tau^2}{4}\right)^2=0 \qquad (3.35)$$

which are respectively solved for

$$\tilde{\Lambda}=-\frac{\tau^2}{4}-\frac{1}{2(6\alpha+\beta)} \qquad \text{and} \qquad \Lambda=\frac{3(12\alpha+\beta)}{8(6\alpha+\beta)^2} \qquad (3.36)$$

provided $(6\alpha+\beta)\neq 0$. Additionaly, if one considers NMG theory, the field equations are simply solved for

$$\tilde{\Lambda}=-\frac{\tau^2}{4}-\frac{2m^2}{5} \qquad \text{and} \qquad \Lambda=\frac{21m^2}{25}\,. \qquad (3.37)$$

3.1.2. Constant curvature solutions without torsion

One can also obtain the constant curvature solutions of the QCG theory by assuming that the space is torsion-free. Similarly, one can take

$$\bar{R}^{ab}=\hat{\Lambda}e^a\wedge e^b \qquad (3.38)$$

for the curvature ansatz (a similar approach is used in [36] for the Lovelock gravity models). Also bear in mind that

$$T^a=0 \qquad (3.39)$$

for this case. Then, the Einstein field equation yields

$$(n-1)(n-4)\hat{\Lambda}^2\left(\alpha\, n(n-1)+\beta+\frac{\eta}{2}(n-2)(n-3)\right)+\hat{\Lambda}(n-1)(n-2)+2\Lambda=0\,. \qquad (3.40)$$

The equation (3.40) can also be written in the form

$$a_2\hat{\Lambda}^2+a_1\hat{\Lambda}+a_0=0\,, \qquad (3.41)$$

where

$$a_2=(n-1)(n-4)\left(\alpha\, n(n-1)+\beta+\frac{\eta}{2}(n-2)(n-3)\right)\,, \qquad (3.42)$$

$$a_1=(n-1)(n-2)\,, \qquad (3.43)$$

$$a_0=2\Lambda\,. \qquad (3.44)$$

Then, the constant curvature $\hat{\Lambda}$ can be solved as

$$\hat{\Lambda}_{\pm}=\frac{-a_1\pm\sqrt{a_1^2-4a_2a_0}}{2a_2}\,. \qquad (3.45)$$

Also, it is obvious that for a physical solution, the parameters should respect the following inequality

$$a_1^2-4a_2a_0\geq 0\,. \qquad (3.46)$$

In this subsection, n-dimensional QCG field equations are obtained in order to see the effect of maximal symmetry onto the field equations of the model. This is accomplished for a cross product structure of two constant curvature manifolds (i.e., $M_n = M_{(n-3)} \times \Omega_{(3)}$). The relation between coupling constants yields solutions with nontrivial torsion. This choice motivates one to investigate the interesting phenomenological consequences of the presence of torsion in higher dimensions. For a comparison, the Levi-Civita constraint case is also examined for the same manifold structure. It can be done by simply taking $\tau = \tilde{\Lambda} = 0$ and $\bar{\Lambda} = \hat{\Lambda}$ in the field equations (3.20)-(3.23) with appropriate spacetime dimension change.
Interestingly, one can also examine constant curvature solution of 3-dimensional quadratic curvature gravity model described by the Lagrangian (3.32). In this case, taking one obtains

$$\Lambda + \hat{\Lambda} - (6\alpha + \beta)\hat{\Lambda}^2 = 0\,, \tag{3.47}$$

which yields

$$\hat{\Lambda}_{\pm} = \frac{1 \pm \sqrt{1 + 4(6\alpha + \beta)\Lambda}}{2(6\alpha + \beta)} \tag{3.48}$$

under the constraint that $1 + 4(6\alpha + \beta)\Lambda \geq 0$ and $(6\alpha + \beta) \neq 0$. In the limit $\alpha \to 0$ and $\beta \to 0$, one recovers TMG. For NMG, one finds

$$\hat{\Lambda} = \frac{2m^2}{5}\left(1 \pm \sqrt{1 + \frac{5\Lambda}{m^2}}\right)\,. \tag{3.49}$$

3.2. Some Exact Solutions Of R^2-Corrected Gravity Theory

In this subsection, some exact solutions of R^2-corrected gravity are examined. This gravity model can be described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}R^{ab} \wedge *(e_a \wedge e_b) + \Lambda * 1 + \frac{\alpha}{2}R^2 * 1\,, \tag{3.50}$$

where Λ is the cosmological constant and α is a coupling constant. The Einstein field equation of R^2 gravity theory under the constraint that the connections are Levi-Civita can be given as

$$f * e_c + \left(\alpha R + \frac{1}{2}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) + 2\alpha D\left(\iota_c * dR\right) = 0\,, \tag{3.51}$$

where

$$f(R) = \frac{1}{2}(\alpha R^2 + R) + \Lambda \tag{3.52}$$

is the gravitational function expressed in terms of scalar curvature and cosmological constant. It can be seen that R^2-corrected gravity theory can admit some classes of solutions including Lifshitz type topological static/stationary solutions, Bianchi type static/stationary solutions and Schrödinger type solutions for R^2-corrected gravity provided that the coupling parameter, cosmological constant and scalar curvature obey the relations

$$\alpha = \frac{1}{8\Lambda}, \qquad R = -4\Lambda\,. \tag{3.53}$$

Here, it should be emphasized that all the analytical solutions presented here are new and exact.

3.2.1. Lifshitz type topological static/stationary solutions

i. Lifshitz type static solutions

In this subsection, Lifshitz type static solutions of R^2-corrected gravity are investigated. Inspired by the work [39], $Li_m \times \Omega_{(n-m)}$ type solutions are investigated where Li_m refers to m-dimensional Lifshitz-type submanifold while $\Omega_{(n-m)}$ corresponds to $(n-m)$-dimensional spherical, hyperboloidal or flat subspace depending on the value of topological constant κ. Requiring the scaling transformations

$$t \to \lambda^z t, \qquad r \to \lambda^{-1} r, \qquad \vec{y} \to \lambda \vec{y}, \qquad \vec{x} \to \lambda \vec{x} \tag{3.54}$$

where z is the so-called dynamical exponent, $\vec{y}$ is a $(m-2)$-dimensional vector and $\vec{x}$ is a $(n-m)$-dimensional vector (for stationary spacetimes, if one identifies $y_{m-2} \equiv \varphi$, then $\vec{y}$ represents an $(m-3)$-dimensional vector), the Einstein field equation (3.51) admits the static vacuum $Li_m \times \Omega_{(n-m)}$ type solution given by

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} dt^2 + \left(\frac{l}{r}\right)^2 dr^2 + \left(\frac{r}{l}\right)^{2\gamma} dy_\alpha dy^\alpha + \left(\frac{r}{l}\right)^{2N} dx_i dx^i \tag{3.55}$$

provided that (3.53) and

$$\begin{aligned} \Lambda &= \frac{1}{4l^2}\bigg((N(n-m) + \gamma(m-2))^2 + 2z\,(z + N(n-m) + \gamma(m-2)) \\ &\quad + N^2(n-m) + \gamma^2(m-2)\bigg)\,. \end{aligned} \tag{3.56}$$

Remarkably, provided that the relations (3.53) and (3.56) hold, R^2-corrected gravity also admit the spacetime solution (which can be identified as black hole solution with spherical, hyperboloidal or flat geometry) given by

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} f^2(r) dt^2 + \left(\frac{l}{r}\right)^2 \frac{dr^2}{f^2(r)} + \left(\frac{r}{l}\right)^{2\gamma} dy_\alpha dy^\alpha + \left(\frac{r}{l}\right)^{2N} \frac{dx_i dx^i}{\left(1 + \kappa\frac{\rho^2}{4}\right)^2}\,, \tag{3.57}$$

where metric function $f^2(r)$ reads

$$f^2(r) = 1 + c_2 \frac{l^{2N}}{r^{2N}} + c_3 \frac{l^{p_+}}{r^{p_+}} + c_4 \frac{l^{p_-}}{r^{p_-}}\,. \tag{3.58}$$

Here,

$$p_{\pm} = \frac{1}{2}\Big(2N(n-m)+3z+2\gamma(m-2) \pm \sqrt{4N(n-m)(z-N)+z^2+4\gamma(m-2)(z-\gamma)}\Big) \tag{3.59}$$

and

$$c_2 = \frac{(n-m)(n-m-1)\kappa l^{2+2N}}{4N^2-2Nd_1+d_2} \tag{3.60}$$

together with

$$d_1 = 2N(n-m)+3z+2\gamma(m-2) \tag{3.61}$$

and

$$d_2 = N^2(n-m)(n-m+1)+2z^2+2z\gamma(m-2)+(m-2)(m-1)\gamma^2+2N(n-m)\left(z+(m-2)\gamma\right), \tag{3.62}$$

while c_3 and c_4 are kept as integration constants. In the solution, the parameter z measures the anisotropic scaling symmetry, the parameter l corresponds to the length scale of corresponding spacetime geometry. The curvature index $\kappa = -1, 0, +1$ corresponds to a hyperbolic, planar and spherical geometry of $(n-m)$-dimensional subspace, respectively, and ρ is the radius of the $(n-m)$-dimensional subspace, which is given by $\rho^2 = x_i x^i$. Here, index α runs on 1, 2, 3,..., $(m-2)$ while index i runs on $(m-3), \ldots, (n-4)$ provided $m < (n-1)$. It is easy to see that the D-dimensional solution of references [40] and [41] corresponds to the particular case $N = \gamma = 1$ for the generic values of z in n-dimensions. In order for the solution given above to describe a physical solution, the function $f(r)$ must be real. This can be satisfied if the inequality expression

$$4N(n-m)(z-N)+z^2+4\gamma(m-2)(z-\gamma) \geq 0 \tag{3.63}$$

holds between the parameters of solution. It implies that if the dynamical exponent z lies in the interval $z \in (-\infty, z_-] \cup [z_+, \infty)$ (keeping N and γ arbitrary) where

$$z_{\mp} = -2N(n-m)-2(m-2)\gamma \mp 2\sqrt{\left(N(n-m)+(m-2)\gamma\right)^2+N^2(n-m)+(m-2)\gamma^2}\,, \tag{3.64}$$

the solution is real. Here, if one considers the branch $z > z_+ > 0$ so that $p_+ \geq p_- \geq z_+ > 0$ and chooses the integration constants c_3 and c_4 properly, the spacetime solution (3.57) can represent either a black hole (a hyperbolic black hole for $\kappa = -1$ and a spherical black hole for $\kappa = 1$) or a black brane for $\kappa = 0$. In any case, assuming that solution has a singularity at $r = r_+$ which is the largest root of $f^2(r)$ such that $f^2(r_+) = 0$, r_+ can be identified as (outer) horizon.

Alternatively, the inequality (3.63) still holds for generic values of scaling parameter z and parameter γ provided that N is restricted to be in the interval $N_- \leq N \leq N_+$ where

$$N_{\mp} = \frac{1}{2}\left(z \mp \sqrt{\frac{(n-m+1)z^2 + 4\gamma(m-2)(z-\gamma)}{(n-m)}}\right), \tag{3.65}$$

that makes exponents p_+ and p_- equal so that solution is real again.
Now, if one takes $z = z_+$, a logarithmic solution can be obtained as

$$f^2(r) = 1 + \bar{c}_2 \frac{l^{2N}}{r^{2N}} + \frac{l^p}{r^p}\left(\bar{c}_3 + \bar{c}_4 \ln\left(\frac{r}{l}\right)\right) \tag{3.66}$$

with

$$p = \frac{1}{2}\Big(2N(n-m) + 3z_+ + 2\gamma(m-2)\Big), \tag{3.67}$$

and

$$\bar{c}_2 \equiv c_2 \mid_{z=z_+} . \tag{3.68}$$

Furthermore, since $z = 0$ lies outside the interval $z \in (-\infty, z_-] \cup [z_+, \infty)$, it seems that a physical solution is not possible for $z = 0$. However, it is not the case. Indeed, if one starts to solve the field equation by taking $z = 0$, one obtains the following solution for the special value of scaling parameter where $z = 0$

$$f^2(r) = 1 + \tilde{c}_2 \frac{l^{2N}}{r^{2N}} + \frac{l^{N(n-m)+\gamma(m-2)}}{r^{N(n-m)+\gamma(m-2)}}\left(\tilde{c}_3 \cos\left(\tilde{N}\ln\left(\frac{r}{l}\right)\right) + \tilde{c}_4 \sin\left(\tilde{N}\ln\left(\frac{r}{l}\right)\right)\right) \tag{3.69}$$

where

$$\tilde{N} = N^2(n-m) + \gamma^2(m-2) \tag{3.70}$$

and

$$\tilde{c}_2 \equiv c_2 \mid_{z=0} . \tag{3.71}$$

At this stage, it is interesting to discuss some thermodynamic properties associated to $Li_m \times \Omega_{(n-m)}$ type solutions. Although, it is not possible to obtain a general, analytical expression for the position of the horizon r_+ (given by the larger positive root of $f^2(r_+) = 0$) as a function of given parameters, one can still proceed to a thermodynamic study. The Wald's formula [43] for the black hole entropy which is given by

$$S_{bh} = -2\pi \int_S \frac{\delta\mathcal{L}}{\delta R_{abcd}} n_{ab} n_{cd} \tag{3.72}$$

where R_{abcd} is the Riemann tensor and n_{ab} is binormal to the horizon which is given by $n_{ab} = (\nabla_a \xi_b)/K$ and it is normalized as $n_{ab}n^{ab} = -2$. It can easily be checked that these constant curvature black hole solutions have a vanishing entropy in the R^2-corrected gravity action since the solution satisfies $1 + 2\alpha R = 0$.
The traditional definition of surface gravity K is based on the idea of of Killing horizon, a hypersurface where a Killing vector of the metric ξ becomes null and it is given in the

literature as

$$K = \left(\frac{-1}{2} (\nabla_b \xi_a)(\nabla^b \xi^a) \right)^{(1/2)} . \tag{3.73}$$

This expression can be written in terms of exterior forms as follows

$$K = \frac{1}{2} \left(*(d\xi \wedge *d\xi) \right)^{(1/2)} . \tag{3.74}$$

For a static, spherically symmetric spacetime in coordinates that the metric can be written as

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{V(r)} + R(r)d\Omega^2, \tag{3.75}$$

the (normalized) time-translational Killing vector has components $(1, 0, 0, 0, ...)$ and thus on the Killing horizon equation becomes

$$K = \lim_{r \to r_+} \left(\frac{1}{2} \sqrt{\frac{V(r)}{U(r)}} U'(r) \right) . \tag{3.76}$$

The thermodynamic temperature of a black hole is proportional to its surface gravity and it is given by

$$T = \frac{K}{2\pi}. \tag{3.77}$$

In the light of these, surface gravity and temperature can be calculated from

$$K = -\frac{\sqrt{n(n-1)}}{2} \lim_{r \to r_+} \left(\frac{r}{l} \right)^{z+1} \left(f^2(r) \right)' \tag{3.78}$$

and

$$T = \frac{\sqrt{n(n-1)}}{4\pi l} \left(2c_2 N \left(\frac{l}{r_+} \right)^{(2N-z)} + c_3 p_+ \left(\frac{l}{r_+} \right)^{(p_+ - z)} + c_4 p_- \left(\frac{l}{r_+} \right)^{(p_- - z)} \right) . \tag{3.79}$$

Interestingly, the specific relation (3.53) of R^2-corrected gravity theory leads the entropy of Lifshitz type static and stationary black holes to vanish while the temperatures of the black holes are nonzero. This feature of Lifshitz black holes is very analogous to BTZ black holes [44]. On the other hand, one should also notice that solutions and temperature expression (3.79) presented above include a very rich solution family due to the presence of curvature index κ which makes the $(n-m)$-dimensional subspace hyperbole, plane or sphere according to the value it takes (i.e., $\kappa = -1, 0, +1$ respectively). A detailed study of the thermodynamics of Lifshitz type black holes is given in [45].
In addition to all these, for some specific relations between constants, the solutions have zero temperature which are also called as extremal black hole solutions. The first examples of asymptotically Lifshitz black hole solutions with extremal horizons are presented in [41]. Inspired by this work, one can find an extremal black hole solution for $\kappa = 0$ and a specific

relation between c_3 and c_4 given by

$$c_3 = -p_- \left(p_- - p_+\right)^{\frac{p_+ - p_-}{p_-}} \left(\frac{c_4}{p_+}\right)^{\frac{p_+}{p_-}}, \tag{3.80}$$

in which the solution (i.e., (3.57)-(3.62)) has zero temperature. In this case, the extremal black hole is

$$\begin{aligned} ds^2 &= -\left(\frac{r}{l}\right)^{2z}\left(1-\frac{p_-}{p_- - p_+}\left(\frac{r_e}{r}\right)^{p_+}+\frac{p_+}{p_- - p_+}\left(\frac{r_e}{r}\right)^{p_-}\right)dt^2 \\ &+\left(\frac{l}{r}\right)^{2}\left(1-\frac{p_-}{p_- - p_+}\left(\frac{r_e}{r}\right)^{p_+}+\frac{p_+}{p_- - p_+}\left(\frac{r_e}{r}\right)^{p_-}\right)^{-1}dr^2+\left(\frac{r}{l}\right)^{2\gamma}dy_\alpha dy^\alpha \\ &+\left(\frac{r}{l}\right)^{2N}dx_i dx^i, \end{aligned} \tag{3.81}$$

where the extremal radius r_e is expressed as

$$r_e = l\left(\frac{p_- - p_+}{c_1 p_+}c_4\right)^{1/2p_-}. \tag{3.82}$$

Besides, for $\kappa = \pm 1$, one encounters another extremal black hole solution which is defined for the particular value of $z = z_+$. This is achieved for the special relation

$$c_3 = -c_4 - N\left(\frac{c_2}{p}\right)^{p/N}(N-p)^{(p-N)/N} \tag{3.83}$$

and the corresponding spacetime geometry yields

$$\begin{aligned} ds^2 &= -\left(\frac{r}{l}\right)^{2z}\left(1+c_2\left(\left(\frac{l}{r}\right)^{2N}-\frac{N}{p}\left(\frac{r_e}{r}\right)^{2p}\left(\frac{l}{r_e}\right)^{2N}\right)\right)dt^2 \\ &+\left(\frac{l}{r}\right)^{2}\left(1+c_2\left(\left(\frac{l}{r}\right)^{2N}-\frac{N}{p}\left(\frac{r_e}{r}\right)^{2p}\left(\frac{l}{r_e}\right)^{2N}\right)\right)^{-1}dr^2+\left(\frac{r}{l}\right)^{2\gamma}dy_\alpha dy^\alpha \\ &+\left(\frac{r}{l}\right)^{2N}\frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2}, \end{aligned} \tag{3.84}$$

where the extremal radius is given by

$$r_e = l\left(\frac{(N-p)}{p}c_2\right)^{1/2N}. \tag{3.85}$$

Before closing this subsection, one can also note that writing spacetime solution (3.57) in the form

$$ds^2 = \left(\frac{r}{l}\right)^{2\theta}\left(-\left(\frac{r}{l}\right)^{2z'}f^2(r)dt^2+\left(\frac{l}{r}\right)^{2}\frac{dr^2}{\tilde{f}^2(r)}+\left(\frac{r}{l}\right)^{2}dy_\alpha dy^\alpha\right)+\left(\frac{r}{l}\right)^{2N}\frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2} \tag{3.86}$$

with the following identifications

$$\theta = \gamma - 1\,, \qquad z' = z + \gamma - 1\,, \qquad \tilde{f}^2(r) = f^2(r)\left(\frac{r}{l}\right)^{2\gamma-2}\,, \tag{3.87}$$

one obtains a hyperscaling violating $Li_m \times \Omega_{(n-m)}$ type static solution for higher dimensional R^2-corrected gravity theory.

ii. Lifshitz type stationary solutions

It should be noted that the Einstein field equation (3.51) can also admit the stationary $Li_m \times \Omega_{(n-m)}$ type vacuum solutions which has the form

$$\begin{aligned} ds^2 &= -\left(\frac{r}{l}\right)^{2z} dt^2 + \left(\frac{r^2}{l^2}\right)\left(d\phi + \frac{\omega l^2}{r^2}dt\right)^2 + \left(\frac{l^2}{r^2}\right)dr^2 \\ &\quad + \left(\frac{r}{l}\right)^{2\gamma} dy_\alpha dy^\alpha + \left(\frac{r}{l}\right)^{2N} \frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2} \end{aligned} \tag{3.88}$$

provided that (3.53) and

$$\begin{aligned} \Lambda &= \frac{2N^2(n-m)(n-m-1) - 4N(z+\gamma(m-2)) + 2m\gamma^2(m-39}{8l^2} \\ &\quad + \frac{z^2(4+3\omega^2) + 2z\gamma(2(m-2)+(2m-3)\omega^2) + \gamma^2(4+3\omega^2)}{8l^2(1+\omega^2)}\,. \end{aligned} \tag{3.89}$$

Similarly as in the static case, provided that the relations (3.53) and (3.89) hold, the Einstein field equation (3.51) also admits the stationary $Li_m \times \Omega_{(n-m)}$ type spacetime solution (which can be referred as stationary black hole solution)

$$\begin{aligned} ds^2 &= -\left(\frac{r}{l}\right)^{2z} f^2(r)dt^2 + \left(\frac{r^2}{l^2}\right)\left(d\phi + \frac{\omega l^2}{r^2}dt\right)^2 + \left(\frac{l^2}{r^2}\right)\frac{dr^2}{f^2(r)} \\ &\quad + \left(\frac{r}{l}\right)^{2\gamma} dy_\alpha dy^\alpha + \left(\frac{r}{l}\right)^{2N} \frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2} \end{aligned} \tag{3.90}$$

for the following metric function

$$f^2(r) = 1 + c_2\frac{l^{2N}}{r^{2N}} + c_3\frac{l^{p_+}}{r^{p_+}} + c_4\frac{l^{p_-}}{r^{p_-}} + c_5\frac{l^{2(z+1)}}{r^{2(z+1)}}\,. \tag{3.91}$$

Here,

$$\begin{aligned} p_\pm &= \frac{1}{2}\Big(2 + 2N(n-m) + 3z + 2\gamma(m-3) \\ &\quad \pm\sqrt{4(z-1) + 4N(n-m)(z-N) + z^2 + 4\gamma(m-3)(z-\gamma)}\Big)\,, \end{aligned} \tag{3.92}$$

and

$$c_2 = \frac{\kappa(n-m)(n-m-1)\, l^{2+2N}}{4N^2 - 2Nd_1 + d_2}\,, \tag{3.93}$$

$$c_5 = \frac{2\omega^2 l^{2+2z}}{4(z+1)^2 - 2(z+1)d_1 + d_2}\,, \tag{3.94}$$

together with

$$d_1 = 2 + 2N(n-m) + 3z + 2\gamma(m-3)\,, \tag{3.95}$$

$$\begin{aligned} d_2 \;=\;& N^2(n-m)(n-m+1) + 2(z^2+z+1) + 2(m-3)(z+1)\gamma \\ &+(m-3)(m-2)\gamma^2 + 2N(n-m)\left(1+z+(m-3)\gamma\right)\,. \end{aligned} \tag{3.96}$$

Here, c_3 and c_4 denote integration constants. As mentioned before, the curvature index $\kappa = -1, 0, +1$ correspond to a hyperbolic, planar and spherical geometry of $(n-m)$-dimensional subspace respectively and ρ is the radius of $(n-m)$-dimensional subspace, which is given by $\rho^2 = x_i x^i$. In the stationary $Li_m \times \Omega_(n-m)$ type solutions, the spacetime index α runs on 1, 2, 3,..., $(m-3)$ and i runs on $(m-2), \ldots, (n-3)$. The parameter z again denotes the dynamical exponent, which measures the anisotropic scaling symmetry, the parameter l represents the radius of Lifshitz spacetime and ω is the rotation parameter. In order to obtain a physical solution, the metric function $f(r)$ should be real. This condition can be achieved by the following inequality:

$$4(z-1) + 4N(n-m)(z-N) + z^2 + 4\gamma(m-3)(z-\gamma) \geq 0\,. \tag{3.97}$$

It is clear from the expression of $p_\pm$ that the dynamical exponent may take the values in the interval $z \in (-\infty, z_-] \cup [z_+, \infty)$ where

$$\begin{aligned} z_\mp \;=\;& -2 - 2N(n-m) - 2\gamma(m-3) \\ &\mp 2\sqrt{1 + N^2(n-m) + \gamma^2(m-3) + \left(1 + N(n-m) + \gamma(m-3)\right)^2}\,. \end{aligned} \tag{3.98}$$

As in the static case, properly choosing the integration constants c_1 and c_2 for $z > z_+ > 0$ side so that $p_+ \geq p_- \geq z_+ \geq 0$, the solution (3.90) expresses a hyperbolic black hole for $\kappa = -1$, spherical black hole for $\kappa = 1$ and a black brane for $\kappa = 0$. In all the cases, the assumption that the solution has a singularity at $r = r_+$ which denotes the largest root of $f^2(r)$ (i.e., $f^2(r_+) = 0$), yields the identification that r_+ represents (outer) horizon.
On the other hand, if one takes

$$N_\mp = \frac{1}{2}\left(z \mp \sqrt{\frac{(-4 + (n-m+1)z^2 - 4\gamma^2(m-3) + 4z(1+\gamma(m-3)))}{(n-m)}}\right) \tag{3.99}$$

p_+ again becomes equal to p_-.
If one takes $z = z_+$, a logarithmic solution can be obtained as

$$\bar{f}^2(r) = 1 + \bar{c}_2 \frac{l^{2N}}{r^{2N}} + \frac{l^p}{r^p}\left(\bar{c}_3 + \bar{c}_4 \ln\left(\frac{r}{l}\right)\right) + \bar{c}_5 \frac{l^{2(z+1)}}{r^{2(z+1)}} \tag{3.100}$$

with

$$p = \frac{1}{2}\Big(2 + 2N(n-m) + 3z_+ + 2\gamma(m-3)\Big) \tag{3.101}$$

and

$$\bar{c}_2 \equiv c_2 \mid_{z=z_+}, \qquad \bar{c}_5 \equiv c_5 \mid_{z=z_+} . \tag{3.102}$$

Furthermore, since $z = 0$ lies outside the interval $z \in (-\infty, z_-] \cup [z_+, \infty)$, it seems that a physical solution is not possible for $z = 0$. However, it is not the case. Indeed, if one starts to solve the field equation by taking $z = 0$, one obtains the following solution for the special value of scaling parameter where $z = 0$

$$\tilde{f}^2(r) = 1 + \tilde{c}_2 \frac{l^{2N}}{r^{2N}} + \tilde{c}_5 \frac{l^2}{r^2} + \frac{l^{N(n-m)+\gamma(m-3)+1}}{r^{N(n-m)+\gamma(m-3)+1}} \left(\tilde{c}_3 \cos\left(\tilde{N} \ln\left(\frac{r}{l}\right)\right) + \tilde{c}_4 \sin\left(\tilde{N} \ln\left(\frac{r}{l}\right)\right)\right), \tag{3.103}$$

where

$$\tilde{N} = N^2(n-m) + \gamma^2(m-3) + 1 \tag{3.104}$$

and

$$\tilde{c}_2 \equiv c_2 \mid_{z=0}, \qquad \tilde{c}_5 \equiv c_5 \mid_{z=0} . \tag{3.105}$$

Besides all these, the standart surface gravity concept K can also be used for thermodynamic temperature of Lifshitz type stationary blackhole. In this case, components of the (normalized) time-translational Killing vector are $(1, 0, 0, ..., \Omega_H, 0)$, where Ω_H denotes angular velocity as $\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}}|_{r=r_+} = -\frac{\omega l^2}{{r_+}^2}$. Then, the surface gravity and thermodynamic temperature of Lifshitz type stationary blackhole can be given by

$$K = -\frac{\sqrt{n(n-1)(n-2)}}{2} \lim_{r \to r_+} \left(\frac{r}{l}\right)^{z+1} \left(f^2(r)\right)' \tag{3.106}$$

and

$$\begin{aligned} T \;\; = \;\; & \frac{\sqrt{n(n-1)(n-2)}}{4\pi l} \left(2c_2 N \left(\frac{l}{r_+}\right)^{(2N-z)} + c_3 p_+ \left(\frac{l}{r_+}\right)^{(p_+-z)} \right. \\ & \left. + c_4 p_- \left(\frac{l}{r_+}\right)^{(p_--z)} + 2(z+1)c_5 \left(\frac{l}{r_+}\right)^{(z+2)} \right). \end{aligned} \tag{3.107}$$

As mentioned in the static case, the solution family of Lifshitz type stationary black holes are quite rich since the rotation parameter ω also contributes to the solutions and the temperatures calculated over the outer event horizon. It is also interesting to note that in both static and stationary case, the dimension of subspace also contributes to temperature. Furthermore, it is noteworthy that although the entropy vanishes for the solutions, the temperatures are not zero.

Furthermore, it can be noted that the extremal stationary Lifshitz type solution can be obtained for $\kappa = 0$, $p_- = (z+1)$ and for the following special relation

$$c_3 = \frac{(z+1)}{p_+ - (z+1)} \left(\frac{(z+1-p_+)(c_4+c_5)}{p_+}\right)^{p_+/(z+1)} . \tag{3.108}$$

Then, the spacetime geometry yields

$$
\begin{aligned}
ds^2 &= -\left(\frac{r}{l}\right)^{2z}\left(1+(c_4+c_5)\left(\left(\frac{l}{r}\right)^{2(z+1)}-\frac{(z+1)}{p_+}\left(\frac{l}{r_e}\right)^{2(z+1)}\left(\frac{r_e}{r}\right)^{2p_+}\right)\right)dt^2 \\
&+\left(\frac{r^2}{l^2}\right)\left(d\phi+\frac{\omega l^2}{r^2}dt\right)^2 \\
&+\left(\frac{l^2}{r^2}\right)\left(1+(c_4+c_5)\left(\left(\frac{l}{r}\right)^{2(z+1)}-\frac{(z+1)}{p_+}\left(\frac{l}{r_e}\right)^{2(z+1)}\left(\frac{r_e}{r}\right)^{2p_+}\right)\right)^{-1}dr^2 \\
&+\left(\frac{r^2}{l^2}\right)^{2\gamma}dy_\alpha dy^\alpha+\left(\frac{r}{l}\right)^{2N}dx_i dx^i\,, \qquad (3.109)
\end{aligned}
$$

where the extremal radius is defined by

$$
r_e = l\left(\frac{(z+1-p_+)(c_4+c_5)}{p_+}\right)^{1/(2z+2)}. \qquad (3.110)
$$

Additionally, one can find another black hole solutions for $\kappa = \pm 1$ for the particular value of $z = z_+$. This is accomplished for the special relation

$$
c_3 = -c_4 - \frac{(z+1)}{(z+1-p)}\left(\frac{(z+1-p)(c_2+c_5)}{p}\right)^{p/(z+1)}. \qquad (3.111)
$$

In this case, the extremal solution is given by

$$
\begin{aligned}
ds^2 &= -\left(\frac{r}{l}\right)^{2z}\left(1+(c_2+c_5)\left(\left(\frac{l}{r}\right)^{2(z+1)}-\frac{(z+1)}{p}\left(\frac{r_e}{r}\right)^{2p}\left(\frac{l}{r_e}\right)^{2(z+1)}\right)\right)dt^2 \\
&+\left(\frac{r^2}{l^2}\right)\left(d\phi+\frac{\omega l^2}{r^2}dt\right)^2 \\
&+\left(\frac{l^2}{r^2}\right)\left(1+(c_2+c_5)\left(\left(\frac{l}{r}\right)^{2(z+1)}-\frac{(z+1)}{p}\left(\frac{r_e}{r}\right)^{2p}\left(\frac{l}{r_e}\right)^{2(z+1)}\right)\right)dr^2 \\
&+\left(\frac{r^2}{l^2}\right)^{2\gamma}dy_\alpha dy^\alpha+\left(\frac{r}{l}\right)^{2N}\frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2}\,, \qquad (3.112)
\end{aligned}
$$

where

$$
r_e = l\left(\frac{(c_2+c_5)(z+1-p)}{p}\right)^{1/(2z+2)}. \qquad (3.113)
$$

When the rotation parameter ω is taken as 0 with appropriate coordinate transformation, the stationary solutions reduce to the static solutions which have been presented in the previous section. Moreover, if one takes $N = 1$ and $g_0 = 1/l$, the solutions are compatible with the solutions given in [40] and [41]. The Lifshitz type topological static/stationary solutions of R^2-corrected gravity will be discussed in a forthcoming work [42].

3.2.2. Bianchi type static/stationary solutions

i. Bianchi type static solutions

Motivated by the work of [12] in which five dimensional R^2-corrected gravity theory is examined for Lifshitz and nine classes of Bianchi type solutions in five dimensions, one can study Bianchi-type static and stationary solutions of the model for torsion-free connection structure in higher dimensions. Therefore, Bianchi Type static solutions are investigated in this subsection for the connection of the spacetime is torsion-free in higher dimensions. General form of Bianchi type metrics are given by

$$ds^2 = -e^{2\beta_t r}dt^2 + dr^2 + \eta_{ij}\, e^{(\beta_i+\beta_j)r}dx^i dx^j\,, \tag{3.114}$$

where β_t, β_i are constants and η_{ij} is a constant matrix which is independent of all coordinates. This metric also has a generalized Lifshitz scaling symmetry

$$r \to r+\epsilon\,, \quad t \to te^{-\beta_t \epsilon}\,, \quad \omega^i \to \omega^i e^{-\beta_i \epsilon}\,, \tag{3.115}$$

where $\omega^i = dx^i$ ($i = 1, 2, 3, \dots, (n-2)$). The Einstein field equation (3.51) for R^2-corrected gravity admits following Bianchi type static vacuum solution:

$$ds^2 = -e^{2\beta_0 r}dt^2 + dr^2 + e^{2\beta_2 r}dx_i dx^i \tag{3.116}$$

provided that

$$\Lambda = \frac{1}{4}\left(2\beta_0^2 + 2(n-2)\beta_0\beta_2 + (n-2)(n-1)\beta_2^2\right) \tag{3.117}$$

and (3.53). On the condition that the relations (3.53) hold, the theory also admits the following spacetime solution

$$ds^2 = -e^{2\beta_0 r}f^2(r)dt^2 + e^{2\beta_1 r}\frac{dr^2}{f^2(r)} + e^{2\beta_2 r}\frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2}\,, \tag{3.118}$$

where β_0, β_1 and β_2 denote constants. Here, the metric function is

$$f^2(r) = c_1 e^{p_+ r} + c_2 e^{p_- r} + c_3 e^{2\beta_1 r} + c_4 e^{2(\beta_1-\beta_2)r}\,, \tag{3.119}$$

where

$$p_\pm = \frac{-\left(3\beta_0 - \beta_1 + 2(n-2)\beta_2\right) \pm \sqrt{(\beta_0+\beta_1)^2 + 4(n-2)\beta_2(\beta_0+\beta_1-\beta_2)}}{2} \tag{3.120}$$

and

$$c_3 = \frac{4\Lambda}{4\beta_1^2 + 2d_1\beta_1 + d_2}\,, \tag{3.121}$$

$$c_4 = \frac{(n-3)(n-4)\kappa}{4(\beta_1-\beta_2)^2 + 2d_1(\beta_1-\beta_2) + d_2} \tag{3.122}$$

together with

$$d_1 = 3\beta_0 - \beta_1 + 2(n-2)\beta_2\,, \tag{3.123}$$

$$d_2 = 2\beta_0(\beta_0 - \beta_1) + 2(n-2)(\beta_0 - \beta_1)\beta_2 + (n-2)(n-1)\beta_2^2\,, \tag{3.124}$$

while c_1, c_2, c_3 and c_4 are integration constants. As stated before, the curvature index $\kappa = -1, 0, +1$ defines the geometry of $(n-2)$-dimensional subspace and ρ denotes the radius of this subspace, which is given by $\rho = x_i x^i$. In the Bianchi type static solution, the spacetime index i takes the values of $2, 3, ..., (n-2)$. In order to get a real metric function,

$$(\beta_0 + \beta_1)^2 + 4(n-2)\beta_2(\beta_0 + \beta_1 - \beta_2) \geq 0 \tag{3.125}$$

should be satisfied between the parameters β_0, β_1 and β_2. Significantly, the solution (3.119) can be identified as a black hole solution in which the geometry is hyperboloidal, flat or spherical according to the value of the curvature index $\kappa = -1, 0, +1$ respectively. As can easily be seen that there are a wide range of solutions where β_0, β_1 and β_2 obey the relations given in (3.120)-(3.124). Moreover, one can assume that the solution has a singularity at $r = r_+$ (the largest root of $f^2(r_+) = 0$) which gives the horizon of Bianchi type static black holes.
The surface gravity and thermodynamic temperature of Bianchi type static black hole can be given by

$$K = \frac{\sqrt{n(n-1)}}{2} \lim_{r \to r_+} e^{(\beta_0 - \beta_1)r} \left(f^2(r)\right)' \tag{3.126}$$

and

$$T = \frac{\sqrt{n(n-1)}}{4\pi} \left(c_1 e^{(p_+ + \beta_0 - \beta_1)r} + c_2 e^{(p_- + \beta_0 - \beta_1)r} + c_3 e^{2\beta_0 r} + c_4 e^{2(\beta_0 - \beta_2)r}\right) \tag{3.127}$$

for vanishing entropy.

ii. Bianchi type stationary solutions

Even though there is no Bianchi type stationary vacuum solution, one can still consider Bianchi Type stationary solutions for the following ansatz

$$ds^2 = -e^{2\beta_0 r} f^2(r) dt^2 + e^{2\beta_1 r}\left(d\phi + \frac{\omega}{r^2}dt\right)^2 + e^{2\beta_2 r}\frac{dr^2}{f^2(r)} + e^{2\beta_3 r}\frac{dx_i dx^i}{\left(1 + \kappa\frac{\rho^2}{4}\right)^2}\,, \tag{3.128}$$

where $f(r)$ is an arbitrary function of r, β_0, β_1 and β_2 are constants, κ is the curvature index which defines the geometry of the $(n-3)$-dimensional subspace and ρ is the radius of this subspace (i.e., $\rho^2 = x_i x^i$). In this case, the spacetime index i runs on $1, 2, 3, \ldots, (n-3)$. The metric function is given by

$$f^2(r) = c_1 e^{p_+ r} + c_2 e^{p_- r} + c_3 e^{2\beta_2 r} + c_4 e^{2(\beta_2 - \beta_3)r} + c_5 \frac{e^{2(\beta_1 - \beta_0)r}}{r^4}\,, \tag{3.129}$$

where

$$\begin{aligned}p_{\pm} &= \frac{1}{2}\Big(-(2(n-3)\beta_3+3\beta_0+2\beta_1-\beta_2) \\ &\pm \left({\beta_0}^2-4{\beta_1}^2+{\beta_2}^2+2\beta_0(2\beta_1+\beta_2)+4\beta_1\beta_2+4(n-3)\beta_3(\beta_0+\beta_2-\beta_3)\right)^{1/2}\Big)\,,\end{aligned} \tag{3.130}$$

$$c_3 = \frac{4\Lambda}{4\beta_2^2+2d_1\beta_2+d_2}\,, \tag{3.131}$$

$$c_4 = \frac{(n-3)(n-4)\kappa}{4(\beta_2-\beta_3)^2+2d_1(\beta_2-\beta_3)+d_2}\,, \tag{3.132}$$

$$c_5 = \omega^2/10 \tag{3.133}$$

together with

$$d_1 = 3\beta_0+2\beta_1-\beta_2+2(n-3)\beta_3\,, \tag{3.134}$$

$$d_2 = 2\left(\beta_0^2+\beta_0\beta_1+\beta_1^2-(\beta_0+\beta_1)\beta_2\right)+2(n-3)(\beta_0+\beta_1-\beta_2)\beta_3+(n-3)(n-2)\beta_3^2\,. \tag{3.135}$$

Here, c_1, c_2, c_3 c_4 and c_5 denote integration constants. The curvature index $\kappa=-1,0,+1$ defines the geometry of $(n-3)$-dimensional subspace and ρ corresponds to the radius of this subspace given by $\rho = x_i x^i$. Bianchi type stationary solutions in n-dimensional spacetime are investigated for (3.53) provided

$$4(\beta_1-\beta_0)+d_1 = 0\,, \tag{3.136}$$

$$4(\beta_1-\beta_0)^2+2d_1(\beta_1-\beta_0)+d_2 = 0\,. \tag{3.137}$$

The real metric condition requires

$${\beta_0}^2-4{\beta_1}^2+{\beta_2}^2+2\beta_0(2\beta_1+\beta_2)+4\beta_1\beta_2+4(n-3)\beta_3(\beta_0+\beta_2-\beta_3)\geq 0\,. \tag{3.138}$$

As in the static case, the solution (3.129) can be identified as Bianchi type stationary black hole solution in which the geometry can be defined according to the curvature index κ. Here, there are a large variety of solutions where all constants of the system should obey. The largest root of $f^2(r_+)=0$, namely singularity at $r=r_+$, leads the horizon of Bianchi type stationay black holes. Finally, taking $\omega=0$ and appropriately changing the constants yields Bianchi type static solutions as expected.
The surface gravity and thermodynamic temperature of Bianchi type stationary black hole in R^2-corrected gravity theory take the following forms for vanishing entropy:

$$K = \frac{\sqrt{n(n-1)(n-2)}}{2}\lim_{r\to r_+} e^{(\beta_0-\beta_2)r}\left(f^2(r)\right)' \tag{3.139}$$

and

$$T = \frac{\sqrt{n(n-1)(n-2)}}{4\pi}\Big(c_1 e^{(p_+ + \beta_0 - \beta_2)r} + c_2 e^{(p_- + \beta_0 - \beta_2)r} + c_3 e^{2\beta_0 r} + c_4 e^{2(\beta_0-\beta_3)r} + c_5 \frac{e^{2(\beta_1-\beta_2)r}}{r^4}\Big)\,. \qquad (3.140)$$

3.2.3. Schrödinger type solutions

Interestingly, the action density (3.50) also has a vacuum solution of the following form

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} dt^2 - \frac{2r^2}{l^2}\, dt d\xi + \frac{l^2}{r^2} dr^2 + \left(\frac{r}{l}\right)^{2\gamma} dy_\alpha dy^\alpha + \left(\frac{r}{l}\right)^{2N} \frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2} \qquad (3.141)$$

when

$$\alpha = \frac{1}{8\Lambda}\,, \qquad R = -4\Lambda\,, \qquad \Lambda = \frac{3}{l^2}\,. \qquad (3.142)$$

It has also been proposed that the corresponding gravity background is a special class of the cosmological pp-wave solution [13, 46]. This metric ansatz obeys the following scaling symmetry

$$t \to \lambda^z t\,, \quad r \to \lambda^{-1} r\,, \quad \xi \to \lambda^{2-z}\xi\,, \quad \vec{y} \to \lambda \vec{y}\,, \quad \vec{x} \to \lambda \vec{x}\,, \quad z \neq 1\,, \qquad (3.143)$$

where z is the dynamical exponent, $\vec{y}$ is a $(m-3)$-dimensional vector and $\vec{x}$ is a $(n-m)$-dimensional vector. Again, provided that the relations given in (3.53) hold, the Einstein field equation (3.51) of R^2-corrected gravity theory admits following Schrödinger type solution

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} f^2(r) dt^2 - \frac{2r^2}{l^2}\, dt d\xi + \frac{l^2}{r^2}\frac{dr^2}{f^2(r)} + \left(\frac{r}{l}\right)^{2\gamma} dy_\alpha dy^\alpha + \left(\frac{r}{l}\right)^{2N} \frac{dx_i dx^i}{\left(1+\kappa\frac{\rho^2}{4}\right)^2}\,, \qquad (3.144)$$

where

$$f^2(r) = 1 + c_1 r^{-\frac{d_2}{d_1}} + c_2 \left(\frac{l}{r}\right)^{2N} \qquad (3.145)$$

together with

$$c_2 = \frac{(n-m)(n-m-1)\kappa l^2}{(d_2 - 2N d_1)}\,, \qquad (3.146)$$

$$d_1 = 2 + (n-m)N + (m-3)\gamma\,, \qquad (3.147)$$

$$d_2 = 6 + (n-m)(n-m-1)N^2 + 2N(n-m)(2+(m-3)\gamma) + (m-3)\gamma(4+(m-2)\gamma)\,. \qquad (3.148)$$

Here, c_1 is an integration constant. Notably, the higher dimensional Schrödinger type solution for R^2-corrected gravity is independent from the dynamical exponent z.

4. pp-WAVE SOLUTIONS OF MINIMAL MASSIVE 3D GRAVITY

In this section, vacuum (no matter coupling) and matter (Maxwell-Chern-Simons coupling) coupled pp-wave solutions of MMG are presented. pp-wave solutions in GR have taken too much attention in recent years. They are an interesting class of exact solutions of Einstein's field equations with some matter sources. The abbreviation "pp"stands for "plane-fronted gravitational waves with parallel rays"[18]. It can be noted that "plane-fronted gravitational waves"implies that there is a geodesic null vector field whose twist, expansion and shear are zero in vacuum. In addition, "parallel rays" implies the requirement that the rotation of the vector field vanishes as well.
With the motivation of [19], one can consider the following 3-dimensional metric ansatz

$$ds^2 = f(u,\rho)du^2 + 2e^{2c\rho}dudv + d\rho^2 \,, \tag{4.1}$$

where $f(u,\rho)$ is any smooth function. In the literature, these type of solutions are also called as an AdS pp-wave. Here, the corresponding connection 1-forms are

$$\omega^0{}_1 = (-\beta + c)\, e^0 - \beta e^2 \,, \qquad \omega^0{}_2 = c\, e^1 \,, \tag{4.2}$$

$$\omega^1{}_2 = -\beta e^0 + (-\beta - c)\, e^2 \,, \tag{4.3}$$

where

$$\beta = \frac{1}{4}f' - \frac{c}{2}f \,. \tag{4.4}$$

Then, the curvature 2-forms can be given by

$$R^{01} = A_1 e^1 \wedge e^0 + A_3 e^1 \wedge e^2 \,, \qquad R^{02} = B_2 e^0 \wedge e^2 \,, \tag{4.5}$$

$$R^{12} = C_1 e^1 \wedge e^0 + C_3 e^1 \wedge e^2 \,, \tag{4.6}$$

where the curvature coefficients are

$$A_1 = -\beta' + c^2 \,, \quad A_3 = C_1 = -\beta' \,, \tag{4.7}$$

$$B_2 = -c^2 \,, \quad C_3 = -\beta' - c^2 \,. \tag{4.8}$$

Note that prime denotes the partial derivative with respect to ρ through this section.

4.1. Vacuum Solutions

In this subsection, the details of the calculation for vacuum pp-wave solutions are presented. The Einstein field equation (2.108) reads

$$
\begin{aligned}
&\sigma A_1 + \Lambda + \frac{1}{\mu(1+\alpha\sigma)}\Big(C_1{}' + c\,(-A_1 + C_1 - C_3) + \beta\,(A_1 + B_2 - C_1)\Big) \\
&-\frac{\alpha}{4\mu^2(1+\alpha\sigma)^3}\,\big(\alpha\Lambda + 3c^2 - 2A_1 + 2B_2\big)\,\big(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3\big) = 0\,,
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
&\sigma A_3 + \frac{1}{\mu(1+\alpha\sigma)}\Big(\,(B_2 + C_3)' + c\,(A_1 - A_3 + B_2 - C_1) + \beta\,(A_1 + B_2 - C_1)\Big) \\
&+\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\,A_3\,\big(\alpha\Lambda + 3c^2 - 2A_1 + C_3\big) = 0\,,
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&-\sigma B_2 + \Lambda + \frac{1}{\mu(1+\alpha\sigma)}\Big(c(-A_3 + C_1) + \beta\,(-A_1 + A_3 + C_1 - C_3)\Big) \\
&-\frac{\alpha}{4\mu^2(1+\alpha\sigma)^3}\Big(\,\big(\alpha\Lambda + 3c^2 - 2A_1 + 2B_2\big)\,\big(\alpha\Lambda + 3c^2 + 2B_2 + 2C_3\big) + 4A_3C_1\Big) = 0\,,
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&-\sigma C_1 + \frac{1}{\mu(1+\alpha\sigma)}\Big(\,(-A_1 + B_2)' + c\,(A_3 + B_2 + C_1 - C_3) + \beta\,(-A_3 - B_2 + C_3)\Big) \\
&-\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\,C_1\,\big(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3\big) = 0\,,
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&-\sigma C_3 + \Lambda - \frac{1}{\mu(1+\alpha\sigma)}\Big(A_3{}' + c\,(-A_1 + A_3 - C_3) + \beta\,(A_3 + 2B_2 - C_3)\Big) \\
&-\frac{\alpha}{4\mu^2(1+\alpha\sigma)^3}\,\big(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3\big)\,\big(\alpha\Lambda + 3c^2 + 2B_2 + 2C_3\big) = 0\,.
\end{aligned} \tag{4.13}
$$

After some straightforward algebra, the field equations can be reduced to

$$
f''' - (2c+\gamma)f'' + 2c\gamma f' = 0\,, \tag{4.14}
$$

where

$$
\gamma = c - \frac{\alpha(\alpha\Lambda - c^2)}{2\mu(1+\alpha\sigma)^2} - \mu\sigma(1+\alpha\sigma)\,. \tag{4.15}
$$

Then, to determine the solution of this third order homogenous equation, one obtains

$$
r\,\big(r^2 - (2c+\gamma)r + 2c\gamma\big) = 0 \tag{4.16}
$$

for the characteristic equation. As can easily be seen (4.16) is a cubic equation with constant coefficients. Apart from $r_1 = 0$, the other roots can be found from the quadratic equation in which the discriminant is

$$\Delta = (\gamma - 2c)^2 \,. \tag{4.17}$$

Since the discriminant is positive (i.e., $\Delta > 0$), the quadratic equation has two real solutions which are $r_2 = 2c$ and $r_3 = \gamma$. Therefore, the vacuum solution is

$$f(u,\rho) = f_1(u) + \frac{f_2(u)}{2c} e^{2c\rho} + \frac{f_3(u)}{\gamma}\, e^{\gamma\rho} \tag{4.18}$$

provided that

$$\Lambda + c^2\sigma - \frac{\alpha(c^2 - \alpha\Lambda)^2}{4\mu^2(1+\alpha\sigma)^3} = 0\,, \tag{4.19}$$

which gives

$$c^2 = \frac{2\mu^2(1+\alpha\sigma)^3}{\alpha}\left(\sigma + \frac{\alpha^2\Lambda}{2\mu^2(1+\alpha\sigma)^3} \pm \sqrt{\sigma^2 + \frac{\alpha\Lambda}{\mu(1+\alpha\sigma)^2}}\right)\,. \tag{4.20}$$

On the other hand, when $\Delta = 0$ (i.e., $\gamma = 2c$), the quadratic equation has only one real solution (i.e., $r_{2,3} = 2c$, repeated roots). Hence, the special vacuum solution can be given by

$$f = f_1(u) + \frac{2c(f_2(u)\rho + f_3(u)) - f_2(u)}{4c^2}\, e^{2c\rho} \tag{4.21}$$

again provided (4.19). In both solutions (i.e., $\Delta > 0$ and $\Delta = 0$), $f_1(u)$, $f_2(u)$ and $f_3(u)$ denote arbitrary functions of their arguments.

4.2. Solutions with Maxwell-Chern-Simons coupling

In this subsection, pp-wave solution of MMG theory coupled with Maxwell-Chern-Simons electromagnetic theory is investigated. General analysis of self-dual solutions for the Einstein-Maxwell-Chern-Simons theory is presented in [47]. A solution of TMG theory coupled with Maxwell-Chern-Simons has been previously given in [48] where the self-duality of Maxwell field is assumed. To this end, the pp-wave metric ansatz (4.1) is used and Maxwell field is taken as

$$F = E(u,\rho)e^0 \wedge e^1 + B(u,\rho)e^1 \wedge e^2 \,. \tag{4.22}$$

Then, the modified Maxwell equations can be written as

$$B' - (\beta - c)\, B - (\beta - c + m)\, E = 0, \tag{4.23}$$

$$E' + (\beta + c)\, E + (\beta + c - m)\, B = 0 \tag{4.24}$$

together with the equation

$$\frac{\partial E}{\partial u} + \frac{\partial B}{\partial u} = 0\,, \tag{4.25}$$

where again $'$ denotes derivative with respect to ρ. Furthermore, Einstein field equations become

$$\sigma A_1 + \Lambda + \frac{1}{\mu(1+\alpha\sigma)}\Big((C_1+\alpha EB)' + c\left(-A_1 + C_1 - C_3 + \alpha\left(E^2+EB+B^2\right)\right)$$
$$+\beta\left(A_1 + B_2 - C_1 - \alpha E(E+B)\right)\Big)$$
$$-\frac{\alpha}{4\mu^2(1+\alpha\sigma)^3}\left(\alpha\Lambda + 3c^2 - 2A_1 + 2B_2 + \frac{\alpha}{2}\left(E^2+3B^2\right)\right)$$
$$\left(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3 + \frac{\alpha}{2}\left(E^2-B^2\right)\right) = -\frac{1}{2}\left(E^2+B^2\right)\,, \tag{4.26}$$

$$\sigma A_3 + \frac{1}{\mu(1+\alpha\sigma)}\Big(\left(B_2 + C_3 - \frac{\alpha}{4}\left(3E^2+B^2\right)\right)'$$
$$+c\left(A_1 - A_3 + B_2 - C_1 - \alpha E(E+2B)\right) + \beta\left(A_1 + B_2 - C_1 - \alpha E(E+B)\right)\Big)$$
$$+\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\left(A_3+\alpha EB\right)\left(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3 + \frac{\alpha}{2}(E^2-B^2)\right) = EB\,, \tag{4.27}$$

$$-\sigma B_2 + \Lambda + \frac{1}{\mu(1+\alpha\sigma)}\Big(c\left(-A_3 + C_1\right) + \beta\left(-A_1 + A_3 + C_1 - C_3 + \alpha(E+B)^2\right)\Big)$$
$$-\frac{\alpha}{4\mu^2(1+\alpha\sigma)^3}\Big(\left(\alpha\Lambda + 3c^2 - 2A_1 + 2B_2 + \frac{\alpha}{2}(E^2+3B^2)\right)$$
$$\left(\alpha\Lambda + 3c^2 + 2B_2 + 2C_3 - \frac{\alpha}{2}(3E^2+B^2)\right)$$
$$+\left(2A_3 + 2\alpha EB\right)\left(2C_1 + 2\alpha EB\right)\Big) = \frac{1}{2}(E^2-B^2)\,, \tag{4.28}$$

$$-\sigma C_1 + \frac{1}{\mu(1+\alpha\sigma)}\Big(\left(-A_1 + B_2 + \frac{\alpha}{4}\left(E^2+3B^2\right)\right)'$$
$$+c\left(A_3 + B_2 + C_1 - C_3 + \alpha B(2E+B)\right) + \beta\left(-A_3 - B_2 + C_3 - \alpha B(E+B)\right)\Big)$$
$$-\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}(C_1+\alpha EB)\left(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3 + \frac{\alpha}{2}(E^2-B^2)\right) = -EB \tag{4.29}$$

and

$$-\sigma C_3 + \Lambda - \frac{1}{\mu(1+\alpha\sigma)}\Big((A_3+\alpha EB)' + c\left(-A_1 + A_3 - C_3 + \alpha(E^2+EB+B^2)\right)$$
$$+\beta(A_3 + B_2 - C_3 + \alpha B(E+B))\Big)$$
$$-\frac{\alpha}{4\mu^2(1+\alpha\sigma)^3}\left(\alpha\Lambda + 3c^2 - 2A_1 + 2C_3 + \frac{\alpha}{2}(E^2-B^2)\right)$$
$$\left(\alpha\Lambda + 3c^2 + 2B_2 + 2C_3 - \frac{\alpha}{2}(3E^2+B^2)\right) = \frac{1}{2}(E^2+B^2)\,. \tag{4.30}$$

Here, the Maxwell equations impose taking anti-self duality condition for the electromagnetic fields (i.e., $E = -B$) where in that case the Maxwell equation simply becomes

$$B' + mB = 0 \tag{4.31}$$

whose solution can be written as

$$B = B_0(u)\, e^{-m\rho} \tag{4.32}$$

where $B_0(u)$ is left arbitrary after the integration with respect to variable ρ. Therefore, substituting the curvature components (4.7) and (4.8) and using (4.15) with the anti-self-duality of the Maxwell fields, the Einstein field equation turns into

$$f''' - (2c+\gamma)f'' + 2c\gamma f' - 4\eta B_0^2 e^{-2m\rho} = 0\,, \tag{4.33}$$

where

$$\eta = \alpha(c+2m) + \mu(1+\alpha\sigma) - \frac{\alpha^2(\alpha\Lambda - c^2)}{2\mu(1+\alpha\sigma)^2}\,. \tag{4.34}$$

It is seen that (4.33) is a third-order non-homogeneous differential equation with constant coefficients such that homogenous solutions are identical with the vacuum solutions (4.6) and (4.8). Note that, the electromagnetic Maxwell-Chern-Simons contributes to particular solution of the Einstein equation. Then, the general solution for the metric function f is

$$f(u,\rho) = f_1(u) + \frac{f_2(u)}{2c}e^{2c\rho} + \frac{f_3(u)}{\gamma}\, e^{\gamma\rho} - \frac{\eta B_0^2(u)}{m(c+m)(2m+\gamma)}e^{-2m\rho} \tag{4.35}$$

for the case where $\gamma \neq 2c$, while it becomes

$$f(u,\rho) = f_1(u) + \frac{2c(f_2(u)\rho + f_3(u)) - f_2(u)}{4c^2}\, e^{2c\rho} - \frac{\eta B_0^2(u)}{2m(c+m)^2}e^{-2m\rho} \tag{4.36}$$

for the case where $\gamma = 2c$. For the case $\gamma = 2c$, there exists the following relation between the cosmological constant and other parameters of MMG theory:

$$\Lambda = \frac{1}{\alpha^2}\left(\alpha c^2 - 2\mu(1+\alpha\sigma)^2(c+\mu\sigma(1+\alpha\sigma))\right)\,. \tag{4.37}$$

In addition to solutions presented above, there exist two more classes of solutions. They are obtained as follows :

1. The case $\gamma = -2m$ and $\gamma \neq 2c$:

In this case, the field equation becomes

$$f''' + 2(m-c)f'' - 4cmf' - 4\eta B_0^2(u)e^{-2m\rho} = 0\,, \tag{4.38}$$

which yields

$$f(u,\rho) = f_1(u) + \frac{f_2(u)}{2c}e^{2c\rho} + \left(-\frac{f_3(u)}{2m} + \frac{\eta B_0^2(u)(c+2m+2m\rho(c+m))}{m^2(c+m)^2}\right)e^{-2m\rho}\,. \tag{4.39}$$

This particular class of solution is valid when the cosmological constant Λ and the other parameters of the theory satisfy the relation

$$\Lambda = \frac{1}{\alpha^2}\left(\alpha c^2 + 2\mu(1+\alpha\sigma)^2(c+2m-\mu\sigma(1+\alpha\sigma))\right) . \tag{4.40}$$

2. The case $c=-m$ and $\gamma \neq 2c$:
For this case, one gets

$$f''' + (2m-\gamma)f'' - 2m\gamma f' - 4\eta_1 B_0^2(u)e^{-2m\rho} = 0 , \tag{4.41}$$

which can be solved as

$$f(u,\rho) = f_1(u) - \frac{f_2(u)}{2m}e^{-2m\rho} + \frac{f_3(u)}{\gamma}e^{\gamma\rho} + \frac{\eta_1 B_0^2(u)\left(4m(1+m\rho)+\gamma(1+2m\rho)\right)}{m^2(2m+\gamma)^2}e^{-2m\rho} , \tag{4.42}$$

where

$$\eta_1 = m\alpha + \mu(1+\alpha\sigma) - \frac{\alpha^2(\alpha\Lambda - m^2)}{2\mu(1+\alpha\sigma)^2} . \tag{4.43}$$

For this special class of solution, the cosmological constant Λ and other parameters are related by

$$\Lambda = \frac{2\mu(1+\alpha\sigma)^2}{\alpha^2}\left(\frac{\alpha m^2}{2\mu(1+\alpha\sigma)^2} - \mu\sigma(1+\alpha\sigma) - \gamma - m\right) . \tag{4.44}$$

Again in all these solutions presented here, $f_1(u)$, $f_2(u)$, $f_3(u)$ are left arbitrary after integration with respect to metric variable ρ. Finally, the electric field solution can be written as

$$E = E_0(u)e^{-m\rho} \tag{4.45}$$

such that $E_0(u) = -B_0(u)$, i.e., the electromagnetic fields satisfy the anti-self duality condition. The pp-wave solutions presented here will be discussed in the forthcoming work [49].

5. CONCLUSION

In this thesis work, using the exterior algebra formalism, generalized gravity models in higher dimensions are constructed in metric-compatible spacetimes. The use of the exterior algebra formalism enables to express the field equations of the corresponding gravity theories in coordinate-free notation. As a result of this, one can manage with less number of indices compared to tensorial notation. Gravitational theories studied in this work are described by the most general gravitational function expressed in terms of the scalar curvature, Ricci-squared and Riemann-squared terms. The theory can be examined in two formalisms. First, the theory is formulated for torsion-free connection structure by viewing the connections and orthonormal co-frames are dependent gravitational variables thus requiring a constraint relation between them. Such a constraint relation is put into the Lagrangian by introducing $(n-2)$-form auxiliary fields (which can also be identified as Lagrange multiplier $(n-2)$-forms). The field equations are obtained by making variations with respect to co-frame, connection and auxiliary field. The auxiliary field can be exactly solved from the field equation obtained from the variation with respect to connection field. Then inserting the Lagrange multiplier term into Einstein field equation, one gets the governing field equation of the corresponding torsion-free theory. The same gravity theory is also investigated in a spacetime with torsion where in this case one can consider the co-frame and connection fields are independent fields. It is shown that the field equations obtained in both formalisms completely differ. Next, as a specific higher curvature model, quadratic curvature gravity models are also studied in spacetimes with and without torsion. Furthermore, three dimensional massive gravity theories namely Topologically Massive Gravity, New Massive Gravity, Generalized Massive Gravity and Minimal Massive Gravity are also examined. Among these theories, it is shown that New Massive Gravity and Generalized Massive Gravity can be formulated in spacetimes with and without torsion.

In the next section, some exact solutions of quadratic curvature gravity are presented. Among them are the constant curvature solutions with and without torsion. The solutions with torsion are product manifold solutions such that total manifold involves two different constant curvature submanifolds. It is seen that, the field equations with torsion have two different classes of solution provided that the coupling constants obey some constraint relations. In addition, constant curvature solutions without torsion are also obtained in $n \geq 3$-dimensional spacetimes.

Next, some exact solutions of R^2-corrected gravity theory are investigated in higher dimensions. It is seen that all the new solutions obtained for this theory are attained provided that $R = -4\Lambda$ and $\alpha = \frac{1}{8\Lambda}$. The first class of solutions are of $Li_m \times \Omega_{(n-m)}$ type where Li_m describes an m-dimensional Lifshitz type spacetime while $\Omega_{(n-m)}$ corresponds to $(n-m)$-dimensional topological space with spherical, hyperboloidal and planar geometry depending on the value of the topological constant κ. By introducing such a metric ansatz, new exact static and stationary solutions are obtained for n-dimensional R^2-corrected gravity for the general values of the dynamical exponent z, which measures the anisotropic scaling symmetry. It can be noted that for static and stationary solutions of Einstein field equation, the scalar curvature is constant although the Ricci 1-forms and Riemann curvature 2-forms are not constant.

In order that Lifshitz type static solutions (3.57) and Lifshitz type stationary solutions (3.90) be physical, the inquality conditions are obtained in terms of z and N. Here, the branch $z > z_+ > 0$ (or equivalently $p_+ \geq p_- \geq z_+ > 0$) with suitably chosen integration constant corresponds to either a black hole (a hyperbolic black hole for $\kappa = -1$ and a spherical black hole for $\kappa = 1$) or a black brane for $\kappa = 0$ in both static and stationary case. In addition, for the special value of $z = z_+$, Einstein field equation yields logarithmic static and stationary Lifshitz type solutions. It ia also recalled that although $z = 0$ lies outside of the given range (i.e., $z = 0 \notin (-\infty, z_-] \cup [z_+, \infty)$), solving the field equation for $z = 0$ results in a periodic solution.

The thermodynamical properties of both static and stationary Lifshitz type solutions are also examined. Employing Wald entropy formulation, one can see that although the surface temperatures calculated at the outer horizon in both cases do not vanish, the solutions have vanishing entropy which is an unusual result from the thermodynamical point of view. Hopefully, such an unusual thermodynamical behavior can be tackled by considering holographic contributions for the entropy [52, 53].

Apart from Lifshitz-type solutions with non-constant temperature, the extremal solutions with zero temperature are also obtained in both static and stationary case.

The other class of the solutions is devoted to n-dimensional Bianchi type static and stationary solutions, which has not been studied in higher dimensions so far. It is found that R^2-corrected gravity theory admits Bianchi type static and stationary solutions provided that some specific relations hold between parameters of solution. In a similar fashion, some thermodynamical properties are also briefly discussed. Since $(1 + 2\alpha R) = 0$ for R^2-corrected gravity, the entropy again vanishes for non-vanishing temperature as in Lifshitz type solutions.

As the last class of solutions for R^2-corrected gravity, the Schrödinger type solutions are presented. It is found that the solution does not depend on the dynamical exponent z.

In the last part of this thesis work, pp-wave solutions of Minimal Massive $3d$-Gravity Theory are obtained for vacuum MMG (without a matter coupling) and for MMG minimally coupled to Maxwell-Chern-Simons. It is found that the pp-wave solution of the resulting field equations require anti-self duality condition for Maxwell field. Within this condition, some specific classes of solutions are presented as well.

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APPENDICES

A. Exterior Algebra

In this appendix, the fundamental exterior algebraic notation of and operators on differential forms used throughout this work with the aid of the concepts given in [50] and [51] are briefly explained.

A.1. Differential forms on manifolds

Let M be an n-dimensional differentiable manifold. This means that M is a topological space and provided with a family of pairs $\{(U_i, \varphi_i)\}$. Here, $\{U_i\}$ is a family of open sets which covers M and φ is a homeomorphism from U_i onto an open subset U'_i of $\mathbb{R}^n$, assigning coordinates to a point p. If $U_i \cap U_j \neq \emptyset$, the transition from one coordinate system to another is smooth; that is, the map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable. The importance of differentiable manifolds resides in the fact that one may use the usual calculus developed in $\mathbb{R}^n$. Since the coordinate transformation is smooth, the calculus is independent of the coordinates which are chosen.
On a manifold, a vector is defined as a tangent vector to a curve in manifold M and all the tangent vectors at q ($q \in M$) form a vector space which is called the tangent space of M at q, denoted by T_qM. In general, $e_\mu = \partial/\partial x^\mu$ ($1 \leq \mu \leq n$) denote the basis vectors of T_qM and dim T_qM = dim M. For a vector $X = X^\mu e_\mu$ ($X \in T_qM$), $\{e_\mu\}$ can be identified as the coordinate basis while X^μ denote the components of vector X with respect to basis e_μ. Since T_qM is a vector space, there exists a dual vector space to T_qM, whose element is a linear function from T_qM to $\mathbb{R}$. Here, the dual space is named as the cotangent space at q and denoted by T_q^*M. An element $\omega : T_qM \to \mathbb{R}$ of T_q^*M is called a dual vector or cotangent vector. In the context of differential forms, this can be identified as a one-form as well. An arbitrary one-form ω can be expressed as $\omega = \omega_\mu dx^\mu$ ($\omega \in T_q^*M$), where $\{dx^\mu\}$ denotes a basis of T_q^*M and ω_μ are the components of ω with respect to that basis. A differential form of order p or a p-form on M is a totally anti-symmetric tensor field of type $(0, p)$ and the vector space of these p-forms are denoted by $\Omega^p(M)$.

A.2. Wedge product

Any p-form can be expressed as a wedge product or antisymmetric tensor product of the basis one-forms:

$$\omega = \frac{1}{p!}\omega_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} , \tag{A.1}$$

where the wedge product $\wedge$ of p one-forms can be expressed by totally antisymmetric tensor product in the form

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} = \sum_{P \in S_p} sgn(P) dx^{\mu_{P(1)}} \otimes dx^{\mu_{P(2)}} \otimes \ldots \otimes dx^{\mu_{P(p)}} , \tag{A.2}$$

while $sgn(P) = +1$ for even permutations, $sgn(P) = -1$ for odd permutations. Here, P is an element of S_p, the symmetric group of order p.
The wedge product also satisfies the following identities for p-forms $\varphi_i \in \Omega^p(M)$ $(i = 1, 2)$ and an r-form $\psi \in \Omega^r(M)$:

1. Wedge product is bilinear.

$$(\varphi_1 + \varphi_2) \wedge \psi = \varphi_1 \wedge \psi + \varphi_2 \wedge \psi \tag{A.3}$$

   and

$$\varphi \wedge (a\psi) = a(\varphi \wedge \psi)\,, \tag{A.4}$$

   where $a \in \mathbb{R}$.

2. Wedge product is associative.

$$(\varphi_1 \wedge \varphi_2) \wedge \psi = \varphi_1 \wedge (\varphi_2 \wedge \psi) \tag{A.5}$$

3. It is graded commutative.

$$\varphi \wedge \psi = (-1)^{pr} \psi \wedge \varphi \tag{A.6}$$

4. When p is odd,

$$\varphi \wedge \varphi = 0 \tag{A.7}$$

   for any p-form φ.

A.3. Exterior product

The exterior product of a p-form ω and r-form ξ is defined as

$$(\omega\wedge\xi) = (X_1, ..., X_{p+r}) = \frac{1}{p!\,r!} \sum_{P \in S_{p+r}} sgn(P)\, \omega\left(X_{P(1)}, ..., X_{P(p)}\right)\, \xi\left(X_{P(p+1)}, ..., X_{P(p+r)}\right)\,, \tag{A.8}$$

where $X_i \in T_qM$ and $(\omega \wedge \xi) \in \Omega^{p+r}(M)$.

A.4. Exterior derivative

The exterior derivative d is a map $\Omega^p(M) \to \Omega^{p+1}(M)$ whose action on (A.1) is defined by

$$d\omega = \frac{1}{p!}\left(\frac{\partial}{\partial x^\mu}\omega_{\mu_1\mu_2...\mu_p}\right) dx^\mu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \tag{A.9}$$

with the properties:

1. For a p-form φ and an r-form ψ

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi\,, \tag{A.10}$$

2.
$$d(d\varphi) \equiv d^2\varphi = 0\,. \tag{A.11}$$

A.5. Interior (inner) derivative

The interior derivative ι_X on a differential form (A.1) with respect to a vector field $X \in T_pM$ is a map $\Omega^p(M) \to \Omega^{p-1}(M)$ defined by

$$\begin{aligned}\iota_X\omega &= \frac{1}{(p-1)!}X^\nu\omega_{\nu\mu_2\ldots\mu_p}dx^{\mu_2}\wedge\ldots\wedge dx^{\mu_p} \\ &= \frac{1}{p!}\sum_{s=1}^{p}X^{\mu_s}\omega_{\mu_1\ldots\mu_s\ldots\mu_p}(-1)^{s-1}dx^{\mu_1}\wedge\ldots\wedge\hat{d}x^{\mu_s}\wedge\ldots\wedge dx^{\mu_p}\,,\end{aligned} \tag{A.12}$$

where the entry below ˆ has been omitted. Some properties of the inner derivative are:

1. ι_X is an anti-derivation. For a p-form φ and an r-form ψ
$$\iota_X(\varphi\wedge\psi) = \iota_X\varphi\wedge\psi + (-1)^p\varphi\wedge\iota_X\psi\,. \tag{A.13}$$

2. It is nilpotent,
$$\iota_X^2\varphi = 0\,. \tag{A.14}$$

3. When acting on differential forms, the Lie derivative can be expressed in terms of exterior and interior derivatives
$$\mathcal{L}_X\omega = (d\,\iota_X + \iota_X d\,)\omega\,. \tag{A.15}$$

A.6. Hodge map

Hodge dual or hodge $*$ map is a linear map $\Omega^p(M) \to \Omega^{n-p}(M)$ in an n-dimensional manifold equipped with the metric g defined over the manifold. The action of hodge map operator on any p-form is defined as

$$*(dx^{\mu_1}\wedge dx^{\mu_2}\wedge\ldots\wedge dx^{\mu_p}) = \frac{\sqrt{|g|}}{(n-p)!}\epsilon^{\mu_1\mu_2\mu_3\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_n}dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_n}\,, \tag{A.16}$$

where the totally anti-symmetric tensor ε is given by

$$\epsilon_{\mu_1\mu_2\mu_3\ldots\mu_n} = \begin{cases} -1 & \text{if } (\mu_1\mu_2\mu_3\ldots\mu_n) \text{ is an odd permutation of (123...n),} \\ 0 & \text{if any two indices are equal,} \\ 1 & \text{if } (\mu_1\mu_2\mu_3\ldots\mu_n) \text{ is an even permutation of (123...n).} \end{cases}$$

For p-form ω given in (A.1), the definition of hodge dual is written by

$$*\omega = \frac{1}{p!}\omega_{\mu_1\mu_2\ldots\mu_p}\epsilon^{\mu_1\mu_2\mu_3\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_n}dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_n}\,. \tag{A.17}$$

In terms of orthonormal co-frames $\{e^i\}$, one can express a p-form ω in the form

$$\omega = \frac{1}{p!}\omega_{abc...p} e^a \wedge e^b \wedge e^c \wedge ... \wedge e^p \,, \tag{A.18}$$

where $\omega_{abc...p}$ denotes components of p-form with respect to co-frame basis e^a. Here, it is convenient to display the following properties of Hodge map $*$ operator.

Hodge $*$ map also has the following properties:

1. The volume n-form in n-dimensional manifold is

$$*1 = \frac{1}{n!}\epsilon_{abc...n}\, e^a \wedge e^b \wedge e^c \wedge ... \wedge e^n \,. \tag{A.19}$$

2.

$$*(e^a \wedge e^b \wedge e^c \wedge ... \wedge e^p) = \frac{1}{(n-p)!}\, \epsilon^{abc...p}{}_{p+1...n} e^{p+1} \wedge ... \wedge e^n \,. \tag{A.20}$$

3.

$$*(e^a \wedge e^b \wedge e^c \wedge ... \wedge e^q) = i^q\, i^{q-1} ...\, i^c\, i^b\, i^a * 1 \,. \tag{A.21}$$

4. For any p-form ω,

$$* * \omega = (-1)^{p(n-p)}\omega \text{ if } (M, g) \text{ is Riemannian and} \tag{A.22}$$

$$* * \omega = (-1)^{1+p(n-p)}\omega \text{ if } (M, g) \text{ is Lorentzian.} \tag{A.23}$$

A.7. Covariant exterior derivative

Given a mixed tensor that is totally anti-symmetric in some subset of r vectors, one can associate a set of r-forms with any basis $\{X_j\}$ with dual basis $\{e^j\}$. Suppose that S is a such tensor of type $(r+q, p)$, one can define a set of r-forms $S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ by

$$S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(X_1, ..., X_r) = S(X_1, ..., X_r, X_{j_1}, ..., X_{j_q}, e^{i_1}, ..., e^{i_p}) \,. \tag{A.24}$$

The covariant exterior D of $S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ can be defined in terms of connection one-forms $\omega^a{}_b$ as

$$\begin{aligned} DS^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} = \quad & dS^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} + \omega^{i_1}{}_{i_s} \wedge S^{i_s \cdots i_p}{}_{j_1 \cdots j_q} + ... + \omega^{i_p}{}_{i_s} \wedge S^{i_1 \cdots i_s}{}_{j_1 \cdots j_q} \\ & - \omega^{i_s}{}_{j_1} \wedge S^{i_1 \cdots i_p}{}_{j_s \cdots j_q} - ... - \omega^{i_s}{}_{j_q} \wedge S^{i_1 \cdots i_p}{}_{j_1 \cdots j_s} \,. \end{aligned} \tag{A.25}$$

Using that definition, one has

$$DS^a{}_b = dS^a{}_b + \omega^a{}_c \wedge S^c{}_b - \omega^d{}_b \wedge S^a{}_d \,. \tag{A.26}$$

B. Miscellaneous Identities

This appendix presents miscellaneous identities used throughout this thesis; some are given together with their proofs.

1. For any p-form φ

$$*(\varphi \wedge e_a) = \iota_a * \varphi \,, \tag{B.1}$$

$$e^a \wedge \iota_a \varphi = p\varphi \,. \tag{B.2}$$

2. For any p-forms α and β, one can use

$$\delta(\alpha \wedge *\beta) = (\delta\alpha) \wedge *\beta + (\delta\beta) \wedge *\alpha - (\delta e^a) \wedge ((\iota_a\beta) \wedge *\alpha - (-1)^p \alpha \wedge \iota_a * \beta) \,. \tag{B.3}$$

3.

$$\iota_{(\delta X^a)} e^b = -\iota_{X^a}(\delta e^b) \,, \tag{B.4}$$

where X^a denotes the frame vector field.

4. For any p-forms α and β, one has the following

$$\alpha \wedge *\beta = \beta \wedge *\alpha \,. \tag{B.5}$$

5.

$$e^a \wedge *e^b = \delta^{ba} * 1 \tag{B.6}$$

Taking the identity (B.1), one can write

$$e^a \wedge *e^b = e^a \wedge \iota^b * 1 \,. \tag{B.7}$$

The effect of the application of the interior product operator onto $(e^a \wedge *1)$ reads

$$\begin{aligned} \iota^b(e^a \wedge *1) &= \delta^{ba} * 1 - e^a \wedge \iota^b * 1 \\ e^a \wedge \iota^b * 1 &= \delta^{ba} * 1 \\ e^a \wedge *e^b &= \delta^{ba} * 1 \end{aligned} \tag{B.8}$$

since $(e^a \wedge *1) = 0$ is an $(n+1)$-form in n-dimensional spacetime.

6.

$$e^a \wedge *(e^b \wedge e^c) = \delta^{ca} * e^b - \delta^{ba} * e^c \tag{B.9}$$

The procedure given in the previous item works for the proof of this identity as well. Therefore, one can write

$$e^a \wedge *(e^b \wedge e^c) = e^a \wedge \iota^c * e^b \,. \tag{B.10}$$

Using (B.1) and (B.30), one can find from the application of the interior product operator on $(e^a \wedge *e^b)$

$$\begin{aligned} \iota^c(e^a \wedge *e^b) &= \delta^{ca} * e^b - e^a \wedge \iota^c * e^b \\ \delta^{ba} * e^c &= \delta^{ca} * e^b - e^a \wedge \iota^c * e^b \\ e^a \wedge \iota^c * e^b &= \delta^{ca} * e^b - \delta^{ba} * e^c \\ e^a \wedge *(e^b \wedge e^c) &= \delta^{ca} * e^b - \delta^{ba} * e^c \,. \end{aligned} \tag{B.11}$$

7. It is necessary to give the following variations

$$\begin{aligned} \delta\left(*(e_a \wedge e_b)\right) &= \frac{1}{(n-2)!}\epsilon_{abc...n}\delta(e^c \wedge e^d \wedge ... \wedge e^n) \\ &= \frac{(n-2)}{(n-2)!}(\delta e^c) \wedge \epsilon_{abc...n}e^d \wedge ... \wedge e^n \\ &= (\delta e^c) \wedge \frac{1}{(n-3)!}\epsilon_{abc...n}e^d \wedge ... \wedge e^n \\ &= (\delta e^c) \wedge *(e_a \wedge e_b \wedge e_c), \end{aligned} \tag{B.12}$$

$$\begin{aligned} \delta(*e_a) &= \frac{1}{(n-1)!}\epsilon_{abc...n}\delta(e^b \wedge e^c \wedge ... \wedge e^n) \\ &= \frac{(n-1)}{(n-1)!}(\delta e^b) \wedge \epsilon_{abc...n}e^c \wedge ... \wedge e^n \\ &= (\delta e^b) \wedge \frac{1}{(n-2)!}\epsilon_{abc...n}e^c \wedge ... \wedge e^n \\ &= (\delta e^b) \wedge *(e_a \wedge e_b) \end{aligned} \tag{B.13}$$

and the variation of the volume n-form defined in (A.19)

$$\begin{aligned} \delta(*1) &= \frac{1}{n!}\epsilon_{abc...n}\delta[e^a \wedge e^b \wedge e^c \wedge ... \wedge e^n] \\ &= \frac{n}{n!}(\delta e^a) \wedge \epsilon_{abc...n}e^b \wedge e^c \wedge ... \wedge e^n \\ &= (\delta e^a) \wedge \frac{1}{(n-1)!}\epsilon_{abc...n}e^b \wedge e^c \wedge ... \wedge e^n \\ &= (\delta e^a) \wedge *e_a \,. \end{aligned} \tag{B.14}$$

8. The following covariant derivative identities also need to be given:

$$\begin{aligned} D * (e_a \wedge e_b) &= D\left(\frac{1}{(n-2)!}\epsilon_{abcd...n}\; e^c \wedge e^d \wedge ... \wedge e^n\right) \\ &= \frac{(n-2)}{(n-2)!}\;\epsilon_{abcd...n}\;(De^c) \wedge e^d \wedge ... \wedge e^n \\ &= (De^c) \wedge \frac{1}{(n-3)!}\;\epsilon_{abcd...n}e^d \;\wedge ... \wedge e^n \\ &= T^c \wedge *(e_a \wedge e_b \wedge e_c)\,, \end{aligned} \tag{B.15}$$

$$
\begin{aligned}
D * (e_a) &= D\left(\frac{1}{(n-1)!}\epsilon_{abcd...n}\; e^b \wedge e^c \wedge ... \wedge e^n\right) \\
&= \frac{(n-1)}{(n-1)!}\; \epsilon_{abcd...n}\; (De^b) \wedge e^c \wedge ... \wedge e^n \\
&= (De^b) \wedge \frac{1}{(n-2)!}\; \epsilon_{abcd...n} e^c\; \wedge ... \wedge e^n \\
&= T^b \wedge *(e_a \wedge e_b)\,, \qquad \text{(B.16)}
\end{aligned}
$$

and the covariant derivative of volume n-form

$$
\begin{aligned}
D * 1 &= D\left(\frac{1}{n!}\epsilon_{abcd...n}\; e^a \wedge e^b \wedge e^c \wedge ... \wedge e^n\right) \\
&= \frac{n}{n!}\; \epsilon_{abcd...n}\; (De^a) \wedge e^b \wedge e^c \wedge ... \wedge e^n \\
&= (De^a) \wedge \frac{1}{(n-1)!}\; \epsilon_{abcd...n} e^b\; \wedge e^c \wedge ... \wedge e^n \\
&= T^a \wedge *e_a\,. \qquad \text{(B.17)}
\end{aligned}
$$

9.

$$\iota_c R_{ab} = R_{abcd} e^d \qquad \text{(B.18)}$$

Applying interior product operator onto definition of curvature 2-forms yields

$$\iota_c R_{ab} = \iota_c \left(\frac{1}{2} R_{abdf} e^d \wedge e^f\right)\,. \qquad \text{(B.19)}$$

Since $\iota_c e^d = \delta_c{}^d$, one can write

$$
\begin{aligned}
\iota_c R_{ab} &= \frac{1}{2} R_{abdf}\left((\iota_c e^d) \wedge e^f - e^d \wedge (\iota_c e^f)\right) \\
&= \frac{1}{2} R_{abdf}\left(\delta_c{}^d e^f - \delta_c{}^f e^d\right) \\
&= \frac{1}{2}\left(R_{abcf} e^f - R_{abdc} e^d\right) \\
&= R_{abcd} e^d \qquad \text{(B.20)}
\end{aligned}
$$

with the appropriate index modification.

10. Similarly, operating the interior product operator on the Ricci 1-forms reads

$$\iota_b P^a = \iota_b (P^a{}_c e^c) = P^a{}_c \delta_b{}^c = P^a{}_b\,, \qquad \text{(B.21)}$$

where $P^a{}_b$ denotes component of the Ricci 1-forms P^a.

11.

$$DT^a = R^a{}_b \wedge e^b \qquad \text{(B.22)}$$

Applying exterior derivative operator d on the first structure equation reads

$$\begin{aligned} dT^a &= d\left(de^a + \omega^a{}_b \wedge e^b\right) \\ dT^a &= (d\omega^a{}_b) \wedge e^b - \omega^a{}_b \wedge (de^b) \end{aligned} \tag{B.23}$$

since $d(de^a{}_b) = 0$. Replacing de^b by $T^b - \omega^b{}_c \wedge e^c$ and using the definition of curvature 2-form (2.5) with the appropriate index change leads to the first Bianchi identity as

$$\begin{aligned} dT^a &= (d\omega^a{}_b) \wedge e^b - \omega^a{}_b \wedge (T^b - \omega^b{}_c \wedge e^c) \\ dT^a + \omega^a{}_b \wedge T^b &= (d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge e^b \\ DT^a &= R^a{}_b \wedge e^b\,. \end{aligned} \tag{B.24}$$

12.

$$DR^a{}_b = 0 \tag{B.25}$$

Similarly applying exterior derivative operator d on the second structure equation gives

$$\begin{aligned} dR^a{}_b &= d\,(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \\ dR^a{}_b &= d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b \end{aligned} \tag{B.26}$$

since $d(d\omega^a{}_b) = 0$. Replacing $d\omega^a{}_c$ by $R^a{}_c - \omega^a{}_d \wedge \omega^d{}_c$ and $d\omega^c{}_b$ by $R^c{}_b - \omega^c{}_d \wedge \omega^d{}_b$ and using the definition of covariant exterior differentiation operator (A.26) simply yields to the second Bianchi identity as

$$\begin{aligned} dR^a{}_b &= \left(R^a{}_c - \omega^a{}_d \wedge \omega^d{}_c\right) \wedge \omega^c{}_b - \omega^a{}_c \wedge \left(R^c{}_b - \omega^c{}_d \wedge \omega^d{}_b\right) \\ dR^a{}_b &= R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b \\ DR^a{}_b &= 0\,. \end{aligned} \tag{B.27}$$

13. When the connection is torsion-free (i.e., $T^a = 0$), one can take

$$\iota^a D(*R_{ab}) = *DP_b \tag{B.28}$$

and

$$2\iota^a DP_{ab} = \iota_b DR\,. \tag{B.29}$$

Here, one can begin to prove the identity (B.28) starting from the left hand side.

Since the connection is Levi-Civita, one can write

$$\begin{aligned}
\iota^a D * R_{ab} &= \frac{1}{2}\iota^a D\left(R_{abcd} * (e^c \wedge e^d)\right) \\
&= \frac{1}{2}\iota^a \left((DR_{abcd}) \wedge *(e^c \wedge e^d) + R_{abcd} \wedge D * (e^c \wedge e^d)\right) \\
&= \frac{1}{2}\iota^a \left((DR_{abcd}) \wedge *(e^c \wedge e^d) + R_{abcd} \wedge T_f \wedge *(e^c \wedge e^d \wedge e^f)\right) \\
&= \frac{1}{2}\iota^a \left((DR_{abcd}) \wedge *(e^c \wedge e^d)\right) \\
&= \frac{1}{2}\underbrace{(\iota^a DR_{abcd}) \wedge *(e^c \wedge e^d)}_{(B15-1)} - \frac{1}{2}\underbrace{(DR_{abcd}) \wedge *(e^c \wedge e^d \wedge e^a)}_{(B15-2)}\,. \qquad \text{(B.30)}
\end{aligned}$$

Since the term $(\iota^a DR_{abcd})$ is a scalar,

$$\begin{aligned}
(B15-1) &= (\iota^a DR_{abcd}) \wedge *(e^c \wedge e^d) \\
&= *\left(e^c \wedge e^d(\iota^a DR_{abcd})\right)\,. \qquad \text{(B.31)}
\end{aligned}$$

Remembering that the connection is torsion-free, one has the following:

$$\begin{aligned}
e^c \wedge e^d(\iota^a DR_{abcd}) &= \iota^a \left(e^c \wedge e^d \wedge (DR_{abcd})\right) - \delta^{ac} e^d \wedge DR_{abcd} + \delta^{ad} e^c \wedge DR_{abcd} \\
&= \iota^a \left(D(e^c \wedge e^d \wedge R_{abcd}) - R_{abcd}\left(D(e^c \wedge e^d)\right)\right) \\
&\quad - e^d \wedge (DR_{ab}{}^a{}_d) + e^c \wedge DR_{abc}{}^a \\
&= 2\iota^a DR_{ab} - 2e^a \wedge (DP_{ba}) \\
&= -2e^a \wedge (DP_{ba}) \\
&= 2DP_b \qquad \text{(B.32)}
\end{aligned}$$

with the suitable index change. Therefore,

$$(B15-1) = 2 * DP_b\,. \qquad \text{(B.33)}$$

On the other hand, trace term $(B15-2)$ can be given as

$$\begin{aligned}
(B15-2) &= (DR_{abcd}) * (e^c \wedge e^d \wedge e^a) \\
&= D\left(R_{abcd} * (e^c \wedge e^d \wedge e^a)\right) - R_{abcd} D * (e^c \wedge e^d \wedge e^a) \\
&= D * (R_{ab} \wedge e^a)\,. \qquad \text{(B.34)}
\end{aligned}$$

Since the connection is Levi-Civita, one can here write $R_{ab} \wedge e^a = DT_b = 0$. Therefore, there is no contribution from $(B15-2)$. Substituting $(B15-1)$ into (B.30) yields

$$\iota^a D(*R_{ab}) = *DP_b\,. \qquad \text{(B.35)}$$

14.

$$R * 1 = R^{ab} \wedge *(e_a \wedge e_b)\,. \qquad \text{(B.36)}$$

Using the identity (B.1) into (B.36) one can write

$$\begin{aligned} R * 1 &= R^{ab} \wedge *(e_a \wedge e_b) \\ &= R^{ab} \wedge \iota_b * e_a \,. \end{aligned} \tag{B.37}$$

Since $R^{ab} \wedge *e_a = 0$ (being an $(n+1)$-form in n-dimensional spacetime), $\iota_b(R^{ab} \wedge *e_a) = 0$ as well. Then, the right hand side of (B.37) can be further simplified by applying the interior product operator as

$$\begin{aligned} \iota_b(R^{ab} \wedge *e_a) &= (\iota_b R^{ab}) \wedge *e_a + R^{ab} \wedge \iota_b * e_a = 0, \\ R^{ab} \wedge \iota_b * e_a &= -(\iota_b R^{ab}) \wedge *e_a \,. \end{aligned} \tag{B.38}$$

Noting that spacetime is metric-compatible ($R^{ab} = -R^{ba}$) and also using $P^a = \iota_b R^{ba}$, one has

$$\begin{aligned} R^{ab} \wedge \iota_b * e_a &= (\iota_b R^{ba}) \wedge *e_a \\ &= P^a \wedge *e_a \\ &= P^a \wedge \iota_a * 1 \,. \end{aligned} \tag{B.39}$$

The right-hand side of (B.39) can further be simplified. For that, using $P^a \wedge *1 = 0$ (again being an $(n+1)$-form in n-dimensional spacetime), one obtains

$$\begin{aligned} \iota_a(P^a \wedge *1) &= (\iota_a P^a) * 1 - P^a \wedge \iota_a * 1 = 0, \\ P^a \wedge \iota_a * 1 &= R * 1 \end{aligned} \tag{B.40}$$

where $\iota_a P^a = R$. Therefore,

$$R^{ab} \wedge *(e_a \wedge e_b) = R * 1 \,.$$

15.

$$P_a \wedge *P^a = U * 1 \,, \tag{B.41}$$

where $U = P_{ab}P^{ab}$ denotes the Ricci-squared scalar. Writing the Ricci 1-forms in terms of its components, one finds

$$P_a \wedge *P^a = P_{ab}P^{ac}e^b \wedge *e_c \,. \tag{B.42}$$

Employing the identity (B.6) easily gives

$$\begin{aligned} P_a \wedge *P^a &= P_{ab}P^{ac}\delta^b{}_c * 1 \\ &= P_{ab}P^{ab} * 1 \\ &= U * 1 \end{aligned} \tag{B.43}$$

with the Ricci-squared scalar $U = P_{ab}P^{ab}$.

16.

$$R^a{}_b \wedge *R_a{}^b = \frac{1}{2} V * 1 \tag{B.44}$$

where $V = R^a{}_{bcd} R_a{}^{bcd}$ is the Kretchmann scalar. Here, the proof of the identity requires writing the curvature 2-forms in terms of Riemann curvature tensor $R^a{}_{bcd}$ as

$$R^a{}_b \wedge *R_a{}^b = \frac{1}{4} R^a{}_{bcd} R_a{}^{bmn} e^c \wedge e^d \wedge *(e_m \wedge e_n)\,. \tag{B.45}$$

Using (B.11) with the appropriate indices, one obtains

$$\begin{aligned} R^a{}_b \wedge *R_a{}^b &= \frac{1}{4} R^a{}_{bcd} R_a{}^{bmn} e^c \wedge \left(\delta^d{}_n * e_m - \delta^d{}_m * e_n \right) \\ &= \frac{1}{4} R^a{}_{bcd} \left(R_a{}^{bmd} e^c \wedge *e_m - R_a{}^{bdn} e^c \wedge *e_n \right) \\ &= \frac{1}{2} R^a{}_{bcd} R_a{}^{bmd} e^c \wedge *e_m\,. \end{aligned} \tag{B.46}$$

Lastly, taking the identity (B.6) gives

$$\begin{aligned} R^a{}_b \wedge *R_a{}^b &= \frac{1}{2} R^a{}_{bcd} R_a{}^{bmd} \delta^c{}_m * 1 \\ &= \frac{1}{2} R^a{}_{bcd} R_a{}^{bcd} * 1 \\ &= \frac{1}{2} V * 1 \end{aligned} \tag{B.47}$$

with the Kretchmann scalar $V = R^a{}_{bcd} R_a{}^{bcd}$.

C. Field Equations of a Generalized Gravity Model in Higher Dimensions

This appendix presents in detail calculations related to field equations of a generalized gravity model in higher dimensions. The field equations are obtained by first considering the co-frame 1-forms e^a and the connection 1-forms $\omega^a{}_b$ as dependent fields with the vanishing torsion constraint, i.e., $T^a = 0$, and by later taking them as independent fields and making variations with respect to each field.

The Lagrangian density of the model under the vanishing torsion constraint is given in (2.8) whose variation with respect to e^a, $\omega^a{}_b$ and λ^a can be written as

$$\begin{aligned} \delta\mathcal{L} &= \underbrace{\left(\frac{\partial f}{\partial R}\right)(\delta R)*1}_{(A)} + \underbrace{\left(\frac{\partial f}{\partial U}\right)(\delta U)*1}_{(B)} + \underbrace{\left(\frac{\partial f}{\partial V}\right)(\delta V)*1}_{(C)} \\ &\quad + \underbrace{f(R,U,V)\delta(*1)}_{(D)} + \underbrace{\delta\mathcal{L}(matter)}_{(E)} + \underbrace{\delta\left(T^a\wedge\lambda_a\right)}_{(F)} . \end{aligned} \tag{C.1}$$

Here, the variation of each term should be considered separately to avoid confusion. Calculations related to variational term (A) can be given as

$$\begin{aligned} \delta(R*1) &= \delta\left(R^{ab}\wedge *(e_a\wedge e_b)\right) \\ (\delta R)*1 + R\,\delta(*1) &= (\delta R^{ab})\wedge *(e_a\wedge e_b) + R^{ab}\wedge\delta\left[*(e_a\wedge e_b)\right] \\ (\delta R)*1 &= \left(\delta(d\omega^{ab}+\omega^a{}_c\wedge\omega^{cb})\right)\wedge *(e_a\wedge e_b) \\ &\quad + R^{ab}\wedge\delta\left(*(e_a\wedge e_b)\right) - R\delta(*1) \\ (\delta R)*1 &= \delta(d\omega^{ab})\wedge *(e_a\wedge e_b) + (\delta\omega^a{}_c)\wedge\omega^{cb}\wedge *(e_a\wedge e_b) \\ &\quad + \omega^a{}_c\wedge(\delta\omega^{cb})\wedge *(e_a\wedge e_b) + R^{ab}\wedge\delta\left(*(e_a\wedge e_b)\right) \\ &\quad - R\,\delta(*1) . \end{aligned} \tag{C.2}$$

Infinitesimal variations and exterior derivative operators on differential forms commute with each other. Therefore,

$$\delta(d\omega^{ab}) = d(\delta\omega^{ab}) . \tag{C.3}$$

Putting the commutation relation (C.3) into (C.2) and appropriately changing some indices result in

$$\begin{aligned} (\delta R)*1 &= d(\delta\omega^{ab})\wedge *(e_a\wedge e_b) + (\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(e_c\wedge e_b) - \omega^c{}_b\wedge *(e_a\wedge e_c)\right) \\ &\quad + R^{ab}\wedge\delta\left(*(e_a\wedge e_b)\right) - R\,\delta(*1) . \end{aligned} \tag{C.4}$$

By making use of (B.12) and (B.14), one finds

$$\begin{aligned} (\delta R)*1 &= d(\delta\omega^{ab})\wedge *(e_a\wedge e_b) + (\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(e_c\wedge e_b) - \omega^c{}_b\wedge *(e_a\wedge e_c)\right) \\ &\quad + (\delta e^c)\wedge\left(R^{ab}\wedge *(e_a\wedge e_b\wedge e_c) - R*e_c\right) . \end{aligned} \tag{C.5}$$

Multiplying with a scalar $\left(\frac{\partial f}{\partial R}\right)$ yields

$$
\begin{aligned}
(A) &= \left(\frac{\partial f}{\partial R}\right)(\delta R) * 1 \\
&= \underbrace{d(\delta\omega^{ab}) \wedge \left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)}_{(A1)} \\
&\quad + \underbrace{\left(\frac{\partial f}{\partial R}\right)(\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *(e_c \wedge e_b) - \omega^c{}_b \wedge *(e_a \wedge e_c))}_{(A2)} \\
&\quad + \underbrace{\left(\frac{\partial f}{\partial R}\right)(\delta e^c) \wedge \Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big)}_{(A3)} \,. \qquad \text{(C.6)}
\end{aligned}
$$

Using the definition of exterior derivative, $(A1)$ can also be expressed as

$$
\begin{aligned}
(A1) &= d(\delta\omega^{ab}) \wedge \left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b) \\
&= \underbrace{d\left(\left(\frac{\partial f}{\partial R}\right)(\delta\omega^{ab}) \wedge *(e_a \wedge e_b)\right)}_{(A1-1)} + \underbrace{(\delta\omega^{ab}) \wedge d\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right)}_{(A1-2)} \,. \qquad \text{(C.7)}
\end{aligned}
$$

Therefore, one can write

$$
\begin{aligned}
(A) &= d\left(\left(\frac{\partial f}{\partial R}\right)(\delta\omega^{ab}) \wedge *(e_a \wedge e_b)\right) + (\delta\omega^{ab}) \wedge D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) \\
&\quad + (\delta e^c) \wedge \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \qquad \text{(C.8)}
\end{aligned}
$$

since

$$
(A1-2) + (A2) = (\delta\omega^{ab}) \wedge D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) \,. \qquad \text{(C.9)}
$$

Here, D denotes the covariant exterior derivative, the definition of which has already been given in Appendix A.

Next, the variation of (B) can be realized by considering

$$
U * 1 = P_a \wedge *P^a \,. \qquad \text{(C.10)}
$$

The variation of both sides gives

$$
\begin{aligned}
\delta(U * 1) &= \delta(P_a \wedge *P^a) \\
(\delta U) * 1 + U\delta(*1) &= \delta(P_a \wedge *P^a) \,. \qquad \text{(C.11)}
\end{aligned}
$$

Hence,

$$
(\delta U) * 1 = \delta(P_a \wedge *P^a) - U\delta(*1) \,. \qquad \text{(C.12)}
$$

Employing identities (B.14) and (B.3), one finds

$$(\delta U) * 1 = 2\underbrace{(\delta P^a) \wedge *P_a}_{(B1)} - \underbrace{(\delta e^b) \wedge ((\iota_b P^a) \wedge *P_a + P^a \wedge \iota_b * P_a)}_{(B2)} - \underbrace{U(\delta e^a) \wedge *e_a}_{(B3)} \,. \quad \text{(C.13)}$$

By using the definition of Ricci 1-form, variational term $(B1)$ can be written as

$$\begin{aligned} (B1) &= (\delta P^a) \wedge *P_a \\ &= \Big(\delta(\iota_b R^{ba})\Big) \wedge *P_a \\ &= \Big(\iota_{(\delta X^b)} R^{ba}\Big) \wedge *P_a + \Big(\iota_b(\delta R^{ba})\Big) \wedge *P_a \,. \end{aligned} \quad \text{(C.14)}$$

Applying the identity (B.4) and suitably changing some indices, one reads

$$\begin{aligned} (B1) &= \left(\iota_{(\delta X^b)} \left(\frac{1}{2} R^{ba}{}_{cd}\, e^c \wedge e^d\right)\right) \wedge *P_a + \iota_b(\delta R^{ba}) \wedge *P_a \\ &= \frac{1}{2} R^{ba}{}_{cd} \left(\iota_{(\delta X^b)} e^c\right) \wedge e^d \wedge *P_a - \frac{1}{2} R^{ba}{}_{cd} e^c \wedge \left(\iota_{(\delta X^b)} e^d\right) \wedge *P_a + \iota_b(\delta R^{ba}) \wedge *P_a \\ &= \underbrace{-R^{ba}{}_{cd}\,(\iota_b(\delta e^c)) \wedge e^d \wedge *P_a}_{(B1-1)} + \underbrace{\Big(\iota_b(\delta R^{ba})\Big) \wedge *P_a}_{(B1-2)} \,. \end{aligned} \quad \text{(C.15)}$$

$(B1-1)$ term can be rather simplified if one takes

$$\begin{aligned} \iota_b \Big(R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge *P_a\Big) &= R^{ba}{}_{cd}\,(\iota_b(\delta e^c))\, e^d \wedge *P_a - R^{ba}{}_{cd}(\delta e^c)(\iota_b e^d) \wedge *P_a \\ &\quad + R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge \iota_b * P_a \,. \end{aligned} \quad \text{(C.16)}$$

Since $\iota_b \left(R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge *P_a\right) = 0$ (this term is $(n+1)$-form in n-dimensional spacetime) and $\iota_b e^d = \delta_b{}^d$, one can rewrite

$$\begin{aligned} (B1-1) &= -R^{ba}{}_{cd}\,(\iota_b(\delta e^c)) \wedge e^d \wedge *P_a \\ &= -R^{ba}{}_{cd}(\delta e^c)\delta_b{}^d \wedge *P_a + R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge \iota_b * P_a \\ &= R^{ba}{}_{bc}(\delta e^c) \wedge *P_a + R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge \iota_b * P_a \,. \end{aligned} \quad \text{(C.17)}$$

With the use of the identity $(B.1)$ and taking $R^{ba}{}_{bc} = P^a{}_c$, the sum of $(B1-1)+(B1-2)$ becomes

$$(B1) = P^a{}_c(\delta e^c) \wedge *P_a + R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge *(P_a \wedge e_b) + \Big(\iota_b(\delta R^{ba})\Big) \wedge *P_a \,. \quad \text{(C.18)}$$

Thence, addition of $(B1)+(B2)+(B3)$ yields

$$\begin{aligned} (\delta U) * 1 &= 2P^a{}_c(\delta e^c) \wedge *P_a + 2R^{ba}{}_{cd}(\delta e^c) \wedge e^d \wedge *(P_a \wedge e_b) + 2\Big(\iota_b(\delta R^{ba})\Big) \wedge *P_a \\ &\quad -(\delta e^b) \wedge ((\iota_b P^a) \wedge *P_a + P^a \wedge \iota_b * P_a) - U(\delta e^a) \wedge *e_a \,. \end{aligned} \quad \text{(C.19)}$$

After appropriately changing some indices, collecting similar terms together, using identities (B.18) and (B.21), one gets

$$
\begin{aligned}
(\delta U) * 1 \quad = \quad & \underbrace{2\Big(\iota_b(\delta R^{ba})\Big) \wedge *P_a}_{(B1-2)} \\
& \underbrace{+ (\delta e^c) \wedge \Big(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big)}_{(B1-3)} .
\end{aligned}
\tag{C.20}
$$

Then, $(B1-2)$ can be further simplified. For that, if one takes

$$
\iota_b\Big((\delta R^{ba}) \wedge *P_a\Big) = \Big(\iota_b(\delta R^{ba})\Big) \wedge *P_a + (\delta R^{ba}) \wedge \iota_b * P_a \,, \tag{C.21}
$$

$(B1-2)$ can alternatively be written by using (B.1) as

$$
(B1-2) = (\delta R^{ab}) \wedge *(P_a \wedge e_b) \tag{C.22}
$$

since $\iota_b\big((\delta R^{ba}) \wedge *P_a\big) = 0$ (again it is $(n+1)$-form in n-dimensional spacetime). Here, the variation of the curvature 2-form leads to

$$
\begin{aligned}
(B1-2) \quad &= \quad \Big(\delta(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb})\Big) \wedge *(P_a \wedge e_b) \\
&= \quad \delta(d\omega^{ab}) \wedge *(P_a \wedge e_b) + (\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *(P_c \wedge e_b) - \omega^c{}_b \wedge *(P_a \wedge e_c))
\end{aligned}
\tag{C.23}
$$

by changing the indices accordingly. After using the commutation relation (C.3), the sum of $(B1-2)+(B1-3)$ and multiplication with $\left(\frac{\partial f}{\partial U}\right)$ reads

$$
\begin{aligned}
(B) \quad &= \quad \left(\frac{\partial f}{\partial U}\right)(\delta U) * 1 \\
&= \quad \underbrace{2\left(\frac{\partial f}{\partial U}\right)\Big(d(\delta\omega^{ab})\Big) \wedge *(P_a \wedge e_b)}_{(B1-4)} \\
&\quad\; +2\left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *(P_c \wedge e_b) - \omega^c{}_b \wedge *(P_a \wedge e_c)) \\
&\quad\; +\left(\frac{\partial f}{\partial U}\right)(\delta e^c) \wedge \Big(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) .
\end{aligned}
\tag{C.24}
$$

Here, $(B1-4)$ can be rearranged by using the definition of exterior derivative together with the antisymmetrization in the first term as

$$
(B1-4) = \frac{1}{2} d\left(\left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge *(P_a \wedge e_b - P_b \wedge e_a)\right) + (\delta\omega^{ab}) \wedge d\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b)\right) . \tag{C.25}
$$

Inserting $(B1-4)$ into (B) and using the following identity

$$\begin{aligned} D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b)\right) &= d\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b)\right) \\ &+ \left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *(P_c \wedge e_b) - \omega^c{}_b \wedge *(P_a \wedge e_c))\ , \end{aligned} \tag{C.26}$$

finally yields

$$\begin{aligned} (B) &= d\left(\left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge *(P_a \wedge e_b - P_b \wedge e_a)\right) \\ &+ (\delta\omega^{ab}) \wedge D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) \\ &+ (\delta e^c) \wedge \left(\frac{\partial f}{\partial U}\right)\Big(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big)\ . \end{aligned} \tag{C.27}$$

Then, making use of the below identity

$$V * 1 = 2R_{ab} \wedge *R^{ab} \tag{C.28}$$

and (B.14) gives for the variation of the next quadratic term as

$$\begin{aligned} \delta(V * 1) &= 2\delta(R_{ab} \wedge *R^{ab}) \\ (\delta V) * 1 &= 2\delta(R_{ab} \wedge *R^{ab}) - V(\delta e^a) \wedge *e_a\ . \end{aligned} \tag{C.29}$$

Employing the identity given in (B.3) yields

$$(\delta V) * 1 = 4\underbrace{(\delta R_{ab}) \wedge *R^{ab}}_{(C1)} - 2(\delta e^c) \wedge \Big((\iota_c R^{ab}) \wedge *R_{ab} - R^{ab} \wedge \iota_c * R_{ab}\Big) - V(\delta e^a) \wedge *e_a\ . \tag{C.30}$$

Applying the similar procedure given in detail with (C.2) also works for the variation of the term $(C1)$ as

$$\begin{aligned} (C1) &= (\delta R^{ab}) \wedge *R_{ab} \\ &= \Big(\delta(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb})\Big) \wedge *R_{ab} \\ &= \delta(d\omega^{ab}) \wedge *R_{ab} + (\delta\omega^a{}_c) \wedge \omega^{cb} \wedge *R_{ab} + \omega^a{}_c \wedge (\delta\omega^{cb}) \wedge *R_{ab}\ . \end{aligned} \tag{C.31}$$

Introducing the commutation relation (C.3) with the appropriate modification of indices, one finds

$$(C1) = d(\delta\omega^{ab}) \wedge *R_{ab} + (\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *R_{cb} - \omega^c{}_b \wedge *R_{ac})\ . \tag{C.32}$$

Substituting the term (C1) into (C.30), multiplying with $\left(\frac{\partial f}{\partial V}\right)$ and collecting together similar terms with the proper indices give

$$
\begin{aligned}
(C) &= (\delta V)\left(\frac{\partial f}{\partial V}\right) * 1 \\
&= \underbrace{4\left(\frac{\partial f}{\partial V}\right) d(\delta\omega^{ab}) \wedge *R_{ab}}_{(C1-1)} \\
&\quad +4\left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *R_{cb} - \omega^c{}_b \wedge *R_{ac}) \\
&\quad +(\delta e^c) \wedge \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big) . \qquad \text{(C.33)}
\end{aligned}
$$

Writing $(C1-1)$ as

$$
\begin{aligned}
(C1-1) &= \left(\frac{\partial f}{\partial V}\right) d(\delta\omega^{ab}) \wedge *R_{ab} \\
&= d\left(\left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge *R_{ab}\right) + (\delta\omega^{ab}) \wedge d\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) \qquad \text{(C.34)}
\end{aligned}
$$

and also using the identity

$$
D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) = d\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + \left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge (\omega_a{}^c \wedge *R_{cb} - \omega^c{}_b \wedge *R_{ac}) \ , \qquad \text{(C.35)}
$$

eventually reads

$$
\begin{aligned}
(C) &= 4d\left(\left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge *R_{ab}\right) + 4(\delta\omega^{ab}) \wedge D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) \\
&\quad +(\delta e^c) \wedge \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big) . \qquad \text{(C.36)}
\end{aligned}
$$

Next, making use of the identity (B.14) easily gives for the variation of (D)

$$
(D) = f\delta(*1) = f(\delta e^a) \wedge *e_a \, . \qquad \text{(C.37)}
$$

The variation of the matter Lagrangian in (C.1) yields

$$
\begin{aligned}
(E) &= \delta\mathcal{L}(matter) \\
&= (\delta e^a) \wedge \frac{\delta\mathcal{L}_m}{\delta e^a} + (\delta\omega^{ab}) \wedge \frac{\delta\mathcal{L}_m}{\delta\omega^{ab}} \\
&= -(\delta e^a) \wedge \tau_a + (\delta\omega^{ab}) \wedge \Omega_{ab} \, , \qquad \text{(C.38)}
\end{aligned}
$$

where $\tau_a \equiv -\frac{\delta\mathcal{L}_m}{\delta e^a}$ is the stress-energy $(n-1)$-form associated with the matter fields and $\Omega_{ab} = \frac{\delta\mathcal{L}_m}{\delta\omega^{ab}}$ is the variational term attained from the connection variation of the matter Lagrangian which can also be defined as the hypermomentum current $(n-1)$-form.

Finally, the contribution to the field equations from the variation of the zero-torsion constraint term in the Lagrangian density (C.1) is

$$\begin{aligned}(F) &= \delta(T^a \wedge \lambda_a) \\ &= (\delta T^a) \wedge \lambda_a + T^a \wedge (\delta \lambda_a) \\ &= \Big(\delta(de^a + \omega^a{}_b \wedge e^b)\Big) \wedge \lambda_a + T^a \wedge (\delta\lambda_a) \\ &= \underbrace{\delta(de^a) \wedge \lambda_a}_{(F1)} + (\delta\omega^a{}_b) \wedge e^b \wedge \lambda_a + \omega^a{}_b \wedge (\delta e^b) \wedge \lambda_a + (\delta\lambda_a) \wedge T^a \,. \end{aligned} \tag{C.39}$$

The commutation relation of infinitesimal variations and exterior derivative operators on differential forms yields

$$\delta(de^a) = d(\delta e^a)\,. \tag{C.40}$$

By taking this and appropriately modifying some indices with the definition of exterior derivative, one can write

$$(F1) = \delta(de^a) \wedge \lambda_a = d\left((\delta e^a) \wedge \lambda_a\right) + (\delta e^a) \wedge (d\lambda_a)\,. \tag{C.41}$$

Hence,

$$\begin{aligned}(F) &= d[(\delta e^a) \wedge \lambda_a] + (\delta e^a) \wedge \underbrace{(d\lambda_a - \omega^b{}_a \wedge \lambda_b)}_{F1-2} \\ &\quad + \underbrace{(\delta\omega^a{}_b) \wedge e^b \wedge \lambda_a}_{(F1-3)} + (\delta\lambda_a) \wedge T^a \end{aligned} \tag{C.42}$$

Here, (F1-2) can be identified as exterior covariant derivative of Lagrange multiplier $(n-2)$-form λ_a, that is

$$D\lambda_a = \lambda_a - \omega^b{}_a \wedge \lambda_b\,. \tag{C.43}$$

Therefore, together with the antisymmetrization of the term $(F1-3)$, the variation term related to the zero-torsion constraint can be expressed as

$$(F) = d\left((\delta e^a) \wedge \lambda_a\right) + (\delta e^a) \wedge D\lambda_a + \frac{1}{2}\,(\delta\omega^a{}_b) \wedge (e_b \wedge \lambda_a - e_a \wedge \lambda_b) + (\delta\lambda_a) \wedge T^a\,. \tag{C.44}$$

Now, collecting together all the related variational terms gives

$$
\begin{aligned}
\delta\mathcal{L} \;=\;& \delta e^c \wedge \Bigg[f * e_c - \tau_c + D\lambda_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\
& + \left(\frac{\partial f}{\partial U}\right)\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) \\
& + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big)\Bigg] \\
& + \delta\omega^{ab} \wedge \Bigg[D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) \\
& + 4D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + \Omega_{ab} + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b)\Bigg] + \delta\lambda_a \wedge T^a \\
& + d\Bigg[\left(\frac{\partial f}{\partial R}\right)(\delta\omega^{ab}) \wedge *(e_a \wedge e_b) + \left(\frac{\partial f}{\partial U}\right)(\delta\omega^{ab}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) \\
& + 4\left(\frac{\partial f}{\partial V}\right)(\delta\omega^{ab}) \wedge *R_{ab} + (\delta e^a) \wedge \lambda_a\Bigg]\,. \qquad (C.45)
\end{aligned}
$$

The variational principle implies

$$
\begin{aligned}
& f * e_c - \tau_c + D\lambda_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\
& + \left(\frac{\partial f}{\partial U}\right)(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c) - U * e_c) \\
& + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big) = 0 \qquad (C.46)
\end{aligned}
$$

and

$$
\begin{aligned}
& D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) + 4D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) \\
& + \Omega_{ab} + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b) = 0 \qquad (C.47)
\end{aligned}
$$

with

$$T^a = de^a + \omega^a{}_b \wedge e^b = 0\,. \qquad (C.48)$$

The Lagrange multiplier $(n-2)$-form λ_a can be obtained from (C.47) by defining

$$
\begin{aligned}
\Lambda_{ab} \;=\;& 2D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + 2D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) \\
& + 8D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + 2\Omega_{ab}\,. \qquad (C.49)
\end{aligned}
$$

Thus, (C.47) takes the following form:

$$\Lambda_{ab} = e_a \wedge \lambda_b - e_b \wedge \lambda_a\,. \qquad (C.50)$$

Applying the inner product operator ι^a on both sides of (C.50), one gets

$$\begin{aligned}\iota^a \Lambda_{ab} &= \iota^a \left(e_a \wedge \lambda_b - e_b \wedge \lambda_a\right) \\ &= (\iota^a e_a) \wedge \lambda_b - e_a \wedge \iota^a \lambda_b - (\iota^a e_b) \wedge \lambda_a + e_b \wedge \iota^a \lambda_a \,. \end{aligned} \tag{C.51}$$

Since $\iota^a e_b = \delta^a{}_b$ and $\iota^a e_a = n$, where n is the dimension of spacetime, one obtains

$$\begin{aligned}\iota^a \Lambda_{ab} &= n\lambda_b - (n-2)\lambda_b - \delta^a{}_b \lambda_a + e_b \wedge \iota^a \lambda_a \\ &= (n - n + 2 - 1)\lambda_b + e_b \wedge \iota^a \lambda_a \\ &= \lambda_b + e_b \wedge \iota^a \lambda_a \end{aligned} \tag{C.52}$$

by using the identity (B.2). The further application of the inner product operator ι^b on (C.52) results in

$$\begin{aligned}\iota^b \iota^a \Lambda_{ab} &= \iota^b (\lambda_b + e_b \wedge \iota^a \lambda_a) \\ &= \iota^b \lambda_b + (\iota^b e_b) \wedge \iota^a \lambda_a - e_b \wedge \iota^b \iota^a \lambda_a \\ &= \iota^b \lambda_b + n \iota^a \lambda_a - (n-3) \iota^a \lambda_a \\ &= (1 + n - n + 3) \iota^a \lambda_a \\ &= 4 \iota^a \lambda_a \,. \end{aligned} \tag{C.53}$$

Substituting this back into (C.52) gives the general expression of the Lagrange multiplier $(n-2)$-forms as

$$\begin{aligned}\lambda_b &= \iota^a \Lambda_{ab} - e_b \wedge \iota^a \lambda_a \\ &= \iota^a \Lambda_{ab} - \frac{1}{4} e_b \wedge \iota^c \iota^a \Lambda_{ac} \,. \end{aligned} \tag{C.54}$$

Then, if one uses (C.49) and expands the exterior covariant derivative, one finds

$$\begin{aligned}\iota^a \Lambda_{ab} = \;& \underbrace{2\,\iota^a D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right)}_{(L1)} + \underbrace{2\,\iota^a D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right)}_{(L2)} \\ & + \underbrace{8\,\iota^a D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right)}_{(L3)} + 2\iota^a \Omega_{ab} \,. \end{aligned} \tag{C.55}$$

Here, one should carefully calculate the operation of inner product separately by using the definition of the exterior derivative operator. Therefore, taking (B.1) and (B.15) one can write

$$
\begin{aligned}
(L1) &= \iota^a D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) \\
&= \iota^a \left(\left(D\left(\frac{\partial f}{\partial R}\right)\right) \wedge *(e_a \wedge e_b) + \left(\frac{\partial f}{\partial R}\right) D * (e_a \wedge e_b)\right) \\
&= \left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_b) + \left(D\left(\frac{\partial f}{\partial R}\right)\right) \iota^a * (e_a \wedge e_b) \\
&\quad + \left(\frac{\partial f}{\partial R}\right) \iota^a \left(T^c \wedge *(e_a \wedge e_b \wedge e_c)\right) \\
&= \left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_b) + \underbrace{\left(D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_b \wedge e^a)}_{(L1-2)} \\
&\quad + \underbrace{\left(\frac{\partial f}{\partial R}\right) \iota^a \left(T^c \wedge *(e_a \wedge e_b \wedge e_c)\right)}_{(L1-3)} . \qquad \text{(C.56)}
\end{aligned}
$$

Here, since the connection is torsion-free (i.e., Levi-Civita connection with $T^c = 0$) and $e^a \wedge e_a = 0$, there is no contribution from $(L1-2)$ and $(L1-3)$. Therefore,

$$
(L1) = \left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_b) . \qquad \text{(C.57)}
$$

The term $(L2)$ can be calculated by operating the covariant exterior derivative and then interior derivative operators respectively

$$
\begin{aligned}
(L2) &= \iota^a D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) \\
&= \iota^a \left(\left(D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_b - P_b \wedge e_a) + \left(\frac{\partial f}{\partial U}\right) D\left(*(P_a \wedge e_b - P_b \wedge e_a)\right)\right) \\
&= \left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_b - P_b \wedge e_a) + \underbrace{\left(D\left(\frac{\partial f}{\partial U}\right)\right) \wedge \iota^a * (P_a \wedge e_b - P_b \wedge e_a)}_{(L2-2)} \\
&\quad + \underbrace{\left(\frac{\partial f}{\partial U}\right) \iota^a D\left({P_a}^c * (e_c \wedge e_b) - {P_b}^c * (e_c \wedge e_a)\right)}_{(L2-3)} . \qquad \text{(C.58)}
\end{aligned}
$$

Employing (B.1) in $(L2-2)$ yields

$$
(L2-2) = \left(D\left(\frac{\partial f}{\partial U}\right)\right) \wedge *(P_a \wedge e_b \wedge e^a - P_b \wedge e_a \wedge e^a) = 0 \qquad \text{(C.59)}
$$

since $e_a \wedge e^a = 0$ and $P_a \wedge e^a = 0$ (for $T^c = 0$). Using the definition of the covariant exterior derivative operator and the equality (B.15), one finds

$$\begin{aligned}(L2-3) &= \left(\frac{\partial f}{\partial U}\right)\iota^a D\left(P_a{}^c * (e_c\wedge e_b) - P_b{}^c * (e_c\wedge e_a)\right)\\ &= \left(\frac{\partial f}{\partial U}\right)\iota^a \left((DP_a{}^c) * (e_c\wedge e_b) + P_a{}^c D * (e_c\wedge e_b)\right.\\ &\quad\left. - (DP_b{}^c) * (e_c\wedge e_a) - P_b{}^c D * (e_c\wedge e_a)\right)\\ &= \left(\frac{\partial f}{\partial U}\right)\iota^a \Big((DP_a{}^c) * (e_c\wedge e_b) + P_a{}^c T^f * (e_c\wedge e_b\wedge e_f)\\ &\quad - (DP_b{}^c) * (e_c\wedge e_a) - P_b{}^c T^f * (e_c\wedge e_a\wedge e_f)\Big)\,. \end{aligned} \tag{C.60}$$

Again since the connection is torsion-free (i.e., $T^f = 0$), this yields

$$(L2-3) = \left(\frac{\partial f}{\partial U}\right)\iota^a \left((DP_a{}^c) * (e_c\wedge e_b) - (DP_b{}^c) * (e_c\wedge e_a)\right)\,. \tag{C.61}$$

Applying the inner product operator reads

$$\begin{aligned}(L2-3) &= \left(\frac{\partial f}{\partial U}\right)\left((\iota^a DP_a{}^c) * (e_c\wedge e_b) + (DP_a{}^c) * (e_c\wedge e_b\wedge e^a)\right.\\ &\quad\left. - (\iota^a DP_b{}^c) * (e_c\wedge e_a) - (DP_b{}^c) * (e_c\wedge e_a\wedge e^a)\right)\\ &= \left(\frac{\partial f}{\partial U}\right)\left((\iota^a DP_a{}^c) * (e_c\wedge e_b) + (DP_a{}^c) * (e_c\wedge e_b\wedge e^a) - (\iota^a DP_b{}^c) * (e_c\wedge e_a)\right)\end{aligned} \tag{C.62}$$

since $e_a\wedge e^a = 0$. Because the term $(\iota^a DP_b{}^c)$ is a scalar, one can write again by using the identity (B.2)

$$(L2-3) = \left(\frac{\partial f}{\partial U}\right)\left((\iota^a DP_a{}^c) * (e_c\wedge e_b) + (DP_a{}^c) * (e_c\wedge e_b\wedge e^a) - *(e_c\wedge e_a(\iota^a DP_b{}^c))\right)\,, \tag{C.63}$$

which can also be rewritten by using the definition of the Ricci 1-forms as

$$\begin{aligned}(L2-3) &= \left(\frac{\partial f}{\partial U}\right)\left((\iota^a DP_a{}^c) * (e_c\wedge e_b) + (DP_a{}^c) * (e_c\wedge e_b\wedge e^a) - *(e_c\wedge (DP_b{}^c))\right)\\ &= \left(\frac{\partial f}{\partial U}\right)\left(*\underbrace{((\iota^a DP_a{}^c) e_c\wedge e_b)}_{(L2-3-1)} + \underbrace{(DP_a{}^c) * (e_c\wedge e_b\wedge e^a)}_{(L2-3-2)} + * DP_b\right)\end{aligned} \tag{C.64}$$

for the reason that $De_c = T_c = 0$ (i.e., the Levi-Civita connection). By using the identity (B.2) and (B.29), $(L2-3-1)$ can also be rewritten as

$$\begin{aligned}(L2-3-1) &= (\iota^a DP_a{}^c) e_c\wedge e_b\\ &= \frac{1}{2}(\iota^c dR) e_c\wedge e_b\\ &= \frac{1}{2} dR\wedge e_b\,.\end{aligned} \tag{C.65}$$

In addition, $(L2-3-2)$ can be further simplified by using the definition of the covariant exterior derivative for Levi-Civita connection as follows:

$$
\begin{aligned}
(L2-3-2) &= D\left(P_a{}^c * (e^a \wedge e_b \wedge e_c)\right) \\
&= (DP_a{}^c) * (e^a \wedge e_b \wedge e_c) + P_a{}^c D * (e^a \wedge e_b \wedge e_c) \\
&= (DP_a{}^c) * (e^a \wedge e_b \wedge e_c) + P_a{}^c T^f * (e^a \wedge e_b \wedge e_c \wedge e_f) \\
&= (DP_a{}^c) * (e^a \wedge e_b \wedge e_c) \\
&= D\left[P_a{}^c * (e^a \wedge e_b \wedge e_c)\right] .
\end{aligned} \tag{C.66}
$$

Since $P_a{}^c$ is a scalar,

$$
\begin{aligned}
(L2-3-2) &= D\left(*(e^a \wedge e_b \wedge P_a{}^c\, e_c)\right) \\
&= D\left(*(P_a \wedge e^a \wedge e_b)\right) \\
&= 0 .
\end{aligned} \tag{C.67}
$$

As can easily be seen, there is no contribution from $(L2-3-2)$ since $P_a \wedge e^a = 0$ (for $T^c = 0$). When $(L2-3-1)$ is substituted into $(L2-3)$, one finds

$$
\begin{aligned}
(L2-3) &= \left(\frac{\partial f}{\partial U}\right)\left(\frac{1}{2} * (dR \wedge e_b) + *DP_b\right) \\
&= \left(\frac{\partial f}{\partial U}\right)\left(\frac{1}{2}\iota_b * dR + *DP_b\right)
\end{aligned} \tag{C.68}
$$

Therefore,

$$
(L2) = \left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_b - P_b \wedge e_a) + \left(\frac{\partial f}{\partial U}\right)\left(\frac{1}{2}\iota_b * dR + *DP_b\right) . \tag{C.69}
$$

On the other hand, the covariant exterior derivative operator in $(L3)$ gives

$$
\begin{aligned}
(L3) &= \iota^a D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) \\
&= \iota^a \left(D\left(\frac{\partial f}{\partial V}\right)\right) * R_{ab} + \left(\frac{\partial f}{\partial V}\right)\iota^a D * R_{ab} \\
&= \iota^a \left(D\left(\frac{\partial f}{\partial V}\right)\right) * R_{ab} + \left(\frac{\partial f}{\partial V}\right) * DP_b
\end{aligned} \tag{C.70}
$$

since $\iota^a D * R_{ab} = *DP_b$. Thus, collecting all the related terms one finds

$$
\begin{aligned}
\iota^a \Lambda_{ab} = \; & 2\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_b) \\
& +2\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_b - P_b \wedge e_a) + \left(\frac{\partial f}{\partial U}\right)(\iota_b * dR + 2 * DP_b) \\
& +8\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right) * R_{ab} + 8\left(\frac{\partial f}{\partial V}\right) * DP_b + 2\iota^a \Omega_{ab} .
\end{aligned} \tag{C.71}
$$

Further application of the inner product operator on (C.71) yields

$$\iota^b \iota^a \Lambda_{ab} = 4\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\iota^b * R_{ab} + 4\left(\frac{\partial f}{\partial V}\right)\iota^b * DP_b + 2\iota^b\iota^a\Omega_{ab}\,. \tag{C.72}$$

Respectively using (B.28) and (B.1), one finds

$$\iota^b \iota^a \Lambda_{ab} = 4\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\iota^b * R_{ab} + 2\iota^b\iota^a\Omega_{ab} \tag{C.73}$$

owing to $\iota^b\iota_b\zeta = 0$ (ζ being any p-form), $e^b \wedge e_b = 0$ and $P_b \wedge e^b = 0$. Substituting (C.71) and (C.73) into (C.54) and changing indices suitably, one obtains

$$\begin{aligned}\lambda_b \;=\; & 2\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_b) + 2\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_b - P_b \wedge e_a)\\ & + \left(\frac{\partial f}{\partial U}\right)(\iota_b * dR + 2 * DP_b) + 8\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right) * R_{ab} + 2\iota^a\Omega_{ab}\\ & + 8\left(\frac{\partial f}{\partial V}\right) * DP_b - \frac{1}{2}e_b \wedge \iota^c\iota^a\Omega_{ac}\,.\end{aligned} \tag{C.74}$$

Finally, the Einstein field equation of the model is given by

$$\begin{aligned}& f * e_c - \tau_c + \left(\frac{\partial f}{\partial R}\right)\left(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\right)\\ & + \left(\frac{\partial f}{\partial U}\right)\left(P^a{}_c * P_a + 2(\iota_c R^{ba}) \wedge *(P_a \wedge e_b) - P^a \wedge *(P_a \wedge e_c) - U * e_c\right)\\ & + \left(\frac{\partial f}{\partial V}\right)\left(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\right)\\ & + D\left[2\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) * (e_a \wedge e_c) + 2\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) * (P_a \wedge e_c - P_c \wedge e_a)\right.\\ & + \left(\frac{\partial f}{\partial U}\right)(\iota_c * dR + 2 * DP_c) + 8\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right) * R_{ac} + 2\iota^a\Omega_{ac}\\ & \left. + 8\left(\frac{\partial f}{\partial V}\right) * DP_c - \frac{1}{2}e_c \wedge \iota^b\iota^a\Omega_{ab}\right] = 0\,.\end{aligned} \tag{C.75}$$

On the other hand, trace of the Einstein field equation of GGM can be obtained by acting e^c on (C.46) as

$$
\begin{aligned}
& f\,\underbrace{e^c \wedge *e_c}_{(T1)} - \underbrace{e^c \wedge \tau_c}_{(T2)} + \left(\frac{\partial f}{\partial R}\right)\left(\underbrace{e^c \wedge R^{ab} \wedge *(e_a \wedge e_b \wedge e_c)}_{(T3)} - R\,\underbrace{e^c \wedge *e_c}_{(T4)}\right) \\
& + \left(\frac{\partial f}{\partial U}\right)\left(\underbrace{P^a{}_c e^c \wedge *P_a}_{(T5)} + \underbrace{e^c \wedge (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a)}_{(T6)}\right. \\
& \left. - \underbrace{e^c \wedge P^a \wedge *(P_a \wedge e_c)}_{(T7)} - U\,\underbrace{e^c * e_c}_{(T8)}\right) + \left(\frac{\partial f}{\partial V}\right)\left(-2\,\underbrace{e^c \wedge (\iota_c R^{ab}) \wedge *R_{ab}}_{(T9)}\right. \\
& \left. +2\,\underbrace{e^c \wedge R^{ab} \wedge *(R_{ab} \wedge e_c)}_{(T10)} - V\,\underbrace{e^c * e_c}_{(T11)}\right) + \underbrace{e^c \wedge D\lambda_c}_{(T12)} = 0\,. \qquad \text{(C.76)}
\end{aligned}
$$

As can easily be seen at the first glance, terms $(T1)$, $(T4)$, $(T8)$ and $(T11)$ are identical and taking the identity (B.2) is enough to get the required expressions, that is $e^c \wedge *e_c = e^c \wedge \iota_c * 1 = n * 1$ where n is the dimension of spacetime. Therefore, one can continue with the stress-energy term $(T2)$:

$$
\begin{aligned}
(T2) &= e^c \wedge \tau_c \\
&= e^c \wedge \tau_{ca} * e^a \\
&= \tau_{ca} e^c \wedge \iota^a * 1 \\
&= \delta^{ca} \tau_{ca} * 1 \\
&= \tau^a{}_a * 1 \\
&= \tau * 1\,. \qquad \text{(C.77)}
\end{aligned}
$$

The other trace term can be calculated by frequently used identities (B.1) and (B.2) as

$$
\begin{aligned}
(T3) &= e^c \wedge R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) \\
&= R^{ab} \wedge e^c \wedge \iota_c * (e_a \wedge e_b) \\
&= (n-2) R^{ab} \wedge *(e_a \wedge e_b) \\
&= (n-2) R * 1\,. \qquad \text{(C.78)}
\end{aligned}
$$

Then, one can easily write

$$
\begin{aligned}
(T5) &= P^a{}_c e^c \wedge *P_a \\
&= P^a \wedge *P_a \\
&= U * 1\,. \qquad \text{(C.79)}
\end{aligned}
$$

The identity $(B.2)$ presents for the next trace term

$$\begin{aligned}(T6) &= e^c \wedge (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) \\ &= 2R^{ba} \wedge *(P_a \wedge e_b - P_b \wedge e_a) \\ &= 2\underbrace{R^{ba} \wedge \iota_b * P_a}_{(T6-1)} - 2\underbrace{R^{ba} \wedge \iota_a * P_b}_{(T6-2)} .\end{aligned} \tag{C.80}$$

Here, one can express $(T6-1)$ with use of the definition of interior product operator as

$$\begin{aligned}\iota_b(R^{ba} \wedge *P_a) &= (\iota_b R^{ba}) \wedge *P_a + R^{ba} \wedge \iota_b * P_a \\ R^{ba} \wedge \iota_b * P_a &= -P^a \wedge *P_a \\ (T6-1) &= -U * 1\end{aligned} \tag{C.81}$$

since $\iota_b(R^{ba} \wedge *P_a) = 0$ (it is $(n+1)$-form in n-dimensional spacetime) and $P^a = \iota_b R^{ba}$. Applying the same procedure for $(T6-2)$ yields

$$\begin{aligned}\iota_a(R^{ba} \wedge *P_b) &= (\iota_a R^{ba}) \wedge *P_b + R^{ba} \wedge \iota_a * P_b \\ R^{ba} \wedge \iota_a * P_b &= -P^b \wedge *P_b \\ (T6-2) &= U * 1 .\end{aligned} \tag{C.82}$$

Therefore,

$$(T6) = -4U * 1 . \tag{C.83}$$

While trace term $(T7)$ can be figured out as

$$\begin{aligned}(T7) &= e^c \wedge P^a \wedge *(P_a \wedge e_c) \\ &= -P^a \wedge e^c \wedge \iota_c * P_a \\ &= -(n-1)P^a \wedge *P_a \\ &= -(n-1)U * 1 ,\end{aligned} \tag{C.84}$$

$(T9)$ can be expressed as

$$\begin{aligned}(T9) &= e^c \wedge (\iota_c R^{ab}) \wedge *R_{ab} \\ &= 2R^{ab} \wedge *R_{ab} \\ &= V * 1\end{aligned} \tag{C.85}$$

again with the help of the identities (B.1) and (B.2). In the same way, one finds

$$\begin{aligned}(T10) &= e^c \wedge R^{ab} \wedge *(R_{ab} \wedge e_c) \\ &= R^{ab} \wedge e^c \wedge \iota_c * R_{ab} \\ &= (n-2)R^{ab} \wedge *R_{ab} \\ &= \frac{(n-2)}{2} V * 1\end{aligned} \tag{C.86}$$

by following the same steps. The calculation of the contribution from term $(T12)$ can be obtained by

$$(T12) = e^c \wedge D\lambda_c = -D(e^c \wedge \lambda_c) \tag{C.87}$$

since the connection is Levi-Civita. Before acting the covariant exterior derivative operator, one should primarily simplify all the terms coming from $(e^c \wedge \lambda_c)$ which yields

$$\begin{aligned}(e^c \wedge \lambda_c) \quad = \quad & 2\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right)\underbrace{e^c \wedge *(e_a \wedge e_c)}_{(T12-1)} + 2\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right)\underbrace{e^c \wedge *(P_a \wedge e_c - P_c \wedge e_a)}_{(T12-2)} \\ & + \left(\frac{\partial f}{\partial U}\right)\left(\underbrace{e^c \wedge \iota_c * dR}_{(T12-3)} + 2\underbrace{e^c \wedge *DP_c}_{(T12-4)}\right) + 8\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\underbrace{e^c \wedge *R_{ac}}_{(T12-5)} \\ & + 2e^c \wedge \iota^a \Omega_{ac} + 8\left(\frac{\partial f}{\partial V}\right)\underbrace{e^c \wedge *DP_c}_{(T12-6)} - \frac{1}{2}\underbrace{e^c \wedge e_c \wedge \iota^b \iota^a \Omega_{ab}}_{(T12-7)}\,.\end{aligned} \tag{C.88}$$

Here, one can start with

$$\begin{aligned}(T12-1) &= e^c \wedge *(e_a \wedge e_c) \\ &= e^c \wedge \iota_c * e_a \\ &= (n-1) * e_a\,.\end{aligned} \tag{C.89}$$

Then, one finds

$$\begin{aligned}(T12-2) &= e^c \wedge *(P_a \wedge e_c - P_c \wedge e_a) \\ &= e^c \wedge \iota_c * P_a - e^c \wedge \iota_a * P_c \\ &= (n-1) * P_a - \underbrace{e^c \wedge \iota_a * P_c}_{(T12-2-2)}\,.\end{aligned} \tag{C.90}$$

Using identity (B.5), one can also write

$$\begin{aligned}(T12-2-2) &= \delta_a{}^c * P_c - \iota_a(e^c \wedge *P_c) \\ &= \delta_a{}^c * P_c - \iota_a(P_c \wedge *e^c) \\ &= *P_a - \iota_a(P_c \wedge \iota^c * 1) \\ &= *P_a - R * e_a\end{aligned} \tag{C.91}$$

since $(P_c \wedge \iota^c * 1) = R * 1$. Thus,

$$(T12-2) = (n-2) * P_a + R * e_a\,. \tag{C.92}$$

If one uses (B.2) for the term $(T12-3)$, one arrives at

$$(T12-3) = e^c \wedge \iota_c * dR = (n-1) * dR\,. \tag{C.93}$$

Since the connection is torsion-free and $D \equiv \nabla_a e^a$, term $(T12-4)$ can also be given as

$$
\begin{aligned}
(T12-4) &= e^c \wedge *DP_c \\
&= e^c \wedge *D(P_{ca} e^a) \\
&= e^c \wedge *[(DP_{ca}) \wedge e^a + P_{ca}(De^a)] \\
&= e^c \wedge *[(\nabla_b P_{ca}) e^b \wedge e^a]
\end{aligned}
\tag{C.94}
$$

where ∇_b is called covariant differentiation with respect to any frame vector field X_b ($\nabla_b \equiv \nabla_{X_b}$). Since the term $(\nabla_b P_{ca})$ is a scalar, by using the identity (B.33), one can also write

$$
\begin{aligned}
(T12-4) &= (\nabla_b P_{ca}) e^c \wedge *(e^b \wedge e^a) \\
&= (\nabla_b P_{ca})(\delta^{ca} * e^b - \delta^{ba} * e^c) \\
&= (\nabla_b P^a{}_a) * e^b - (\nabla_b P^b{}_a) * e^a \\
&= *(\nabla_b R e^b) - *(\nabla_b P^b{}_a e^a) \,.
\end{aligned}
\tag{C.95}
$$

For the reason that $2\nabla_a P^a{}_c = \nabla_c R$ and remembering $D \equiv \nabla_a e^a$, one finally finds

$$
\begin{aligned}
(T12-4) &= *dR - *(\nabla_a R e^a) \\
&= *dR - \frac{1}{2} * dR \\
&= \frac{1}{2} * dR \,.
\end{aligned}
\tag{C.96}
$$

Expressing the curvature 2-form in terms of Riemann tensor gives for the next term

$$
\begin{aligned}
(T12-5) &= e^c \wedge *R_{ac} \\
&= \frac{1}{2} R_{acmn} e^c \wedge *(e^m \wedge e^n) \,.
\end{aligned}
\tag{C.97}
$$

Here, the identity (B.33) leads with the appropriate index change

$$
\begin{aligned}
(T12-5) &= \frac{1}{2} R_{acmn} \left(\delta^{nc} * e^m - \delta^{mc} * e^n\right) \\
&= \frac{1}{2} \left(R_{acm}{}^c * e^m - R_{ac}{}^c{}_n * e^n\right) \\
&= R_{acm}{}^c * e^m \\
&= P_{am} * e^m \\
&= *P_a \,.
\end{aligned}
\tag{C.98}
$$

$(T12-6)$ can be express by using the result of the term $(T12-4)$ as follows

$$
\begin{aligned}
(T12-6) &= e^c \wedge *DP_c \\
&= \frac{1}{2} * DR \,.
\end{aligned}
\tag{C.99}
$$

Consequently, the last term, namely (T12-7), is identically zero since $e^c \wedge e_c = 0$. Therefore, one can write

$$\begin{aligned}(e^c \wedge \lambda_c) &= 2\left((n-1)\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) + R\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right)\right) * e_a \\ &\quad +2\left((n-2)\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 4\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right) * P_a \\ &\quad +\left(n\left(\frac{\partial f}{\partial U}\right) + 4\left(\frac{\partial f}{\partial V}\right)\right) * dR + 2e^c \wedge \iota^a \Omega_{ac}\,. \end{aligned} \tag{C.100}$$

Operating the exterior derivative operator gives

$$\begin{aligned}(T12) &= -D(e^c \wedge \lambda_c) \\ &= -2\left[(n-1)D\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) + \left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) dR + RD\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right)\right] \wedge * e_a \\ &\quad -2\left[(n-2)D\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 4D\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right] \wedge * P_a \\ &\quad -2\left[(n-2)\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 4\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right] D * P_a \\ &\quad -\left[nD\left(\frac{\partial f}{\partial U}\right) + 4D\left(\frac{\partial f}{\partial V}\right)\right] \wedge * dR \\ &\quad -\left[n\left(\frac{\partial f}{\partial U}\right) + 4\left(\frac{\partial f}{\partial V}\right)\right] D * dR + 2e^c \wedge D(\iota^a \Omega_{ac}) \end{aligned} \tag{C.101}$$

for the Levi-Civita connection. Eventually, substituting all the calculated terms yields

$$\begin{aligned}&\left[nf - 2R\left(\frac{\partial f}{\partial R}\right) - 4U\left(\frac{\partial f}{\partial U}\right) - 4V\left(\frac{\partial f}{\partial V}\right) - \tau\right] * 1 \\ &-2\left[(n-1)D\left(\iota^a D\left(\frac{\partial f}{\partial R}\right)\right) + \left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) dR + RD\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right)\right] \wedge * e_a \\ &-2\left[(n-2)D\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 4D\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right] \wedge * P_a \\ &-2\left[(n-2)\left(\iota^a D\left(\frac{\partial f}{\partial U}\right)\right) + 4\left(\iota^a D\left(\frac{\partial f}{\partial V}\right)\right)\right] D * P_a \\ &-\left[nD\left(\frac{\partial f}{\partial U}\right) + 4D\left(\frac{\partial f}{\partial V}\right)\right] \wedge * dR \\ &-\left[n\left(\frac{\partial f}{\partial U}\right) + 4\left(\frac{\partial f}{\partial V}\right)\right] D * dR + 2e^c \wedge D(\iota^a \Omega_{ac}) = 0 \end{aligned} \tag{C.102}$$

for the trace of the Einstein field equation of n-dimensional GGM.

On the other hand, taking co-frame 1-forms and the connection 1-forms as independent fields and making variations with respect to each field give the following field equations:

$$
\begin{aligned}
& f * e_c - \tau_c + \left(\frac{\partial f}{\partial R}\right)\Big(R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - R * e_c\Big) \\
& + \left(\frac{\partial f}{\partial U}\right)\Big(P^a{}_c * P_a + (\iota_c R^{ba}) \wedge *(P_a \wedge e_b - P_b \wedge e_a) - P^a \wedge *(P_a \wedge e_c) - U * e_c\Big) \\
& + \left(\frac{\partial f}{\partial V}\right)\Big(-2(\iota_c R^{ab}) \wedge *R_{ab} + 2R^{ab} \wedge *(R_{ab} \wedge e_c) - V * e_c\Big) = 0
\end{aligned}
\tag{C.103}
$$

and

$$
D\left(\left(\frac{\partial f}{\partial R}\right) * (e_a \wedge e_b)\right) + D\left(\left(\frac{\partial f}{\partial U}\right) * (P_a \wedge e_b - P_b \wedge e_a)\right) + 4D\left(\left(\frac{\partial f}{\partial V}\right) * R_{ab}\right) + \Omega_{ab} = 0\,. \tag{C.104}
$$

D. Field Equations of a Quadratic Curvature Gravity Model in Higher Dimensions

In this appendix, the field equations of higher dimensional Quadratic Curvature Gravity (QCG) Model are derived in detail. As in the previous appendix, the field equations of the model are obtained for two cases in which the connection of the spacetime involves torsion and the connection of the spacetime is torsion free. Note that, some of the calculations are very similar or identical to the ones given in the previous appendix. However, it is preferred to be presented in a separate appendix for calculations to be clearly seen and easily followed.

The Lagrangian density of QCG under the Levi-Civita connection structure is given in (2.38) whose variation with respect to e^a, $\omega^a{}_b$ and λ^a can be given by

$$\begin{aligned}\delta\mathcal{L}_{QCG} &= \frac{1}{2}\underbrace{\delta\left(R^{ab}\wedge *(e_a\wedge e_b)\right)}_{(a)}+\frac{\alpha}{2}\underbrace{\delta\left(R^2 * 1\right)}_{(b)}+\underbrace{\delta(\Lambda * 1)}_{(c)}\\ &\quad +\frac{\beta}{2}\underbrace{\delta\left(P_a\wedge *P^a\right)}_{(d)}+\frac{\gamma}{4}\underbrace{\delta\left(R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right)}_{(e)}+\underbrace{\delta\left(T^a\wedge\lambda_a\right)}_{(f)}.\end{aligned} \tag{D.1}$$

Here, Λ is a cosmological constant and α, β, γ denote constants. In order to avoid confusion, the variation of each term should be considered separately. Using the second Cartan structure equation and the identity (B.12), the variation of well-known Einstein-Hilbert term can be given as

$$\begin{aligned}(a) &= \delta\left(R^{ab}\wedge *(e_a\wedge e_b)\right)\\ &= (\delta R^{ab})\wedge *(e_a\wedge e_b)+R^{ab}\wedge\delta\left(*(e_a\wedge e_b)\right)\\ &= \left(\delta(d\omega^{ab}+\omega^a{}_c\wedge\omega^{cb})\right)\wedge *(e_a\wedge e_b)+R^{ab}\wedge\delta\left[*(e_a\wedge e_b)\right]\\ &= \delta(d\omega^{ab})\wedge *(e_a\wedge e_b)+(\delta\omega^a{}_c)\wedge\omega^{cb}\wedge *(e_a\wedge e_b)\\ &\quad +\omega^a{}_c\wedge(\delta\omega^{cb})\wedge *(e_a\wedge e_b)+R^{ab}\wedge(\delta e^c)\wedge *(e_a\wedge e_b\wedge e_c).\end{aligned} \tag{D.2}$$

The commutation relation (C.3) yields

$$\begin{aligned}\delta\left(R^{ab}\wedge *(e_a\wedge e_b)\right) &= \underbrace{d(\delta\omega^{ab})\wedge *(e_a\wedge e_b)}_{(a1)}+(\delta\omega^a{}_c)\wedge\omega^{cb}\wedge *(e_a\wedge e_b)\\ &\quad +\omega^a{}_c\wedge(\delta\omega^{cb})\wedge *(e_a\wedge e_b)+R^{ab}\wedge(\delta e^c)\wedge *(e_a\wedge e_b\wedge e_c).\end{aligned} \tag{D.3}$$

The definition of exterior derivative in the term $(a1)$ leads to

$$d(\delta\omega^{ab})\wedge *(e_a\wedge e_b)=d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge d*(e_a\wedge e_b). \tag{D.4}$$

Changing some indices appropriately, the variation of the Einstein-Hilbert term (a) finally calculated as

$$\begin{aligned}(a) &= \delta\left(R^{ab}\wedge *(e_a\wedge e_b)\right)\\ &= d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge d*(e_a\wedge e_b)\\ &\quad +(\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(e_c\wedge e_b)-\omega^c{}_b\wedge *(e_a\wedge e_c)\right)\\ &\quad +R^{ab}\wedge(\delta e^c)\wedge *(e_a\wedge e_b\wedge e_c)\\ &= d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge D*(e_a\wedge e_b)\\ &\quad +(\delta e^c)\wedge R^{ab}\wedge *(e_a\wedge e_b\wedge e_c) \end{aligned} \tag{D.5}$$

since

$$D\left(*(e_a\wedge e_b)\right)=d*(e_a\wedge e_b)+\omega_a{}^c\wedge *(e_c\wedge e_b)-\omega^c{}_b\wedge *(e_a\wedge e_c)\,. \tag{D.6}$$

Calculations related to variational term (b) can be given as

$$\begin{aligned}(b) &= \delta\left(R^2 * 1\right)\\ &= 2R(\delta R)*1+R^2(\delta e^a)\wedge e_a\,. \end{aligned} \tag{D.7}$$

Since the variational term $(\delta R)*1$ has already been calculated in (C.5), one has

$$\begin{aligned}(b) &= 2R\underbrace{d(\delta\omega^{ab})\wedge *(e_a\wedge e_b)}_{(b1)}+2R(\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(e_c\wedge e_b)-\omega^c{}_b\wedge *(e_a\wedge e_c)\right)\\ &\quad +(\delta e^c)\wedge\left(2RR^{ab}\wedge *(e_a\wedge e_b\wedge e_c)-R^2*e_c\right) \end{aligned} \tag{D.8}$$

with the help of commutation relation (C.3). Here, the term $(b1)$ can also be written as

$$\begin{aligned}(b1) &= d(\delta\omega^{ab})\wedge *(e_a\wedge e_b)\\ &= d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge d*(e_a\wedge e_b)\,. \end{aligned} \tag{D.9}$$

Thus,

$$\begin{aligned}(b) &= \underbrace{2\,Rd\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)}_{(b1-1)}\\ &\quad +2R(\delta\omega^{ab})\wedge\left(d*(e_a\wedge e_b)+\omega_a{}^c\wedge *(e_c\wedge e_b)-\omega^c{}_b\wedge *(e_a\wedge e_c)\right)\\ &\quad +(\delta e^c)\wedge\left(2RR^{ab}\wedge *(e_a\wedge e_b\wedge e_c)-R^2*e_c\right)\,. \end{aligned} \tag{D.10}$$

To get a more compact expression, one can again use the following commutation relation for term $(b1-1)$:

$$\begin{aligned}(b1-1) &= Rd\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)\\ &= d\left(R(\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+\left(\delta\omega^{ab}\right)\wedge(dR)\wedge *(e_a\wedge e_b)\,. \end{aligned} \tag{D.11}$$

Substituting $(b1-1)$ into the variation of the curvature scalar squared term (b), one finds

$$\begin{aligned}(b) &= 2d\left(R(\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+2(\delta\omega^{ab})\wedge D\left(R*(e_a\wedge e_b)\right)\\ &\quad+(\delta e^c)\wedge\left(2RR^{ab}\wedge *(e_a\wedge e_b\wedge e_c)-R^2*e_c\right)\,. \end{aligned}\tag{D.12}$$

Since the next variational term (c) can be easily obtained by using the identity (B.14) (i.e., $\delta*1=(\delta e^a)\wedge *e_a$), one can continue with the variation of the next term. Applying identities (B.14) and (B.3) onto the Ricci-squared term, one gets

$$\begin{aligned}(d) &= \delta\left(P^a\wedge *P_a\right)\\ &= 2\underbrace{(\delta P^a)\wedge *P_a}_{(d1)}-(\delta e^b)\wedge\left((\iota_b P^a)\wedge *P_a+P^a\wedge\iota_b*P_a\right)\,. \end{aligned}\tag{D.13}$$

Since the final form of the variational term $(d1)$ has already been given in (C.18), the variation of the Ricci-squared term (d) takes the following form:

$$\begin{aligned}(d) &= 2P^a{}_c(\delta e^c)\wedge *P_a+2R^{ba}{}_{cd}(\delta e^c)\wedge e^d\wedge *(P_a\wedge e_b)+2\left(\iota_b(\delta R^{ba})\right)\wedge *P_a\\ &\quad-(\delta e^b)\wedge\left((\iota_b P^a)\wedge *P_a+P^a\wedge\iota_b*P_a\right)\,. \end{aligned}\tag{D.14}$$

Following the same procedure outlined in (C.20)-(C.23), one finds

$$\begin{aligned}(d) &= 2\underbrace{\left(d(\delta\omega^{ab})\right)\wedge *(P_a\wedge e_b)}_{(d2)}\\ &\quad+(\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(P_c\wedge e_b)-\omega^c{}_b\wedge *(P_a\wedge e_c)\right)\\ &\quad+(\delta e^c)\wedge\left(P^a{}_c*P_a+2(\iota_c R^{ba})\wedge *(P_a\wedge e_b)-P^a\wedge *(P_a\wedge e_c)\right)\,. \end{aligned}\tag{D.15}$$

Here, $(d2)$ can also be rearranged by using the definition of exterior derivative together with the antisymmetrization in the first term as

$$\begin{aligned}(d2) &= \left(d(\delta\omega^{ab})\right)\wedge *(P_a\wedge e_b)\\ &= \frac{1}{2}d\left((\delta\omega^{ab}\wedge *(P_a\wedge e_b-P_b\wedge e_a))\right)+(\delta\omega^{ab})\wedge d\left(*(P_a\wedge e_b)\right)\,. \end{aligned}\tag{D.16}$$

After inserting $(d2)$ in (d) and using the following identity

$$D\left(*(P_a\wedge e_b)\right)=d\left(*(P_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(P_c\wedge e_b)-\omega^c{}_b\wedge *(P_a\wedge e_c)\right)\,,\tag{D.17}$$

one eventually reaches

$$\begin{aligned}(d) &= d\left((\delta\omega^{ab})\wedge *(P_a\wedge e_b-P_b\wedge e_a)\right)\\ &\quad+(\delta\omega^{ab})\wedge D\left(*(P_a\wedge e_b-P_b\wedge e_a)\right)\\ &\quad+(\delta e^c)\wedge\left(P^a{}_c*P_a+2(\iota_c R^{ba})\wedge *(P_a\wedge e_b)-P^a\wedge *(P_a\wedge e_c)\right)\,. \end{aligned}\tag{D.18}$$

The variation of the Gauss-Bonnet term with respect to orthonormal 1-form e^a and connection 1-forms $\omega^a{}_b$ yield

$$\begin{aligned}(e) &= \delta\Big(R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\Big)\\ &= (\delta R^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\\ &\quad +R^{ab}\wedge(\delta R^{cd})\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\\ &\quad +R^{ab}\wedge R^{cd}\wedge\delta\left(*(e_a\wedge e_b\wedge e_c\wedge e_d)\right)\,. \end{aligned}\tag{D.19}$$

Appropriately changing some indices gives

$$\begin{aligned}\delta\Big(R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\Big) &= 2\,\underbrace{(\delta R^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)}_{(e1)}\\ &\quad +R^{ab}\wedge R^{cd}\wedge\underbrace{\delta\left(*(e_a\wedge e_b\wedge e_c\wedge e_d)\right)}_{(e2)}\,. \end{aligned}\tag{D.20}$$

The variation of the curvature 2-form, can be written as

$$\begin{aligned}\delta R^{ab} &= \delta(d\omega^{ab}+\omega^a{}_f\wedge\omega^{fb})\\ &= \delta(d\omega^{ab})+(\delta\omega^a{}_f)\wedge\omega^{fb}+\omega^a{}_f\wedge\delta(\omega^{fb})\,. \end{aligned}\tag{D.21}$$

Commutation relation between infinitesimal variations and exterior derivative operators leads to

$$\delta R^{ab}=d(\delta\omega^{ab})+(\delta\omega^a{}_f)\wedge\omega^{fb}+\omega^a{}_f\wedge\delta(\omega^{fb})\,.\tag{D.22}$$

After inserting variation of the curvature in $(e1)$, one finds

$$\begin{aligned}(e1) &= 2\,\underbrace{d(\delta\omega^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)}_{(e1-1)}\\ &\quad +2(\delta\omega^a{}_f)\wedge\omega^{fb}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\\ &\quad +\omega^a{}_f\wedge\delta(\omega^{fb})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\,. \end{aligned}\tag{D.23}$$

Here, the term $(e1-1)$ can be rearranged by the definition of exterior derivative operator as

$$\begin{aligned}(e1-1) &= d(\delta\omega^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\\ &= d\Big[(\delta\omega^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\Big]\\ &\quad +(\delta\omega^{ab})\wedge d\Big[R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\Big]\,. \end{aligned}\tag{D.24}$$

Substituting $(e1-1)$ back into $(e1)$, changing some indices and using the definition of covariant exterior derivative operator read

$$
\begin{aligned}
(e1) &= 2d\left[(\delta\omega^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right] \\
&\quad +2(\delta\omega^{ab})\wedge d\left[R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right] \\
&\quad +2(\delta\omega^{ab})\wedge R^{cd}\wedge\left(\omega^f{}_b\wedge *(e_a\wedge e_f\wedge e_c\wedge e_d)+\omega^f{}_a\wedge *(e_f\wedge e_b\wedge e_c\wedge e_d)\right) \\
&= 2d\left[(\delta\omega^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right] \\
&\quad +2(\delta\omega^{ab})\wedge D\left(R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right) \qquad (D.25)
\end{aligned}
$$

It is easy to write the variation $(e2)$ with the help of third item in this appendix part as

$$
\begin{aligned}
(e2) &= \delta\left(*(e_a\wedge e_b\wedge e_c\wedge e_d)\right) \\
&= \left(\delta e^f\right)\wedge R^{ab}\wedge R^{cd}\wedge *\left(e_a\wedge e_b\wedge e_c\wedge e_d\wedge e_f\right) \qquad (D.26)
\end{aligned}
$$

Finally, inserting back the terms $(e1)$ and $(e2)$ results in as follows for the variation of the Gauss-Bonnet term:

$$
\begin{aligned}
(e) &= 2d\left[(\delta\omega^{ab})\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right] \\
&\quad +2(\delta\omega^{ab})\wedge D\left(R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right) \\
&\quad +\left(\delta e^f\right)\wedge R^{ab}\wedge R^{cd}\wedge *\left(e_a\wedge e_b\wedge e_c\wedge e_d\wedge e_f\right). \qquad (D.27)
\end{aligned}
$$

Since the variation of vanishing torsion constraint term (f) has already been obtained in (C.44), collection of all the related terms gives

$$
\begin{aligned}
\delta\mathcal{L}_{QCG} &= \delta e^f\wedge\left[\left(\alpha R+\frac{1}{2}\right)R^{ab}\wedge *(e_a\wedge e_b\wedge e_f)+\left(\Lambda-\frac{\alpha}{2}R^2\right)*e_f\right. \\
&\quad +\frac{\beta}{2}\left[P^a{}_f * P_a+2(\iota_f R^{ba})\wedge *(P_a\wedge e_b)-P^a\wedge *(P_a\wedge e_f)\right] \\
&\quad \left.+\frac{\gamma}{4}R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d\wedge e_f)+D\lambda_f\right] \\
&\quad +\omega^{ab}\wedge\left[\left(\alpha R+\frac{1}{2}\right)D*(e_a\wedge e_b)+\alpha dR\wedge *(e_a\wedge e_b)\right. \\
&\quad +\frac{\beta}{2}D\left(*(P_a\wedge e_b-P_b\wedge e_a)\right)+\frac{\gamma}{2}D\left(R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right) \\
&\quad \left.+\frac{1}{2}(e_b\wedge\lambda_a-e_a\wedge\lambda_b)\right]+\delta\lambda_a\wedge T^a \\
&\quad +d\left[\left(\alpha R+\frac{1}{2}\right)(\delta\omega^{ab})\wedge *(e_a\wedge e_b)+\frac{\beta}{2}(\delta\omega^{ab})\wedge *(P_a\wedge e_b-P_b\wedge e_a)\right. \\
&\quad \left.+\frac{\gamma}{2}R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)+(\delta e^a)\wedge\lambda_a\right]. \qquad (D.28)
\end{aligned}
$$

The variational principle indicates

$$
\begin{aligned}
&\left(\alpha R+\frac{1}{2}\right) R^{ab} \wedge *(e_a \wedge e_b \wedge e_f)+\left(\Lambda-\frac{\alpha}{2} R^2\right) * e_f \\
&+\frac{\beta}{2}\left[P^a{}_f * P_a+2(\iota_f R^{ba}) \wedge *(P_a \wedge e_b)-P^a \wedge *(P_a \wedge e_f)\right] \\
&+\frac{\gamma}{4} R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d \wedge e_f)+D \lambda_f=0 \qquad \text{(D.29)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2} D *(e_a \wedge e_b)+\alpha D\left(R *(e_a \wedge e_b)\right)+\frac{\beta}{2} D\left(*(P_a \wedge e_b-P_b \wedge e_a)\right) \\
&+\frac{\gamma}{2} D\left(R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)\right)+\frac{1}{2}(e_b \wedge \lambda_a-e_a \wedge \lambda_b)=0 \qquad \text{(D.30)}
\end{aligned}
$$

again with

$$
T^a=d e^a+\omega^a{}_b \wedge e^b=0 \,. \qquad \text{(D.31)}
$$

The Lagrange multiplier $(n-2)$-forms λ_a can be easily found by defining

$$
\begin{aligned}
\Lambda_{ab} \;=\;& D *(e_a \wedge e_b)+2 \alpha D\left(R *(e_a \wedge e_b)\right)+\beta D\left(*(P_a \wedge e_b-P_b \wedge e_a)\right) \\
&+\gamma D\left(R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)\right) . \qquad \text{(D.32)}
\end{aligned}
$$

The field equation coming from the variation with respect to connection fields takes the same form as given in (C.50) with the previous definition. Thus, one can use (C.54) (i.e., $\lambda_b=\iota^a \Lambda_{ab}-\frac{1}{4} e_b \wedge \iota^c \iota^a \Lambda_{ac}$) to obtain corresponding Lagrange multiplier $(n-2)$-forms. To this end, one should first deal with the operation of inner product operator onto Λ_{ab} which can be written as follows:

$$
\begin{aligned}
\iota^a \Lambda_{ab} \;=\;& \underbrace{\iota^a D *(e_a \wedge e_b)}_{(K1)}+2 \alpha \underbrace{\iota^a D\left(R *(e_a \wedge e_b)\right)}_{(K2)}+\beta \underbrace{\iota^a D\left(*(P_a \wedge e_b-P_b \wedge e_a)\right)}_{(K3)} \\
&+\gamma \underbrace{\iota^a D\left(R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)\right)}_{(K4)} . \qquad \text{(D.33)}
\end{aligned}
$$

Here, each term should be considered separately again. Taking the identity (B.15) in the term $(K1)$ easily yields

$$
\begin{aligned}
(K1) \;=\;& \iota^a D *(e_a \wedge e_b) \\
&=\iota^a\left(T^c \wedge *(e_a \wedge e_b \wedge e_c)\right) \\
&=0 \qquad \text{(D.34)}
\end{aligned}
$$

since the Levi-Civita connection (i.e., $T^c=0$) is assumed in the theory. Then, the calculation of inner product operator onto the next term $(K2)$ can be given as

$$\begin{aligned}
(K2) &= \iota^a D\left(R * (e_a \wedge e_b)\right) \\
&= \iota^a \left(dR \wedge *(e_a \wedge e_b) + RD * (e_a \wedge e_b)\right) \\
&= \iota^a \left(dR \wedge *(e_a \wedge e_b) + R\left(T^c \wedge *(e_a \wedge e_b \wedge e_c)\right)\right) \\
&= (\iota^a dR) \wedge *(e_a \wedge e_b) - dR \wedge \iota^a * (e_a \wedge e_b) \\
&= * \left(e_a(\iota^a dR) \wedge e_b\right) - dR * (e_a \wedge e_b \wedge e^a) \\
&= *(dR \wedge e_b) \\
&= \iota_b * dR
\end{aligned} \tag{D.35}$$

by using (B.1), (B.2) and $e^a \wedge e_a = 0$. Employing the definition of the Ricci 1-forms and covariant exterior derivative operator for the term $(K3)$ leads

$$\begin{aligned}
(K3) &= \iota^a D\left(*(P_a \wedge e_b - P_b \wedge e_a)\right) \\
&= \iota^a D\left({P_a}^c * (e_c \wedge e_b) - {P_b}^c * (e_c \wedge e_a)\right) \\
&= \iota^a \Big((D{P_a}^c) \wedge *(e_c \wedge e_b) + {P_a}^c T^f \wedge *(e_c \wedge e_b \wedge e_f) \\
&\quad - (D{P_b}^c) \wedge *(e_c \wedge e_a) - {P_b}^c T^f \wedge *(e_c \wedge e_a \wedge e_f) \Big) .
\end{aligned} \tag{D.36}$$

The torsion-free connection reads

$$(K3) = \iota^a \left((D{P_a}^c) \wedge *(e_c \wedge e_b) - (D{P_b}^c) \wedge *(e_c \wedge e_a)\right) . \tag{D.37}$$

Applying the inner product operator yields

$$\begin{aligned}
(K3) &= (\iota^a D{P_a}^c) * (e_c \wedge e_b) + (D{P_a}^c) * (e_c \wedge e_b \wedge e^a) \\
&\quad -(\iota^a D{P_b}^c) * (e_c \wedge e_a) - (D{P_b}^c) * (e_c \wedge e_a \wedge e^a) \\
&= (\iota^a D{P_a}^c) * (e_c \wedge e_b) + (D{P_a}^c) * (e_c \wedge e_b \wedge e^a) \\
&\quad -(\iota^a D{P_b}^c) * (e_c \wedge e_a)
\end{aligned} \tag{D.38}$$

since $e_a \wedge e^a = 0$. Due to the term $(\iota^a D{P_b}^c)$ is a scalar, one can write

$$\begin{aligned}
(K3) &= (\iota^a D{P_a}^c) * (e_c \wedge e_b) + (D{P_a}^c) * (e_c \wedge e_b \wedge e^a) - *(e_c \wedge e_a(\iota^a D{P_b}^c)) \\
&= (\iota^a D{P_a}^c) * (e_c \wedge e_b) + (D{P_a}^c) * (e_c \wedge e_b \wedge e^a) - *(e_c \wedge (D{P_b}^c))
\end{aligned} \tag{D.39}$$

with the help of (B.2) and the definition of Ricci 1-forms. Using the identity $De_c = T_c = 0$ which is valid only for the Levi-Civita connection, one finds

$$(K3) = \underbrace{*\left((\iota^a D{P_a}^c)e_c \wedge e_b\right)}_{(K3-1)} + \underbrace{(D{P_a}^c) * (e_c \wedge e_b \wedge e^a)}_{(K3-2)} + * DP_b \tag{D.40}$$

As can clearly be seen that while $(K3-1)$ is identical with $(L2-3-1)$, $(K3-2)$ is the same as $(L2-3-2)$. Since there is no contribution from $(K3-2)$ (details are given in

(C.66)-(C.67) for $(L2-3-2)$), one can write

$$(K3) = \frac{1}{2}\iota_b * dR + *DP_b\,. \tag{D.41}$$

Final term $(K4)$ can also be given as

$$\begin{aligned}
(K4) &= \iota^a D\left(R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right)\\
&= \iota^a\left((DR^{cd})\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d) + R^{cd}\wedge D\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d)\right)\\
&= \iota^a\left((DR^{cd})\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d) + R^{cd}\wedge T^f\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d\wedge e_f)\right)\\
&= 0\,.
\end{aligned} \tag{D.42}$$

Owing to the identity (B.25) and the Levi-Civita connection structure, one finds that $(K4)$ has no contribution to Λ_{ab}. Therefore,

$$\iota^a\Lambda_{ab} = \left(2\alpha+\frac{\beta}{2}\right)\iota_b * dR + \beta * DP_b\,. \tag{D.43}$$

Further application of the inner product operator and taking (B.1) yields

$$\begin{aligned}
\iota^b\iota^a\Lambda_{ab} &= \left(2\alpha+\frac{\beta}{2}\right)\iota^b\iota_b * dR + \beta\iota^b * DP_b\\
&= \beta * (DP_b\wedge e^b)\\
&= 0
\end{aligned} \tag{D.44}$$

since $\iota^b\iota_b\zeta = 0$ (ζ denotes any p-form), $T^b = 0$ and $P_b\wedge e^b = 0$. Consequently, Lagrange multiplier $(n-2)$-form of QCG model can be given by

$$\lambda_b = \left(2\alpha+\frac{\beta}{2}\right)\iota_b * dR + \beta * DP_b\,. \tag{D.45}$$

Then, the Einstein field equation of the model is

$$\begin{aligned}
&\left(\alpha R+\frac{1}{2}\right)R^{ab}\wedge *(e_a\wedge e_b\wedge e_f) + \left(\Lambda-\frac{\alpha}{2}R^2\right) * e_f\\
&+\frac{\beta}{2}\left[P^a{}_f * P_a + 2(\iota_f R^{ba})\wedge *(P_a\wedge e_b) - P^a\wedge *(P_a\wedge e_f)\right]\\
&+\frac{\gamma}{4}R^{ab}\wedge R^{cd}\wedge *(e_a\wedge e_b\wedge e_c\wedge e_d\wedge e_f)\\
&+D\left(\left(2\alpha+\frac{\beta}{2}\right)\iota_f * dR + \beta * DP_f\right) = 0\,.
\end{aligned} \tag{D.46}$$

E. Minimal Massive Gravity Field Equations

In this appendix, the field equations of Minimal Massive Gravity (MMG) with and without matter coupling are derived.

E.1. Field equations without matter (Vacuum field equations)

The variation of MMG Lagrangian density with respect to the independent variables e^a, $\omega^a{}_b$ and λ^a yields (without matter coupling)

$$\begin{aligned}\delta\mathcal{L}_{MMG} \;=\;& -\frac{\sigma}{2}\,\underbrace{\delta\left(R^{ab}(\omega)\wedge *(e_a\wedge e_b)\right)}_{(a)}+\frac{1}{4\mu}\,\underbrace{\delta\left(\omega^a{}_b\wedge d\omega^b{}_a+\frac{2}{3}\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a\right)}_{(b)}\\ &+\underbrace{\delta\,(\Lambda * 1)}_{(c)}+\delta\left[\lambda_a\wedge T^a(\omega)\right]+\frac{\alpha}{2}\,\underbrace{\delta\left(\lambda^a\wedge\lambda^b\wedge *(e_a\wedge e_b)\right)}_{(d)}\,. \end{aligned} \tag{E.1}$$

The variation of Einstein-Hilbert term (a) can be obtained by using identity (B.12) and the commutation relation (C.3)

$$\begin{aligned}(a) &= \delta\left(R^{ab}(\omega)\wedge *(e_a\wedge e_b)\right)\\ &= \left(\delta R^{ab}(\omega)\right)\wedge *(e_a\wedge e_b)+R^{ab}(\omega)\wedge\delta\left(*(e_a\wedge e_b)\right)\\ &= \left(\delta(d\omega^{ab}+\omega^a{}_c\wedge\omega^{cb})\right)\wedge *(e_a\wedge e_b)+R^{ab}(\omega)\wedge(\delta e^c)\wedge *(e_a\wedge e_b\wedge e_c)\\ &= \delta(d\omega^{ab})\wedge *(e_a\wedge e_b)+(\delta\omega^a{}_c)\wedge\omega^{cb}\wedge *(e_a\wedge e_b)+\omega^a{}_c\wedge(\delta\omega^{cb})\wedge *(e_a\wedge e_b)\\ &\quad +(\delta e^c)\wedge R^{ab}(\omega)\wedge *(e_a\wedge e_b\wedge e_c)\,. \end{aligned} \tag{E.2}$$

By taking the definition of covariant exterior derivative and appropriately changing indices, this can also be rewritten as

$$\begin{aligned}(a) &= d(\delta\omega^{ab})\wedge *(e_a\wedge e_b)+(\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(e_c\wedge e_b)-\omega^c{}_b\wedge *(e_a\wedge e_c)\right)\\ &\quad +(\delta e^c)\wedge R^{ab}(\omega)\wedge *(e_a\wedge e_b\wedge e_c)\\ &= d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge d*(e_a\wedge e_b)\\ &\quad +(\delta\omega^{ab})\wedge\left(\omega_a{}^c\wedge *(e_c\wedge e_b)-\omega^c{}_b\wedge *(e_a\wedge e_c)\right)+(\delta e^c)\wedge R^{ab}(\omega)\wedge *(e_a\wedge e_b\wedge e_c)\\ &= d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+\delta\omega^{ab}\wedge D*(e_a\wedge e_b)+(\delta e^c)\wedge R^{ab}(\omega)\wedge *(e_a\wedge e_b\wedge e_c)\,. \end{aligned} \tag{E.3}$$

Then, considering (B.15) and (C.3), one can finally obtain

$$(a)=d\left((\delta\omega^{ab})\wedge *(e_a\wedge e_b)\right)+(\delta\omega^{ab})\wedge T^c\wedge *(e_a\wedge e_b\wedge e_c)-2(\delta e^a)\wedge G_a(\omega) \tag{E.4}$$

where

$$G_a(\omega)=-\frac{1}{2}R^{bc}(\omega)\wedge *(e_a\wedge e_b\wedge e_c)=-\frac{1}{2}\epsilon_{abc}R^{bc}(\omega) \tag{E.5}$$

denotes the Einstein 2-forms where $*(e_a\wedge e_b\wedge e_c)=\epsilon_{abc}$ in 3-dimensions.

Next, the variation of Chern-Simons term (b) can be obtained in a similar manner. Again by considering (C.3), one can obtain

$$\begin{aligned}
(b) &= \delta\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right) \\
&= \underbrace{\delta\left(\omega^a{}_b \wedge d\omega^b{}_a\right)}_{(b-1)} + \frac{2}{3}\underbrace{\delta\left(\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right)}_{(b-2)} . \qquad \text{(E.6)}
\end{aligned}$$

Here the variational term $(b-1)$ can be calculated as

$$\begin{aligned}
(b-1) &= \delta\left(\omega^a{}_b \wedge d\omega^b{}_a\right) \\
&= (\delta\omega^a{}_b) \wedge d\omega^b{}_a + \omega^a{}_b \wedge d\left(\delta\omega^b{}_a\right) . \qquad \text{(E.7)}
\end{aligned}$$

Using

$$d\left[\omega^a{}_b \wedge (\delta\omega^b{}_a)\right] = (d\omega^a{}_b) \wedge (\delta\omega^b{}_a) - \omega^a{}_b \wedge d(\delta\omega^b{}_a) , \qquad \text{(E.8)}$$

one can get

$$(b-1) = d\left((\delta\omega^{ab}) \wedge \omega_{ba}\right) + 2\left(\delta\omega^{ab}\right) \wedge d\omega_{ba} . \qquad \text{(E.9)}$$

By appropriately changing indices, $(b-2)$ can be calculated as

$$\begin{aligned}
(b-2) &= \delta\left(\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right) \\
&= (\delta\omega^a{}_b) \wedge \omega^b{}_c \wedge \omega^c{}_a + \omega^a{}_b \wedge \left(\delta\omega^b{}_c\right) \wedge \omega^c{}_a + \omega^a{}_b \wedge \omega^b{}_c \wedge (\delta\omega^c{}_a) \\
&= 3\left(\delta\omega^{ab}\right) \wedge \omega_{bc} \wedge \omega^c{}_a . \qquad \text{(E.10)}
\end{aligned}$$

Inserting $(b-1)$ and $(b-2)$ into (E.6), the variation of Chern-Simon term (b) finally results in

$$\begin{aligned}
(b) &= d\left((\delta\omega^{ab}) \wedge \omega_{ba}\right) + 2\left(\delta\omega^{ab}\right) \wedge (d\omega_{ba} + \omega_{bc} \wedge \omega^c{}_a) \\
&= d\left((\delta\omega^{ab}) \wedge \omega_{ba}\right) + 2\left(\delta\omega^{ab}\right) \wedge R_{ba}(\omega) . \qquad \text{(E.11)}
\end{aligned}$$

The variational term (c) which contains cosmological constant can be simply given by using identity (B.14) as

$$(c) = \Lambda * 1 = \Lambda(\delta e^a) \wedge *e_a . \qquad \text{(E.12)}$$

Since the calculations for the variation of zero-torsion constraint term has already been given in (C.44), one can continue with the variational term (d). Using (B.12) and noting that $*(e^a \wedge e^b \wedge e^c) = \epsilon^{abc}$ in three dimensions, one obtains

$$\begin{aligned}
(d) &= \delta\left(\lambda^a \wedge \lambda^b \wedge *(e_a \wedge e_b)\right) \\
&= 2(\delta\lambda^a) \wedge \lambda^b \wedge *(e_a \wedge e_b) + \lambda^a \wedge \lambda^b \wedge (\delta e^c) \wedge *(e_a \wedge e_b \wedge e_c) \\
&= 2\epsilon_{abc}(\delta\lambda^a) \wedge e^b \wedge \lambda^c + \epsilon_{abc}(\delta e^a) \wedge \lambda^b \wedge \lambda^c . \qquad \text{(E.13)}
\end{aligned}$$

Collecting all the variational terms, one can finally obtain

$$\begin{aligned}
\delta\mathcal{L}_{MMG} &= \delta e^a \wedge \Big(\sigma G_a(\omega) + \Lambda * e_a + D(\omega)\lambda_a + \frac{\alpha}{2}\epsilon_{abc}\lambda^b \wedge \lambda^c\Big) \\
&\quad + \delta\omega^{ab} \wedge \Big(-\frac{\sigma}{2}D(\omega) * (e_a \wedge e_b) + \frac{1}{2\mu}R_{ba}(\omega) - \frac{1}{2}(e_a \wedge \lambda_b - e_b \wedge \lambda_a)\Big) \\
&\quad + \delta\lambda^a \wedge \Big(T_a(\omega) + \alpha\epsilon_{abc}e^b \wedge \lambda^c\Big) \\
&\quad + d\left(-\frac{\sigma}{2}\Big[(\delta\omega^{ab}) \wedge *(e_a \wedge e_b)\Big] + \frac{1}{4\mu}\Big(\delta\omega^{ab}\Big) \wedge \omega_{ba} + (\delta e^a) \wedge \lambda_a\right) . \qquad \text{(E.14)}
\end{aligned}$$

The variational principle (2.11) implies:

$$\sigma G^a(\omega) + \Lambda * e^a + D(\omega)\lambda^a + \frac{\alpha}{2}\epsilon^{abc}\lambda_b \wedge \lambda_c = 0\,, \qquad \text{(E.15)}$$

$$-\frac{\sigma}{2}D(\omega) * (e^a \wedge e^b) + \frac{1}{2\mu}R^{ba}(\omega) - \frac{1}{2}(e^a \wedge \lambda^b - e^b \wedge \lambda^a) = 0 \qquad \text{(E.16)}$$

and

$$T^a(\omega) + \alpha\epsilon^{abc}e_b \wedge \lambda_c = 0\,. \qquad \text{(E.17)}$$

Here, it is convenient to start with (E.17), the field equation for the auxiliary field 1-form λ^a. As can easily be seen, the equation yields a torsion in terms of λ^a. By using (2.3), this can be further written in the following form:

$$de^a + ({\omega^a}_b - \alpha{\epsilon^a}_{bc}\lambda^c) \wedge e^b = 0\,. \qquad \text{(E.18)}$$

Therefore, the expression $({\omega^a}_b - \alpha{\epsilon^a}_{bc}\lambda^c)$ satisfies the structure equations with vanishing torsion. Then, the connection 1-form can be decomposed into the Riemannian part denoted by ${\Gamma^a}_b$ and the contorsion part identified by

$${K^a}_b = \alpha{\epsilon^a}_{bc}\lambda^c \qquad \text{(E.19)}$$

as

$${\omega^a}_b = {\Gamma^a}_b + \alpha{\epsilon^a}_{bc}\lambda^c\,. \qquad \text{(E.20)}$$

The decomposition of the connection 1-form (E.20) can be further applied for the decomposition of the curvature 2-form. Using (2.4), (E.19) and (E.20) gives

$$\begin{aligned}
{R^a}_b(\omega) &= d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b \\
&= d\left({\Gamma^a}_b + {K^a}_b\right) + ({\Gamma^a}_c + {K^a}_c) \wedge ({\Gamma^c}_b + {K^c}_b) \\
&= d{\Gamma^a}_b + d{K^a}_b + {\Gamma^a}_c \wedge {\Gamma^c}_b + {\Gamma^a}_c \wedge {K^c}_b + {K^a}_c \wedge {\Gamma^c}_b + {K^a}_c \wedge {K^c}_b \\
&= {R^a}_b(\Gamma) + d{K^a}_b + {\Gamma^a}_c \wedge {K^c}_b - {\Gamma^c}_b \wedge {K^a}_c + {K^a}_c \wedge {K^c}_b \\
&= {R^a}_b(\Gamma) + D(\Gamma){K^a}_b + {K^a}_c \wedge {K^c}_b \\
&= {R^a}_b(\Gamma) + \alpha{\epsilon^a}_{bc}D(\Gamma)\lambda^c + \alpha^2({\epsilon^a}_{cd}\lambda^d) \wedge ({\epsilon^c}_{bf}\lambda^f) \\
&= {R^a}_b(\Gamma) + \alpha{\epsilon^a}_{bc}D(\Gamma)\lambda^c - \alpha^2{\epsilon_c}^a{}_d{\epsilon^c}_{bf}\lambda^d \wedge \lambda^f\,. \qquad \text{(E.21)}
\end{aligned}$$

Using the following identity in 3-dimensions

$$\epsilon_c{}^a{}_d \epsilon^c{}_{bf} = -\left(\delta^a{}_b\, \delta_{df} - \delta^a{}_f\, \delta_{db}\right) \tag{E.22}$$

gives

$$\begin{aligned} R^a{}_b(\omega) &= R^a{}_b(\Gamma) + \alpha\epsilon^a{}_{bc} D(\Gamma)\lambda^c + \alpha^2 \left(\delta^a{}_b\, \delta_{df} - \delta^a{}_f \delta_{db}\right) \lambda^d \wedge \lambda^f \\ &= R^a{}_b(\Gamma) + \alpha\epsilon^a{}_{bc} D(\Gamma)\lambda^c + \alpha^2 \left(\delta^a{}_b \lambda^d \wedge \lambda_d - \lambda_b \wedge \lambda^a\right) \\ &= R^a{}_b(\Gamma) + \alpha\epsilon^a{}_{bc} D(\Gamma)\lambda^c + \alpha^2 \lambda^a \wedge \lambda_b \end{aligned} \tag{E.23}$$

since $\lambda^d \wedge \lambda_d = 0$. The connection equation (E.16) can be rewritten in terms of the Levi-Civita connection $\Gamma^a{}_b$ with the help of equation (E.23) and the definition of covariant exterior derivative. Therefore, one should start with the calculation of the covariant exterior term by using the identity (E.22):

$$\begin{aligned} D(\omega) * (e^a \wedge e^b) &= \epsilon^{ab}{}_c T^c(\omega) \\ &= -\alpha\epsilon^{ab}{}_c\, \epsilon^c{}_{df} e^d \wedge \lambda^f \\ &= \alpha \left(\delta^a{}_d \delta^b{}_f - \delta^a{}_f \delta^b{}_d\right) e^d \wedge \lambda^f \\ &= \alpha \left(e^a \wedge \lambda^b - e^b \wedge \lambda^a\right) . \end{aligned} \tag{E.24}$$

The combination of (E.21) and (E.24) into the connection equation (E.16) yields

$$-\frac{1}{2\mu} R^{ab}(\Gamma) - \frac{\alpha}{2\mu}\epsilon^{ab}{}_c D(\Gamma)\lambda^c - \frac{\alpha^2}{2\mu}\lambda^a \wedge \lambda^b - \frac{1}{2}(1+\alpha\sigma)(e^a \wedge \lambda^b - e^b \wedge \lambda^a) = 0\,. \tag{E.25}$$

Note that, by this way, the connection equation is rewritten in terms of Riemannian quantities. This can also be given in the equivalent form as

$$\epsilon^{ab}{}_c D(\Gamma)\lambda^c = -\frac{1}{\alpha} R^{ab}(\Gamma) - \alpha\lambda^a \wedge \lambda^b - \frac{\mu}{\alpha}(1+\alpha\sigma) e^a \wedge e^b\,. \tag{E.26}$$

On the other hand, the equation for basis co-frame 1-forms can be expressed in terms of Riemannian quantities as well. To this end, taking the definition of Einstein 2-forms (2.53) and covariant exterior derivative of auxiliary field $D(\omega)\lambda^a = D(\Gamma)\lambda^a + K^a{}_b \wedge \lambda^b$ in (E.15) with $K^a{}_b = \alpha\epsilon^a{}_{bc}\lambda^c$, one finds

$$-\frac{\sigma}{2}\epsilon^{abc} R_{bc}(\Gamma) + \Lambda * e^a + (1+\alpha\sigma) D(\Gamma)\lambda^a - \frac{\alpha}{2}(1+\alpha\sigma)\epsilon^{abc}\lambda_b \wedge \lambda_c = 0\,. \tag{E.27}$$

Multiplication with the term $\epsilon^{df}{}_a$ with accordingly changing indices leads

$$\sigma R^{ab}(\Gamma) - \Lambda e^a \wedge e^b + (1+\alpha\sigma)\epsilon^{ab}{}_c D(\Gamma)\lambda^c + \alpha(1+\alpha\sigma)\lambda^a \wedge \lambda^b = 0\,. \tag{E.28}$$

Substituting the expression for $\epsilon^{ab}{}_c D(\Gamma)\lambda^c$ in (E.28) one obtains

$$-R^{ab}(\Gamma) - \alpha\Lambda e^a \wedge e^b - \mu(1+\alpha\sigma)^2 \left(e^a \wedge \lambda^b - e^b \wedge \lambda^a\right) = 0\,. \tag{E.29}$$

This can be solved for the auxiliary field 1-forms by taking (C.50). In this case, one gets

$$\Lambda^{ab} = -\frac{1}{\mu(1+\alpha\sigma)^2}\left(R^{ab}(\Gamma) + \alpha\Lambda e^a \wedge e^b\right) . \tag{E.30}$$

To find the auxiliary 1-form field λ^a of MMG theory, one can follow the procedure given in (C.50)-(C.54); to this end, one begins with the application of the inner product operator on (E.30):

$$\begin{aligned}
\iota^a\Lambda_{ab} &= -\frac{1}{\mu(1+\alpha\sigma)^2}\iota^a\left(R_{ab}(\Gamma) + \alpha\Lambda e_a \wedge e_b\right) \\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\iota^a\left(\frac{1}{2}R_{ab}{}^{cd}e_c \wedge e_d + \alpha\Lambda e_a \wedge e_b\right) \\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(\frac{1}{2}R_{ab}{}^{cd}\delta^a{}_c e_d - \frac{1}{2}R_{ab}{}^{cd}\delta^a{}_d e_c + \alpha\Lambda\delta^a{}_a e_b - \alpha\Lambda\delta^a{}_b e_a\right) \\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(R_{ab}{}^{ad}e_d + 2\alpha\Lambda e_b\right) \\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(P_b{}^d e_d + 2\alpha\Lambda e_b\right) \\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(P_b + 2\alpha\Lambda e_b\right) .
\end{aligned} \tag{E.31}$$

By further applying the inner product operator into equation (E.31), one finds

$$\begin{aligned}
\iota^b\iota^a\Lambda_{ab} &= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(\iota^b P_b + 2\alpha\Lambda\delta^b{}_b\right) \\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(R + 6\alpha\Lambda\right) .
\end{aligned} \tag{E.32}$$

Back substitution of (E.31) and (E.32) into equation (C.54) with suitable index change yields

$$\lambda_a = -\frac{1}{2\mu(1+\alpha\sigma)^2}\left(2P_a + \left(\alpha\Lambda - \frac{1}{2}R\right)e_a\right) . \tag{E.33}$$

The auxiliary 1-form can also be rewritten as

$$\lambda_a = -\frac{1}{2\mu(1+\alpha\sigma)^2}\left(2Y_a(\Gamma) + \alpha\Lambda e_a\right) , \tag{E.34}$$

where $Y_a(\Gamma)$ denotes the Schouten 1-form defined as

$$Y_a(\Gamma) = P_a - \frac{1}{4}Re_a . \tag{E.35}$$

Consequently, the auxiliary vector-valued 1-form λ^a is eliminated from the metric equations which are attained by the coframe and basis 1-form variations in terms of gravitational variables the theory contains. When the auxiliary 1-form field is written in equation (E.27), the Einstein field equation in terms of Riemannian quantities simply takes the following form

$$-\frac{\sigma}{2}\epsilon^{abc}R_{bc}(\Gamma)+\Lambda*e^a-\frac{1}{2\mu(1+\alpha\sigma)}D(\Gamma)\left(2Y^a(\Gamma)+\alpha\Lambda e^a\right)$$
$$-\frac{\alpha}{8\mu^2(1+\alpha\sigma)^3}\epsilon^{abc}\left(2Y_b(\Gamma)+\alpha\Lambda e_b\right)\wedge\left(2Y_c(\Gamma)+\alpha\Lambda e_c\right)=0\,. \tag{E.36}$$

Einstein field equation of the model can be equivalently given by using the definition $C^a = D(\Gamma)Y^a(\Gamma)$ of the Cotton 2-form and the identity $R_{bc}(\Gamma) = e_b \wedge Y_c(\Gamma) - e_c \wedge Y_b(\Gamma)$ of the curvature 2-form as

$$-\left(\frac{\sigma}{2}+\frac{\alpha^2\Lambda}{4\mu^2(1+\alpha\sigma)^3}\right)\epsilon^{abc}R_{bc}(\Gamma)+\left(\Lambda-\frac{\alpha^3\Lambda^2}{4\mu^2(1+\alpha\sigma)^3}\right)*e^a$$
$$-\frac{1}{\mu(1+\alpha\sigma)}C^a-\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\epsilon^{abc}Y_b(\Gamma)\wedge Y_c(\Gamma)=0 \tag{E.37}$$

since $D(\Gamma)e^a = 0$ for Levi-Civita connection.

E.2. Field equations with Maxwell-Chern-Simons coupling

One can consider Lagrangian density in the following form

$$\begin{aligned}\bar{\mathcal{L}}_{MMG} &= \mathcal{L}_{MMG}+\mathcal{L}(matter)\\ &= \mathcal{L}_{MMG}-\frac{1}{2}F\wedge *F-\frac{1}{2}mA\wedge F\,,\end{aligned} \tag{E.38}$$

which contains the standard MMG lagrangian (E.1) and the standard Maxwell field $F = dA$ along with the Chern-Simons term with the coupling constant m. Since the variations of $\mathcal{L}_{MMG}$ have been calculated, one can continue with the variation of the Maxwell term which yields

$$\delta\bar{\mathcal{L}}_{MMG}=\delta\mathcal{L}_{MMG}-\frac{1}{2}\underbrace{\delta\left(F\wedge *F\right)}_{(e)}-\frac{1}{2}m\underbrace{\delta\left(A\wedge F\right)}_{(f)}\,. \tag{E.39}$$

The variational term (e) can be calculated by first considering the identity (B.3)

$$\begin{aligned}(e) &= \delta(F\wedge *F)\\ &= 2(\delta F)\wedge *F-(\delta e_a)\wedge\left[(\iota^a F)\wedge *F-F\wedge\iota^a *F\right]\\ &= 2(\delta F)\wedge *F-2(\delta e_a)\wedge\tau^a[F]\end{aligned} \tag{E.40}$$

where

$$\tau^a[F]=\frac{1}{2}\left(\iota^a F\wedge *F-F\wedge\iota^a *F\right) \tag{E.41}$$

is the electromagnetic field energy momentum 2-form. When the commutation relation between infinitesimal variations and exterior derivative operators is used, one gets

$$\begin{aligned}(\delta F)\wedge *F &= \delta(dA)\wedge *F\\ &= d(\delta A)\wedge *F\,,\end{aligned} \tag{E.42}$$

which can also be written as

$$d(\delta A) \wedge *F = d[(\delta A) \wedge *F] + (\delta A) \wedge d * F\,. \tag{E.43}$$

Therefore,

$$(e) = 2d\,[(\delta A) \wedge *F] + 2(\delta A) \wedge d * F - 2(\delta e_a) \wedge \tau^a[F]\,. \tag{E.44}$$

Then, applying the same procedure on variational term (f) gives

$$\begin{aligned}
(f) &= \delta(A \wedge F) \\
&= (\delta A) \wedge F + A \wedge \delta(dA) \\
&= (\delta A) \wedge F + A \wedge d(\delta A) \\
&= (\delta A) \wedge F + d(\delta A) \wedge A\,.
\end{aligned} \tag{E.45}$$

Using

$$d(\delta A) \wedge A = d\,[(\delta A) \wedge A] + (\delta A) \wedge F \tag{E.46}$$

leads

$$(f) = d\,[(\delta A) \wedge A] + 2(\delta A) \wedge F\,. \tag{E.47}$$

Now, collecting all the variations yields

$$\begin{aligned}
\delta\bar{\mathcal{L}}_{MMG} = {} & \delta e^a \wedge \left(\sigma G_a(\omega) + \Lambda * e_a + D(\omega)\lambda_a + \frac{\alpha}{2}\epsilon_{abc}\lambda^b \wedge \lambda^c + \tau_a[F]\right) \\
& + \delta\omega^{ab} \wedge \left(-\frac{\sigma}{2}D(\omega) * (e_a \wedge e_b) + \frac{1}{2\mu}R_{ba}(\omega) - \frac{1}{2}(e_a \wedge \lambda_b - e_b \wedge \lambda_a)\right) \\
& + \delta\lambda^a \wedge \left(T_a(\omega) + \alpha\epsilon_{abc}e^b \wedge \lambda^c\right) + (\delta A) \wedge (d * F + mF) \\
& + d\left(-\frac{\sigma}{2}\left(\delta\omega^{ab} \wedge *(e_a \wedge e_b)\right) + \frac{1}{4\mu}\left(\delta\omega^{ab}\right) \wedge \omega_{ba} + (\delta e^a) \wedge \lambda_a + \frac{1}{2}(\delta A) \wedge A\right)\,.
\end{aligned} \tag{E.48}$$

Assuming that the fields vanish on the boundary of the spacetime manifold, the closed forms do not give any contribution to the field equations. Hence, the variational principle (2.11) implies

$$\sigma G_a(\omega) + \Lambda * e_a + D(\omega)\lambda_a + \frac{\alpha}{2}\epsilon_{abc}\lambda^b \wedge \lambda^c + \tau_a[F] = 0\,, \tag{E.49}$$

$$-\frac{\sigma}{2}D(\omega) * (e_a \wedge e_b) + \frac{1}{2\mu}R_{ba}(\omega) - \frac{1}{2}(e_a \wedge \lambda_b - e_b \wedge \lambda_a)\,, \tag{E.50}$$

$$T_a(\omega) + \alpha\epsilon_{abc}e^b \wedge \lambda^c = 0\,, \tag{E.51}$$

$$d * F + mF = 0\,. \tag{E.52}$$

As can easily be seen, the field equations (E.15) and (E.16), which are given in the vacuum field equations of MMG, are identical with (E.49) and (E.50). Therefore, the decomposition of connection 1-form (E.20) and curvature 2-form (E.21) are still valid in this case. The expression of connection equation in terms of Riemannian quantities has already been found in (E.25). Therefore, to express the equation for basis coframe 1-forms in terms of Riemannian quantities, one should carefully add the contribution, which comes from the electromagnetic field energy momentum 2-form as

$$-\frac{\sigma}{2}\epsilon_{abc}R^{bc}(\Gamma)+\Lambda * e_a+(1+\alpha\sigma)D(\Gamma)\lambda_a-\frac{\alpha}{2}(1+\alpha\sigma)\epsilon_{abc}\lambda^b\wedge\lambda^c+\tau_a[F]=0\,. \quad \text{(E.53)}$$

Multiplication with the term ϵ^{dfa} with accordingly changing indices leads

$$\sigma R^{ab}(\Gamma)-\Lambda e^a\wedge e^b+(1+\alpha\sigma)\epsilon^{ab}{}_cD(\Gamma)\lambda^c+\alpha(1+\alpha\sigma)\lambda^a\wedge\lambda^b+\epsilon^{abc}\tau_c[F]=0\,. \quad \text{(E.54)}$$

Substituting (E.26) into (E.54) yields

$$-R^{ab}(\Gamma)-\alpha\Lambda e^a\wedge e^b-\mu(1+\alpha\sigma)^2\left(e^a\wedge\lambda^b-e^b\wedge\lambda^a\right)+\alpha\epsilon^{abc}\tau_c[F]=0\,. \quad \text{(E.55)}$$

Here, to find the solution of the auxiliary field 1-forms, (E.55) should be solved. Identifying

$$\Lambda^{ab}=-\frac{1}{\mu(1+\alpha\sigma)^2}\left(R^{ab}(\Gamma)+\alpha\Lambda e^a\wedge e^b-\alpha\epsilon^{ab}{}_c\tau^c[F]\right)\,, \quad \text{(E.56)}$$

one can obtain equation for λ_a in the form (C.50). Auxiliary 1-form field λ^a of matter coupled MMG can be calculated by following the same steps given in (C.51)-(C.53). For this, one should first calculate

$$\begin{aligned}
\iota_a\Lambda^{ab} &= -\frac{1}{\mu(1+\alpha\sigma)^2}\iota_a\left(R^{ab}(\Gamma)+\alpha\Lambda e^a\wedge e^b-\alpha\epsilon^{ab}{}_c\tau^c[F]\right)\\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\iota_a\left(\frac{1}{2}R^{abcd}e_c\wedge e_d+\alpha\Lambda e^a\wedge e^b-\alpha\epsilon^{ab}{}_c\tau^c[F]\right)\\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(\frac{1}{2}R^{abcd}\delta_{ac}e_d-\frac{1}{2}R^{abcd}\delta_{ad}e_c+\alpha\Lambda\delta_a{}^ae^b-\alpha\Lambda\delta_a{}^be^a-\alpha\epsilon^{ab}{}_c\iota_a\tau^c[F]\right)\\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(R^{ab}{}_a{}^de_d+2\alpha\Lambda e^b-\alpha\epsilon^{ab}{}_c\iota_a\tau^c[F]\right)\\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(P^{bd}e_d+2\alpha\Lambda e^b-\alpha\epsilon^{ab}{}_c\iota_a\tau^c[F]\right)\\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(P^b+2\alpha\Lambda e^b-\alpha\epsilon^{ab}{}_c\iota_a\tau^c[F]\right)\,.
\end{aligned} \quad \text{(E.57)}$$

Further application of the inner product operator into equation (E.57) reads

$$\begin{aligned}
\iota_b\iota_a\Lambda^{ab} &= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(\iota_bP^b+2\alpha\Lambda\delta_b{}^b-\alpha\epsilon^{ab}{}_c\iota_b\iota_a\tau^c[F]\right)\\
&= -\frac{1}{\mu(1+\alpha\sigma)^2}\left(R+6\alpha\Lambda-\alpha\epsilon^{ab}{}_c\iota_b\iota_a\tau^c[F]\right)\,.
\end{aligned} \quad \text{(E.58)}$$

Back substitution of (E.57) and (E.58) into equation (C.55) with suitable change of index leads to

$$\lambda^b = -\frac{1}{\mu(1+\alpha\sigma)^2}\left(P^b - \frac{1}{4}Re^b + \frac{1}{2}\alpha\Lambda e^b - \alpha\epsilon^{ab}{}_c\iota_a\tau^c[F] + \frac{1}{4}\alpha\epsilon^{ac}{}_d e^b\wedge\iota_c\iota_a\tau^d[F]\right). \tag{E.59}$$

It is noted that auxiliary 1-form field can also be expressed as

$$\lambda^b = -\frac{1}{\mu(1+\alpha\sigma)^2}\left(Y^b(\Gamma) + \frac{1}{2}\alpha\Lambda e^b - \alpha\epsilon^{ab}{}_c\iota_a\tau^c[F] + \frac{1}{4}\alpha\epsilon^{ac}{}_d e^b\wedge\iota_c\iota_a\tau^d[F]\right), \tag{E.60}$$

where $Y^b(\Gamma)$ again represents the Schouten 1-form which is given by equation (E.35). Note that the electromagnetic field energy 2-form can also be written as $\tau^c[F] = \tau^{cf}[F] * e_f$ where $\tau^{cf}[F]$ denotes the components of the stress-energy tensor. Taking the trace of the stress-energy tensor $\tau[F]*1 = \tau_c[F]\wedge e^c$, identities $*\tau_c = -\tau_{ca}e^a$ and (E.22) simplifies the auxiliary field as

$$\lambda^b = -\frac{1}{\mu(1+\alpha\sigma)^2}\left(Y^b(\Gamma) + \frac{1}{2}\alpha\Lambda e^b + \alpha\left(-\tau^{cb}[F]e_c + \frac{1}{2}\tau[F]e^b\right)\right). \tag{E.61}$$

A new stress-energy tensor can be defined as

$$-\tau^{cb}[F]e_c + \frac{1}{2}\tau[F]e^b = -*\hat{\tau}^b[F] \tag{E.62}$$

which also points out the following

$$\hat{\tau}^b[F] = -\tau^b[F] + \frac{1}{2}\tau[F]*e^b. \tag{E.63}$$

Therefore, the final form of the auxiliary field 1-form is

$$\lambda^b = -\frac{1}{\mu(1+\alpha\sigma)^2}\left(Y^b(\Gamma) + \frac{1}{2}\alpha\Lambda e^b + \alpha*\hat{\tau}^b[F]\right). \tag{E.64}$$

After finding the expression of auxiliary field 1-form, one can write the Einstein field equation of the theory (E.53) in terms of Riemannian quantities as

$$\begin{aligned}
&-\frac{\sigma}{2}\epsilon_{abc}R^{bc}(\Gamma) + \Lambda * e_a - \frac{1}{\mu(1+\alpha\sigma)}D(\Gamma)\left(Y_a(\Gamma) + \frac{1}{2}\alpha\Lambda e_a + \alpha*\hat{\tau}_a[F]\right)\\
&-\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\left(Y^b(\Gamma) + \frac{1}{2}\alpha\Lambda e^b + \alpha*\hat{\tau}^b[F]\right)\\
&\wedge\left(Y^c(\Gamma) + \frac{1}{2}\alpha\Lambda e^c + \alpha*\hat{\tau}^c[F]\right) + \tau_a[F] = 0.
\end{aligned} \tag{E.65}$$

Einstein field equation of the matter coupled case can be equivalently rewritten by using the definition $C^a = D(\Gamma)Y^a(\Gamma)$ of the Cotton 2-form and the identity $R_{bc}(\Gamma) = e_b\wedge Y_c(\Gamma)-$

$e_c \wedge Y_b(\Gamma)$ of the curvature 2-form as

$$
\begin{aligned}
&\left(-\frac{\sigma}{2} - \frac{\Lambda\alpha^2}{4\mu^2(1+\alpha\sigma)^3}\right)\epsilon_{abc}R^{bc} - \frac{1}{\mu(1+\alpha\sigma)}C_a + \tau_a[F] \\
&+\left(\Lambda - \frac{\Lambda^2\alpha^3}{4\mu^2(1+\alpha\sigma)^3}\right) * e_a - \frac{\alpha}{2\mu(1+\alpha\sigma)^3}\epsilon_{abc}Y^b(\Gamma)\wedge Y^c(\Gamma) \\
&-\frac{\alpha}{\mu(1+\alpha\sigma)^3}D(\Gamma) * \hat{\tau}_a[F] - \frac{\alpha^3}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc} * \hat{\tau}^b[F] \wedge *\hat{\tau}^c[F] \\
&-\frac{\alpha^2}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\left(Y^b(\Gamma)\wedge *\hat{\tau}^c[F] - Y^c(\Gamma)\wedge *\hat{\tau}^b[F]\right) \\
&-\frac{\alpha^3\Lambda}{4\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\left(e^b \wedge *\hat{\tau}^c[F] - e^c \wedge *\hat{\tau}^b[F]\right) = 0\,.
\end{aligned}
\tag{E.66}
$$

The field equation can also be considered in the following form

$$
\mathcal{E}_a + \tau_a[F] + \mathcal{O}_a[\hat{\tau}] = 0\,, \tag{E.67}
$$

where

$$
\begin{aligned}
\mathcal{E}_a \;=\;& \left(-\frac{\sigma}{2} - \frac{\Lambda\alpha^2}{4\mu^2(1+\alpha\sigma)^3}\right)\epsilon_{abc}R^{bc} - \frac{1}{\mu(1+\alpha\sigma)}C_a + \left(\Lambda - \frac{\Lambda^2\alpha^3}{4\mu^2(1+\alpha\sigma)^3}\right) * e_a \\
&-\frac{\alpha}{2\mu^2(1+\alpha\sigma)^3}\,\epsilon_{abc}Y^b(\Gamma)\wedge Y^c(\Gamma)
\end{aligned}
\tag{E.68}
$$

and

$$
\begin{aligned}
\mathcal{O}_a[\hat{\tau}] \;=\;& -\frac{\alpha}{\mu(1+\alpha\sigma)^3}D(\Gamma) * \hat{\tau}_a[F] - \frac{\alpha^2}{2\mu^2(1+\alpha\sigma)^3}\epsilon_{abc}\Big(\alpha * \hat{\tau}^b[F] \wedge *\hat{\tau}^c[F] \\
&+\left(Y^b(\Gamma)\wedge *\hat{\tau}^c[F] - Y^c(\Gamma)\wedge *\hat{\tau}^b[F]\right) + \frac{\alpha\Lambda}{2}\left(e^b \wedge *\hat{\tau}^c[F] - e^c \wedge *\hat{\tau}^b[F]\right)\Big)\,.
\end{aligned}
\tag{E.69}
$$

Here, $\mathcal{E}_a$ denotes the stress tensor of pure MMG theory and $\mathcal{O}_a[\hat{\tau}]$ is the stress tensor contribution coming from the electromagnetic field. To derive the consistency relation for the electromagnetic field coupled MMG theory, one should first take the covariant exterior derivative of the field equation (E.66). Using the relations $DR^{bc} = 0$ (identity (B.25)), $DC_a = 0$, $D\tau_a[F] = 0$ and $D * e_a = 0$ (for the torsion-free connection) yields

$$
-\frac{\alpha}{\mu^2(1+\alpha\sigma)^3}\,\epsilon_{abc}C^b \wedge Y^c + D\mathcal{O}_a[\hat{\tau}] = 0\,. \tag{E.70}
$$

Eliminating the Cotton 2-form C_a from the Einstein field equation (E.66) reads

$$
\begin{aligned}
C_a \;=\;& \mu(1+\alpha\sigma)\left(\tau_a[F] + \mathcal{O}_a[\hat{\tau}]\right) + \mu(1+\alpha\sigma)\left(-\frac{\sigma}{2} - \frac{\Lambda\alpha^2}{4\mu^2(1+\alpha\sigma)^3}\right)\epsilon_{abc}R^{bc} \\
&+\mu(1+\alpha\sigma)\left(\Lambda - \frac{\Lambda^2\alpha^3}{4\mu^2(1+\alpha\sigma)^3}\right) * e_a - \frac{\alpha}{2(1+\alpha\sigma)^2}\epsilon_{abc}Y^b(\Gamma)\wedge Y^c(\Gamma)\,.
\end{aligned}
\tag{E.71}
$$

Then, substituting (E.71) into (E.70) with the suitable index change gives

$$
\begin{aligned}
&-\frac{\alpha}{\mu(1+\alpha\sigma)^2}\epsilon_{abc}\left(\tau^b[F]+\mathcal{O}^b[\hat{\tau}]\right)\wedge Y^c(\Gamma) \\
&+\frac{\alpha}{\mu(1+\alpha\sigma)^2}\left(\frac{\sigma}{2}+\frac{\Lambda\alpha^2}{4\mu^2(1+\alpha\sigma)^3}\right)\underbrace{\epsilon_{abc}\epsilon^{bmn}R_{mn}\wedge Y^c(\Gamma)}_{(C1)} \\
&-\frac{\alpha}{\mu(1+\alpha\sigma)^2}\left(\Lambda-\frac{\Lambda^2\alpha^3}{4\mu^2(1+\alpha\sigma)^3}\right)\underbrace{\epsilon_{abc}(*e^b)\wedge Y^c(\Gamma)}_{(C2)} \\
&-\frac{\alpha^2}{2\mu^2(1+\alpha\sigma^5)}\underbrace{\epsilon_{abc}\epsilon^{bmn}Y_m(\Gamma)\wedge Y_n(\Gamma)\wedge Y^c(\Gamma)}_{(C3)}+D\mathcal{O}_a[\hat{\tau}]=0\,. \qquad \text{(E.72)}
\end{aligned}
$$

The contribution of each term in (E.71) should be evaluated separately. Therefore, by using identities (E.35), (E.22) and $e_a\wedge P^a=0$ (for Levi-Civita connection) with $R_{mn}(\Gamma)=e_m\wedge Y_n(\Gamma)-e_n\wedge Y_m(\Gamma)$ of the curvature 2-form, one can see that there is no contribution coming from the term $(C1)$:

$$
\begin{aligned}
(C1) &= \epsilon_{abc}\epsilon^{bmn}R_{mn}\wedge Y^c(\Gamma) \\
&= -\epsilon_{bac}\epsilon^{bmn}\left(e_m\wedge Y_n(\Gamma)-e_n\wedge Y_m(\Gamma)\right)\wedge Y^c(\Gamma) \\
&= \left(\delta_a{}^m\delta_c{}^n-\delta_a{}^n\delta_c{}^m\right)\left(e_m\wedge Y_n(\Gamma)-e_n\wedge Y_m(\Gamma)\right)\wedge Y^c(\Gamma) \\
&= \left(e_a\wedge Y_c(\Gamma)-e_c\wedge Y_a(\Gamma)\right)\wedge Y^c(\Gamma)-\left(e_c\wedge Y_a(\Gamma)-e_a\wedge Y_c(\Gamma)\right)\wedge Y^c(\Gamma) \\
&= 0\,. \qquad \text{(E.73)}
\end{aligned}
$$

Then, the calculation for the next term can be achieved as:

$$
\begin{aligned}
(C2) &= \epsilon_{abc}(*e^b)\wedge Y^c(\Gamma) \\
&= \frac{1}{2}\epsilon_{abc}\epsilon^{bmn}e_m\wedge e_n\wedge Y^c(\Gamma) \\
&= -\frac{1}{2}\epsilon_{bac}\epsilon^{bmn}e_m\wedge e_n\wedge Y^c(\Gamma) \\
&= \frac{1}{2}\left(\delta_a{}^m\delta_c{}^n-\delta_a{}^n\delta_c{}^m\right)e_m\wedge e_n\wedge Y^c(\Gamma) \\
&= \frac{1}{2}\left(e_a\wedge e_c\wedge Y^c(\Gamma)-e_c\wedge e_a\wedge Y^c(\Gamma)\right) \\
&= e_a\wedge e_c\wedge Y^c(\Gamma) \\
&= e_a\wedge e_c\wedge\left(P^c-\frac{1}{4}Re^c\right) \\
&= 0\,. \qquad \text{(E.74)}
\end{aligned}
$$

Similarly, one attains that the last term does not have any contribution to the consistency relation:

$$
\begin{aligned}
(C2) &= \epsilon_{abc}\epsilon^{bmn} Y_m(\Gamma) \wedge Y_n(\Gamma) \wedge Y^c(\Gamma) \\
&= -\epsilon_{bac}\epsilon^{bmn} Y_m(\Gamma) \wedge Y_n(\Gamma) \wedge Y^c(\Gamma) \\
&= \left(\delta_a{}^m \delta_c{}^n - \delta_a{}^n \delta_c{}^m\right) Y_m(\Gamma) \wedge Y_n(\Gamma) \wedge Y^c(\Gamma) \\
&= Y_a(\Gamma) \wedge Y_c(\Gamma) \wedge Y^c(\Gamma) - Y_c(\Gamma) \wedge Y_a(\Gamma) \wedge Y^c(\Gamma) \\
&= 0\,.
\end{aligned}
\tag{E.75}
$$

Finally, the consistency condition for electromagnetic field coupled MMG theory becomes

$$
D\mathcal{O}_a[\hat{\tau}] = \frac{\alpha}{\mu(1+\alpha\sigma)^2} \epsilon_{abc} \left(\tau^b[F] + \mathcal{O}^b[\hat{\tau}]\right) \wedge Y^c(\Gamma)\,. \tag{E.76}
$$

The direct contribution coming from electromagnetic field ensures that the field equation of matter-coupled MMG theory to be consistent on shell.

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