ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON r-MIN AND r-MAX MATRICES AND THEIR PROPERTIES

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

NAZLIHAN TERZİOĞLU

JULY 2022

ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON r-MIN AND r-MAX MATRICES AND THEIR PROPERTIES

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

Nazlıhan TERZİOĞLU

ADVISOR : Assoc. Prof. Dr. Can KIZILATEŞ

ZONGULDAK
JULY 2022

APPROVAL OF THE THESIS:

The thesis entitled “On r-Min And r-Max Matrices And Their Properties” and submitted by Nazlıhan Terzioğlu has been examined and accepted by the jury as a Master of Science thesis in Department of Mathematics, Graduate School of Natural and Applied Sciences, Zonguldak Bülent Ecevit University. 19/07/2022

Advisor: Assoc. Prof. Dr. Can KIZILATEŞ ....
Zonguldak Bülent Ecevit University, Faculty of Arts and Science,
Department of Mathematics

Member: Prof. Dr. Naim TUGLU ....
Gazi University, Faculty of Science, Department of Mathematics

Member: Assist. Prof. Dr. Emrah POLATLI ....
Zonguldak Bülent Ecevit University, Faculty of Arts and Science,
Department of Mathematics

Approved by the Graduate School of Natural and Applied Sciences. …./…./2022

Prof. Dr. Ahmet ÖZARSLAN
Director

"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly."

Nazlıhan TERZİOĞLU

ABSTRACT

Master of Science Thesis

ON r-MIN AND r-MAX MATRICES AND THEIR PROPERTIES

Nazlıhan TERZİOĞLU

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Dr. Can KIZILATEŞ
July 2022, 43 pages

In this thesis, r-min and r-max matrices, which are the general forms of min and max matrices, are defined such that min and max matrices are obtained for r=1 in these matrices. In the first part, matrix theory and number theory were mentioned and the studies in the literature were listed. The next chapter is devoted to the basic definitions and theorems utilized throughout the thesis. In the third chapter, in which the basic results we obtained about r-min and r-max matrices are given, in addition to the determinant, inverse, norm and factorization of these matrices, linear algebraic properties are also obtained for Hadamard inverses. Finally, these theoretical results were coded in the Maple 18 program and compared with the program's own results and found to be provided. For numerical examples of the obtained results, r-min and r-max matrices, whose entries are Fibonacci numbers and hyperharmonic Fibonacci numbers, are taken and their determinants, inverses and norms are shown.

Keywords: Max matrix, min matrix, circulant matrix, determinant, matrix inverse, matrix norm

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

r-MIN VE r-MAX MATRİSLER VE ONLARIN ÖZELLİKLERİ ÜZERİNE

Nazlıhan TERZİOĞLU

Zonguldak Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Doç. Dr. Can KIZILATEŞ
Temmuz 2022, 43 sayfa

Bu tezde, min ve max matrislerinin genel formu olan *r*-min ve *r*-max matrisleri tanımlandı öyle ki bu matrislerde *r*=1 için min ve max matrisleri elde edilmektedir. İlk bölümde matris teori ve sayılar teorisine değinilerek, literatürde yer alan çalışmalar listelendi. Sonraki bölüm, tez boyunca faydalanılan temel tanım ve teoremlere ayrıldı. *r*-min ve *r*-max matrisleriyle ilgili elde ettiğimiz temel sonuçların verildiği üçüncü bölümde, bu matrislerin determinantına, tersine, normuna ve çarpanlara ayrılmasına ek olarak, Hadamard tersleri için de lineer cebirsel özellikler elde edildi. Son olarak, bu teorik sonuçlar Maple 18 programında kodlandı ve programın kendi sonuçlarıyla kıyaslandı, sağlandıkları görüldü. Elde edilen sonuçların sayısal örnekleri için ise elemanları Fibonacci sayıları ve hiperharmonik Fibonacci sayıları olan *r*-min ve *r*-max matrisleri ele alınarak, determinantları, tersleri ve normları gösterildi.

Anahtar Kelimeler: Max matris, min matris, circulant matris, determinant, matris tersi, matris normu

Bilim Kodu: 403.01.01

ACKNOWLEDGEMENTS

I want to specify my thanks to my advisor Assoc. Prof. Dr. Can KIZILATEŞ for his support, and guidance during my master's process and all stages of writing this thesis. I also thank my teachers in the Department of Mathematics, Çanakkale 18 Mart University, especially, Prof. Dr. Neşet AYDIN and Assist. Prof. Dr. Didem YEŞİL.

And, of course, I am grateful to my family for the support and tolerance they have given me throughout my education life. I would also like to thank Dr. Taibe KULAKSIZ, my friend of 16 years for now, for her support and realistic comments during this difficult process.

Finally, my special thanks go to nature for the peace, calmness, chaos and inspiration it provides.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{C}$	: Complex numbers
$A=[a_{ij}]_{i,j=1}^{n}$	: Matrix that its entries are a_{ij} for i,j=1,2,…,n
A^T	: Transpose of the A
AoB	: Hadamard product of A and B
A^{o-1}	: Hadamard inverse of A
$\Vert A\Vert_E$	: Euclidean norm of A
$\Vert A\Vert_2$	: Spectral norm of A
F_n	: n-th Fibonacci number
$\mathbb{F}_n^{(\tau)}$	: n-th hyperharmonic Fibonacci number
$\boldsymbol{P}_{r,n}$	: r-min matrix
$\boldsymbol{Q}_{r,n}$	: r-max matrix

CHAPTER 1

INTRODUCTION

English mathematician J.Sylvester introduced the term matrix to literature. Afterward, in the 1850s, A. Cayley developed its algebraic aspect of matrices such that they are still very useful. From Cayley to nowadays, different types of matrices and operations on them have been described by mathematicians. Today, matrix theory has a big importance in various areas such as natural sciences, social sciences, technology, art.

Especially in applied mathematics and engineering, a problem can be solved more easily and clearly by using matrices. Concepts such as eigenvalue, singular value, inverse, norm, and determinant provide us with important information about the properties of matrices and therefore the problem being studied. However, it is very difficult to calculate the numerical values of these concepts in large dimensional matrices. Therefore, in matrix theory studies, choosing special types of matrices or the entries of the matrix with special values; is an effective tool for researchers both in terms of convenience and in terms of obtaining remarkable findings in their problems.

At this point, number theory, an important branch of mathematics, plays a very essential role. Many sequences have been created and new properties have been revealed from the relations between the numbers. One of the most famous sequences is the Fibonacci numbers which was introduced by L. Fibonacci in 200 BC, and today it is used in science, contemporary art, cipher technologies, etc. used in professions. Many mathematicians also have been interested in Fibonacci numbers and introduced new number sequences that related to them such as Lucas numbers, harmonic Fibonacci numbers, hyperharmonic Fibonacci numbers, Horadam numbers, etc. In recent years, special types of matrices whose elements are these numbers have been studied by many researchers. For example, linear algebra properties, such as determinant, inverse, norm, the eigenvalue of circulant, r-circulant, geometric circulant, Hankel, Toeplitz, min and max matrices, whose entries are these numbers have been studied by many researchers.

1.1 AIM OF THE THESIS

Our aim in this thesis is to present the r-min and r-max matrices, which are the general forms of the min and max matrices, thanks to the relationship between the circulant and r-circulant matrices. In fact, this matrices are the general forms of the min and max matrices. Then we gave determinant, inverse, norm, and factorization properties of these newly defined matrices and their Hadamard inverse matrices. Finally, our results are confirmed with several examples.

1.2 REVIEW OF THE LITERATURE

In this section, some information will be given about references used in the thesis.

Reams gave that Hadamard inverse of A, given by $A^{\circ -1}$ is positive definite if the symmetric matrix $A = [a_{ij}]_{i,j=1}^{n}$, which all its entries positive and only one positive eigenvalue, is invertible (Reams, 1999).

Solak found out some bounds for the spectral and Euclidean norms of the $A = [a_{ij}] = \left[F_{(\mathrm{mod}(j-i,n))}\right]$ and $B = [b_{ij}] = \left[L_{(\mathrm{mod}(j-i,n))}\right]$ matrices (Solak, 2005).

Bhatia gave some examples on infinitely divisible matrices (Bhatia, 2006).

da Fonseca presented a method for solving a problem on some symmetric tridiagonal matrices (Fonseca, 2007).

Kocer gave some properties of the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers. Then, she defined the circulant, negacyclic and semicirculant matrices with these numbers and investigate the norms, eigenvalues and determinants of these matrices (Kocer, 2007).

Kocer obtained the spectral norm and eigenvalues of circulant matrices with Horadam numbers (Kocer et al., 2007).

Shen and Cen found upper and lower bounds for the spectral norms of r-circulant matrices which entries are Fibonacci numbers Lucas numbers (Shen and Cen, 2010).

Bhatia proved six different way that min matrix is positive semidefine (Bhatia, 2011).

Bahsi and Solak computed the norms of circulant matrices whose components are the hyper-Fibonacci numbers and hyper-Lucas numbers (Bahsi and Solak, 2014).

Petroudi and Pirouz studied on $n \times n$ ordered new matrix $A = \left[a^{\min(i,j)-1}\right]_{i,j=1}^{n}$, for $a \in \mathbb{R}^+$ constant. They found some linear algebraic properties of A and $A^{\circ -1}$ (Petroudi and Pirouz, 2015).

Tuglu and Kızılateş studied norms of circulant and r-circulant matrices involving harmonic Fibonacci and hyperharmonic Fibonacci numbers (Tuglu and Kızılateş, 2015a).

Tuglu and Kızılateş presented norms of some circulant matrices and some special matrices, which entries consist of harmonic Fibonacci numbers (Tuglu and Kızılateş, 2015b).

Tuglu et al. defined the harmonic and hyperharmonic Fibonacci numbers and obtained some algebraic identities like as harmonic and hyperharmonic numbers (Tuglu, Kızılateş and Kesim, 2015).

Bahsi and Solak derived relationships between generalized Fibonacci numbers and the characteristic polynomials of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ whose entries are $a_{ij} = k + \min(i,j) - 1$ and $b_{ij} = k + \max(i,j) - 1$, respectively, where k is a real number (Bahsi and Solak, 2015).

Radičić investigated the eigenvalues, the determinant, the Euclidean norm and bounds for the spectral norm of k-circulant matrix (for $0 \neq k \in \mathbb{C}$) with geometric sequence, where k is a nonzero complex number (Radičić, 2016).

Kızılateş and Tuglu defined a geometric circulant matrix whose entries are the generalized Fibonacci numbers and hyperharmonic Fibonacci numbers. Additionally, they gave upper and lower bounds for the spectral norms of these matrices (Kızılateş and Tuglu, 2016).

Radičić studied k-circulant matrices, for $0 \neq k \in \mathbb{C}$, with arithmetic sequence and investigate the eigenvalues, determinants and Euclidean norms of such matrices. Additionally, for $k = 1$, obtained the inverses of such (invertible) matrices and derived the Moore-Penrose inverses of such (singular) matrices (Radičić, 2017).

Kızılateş and Tuglu studied the spectral norms of the geometric circulant matrices and the symmetric geometric circulant matrices with the Tribonacci numbers (Kızılateş and Tuglu, 2018).

Solak and Bahsi examined some properties of the matrix $D(C_n)$, which results from applying the Ducci map to the rows of the circulant matrix C_n with first row $(c_0, c_1, \ldots, c_{n-1})$ (Solak and Bahsi, 2018).

Shi gave a new estimation for the norms of a r-circulant matrix involving the generalized k-Horadam numbers (Shi, 2018).

Kılıç and Arıkan defined matrices $\left[a_{\max(i,j)}\right]_{i,j\geq 1}$ and $\left[a_{\min(i,j)}\right]_{i,j\geq 1}$, and studied reciprocal analogues of sequence $\{a_n\}$ (Kılıç and Arıkan, 2019).

Ozgul and Bahsi computed determinants, inverses of L and its Hadamard inverse for min matrix $L = \left[L^r_{k+\min(i,j)-1}\right]_{i,j=1}^{n}$, where L^r_ndenotes the nth hyper-Lucas number of order r (Ozgul and Bahsi, 2020).

Shi studied the spectral norms of RFMLR-circulant matrices involving exponential forms and trigonometric functions (Shi, 2021).

Akbiyik et al. defined harmonic Pell numbers and studied on a new type symmetric matrix which includes this numbers, and found some inequalities by using matrix norms (Akbiyik et al., 2021).

Andrade et al. introduced circulant-like matrices and gave their inverse and other linear algebraic properties (Andrade, Carrasco-Olivera and Manzaneda, 2021).

CHAPTER 2

SOME BASIC DEFINITIONS

This part includes some basic definitions and properties related to our thesis.

2.1 BASIC DEFINITIONS AND PROPERTIES OF MATRICES

Definition 2.1 (*Hungerford, 1974*) *Matrices are literally defined as a rectangular area that has m rows and n columns formed by numbers from a ring R and called $m \times n$ matrix. A matrix is represented as follow:*

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In the case of $m = n$, it is called $n \times n$ square matrix. *An $n \times n$ matrix with $a_{ij} = 0$ for all $i \neq j$ is called a* diagonal matrix. *If R has an identity element and this element is a_{ii} entries for $n \times n$ diagonal matrix, it is called* identity matrix *and denoted by I_n. Transpose of an $m \times n$ matrix $A = [a_{ij}]$ is $n \times m$ matrix $A^T = [a_{ji}]$. Two $n \times m$ matrices $[a_{ij}]$ and $[b_{ij}]$ are equal if and only if $a_{ij} = b_{ij}$ for all $i, j = 1, 2, \ldots, n$.*

Definition 2.2 (*Hungerford, 1974*) *Let A, B be two $m \times n$ matrices. Then sum $A + B$ is equal to $C = [c_{ij}]$ matrix formed with entries $c_{ij} = a_{ij} + b_{ij}$ and obviously, C is $m \times n$ matrix too. Also, ordinary matrix product $AB = D = [d_{ij}]$ is found with*

$$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

equation, which A, B, D are matrices ordered $m \times n$, $n \times p$, $m \times p$, respectively. Note that the product of matrices is assosiative but it is not commutative in all case.

Also, matrices are useful for solving linear equation systems. A row of matrix is formed by coefficent of a linear equation. By using *elementary row (column) operations* defined as following on a matrix, a linear equation system is solving easily.

- *Type1.* Interchange of two rows.
- *Type2.* Multiplication of a row (column) by a nonzero scalar c.
- *Type3.* Addition of a nonzero scalar c multiple of one row (column) to another row (column).

Definition 2.3 (*Garcia and Horn, 2019*) *Let $M_n(\mathbb{C})$ be set of $n \times n$ matrices that their entries are complex number. The determinant function is* $\det : M_n(\mathbb{C}) \to \mathbb{C}$.

Determinant can be computed by using Laplace expansion.
Laplace expansion Determinant of an $n \times n$ matrix as a certain sum of determinants of $(n-1) \times (n-1)$ matrices. Let $\det [a_{11}] = a_{11}$, let $n \geq 2$, let $A \in M_n$, and let $A_{ij} \in M_{n-1}$ denote the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A. Then for any $i, j = \{1, 2, \ldots, n\}$, we have

$$\begin{aligned} \det A &= \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{ik} \\ &= \sum_{k=1}^{n} (-1)^{j+k} a_{kj} \det A_{kj}. \end{aligned}$$

Determinant of a matrix can be solved more easily thanks to elementary operations.

- The effect of one operation *Type1* on the determinant is to multiply it by -1;
- The effect of one operation *Type2* is to multiply it by the c;
- The effect of operation *Type3* does not change the determinant.

Definition 2.4 (*Hungerford, 1974*) *Let A_n be a square matrix. A_n is* nonsingular (invertible) *if and only if there is a square matrix B_n where $AB = BA = I_n$. B is called by* inverse of matrix A *and denoted by $B = A^{-1}$.*

Some Properties:

- $\det(AB) = (\det A)(\det B)$

- $\det A \neq 0$ if and only if A is invertible matrix.

- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

- If $a_{ij} = 0$ holds for the case of $j > i$ (or $i > j$), $\det A_n = \prod_{i=1}^{n} a_{ii}$.

  Then, in the view of such basic definitions and properties on matrix theory, various and important matrices are introduced to literature by mathematicians. Two of these are min and max matrices and they are defined as follows.

Definition 2.5 (*Bhatia, 2006*) *The min and max matrices are defined by*

$$\mathbf{M} = [\min(i,j)]_{i,j=1}^{n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}, \tag{2.1}$$

and

$$\mathbb{M} = [\max(i,j)]_{i,j=1}^{n} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{pmatrix}. \tag{2.2}$$

Neudecker et al. defined a more general matrix than the matrix $\mathbf{M}$ (Neudecker et al, 2009). Then, Mattila and Haukkanen studied some properties of following type of min and max matrices (Mattila and Haukkanen, 2016).

Definition 2.6 *Let $T = \{x_1, x_2, \ldots, x_n\}$ be a finite multiset of real numbers, where $x_1 \leq x_2 \leq \cdots \leq x_n$ (in some cases, assume that $x_1 < x_2 < \cdots < x_n$). The min matrix $(T)_{\min}$ of the set T has $\min(x_i, x_j)$ as its ij entry, whereas the max matrix of the set T*

has $\max(x_i, x_j)$ *as its* ij *entry and is denoted by* $(T)_{\max}$. *Both matrices are square and symmetric and they can be written as*

$$(T)_{\min} = \begin{pmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}, \tag{2.3}$$

and

$$(T)_{\max} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_2 & x_3 & \cdots & x_n \\ x_3 & x_3 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & x_n & \cdots & x_n \end{pmatrix}. \tag{2.4}$$

Definition 2.7 (*Shen and Cen, 2010*) *An* r*-circulant matrix with the first row* $(c_0, c_1, c_2, \ldots, c_{n-1})$ *is a square matrix as the following form*

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{pmatrix},$$

where r *is a nonzero complex number.*

If $r = 1$, then r-circulant matrices are called circulant matrix. If $r = -1$, then r-circulant matrices are called skew circulant matrix.

Definition 2.8 (*Horn, 1990*) *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be* $n \times n$ *matrices with complex entries.*

- *The Hadamard product of* A *and* B *is defined by*

  $$AoB = (a_{ij} b_{ij}). \tag{2.5}$$

- *The Hadamard inverse of the* A, *denoted by* $A^{\circ -1} = \left[a_{ij}^{-1}\right]$, *provided that* $a_{i,j} \neq 0$ *for all* $i, j = 1, 2, \ldots, n$.

Definition 2.9 (*Horn, 1990*) *The Euclidean norm of matrix A is*

$$\|A\|_E = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}. \tag{2.6}$$

Definition 2.10 (*Horn, 1990*) *The spectral norm of matrix A is*

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i(A^H A)}, \tag{2.7}$$

where $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$ and A^H is the conjugate transpose of matrix A.

Lemma 2.1 *(Horn and Johnson, 1991) Let $E = [e_{ij}]$ and $F = [f_{ij}]$ be two $m \times n$ matrices. Thus*

$$\|E \circ F\|_2 \leq r_1(E)\, c_1(F), \tag{2.8}$$

where

$$r_1(E) = \max_{1\leq i\leq m} \sqrt{\sum_{j=1}^{n} |e_{ij}|^2},$$

and

$$c_1(F) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{m} |f_{ij}|^2}.$$

2.2 FIBONACCI AND HYPERHARMONIC FIBONACCI NUMBERS

Now, we give two sequences used frequently in number theory, matrix theory, algebra and related areas.

Definition 2.11 (*Koshy, 2001*) *The Fibonacci numbers which are defined by the recurrence relation, for $n \geq 0$:*

$$F_{n+2} = F_{n+1} + F_n,$$

where $F_0 = 0$ and $F_1 = 1$.

Fibonacci numbers continue as $F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13, \ldots$.

Definition 2.12 (*Tuglu et al., 2015*) *Hyperharmonic Fibonacci numbers defined by Tuglu et al. for $n, \tau \geq 1$,*

$$\mathbb{F}_n^{(\tau)} = \sum_{k=1}^{n} \mathbb{F}_k^{(\tau-1)} \tag{2.9}$$

with $\mathbb{F}_n^{(0)} = \frac{1}{F_n}$ and $\mathbb{F}_0^{(\tau)} = 0$.

If we take $\tau = 1$ in equation (2.9), it gives as harmonic Fibonacci numbers and computed with following equation.

$$\mathbb{F}_n = \sum_{k=1}^{n} \frac{1}{F_k}.$$

Table 2.1 The first few hyperharmonic Fibonacci numbers for $\tau = 0, 1, \ldots, 5$.

n	2	3	4	5	6	7	8	9	10	11
$\mathbb{F}_n^{(0)}$	1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{13}$	$\frac{1}{21}$	$\frac{1}{34}$	$\frac{1}{55}$
$\mathbb{F}_n^{(1)}$	2	$\frac{5}{2}$	$\frac{17}{6}$	$\frac{91}{30}$	$\frac{379}{120}$	$\frac{5047}{1560}$	$\frac{35849}{10920}$	$\frac{614893}{185640}$	$\frac{6800951}{2042040}$	$\frac{607326679}{181741560}$
$\mathbb{F}_n^{(2)}$	3	$\frac{11}{2}$	$\frac{25}{3}$	$\frac{341}{30}$	$\frac{581}{40}$	$\frac{13853}{780}$	$\frac{76597}{3640}$	$\frac{226067}{9282}$	$\frac{56535691}{2042040}$	$\frac{939833863}{30290260}$
$\mathbb{F}_n^{(3)}$	4	$\frac{19}{2}$	$\frac{107}{6}$	$\frac{146}{5}$	$\frac{1749}{40}$	$\frac{95917}{1560}$	$\frac{90121}{1092}$	$\frac{661397}{6188}$	$\frac{274796701}{2042040}$	$\frac{30095909567}{181741560}$
$\mathbb{F}_n^{(4)}$	5	$\frac{29}{2}$	$\frac{97}{3}$	$\frac{923}{15}$	$\frac{12631}{120}$	$\frac{6503}{39}$	$\frac{90735}{364}$	$\frac{550973}{1547}$	$\frac{1002081061}{2042040}$	$\frac{29820280999}{45435390}$
$\mathbb{F}_n^{(5)}$	6	$\frac{41}{2}$	$\frac{317}{6}$	$\frac{3431}{30}$	$\frac{1757}{8}$	$\frac{120547}{312}$	$\frac{1388239}{2184}$	$\frac{36823415}{37128}$	$\frac{504561481}{340340}$	$\frac{5553099355}{2596308}$

CHAPTER 3

r-MIN AND r-MAX MATRICES AND THEIR PROPERTIES

In this section the thesis, we introduce two new matrices that we defined. These matrices are generalizations of the min and max matrices, and we call them r-min and r-max matrix, respectively. Next, we give our results for the basic linear algebraic properties, such as the determinant, inverse, Euclidean norm, spectral norm and factorization, of both r-min, r-max matrices, and their Hadamard inverses.

Finally, we state that the results of this section are quoted from the article published by us. (Kızılateş and Terzioğlu, 2022).

Definition 3.1 *Let r be a complex number and $x_1 < x_2 < \cdots < x_n$. Then r-min matrix $\mathbf{P}_{r,n} = [p_{ij}]_{i,j=1}^{n}$ and r-max matrix $\mathbf{Q}_{r,n} = [q_{ij}]_{i,j=1}^{n}$ are as follows:*

$$p_{ij} = \begin{cases} rx_{\min\{i,j\}} & i > j \\ x_{\min\{i,j\}} & i \leq j \end{cases}, \tag{3.1}$$

$$q_{ij} = \begin{cases} rx_{\max\{i,j\}} & i > j \\ x_{\max\{i,j\}} & i \leq j \end{cases}. \tag{3.2}$$

That is, these matrices are presented as below:

$$\mathbf{P}_{r,n} = \begin{pmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ rx_1 & x_2 & x_2 & \cdots & x_2 \\ rx_1 & rx_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rx_1 & rx_2 & rx_3 & \cdots & x_n \end{pmatrix}, \tag{3.3}$$

and

$$\mathbf{Q}_{r,n} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ rx_2 & x_2 & x_3 & \cdots & x_n \\ rx_3 & rx_3 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rx_n & rx_n & rx_n & \cdots & x_n \end{pmatrix}. \tag{3.4}$$

Note that if we take $r = 1$ in (3.3) and (3.4), we get the $(T)_{\min}$ and $(T)_{\max}$ matrices in the (2.3) and (2.4), respectively.

We now give the determinants and inverses for the $\mathbf{P}_{r,n}$ and $\mathbf{Q}_{r,n}$ matrices. Additionally, we found Euclidean norms and upper bounds for the spectral norms of these matrices.

Theorem 3.1 *The determinant for matrix $\mathbf{P}_{r,n}$ in (3.3) as follows*

$$\det(\mathbf{P}_{r,n}) = x_1 \prod_{i=1}^{n-1} (x_{i+1} - rx_i).$$

Proof. For $1 \leq j \leq n-1$, if we apply in order $-C_j + C_{j+1} \to C_{j+1}$ elementary column operators into $\mathbf{P}_{r,n}$, starting $(n-1)$th column, we find that

$$\det(\mathbf{P}_{r,n}) = \det \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ rx_1 & x_2 - rx_1 & 0 & \cdots & 0 & 0 \\ rx_1 & r(x_2 - x_1) & x_3 - rx_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rx_1 & r(x_2 - x_1) & r(x_3 - x_2) & \cdots & x_{n-1} - rx_{n-2} & 0 \\ rx_1 & r(x_2 - x_1) & r(x_3 - x_2) & \cdots & r(x_{n-1} - x_{n-2}) & x_n - rx_{n-1} \end{pmatrix}.$$

Hence, we get

$$\det(\mathbf{P}_{r,n}) = x_1 \prod_{i=1}^{n-1} (x_{i+1} - rx_i).$$

■

Note that in there, let (x_i) be an increasing sequence for $i = 1, 2, \ldots, n-1$. We can obtain that $\det(\mathbf{P}_{r,n}) = 0$ in case $x_1 = 0$ or $r = \frac{x_{i+1}}{x_i}$. For instance, when $r = 3$ if $x_{11} = 7$ and $x_{12} = 21$, $\det(\mathbf{P}_{r,n}) = 0$ is achieved. This indicates that matrix $\mathbf{P}_{r,n}$ may be singular.

Theorem 3.2 *Suppose that $\mathbf{P}_{r,n}$ is a matrix as in (3.3) and*

$$\mathbf{P}_{r,n} = \begin{pmatrix} \mathbf{P}_{r,n-1} & E_{(n-1)\times 1} \\ rE^T_{1\times(n-1)} & x_n \end{pmatrix}$$

with $E_{(n-1)\times 1} = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{pmatrix}^T$ *(shortly, denoted by* E*) is a column matrix. If* $\mathbf{P}_{r,n}$ *is nonsingular matrix, then the inverse of* $\mathbf{P}_{r,n}$ *is*

$$\mathbf{P}_{r,n}^{-1} = \begin{pmatrix} \mathbf{P}_{r,n-1}^{-1} + ru_n\mathbf{P}_{r,n-1}^{-1}EE^T\mathbf{P}_{r,n-1}^{-1} & -u_n\mathbf{P}_{r,n-1}^{-1}E \\ \\ -ru_nE^T\mathbf{P}_{r,n-1}^{-1} & u_n \end{pmatrix}, \tag{3.5}$$

where

$$u_n = \frac{1}{x_n - rx_{n-1}}. \tag{3.6}$$

Proof. We can prove the theorem by using mathematical induction method on n. For $n = 2$, we obtain

$$\begin{aligned} \mathbf{P}_{r,2}^{-1} &= \frac{1}{\det(\mathbf{P}_{r,2})}\begin{pmatrix} x_2 & -x_1 \\ -rx_1 & x_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_2}{x_1(x_2-rx_1)} & -\frac{1}{x_2-rx_1} \\ -\frac{r}{x_2-rx_1} & \frac{1}{x_2-rx_1} \end{pmatrix}. \end{aligned} \tag{3.7}$$

Besides, it provides in our assertion of Equality (3.5) for $n = 2$:

$$\begin{aligned} \begin{pmatrix} \mathbf{P}_{r,1}^{-1} + ru_2\mathbf{P}_{r,1}^{-1}EE^T\mathbf{P}_{r,1}^{-1} & -u_2\mathbf{P}_{r,1}^{-1}E \\ \\ -ru_2E^T\mathbf{P}_{r,1}^{-1} & u_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{x_1} + r\frac{1}{-rx_1+x_2}\frac{1}{x_1}x_1x_1\frac{1}{x_1} & -\frac{1}{-rx_1+x_2}\frac{1}{x_1}x_1 \\ \\ -\frac{1}{x_2-rx_1}rx_1\frac{1}{x_1} & \frac{1}{x_2-rx_1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{x_1} + \frac{r}{-rx_1+x_2} & -\frac{1}{-rx_1+x_2} \\ \\ -\frac{r}{x_2-rx_1} & \frac{1}{x_2-rx_1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_2}{x_1(x_2-rx_1)} & -\frac{1}{x_2-rx_1} \\ \\ -\frac{r}{x_2-rx_1} & \frac{1}{x_2-rx_1} \end{pmatrix}. \end{aligned} \tag{3.8}$$

Thus, Equality (3.5) holds by looking at Equalities (3.7) and (3.8). Assume that our assertion is true for $n = t - 1$.

By virtue of $\mathbf{P}_{r,t-1}^{-1}\mathbf{P}_{r,t-1} = I_{(t-1)\times(t-1)}$, we get

$$(\mathbf{P}_{r,t-1})^{-1}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{3.9}$$

For $n = t$, by multiping $\mathbf{P}_{r,t}^{-1}$ and $\mathbf{P}_{r,t}$, we have

$$
\begin{pmatrix} \mathbf{P}_{r,t-1}^{-1} + ru_t\mathbf{P}_{r,t-1}^{-1}EE^T\mathbf{P}_{r,t-1}^{-1} & -u_t\mathbf{P}_{r,t-1}^{-1}E \\ \\ -ru_tE^T\mathbf{P}_{r,t-1}^{-1} & u_t \end{pmatrix} \begin{pmatrix} \mathbf{P}_{r,t-1} & E \\ rE^T & x_t \end{pmatrix}
$$

$$
= \begin{pmatrix} \begin{matrix} (\mathbf{P}_{r,t-1}^{-1} + ru_t\mathbf{P}_{r,t-1}^{-1}EE^T\mathbf{P}_{r,t-1}^{-1})\mathbf{P}_{r,t-1} \\ -ru_t\mathbf{P}_{r,t-1}^{-1}EE^T \end{matrix} & \begin{matrix} (\mathbf{P}_{r,t-1}^{-1} + u_t\mathbf{P}_{r,t-1}^{-1}ErE^T\mathbf{P}_{r,t-1}^{-1})E \\ -u_t\mathbf{P}_{r,t-1}^{-1}Ex_t \end{matrix} \\ \\ -u_trE^T\mathbf{P}_{r,t-1}^{-1}\mathbf{P}_{r,t-1} + u_trE^T & -u_trE^T\mathbf{P}_{r,t-1}^{-1}E + u_tx_t \end{pmatrix}
$$

$$
= \begin{pmatrix} I_{(t-1)\times(t-1)} & \begin{matrix} \mathbf{P}_{r,t-1}^{-1}E + ru_t\mathbf{P}_{r,t-1}^{-1}EE^T\mathbf{P}_{r,t-1}^{-1}E \\ -x_tu_t\mathbf{P}_{r,t-1}^{-1}E \end{matrix} \\ \\ (0)_{1\times(t-1)} & -u_t(rE^T\mathbf{P}_{r,t-1}^{-1}E + x_t) \end{pmatrix}.
$$

Then, from (3.9), we have

$$
= \begin{pmatrix} I_{(t-1)\times(t-1)} & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + ru_t \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_{t-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} - x_tu_t \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ \\ (0)_{1\times(t-1)} & -u_t \left\{ r \begin{pmatrix} x_1 & x_2 & \cdots & x_{t-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + x_t \right\} \end{pmatrix}
$$

$$
= \begin{pmatrix} I_{(t-1)\times(t-1)} & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \left\{ 1 + ru_t \begin{pmatrix} x_1 & x_2 & \cdots & x_{t-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} - x_tu_t \right\} \\ \\ (0)_{1\times(t-1)} & -u_t(rx_{t-1} + x_t) \end{pmatrix}
$$

$$
\begin{aligned}
&= \begin{pmatrix} I_{(t-1)\times(t-1)} & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \{1 + r u_t x_{t-1} - x_t u_t\} \\ & \\ (0)_{1\times(t-1)} & -u_t(r x_{t-1} + x_t) \end{pmatrix} \\
&= \begin{pmatrix} I_{(t-1)\times(t-1)} & (0)_{(t-1)\times 1} \\ & \\ (0)_{1\times(t-1)} & 1 \end{pmatrix} \\
&= I_{t\times t}.
\end{aligned}
$$

On the other hand $\mathbf{P}_{r,t}\mathbf{P}_{r,t}^{-1}$ can be similarly obtained. So the proof is completed. ■

Theorem 3.3 *Assume that $\mathbf{P}_{r,n}$ is a matrix as in (3.3). The Euclidean norm of $\mathbf{P}_{r,n}$ is*

$$
\|\mathbf{P}_{r,n}\|_E = \left[\sum_{s=1}^{n} \left((n-s)\left(|r|^2+1\right)+1\right) x_s^2\right]^{\frac{1}{2}}.
$$

Proof. By using the definition of Euclidean norm, we obtain

$$
\begin{aligned}
\|\mathbf{P}_{r,n}\|_E &= \left[\sum_{s=1}^{n}(n-s+1)x_s^2 + \sum_{s=1}^{n}(n-s)\,|r|^2\, x_s^2\right]^{\frac{1}{2}} \\
&= \left[\sum_{s=1}^{n} \left((n-s)\left(|r|^2+1\right)+1\right) x_s^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

■

Theorem 3.4 *Let $\mathbf{P}_{r,n}$ be a matrix as in the matrix form (3.3). We have*

i) If $|r| \geq 1$, then

$$
\|\mathbf{P}_{r,n}\|_2 \leq \sqrt{\left((n-1)\,|r|^2+1\right)\sum_{s=1}^{n} x_s^2}.
$$

ii) If $|r| < 1$, then

$$
\|\mathbf{P}_{r,n}\|_2 \leq \sqrt{n\sum_{s=1}^{n} x_s^2}.
$$

Proof. *i*) Let $\mathbf{A} = [a_{ij}]_{i,j=1}^{n}$ and $\mathbf{B} = [b_{ij}]_{i,j=1}^{n}$ be the following matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \dots & r & 1 \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

Then we obtain

$$r_1(\mathbf{A}) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{nj}|^2} = \sqrt{\left((n-1)\,|r|^2 + 1\right)},$$

and

$$c_1(\mathbf{B}) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{in}|^2} = \sqrt{\sum_{s=1}^{n} x_s^2}.$$

Since $\mathbf{P}_{r,n} = \mathbf{A} \circ \mathbf{B}$, based on (2.8), we have

$$\|\mathbf{P}_{r,n}\|_2 \le \sqrt{\left((n-1)\,|r|^2 + 1\right) \sum_{s=1}^{n} x_s^2}.$$

ii) For $|r| < 1$, we shall obtain the upper bound for the spectral norm of $\mathbf{P}_{r,n}$. Since $\mathbf{P}_{r,n} = \mathbf{A} \circ \mathbf{B}$, then we obtain

$$r_1(\mathbf{A}) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{n},$$

and

$$c_1(\mathbf{B}) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{s=1}^{n} x_s^2}.$$

Therefore, by virtue of (2.8), we have

$$\|\mathbf{P}_{r,n}\|_2 \le \sqrt{n \sum_{s=1}^{n} x_s^2}.$$

■

Lemma 3.5 *Let $\mathbf{P}_{r,n}$ be the matrix defined as in (3.3). Then, for $n \geq 3$, $r \neq 0$ and $\boldsymbol{K} = [\mathbf{k}_{ij}]_{i,j=1}^{n}$, $\boldsymbol{L} = [\mathbf{l}_{ij}]_{i,j=1}^{n}$, $\boldsymbol{S} = [\mathbf{s}_{ij}]_{i,j=1}^{n}$ the factorization of $\mathbf{P}_{r,n}$ is as follows:*

$$\mathbf{P}_{r,n} = \boldsymbol{K}.\boldsymbol{L}.\boldsymbol{S},$$

where

$$\mathbf{k}_{ij} = \begin{cases} 0 & , j > i \\ r^2 & , i > 1 \text{ and } j = 1 \\ r & , \text{otherwise} \end{cases},$$

$$\mathbf{l}_{ij} = \begin{cases} x_1 & , i = j = 1 \\ x_i - r x_{i-1} & , i = j \geq 2 \\ (r-1)x_j & , i \geq 2 \text{ and } i - j \geq 1 \\ 0 & , \text{otherwise} \end{cases},$$

$$\mathbf{s}_{ij} = \begin{cases} \frac{1}{r} & , j \geq i \\ 0 & , \text{otherwise} \end{cases}.$$

Similarly, we can give the factorization of $\mathbf{P}_{r,n}$ as

$$\mathbf{P}_{r,n} = \begin{pmatrix} r & 0 & 0 & \cdots & 0 \\ r^2 & r & 0 & \cdots & 0 \\ r^2 & r & r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^2 & r & r & \cdots & r \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 - r x_1 & 0 & \cdots & 0 \\ 0 & (r-1)x_2 & x_3 - r x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n - r x_{n-1} \end{pmatrix} \begin{pmatrix} \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & \cdots & \frac{1}{r} \\ 0 & \frac{1}{r} & \frac{1}{r} & \cdots & \frac{1}{r} \\ 0 & 0 & \frac{1}{r} & \cdots & \frac{1}{r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{r} \end{pmatrix}.$$

Theorem 3.6 *The determinant of the matrix $\mathbf{Q}_{r,n}$ is*

$$\det(\mathbf{Q}_{r,n}) = x_n \prod_{i=1}^{n-1} (x_i - r x_{i+1}).$$

Proof. For $2 \leq i \leq n$, if we apply some elementary row operators to $\mathbf{Q}_{r,n}$, we find that

$$\det(\mathbf{Q}_{r,n}) = \det \begin{pmatrix} x_1 - r x_2 & 0 & 0 & \cdots & 0 & 0 \\ r(x_2 - x_3) & x_2 - r x_3 & 0 & \cdots & 0 & 0 \\ r(x_3 - x_4) & r(x_3 - x_4) & x_3 - r x_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r(x_{n-1} - x_n) & r(x_{n-1} - x_n) & r(x_{n-1} - x_n) & \cdots & x_{n-1} - r x_n & 0 \\ r x_n & r x_n & r x_n & \cdots & r x_n & x_n \end{pmatrix}.$$

Hence, we obtain

$$\det(\mathbf{Q}_{r,n}) = x_n \prod_{i=1}^{n-1} (x_i - r x_{i+1}).$$

■

Note that in there, let (x_i) be an increasing sequence for $i = 1, 2, \ldots, n-1$. We can obtain that $\det(\mathbf{Q}_{r,n}) = 0$ in case $x_1 = 0$ or $r = \frac{x_i}{x_{i+1}}$. For instance, when $r = \frac{1}{5}$, if $x_6 = 15$ and $x_7 = 75$, $\det(\mathbf{Q}_{r,n}) = 0$ is achieved. This indicates that matrix $\mathbf{Q}_{r,n}$ may be singular.

Theorem 3.7 *Suppose that $\mathbf{Q}_{r,n}$ is a matrix as in (3.4) and*

$$\mathbf{Q}_{r,n} = \begin{pmatrix} \mathbf{Q}_{r,n-1} & F_{(n-1)\times 1} \\ rF^T_{1\times(n-1)} & x_n \end{pmatrix}$$

with $F_{(n-1)\times 1} = \begin{pmatrix} x_n & x_n & \cdots & x_n \end{pmatrix}^T$ (shortly, denoted by F) is a column matrix. If $\mathbf{Q}_{r,n}$ is nonsingular matrix, then the inverse of $\mathbf{Q}_{r,n}$ is

$$\mathbf{Q}^{-1}_{r,n} = \begin{pmatrix} \mathbf{Q}^{-1}_{r,n-1} + v_n r \mathbf{Q}^{-1}_{r,n-1} F F^T \mathbf{Q}^{-1}_{r,n-1} & -v_n \mathbf{Q}^{-1}_{r,n-1} F \\ \\ -v_n r F^T \mathbf{Q}^{-1}_{r,n-1} & v_n \end{pmatrix}, \tag{3.10}$$

where

$$v_n = \frac{x_{n-1}}{x_n(x_{n-1} - r x_n)}. \tag{3.11}$$

Proof. For the proof, we use induction method on n. For $n = 2$, we obtain

$$\begin{aligned} \mathbf{Q}^{-1}_{r,2} &= \frac{1}{\det(\mathbf{Q}_{r,2})} \begin{pmatrix} x_2 & -x_2 \\ -r x_2 & x_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{x_1 - r x_2} & -\frac{1}{x_1 - r x_2} \\ -\frac{r}{x_1 - r x_2} & \frac{x_1}{x_2(x_1 - r x_2)} \end{pmatrix}. \end{aligned} \tag{3.12}$$

On the other hand, for $n = 2$ in our assertion of Equality (3.10), we have

$$\begin{pmatrix} \mathbf{Q}^{-1}_{r,1} + r v_2 \mathbf{Q}^{-1}_{r,1} F F^T \mathbf{Q}^{-1}_{r,1} & -v_2 \mathbf{Q}^{-1}_{r,1} F \\ \\ -v_2 r F^T \mathbf{Q}^{-1}_{r,1} & v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{x_1} + r \frac{x_1}{x_2(x_1 - r x_2)} \frac{1}{x_1} x_2 x_2 \frac{1}{x_1} & -\frac{x_1}{x_2(x_1 - r x_2)} \frac{1}{x_1} x_2 \\ \\ -\frac{x_1}{x_2(x_1 - r x_2)} r x_2 \frac{1}{x_1} & \frac{x_1}{x_2(x_1 - r x_2)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{x_1} + \frac{rx_2}{x_1(x_1-rx_2)} & -\frac{1}{x_1-rx_2} \\ \\ -\frac{r}{x_1-rx_2} & \frac{x_1}{x_2(x_1-rx_2)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{x_1-rx_2} & -\frac{1}{x_1-rx_2} \\ \\ -\frac{r}{x_1-rx_2} & \frac{x_1}{x_2(x_1-rx_2)} \end{pmatrix}. \tag{3.13}$$

Thus, from Equalities (3.12) and (3.13), Equality (3.10) holds. Assume that our assertion is true for $n = t-1$. Then, since $\mathbf{Q}_{r,t-1}^{-1}\mathbf{Q}_{r,t-1} = I_{(t-1)\times(t-1)}$, we get

$$\mathbf{Q}_{r,t-1}^{-1} \begin{pmatrix} x_{t-1} \\ x_{t-1} \\ \vdots \\ x_{t-1} \end{pmatrix}_{(t-1)\times 1} = \mathbf{Q}_{r,t-1}^{-1} x_{t-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(t-1)\times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{(t-1)\times 1}. \tag{3.14}$$

If we multipy with $\frac{x_t}{x_{t-1}}$ both sides of Equation (3.14), we have

$$\mathbf{Q}_{r,t-1}^{-1} x_t \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(t-1)\times 1} = \frac{x_t}{x_{t-1}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{(t-1)\times 1}. \tag{3.15}$$

For $n = t$, we get

$$\begin{aligned}
(\mathbf{Q}_{r,t}^{-1})(\mathbf{Q}_{r,t}) &= \begin{pmatrix} \mathbf{Q}_{r,t-1}^{-1} + v_t r \mathbf{Q}_{r,t-1}^{-1} F F^T \mathbf{Q}_{r,t-1}^{-1} & -v_t \mathbf{Q}_{r,t-1}^{-1} F \\ \\ -v_t r F^T \mathbf{Q}_{r,t-1}^{-1} & v_t \end{pmatrix} \begin{pmatrix} \mathbf{Q}_{r,t-1} & F \\ \\ rF^T & x_t \end{pmatrix} \\
&= \begin{pmatrix} \begin{matrix}(\mathbf{Q}_{r,t-1}^{-1} + r v_t \mathbf{Q}_{r,t-1}^{-1} F F^T \mathbf{Q}_{r,t-1}^{-1})\mathbf{Q}_{r,t-1} \\ -v_t r \mathbf{Q}_{r,t-1}^{-1} F F^T\end{matrix} & \begin{matrix}(\mathbf{Q}_{r,t-1}^{-1} + r v_t \mathbf{Q}_{r,t-1}^{-1} F F^T \mathbf{Q}_{r,t-1}^{-1})F \\ -v_t \mathbf{Q}_{r,t-1}^{-1} F x_t\end{matrix} \\ \\ -v_t r F^T \mathbf{Q}_{r,t-1}^{-1} \mathbf{Q}_{r,t-1} + v_t r F^T & -v_t r F^T \mathbf{Q}_{r,t-1}^{-1} F + v_t x_t \end{pmatrix} \\
&= \begin{pmatrix} I_{(t-1)\times(t-1)} & \mathbf{Q}_{r,t-1}^{-1} F + v_t r \mathbf{Q}_{r,t-1}^{-1} F F^T \mathbf{Q}_{r,t-1}^{-1} F - v_t x_t \mathbf{Q}_{r,t-1}^{-1} F \\ \\ (0)_{1\times(t-1)} & -v_t (r F^T \mathbf{Q}_{r,t-1}^{-1} F - x_t) \end{pmatrix}.
\end{aligned}$$

From (3.15) and (3.11), we obtain

$$= \begin{pmatrix} I_{(t-1)} & \frac{x_t}{x_{t-1}}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} + \frac{v_t r x_t^2}{x_{t-1}^2}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\begin{pmatrix} x_t & x_t & \cdots & x_t\end{pmatrix}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} - \frac{v_t x_t^2}{x_{t-1}}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \\ (0)_{1\times(t-1)} & -v_t\left\{ r\frac{x_t}{x_{t-1}}\begin{pmatrix} x_t & x_t & \cdots & x_t\end{pmatrix}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} - x_t\right\} \end{pmatrix}$$

$$= \begin{pmatrix} I_{(t-1)\times(t-1)} & \frac{x_t}{x_{t-1}}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\left\{1 + v_t r\frac{x_t}{x_{t-1}}\begin{pmatrix} x_t & x_t & \cdots & x_t\end{pmatrix}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} - v_t x_t\right\} \\ (0)_{1\times(t-1)} & -v_t(r\frac{x_t^2}{x_{t-1}} - x_t) \end{pmatrix}$$

$$= \begin{pmatrix} I_{(t-1)\times(t-1)} & \frac{x_t}{x_{t-1}}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}(1 + v_t(r\frac{x_t^2}{x_{t-1}} - x_t)) \\ (0)_{1\times(t-1)} & -v_t\frac{x_t(rx_t - x_{t-1})}{x_{t-1}} \end{pmatrix}$$

$$= \begin{pmatrix} I_{(t-1)\times(t-1)} & \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}(1 + v_t\frac{x_t(rx_t - x_{t-1})}{x_{t-1}}) \\ (0)_{1\times(t-1)} & -v_t\frac{x_t(rx_t - x_{t-1})}{x_{t-1}} \end{pmatrix}$$

$$= \begin{pmatrix} I_{(t-1)\times(t-1)} & (0)_{(t-1)\times 1} \\ (0)_{1\times(t-1)} & 1 \end{pmatrix}$$

$$= I_{t\times t}.$$

On the other hands $\mathbf{Q}_{r,t}\mathbf{Q}_{r,t}^{-1}$ can be similarly obtained. So the proof is completed. ■

Theorem 3.8 *Assume that* $\mathbf{Q}_{r,n}$ *is a matrix as in (3.4). The Euclidean norm of* $\mathbf{Q}_{r,n}$ *is*

$$\|\mathbf{Q}_{r,n}\|_E = \left[\sum_{s=1}^{n}\left(s\left(|r|^2+1\right) - |r|^2\right)x_s^2\right]^{\frac{1}{2}}.$$

Proof. From the definition of Euclidean norm we have,

$$\begin{aligned}\|\mathbf{Q}_{r,n}\|_E &= \left[\sum_{s=1}^{n} s x_s^2 + \sum_{s=1}^{n} (s-1)\,|r|^2\, x_s^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{s=1}^{n} \left(s\left(|r|^2+1\right) - |r|^2\right) x_s^2\right]^{\frac{1}{2}}.\end{aligned}$$

■

Theorem 3.9 *Let* $\mathbf{Q}_{r,n}$ *be a matrix as in the matrix form (3.4). We have*

i) If $|r| \geq 1$, *then*

$\|\mathbf{Q}_{r,n}\|_2 \leq \sqrt{n\left((n-1)|r|^2+1\right)x_n^2}.$

ii) If $|r| < 1$, *then*

$\|\mathbf{Q}_{r,n}\|_2 \leq n|x_n|.$

Proof. *i)* Let $\mathbf{C} = [c_{ij}]_{i,j=1}^{n}$ and $\mathbf{D} = [d_{ij}]_{i,j=1}^{n}$ be the following matrices

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ r & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \dots & 1 \end{pmatrix},$$

and

$$\mathbf{D} = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ x_2 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n & x_n & x_n & \dots & x_n & x_n \end{pmatrix}.$$

Then we obtain

$$r_1(\mathbf{C}) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{(n-1)|r|^2+1},$$

and

$$c_1(\mathbf{D}) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |d_{in}|^2} = \sqrt{n x_n^2}.$$

Since $\mathbf{Q}_{r,n} = \mathbf{C} \circ \mathbf{D}$, based on (2.8), we have

$$\|\mathbf{Q}_{r,n}\|_2 \leq \sqrt{n\left((n-1)|r|^2+1\right)x_n^2}.$$

ii) For $|r| < 1$, we shall obtain the upper bound for the spectral norm of $\mathbf{Q}_{r,n}$. We obtain

$$r_1(\mathbf{C}) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{n},$$

and

$$c_1(\mathbf{D}) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |d_{ij}|^2} = \sqrt{nx_n^2}.$$

Therefore, from (2.8), we have

$$\|\mathbf{Q}_{r,n}\|_2 \leq n|x_n|.$$

■

Lemma 3.10 *Let $\mathbf{Q}_{r,n}$ be the matrix defined as in (3.4). Then, for $n \geq 3$, $r \neq 0$ and $\mathbb{K} = [k_{ij}]_{i,j=1}^{n}$, $\mathbb{L} = [l_{ij}]_{i,j=1}^{n}$, $\mathbb{S} = [s_{ij}]_{i,j=1}^{n}$, the factorization of $\mathbf{Q}_{r,n}$ is as follows:*

$$\mathbf{Q}_{r,n} = \mathbb{K}.\mathbb{L}.\mathbb{S},$$

where

$$\mathbb{K} = [k_{ij}]_{i,j=1}^{n} = \begin{cases} r & , j \geq i \\ 0 & , \textit{otherwise} \end{cases},$$

$$\mathbb{L} = [l_{ij}]_{i,j=1}^{n} = \begin{cases} x_n & , i = j = n \\ x_i - rx_{i+1} & , n > i = j \geq 1 \\ (r-1)x_i & , i - j = 1 \\ 0 & , \textit{otherwise} \end{cases},$$

$$\mathbb{S} = [s_{ij}]_{i,j=1}^{n} = \begin{cases} \frac{1}{r} & , i \geq j \\ 0 & , \textit{otherwise} \end{cases}.$$

In other words, we can give the factorization of $\mathbf{Q}_{r,n}$ as follows:

$$\mathbf{Q}_{r,n} = \begin{pmatrix} r & r & r & \cdots & r \\ 0 & r & r & \cdots & r \\ 0 & 0 & r & \cdots & r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r \end{pmatrix} \begin{pmatrix} x_1 - rx_2 & 0 & 0 & \cdots & 0 \\ (r-1)x_2 & x_2 - rx_3 & 0 & \cdots & 0 \\ 0 & (r-1)x_3 & x_3 - rx_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \frac{1}{r} & 0 & 0 & \cdots & 0 \\ \frac{1}{r} & \frac{1}{r} & 0 & \cdots & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & \cdots & \frac{1}{r} \end{pmatrix}.$$

Theorem 3.11 *Let $\mathbf{P}_{r,n}$ is a matrix in (3.3), the determinant of $\mathbf{P}_{r,n}^{\circ -1}$ is as follows*

$$\det(\mathbf{P}_{r,n}^{\circ -1}) = \frac{1}{x_1} \prod_{i=1}^{n-1} \left(\frac{1}{x_{i+1}} - \frac{1}{rx_i} \right).$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. ■

Note that in there, let (x_i) be an increasing sequence for $i = 1, 2, \ldots, n-1$. We can obtain that $\det(\mathbf{P}_{r,n}^{\circ -1}) = 0$ in same conditions as the $\det(\mathbf{P}_{r,n}) = 0$. This indicates that matrix $\mathbf{P}_{r,n}^{\circ -1}$ may be singular.

Theorem 3.12 *Assume that $\mathbf{P}_{r,n}$ is a matrix in (3.3),*

$$\mathbf{P}_{r,n}^{\circ -1} = \begin{pmatrix} \mathbf{P}_{r,n-1}^{\circ -1} & \mathbf{G}_{(n-1)\times 1} \\ & \\ r\mathbf{G}_{1\times(n-1)}^{T} & \frac{1}{x_n} \end{pmatrix},$$

with $\mathbf{G}_{(n-1)\times 1} = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_{n-1}} \end{pmatrix}^T$ (shortly, denoted by $\mathbf{G}$) is a column matrix. If $\mathbf{P}_{r,n}^{\circ -1}$ is nonsingular matrix, then the inverse of $\mathbf{P}_{r,n}^{\circ -1}$ is

$$\left(\mathbf{P}_{r,n}^{\circ -1}\right)^{-1} = \begin{pmatrix} \left(\mathbf{P}_{r,n-1}^{\circ -1}\right)^{-1} + z_n r \left(\mathbf{P}_{r,n-1}^{\circ -1}\right)^{-1} \mathbf{G}\mathbf{G}^T \left(\mathbf{P}_{r,n-1}^{\circ -1}\right)^{-1} & -z_n \left(\mathbf{P}_{r,n-1}^{\circ -1}\right)^{-1} \mathbf{G} \\ & \\ -z_n r \mathbf{G}^T \left(\mathbf{P}_{r,n-1}^{\circ -1}\right)^{-1} & z_n \end{pmatrix},$$

where

$$z_n = \frac{rx_n x_{n-1}}{rx_{n-1} - x_n}.$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. ■

Theorem 3.13 *Assume that $\mathbf{P}_{r,n}$ is a matrix as in (3.3). The Euclidean norm of $\mathbf{P}_{r,n}^{\circ -1}$ is*

$$\left\|\mathbf{P}_{r,n}^{\circ -1}\right\|_E = \left[\sum_{s=1}^{n} \left((n-s) \left(\frac{1}{|r|^2} + 1 \right) + 1 \right) \frac{1}{x_s^2} \right]^{\frac{1}{2}}$$

Proof. By virtue of (2.6), we obtain

$$
\begin{aligned}
\left\|\mathbf{P}_{r,n}^{\circ-1}\right\|_E &= \left[\sum_{s=1}^{n}(n-s+1)\frac{1}{x_s^2}+\sum_{s=1}^{n}(n-s)\frac{1}{|r|^2 x_s^2}\right]^{\frac{1}{2}} \\
&= \left[\sum_{s=1}^{n}\left((n-s)\left(\frac{1}{|r|^2}+1\right)+1\right)\frac{1}{x_s^2}\right]^{\frac{1}{2}}.
\end{aligned}
$$

■

Theorem 3.14 *Let $\mathbf{P}_{r,n}$ be a matrix as in the matrix form (3.3). We have*

i) If $|r|\geq 1$, then

$$\left\|\mathbf{P}_{r,n}^{\circ-1}\right\|_2 \leq \frac{n}{|x_1|}.$$

ii) If $|r|<1$, then

$$\left\|\mathbf{P}_{r,n}^{\circ-1}\right\|_2 \leq \frac{\sqrt{n}}{|r|\,|x_1|}\sqrt{n+|r|^2-1}.$$

Proof. *i)* Let $\mathbf{E}=[e_{ij}]_{i,j=1}^{n}$ and $\mathbf{F}=[f_{ij}]_{i,j=1}^{n}$ be the following $n\times n$ matrices

$$
\mathbf{E}=\begin{pmatrix}
1 & 1 & 1 & \dots & 1 & 1 \\
\frac{1}{r} & 1 & 1 & \dots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{r} & \frac{1}{r} & \frac{1}{r} & \dots & \frac{1}{r} & 1
\end{pmatrix},
$$

and

$$
\mathbf{F}=\begin{pmatrix}
\frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} & \cdots & \frac{1}{x_1} \\
\frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_2} & \cdots & \frac{1}{x_2} \\
\frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_n}
\end{pmatrix}.
$$

Then, we obtain

$$r_1(\mathbf{E})=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|e_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|e_{1j}|^2}=\sqrt{n},$$

and

$$c_1(\mathbf{F})=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|f_{ij}|^2}=\sqrt{\sum_{i=1}^{n}|f_{i1}|^2}=\sqrt{\sum_{s=1}^{n}\frac{1}{x_1^2}}=\frac{1}{|x_1|}\sqrt{n}.$$

Since $\mathbf{P}_{r,n}^{\circ -1} = \mathbf{E} \circ \mathbf{F}$, based on (2.8), we have

$$\left\|\mathbf{P}_{r,n}^{\circ -1}\right\|_2 \leq \frac{n}{|x_1|}$$

ii) For $|r| < 1$, we shall obtain the upper bound for the spectral norm of $\mathbf{P}_{r,n}^{\circ -1}$. Then we obtain

$$r_1(\mathbf{E}) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |e_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |e_{nj}|^2} = \sqrt{\frac{n-1}{|r|^2} + 1},$$

and

$$c_1(\mathbf{F}) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |f_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |f_{i1}|^2} = \sqrt{\sum_{s=1}^{n} \frac{1}{x_1^2}} = \frac{1}{|x_1|}\sqrt{n}.$$

Therefore, from Lemm (2.8), we have

$$\left\|\mathbf{P}_{r,n}^{\circ -1}\right\|_2 \leq \frac{\sqrt{n}}{|r|\,|x_1|}\sqrt{n + |r|^2 - 1}.$$

■

Theorem 3.15 *Let* $\mathbf{Q}_{r,n}$ *is a matrix in (3.4), determinant of* $\mathbf{Q}_{r,n}^{\circ -1}$ *is as follows*

$$\det(\mathbf{Q}_{r,n}^{\circ -1}) = \frac{1}{x_n}\prod_{i=1}^{n-1}\left(\frac{1}{x_i} - \frac{1}{r x_{i+1}}\right).$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.6. ■

Note that in there, let (x_i) be an increasing sequence for $i = 1, 2, \ldots, n-1$. We can obtain $\det(\mathbf{Q}_{r,n}^{\circ -1}) = 0$ in same conditions with $\det(\mathbf{Q}_{r,n}) = 0$. This indicates that matrix $\mathbf{Q}_{r,n}^{\circ -1}$ may be singular.

Theorem 3.16 *Assume that* $\mathbf{Q}_{r,n}$ *is a matrix in (3.4) and*

$$\mathbf{Q}_{r,n}^{\circ -1} = \begin{pmatrix} \mathbf{Q}_{r,n-1}^{\circ -1} & H \\ & \\ rH^T & \frac{1}{x_n} \end{pmatrix}$$

with $H = \begin{pmatrix} \frac{1}{x_n} & \frac{1}{x_n} & \cdots & \frac{1}{x_n} \end{pmatrix}^T$ *(shortly, denoted by* H*) is* $(n-1) \times 1$ *matrix. If* $\mathbf{Q}_{r,n}^{\circ -1}$ *is nonsingular matrix, then the inverse of* $\mathbf{Q}_{r,n}^{\circ -1}$ *is*

$$\left(\mathbf{Q}_{r,n}^{\circ -1}\right)^{-1} = \begin{pmatrix} \left(\mathbf{Q}_{r,n-1}^{\circ -1}\right)^{-1} + y_n r \left(\mathbf{Q}_{r,n-1}^{\circ -1}\right)^{-1} H H^T \left(\mathbf{Q}_{r,n-1}^{\circ -1}\right)^{-1} & -y_n \left(\mathbf{Q}_{r,n-1}^{\circ -1}\right)^{-1} H \\ & \\ -y_n r H^T \left(\mathbf{Q}_{r,n-1}^{\circ -1}\right)^{-1} & y_n \end{pmatrix}.$$

where

$$y_n = \frac{rx_n^2}{rx_n - x_{n-1}}.$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.7. ■

Theorem 3.17 *Assume that $\mathbf{Q}_{r,n}$ is a matrix as in (3.4). The Euclidean norm of $\mathbf{Q}_{r,n}^{\circ -1}$ is*

$$\left\|\mathbf{Q}_{r,n}^{\circ -1}\right\|_E = \left[\sum_{s=1}^{n}\left(s\left(\frac{1}{|r|^2}+1\right)-\frac{1}{|r|^2}\right)\frac{1}{x_s^2}\right]^{\frac{1}{2}}.$$

Proof. By virtue of (2.6), we get

$$\begin{aligned}\left\|\mathbf{Q}_{r,n}^{\circ -1}\right\|_E &= \left[\sum_{s=1}^{n} s\frac{1}{x_s^2}+\sum_{s=1}^{n}(s-1)\frac{1}{|r|^2 x_s^2}\right]^{\frac{1}{2}} \\ &= \left[\sum_{s=1}^{n}\left(s\left(\frac{1}{|r|^2}+1\right)-\frac{1}{|r|^2}\right)\frac{1}{x_s^2}\right]^{\frac{1}{2}}.\end{aligned}$$

■

Theorem 3.18 *Let $\mathbf{Q}_{r,n}$ be a matrix as in the matrix form (3.4). We have*

i) If $|r| \geq 1$, then

$$\left\|\mathbf{Q}_{r,n}^{\circ -1}\right\|_2 \leq \sqrt{n\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

ii) If $|r| < 1$, then

$$\left\|\mathbf{Q}_{r,n}^{\circ -1}\right\|_2 \leq \sqrt{\left((n-1)\frac{1}{|r|^2}+1\right)\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

Proof. *i)* Let $\mathbb{G} = [g_{ij}]_{i,j=1}^{n}$ and $\mathbb{H} = [h_{ij}]_{i,j=1}^{n}$ be the following matrices

$$\mathbb{G} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ \frac{1}{r} & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & \dots & \frac{1}{r} & 1 \end{pmatrix},$$

and

$$\mathbb{H}=\begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_n} \\ \frac{1}{x_2} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_n} \\ \frac{1}{x_3} & \frac{1}{x_3} & \frac{1}{x_3} & \cdots & \frac{1}{x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n} & \frac{1}{x_n} & \frac{1}{x_n} & \cdots & \frac{1}{x_n} \end{pmatrix}.$$

Then we obtain

$$r_1(\mathbb{G})=\max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|g_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|g_{1j}|^2}=\sqrt{n},$$

and

$$c_1(\mathbb{H})=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|h_{ij}|^2}=\sqrt{\sum_{i=1}^{n}|h_{i1}|^2}=\sqrt{\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

Since $\mathbf{Q}_{r,n}^{\circ -1}=\mathbb{G}\circ\mathbb{H}$, based on (2.8), we have

$$\left\|\mathbf{Q}_{r,n}^{\circ -1}\right\|_2\le\sqrt{n\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

$ii)$ For $|r|<1$, we shall obtain the upper bound for the spectral norm of $\mathbf{Q}_{r,n}^{\circ -1}$. Thus $\mathbf{Q}_{r,n}^{\circ -1}=\mathbb{G}\circ\mathbb{H}$, then we obtain

$$r_1(\mathbb{G})=\max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|g_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|g_{nj}|^2}=\sqrt{(n-1)\frac{1}{|r|^2}+1},$$

and

$$c_1(\mathbb{H})=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|h_{ij}|^2}=\sqrt{\sum_{i=1}^{n}|h_{i1}|^2}=\sqrt{\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

Therefore, from (2.8), we have

$$\left\|\mathbf{Q}_{r,n}^{\circ -1}\right\|_2\le\sqrt{\left((n-1)\frac{1}{|r|^2}+1\right)\sum_{i=1}^{n}\frac{1}{x_i^2}}.$$

■

CHAPTER 4

SOME NUMERICAL EXAMPLES

In this chapter, we present four numerical examples for verify our theoretical results. The first and second examples related to the $\mathbf{P}_{r,n}$ and $\mathbf{Q}_{r,n}$ matrices whose entries are Fibonacci numbers, respectively. The third and fourth of the examples associated with the $\mathbf{P}_{r,n}$ and $\mathbf{Q}_{r,n}$ matrices whose entries are hyperharmonic Fibonacci numbers, respectively. This chapter is cited from article which has been published by us (Kızılateş and Terzioğlu, 2022).

Now, we give some illustrative examples for r-min and r-max matrices with Fibonacci and hyperharmonic Fibonacci numbers.

Example 4.1 *Let be*

$$\mathbf{F}_{r,n} = \begin{cases} rF_{\min\{i,j\}} & i > j \\ F_{\min\{i,j\}} & i \leq j \end{cases},$$

a matrix as in matrix form (3.3) for $n = 5$ and $r = 87 - 64\sqrt{3}$. Then, we have

$$\begin{aligned}
\mathbf{F}_{87-64\sqrt{3},5} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 87-64\sqrt{3} & 1 & 1 & 1 & 1 \\ 87-64\sqrt{3} & 87-64\sqrt{3} & 2 & 2 & 2 \\ 87-64\sqrt{3} & 87-64\sqrt{3} & 2(87-64\sqrt{3}) & 3 & 3 \\ 87-64\sqrt{3} & 87-64\sqrt{3} & 2(87-64\sqrt{3}) & 3(87-64\sqrt{3}) & 5 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -23.8512517 & 1 & 1 & 1 & 1 \\ -23.8512517 & -23.8512517 & 2 & 2 & 2 \\ -23.8512517 & -23.8512517 & -47.7025034 & 3 & 3 \\ -23.8512517 & -23.8512517 & -47.7025034 & -71.5537551 & 5 \end{pmatrix}.
\end{aligned}$$

From the Theorem 3.1, the determinant of $\mathbf{F}_{87-64\sqrt{3},5}$ *can be calculated as*

$$\begin{aligned}\det\left(\mathbf{F}_{87-64\sqrt{3},5}\right) &= \prod_{i=1}^{4}(F_{i+1}-rF_i)\\ &= 4456622592-2571592576\sqrt{3}\\ &\approx 2.493593\times 10^{6}.\end{aligned}$$

By virtue of Theorem 3.2, the inverse of the matrix $\mathbf{F}_{87-64\sqrt{3},5}$ *can be written as*

$$\begin{aligned}\mathbf{F}^{-1}_{87-64\sqrt{3},5} &= \begin{pmatrix} -\frac{1}{2}\frac{1025\sqrt{3}-1836}{(-171+128\sqrt{3})(-4+3\sqrt{3})(-43+32\sqrt{3})} & \cdots & 0\\ \frac{33}{2}\frac{-10804+6263\sqrt{3}}{\left(-171+128\sqrt{3}\right)\left(-4+3\sqrt{3}\right)\left(-43+32\sqrt{3}\right)\left(-85+64\sqrt{3}\right)} & \cdots & 0\\ \frac{-11\left(47\sqrt{3}-84\right)}{\left(-171+128\sqrt{3}\right)\left(-85+64\sqrt{3}\right)\left(-4+3\sqrt{3}\right)} & \cdots & 0\\ -\frac{1}{8}\frac{-9885+5536\sqrt{3}}{\left(-171+128\sqrt{3}\right)\left(-85+64\sqrt{3}\right)\left(-4+3\sqrt{3}\right)} & \cdots & -\frac{1}{64\left(-4+3\sqrt{3}\right)}\\ \frac{3}{8}\frac{554368\sqrt{3}-956511}{\left(-171+128\sqrt{3}\right)\left(-85+64\sqrt{3}\right)\left(-4+3\sqrt{3}\right)} & \cdots & \frac{1}{64\left(-4+3\sqrt{3}\right)} \end{pmatrix}\\ &\approx \begin{pmatrix} 0.04023942156 & -0.04023942179 & \cdots & 0\\ 0.03712627281 & 0.04023942179 & \cdots & 0\\ 0.01819701662 & 0 & \cdots & 0\\ 0.02362881560 & 0 & \cdots & -0.1306271650\\ 0.8808085226 & 0 & \cdots & 0.01306271650 \end{pmatrix}.\end{aligned}$$

By virtue of Theorem 3.3, one can calculate the Euclidean norm of the matrix $\mathbf{F}_{87-64\sqrt{3},5}$ *as*

$$\begin{aligned}\left\|\mathbf{F}_{87-64\sqrt{3},5}\right\|_E &= 2\sqrt{119158-66816\sqrt{3}}\\ &\approx 117.1203355.\end{aligned}$$

For the spectral norm, we can give the following inequality from Theorem 3.4,

$$\left\|\mathbf{F}_{87-64\sqrt{3},5}\right\|_2 = 101.6645097 \leq 301.7634061.$$

The Hadamard inverse of $\mathbf{F}_{87-64\sqrt{3},5}$ *can be written as*

$$\mathbf{F}^{\circ-1}_{87-64\sqrt{3},5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ \frac{1}{\left(87-64\sqrt{3}\right)} & 1 & 1 & 1 & 1\\ \frac{1}{\left(87-64\sqrt{3}\right)} & \frac{1}{\left(87-64\sqrt{3}\right)} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{\left(87-64\sqrt{3}\right)} & \frac{1}{\left(87-64\sqrt{3}\right)} & \frac{1}{174-128\sqrt{3}} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{\left(87-64\sqrt{3}\right)} & \frac{1}{\left(87-64\sqrt{3}\right)} & \frac{1}{174-128\sqrt{3}} & \frac{1}{261-192\sqrt{3}} & \frac{1}{5} \end{pmatrix}.$$

By virtue of Theorem 3.11, the determinant of $\mathbf{F}^{\circ -1}_{87-64\sqrt{3},5}$ *can be calculated as*

$$\det(\mathbf{F}^{\circ -1}_{87-64\sqrt{3},5}) = -\frac{32}{45}\frac{\left(-34817364+20090567\sqrt{3}\right)}{\left(-87+64\sqrt{3}\right)^4} \approx 0.04280635458.$$

Also since Theorems 3.13 and 3.14, the Euclidean norm and spectral norm of $\mathbf{F}^{\circ -1}_{87-64\sqrt{3},5}$ *can be calculated as*

$$\left\|\mathbf{F}^{\circ -1}_{87-64\sqrt{3},5}\right\|_E = \frac{1}{30}\frac{\sqrt{178938277-100346496\sqrt{3}}}{\left(-87+64\sqrt{3}\right)} \simeq 3.166322991,$$

and

$$\left\|\mathbf{F}^{\circ -1}_{87-64\sqrt{3},5}\right\|_2 = 3.013229215 \leq 5.$$

Example 4.2 *Let be*

$$\mathcal{F}_{r,n} = \begin{cases} rF_{\max\{i,j\}} & i > j \\ F_{\max\{i,j\}} & i \leq j \end{cases},$$

a matrix as in matrix form (3.4) for $n = 5$ *and* $r = 87 - 64\sqrt{3}$*. Then, we obtain*

$$\mathcal{F}_{87-64\sqrt{3},5} = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ 87-64\sqrt{3} & 1 & 2 & 3 & 5 \\ 174-128\sqrt{3} & 174-128\sqrt{3} & 2 & 3 & 5 \\ 261-192\sqrt{3} & 261-192\sqrt{3} & 261-192\sqrt{3} & 3 & 5 \\ 435-320\sqrt{3} & 435-320\sqrt{3} & 435-320\sqrt{3} & 435-320\sqrt{3} & 5 \end{pmatrix}$$
$$\approx \begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ -23.8512517 & 1 & 2 & 3 & 5 \\ -47.7025034 & -47.7025034 & 2 & 3 & 5 \\ -71.5537551 & -71.5537551 & -71.5537551 & 3 & 5 \\ -119.2562586 & -119.2562586 & -119.2562586 & -119.2562586 & 5 \end{pmatrix}.$$

From the Theorem 3.6, the determinant of $\mathcal{F}_{87-64\sqrt{3},5}$ *can be calculated as*

$$\begin{aligned} \det\left(\mathcal{F}_{87-64\sqrt{3},5}\right) &= F_4\prod_{i=1}^{4}(F_i - rF_{i+1}) \\ &= 113353860960 - 65413463680\sqrt{3} \\ &\approx 5.44184 \times 10^7. \end{aligned}$$

Since Theorem 3.7, the inverse of the matrix $\mathcal{F}_{87-64\sqrt{3},5}$ can be written as

$$\mathcal{F}^{-1}_{87-64\sqrt{3},5} = \begin{pmatrix} -\frac{1}{2}\frac{10364\sqrt{3}-18513}{(192\sqrt{3}-259)(20\sqrt{3}-27)(-43+32\sqrt{3})} & \cdots & 0 \\ -\frac{39}{2}\frac{53500\sqrt{3}-92321}{(192\sqrt{3}-259)(-173+128\sqrt{3})(20\sqrt{3}-27)(-43+32\sqrt{3})} & \cdots & 0 \\ \frac{3(1156\sqrt{3}-2063)}{(192\sqrt{3}-259)(-173+128\sqrt{3})(20\sqrt{3}-27)} & \cdots & 0 \\ \frac{1}{2}\frac{-9885+5536\sqrt{3}}{(192\sqrt{3}-259)(-173+128\sqrt{3})(20\sqrt{3}-27)} & \cdots & -\frac{1}{16(20\sqrt{3}-27)} \\ \frac{3}{2}\frac{554368\sqrt{3}-956511}{(192\sqrt{3}-259)(-173+128\sqrt{3})(20\sqrt{3}-27)} & \cdots & \frac{3}{80(20\sqrt{3}-27)} \end{pmatrix},$$

$$\approx \begin{pmatrix} 0.04023942170 & -0.04023942179 & \cdots & 0 \\ -0.01970659776 & 0.04023942179 & \cdots & 0 \\ -0.006658172409 & 0 & \cdots & 0 \\ -0.005413675190 & 0 & \cdots & -0.008179540350 \\ 0.2018049201 & 0 & \cdots & 0.004907724210 \end{pmatrix}.$$

By virtue of Theorem 3.8, one can calculate the Euclidean norm of the matrix $\mathcal{F}_{87-64\sqrt{3},5}$ as

$$\begin{aligned} \left\|\mathcal{F}_{87-64\sqrt{3},5}\right\|_E &= 2\sqrt{675182-378624\sqrt{3}} \\ &\approx 278.4671980. \end{aligned}$$

For the spectral norm, we can give the following inequality from Theorem 3.9,

$$\left\|\mathcal{F}_{87-64\sqrt{3},5}\right\|_2 = 267.1531432 \leq 533.4473765.$$

The Hadamard inverse of $\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5}$ can be written as

$$\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{87-64\sqrt{3}} & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{174-128\sqrt{3}} & \frac{1}{174-128\sqrt{3}} & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{261-192\sqrt{3}} & \frac{1}{261-192\sqrt{3}} & \frac{1}{261-192\sqrt{3}} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{435-320\sqrt{3}} & \frac{1}{435-320\sqrt{3}} & \frac{1}{435-320\sqrt{3}} & \frac{1}{435-320\sqrt{3}} & \frac{1}{5} \end{pmatrix}.$$

By virtue of Theorem 3.15, the determinant of $\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5}$ can be calculated as

$$\begin{aligned} \det(\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5}) &= -\frac{8}{225}\frac{-708461631+408834148\sqrt{3}}{\left(-87+64\sqrt{3}\right)^4}, \\ &\approx 0.03736698365. \end{aligned}$$

Also the Euclidean norm and spectral norms of $\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5}$ *can be calculated as*

$$\begin{aligned}\left\|\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5}\right\|_E &= \frac{1}{30}\frac{\sqrt{78536229-44042880\sqrt{3}}}{-87+64\sqrt{3}} \\ &\approx 2.097128606,\end{aligned}$$

and

$$\left\|\mathcal{F}^{\circ-1}_{87-64\sqrt{3},5}\right\|_2 = 1.856931660 \leq 3.464903396.$$

Example 4.3 *Let be*

$$\mathbf{H}_{r,n} = \begin{cases} r\mathbb{F}_{\min\{i,j\}} & i > j \\ \mathbb{F}_{\min\{i,j\}} & i \leq j \end{cases},$$

a matrix as in matrix form (3.3) for $\tau = 4$, $n = 5$ *and* $r = \frac{1}{5} - \frac{1}{3}\sqrt{2}$. *Then, we have*

$$\begin{aligned}\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \frac{1}{5}-\frac{1}{3}\sqrt{2} & 5 & 5 & 5 & 5 \\ \frac{1}{5}-\frac{1}{3}\sqrt{2} & 1-\frac{5}{3}\sqrt{2} & \frac{29}{2} & \frac{29}{2} & \frac{29}{2} \\ \frac{1}{5}-\frac{1}{3}\sqrt{2} & 1-\frac{5}{3}\sqrt{2} & \frac{29}{10}-\frac{29}{6}\sqrt{2} & \frac{97}{3} & \frac{97}{3} \\ \frac{1}{5}-\frac{1}{3}\sqrt{2} & 1-\frac{5}{3}\sqrt{2} & \frac{29}{10}-\frac{29}{6}\sqrt{2} & \frac{97}{15}-\frac{97}{9}\sqrt{2} & \frac{923}{15} \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -0.2714045206 & 5 & 5 & 5 & 5 \\ -0.2714045206 & -1.357022604 & 14.5 & 14.5 & 14.5 \\ -0.2714045206 & -1.357022604 & -3.935365549 & 32.33333333 & 32.333333 \\ -0.2714045206 & -1.357022604 & -3.935365549 & -8.775412833 & 61.533333 \end{pmatrix}.\end{aligned}$$

From Theorem 3.1, the determinant of $\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be calculated as*

$$\begin{aligned}\det\left(\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right) &= \prod_{i=1}^{4}(\mathbb{F}^{(4)}_{i+1} - r\mathbb{F}^{(4)}_{i}) \\ &= \frac{7793006009}{60750} + \frac{364582504}{6075}\sqrt{2} \\ &\approx 2.131519543 \times 10^5.\end{aligned}$$

The inverse of the matrix $\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be written as*

$$\mathbf{H}^{-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5} = \begin{pmatrix} \frac{75(204377944+87080005\sqrt{2})}{(145\sqrt{2}+883)(10\sqrt{2}+81)(72+5\sqrt{2})(485\sqrt{2}+2478)} & \cdots & 0 \\ \frac{57(889478+9280925\sqrt{2})}{(145\sqrt{2}+883)(10\sqrt{2}+81)(72+5\sqrt{2})(485\sqrt{2}+2478)} & \cdots & 0 \\ \frac{2140(59150\sqrt{2}+39171)}{(145\sqrt{2}+883)(10\sqrt{2}+81)(72+5\sqrt{2})(485\sqrt{2}+2478)} & \cdots & 0 \\ \frac{152424(305\sqrt{2}+309)}{(145\sqrt{2}+883)(10\sqrt{2}+81)(72+5\sqrt{2})(485\sqrt{2}+2478)} & \cdots & -\frac{45}{(485\sqrt{2}+2478)} \\ \frac{11252(6758+5205\sqrt{2})}{(145\sqrt{2}+883)(10\sqrt{2}+81)(72+5\sqrt{2})(485\sqrt{2}+2478)} & \cdots & \frac{45}{(485\sqrt{2}+2478)} \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.9485138130 & -0.1897027626 & \cdots & 0 \\ 0.03084556240 & 0.1767151573 & \cdots & 0 \\ 0.01014900332 & 0.006386010530 & \cdots & 0 \\ 0.004357286528 & 0.002741715295 & \cdots & -0.01422298156 \\ 0.006134335132 & 0.003859879387 & \cdots & 0.01422298156 \end{pmatrix}.$$

By virtue of Theorem 3.3, one can calculate the Euclideam norm of the matrix $\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *as*

$$\begin{aligned} \left\|\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_E &= \frac{1}{90}\sqrt{56846681 - 1668540\sqrt{2}} \\ &\approx 82.01701803. \end{aligned}$$

For the spectral norm, we can give the following inequality from Theorem 3.4

$$\left\|\mathbf{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_2 = 72.22317730 \le 159.1861419.$$

The Hadamard inverse of $\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be written as*

$$\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -\frac{15}{-3+5\sqrt{2}} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -\frac{15}{-3+5\sqrt{2}} & -\frac{3}{-3+5\sqrt{2}} & \frac{2}{29} & \frac{2}{29} & \frac{2}{29} \\ -\frac{15}{-3+5\sqrt{2}} & -\frac{3}{-3+5\sqrt{2}} & -\frac{30}{29\left(-3+5\sqrt{2}\right)} & \frac{3}{97} & \frac{3}{97} \\ -\frac{15}{-3+5\sqrt{2}} & -\frac{3}{-3+5\sqrt{2}} & -\frac{30}{29\left(-3+5\sqrt{2}\right)} & -\frac{45}{97\left(-3+5\sqrt{2}\right)} & \frac{15}{923} \end{pmatrix}.$$

By virtue of Theorem 3.11, the determinant of $\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be calculated as*

$$\begin{aligned} \det\left(\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right) &= \frac{18}{7303670387}\frac{7793006009 + 3645825040\sqrt{2}}{(-3+5\sqrt{2})^4} \\ &\approx 0.1161805948. \end{aligned}$$

Also the Euclidean norm and spectral norm of $\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be calculated as*

$$\begin{aligned} \left\|\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_E &= \frac{4}{12981995}\frac{\sqrt{13006112232755946 - 1635745846758570\sqrt{2}}}{\left(-3+5\sqrt{2}\right)} \\ &\approx 7.826296744, \end{aligned}$$

and

$$\left\|\mathbf{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_2 = 7.529662896 \le 16.62877804.$$

Example 4.4 *Let be*

$$\mathcal{H}_{r,n} = \begin{cases} r\mathbb{F}_{\max\{i,j\}} & i > j \\ \mathbb{F}_{\max\{i,j\}} & i \leq j \end{cases},$$

a matrix as in matrix form (3.3) for $\tau = 4$, $n = 5$ *and* $r = \frac{1}{5} - \frac{1}{3}\sqrt{2}$. *Then, we have*

$$\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5} = \begin{pmatrix} 1 & 5 & \frac{29}{2} & \frac{97}{3} & \frac{923}{15} \\ 1-\frac{5}{3}\sqrt{2} & 5 & \frac{29}{2} & \frac{97}{3} & \frac{923}{15} \\ \frac{29}{10}-\frac{29}{6}\sqrt{2} & \frac{29}{10}-\frac{29}{6}\sqrt{2} & \frac{29}{2} & \frac{97}{3} & \frac{923}{15} \\ \frac{97}{15}-\frac{97}{9}\sqrt{2} & \frac{97}{15}-\frac{97}{9}\sqrt{2} & \frac{97}{15}-\frac{97}{9}\sqrt{2} & \frac{97}{3} & \frac{923}{15} \\ \frac{923}{75}-\frac{923}{45}\sqrt{2} & \frac{923}{75}-\frac{923}{45}\sqrt{2} & \frac{923}{75}-\frac{923}{45}\sqrt{2} & \frac{923}{75}-\frac{923}{45}\sqrt{2} & \frac{923}{15} \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 5 & 14.5 & 32.33333333 & 61.53333333 \\ -1.357022604 & 5 & 14.5 & 32.33333333 & 61.53333333 \\ -3.935365549 & -3.935365549 & 14.5 & 32.33333333 & 61.53333333 \\ -8.775412833 & -8.775412833 & -8.775412833 & 32.33333333 & 61.53333333 \\ -16.70042483 & -16.70042483 & -16.70042483 & -16.70042483 & 61.53333333 \end{pmatrix}.$$

From the Theorem 3.6, the determinant of $\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be calculated as*

$$\begin{aligned} \det\left(\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right) &= \mathbb{F}_5^{(4)} \prod_{i=1}^{4} (\mathbb{F}_i^{(4)} - r\mathbb{F}_{i+1}^{(4)}) \\ &= \frac{416486453903}{546750} + \frac{950998282}{1875}\sqrt{2} \\ &\approx 1.479037167 \times 10^6. \end{aligned}$$

The inverse of the matrix $\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be written as*

$$\mathcal{H}^{-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5} = \begin{pmatrix} \frac{339}{5}\frac{13294044\sqrt{2}+19965985}{(970\sqrt{2}+723)(63+145\sqrt{2})(4615\sqrt{2}+4506)} & \cdots & 0 \\ -\frac{57}{10}\frac{(40441836+37932295\sqrt{2})\sqrt{2}}{(970\sqrt{2}+723)(63+145\sqrt{2})(4615\sqrt{2}+4506)} & \cdots & 0 \\ -\frac{642(239217\sqrt{2}+267380)}{(970\sqrt{2}+723)(63+145\sqrt{2})(4615\sqrt{2}+4506)} & \cdots & 0 \\ -\frac{228636(309\sqrt{2}+610)}{(970\sqrt{2}+723)(63+145\sqrt{2})(4615\sqrt{2}+4506)} & \cdots & -\frac{225}{(4615\sqrt{2}+4506)} \\ \frac{33756(5205+3379\sqrt{2})}{(970\sqrt{2}+723)(63+145\sqrt{2})(4615\sqrt{2}+4506)} & \cdots & \frac{109125}{923(4615\sqrt{2}+4506)} \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.4242640688 & -0.4242640688 & \cdots & 0 \\ -0.1224234491 & 0.2343382884 & \cdots & 0 \\ -0.06276678671 & 0.03949445556 & \cdots & 0 \\ -0.03863997109 & 0.02431325070 & \cdots & -0.02039411290 \\ 0.05439865626 & -0.03422901548 & \cdots & 0.01071629984 \end{pmatrix}.$$

By virtue of Theorem 3.8, one can calculate the Euclidean norm of the matrix $\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *as*

$$\begin{aligned}\left\|\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_E &= \frac{1}{450}\sqrt{5812961201-505635420\sqrt{2}}\\ &\approx 158.6654953.\end{aligned}$$

For the spectral norm, we can give the following inequality from Theorem 3.9

$$\left\|\mathcal{H}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_2 = 148.3852561 \leq 307.6666667.$$

The Hadamard inverse of $\mathcal{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be written as*

$$\mathcal{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5} = \begin{pmatrix} 1 & \frac{1}{5} & \frac{2}{29} & \frac{3}{97} & \frac{15}{923} \\ -\frac{3}{-3+5\sqrt{2}} & \frac{1}{5} & \frac{2}{29} & \frac{3}{97} & \frac{15}{923} \\ -\frac{30}{29\left(-3+5\sqrt{2}\right)} & -\frac{30}{29\left(-3+5\sqrt{2}\right)} & \frac{2}{29} & \frac{3}{97} & \frac{15}{923} \\ -\frac{45}{97\left(-3+5\sqrt{2}\right)} & -\frac{45}{97\left(-3+5\sqrt{2}\right)} & -\frac{45}{97\left(-3+5\sqrt{2}\right)} & \frac{3}{97} & \frac{15}{923} \\ -\frac{225}{923\left(-3+5\sqrt{2}\right)} & -\frac{225}{923\left(-3+5\sqrt{2}\right)} & -\frac{225}{923\left(-3+5\sqrt{2}\right)} & -\frac{225}{923\left(-3+5\sqrt{2}\right)} & \frac{15}{923} \end{pmatrix}.$$

By virtue of Theorem 3.15, the determinant of $\mathcal{H}^{\circ-1}_{r,\max}$ *can be calculated as*

$$\begin{aligned}\det(\mathcal{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}) &= \frac{1}{\mathbb{F}_n^{(k)}}\prod_{i=1}^{n-1}\left(\frac{1}{\mathbb{F}_i^{(k)}}-\frac{1}{r\mathbb{F}_{i+1}^{(k)}}\right)\\ &= \frac{5085}{6741287767201}\frac{\sqrt{2}(26588088+19965985\sqrt{2})}{(-3+5\sqrt{2})^4}\\ &\approx 0.0002129131476.\end{aligned}$$

Also the Euclidean norm and spectral norms of $\mathcal{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}$ *can be calculated as*

$$\begin{aligned}\left\|\mathcal{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_E &= \left[\sum_{s=1}^{n}\left(s\left(\frac{1}{|r|^2}+1\right)-\frac{1}{|r|^2}\right)\frac{1}{\left(\mathbb{F}_s^{(4)}\right)^2}\right]^{\frac{1}{2}}\\ &= \frac{4}{12981995}\frac{\sqrt{809893794641298-347412913154910\sqrt{2}}}{(-3+5\sqrt{2})}\\ &\approx 1.350883607,\end{aligned}$$

and

$$\left\|\mathcal{H}^{\circ-1}_{\frac{1}{5}-\frac{1}{3}\sqrt{2},5}\right\|_2 = 1.279274986 \leq 7.605650716.$$

CHAPTER 5

CONCLUSION

As it is known, matrix theory and number theory are used in many fields from science to visual arts such as cryptography, computer science, statistics, illustration, software development. For example, in viral marketing, an infinite linear model can be studied by reducing it to finite Hankel min-max matrices (Tomášková and Gavalec, 2010).
In this study, we also aimed to generalize the min and max matrices in the literature and present some linear algebraic properties. Because of that, we defined the r-min and r-max matrices. We also gave the results which we obtained and four numerical examples. We think that this generalization we made on min and max matrices will both inspire those who work on matrix theory and add flexibility and diversity to applications in other fields. With this motivation, we will try to generalize other matrices in the literature as well as min-max matrices.

REFERENCES

[1] **Andrade E, Carrasco-Olivera D and Manzaneda C** (2021) On circulant like matrices properties involving Horadam, Fibonacci, Jacobsthal and Pell numbers. Linear Algebra Appl., 617:100-120.

[2] **Bahsi M and Solak S** (2014) On the norms of r-circulant matrices with the hyper-Fibonacci and Lucas numbers. J Math Inequal., 8(4):693-705.

[3] **Bahsi M and Solak S** (2015) Some particular matrices and their characteristic polynomials, Linear Multilinear Algebra, 63:2071-2078.

[4] **Bertaccini D and Ng M K** (2001) Skew-circulant preconditioners for systems of LMF-based ODE codes. Numer Anal Appl. LNCS, 93-101.

[5] **Bhatia R** (2006) Infinitely divisible matrices. Amer. Math. Monthly, 113(2):221-235.

[6] **Bhatia R** (2011) Min matrices and mean matrices. Math. Intelligencer, 33(2): 22-28.

[7] **da Fonseca C M** (2007) On the eigenvalues of some tridiagonal matrices. J. Comput. Appl. Math., 200(1):283-286.

[8] **Horn R A** (1990) The Hadamard product. Proc Sympos Appl Math., 40:87-169.

[9] **Horn R A and Johnson C R** (1991) Topics in matrix analysis. Cambridge University Press, Cambridge.

[10] **Horn R A and Johnson C R** (2013) Matrix analysis. Cambridge University Press, Cambridge.

[11] **Kızılateş C and Terzioğlu N** (2022) On the r-min and r-max matrices. J Appl Math Comput., doi: 10.1007/s12190-022-01717-y.

[12] **Kızılateş C and Tuglu N** (2016) On the bounds for the spectral norms of geometric circulant matrices. J Inequal Appl., 2016:312.

[13] **Kızılateş C and Tuglu N** (2018) On the norms of geometric and symmetric geometric circulant matrices with the Tribonacci number. Gazi Univ J Sci., 31(2):555-567.

[14] **Kocer E G** (2007) Circulant, negacyclic and semicirculant matrices with the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers. Hacet J Math Stat., 36(2):133-142.

[15] **Kocer E G, Mansour T and Tuglu N** (2007) Norms of circulant and semicirculant matrices with Horadam's numbers. Ars Comb., 85:353-359.

REFERENCES (continued)

[16] **Lyness J N and Sørevik T** (2004) Four-dimensional lattice rules generated by skew-circulant matrices. Math Comput., 73(245):279-295.

[17] **Mattila M and Haukkanen P** (2016) Studying the various properties of min and max matrices-elementary vs. more advanced methods. Spec. Matrices, 4:101-109.

[18] **Neudecker H, Trenkler G and Liu S** (2009) Problem section. Stat Papers;50:221-223.

[19] **Ozgul D and Bahsi M** (2020) Min matrices with hyper Lucas numbers. Journal of Science and Arts., 4(53);855-864.

[20] **Petroudi S H J and Pirouz B** (2015) A particular matrix, Its inversion and some norms. Appl. Comput. Math., 4:47-52.

[21] **Pólya G and Szegö G** (1971) Problems and Theorems in Analysis II. Vol. II, 4th ed., Springer.

[22] **Radicic B** (2016) On k-circulant matrices (with geometric sequence). Quaest Math., 39(1);135-144.

[23] **Radicic B** (2017) On k-circulant matrices with arithmetic sequence. Filomat, 31(8):2517-2525.

[24] **Reams R** (1999) Hadamard inverses, square roots and products of almost semidefinite matrices. Linear Algebra Appl., 288:35-43.

[25] **Shen S Q and Cen J M** (2010) On the bounds for the norms of r-circulant matrices with Fibonacaci and Lucas numbers, Appl. Math. Comput., 216:2891-2897.

[26] **Shi B** (2018) The spectral norms of geometric circulant matrices with the generalized k-Horadam numbers. J Inequal Appl., 2018:14.

[27] **Shi B** (2021) On the Norms of RFMLR-Circulant Matrices with the Exponential and Trigonometric Functions. J. Math., 2021:1-9.

[28] **Solak S** (2005) On the norms of circulant matrices with the Fibonacci and Lucas numbers, Appl. Math. Comput., 160: 125-132.

[29] **Solak S and Bahsi M** (2015) Some Properties of Circulant Matrices with Ducci Sequences. Linear Algebra Appl., 542:557-568.

[30] **Tomášková H and Gavalec M** (2012) Hankel min-max matrices and their aplications. 30th International Conference Mathematical Methods in Economics, 11-13 September 2012, Karviná, Czech Republic.

REFERENCES (continued)

[31] **Tuglu N and Kızılateş C** (2015) On the norms of circulant and r-circulant matrices with the hyperharmonic Fibonacci numbers. J Inequal Appl., 2015:253.

[32] **Tuglu N and Kızılateş C** (2015) On the norms of some special matrices with the harmonic Fibonacci numbers. Gazi Univ J Sci., 28(3):497-501.

[33] **Tuglu N, Kızılateş C and Kesim S** (2015) On the harmonic and hyperharmonic Fibonacci numbers. Adv Differ Equ., 2015:297.

[34] **Zhang F** (1999) Matrix Theory, Basic results and techniques. Springer-Verlag, New York.

[35] **Zhao G** (2009) The improved nonsingularity on the r -circulant matrices in signal processing. Proc Int Conf Comp Tech Deve (Kota Kinabalu/Malaysia), 564-567.

CURRICULUM VITAE

Nazlıhan Terzioğlu was educated in Istanbul until university education. After graduating from high school in the department of information technologies, she studied mathematics at Çanakkale 18 Mart University. She received her undergraduate degree in 2019 and completed her graduate education at Zonguldak Bülent Ecevit University in 2022.