ON MODULES WHICH SATISFY THE RADICAL FORMULA

by

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ABSTRACT

ON MODULES WHICH SATISFY THE RADICAL FORMULA

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Let R be a commutative ring with identity and M be an R-module. M satisfies the radical formula if for every submodule N of M, the radical of N is the submodule generated by its envelope.

Prime submodules should be studied before we examine the radical formula. So definition of prime submodule and some characterizations of prime submodules are given in chapter I. In chapter II, we discussed under what conditions prime radical of a module is distributive over intersection.

In chapter III, envelope of a module is introduced and modules which satisfy the radical formula are investigated.

In last chapter, generalization of Cohen's theorem, which gives the relationship between a Noetherian ring and prime ideals, to modules is given and we discussed ascending chain condition on modules.

Keywords: Prime submodule, prime radical, radical formula.

ÖZ

RADİKAL FORMÜLÜNÜ SAĞLAYAN MODÜLLER

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R birimli, değişmeli bir halka ve M bir R-modül olsun. M'nin her N alt modülü için zarfı tarafından üretilen alt modül asal radikaline eşit ise M radikal formülünü sağlar denir.

Radikal formülü konusu incelenmeden önce asal alt modül kavramı verilmelidir. Bu nedenle asal alt modül tanımı ve karakterizasyonları I. bölümde verilmiştir. II. bölümde bir modülün asal radikalinin hangi koşullar altında arakesit üzerine dağıldığı incelenmiştir.

III. Bölümde, zarf kavramı verilerek radikal formülünü sağlayan modüller incelenmiştir.

Son bölümde, Cohen'in Noether bir halka ile asal idealler arasındaki ilişkiyi kuran teoreminin modüllere genellemesi verilmiş ve radikal alt modüller üzerinde artan zincir kuralı incelenmiştir.

Anahtar Kelimeler: Asal alt modül, asal radikal, radikal formülü.

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To My Family

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CHAPTER 1

PRELIMINARIES

1.1 Introduction

Throughout R will denote a commutative ring with identity and M will be a unitary R-module.

Definition 1.1 *Let R be a ring and M an R-module, a proper submodule N of M is called prime if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$.*

Definition 1.2 *For any submodule N of M, $(N : M)$ denotes the set of elements $r \in R$ such that $rM \subseteq N$. This is usually called residual by N. Note that $(N : M)$ is the annihilator of the module M/N and $(N : M)$ is an ideal of R. For any ideal J of R $(N :_M J)$ denotes the set of elements $m \in M$ such that $Jm \subseteq N$.*

If N is a prime submodule of M with $\mathcal{P} = (N : M)$, then N is called $\mathcal{P}$-prime submodule.

Example 1.3 *Let F be a field and V be a vector space over the field F. Then every proper subspace W of V is a $\langle 0\rangle$-prime submodule of V.*

Proof. Let W be a proper subspace of V and $f \in F$, $v \in V$ such that $fv \in W$. If $f = 0$, then $fV \subseteq W$. If $f \neq 0$, then f^{-1} exists and $v \in W$. Thus, W is a prime submodule. Since W is proper subspace of V, $(W : V) = \langle 0\rangle$. $\square$

Example 1.4 *[[4], Result 1] Every direct summand of a torsion-free module is prime.*

Proof. Let M be a torsion free R-module and $M = N \bigoplus N'$ where N, N' are submodules of M. First note that $Ann_R(x) \neq 0$ implies that $x = 0$ for any $x \in M$. Now let $0 \neq r \in R$, $m = n + n' \in M$ and $rm \in N$ for some $n \in N$, $n' \in N'$. Thus $rn + rn' \in N$. Then $rn' = 0$ and $r \in Ann_R(n')$. Since $r \neq 0$, $n' = 0$. Therefore $m = n + 0 = n \in N$. □

Example 1.5 *The zero submodule is the only prime submodule of the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers.*

Proof. Let N be a non-zero prime submodule of $\mathbb{Q}$. Then $N \neq \mathbb{Q}$. Let $x = k/l \in \mathbb{Q} - N$ and $y = r/s$ be a non-zero element of N for some non-zero integers k, l, r and s. Hence $rlx = rk = (r/s)sk \in N$. Since N is a prime submodule and $x \notin N$, $rl \in (N : \mathbb{Q})$. So $rl\mathbb{Q} \subseteq N$. Since $rl\mathbb{Q} = \mathbb{Q}$, we have $N = \mathbb{Q}$. Therefore zero is the only prime submodule. □

Lemma 1.6 *[[1], Lemma 1] Let R be a ring and M an R-module. Then a submodule N of M is prime if and only if $\mathcal{P} = (N : M)$ is a prime ideal of R and the $R/\mathcal{P}$-module M/N is torsion-free .*

Proof. ($\Rightarrow$) Let N be a prime submodule of M and let $ab \in \mathcal{P} = (N : M)$ where $a, b \in R$. Then $(ab)M \subseteq N$ so $a(bM) \subseteq N$ implies that $aM \subseteq N$ or $bM \subseteq N$ since N is prime. Hence $\mathcal{P} = (N : M)$ is a prime ideal of R.

Since $\mathcal{P}M \subseteq N$, M/N is a $R/\mathcal{P}$-module. Let $0 \neq r + \mathcal{P} \in R/\mathcal{P}$ and $m + N \in M/N$ such that $(r + \mathcal{P})(m + N) = 0$. Then $rm + N = 0$ implies that $rm \in N$. Hence either $m \in N$ or $rM \subseteq N$. Since $r \notin \mathcal{P}$, we have $m \in N$. Hence M/N is a torsion-free $R/\mathcal{P}$-module.

($\Leftarrow$) Suppose $\mathcal{P} = (N : M)$ is a prime ideal and M/N is a torsion free $R/\mathcal{P}$-module. Let $r \in R$, $m \notin N$ and $rm \in N$; then $(r+\mathcal{P})(m+N) = 0$. Since $m \notin N$, we have $r \in \mathcal{P}$. Hence $rM \subseteq N$. Therefore N is a prime submodule of M. □

Corollary 1.7 *[[1], Corollary 2] Let R be a domain and M be a non-torsion R-module, that is, $T(M) \neq M$ where $T(M)$ denotes the torsion submodule of M. Then*

(i) $T(M)$ is prime, and

(ii) $\mathcal{P}M = M$ or $\mathcal{P}M$ is a prime submodule of M for each maximal ideal $\mathcal{P}$ of R.

Proof. (i) We must show that $\mathcal{P} = (T(M) : M)$ is a prime ideal and $M/T(M)$ is a torsion free $R/\mathcal{P}$-module. Since $T(M) \neq M$, there exists $m_1 \in M - T(M)$ such that $Ann_R(m_1) = 0$. Let $p \in \mathcal{P}$. Then $pm_1 \in T(M)$ and $Ann_R(pm_1) \neq 0$. Thus there exists $0 \neq r \in R$ such that $r(pm_1) = 0$. Then $p = 0$ since R is a domain and $T(M) \neq M$. Hence $\mathcal{P} = (T(M) : M) = 0$. Since R is a domain, $\mathcal{P}$ is a prime ideal. Clearly $M/T(M)$ is an $R/\mathcal{P}$-module. Let $0 \neq r + \mathcal{P} \in R/\mathcal{P}$ and $m + T(M) \in M/T(M)$ such that $(r + \mathcal{P})(m + T(M)) = 0$. Then $rm \in T(M)$ so there exists a non-zero $s \in R$ such that $(sr)m = 0$. Since R is a domain, $m \in T(M)$. Therefore $M/T(M)$ is torsion-free.

(ii) Suppose $\mathcal{P}M \neq M$. Clearly $\mathcal{P} \subseteq (\mathcal{P}M : M)$. Since $\mathcal{P}$ is a maximal ideal, $\mathcal{P} = (\mathcal{P}M : M)$. Hence $(\mathcal{P}M : M)$ is a prime ideal. $\mathcal{P}(M/\mathcal{P}M) = 0$ implies that $M/\mathcal{P}M$ is an $R/\mathcal{P}$-module. Now we will show that $T(M/\mathcal{P}M) = 0$. Let $m + \mathcal{P}M \in T(M/\mathcal{P}M)$. Then there exists $0 \neq r + \mathcal{P} \in R/\mathcal{P}$ such that $(r + \mathcal{P})(m + \mathcal{P}M) = 0$. Since $R/\mathcal{P}$ is a field, $(r + \mathcal{P})^{-1}$ exists. So $m \in \mathcal{P}M$. This means that $T(M/\mathcal{P}M) = 0$. Therefore $\mathcal{P}M$ is a prime submodule of M. $\square$

We have given some examples of modules which have prime submodules. However it is not difficult at all to give examples of modules which have no prime submodules.

Example 1.8 *As a $\mathbb{Z}$-module, the Prüfer group $\mathbb{Z}(p^\infty)$ has no prime submodules.*

Proof. Suppose K is a prime submodule of $\mathbb{Z}(p^\infty)$. Then $\mathcal{P} = (K : \mathbb{Z}(p^\infty))$ is a prime ideal of $\mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain, $\mathcal{P} = 0$ or $\mathcal{P} = q\mathbb{Z}$ for some prime $q \in \mathbb{Z}$. If $\mathcal{P} = 0$, then $\mathbb{Z}(p^\infty)/K$ is a torsion-free $\mathbb{Z}$-module but we know that $\mathbb{Z}(p^\infty)$ is a torsion module. This gives a contradiction $\mathbb{Z}(p^\infty) = K$. If $\mathcal{P} = q\mathbb{Z}$, then $\mathbb{Z}(p^\infty) = q\mathbb{Z}(p^\infty) \subseteq K$. We get another contradiction that $\mathbb{Z}(p^\infty) = K$. Therefore $\mathbb{Z}(p^\infty)$ has no prime submodules. □

Lemma 1.9 *Let $\{N_i : i \in I\}$ be a family of prime submodules. Then $(\bigcap_{i\in I} N_i : M) = \bigcap_{i\in I}(N_i : M)$.*

Proof. Let $x \in \bigcap_{i\in I}(N_i : M)$. Then $x \in (N_i : M)$ for all $i \in I$ and $xM \subseteq \bigcap_{i\in I} N_i$. Thus $x \in (\bigcap_{i\in I} N_i : M)$. Conversely let $y \in (\bigcap_{i\in I} N_i : M)$. Then $yM \subseteq \bigcap_{i\in I} N_i \subseteq N_i$ for all $i \in I$. Hence $y \in (N_i : M)$ for all $i \in I$, so that $y \in \bigcap_{i\in I}(N_i : M)$. □

Lemma 1.10 *Let M be a finitely generated module and $\{N_i : i \in I\}$ be a totally ordered family of prime submodules. Then $(\bigcup_{i\in I} N_i : M) = \bigcup_{i\in I}(N_i : M)$.*

Proof. Let $x \in (\bigcup_{i\in I} N_i : M)$. Then $xM \subseteq \bigcup_{i\in I} N_i$. Since M is finitely generated, $M = Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$. Hence for all $m_j \in M$, $xm_j \in N_{i_j}$ for some $i_j \in I$. Since $\{N_i : i \in I\}$ is totally ordered, there exist $k \in I$ such that $N_{i_1}, N_{i_2}, \ldots, N_{i_n} \subseteq N_k$. Thus $xM \subseteq N_k$ and $x \in \bigcup_{i\in I}(N_i : M)$. Now let $x' \in \bigcup_{i\in I}(N_i : M)$. Then $x' \in (N_i : M)$ for some $i \in I$ and $x'M \subseteq N_i \subseteq \bigcup_{i\in I} N_i$. Hence $x' \in (\bigcup_{i\in I} N_i : M)$. □

Lemma 1.11 *Let $M = K \oplus L$ be a direct sum of its submodules K and L and let N be a prime submodule of K. Then $N \oplus L$ is a prime submodule of M.*

Proof. Let $r \in R$, $m \in M$ and $rm \in N \oplus L$. Then $rm = n + l$ for some $n \in N$ and $l \in L$. Since $M = K \oplus L$, $m = k + l_1$ for some $k \in K$ and $l_1 \in L$. So that $rm = r(k + l_1) = rk + rl_1 = n + l$. Then $rk - n = l - rl_1 \in K \cap L = 0$ and

$l = rl_1$, $rk = n \in N$. Since N is prime, $k \in N$ or $rK \subseteq N$. If $k \in N$, then $m = k + l_1 \in N \oplus L$. If $rK \subseteq N$, then $rM = r(K+L) = rK + rL \subseteq N \oplus L$. Hence $N \oplus L$ is a prime submodule of M. □

Lemma 1.12 *Let M be an R-module, I be an ideal of R such that $I \subseteq Ann(M)$ and P be a prime submodule of M. Then P is a prime submodule of the R/I-module M.*

Proof. Let $r + I \in R/I$, $m \in M$ and $(r+I)m \in P$. Then $rm \in P$. Since P is a prime submodule, $m \in P$ or $rM \subseteq P$. Hence $m \in P$ or $(r+I)M \subseteq P$. □

The following lemma, proof of which is routine, is well-known.

Lemma 1.13 *[[2], Result 1.1] Let M, M' be R-modules with $\varphi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $N \supseteq K = Ker\varphi$. Then there exists a one-to-one order preserving correspondence between the proper submodules of M containing N and the proper submodules of M' containing $\varphi(N)$. Furthermore, for any submodule N' of M', there exists a submodule L of M such that $L \supseteq K$ and $\varphi(L) = N'$.*

Lemma 1.14 *[[2], Result 1.2] Assume the hypothesis given in Lemma 1.13. Then*

(i) If P is a prime submodule of M containing N, then $\varphi(P)$ is a prime submodule of M' containing $\varphi(N)$.

(ii) If P' is a prime submodule of M' containing $\varphi(N)$, then $\varphi^{-1}(P')$ is a prime submodule of M containing N.

Proof. (i) Since P is a prime submodule of M containing N, by Lemma 1.13 $\varphi(N) \subseteq \varphi(P)$ and since $P \neq M$, $\varphi(P) \neq \varphi(M) = M'$. Let $r \in R$, $m' \in M'$ and $rm' \in \varphi(P)$. There exists $m \in M$ such that $\varphi(m) = m'$. So $r\varphi(m) = \varphi(rm) \in \varphi(P)$. Then $rm \in P$ and since P is a prime submodule,

$m \in P$ or $rM \subseteq P$. If $m \in P$, then $\varphi(m) = m' \in \varphi(P)$. If $rM \subseteq P$, then $rM' \subseteq \varphi(P)$. Therefore $\varphi(P)$ is a prime submodule of M'.

(ii) Since P' is a prime submodule of M', by Lemma 1.13 there exists a submodule L of M such that $L \supseteq K$ and $\varphi(L) = P'$. Let $n \in N$. Then $\varphi(n) \in \varphi(N) \subseteq \varphi(L)$, so $\varphi(n) = \varphi(l)$ for some $l \in L$. Then $(n - l) \in K \subseteq L$ and hence $n \in L = \varphi^{-1}(P')$. Let $r \in R$, $m \in M$ and $rm \in \varphi^{-1}(P')$. Then $\varphi(rm) = r\varphi(m) \in P'$. Since P' is a prime submodule of M' we have $\varphi(m) \in P'$ or $rM' \subseteq P'$. If $\varphi(m) \in P' = \varphi(L)$, then there exists $x \in L$ such that $\varphi(m) = \varphi(x)$. Then $\varphi(m - x) = 0_{M'}$ so that $m - x \in K \subseteq L$ and $m = x + y$ for some $y \in L$. Hence $m \in L = \varphi^{-1}(P')$. If $rM' \subseteq P'$, then $rm' \in P'$ for all $m' \in M'$. Then there exists $\bar{m} \in M$ such that $\varphi(\bar{m}) = m'$ for all $m' \in M'$. It gives that $\varphi(r\bar{m}) \in \varphi(L)$ and $r\bar{m} \in L$. Hence $rM \subseteq L = \varphi^{-1}(P')$. □

Definition 1.15 *Let I be an ideal of a ring R. Then the ideal $\sqrt{I} = \{r \in R : r^n \in I$ for some positive integer $n\}$ is called the radical of I.*

Definition 1.16 *Let R be a ring with identity and M be an R-module. A proper submodule N of M is $\mathcal{P}$-primary provided that $r \in R$, $m \notin N$ and $rm \in N$ implies $r^nM \subseteq N$ for some positive integer n and $\sqrt{(N : M)} = \mathcal{P}$.*

Clearly every prime submodule of a module is primary.

Proposition 1.17 *[[4], Proposition 1] (i) Let N be a primary submodule of an R-module M. Then N is prime if and only if $(N : M)$ is a prime ideal.*

(ii) If K is a $\mathcal{P}$-primary submodule of M containing a $\mathcal{P}$-prime submodule then K is $\mathcal{P}$-prime

Proof. (i) Necessity is clear by Lemma 1.6.

Now suppose $(N : M)$ is a prime ideal of R. Let $r \in R$, $m \in M$ and $rm \in N$. Since N is primary, we have $m \in N$ or $r^n M \subseteq N$ for some positive integer n. If $m \notin N$, then $r^n \in (N : M)$. So $r \in (N : M)$.

(ii) Let K be a primary submodule of M with $\sqrt{(K : M)} = \mathcal{P}$ and let N be a prime submodule of M where $(N : M) = \mathcal{P}$ and $N \subseteq K$. Let $r \in R$, $m \in M$ and $rm \in K$. Since K is a primary submodule, $r^n \in (K : M)$ or $m \in K$ for some positive integer n. Then $r \in \sqrt{(K : M)} = \mathcal{P} = (N : M)$ or $m \in K$. This gives $rM \subseteq K$ or $m \in K$. Therefore K is a prime submodule.

We have $\sqrt{(K : M)} = (N : M) = \mathcal{P}$. Since $N \subseteq K$, we have $(N : M) \subseteq (K : M)$. Hence $\mathcal{P} \subseteq (K : M)$. The reverse inclusion is also satisfied since $(K : M) \subseteq \sqrt{(K : M)}$. This yields that K is a $\mathcal{P}$-prime submodule.

□

Now we will give a characterization of prime submodules.

Theorem 1.18 *[[4], Theorem 1] Let N be a proper submodule of an R-module M with $(N : M) = \mathcal{P}$. Then the following statements are equivalent.*

(i) N is a prime submodule of M;

(ii) M/N is torsion-free $R/\mathcal{P}$-module;

(iii) $(N :_M \langle r \rangle) = N$ for every $r \in R - \mathcal{P}$;

(iv) $(N :_M J) = N$ for every ideal $J \nsubseteq \mathcal{P}$;

(v) $(N : \langle e \rangle) = \mathcal{P}$ for every $e \in M - N$;

(vi) $(N : L) = \mathcal{P}$ for every submodule L of M properly containing N;

(vii)$Ass(M/N) = \mathcal{P}$;

(viii) $\mathcal{P} = Z_R(M/N)$ where $Z_R(M/N)$ is the set of all zero divisors of M/N.

Proof. $(i) \Rightarrow (ii)$ By Lemma 1.6.

$(ii) \Rightarrow (iii)$ Clearly $N \subseteq (N :_M \langle r \rangle)$. Now suppose M/N is a torsion free $R/\mathcal{P}$-module. Then for any $r \in R$ and for all $m \in M - N$, $(r + \mathcal{P})(m + N) = 0$

implies that $r \in \mathcal{P}$. Now let $m \in (N :_M \langle r \rangle)$ for some $r \in R - \mathcal{P}$. Then $rm \in N$ and $(r + \mathcal{P})(m + N) = 0$ gives $m \in N$ since $r \notin \mathcal{P}$. Hence $(N :_M \langle r \rangle) \subseteq N$.

$(iii) \Rightarrow (iv)$ Clearly $N \subseteq (N :_M J)$. Suppose $(N :_M \langle r \rangle) = N$ for every $r \in R - \mathcal{P}$. Let J be an ideal of R such that $J \nsubseteq \mathcal{P}$. Then there exists $x \in J - \mathcal{P}$ such that $(N :_M \langle x \rangle) = N$. Now let $m \in (N :_M J)$. Then $Jm \subseteq N$. In particular $xm \in N$. Hence $m \in N$ since $(N :_M \langle x \rangle) = N$. Thus $(N :_M J) \subseteq N$. Therefore $(N :_M J) = N$.

$(iv) \Rightarrow (v)$ Suppose $(N :_M J) = N$ for every ideal $J \nsubseteq \mathcal{P}$. Let $e \in M - N$. Clearly $\mathcal{P} \subseteq (N : \langle e \rangle)$. Now let $r \in (N : \langle e \rangle)$. Then $re \in N$. If $r \notin \mathcal{P}$, then $e \in (N :_M \langle r \rangle) = N$, which is a contradiction. Hence $r \in \mathcal{P}$, that is, $(N : \langle e \rangle) \subseteq \mathcal{P}$.

$(v) \Rightarrow (vi)$ Suppose $(N : \langle e \rangle) = \mathcal{P}$ for every $e \in M - N$. Clearly $\mathcal{P} \subseteq (N : L)$ for all submodules L of M such that $N \subsetneq L$. Let $r \in (N : L)$. Then
$(N : L) = (N : (\sum\limits_{l \in L} Rl)) = \bigcap\limits_{l \in L} (N : Rl) = \mathcal{P}$.

$(vi) \Rightarrow (vii)$ Suppose $(N : L) = \mathcal{P}$ for all submodule L of M such that $N \subsetneq L$. Let $Q \in Ass(M/N)$. Then $Q = Ann_R(x + N)$ for some $x \in M - N$. Note that $N \subsetneq \langle x \rangle + N$ and $\langle x \rangle + N$ is a submodule of M. By hypothesis, $(N : (\langle x \rangle + N)) = \mathcal{P}$. So $\mathcal{P}(\langle x \rangle + N) \subseteq N$ and $\mathcal{P}x \subseteq N$. Hence $\mathcal{P} \subseteq Ann_R(x + N) = Q$. Conversely let $r \in Q = Ann_R(x + N)$. Then $rx \in N$ for some $x \in M - N$. So $r(\langle x \rangle + N) \subseteq N$ and $r \in (N : (\langle x \rangle + N)) = \mathcal{P}$. Hence $Q \subseteq \mathcal{P}$.

$(vii) \Rightarrow (viii)$ Suppose $Ass(M/N) = \mathcal{P}$, that is, $\mathcal{P} = Ann_R(m + N)$ for all $m \in M - N$. Clearly $\mathcal{P} \subseteq Z_R(M/N)$. Now let $x \in Z_R(M/N)$. Then $xm \in N$ for some $m \in M - N$. Hence $x \in Ann_R(m + N)$ for some $m \in M - N$. So $x \in \mathcal{P}$ and hence $Z_R(M/N) \subseteq \mathcal{P}$.

$(viii) \Rightarrow (i)$ Suppose $\mathcal{P} = Z_R(M/N)$. Let $r \in R$, $m \in M - N$ such that $rm \in N$. Then $r \in Z_R(M/N) = \mathcal{P} = (N : M)$. Thus $rM \subseteq N$. Therefore N is a prime submodule of M. $\square$

Proposition 1.19 *[[4], Proposition 2] If N is a submodule of an R-module M whose residual $(N : M)$ is a maximal ideal of R, then N is a prime submodule. In particular, $\mathcal{M}M$ is a prime submodule of an R-module M for every maximal ideal $\mathcal{M}$ of R such that $\mathcal{M}M \neq M$.*

Proof. M/N is an $R/\mathcal{P}$-module since $\mathcal{P} = (N : M) \subseteq Ann(M/N)$. So M/N satisfies all vector space axioms. Hence M/N is a vector space over the field $R/\mathcal{P}$. Since M/N is torsion-free, by Theorem 1.18 N is a prime submodule of M.

In particular $(\mathcal{M}M : M) = \mathcal{M}$ since $\mathcal{M}$ is a maximal ideal. By the first part $\mathcal{M}M$ is a prime submodule. $\square$

The following corollary is an easy result of Proposition 1.19.

Corollary 1.20 *[[4], Proposition 3] Let N be a proper submodule of an R-module M and let $\mathcal{M}$ be a maximal ideal of R. Then N is $\mathcal{M}$-prime if and only if $\mathcal{M}M \subseteq N$. Consequently, if N is an $\mathcal{M}$-prime submodule of M, then so is every proper submodule of M containing N.*

Proof. Let N be a prime submodule of M such that $(N : M) = \mathcal{M}$. Then $\mathcal{M}M \subseteq N$. Conversely suppose $\mathcal{M}M \subseteq N$. Then $\mathcal{M} \subseteq (N : M)$. Since $\mathcal{M}$ is a maximal ideal, we have $(N : M) = \mathcal{M}$. Hence by Proposition 1.19, N is an $\mathcal{M}$-prime submodule.

Now let N be an $\mathcal{M}$-prime submodule of M and K be a submodule of M such that $N \subseteq K$. By the first part of the proof, $\mathcal{M}M \subseteq N \subseteq K$. Hence K is an $\mathcal{M}$-prime submodule of M again by the first part. $\square$

Proposition 1.21 *[[4], Proposition 4] If N is a maximal submodule of an R-module M, then N is a prime submodule and $(N : M)$ is a maximal ideal of R.*

Proof. Let N be a maximal submodule of M, then M/N is a simple R-module. Since every simple module is cyclic, M/N is a cyclic R-module, that is, $M/N = R\bar{m}$ for some $\bar{m} \in M/N$, and $Ann_R(\bar{m}) = Ann_R(M/N) = (N : M)$ is a maximal ideal. Hence N is a prime submodule by Proposition 1.19. □

Corollary 1.22 *[[4], Corollary] If M is a finitely generated module , then every proper submodule of M is contained in a prime submodule.*

Proof. Let M be a finitely generated module and N be a proper submodule of M. N is contained in a maximal submodule of M since any proper submodule of a finitely generated module is contained in a maximal submodule. Hence every proper submodule of M is contained in a prime submodule by Proposition 1.21. □

Definition 1.23 *Let M be an R-module, a submodule N of M is said to be maximal $\mathcal{P}$-prime submodule of M if N is a $\mathcal{P}$-prime submodule and there does not exist any $\mathcal{P}$-prime submodule containing N.*

In the same manner we can define maximal $\mathcal{P}$-primary submodule . N is a maximal $\mathcal{P}$-primary submodule provided that N is $\mathcal{P}$-primary and there does not exist any $\mathcal{P}$-primary submodule containing N.

Proposition 1.24 *[[4], Proposition 6] N is a maximal $\mathcal{P}$-primary submodule of a module M if and only if N is a maximal $\mathcal{P}$-prime submodule of M.*

Proof. Suppose first that N is a maximal primary submodule and $\sqrt{(N : M)} = \mathcal{P}$. Let us first show that for all $r \in R - (N : M)$, $(N :_M \langle r \rangle)$ is a $\mathcal{P}$-primary submodule. Let $r \in R - (N : M)$, $x \in R$, $y \notin (N :_M \langle r \rangle)$ and $xy \in (N :_M \langle r \rangle)$. Then $xyr \in N$ and we have $yr \in N$ or $x^n M \subseteq N$ for some positive integer n. If $yr \in N$, then $y \in (N :_M \langle r \rangle)$. This contradicts the choice of y. Hence $x^n M \subseteq N$. Then $x^n rM \subseteq N$ and $x^n M \subseteq (N :_M \langle r \rangle)$. Hence

$(N :_M \langle r\rangle)$ is a primary submodule. Let us show that $\sqrt{((N :_M \langle r\rangle) : M)} = \mathcal{P}$. Let $m \in \sqrt{((N :_M \langle r\rangle) : M)}$. Then $m^t \in ((N :_M \langle r\rangle) : M)$ for some positive integer t. Hence $m^t rM \subseteq N$. Since N is a primary submodule, we have $rm' \in N$ or $m^{tk}M \subseteq N$ for all $m' \in M$ and for some positive integer k. This gives $rM \subseteq N$ or $m^{tk} \in (N : M)$. Since $r \in R - \mathcal{P}$ we have $m \in \sqrt{(N : M)}$. Therefore $\sqrt{((N :_M \langle r\rangle) : M)} \subseteq \sqrt{(N : M)} = \mathcal{P}$. Now let $s \in \sqrt{(N : M)}$. Then $s^l \in (N : M)$ for some positive integer l. So $s^l M \subseteq N$ and $s^l rM \subseteq N$. This implies that $s^l M \subseteq (N :_M \langle r\rangle)$ and $s \in \sqrt{((N :_M \langle r\rangle) : M)}$. Thus $\mathcal{P} = \sqrt{(N : M)} \subseteq \sqrt{((N :_M \langle r\rangle) : M)}$. Therefore $(N :_M \langle r\rangle)$ is a $\mathcal{P}$-primary submodule. Since $N \subseteq (N :_M \langle r\rangle)$ and N is maximal, we have $N = (N :_M \langle r\rangle)$. Hence N is a prime submodule by Theorem 1.18.

Since $(N : M)$ is a prime ideal, $\sqrt{(N : M)} = (N : M) = \mathcal{P}$. Suppose K is a $\mathcal{P}$-prime submodule of M such that $N \subsetneqq K \subseteq M$. Since every prime submodule of a module is primary, K is a $\mathcal{P}$-primary submodule and maximality of N implies that K must be M. Therefore N is a maximal $\mathcal{P}$-prime submodule.

Conversely suppose N is a maximal $\mathcal{P}$-prime submodule of M. Then N is a $\mathcal{P}$-primary submodule of M by the definition. In order to complete the proof let us show N is maximal. Let K be a $\mathcal{P}$-primary submodule and $N \subsetneqq K \subseteq M$. By Proposition 1.17, K is a $\mathcal{P}$-prime submodule. Since N is a maximal $\mathcal{P}$-prime submodule, $K = M$. Hence N is a maximal $\mathcal{P}$-primary submodule of M. $\square$

Definition 1.25 *An ideal I of the ring R is said to be radical ideal if $I = \sqrt{I}$.*

Proposition 1.26 *[[4], Proposition 8] Let M be a finitely generated R-module and let I be a radical ideal of R. Then $(IM : M) = I$ if and only if $Ann_R(M) \subseteq I$.*

Proof. Suppose that $(IM : M) = I$ and $x \in Ann_R(M)$. Then $xM = 0 \subseteq IM$. Hence $x \in (IM : M) = I$.

Conversely assume that $Ann_R(M) \subseteq I$ and let $r \in R$ with $r \in (IM : M)$. If M is generated by n elements, then $r^n + y \in Ann_R(M) \subseteq I$ for some $y \in I$. Then $r^n + y = y'$ for some $y' \in I$. Hence $r^n \in I$ and $r \in \sqrt{I} = I$. Thus $(IM : M) \subseteq I$ and $I \subseteq (IM : M)$ is also satisfied. Therefore $(IM : M) = I$. □

Corollary 1.27 *[[4], Corollary 2] Let $\mathcal{P}$ be a finitely generated prime ideal of a ring R such that $Ann_R(\mathcal{P}) \subseteq \mathcal{P}$. Then the following statements are equivalent*

(i) $\mathcal{P}^2$ is a $\mathcal{P}$-primary ideal of R;

(ii) $\mathcal{P}^2$ is a $\mathcal{P}$-primary submodule of the R-module $\mathcal{P}$;

(iii) $\mathcal{P}^2$ is a $\mathcal{P}$-prime submodule of the R-module $\mathcal{P}$.

Proof. Note that since $Ann_R(\mathcal{P}) \subseteq \mathcal{P}$, $(\mathcal{P}^2 : \mathcal{P}) = \mathcal{P}$ by Proposition 1.26.

$(i) \Rightarrow (ii)$ Let $r \in R$, $p \notin \mathcal{P}^2$ and $rp \in \mathcal{P}^2$. Since $\mathcal{P}^2$ is a primary ideal, we have $r^n \in \mathcal{P}^2$ for some positive integer n. Hence $r^n\mathcal{P} \subseteq \mathcal{P}^2$ and $\mathcal{P}^2$ is a primary submodule. Since $(\mathcal{P}^2 : \mathcal{P}) = \mathcal{P}$ and $\mathcal{P}$ is a prime ideal of R, $\sqrt{\mathcal{P}} = \mathcal{P}$ and this implies $\sqrt{(\mathcal{P}^2 : \mathcal{P})} = \mathcal{P}$.

$(ii) \Rightarrow (iii)$ Since $\sqrt{(\mathcal{P}^2 : \mathcal{P})} = (\mathcal{P}^2 : \mathcal{P}) = \mathcal{P}$ is a prime ideal of R, $\mathcal{P}^2$ is a $\mathcal{P}$-prime submodule of $\mathcal{P}$ by Proposition 1.17.

$(iii) \Rightarrow (i)$ It is clear that $\sqrt{\mathcal{P}^2} = \mathcal{P}$. For; $\mathcal{P}^2 \subseteq \mathcal{P}$ we have $\sqrt{\mathcal{P}^2} \subseteq \sqrt{\mathcal{P}} = \mathcal{P}$ by the definition of radical. To prove the reverse inclusion let $x \in \mathcal{P}$, then $x^m \in \mathcal{P}$ for some positive integer m. Hence $x^{2m} = x^m x^m \in \mathcal{P}^2$, so $x \in \sqrt{\mathcal{P}^2}$. Let $r \in R$, $s \notin \mathcal{P}^2$ such that $rs \in \mathcal{P}^2$. This implies that $r\mathcal{P} \subseteq \mathcal{P}^2$ or $s \in \mathcal{P}^2$ since $\mathcal{P}^2$ is prime submodule. Since $s \notin \mathcal{P}^2$ we have $r\mathcal{P} \subseteq \mathcal{P}^2$. Then $r \in (\mathcal{P}^2 : \mathcal{P}) = \mathcal{P}$. Since we have showed that $\sqrt{\mathcal{P}^2} = \mathcal{P}$, there exists a positive integer n such that $r^n \in \mathcal{P}^2$. Therefore $\mathcal{P}^2$ is a primary ideal of R. □

Corollary 1.28 *[[4], Corollary 3] If M is a finitely generated R-module and $\mathcal{M}$ is a maximal ideal of R containing $Ann_R(M)$, then $\mathcal{M}M \neq M$ so that $\mathcal{M}M$ is a prime submodule of M. In particular, if M is a finitely generated faithful*

R-module , then $\mathcal{M}M$ is a prime submodule of M for every maximal ideal $\mathcal{M}$ of R.

Proof. Let $\mathcal{M}$ be a maximal ideal of R containing $Ann_R(M)$. By Proposition 1.26, $(\mathcal{M}M : M) = \mathcal{M}$. If $\mathcal{M}M = M$, then we get $(\mathcal{M}M : M) = (M : M)$ and this implies the contradiction $\mathcal{M} = R$. Hence by Proposition 1.19, $\mathcal{M}M$ is a prime submodule of M.

Now assume that M is a finitely generated faithful R-module. Since $Ann_R(M) = 0$, $Ann_R(M) \subseteq \mathcal{M}$ for every maximal ideal $\mathcal{M}$ of R. Hence by the first part of the proof $\mathcal{M}M$ is a prime submodule of M. □

Definition 1.29 *Let R be a ring and M be an R-module. Let N be a proper submodule of M and suppose that $N = P_1 \cap \cdots \cap P_n$, where $P_1, ..., P_n$ are primary submodules of M. This is called a primary decomposition of N.*

Corollary 1.30 *[[4], Corollary 4] Let M be a finitely generated R-module having primary decomposition for submodules and let $\mathcal{P}$ be a prime ideal containing $Ann_R(M)$. Then $(\mathcal{P}M : M) = \mathcal{P}$, consequently there exists a submodule of M which is $\mathcal{P}$-prime.*

Proof. By Proposition 1.26, $(\mathcal{P}M : M) = \mathcal{P}$. Since submodules of M have a primary decomposition, we can write $\mathcal{P}M = \bigcap_{i=1}^{k} N_i$ where each N_i is a $\mathcal{P}_i$-primary submodule of M. Then $\mathcal{P} = (\mathcal{P}M : M) = (\bigcap_{i=1}^{k} N_i : M) = \bigcap_{i=1}^{k}(N_i : M)$ by Lemma 1.9. Since $\mathcal{P}$ is a prime ideal and $\bigcap_{i=1}^{k}(N_i : M) \subseteq \mathcal{P}$, we have $(N_j : M) \subseteq \mathcal{P}$ for some j. Also $\mathcal{P} \subseteq (N_i : M)$ for every $i = 1, 2, \ldots, k$. Hence $\mathcal{P} = (N_j : M)$ for some j. Therefore N_j is a $\mathcal{P}$-prime submodule of M by Proposition 1.17. □

Applying Corollary 1.30 to Proposition 1.26 we can give some characterization of faithful Noetherian modules.

Theorem 1.31 *[[4], Theorem 2] Let M be a faithful Noetherian R-module. Then, for every prime ideal $\mathcal{P}$ of R, there exists a prime submodule N of M such that $(N : M) = \mathcal{P}$.*

Proof. Since M is a Noetherian module, M has a primary decomposition and M is a finitely generated module. Since $Ann_R(M) = 0$, we have $Ann_R(M) \subseteq \mathcal{P}$ for all prime ideals $\mathcal{P}$ of R. By Proposition 1.26, $(\mathcal{P}M : M) = \mathcal{P}$. Hence there exists a submodule of M which is $\mathcal{P}$-prime by Corollary 1.30. □

1.2 Prime Submodules Under Localization

In this section M will denote an R-module and S will denote a multiplicatively closed subset of R which contains 1 but not 0. Let M_S be the localization of M at S and $f : M \to M_S$ be a natural map. For any R_S-submodule Q of M_S, we define $Q^c = \{m \in M : f(m) \in Q\}$.

Proposition 1.32 *(i) For any prime R-submodule P of M with $M_S \neq P_S$, $P_S^c = P$.*

(ii) For any R_S-submodule Q of M_S, $Q_S^c = Q$.

Proof. (i) Let $x \in P$. Then $f(x) \in P_S$ and hence $x \in P_S^c$. Thus $P \subseteq P_S^c$. Now let $y \in P_S^c$. Then $f(y) \in P_S$, that is, $sy/s \in P_S$. There exist $p \in P$ and $t \in S$ such that $sy/s = p/t$. Then $usty = usp \in P$ for some $u \in S$. Hence $y \in P$ or $(ust)M \subseteq P$. If $y \in P$, then it is clear. If $(ust)M \subseteq P$, then for all $m \in M$ we have $(ust)m \in P$. Then we can write for any $x \in M$, $x = x.1/1 = ustx/ust \in P_S$. Thus $M_S = P_S$. This contradicts with our assumption. Therefore $P_S^c \subseteq P$.

(ii) Let $a/s \in Q_S^c$. Then there exist $b \in Q^c$ and $s' \in S$ such that $a/s = b/s'$. Then $us'a = usb \in Q^c$ for some $u \in S$. Since $us'a \in Q^c$, $f(us'a) = us'at/t \in Q$ for some $t \in S$. Hence $a/s = us'ta/sus't \in Q$. Conversely let $x \in Q$. Then $x = sx/s \in Q$, that is, $f(x) \in Q$. Hence $x \in Q^c$ and $sx/s \in Q_S^c$. □

Proposition 1.33 *[[7], Proposition 2.1] (i) If P is a prime submodule of M with $M_S \neq P_S$, then P_S is a prime R_S-submodule of M_S.*

(ii) If Q is a prime R_S-submodule of M_S, then Q^c is a prime R-submodule of M.

Proof. (i) Let $r \in R_S$, $m \in M_S$ and $rm \in P_S$ such that $r = x/s$, $m = y/s'$ for some $x \in R$, $y \in M$ and $s, s' \in S$. Then $rm = xy/ss' \in P_S$. There exist $z \in P$, $t \in S$ such that $xy/ss' = z/t$ and $u \in S$ with $(ut)xy = (uss')z \in P$. Then $y \in P$ or $(utx)M \subseteq P$ since P is a prime submodule. If $y \in P$, then $m = y/s' \in P_S$. If $(utx)M \subseteq P$, then $(utx)m_1 \in P$ for all $m_1 \in M$. Hence $(utx)m_1/s \in P_S$ and $(x/s)(uts'm_1/s') \in P_S$ for all $m_1 \in M$. Thus $rM_S \subseteq P_S$. Therefore P_S is a prime submodule of M_S.

(ii) Suppose $Q^c = M$. Then for all $m \in M$, $f(m) \in Q$. Hence $Q = M_S$. Therefore $Q^c \neq M$. Let $r \in R$, $m \in M$ and $rm \in Q^c$. Then

$$f(rm) = s(rm)/s = (rs/s)(sm/s) \in Q.$$

Thus $(rs/s)M_S \subseteq Q$ or $sm/s \in Q$. If $(rs/s)M_S \subseteq Q$, then $(rs/s)(tm_1/t) \in Q$ for all $m_1 \in M$. Then $(rst/st)m_1 = st(rm_1)/st = f(rm_1) \in Q$ and $rm_1 \in Q^c$ for all $m_1 \in M$. Hence $rM \subseteq Q^c$. If $(rs/s)M_S \nsubseteq Q$, then $sm/s \in Q$. Hence $f(m) \in Q$, that is, $m \in Q^c$. Therefore Q^c is a prime submodule of M. □

Corollary 1.34 *[[7], Corollary 2.2] Let $A = \{P : P$ is a prime R-submodule of M with $M_S \neq P_S\}$ and $B = \{Q : Q$ is a prime R_S-submodule of $M_S\}$. Then φ which is defined by $\varphi(P) = P_S$ is a bijective order preserving map from A to B. Its inverse map is given by $\varphi^{-1}(Q) = Q^c$.*

1.3 Usual Determinant Argument

Proposition 1.35 *Let R be a subring of the commutative ring S and let M be an S-module which, when considered as an R-module by restriction of scalars, can be finitely generated by n elements ($n \geq 1$). Let $s \in S$ and let I be an ideal of R such that $sM \subseteq IM$. Then there exist $a_i \in I^i$ for $i = 1, \dots, n$ such that*

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \in Ann_S(M).$$

Proof. Suppose that M is generated as an R-module by $g_1, \dots, g_n$. Then for each $i = 1, \dots, n$, there exist $b_{i1}, \dots, b_{in} \in I$ such that

$$sg_i = \sum_{j=1}^{n} b_{ij} g_j.$$

Then we have a system of homogenous equations such that

$$\sum_{j=1}^{n} (b_{ij} - s\delta_{ij}) g_j = 0,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We can write the system in the matrix form as

$$\begin{bmatrix} b_{11} - s & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} - s & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} - s \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Say C is the coefficient matrix of this system and multiply both sides of the system with $adj(C)$. Now let us use the fact $(adjC)C = C(adjC) = (\det C)I_n$ to deduce that

$$(\det C)g_i = 0 \quad \text{for all} \quad i = 1, \dots, n.$$

Hence $\det C \in Ann_S(M)$. It follows from the definition of determinant that

$$\det C = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n,$$

where $a_i \in I^i$ for $i = 1, \ldots, n$. $\square$

Corollary 1.36 *Let I and J be ideals of a ring R such that $IJ = I$ and I is finitely generated. Then there exists an element $y \in J$ such that $x = xy$ for each $x \in I$.*

Proof. Suppose that I is generated by n elements. Since every ideal of R can be considered as an R-module and I is finitely generated, we can use the usual determinant argument with $s = 1$. By Proposition 1.35, there exist $y_i \in J^i$ for all $i = 1, 2, \ldots, n$ such that

$$1 + y_1 + y_2 + \cdots + y_n \in Ann_R(I).$$

Hence $(1 + y_1 + y_2 + \cdots + y_n)x = 0$ for all $x \in I$. Then $x = xy$ where $y = (-y_1 - y_2 - \cdots - y_n) \in J$. $\square$

CHAPTER 2

PRIME RADICAL OF SUBMODULES

2.1 Properties of the Prime Radical

In this chapter we will deal with the prime radical of submodules and its properties.

Definition 2.1 *Let M be a module over a ring R and N be a submodule of M with $N \neq M$. The prime radical of N in M, $rad_M(N)$, is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then $rad_M(N) = M$. In particular $rad_M(M) = M$. We also define the radical $RadM$ of a module M as the intersection of all maximal proper submodules.*

Definition 2.2 *Let R be a commutative ring. Then the Jacobson radical of R is the intersection of all the maximal ideals of R. It is denoted by $J(R)$.*

The intersection of all prime ideals of R, $nil(R)$, is said to be the nil radical of R.

Lemma 2.3 *[5] Let M be an R-module. Then*

(i) $J(R)M \subseteq RadM$,

(ii) $nil(R)M \subseteq rad_M(0) \subseteq RadM$.

Proof. (i) Let P be any maximal submodule of M. Then P is a prime submodule of M and $(P : M) = \mathcal{M}$ is a maximal ideal of R by Proposition

1.21. Hence $J(R) \subseteq (P : M)$. Then $J(R)M \subseteq (P : M)M \subseteq P$. Hence $J(R)M \subseteq RadM$.

(ii) Let K be any prime submodule of M. Since $(K : M)$ is a prime ideal of R, $nil(R) \subseteq (K : M)$. Then $nil(R)M \subseteq (K : M)M \subseteq K$ and hence $nil(R)M \subseteq rad_M(0)$. We know that if L is any maximal submodule of M, then L is a prime submodule. Then $rad_M(0) \subseteq L$. Hence $rad_M(0) \subseteq RadM$. □

Theorem 2.4 *[[5], Theorem 1] Let M be a module over an Artinian ring R. Then*

(i) $RadM = J(R)M$,

(ii) $RadM = rad_M(0)$.

Proof. (i) $R/J(R)$ is a semisimple ring since R is an Artinian ring and $J(R/J(R)) = 0$. Then $M/J(R)M$ is a semisimple module as an $R/J(R)$-module. Hence $M/J(R)M$ is a semisimple module as an R-module. So $Rad(M/J(R)M) = RadM/J(R)M = 0$. Therefore $RadM = J(R)M$.

(ii) Since R is Artinian, $J(R) = nil(R)$. Then by the first part, $RadM = J(R)M = nil(R)M \subseteq rad_M(0) \subseteq RadM$. Hence $RadM = rad_M(0)$. □

Definition 2.5 *Let N be a submodule of an R-module M. A prime submodule P of M is called a minimal prime submodule of N if $N \subseteq P$ and there is no smaller prime submodule with this property.*

A minimal prime submodule of $\langle 0 \rangle$ is a minimal prime submodule of M.

Lemma 2.6 *[[5], Lemma 1] Let $\{P_i : i \in I\}$ be a non-empty family of prime submodules of an R-module M and suppose that the family is totally ordered by inclusion. Then $\bigcap_{i\in I} P_i$ is a prime submodule of M. Similarly, $\bigcup_{i\in I} P_i$ is a prime submodule of M if M is finitely generated.*

Proof. Say $(P_i : M) = \mathcal{P}_i$ for every $i \in I$. Then $\{\mathcal{P}_i : i \in I\}$ is totally ordered family of prime ideals of R. Thus, $\mathcal{P} = \bigcap_{i\in I} \mathcal{P}_i$ and $\mathcal{P}' = \bigcup_{i\in I} \mathcal{P}_i$ are prime ideals of R. Let $r \in R$, $m \in M$ and $rm \in \bigcap_{i\in I} P_i$. Then $rm \in P_i$, for every $i \in I$. Since P_i is a prime submodule, $m \in P_i$ or $rM \subseteq P_i$ for every $i \in I$. Also $(\bigcap_{i\in I} P_i : M) = \bigcap_{i\in I}(P_i : M) = \bigcap_{i\in I} \mathcal{P}_i = \mathcal{P}$, by Lemma 1.9. Hence $\bigcap_{i\in I} P_i$ is a $\mathcal{P}$-prime submodule of M. Also $\bigcup_{i\in I} P_i$ is a $\mathcal{P}'$-prime submodule of M. For, let $r'm \in \bigcup_{i\in I} P_i$. Then $r'm \in P_i$ for some $i \in I$. Since P_i is a prime submodule, $m \in P_i$ or $r'M \subseteq P_i$. Hence $m \in \bigcup_{i\in I} P_i$ or $r'M \subseteq \bigcup_{i\in I} P_i$ and $(\bigcup_{i\in I} P_i : M) = \mathcal{P}'$ by Lemma 1.10. □

Proposition 2.7 *[[5], Proposition 1] Let M be an R-module. If a submodule N of M is contained in a prime submodule P, then P contains a minimal prime submodule of N.*

Proof. Let $A = \{K : K$ is a prime submodule of M and $N \subseteq K \subseteq P\}$. $A \neq \emptyset$ since $P \in A$. If $K', K'' \in A$, then let us define a relation $\leq$ such that $K' \leq K''$ if $K'' \subseteq K'$. This gives a partial order on A. Let A' be a non-empty subset of A. By Lemma 2.6, intersection of all members of A' is a prime submodule; say $\bar{K}$. Then $N \subseteq \bar{K} \subseteq P$. Hence $\bar{K} \in A$ and $\bar{K} \subseteq K'$ for all $K' \in A'$. We have $K' \leq \bar{K}$. Thus $\bar{K}$ is an upper bound for A'. By Zorn's lemma, A contains a maximal element K^*. Since $K^* \in A$ we have $N \subseteq K^* \subseteq P$ and K^* is a prime submodule of M. To complete the proof we will show that K^* is a minimal prime submodule of N. Now suppose K_1 is a prime submodule of M such that $N \subseteq K_1 \subseteq P$ and $K_1 \subseteq K^*$. Then $K_1 \in A$ and $K^* \leq K_1$. Thus, $K^* = K_1$ since K^* is maximal in A. Therefore K^* is a minimal prime submodule of N. □

Corollary 2.8 *[[5], Corollary 1] Every proper submodule of a finitely generated module possesses at least one minimal prime submodule.*

Proof. Let M be a finitely generated module and N be a proper submodule of M. Then there exists a maximal submodule N' of M such that $N \subseteq N'$. Since N' is maximal, N' is a prime submodule by Proposition 1.21. Then by Proposition 2.7, N' contains a minimal prime submodule of N. Hence N has at least one minimal prime submodule. □

Corollary 2.9 *[[5], Corollary 2] Every prime submodule of a module M contains at least one minimal prime submodule of M.*

Proof. Let P be a prime submodule of M. In Proposition 2.7 take $N = 0$, then apply Proposition 2.7 to P. □

The above facts are summarized in the following theorem which gives another characterization of the prime radical of a submodule.

Theorem 2.10 *[[5], Theorem 2] Let M be a finitely generated R-module. Then the prime radical of a proper submodule N of M is the intersection of its minimal prime submodules.*

Proof. Let N be a proper submodule of M. Then by Corollary 2.8 N has at least one minimal prime submodule, say P_i. Let L be the intersection of all minimal prime submodules of N. Since $rad_M(N)$ is the intersection of all prime submodules of M containing N, $rad_M(N) \subseteq \bigcap\limits_{i\in I} P_i = L$. On the other hand, let P be any prime submodule containing N. Then P contains some minimal prime submodule Q_i of N by Proposition 2.7. Hence $L = \bigcap\limits_{i\in I} P_i \subseteq rad_M(N) \subseteq L$.

Lemma 2.11 *[[2], Corollary 1.3] Let M and M' be R-modules with $\varphi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $N \supseteq K = Ker\varphi$. Then*

(i) $\varphi(rad_M(N)) = rad_{M'}(\varphi(N))$.

(ii) If N' is a submodule of M', then $\varphi^{-1}(rad_{M'}(N')) = rad_M(\varphi^{-1}(N'))$.

Proof. (i) Let $x \in \varphi(rad_M(N))$. Then $x = \varphi(y)$ for some $y \in rad_M(N)$. So that $y \in P_i$ where P_i is a prime submodule of M containing N for all i. Then $\varphi(y) \in \varphi(P_i)$. By Lemma 1.14, $\varphi(P_i)$ is a prime submodule of M' containing $\varphi(N)$. Hence $\varphi(y) \in \bigcap_{i \in I} \varphi(P_i) = rad_{M'}(\varphi(N))$. Hence $x \in rad_{M'}(\varphi(N))$.

Let $m' \in rad_{M'}(\varphi(N))$. Then $m' \in Q_j$ for all prime submodules Q_j of M' containing $\varphi(N)$. There exists $m \in M$ such that $\varphi(m) = m' \in Q_j$. Then $m \in \varphi^{-1}(Q_j)$ such that $\varphi^{-1}(Q_j)$ is a prime submodule of M containing N by Lemma 1.14. Hence $m \in rad_M(N)$. So that $m' = \varphi(m) \in \varphi(rad_M(N))$.

(ii) Let $n' \in \varphi^{-1}(rad_{M'}(N'))$. Then $\varphi(n') \in rad_{M'}(N')$, that is, $\varphi(n') \in P$ for all prime submodule of M' containing N'. Hence $n' \in \varphi^{-1}(P)$ and $n' \in rad_M(\varphi^{-1}(N'))$. Let $\bar{n} \in rad_M(\varphi^{-1}(N'))$. Then $\bar{n} \in P'$ for all prime submodules P' of M containing $\varphi^{-1}(N')$ and $\varphi(\bar{n}) \in \varphi(P')$. Since $\varphi(P')$ is a prime submodule of M' containing N', $\varphi(\bar{n}) \in rad_{M'}(N')$. Then $\bar{n} \in \varphi^{-1}(rad_{M'}(N'))$. Hence $rad_M(\varphi^{-1}(N')) = \varphi^{-1}(rad_{M'}(N'))$. $\square$

Proposition 2.12 *[[6], Proposition 1] Let N be a submodule of an R-module M and I be an ideal of R. Then $Irad_M(N) \subseteq rad_M(IN)$.*

Proof. Let $\Lambda_1 = \{P : P$ is a prime submodule of M containing $N\}$, $\Lambda_2 = \{P' : P'$ is a prime submodule of M such that $I \subseteq (P' : M)\}$, and $\Lambda_3 = \{P^* : P^*$ is a prime submodule of M containing $IN\}$. Since each $P^* \in \Lambda_3$ is prime, $IN \subseteq P^*$ if and only if $N \subseteq P^*$ or $I \subseteq (P^* : M)$. Therefore $\Lambda_3 = \Lambda_1 \cup \Lambda_2$ and $rad_M(IN) = \bigcap_{P^* \in \Lambda_3} P^* = (\bigcap_{P \in \Lambda_1} P) \bigcap (\bigcap_{P' \in \Lambda_2} P') = rad_M(N) \cap rad_M(IM)$. It is clear that $Irad_M(N) \subseteq rad_M(N)$ and $Irad_M(N) \subseteq IM \subseteq rad_M(IM)$. Then $Irad_M(N) \subseteq rad_M(N) \cap rad_M(IM) = rad_M(IN)$. $\square$

In the following proposition we will give some basic rules concerning $rad_M(N)$.

Proposition 2.13 *[[5], Proposition 2 and [6], Corollary 1] Let M be an R-module, J be an index set and let N, L and N_j for $j \in J$ be submodules of*

M and I be an ideal of R. Then

(i) $N \subseteq rad_M(N)$,

(ii) $rad_M(rad_M(N)) = rad_M(N)$,

(iii) $rad_M(\bigcap_{j\in J} N_j) \subseteq \bigcap_{j\in J} rad_M(N_j) = rad_M(\bigcap_{j\in J} rad_M(N_j))$,

(iv) $\sum_{j\in J} rad_M(N_j) \subseteq rad_M(\sum_{j\in J} N_j) = rad_M(\sum_{j\in J} rad_M(N_j))$,

(v) $rad_M(IM) = rad_M(\sqrt{I}M)$ *for every ideal I of R,*

(vi) $\sqrt{(N : M)} \subseteq (rad_M(N) : M)$,

(vii) if M is finitely generated, then $rad_M(N) = M$ if and only if $N = M$,

(viii) $(rad_M(N) : rad_M(L)) = (rad_M(N) : L)$,

(ix) $rad_M(N :_M I) \subseteq (rad_M(N) :_M I) = rad_M((rad_M(N) :_M I))$,

(x) $rad_M(IN) = rad_M(Irad_M(N))$.

Proof. (i) Let $x \in N$. Then $x \in P$ for all prime submodules P of M containing N. Hence $x \in rad_M(N)$.

(ii) $rad_M(N) \subseteq rad_M(rad_M(N))$ is clear by (i). Since $N \subseteq rad_M(N) \subseteq Q_i$ for all prime submodules Q_i containing $rad_M(N)$, we have $\bigcap_{i\in I} Q_i \subseteq P_j$ for all prime submodules P_j containing N. Hence $rad_M(rad_M(N)) = \bigcap_{i\in I} Q_i \subseteq rad_M(N)$.

(iii) $rad_M(\bigcap_{j\in J} N_j) = \bigcap_{i\in I} P_i$ for all prime submodules P_i containing $\bigcap_{j\in J} N_j$ and let $\{Q_{ij}\}$ be the set of all prime submodules containing N_j. Since $\bigcap_{j\in J} N_j \subseteq N_j \subseteq Q_{ij}$ for all i and j, $rad_M(\bigcap_{j\in J} N_j) \subseteq rad_M(N_j)$. Hence $rad_M(\bigcap_{j\in J} N_j) \subseteq \bigcap_{j\in J} rad_M(N_j)$.

Since $\bigcap_{j\in J} rad_M(N_j) \subseteq rad_M(N_j)$ for all $j \in J$, $rad_M(\bigcap_{j\in J} rad_M(N_j)) \subseteq rad_M(rad_M(N_j)) = rad_M(N_j)$ for all $j \in J$. Hence $rad_M(\bigcap_{j\in J} rad_M(N_j)) \subseteq \bigcap_{j\in J} rad_M(N_j)$. By (i), $\bigcap_{j\in J} rad_M(N_j) = rad_M(\bigcap_{j\in J} rad_M(N_j))$ is clear.

(iv) Since $N_j \subseteq \sum_{j\in J} N_j$ for all j, $rad_M(N_j) \subseteq rad_M(\sum_{j\in J} N_j)$. Then $\sum_{j\in J} rad_M(N_j) \subseteq rad_M(\sum_{j\in J} N_j)$. Let us show that $rad_M(\sum_{j\in J} N_j) = rad_M(\sum_{j\in J} rad_M(N_j))$. It is clear that $\sum_{j\in J} N_j \subseteq \sum_{j\in J} rad_M(N_j)$. Then $rad_M(\sum_{j\in J} N_j) \subseteq rad_M(\sum_{j\in J} rad_M(N_j))$.

Let $\{Q_i\}$ be the set of all prime submodules of M containing $\sum_{j\in J} rad_M(N_j)$ and let P_k be any prime submodule of M such that $\sum_{j\in J} N_j \subseteq P_k$. Since $N_j \subseteq \sum_{j\in J} N_j$, $rad_M(N_j) \subseteq P_k$ for all j. Hence $\sum_{j\in J} rad_M(N_j) \subseteq P_k$ for all prime submodules P_k of M containing $\sum_{j\in J} N_j$. Then $\bigcap_{i\in I} Q_i \subseteq P_k$, for all P_j. Therefore $rad_M(\sum_{j\in J} rad_M(N_j)) \subseteq rad_M(\sum_{j\in J} N_j)$.

(v) This is trivially true if $rad_M(IM) = M$. If there exists a p-prime submodule P such that $IM \subseteq P$, then $I \subseteq (IM : M) \subseteq (P : M) = \mathcal{P}$. Then $\sqrt{I} \subseteq \mathcal{P}$. For; if $x \in \sqrt{I}$, then there exists a positive integer n such that $x^n \in I \subseteq \mathcal{P}$. Hence $x \in \mathcal{P}$. So that $\sqrt{I}M \subseteq \mathcal{P}M \subseteq P$. It follows that $rad_M(\sqrt{I}M) \subseteq P$, for every prime submodule P containing IM. Thus $rad_M(\sqrt{I}M) \subseteq rad_M(IM)$. Since $I \subseteq \sqrt{I}$, $IM \subseteq \sqrt{I}M$ and $rad_M(IM) \subseteq rad_M(\sqrt{I}M)$.

(vi) Let $x \in \sqrt{(N : M)}$. Then there exists a positive integer n such that $x^n \in (N : M)$. So that $x^n M \subseteq N \subseteq P$ for all prime submodules P containing N. Since P is a prime submodule, $x \in (rad_M(N) : M)$.

(vii) Let $N = M$, then $rad_M(N) = rad_M(M) = M$. Conversely assume $rad_M(N) = M$ and $N \neq M$. Since M is finitely generated, N has at least one minimal prime submodule P_i such that $N \subseteq P_i$ by Corollary 2.8. Hence $M = rad_M(N) = \bigcap_{i\in I} P_i$ by Theorem 2.10 and $M \subseteq P_i$. This contradicts the fact that P_i is a prime submodule. Therefore $N = M$.

(viii) Let $x \in (rad_M(N) : rad_M(L))$. Then $x rad_M(L) \subseteq rad_M(N)$ and $xL \subseteq rad_M(N)$. Hence $x \in (rad_M(N) : L)$. Let $r \in (rad_M(N) : L)$. Then $rL \subseteq rad_M(N)$ and $rM \subseteq P$ or $L \subseteq P$ for all prime submodule of M containing N. If $rM \subseteq P$, then $r rad_M(L) \subseteq rad_M(N)$. Hence $r \in (rad_M(N) : rad_M(L))$. If $L \subseteq P$, then $rad_M(L) \subseteq rad_M(N)$ and $r \in (rad_M(N) : rad_M(L))$. Hence $(rad_M(N) : L) \subseteq (rad_M(N) : rad_M(L))$.

(ix) Since $(N :_M I)$ is a submodule of M, we can write $Irad_M(N :_M I) \subseteq rad_M(I(N :_M I))$ by Proposition 2.12. Then $Irad_M(N :_M I) \subseteq rad_M(N)$ and $rad_M(N :_M I) \subseteq (rad_M(N) :_M I)$. Let P be any prime submodule of M containing N. Then $rad_M(rad_M(N) :_M I) \subseteq rad_M(P :_M I) \subseteq (rad_M(P) :_M I)$. This gives that $rad_M((rad_M(N) :_M I)) = (rad_M(N) :_M I)$.

(x) By Proposition 2.12, $Irad_M(N) \subseteq rad_M(IN)$ and then $rad_M(Irad_M(N)) \subseteq rad_M(rad_M(IN)) = rad_M(IN)$. Now $IN \subseteq Irad_M(N)$ gives that $rad_M(IN) \subseteq rad_M(Irad_M(N))$. □

Corollary 2.14 *[[5], Corollary 1] Let M be a finitely generated module and let N and L be submodules of M. Then $rad_M(N) + rad_M(L) = M$ if and only if $N + L = M$.*

Proof. Assume that $rad_M(N) + rad_M(L) = M$. Then by Proposition 2.13, $rad_M(rad_M(N) + rad_M(L)) = rad_M(N) + rad_M(L) = M$. By the same proposition $rad_M(N + L) = M$ and $N + L = M$.

Conversely suppose $N + L = M$. By Proposition 2.13, we get $M = rad_M(N + L) = rad_M(rad_M(N) + rad_M(L))$. Hence $rad_M(N) + rad_M(L) = M$. □

Corollary 2.15 *[[5], Corollary 2] Let M be an R-module and I be an ideal of R. Then $rad_M(I^n M) = rad_M(IM)$ for every positive integer n.*

Proof. Since $\sqrt{I^n} = \sqrt{I}$ we have $rad_M(I^n M) = rad_M(\sqrt{I^n} M) = rad_M(\sqrt{I} M) = rad_M(IM)$ by Proposition 2.13. □

Lemma 2.16 *[[1], Lemma 4] Let R be a ring and M an R-module. If N is a submodule of M, then $rad_N(0) \subseteq rad_M(0)$.*

Proof. Let K be any prime submodule of M. If $N \subseteq K$, then $rad_N(0) \subseteq K$. If $N \nsubseteq K$, then $N \cap K$ is a prime submodule of N. For, let $r \in R$, $m \in N$

and $rm \in N \cap K \subseteq K$. Since K is a prime submodule, $m \in K$ or $rM \subseteq K$. If $m \in K$, then $m \in N \cap K$. If $rM \subseteq K$, then $rN \subseteq K$. Thus $rN \subseteq N \cap K$ and hence $rad_N(0) \subseteq N \cap K \subseteq K$. In any case $rad_N(0) \subseteq K$. This implies that $rad_N(0) \subseteq rad_M(0)$. $\square$

Lemma 2.17 *[[1], Lemma 6] Let R be a ring and M be an R-module such that $M = \bigoplus_{\Lambda} M_\lambda$ is a direct sum of submodules M_λ for all $\lambda \in \Lambda$. Then $rad_M(0) = \bigoplus_{\Lambda} rad_{M_\lambda}(0)$.*

Proof. Since M_λ is a submodule of M, by Lemma 2.16 $rad_{M_\lambda}(0) \subseteq rad_M(0)$ for all $\lambda \in \Lambda$, so that $\bigoplus_{\Lambda} rad_{M_\lambda}(0) \subseteq rad_M(0)$. Let $m \in M$ and $m \notin \bigoplus_{\Lambda} rad_{M_\lambda}(0)$. Then there exists $\mu \in \Lambda$ such that $\pi_\mu(m) \notin rad_{M_\mu}(0)$, where $\pi_\mu : M \longrightarrow M_\mu$ denotes the canonical projection. Since $\pi_\mu(m) \notin rad_{M_\mu}(0)$, there exists a prime submodule N_μ of M_μ such that $\pi_\mu(m) \notin N_\mu$. Let $K = N_\mu \bigoplus (\bigoplus_{\lambda \neq \mu} M_\mu)$. By Lemma 1.11, K is a prime submodule of M and $m \notin K$ since $\pi_\mu(m) \notin N_\mu$. This implies that $m \notin rad_M(0)$. Therefore $rad_M(0) \subseteq \bigoplus_{\Lambda} rad_{M_\lambda}(0)$. $\square$

Lemma 2.18 *[[1], Lemma 3] Let R be a domain of dimension one and let M be an R-module. Then $rad_M(0) = T(M) \cap [\cap\{\mathcal{M}M : \mathcal{M}$ is a maximal ideal of $R\}]$, where $T(M)$ denotes the torsion submodule of M.*

Proof. By Corollary 1.7, $rad_M(0) \subseteq T(M) \cap [\cap\{\mathcal{M}M : \mathcal{M}$ is maximal $\}]$. Now let N be any prime submodule of M and let $\mathcal{P}' = (N : M)$. By Lemma 1.6, $\mathcal{P}'$ is a prime ideal of R. Since R is a domain of dimension one, $\mathcal{P}' = 0$ or $\mathcal{P}'$ is a maximal ideal. If $P' = 0$, then M/N is a torsion free R-module and hence $T(M) \subseteq N$. For let $m \in T(M)$. Then there exists $0 \neq r \in R$ such that $rm = 0 \in N$. Since M/N is torsion free, $m \in N$. Hence $T(M) \cap [\cap\{\mathcal{M}M\}] \subseteq T(M) \subseteq N$. If $\mathcal{P}' \neq 0$, then $\mathcal{P}'$ is a maximal ideal and $\mathcal{P}'M \subseteq N$. So that $T(M) \cap [\cap\{\mathcal{M}M\}] \subseteq \mathcal{P}'M \subseteq N$. Hence in any case, $T(M) \cap [\cap\{\mathcal{M}M\}] \subseteq N$. Therefore $T(M) \cap [\cap\{\mathcal{M}M\}] \subseteq rad_M(0)$.

$\square$

Proposition 2.19 *Let R be a Noetherian domain of dimension one and r be any element of R. Then there exist finite number of maximal ideals of R containing r.*

Proof. Let $\Phi = \{I : I$ is an ideal of R and $I = \sum_{i=1}^{n} \mathcal{M}_i$ where $\mathcal{M}_i$ is a maximal ideal containing $r\}$. Since R is Noetherian, Φ has a maximal element say, $J = \mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_n$. Assume that $\mathcal{M}$ is a maximal ideal of R containing r and $\mathcal{M} \neq \mathcal{M}_i$ for all $i = 1, \ldots, n$. Since J is maximal, $\mathcal{M} + \mathcal{M}_1 + \cdots + \mathcal{M}_n = \mathcal{M}_1 + \cdots + \mathcal{M}_n$. Hence $\mathcal{M}_i = \mathcal{M}$ for all $i = 1, \ldots, n$. Therefore there exist only a finite number of maximal ideals containing r. $\square$

The following lemma states that for Noetherian domains of dimension one, the study of prime radicals of general modules reduces to that for finitely generated modules.

Lemma 2.20 *[[1], Lemma 5] Let R be a Noetherian domain of dimension one and let M be any R-module. Then $rad_M(0) = \bigcup rad_L(0)$, where the union is taken over all finitely generated submodules L of M.*

Proof. By Lemma 2.16, $rad_L(0) \subseteq rad_M(0)$ and $\bigcup rad_L(0) \subseteq rad_M(0)$ for any finitely generated submodule L of M. On the other hand, let $m \in rad_M(0)$. By Lemma 2.18, $m \in T(M) \cap [\cap\{\mathcal{M}M\}]$. Hence there exists $0 \neq r \in R$ such that $rm = 0$. By Proposition 2.19, there exists a positive integer n such that $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n$ are maximal ideals containing r. By Lemma 2.18, $m \in \mathcal{M}_i M$ for each $i = 1, \ldots, n$. Let L_i be a finitely generated submodule of M, say $L_i = Rm_i$. Then $\mathcal{M}_i L_i = \mathcal{M}_i R m_i$. Since $m \in \mathcal{M}_i M$, $m = x_1 m_1 + x_2 m_2 + \cdots + x_n m_n$ such that $x_i \in \mathcal{M}_i$ and $m_i \in M$. Hence $m \in \mathcal{M}_i L_i$. Let $L = L_1 + L_2 + \cdots + L_n$. Then L is a finitely generated submodule of M and $m \in \mathcal{M}_i L$ since $\mathcal{M}_i L_i \subseteq \mathcal{M}_i L$. Now let $\mathcal{M}$ be any maximal ideal of R. Suppose that $\mathcal{M} \neq \mathcal{P}_i$ for all i. Then $Rr + \mathcal{M} = R$ and

$Rm = Rrm + \mathcal{M}m$. Thus $Rm = \mathcal{M}m$ and $m \in Rm = \mathcal{M}m \subseteq \mathcal{M}\mathcal{M}_i L \subseteq \mathcal{M}L$. By Lemma 2.18, $m \in rad_L(0)$. It gives us that $rad_M(0) \subseteq \bigcup rad_L(0)$. $\square$

Lemma 2.21 *[[7], Corollary 2.3] Let M be an R-module and S be a multiplicatively closed subset of R which contains* 1 *but not* 0. *Suppose M is a Noetherian R-module and N is an R-submodule of M. Then $(rad_M(N))_S = rad_{M_S}(N_S)$.*

Proof. Suppose $M_S = N_S$. Then $(rad_M(N))_S = (\bigcap_{i=1}^{l} P_i)_S = \bigcap_{i=1}^{l} P_{i_S}$ where P_i is a prime submodule of M containing N for all i. Without loss of generality, we may assume each $P_{i_S} \neq M_S$. By Corollary 1.34, P_{i_S} are all minimal prime R_S-submodule of M_S which contains N_S for all $i = 1, ..., l$. Then $rad_{M_S}(N_S) = \bigcap_{i=1}^{l} P_{i_S} M_S = (rad_M(N))_S$. Now assume $M_S \neq N_S$. By using the preceding argument equality occurs. $\square$

2.2 The Prime Radical of an Intersection

It is not always true for submodules N and L that $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$. Vector spaces and multiplication modules (see, Definition 3.13) satisfy this property. In particular every cyclic module enjoys this property. Now we will investigate under what conditions this property holds.

Proposition 2.22 *[[6], Proposition 2] Let M be a finitely generated R-module, $\mathcal{M}$ be a maximal ideal of R, and Q be an $\mathcal{M}$-primary submodule of M. Then $rad_M(Q)$ is an $\mathcal{M}$-prime submodule and $rad_M(Q) = rad_M(Q + \mathcal{M}M) = Q + \mathcal{M}M$.*

Proof. By Proposition 2.13, $\mathcal{M} = \sqrt{(Q : M)} \subseteq (rad_M(Q) : M)$. Since $\mathcal{M}$ is a maximal ideal, $(rad_M(Q) : M) = R$ or $\mathcal{M}$. If $(rad_M(Q) : M) = R$, then $M = RM \subseteq rad_M(Q)$. By Proposition 2.13, $Q = M$ and this gives a contradiction. Hence $(rad_M(Q) : M) = \mathcal{M}$. Thus by Proposition 1.19, $rad_M(Q)$ is an $\mathcal{M}$-prime submodule.

Every prime submodule containing $Q + \mathcal{M}M$ also contains Q. Hence $rad_M(Q) \subseteq rad_M(Q + \mathcal{M}M)$. Now put $rad_M(Q) = \bigcap_{i \in I} P_i$ for every $\mathcal{P}_i$-prime submodule P_i of M containing Q. Then $\mathcal{M} = \sqrt{(Q : M)} \subseteq (rad_M(Q) : M) \subseteq (P_i : M) = \mathcal{P}_i$ for every i by Proposition 2.13. Hence $\mathcal{M}M \subseteq \mathcal{P}_i M$, so that $Q + \mathcal{M}M \subseteq P_i + \mathcal{P}_i M = P_i$ for every i. Thus $rad_M(Q + \mathcal{M}M) \subseteq \bigcap_{i \in I} P_i = rad_M(Q)$. In order to complete the proof it is enough to show that $rad_M(Q) = Q + \mathcal{M}M$. $Q + \mathcal{M}M \subseteq rad_M(Q + \mathcal{M}M) = rad_M(Q)$ is clear by Proposition 2.13. Since $\mathcal{M}M \subseteq Q + \mathcal{M}M$, we have $\mathcal{M} \subseteq ((Q + \mathcal{M}M) : M) \subseteq (rad_M(Q) : M) = \mathcal{M}$. Thus $((Q + \mathcal{M}M) : M) = \mathcal{M}$, so that $Q + \mathcal{M}M$ is a prime submodule of M by Proposition 1.19. Then $rad_M(Q) \subseteq Q + \mathcal{M}M$ since $Q \subseteq Q + \mathcal{M}M$. Therefore $rad_M(Q) = Q + \mathcal{M}M$. □

Lemma 2.23 *[[5], Lemma 2] Let N and L be submodules of an R-module M, and P be a $\mathcal{P}$-prime submodule of M such that $N \cap L \subseteq P$. If $(N : M) \nsubseteq \mathcal{P}$, then $L \subseteq P$.*

Proof. Suppose $N \cap L \subseteq P$, $(N : M) \nsubseteq \mathcal{P}$ and $L \nsubseteq P$. Since $N \cap L \subseteq P$, $((N \cap L) : M) \subseteq (P : M)$ so that $(N : M) \cap (L : M) \subseteq (P : M)$. Then $(N : M) \subseteq (P : M)$ or $(L : M) \subseteq (P : M)$ since $(P : M)$ is a prime ideal. This gives us that $(N : M) \subseteq (P : M)$ since $L \nsubseteq P$. Hence $(N : M) \subseteq \mathcal{P}$. This is a contradiction. Therefore $L \subseteq P$. □

Proposition 2.24 *[[5], Proposition 5] Let I be an ideal of R and N be a submodule of an R-module M. If P is a $\mathcal{P}$-prime submodule of M such that $IM \cap N \subseteq P$, then $IM \subseteq P$ or $N \subseteq P$.*

Proof. Since $IM \cap N \subseteq P$, $(IM : M) \cap (N : M) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal of R, either $(IM : M) \subseteq \mathcal{P}$ or $(N : M) \subseteq \mathcal{P}$. If $(IM : M) \subseteq \mathcal{P}$, then

$I \subseteq (IM : M) \subseteq \mathcal{P}$. This gives that $IM \subseteq \mathcal{P}M \subseteq P$. If $(IM : M) \nsubseteq \mathcal{P}$, then $N \subseteq P$ by Lemma 2.23. $\square$

Proposition 2.25 *[[5], Proposition 6] Let N and L be submodules of an R-module M such that whenever $N \cap L \subseteq P$ we have $N \subseteq P$ or $L \subseteq P$ for any prime submodule P of M. Then $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$.*

Proof. If $rad_M(N \cap L) = M$, then $rad_M(N) = rad_M(L) = M$ and $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$. If $rad_M(N \cap L) \neq M$, then there exists a prime submodule P such that $N \cap L \subseteq P$. By the hypothesis, $N \subseteq P$ or $L \subseteq P$. So that $rad_M(N) \subseteq P$ or $rad_M(L) \subseteq P$. Since this is true for all prime submodules P containing $N \cap L$, $rad_M(N) \cap rad_M(L) \subseteq rad_M(N \cap L)$. Therefore $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$. $\square$

Combining Proposition 2.24 and Proposition 2.25 we have the following result.

Corollary 2.26 *[[6], Corollary 2] For any ideal I of a ring R and a submodule N of an R-module M, $rad_M(IM \cap N) = rad_M(IM) \cap rad_M(N) = rad_M(IN)$.*

Corollary 2.27 *[[6], Corollary 3] For any ideal I of a ring R and a submodule N of an R-module M, $rad_M(IN) = rad_M(\sqrt{I}N) = rad_M(I rad_M(N))$.*

Proof. By Corollary 2.26 and Proposition 2.13, $rad_M(IN) = rad_M(IM) \cap rad_M(N) = rad_M(\sqrt{I}M) \cap rad_M(N) = rad_M(\sqrt{I}N)$. Also we know that $rad_M(IN) = rad_M(I rad_M(N))$ by Proposition 2.13. $\square$

Proposition 2.28 *[[6], Proposition 3] If N and L are submodules of an R-module M such that $\sqrt{(N : M)} + \sqrt{(L : M)} = R$, then $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$.*

Proof. Suppose P is a $\mathcal{P}$-prime submodule containing $N \cap L$, then $(N : M) \cap (L : M) \subseteq \mathcal{P}$ so that $(N : M) \subseteq \mathcal{P}$ or $(L : M) \subseteq \mathcal{P}$. If $(N : M) \subseteq \mathcal{P}$, then $(L : M) \nsubseteq \mathcal{P}$. Otherwise $\sqrt{(N : M)} + \sqrt{(L : M)} = R \subseteq \mathcal{P}$ which gives a contradiction. Hence $N \subseteq P$ by Lemma 2.23. Therefore we have $N \subseteq P$ or $L \subseteq P$. By Proposition 2.25, $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$. □

Corollary 2.29 *[[6], Corollary] Let R be a ring in which every non-zero prime ideal is maximal and M be an R-module. If N and L are primary submodules of M such that $\sqrt{(N : M)} \nsubseteq \sqrt{(L : M)}$ and that $(L : M) \neq \langle 0 \rangle$, then $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$.*

Proof. Say $\mathcal{P}_1 = \sqrt{(N : M)}$ and $\mathcal{P}_2 = \sqrt{(L : M)}$. Since $\mathcal{P}_1 \nsubseteq \mathcal{P}_2$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are both distinct non-zero prime ideals. Hence $\mathcal{P}_1 + \mathcal{P}_2 = R$ and we have $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$ by Proposition 2.28. □

Proposition 2.30 *[[5], Proposition 7] If V is a vector space, then $rad_V(W_1 \cap W_2) = rad_V(W_1) \cap rad_V(W_2)$ for every pair of subspaces W_1 and W_2.*

Proof. Since every proper subspace of a vector space is $\langle 0 \rangle$-prime, W_1, W_2 and $W_1 \cap W_2$ are prime submodules of V. Then $rad_V(W_1) = W_1$, $rad_V(W_2) = W_2$ and $rad_V(W_1 \cap W_2) = W_1 \cap W_2$. Hence, $rad_V(W_1 \cap W_2) = rad_V(W_1) \cap rad_V(W_2)$. □

The converse of Proposition 2.25 is not necessarily true in view of Proposition 2.30 and the following remark.

Remark 2.31 *Let V be the Euclidean $3-$space $\mathbb{R}^3$. If W_1 and W_2 are non-identical one dimensional subspaces (lines), then $W_1 \cap W_2 = \langle 0 \rangle \subseteq U$ for every subspace U of V, even though $W_1 \nsubseteq U$ and $W_2 \nsubseteq U$.*

Now we will give an example of a module which does not satisfy the property $rad_M(N \cap L) = rad_M(N) \cap rad_M(L)$.

Example 2.32 ([6], Example) *Let $M = \mathbb{Z} \times \mathbb{Z}$, a free $\mathbb{Z}$-module, $N = (2,3)\mathbb{Z}$, a cyclic module of M generated by $(2,3)$ and $L = (4,0)\mathbb{Z} + (0,1)\mathbb{Z}$. Then $rad_M(N \cap L)$ is not equal to $rad_M(N) \cap rad_M(L)$.*

Proof. Firstly, we show that N is a $\langle 0\rangle$-prime submodule of M. Clearly, $(N : M) = \langle 0\rangle$. Let $r \in \mathbb{Z}$ and $(a,b) \in M$ such that $r(a,b) \in N$. Then $r(a,b) = (2,3)x$ for some $x \in \mathbb{Z}$. If $r = 0$, then $rM \subseteq N$. Assume that $r \neq 0$. If $a = b = 0$, then $(a,b) = (0,0) \in N$. If $a \neq 0$ (or $b \neq 0$), then $ra = 2x \neq 0$ so that $x \neq 0$ and $rb = 3x \neq 0$. Then $r = 2x/a = 3x/b$ and hence $(a,b) = (2,3)c \in N$ for some $c \in \mathbb{Z}$. Therefore N is a $\langle 0\rangle$-prime submodule of M and hence $rad_M(N) = N$. Similarly, we can verify that L is a $\langle 2\rangle$-primary submodule and that $2M$ is a $\langle 2\rangle$-prime submodule. Let $n \in \mathbb{Z}$, $(z,t) \in M$ such that $n(z,t) \in L$ and $(z,t) \notin L$. Since $(z,t) \notin L$, $z \neq 4u$ for some integer u. Hence n is even or $4 \mid n$. Therefore L is primary submodule. It is clear that $\sqrt{(L : M)} = \langle 2\rangle$. Since $\langle 2\rangle$ is a maximal ideal of $\mathbb{Z}$, it is enough to show that $(2M : M) = \langle 2\rangle$ for $2M$ is a prime submodule. It is clear that $2M = 2(\mathbb{Z} \times \mathbb{Z}) = 2\mathbb{Z} \times 2\mathbb{Z}$. Since $\langle 2\rangle M \subseteq 2M$, $\langle 2\rangle \subseteq (2M : M)$. Let $y \in (2M : M)$. Then $y(e,f) = (2g, 2h)$ for some $g, h \in \mathbb{Z}$ and for all $(e,f) \in M$. $ye = 2g$ and $yf = 2h$. Then we have $(2 \mid y$ or $2 \mid e)$ and $(2 \mid y$ or $2 \mid f)$. Since this is true for all $(e,f) \in M$, we have $2 \mid y$. Hence $y \in \langle 2\rangle = 2\mathbb{Z}$. Therefore $2M$ is a $\langle 2\rangle$-prime submodule.

Applying Proposition 2.22, we have

$$rad_M(L) = L + 2M = (4\mathbb{Z} \times \mathbb{Z}) + (2\mathbb{Z} \times 2\mathbb{Z}) = (2,0)\mathbb{Z} + (0,1)\mathbb{Z}.$$

Consequently $rad_M(N) \cap rad_M(L) = (2,3)\mathbb{Z} \cap (2\mathbb{Z} \times \mathbb{Z}) = (2,3)\mathbb{Z}$ because $(2,3)\mathbb{Z} \subseteq (2\mathbb{Z} \times \mathbb{Z})$. On the other hand, $N \cap L = (2,3)\mathbb{Z} \cap (4\mathbb{Z} \times \mathbb{Z}) = (4,6)\mathbb{Z}$. For; $(u,v) \in N \cap L$, $(u,v) = (2k, 3k)$ and $(u,v) = (4t, l)$ for some $k, t, l \in \mathbb{Z}$. Since $(2k, 3k) = (4t, l)$, $(u,v) = (4t, 6t) = (4,6)t$. Since $(2,3)\mathbb{Z}$ and $2M$ are prime submodules of M containing $(4,6)\mathbb{Z}$,

$$(4,6)\mathbb{Z} \subseteq rad_M(N \cap L) = rad_M((4,6)\mathbb{Z}) \subseteq (4,6)\mathbb{Z}$$

which means that $(4,6)\mathbb{Z}$ is a radical submodule. Thus

$$rad_M(N \cap L) = rad_M((4,6)\mathbb{Z}) = (4,6)\mathbb{Z} \subsetneqq (2,3)\mathbb{Z} = rad_M(N) \cap rad_M(L).$$

□

CHAPTER 3

THE RADICAL FORMULA

3.1 The Envelope of a Module

Let M be an R-module, I be an ideal of R and N be a submodule of M. As is well-known, the radical of I defined as the intersection of all prime ideals containing I, has the characterization $\sqrt{I} = \{r \in R : r^n \in I$, for some $n \in \mathbb{Z}^+\}$. A natural question arises as to whether there is a similar characterization for the prime radical of a submodule, in particular, a characterization in which knowledge of prime submodules (indeed even prime ideals) is not necessary. We will show that under certain conditions such a characterization is provided by the concept of the envelope of a submodule.

Definition 3.1 *Let M be an R-module and N a submodule of M. The envelope of N in M which is denoted by $E_M(N)$ is defined to be the set*

$$\{rm : r \in R, m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbb{Z}^+\}.$$

The envelope of a submodule is not a submodule in general. It is clear that $\langle E_M(N)\rangle$, the submodule generated by $E_M(N)$, is always contained in $rad_M(N)$.

In Lemma 2.11, we showed that prime radical of a submodule is linear but the envelope is not linear in the sense that if φ is an R-module homomorphism on M and N is a submodule of M, it is not necessary that $\varphi(E_M(N)) = E_{M'}(\varphi(N))$. As the following examples show the results in Lemma 2.11 are not true for the envelope if $N \not\supseteq Ker\varphi$ or φ is not onto.

Example 3.2 ([2], Remark) *Let $\varphi : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ be the natural $\mathbb{Z}$-module epimorphism and $N = \langle 0 \rangle$. Then $\varphi(E_M(N))$ is not equal to $E_{M'}(\varphi(N))$.*

Proof. We know that $E_M(\langle 0 \rangle) = \{rx : r, x \in \mathbb{Z}$ such that $r^n x \in \langle 0 \rangle$ for some $n \in \mathbb{Z}\}$. Let $rx \in E_M(\langle 0 \rangle)$ where $0 \neq r \in \mathbb{Z}$ and $x \in \mathbb{Z}$. Then $r^n x = 0$. This gives that $r = 0$ or $rx = 0$. Since $r \neq 0$, $E_M(N) = \langle 0 \rangle$ and $\varphi(E_M(N)) = \langle 0 \rangle + 4\mathbb{Z} = \langle \bar{0} \rangle$. On the other hand, $\varphi(N) = \varphi(\langle 0 \rangle) = \langle \bar{0} \rangle$ and

$$E_{M'}(\varphi(N)) = E_{M'}(\langle \bar{0} \rangle) = \{r\bar{x} : r \in \mathbb{Z},\ \bar{x} \in \mathbb{Z}/4\mathbb{Z} \text{ and } r^n \bar{x} = \bar{0} \text{ for some } n \in \mathbb{Z}^+\}.$$

It is clear that $E_{M'}(\varphi(N)) = \langle \bar{2} \rangle$. Hence $E_{M'}(\varphi(N)) \neq \varphi(E_M(N))$. $\square$

Example 3.3 ([2], Remark) *Let $\varphi : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ be a homomorphism such that $\varphi(x) = 2x$ for some $x \in \mathbb{Z}$ and $N = \langle 2 \rangle = Ker\varphi$. Then $\varphi(E_M(N))$ is not equal to $E_{M'}(\varphi(N))$.*

Proof. Note that $E_M(N) = E_M(\langle 2 \rangle) = \{ra : r, a \in \mathbb{Z},\ r^n a = 2t,\ n, t \in \mathbb{Z}^+\}$. If either r or a is even, then $r^n a = 2t$ for some $n \in \mathbb{Z}^+$. Now suppose $r = 2m + 1$ and $a = 2k + 1$ for some $m, k \in \mathbb{Z}$. Then $r^n a$ is not even. If any of r and a is odd, then the number $r^n a$ is even; so ra is an even number. Hence $ra \in \langle 2 \rangle$. Therefore $E_M(\langle 2 \rangle) = \langle 2 \rangle$. Also $\varphi(E_M(N)) = \varphi(\langle 2 \rangle) = \langle \bar{0} \rangle$. Conversely, $E_{M'}(\varphi(\langle 2 \rangle)) = E_{M'}(\langle \bar{0} \rangle) = \langle \bar{2} \rangle$. Therefore $\varphi(E_M(N)) \neq E_{M'}(\varphi(N))$. $\square$

Lemma 3.4 *[[2], Lemma 1.4] Assume the hypothesis given in Lemma 1.13. Then*

(i) $\varphi(E_M(N)) = E_{M'}(\varphi(N))$;

(ii) $\langle \varphi^{-1}(E_{M'}(N')) \rangle = \langle E_M(\varphi^{-1}(N')) \rangle$.

Proof. (i) Let $x \in \varphi(E_M(N))$. Then $x = \varphi(y)$ for some $y \in E_M(N)$. So $y = rm$ such that $r \in R$, $m \in M$ and $r^n m \in N$ for some positive integer n. Then $x = \varphi(rm) = r\varphi(m)$. Since $r^n m \in N$ we have $\varphi(r^n m) = r^n \varphi(m) \in \varphi(N)$ and $r\varphi(m) \in E_{M'}(\varphi(N))$. Hence $x = r\varphi(m) \in E_{M'}(\varphi(N))$. Now assume that

$z \in E_{M'}(\varphi(N))$. Then $z = sm'$ such that $s \in R$, $m' \in M'$ and $s^t m' \in \varphi(N)$ for some positive integer t. Since φ is onto, there exists $\bar{m} \in M$ such that $\varphi(\bar{m}) = m'$. Then $z = s\varphi(\bar{m}) = \varphi(s\bar{m})$ and $s^t\varphi(\bar{m}) = \varphi(s^t\bar{m}) \in \varphi(N)$. This implies that $s^t\bar{m} \in N$, that is, $s\bar{m} \in E_M(N)$. Hence $z \in \varphi(E_M(N))$.

(ii) Assume that $x \in E_M(\varphi^{-1}(N'))$. Then there exist $r \in R$, $m \in M$ such that $x = rm$ and $r^n m \in \varphi^{-1}(N')$. This gives that $\varphi(r^n m) = r^n\varphi(m) \in N'$ and so $r\varphi(m) = \varphi(rm) \in E_{M'}(N')$. Hence $x = rm \in \varphi^{-1}(E_{M'}(N'))$ and thus $\langle E_M(\varphi^{-1}(N'))\rangle \subseteq \langle \varphi^{-1}(E_{M'}(N'))\rangle$. Now assume $x' \in \varphi^{-1}(E_{M'}(N'))$. Then $\varphi(x') \in E_{M'}(N')$, so there exist $s \in R$, $m' \in M'$ and $k \in \mathbb{Z}^+$ such that $\varphi(x') = sm'$ and $s^k m' \in N'$ and also $\varphi(y) = m'$ for some $y \in M$. We get $s^k\varphi(y) = \varphi(s^k y) \in N'$. Then $s^k y \in \varphi^{-1}(N')$ and $sy \in E_M(\varphi^{-1}(N'))$. Since $\varphi(x') = s\varphi(y)$, $x' - sy \in ker\varphi \subseteq \varphi^{-1}(N') \subseteq E_M(\varphi^{-1}(N')) \subseteq \langle E_M(\varphi^{-1}(N'))\rangle$ and $x' \in \langle E_M(\varphi^{-1}(N'))\rangle$. Hence $\langle \varphi^{-1}(E_{M'}(N'))\rangle \subseteq \langle E_M(\varphi^{-1}(N'))\rangle$. □

Corollary 3.5 *Assume the hypothesis given in Lemma 1.13. Let S be any subset of M' containing* 0*. Then $\varphi^{-1}(\langle S\rangle) = \langle \varphi^{-1}(S)\rangle$.*

Proof. Let $x \in \varphi^{-1}(\langle S\rangle)$. Then $\varphi(x) \in \langle S\rangle$ and thus $\varphi(x) \in N$ for all submodules N of M' containing S and $x \in \varphi^{-1}(N)$. Hence $x \in \langle \varphi^{-1}(S)\rangle$. Now let $y \in \langle \varphi^{-1}(S)\rangle$. Then $y = r_1m_1 + \cdots + r_nm_n$ for some $r_i \in R$ and $m_i \in \varphi^{-1}(S)$ $(1 \leq i \leq n)$. Hence $\varphi(y) = \varphi(r_1m_1 + \cdots + r_nm_n) = r_1\varphi(m_1) + \cdots + r_n\varphi(m_n)$ such that $\varphi(m_i) \in S$. Then $\varphi(y) \in \langle S\rangle$ and $y \in \varphi^{-1}(\langle S\rangle)$. □

Now using the above facts we will give the relationship between the submodule generated by the envelope of a given submodule and the prime radical of that submodule.

Theorem 3.6 *[[2], Lemma 1.5] Assume the hypothesis given in Lemma 1.13.*

(i) If $rad_M(N) = \langle E_M(N)\rangle$, then $rad_{M'}(\varphi(N)) = \langle E_{M'}(\varphi(N))\rangle$,

(ii) If N' is a submodule of M' and $rad_{M'}(N') = \langle E_{M'}(N')\rangle$, then $rad_M(\varphi^{-1}(N')) = \langle E_M(\varphi^{-1}(N'))\rangle$.

Proof. (i) Suppose that $rad_M(N) = \langle E_M(N)\rangle$. We know that $rad_{M'}(\varphi(N)) = \varphi(rad_M(N))$. So we must show that $\varphi(\langle E_M(N)\rangle) = \langle E_{M'}(\varphi(N))\rangle$. Let $x \in \varphi(\langle E_M(N)\rangle)$. Then there exist $y \in \langle E_M(N)\rangle$, $r_i \in R$ $(1 \le i \le n)$ and $x_i \in E_M(N)$ such that $x = \varphi(y)$ and $y = r_1x_1 + \cdots + r_nx_n$. Also we have $x_i = a_ib_i$ such that $a_i \in R$, $b_i \in M$ and $a_i^{t_i}b_i \in N$ for some positive integer t_i and for all i. Then $x = \varphi(y) = \varphi(r_1x_1 + \cdots + r_nx_n) = r_1\varphi(a_1b_1) + \cdots + r_n\varphi(a_nb_n) = r_1a_1\varphi(b_1) + \cdots + r_na_n\varphi(b_n)$. $x \in \langle E_{M'}(\varphi(N))\rangle$ since each $a_i\varphi(b_i) \in E_{M'}(\varphi(N))$. Now let $x' \in \langle E_{M'}(\varphi(N))\rangle$. Then $x' = s_1m_1 + \cdots + s_km_k$ for some $s_i \in R$ and $m_i \in E_{M'}(\varphi(N))$ $(1 \le i \le k)$. Then $m_i = d_iy_i$ such that $d_i \in R$, $y_i \in M'$ and $d_i^{m_i}y_i \in \varphi(N)$ for some $m_i \in \mathbb{Z}^+$. Since $y_i \in M'$, there exist $y_i' \in M$ such that $y_i = \varphi(y_i')$ for all i. Hence we get $x' = s_1d_1y_1 + \cdots + s_kd_ky_k = s_1d_1\varphi(y_1') + \cdots + s_kd_k\varphi(y_k') = \varphi(s_1d_1y_1' + \cdots + s_kd_ky_k')$. Since $d_i^{m_i}y_i \in \varphi(N)$, $d_i^{m_i}y_i' \in N$ and hence $d_iy_i' \in E_M(N)$. Therefore $x' \in \varphi(\langle E_M(N)\rangle)$.

(ii) Suppose that $rad_{M'}(N') = \langle E_{M'}(N')\rangle$. By Lemma 2.11, $rad_M(\varphi^{-1}(N')) = \varphi^{-1}(rad_{M'}(N')) = \varphi^{-1}(\langle E_{M'}(N')\rangle)$. It is enough to show that $\varphi^{-1}(\langle E_{M'}(N')\rangle) = \langle E_M(\varphi^{-1}(N'))\rangle$. Let $x \in \varphi^{-1}(\langle E_{M'}(N')\rangle)$. Then $\varphi(x) \in \langle E_{M'}(N')\rangle$, so $\varphi(x) = r_1x_1 + \cdots + r_nx_n$ for some $r_i \in R$, $x_i \in E_{M'}(N')$ $(1 \le i \le n)$. Since $x_i \in E_{M'}(N')$, there exist $a_i \in R$ and $b_i \in M'$ such that $x_i = a_ib_i$ and $a_i^{t_i}b_i \in N'$ for some positive integer t_i. Then $x_i = a_i\varphi(c_i)$ for some $c_i \in M$ and $a_i^{t_i}\varphi(c_i) \in N'$. Hence $a_i^{t_i}c_i \in \varphi^{-1}(N')$ and so $a_ic_i \in E_M(\varphi^{-1}(N'))$. Then $\varphi(x) = r_1x_1 + \cdots + r_nx_n = \varphi(r_1a_1c_1 + \cdots + r_na_nc_n)$. Hence

$$x - (r_1a_1c_1 + \cdots + r_na_nc_n) \in ker\varphi \subseteq \langle E_M(\varphi^{-1}(N'))\rangle.$$

Thus $x \in \langle E_M(\varphi^{-1}(N'))\rangle$. Now let $y \in \langle E_M(\varphi^{-1}(N'))\rangle$. Then $y = s_1y_1 + \cdots + s_ky_k$ for some $s_i \in R$, $y_i \in E_M(\varphi^{-1}(N'))$ $(1 \le i \le k)$. Then $y_i = u_im_i$ for some $u_i \in R$,

$m_i \in M$ such that $u_i^{l_i} m_i \in \varphi^{-1}(N')$ for some positive integer l_i. This gives that $u_i\varphi(m_i) \in E_{M'}(N')$. Then $y = s_1u_1m_1 + \cdots + s_ku_km_k$ and $\varphi(y) = s_1u_1\varphi(m_1) + \cdots + s_ku_k\varphi(m_k)$. Since $u_i\varphi(m_i) \in E_{M'}(N')$ for all i, $\varphi(y) \in \langle E_{M'}(N')\rangle$. Hence $y \in \varphi^{-1}(\langle E_{M'}(N')\rangle)$. □

Lemma 3.7 *Let M be an R-module and S be a multiplicatively closed subset of R. Then $\langle E_M(N)\rangle_S = \langle E_{M_S}(N_S)\rangle$*

Proof. Let $x \in E_{M_S}(N_S)$. Then $x = rm$ for some $r \in R_S$, $m \in M_S$ and $r^n m \in N_S$ for some positive integer n. Then $x = (r_1/s_1)(m_1/s_2)$ and $(r_1^n m_1/s_1^n s_2) \in N_S$ for some $r_1 \in R$, $m_1 \in M$ and $s_1, s_2 \in S$. There exist $v \in N$ and $s_3 \in S$ such that $(r_1^n m_1/s_1^n s_2) = v/s_3$. Then there exists $u \in S$ such that $(us_1^n s_2)v = (us_3)r_1^n m_1 \in N$. This implies that $(us_3r_1)^n m_1 \in N$ and hence $us_3r_1m_1 \in E_M(N)$. Thus $(us_3r_1m_1/us_3s_1s_2) = (r_1m_1/s_1s_2) = x \in \langle E_M(N)\rangle_S$. Therefore $\langle E_{M_S}(N_S)\rangle \subseteq \langle E_M(N)\rangle_S$. Now let $y \in \langle E_M(N)\rangle_S$. Then $y = a/s$ for some $a \in \langle E_M(N)\rangle$ and $s \in S$. There exist $r_i' \in R$, $m_i' \in M$ such that $a = r_1'm_1' + \cdots + r_k'm_k'$ with $(r_i')^{n_i} m_i' \in N$ $(1 \leq i \leq k)$. Then $(r_i')^{n_i} m_i'/s^{n_i} = (r_i'/s)^{n_i} m_i \in N_S$ for all i. Hence $(r_i'/s)m_i \in E_{M_S}(N_S) \subseteq \langle E_{M_S}(N_S)\rangle$ and $y = a/s \in \langle E_{M_S}(N_S)\rangle$. Therefore $\langle E_M(N)\rangle_S = \langle E_{M_S}(N_S)\rangle$. □

3.2 Modules Which Satisfy The Radical Formula

Definition 3.8 *A module M satisfies the radical formula (M s.t.r.f.), if for every submodule N of M, the prime radical of N is the submodule generated by its envelope, that is, $rad_M(N) = \langle E_M(N)\rangle$. A ring R s.t.r.f. provided that for every R-module M, M s.t.r.f.*

Lemma 3.9 *Let M be an R-module which s.t.r.f. and I be an ideal of R such that $I \subseteq Ann_R(M)$. Then M s.t.r.f. as an R/I-module.*

Proof. Since M s.t.r.f, $rad_M(N) = \langle E_M(N) \rangle$ for all submodules N of M. By Lemma 1.12, every prime submodule P of M as an R-module is a prime submodule of M as an R/I-module. Similarly we can show that if P' is a prime submodule of the R/I-module M, then P' is a prime submodule of M as an R-module. Therefore the prime radical of N in M as an R-module is the same as the prime radical of N in M as an R/I-module. Now by the definition of envelope we can conclude that $\langle E_M(N) \rangle_R = \langle E_M(N) \rangle_{R/I}$. Therefore M s.t.r.f. as an R/I-module. □

Lemma 3.10 *Let M be an R-module which s.t.r.f.. Then every homomorphic image of M s.t.r.f. as an R-module.*

Proof. Suppose M s.t.r.f.. Let N be any submodule of M. Consider the canonical epimorphism $\pi : M \to M/N$. Let K be any submodule of M containing N. Then by Theorem 3.6, $rad_{M/N}(\pi(K)) = \langle E_{M/N}(\pi(K)) \rangle$. Hence $rad_{M/N}(K/N) = \langle E_{M/N}(K/N) \rangle$. Therefore M/N s.t.r.f. as an R-module. □

Proposition 3.11 *[[8], Proposition 2.1] Let R be a ring, M be an R-module which s.t.r.f. and I be an ideal of R. Then M/IM s.t.r.f. as an R/I-module. Consequently, if R s.t.r.f., then the quotient ring R/I s.t.r.f., for any ideal I of R.*

Proof. Suppose that M s.t.r.f. as an R-module. By Lemma 3.10, M/IM s.t.r.f. as an R-module. Since $I \subseteq Ann_R(M/IM)$, M/IM s.t.r.f. as an R/I-module by Lemma 3.9. □

In the following theorem we will give some equivalent conditions for a ring to s.t.r.f..

Theorem 3.12 *[[2], Theorem 1] Let R be a ring. Then R s.t.r.f. provided that any one of the following is satisfied.*

(i) For every free R-module F, F s.t.r.f.,

(ii) For every faithful R-module M, M s.t.r.f.,

(iii) For every R-module M, $rad_M(0) = \langle E_M(0) \rangle$,

(iv) R is a ring homomorphic image of S, where S s.t.r.f.

Proof. It is evident that (i) and (ii) are true since every module M is the homomorphic image of both a free R-module and a faithful R-module and by Theorem 3.10.

(iii) Let N be a submodule of M and apply Theorem 3.6, letting $M' = M/N$, $N' = N$ and $\varphi = \pi$ where $\pi : M \to M/N$ is the canonical epimorphism.

(iv) Let $\varphi : S \to R$ be an epimorphism. Suppose S s.t.r.f.. An R-module M is realized as an S-module by the standard pullback along φ. Since φ is an epimorphism, every S-submodule of M is also an R-submodule, and vice-versa. Indeed, in this case, a submodule of M is prime as an S-submodule if and only if it is prime as an R-submodule. Furthermore, it is easily shown that for any submodule N of M, $E_M({}_SN) = E_M({}_RN)$. Hence we have $rad_M({}_RN) = rad_M({}_SN) = \langle E_M({}_SN) \rangle = \langle E_M({}_RN) \rangle$, since S-linear combinations coincide with R-linear combinations. Therefore R s.t.r.f.. □

Definition 3.13 *An R-module M is called a multiplication module if for every submodule N there exists an ideal I of R such that $N = IM$.*

It is clear that every cyclic module is a multiplication module.

Lemma 3.14 *Let M be a multiplication module. Then $rad_M(N) = \sqrt{N : M}M$ for all submodules N of M.*

Proof. Let P_i be a prime submodule of M containing N. Then $(N : M) \subseteq (P_i : M)$ and this gives that $\sqrt{(N : M)} \subseteq \sqrt{(P_i : M)} = (P_i : M)$. Hence $\sqrt{(N : M)} \subseteq rad_M(N)$. Now let K denotes the collection of all prime ideals $\mathcal{P}$ of

R containing $(N : M)$. Then $\sqrt{(N:M)} = \bigcap_{\mathcal{P}\in K} \mathcal{P}$ and $\sqrt{(N:M)}M = \bigcap_{\mathcal{P}\in K} \mathcal{P}M$ by [[22], Theorem 1.6]. If $M = \mathcal{P}M$, then $rad_M(N) \subseteq \mathcal{P}M$. If $M \neq \mathcal{P}M$, then $N = (N : M)M \subseteq \mathcal{P}M$. Hence in any case $rad_M(N) \subseteq \sqrt{(N:M)}M$. □

Proposition 3.15 *Every multiplication module satisfies the radical formula.*

Proof. Let M be a multiplication module and N be a submodule of M. Let $x \in rad_M(N)$. By Lemma 3.14, $x \in \sqrt{(N:M)}M$. Then $x = a_1m_1 + \cdots + a_nm_n$ for some $a_i \in \sqrt{(N:M)}$, $m_i \in M$. Since $a_i \in \sqrt{(N:M)}$, $a_im_i \in \langle E_M(N)\rangle$ for all i. Hence $x \in \langle E_M(N)\rangle$. Therefore M s.t.r.f.. □

Prior to [2] the only rings known to s.t.r.f. were fields and the only modules (other than vector spaces) known to s.t.r.f. were multiplication modules. Later Jenkins and Smith showed that Dedekind domains also s.t.r.f.. In [1], Jenkins and Smith give a conjuncture that Dedekind domains are the only Noetherian domains which satisfy the radical formula. Then in [7], Man tried to tackle this problem. Now we will give some results about modules which s.t.r.f..

Proposition 3.16 *[[23], Corollary 1.2.] Any cyclic module satisfies the radical formula.*

Proof. Let M be a cyclic module. Then $M = Rm$ for some $m \in M$. We know that $Rm \cong R/Ann_R(m)$ and ${}_RR$ s.t.r.f.. Hence by Lemma 3.10, M s.t.r.f.. □

Lemma 3.17 *[[1], Lemma 7] Let R be a ring and M be an R-module such that $rad_M(0) = \langle E_M(0)\rangle$. Then $rad_N(0) = \langle E_N(0)\rangle$ for any direct summand N of M.*

Proof. Suppose that $M = N \oplus N'$ for some submodule N' of M. We have already remarked that $\langle E_N(0)\rangle \subseteq rad_N(0)$. Let us suppose that $m \in rad_N(0)$. By Lemma 2.16, $m \in rad_M(0) = \langle E_M(0)\rangle$. Then there exist positive integers n, k_i and elements $r_i \in R$, $m_i \in M$ $(1 \leq i \leq n)$ such that $m = r_1m_1 + \cdots + r_nm_n$ and

$r_i^{k_i} m_i = 0$. For each $1 \le i \le n$, there exist elements $x_i \in N$, $y_i \in N'$ such that $m_i = x_i + y_i$. Then $m = r_1(x_1+y_1)+\cdots+r_n(x_n+y_n)$ and $m-(r_1x_1+\cdots+r_nx_n) = r_1y_1+\cdots+r_ny_n \in N \cap N' = 0$. Hence $m = r_1x_1+\cdots+r_nx_n$ and since $r_i^{k_i}m_i = 0$, $r_i^{k_i}x_i = 0$. Thus $m \in \langle E_N(0)\rangle$. It follows that $rad_N(0) \subseteq \langle E_N(0)\rangle$. $\Box$

Corollary 3.18 *[[1], Lemma 8] Let R be any ring and M any projective R-module then $rad_M(0) = \langle E_M(0)\rangle$.*

Proof. Let M be a projective R-module. Then there exists a free R-module F such that M is a direct summand of F. There exist an index set Λ and cyclic submodules F_λ, $(\lambda \in \Lambda)$ of F such that $F = \oplus_\lambda F_\lambda$. By Lemma 2.17, $rad_F(0) = \oplus_\lambda rad_{F_\lambda}(0)$. By Proposition 3.16, $rad_{F_\lambda}(0) = \langle E_{F_\lambda}(0)\rangle \subseteq \langle E_F(0)\rangle$ for each $\lambda \in \Lambda$. Then $rad_F(0) = \oplus_\Lambda rad_{F_\lambda}(0) \subseteq \oplus_\Lambda \langle E_{F_\lambda}(0)\rangle \subseteq \langle E_F(0)\rangle$ for all $\lambda \in \Lambda$. Since $\langle E_F(0)\rangle \subseteq rad_F(0)$ always holds, $rad_F(0) = \langle E_F(0)\rangle$. Hence by Lemma 3.17, $rad_M(0) = \langle E_M(0)\rangle$. $\Box$

The following theorem shows that Dedekind domains satisfy the radical formula.

Theorem 3.19 *[[1], Lemma 9] Let R be a Dedekind domain and M be any R-module. Then $rad_M(0) = \langle E_M(0)\rangle$.*

Proof. As we have already remarked $\langle E_M(0)\rangle \subseteq rad_M(0)$. Let $m \in rad_M(0)$. By Lemma 2.20, $m \in rad_L(0)$ for some finitely generated submodule L of M. Now $L = L_1 \oplus \cdots \oplus L_k$ for some positive integer k and submodules L_i $(1 \le i \le k)$ of L such that L_i is either projective or cyclic for each i [see, [18]]. By Lemma 2.17, Proposition 3.16 and Corollary 3.18, $m \in rad_{L_1}(0) \oplus rad_{L_2}(0) \oplus \cdots \oplus rad_{L_k}(0) \subseteq \langle E_{L_1}(0)\rangle \oplus \langle E_{L_2}(0)\rangle \oplus \cdots \oplus \langle E_{L_k}(0)\rangle \subseteq \langle E_L(0)\rangle \subseteq \langle E_M(0)\rangle$. Thus $rad_M(0) \subseteq \langle E_M(0)\rangle$. It follows that $rad_M(0) = \langle E_M(0)\rangle$. $\Box$

Proposition 3.20 *[[7], Proposition 2.4] Suppose M is a Noetherian R-module. The following are equivalent.*

(i) M s.t.r.f.

(ii) $M_{\mathcal{M}}$ s.t.r.f. as an $R_{\mathcal{M}}$-module for every maximal ideal $\mathcal{M}$ in R.

Proof. Let $\mathcal{M}$ be a maximal ideal of R. By Lemma 2.21 and Lemma 3.7, $rad_M(N) = \langle E_M(N)\rangle$ if and only if $rad_{M_{\mathcal{M}}}(N_{\mathcal{M}}) = (rad_M(N))_{\mathcal{M}} = (\langle E_M(N)\rangle)_{\mathcal{M}} = \langle E_{M_{\mathcal{M}}}(N_{\mathcal{M}})\rangle$ for any submodule N of M. □

Proposition 3.21 *[[7], Proposition 3.1] Suppose $(R, \mathcal{M})$ is a commutative Noetherian local domain of dimension one. Let $x, y \in R$ such that $x \in \mathcal{M} - Ry$ and $y \in \mathcal{M} - Rx$. Then the prime radical of $R(x, y)$ in $R^2 = R \oplus R$ is $\Lambda(x, y) = \{(r_1, r_2) \in R^2 : r_1 y = r_2 x\}$.*

Proof. Let us first show that $\Lambda(x, y)$ is a prime submodule of R^2 containing $R(x, y)$. It is clear that $R(x, y) \subseteq \Lambda(x, y)$. Suppose $R^2 = \Lambda(x, y)$. Then $(a, b) \in \Lambda(x, y)$ for all $(a, b) \in R^2$ and $ay = bx$. Hence it gives a contradiction that $y = a^{-1}bx \in Rx$. Thus $\Lambda(x, y) \subsetneq R^2$. Therefore $\Lambda(x, y)$ is a proper submodule of R^2. Let $r \in R$, $(a, b) \in R^2$ such that $r(a, b) \in \Lambda(x, y)$. Then $r(ay - bx) = 0$. Since R is a domain, $r = 0$ or $ay - bx = 0$. If $r = 0$, then $rR^2 \subseteq \Lambda(x, y)$. If $ay - bx = 0$, then $(a, b) \in \Lambda(x, y)$. Hence $\Lambda(x, y)$ is a prime submodule of R^2 containing $R(x, y)$.

Now suppose that N is a prime submodule of R^2 containing $R(x, y)$ with $N \subseteq \Lambda(x, y)$. Since $x \neq 0$, we have $x\Lambda(x, y) \subseteq N$ and $xR^2 \nsubseteq \Lambda(x, y)$. For let $(s_1, s_2) \in \Lambda(x, y)$. Then $x(s_1, s_2) = (s_1 x, s_2 x) = (s_1 x, s_1 y) = s_1(x, y) \in R(x, y) \subseteq N$. Hence $xR^2 \nsubseteq N$ since $N \subseteq \Lambda(x, y)$ and then $\Lambda(x, y) \subseteq N$. Therefore $\Lambda(x, y)$ is a minimal prime submodule of R^2 containing $R(x, y)$. It sufficies to show $\Lambda(x, y)$ is the only minimal prime submodule. Let P be a prime R-submodule in R^2 which contains $R(x, y)$. Put $\mathcal{P} = Ann_R(R^2/P) = (P : R^2)$. As P is a prime submodule of R^2, $\mathcal{P}$ is a prime ideal. We now show $\Lambda(x,y) \subseteq P$. Since R is a local ring, $\mathcal{P} = \mathcal{M}$ or $\mathcal{P} = 0$.

Suppose $\mathcal{P} = \mathcal{M}$. Then $\mathcal{M}(R \oplus R) = \mathcal{M} \oplus \mathcal{M} \subseteq P$. As $y \in \mathcal{M} - Rx$ and $x \in \mathcal{M} - Ry$, we have $\Lambda(x,y) \subseteq \mathcal{M} \oplus \mathcal{M}$. To see that let $(r_1, r_2) \in \Lambda(x,y)$. If $r_1 \notin \mathcal{M}$, then r_1 is a unit. This implies that $y \in Rx$ since $r_1 y = r_2 x$. Thus $r_1 \in \mathcal{M}$. Similarly if $r_2 \notin \mathcal{M}$, then $x \in Ry$. Hence $r_2 \in \mathcal{M}$. Therefore $\Lambda(x,y) \subseteq \mathcal{M} \oplus \mathcal{M} \subseteq P$.

Suppose $\mathcal{P} = 0$. Since $P \neq R^2$, we may assume $(1,0) \notin P$. Let $(r_1, r_2) \in P$ be given. Then $(yr_1 - r_2x)(1,0) = y(r_1,r_2) - r_2(x,y) \in P$. Since $(1,0) \notin P$, we have $(yr_1 - r_2x)R^2 \subseteq P$. Then $(yr_1 - r_2x) \in (P : R^2) = \mathcal{P} = 0$. Hence $(yr_1 - r_2x) = 0$. Then $(r_1, r_2) \in \Lambda(x,y)$ and $P \subseteq \Lambda(x,y)$. Since $\Lambda(x,y)$ is minimal, $\Lambda(x,y) = P$. Therefore the prime radical of $R(x,y)$ is $\Lambda(x,y)$. $\Box$

Theorem 3.22 *[[7], Theorem 3.2 and Theorem 3.3] Suppose R is a Noetherian domain of dimension one and R^2 s.t.r.f. as an R-module. Then R is a Dedekind domain.*

Proof. It sufficies to show that $R_{\mathcal{M}}$ is a discrete valuation ring for each maximal ideal $\mathcal{M}$. Since R^2 s.t.r.f. as an R-module, $R^2_{\mathcal{M}}$ s.t.r.f. as an $R_{\mathcal{M}}$-module for all maximal ideal $\mathcal{M}$ of R by Proposition 3.20.

Let us suppose that $\mathcal{P}$ is the unique maximal ideal of the local ring $R_{\mathcal{M}}$. Choose $x \in \mathcal{P} - \mathcal{P}^2$. If we can show that $\mathcal{P} = R_{\mathcal{M}}x$, then the proof will be completed. As $R_{\mathcal{M}}$ has dimension one and $x \neq 0$, $\mathcal{P}$ is the only associated prime ideal of $R_{\mathcal{M}}/(R_{\mathcal{M}}x)$. Hence every element of $\mathcal{P}$ is a zero divisor in $R_{\mathcal{M}}/(R_{\mathcal{M}}x)$. Now we must show that $\mathcal{P} \subseteq R_{\mathcal{M}}x + \mathcal{P}^2$. Let $s \in \mathcal{P}$ be given. By the above discussion, there exist $y \in R_{\mathcal{M}} - R_{\mathcal{M}}x$ and $r \in R_{\mathcal{M}}$ with $s(y + R_{\mathcal{M}}x) = 0$. This gives that $sy = xr$. If $s \in R_{\mathcal{M}}x$, then $s \in R_{\mathcal{M}}x + \mathcal{P}^2$. From now on, we assume $s \in \mathcal{P} - R_{\mathcal{M}}x$. Then y is not a unit. For if y is a unit, then $sy = xr$ would give a contradiction that $s \in R_{\mathcal{M}}x$. Hence $y \in \mathcal{P} - R_{\mathcal{M}}x$.

As $x \in \mathcal{P} - \mathcal{P}^2$ and $y \in \mathcal{P} - R_{\mathcal{M}}x$, we have $x \in \mathcal{P} - R_{\mathcal{M}}y$. Since $x \in \mathcal{P} - R_{\mathcal{M}}y$ we have $y \in \mathcal{P} - R_{\mathcal{M}}x$ and $sy = rx$, $(s,r) \in \Lambda(x,y)$. We also have that $\Lambda(x,y) = \langle E_{R^2_{\mathcal{M}}}(R_{\mathcal{M}}(x,y))\rangle$ since $R^2_{\mathcal{M}}$ s.t.r.f.. By the definition of $E_{R^2_{\mathcal{M}}}(R_{\mathcal{M}}(x,y))$, there exist

(a) $r_1, r_2, \ldots, r_k \in R_{\mathcal{M}} - \{0\}$,

(b) $(c_1,d_1),(c_2,d_2),\ldots,(c_k,d_k) \in R^2_{\mathcal{M}} - \{(0,0)\}$, and

(c) non-negative integers $n_1, n_2, \ldots, n_k$

such that

(i) $(s,r) = \sum_{i=1}^{k} r_i(c_i,d_i)$, and

(ii) for each $1 \leq i \leq k$, $r_i^{n_i}(c_i,d_i) = f_i(x,y)$ for some $f_i \in R_{\mathcal{M}}$.

Since each $r_i \neq 0$ and $R_{\mathcal{M}}$ is a domain, (ii) gives that $c_i y = x d_i$ for each i. Also since $y \in \mathcal{P} - R_{\mathcal{M}}x$, every $c_i \in \mathcal{P}$. If r_i is a unit, then (ii) gives $r_i c_i \in R_{\mathcal{M}}x$. If not $r_i \in \mathcal{P}$ and then $r_i c_i \in \mathcal{P}^2$. Hence $s = \sum_{i=1}^{k} r_i c_i \in R_{\mathcal{M}}x + \mathcal{P}^2$. Hence $\mathcal{P} \subseteq R_{\mathcal{M}}x + \mathcal{P}^2$. Since $\mathcal{P}$ is a maximal ideal of $R_{\mathcal{M}}$, $\mathcal{P} = R_{\mathcal{M}}x + \mathcal{P}^2$. Therefore $\mathcal{P} = R_{\mathcal{M}}x$ by Nakayama's Lemma. Hence $R_{\mathcal{M}}$ is a discrete valuation ring for all maximal ideal $\mathcal{M}$ of R and thus R is a Dedekind domain. □

Theorem 3.23 *Let R be a Noetherian domain of dimension one. Then R s.t.r.f. if and only if R is a Dedekind domain.*

Proof. Sufficiency follows from Theorem 3.19. Now suppose R s.t.r.f. Then R^2 s.t.r.f. as an R-module. Then R is a Dedekind domain by Theorem 3.22. □

In [8], Man have gives an answer or the following question (see, Theorem 3.40).

Does there exist a Noetherian ring of dimension greater than one which s.t.r.f.?

Theorem 3.24 *[[8], Theorem 2.2] Suppose that $(R,\mathcal{M})$ is a local Noetherian ring and R^2 s.t.r.f. as an R-module. Let $a,b \in \mathcal{M}$ and I be an ideal of R such* *that*

$$b \notin Ra + Ann_R(x^n) \cap (Ra + Rb)$$

for any $x \in \sqrt{Ra} - I$, $n \in \mathbb{N}$. *Then* $a \in I + \mathcal{M}^2$.

Proof. Let $J = Ra + Rb$. It is easy to verify that a prime submodule of R^2 contains $J(a,b)$ if and only if it contains (a,b). Thus $rad_{R^2}(J(a,b)) = rad_{R^2}(R(a,b))$. Since R^2 s.t.r.f., $rad_{R^2}(R(a,b)) = \langle E_{R^2}(J(a,b))\rangle$. Hence $(a,b) \in \langle E_{R^2}(J(a,b))\rangle$. By the definition of $E_{R^2}(J(a,b))$, there exist

(a) $r_1, r_2, \ldots, r_k \in R - \{0\}$,

(b) $(c_1, d_1), (c_2, d_2), \ldots, (c_k, d_k) \in R^2 - \{(0,0)\}$, and

(c) non-negative integers $n_1, n_2, \ldots, n_k$,

such that

(i) $(a,b) = \sum_{i=1}^{k} r_i(c_i, d_i)$, and

(ii) for each $1 \leq i \leq k$, $r_i^{n_i}(c_i, d_i) = f_i(a,b)$, for some $f_i \in J$.

By (i), we have $a = \sum_{i=1}^{k} r_i c_i$. If we can show that $r_i c_i \in I + \mathcal{M}^2$, then it is done. If r_i is a unit, then from (ii) $r_i c_i \in Ja \subseteq \mathcal{M}^2$. If $r_i \in I$, then $r_i c_i \in I$. Now suppose that $r_i \in \mathcal{M} - I$. We must show that $c_i \in \mathcal{M}$. Suppose not. From (ii), we have $r_i \in \sqrt{Ra}$ and $r_i^{n_i}(ad_i - bc_i) = 0$. Hence $ad_i - bc_i \in Ann_R(r_i^{n_i}) \cap (Ra + Rb)$ and this gives the contradiction that $b \in Ra + Ann_R(r_i^{n_i}) \cap (Ra + Rb)$. Therefore $c_i \in \mathcal{M}$ and hence $r_i c_i \in \mathcal{M}^2$. This implies that $a \in \mathcal{M}^2 + I$. □

Theorem 3.25 *[[8], Corollary 2.3] Suppose that R is a Noetherian ring, R^2 s.t.r.f. as an R-module, dim $R \geq 1$ and $\mathcal{P}$ is a minimal prime ideal of R. Then $\mathcal{P}$ is the only $\mathcal{P}$-primary ideal of R and $R/\mathcal{P}$ is a Dedekind domain. In particular, R is a Dedekind domain if R is a domain.*

Proof. Let us first assume that R is local with maximal ideal $\mathcal{M}$ and $\sqrt{0} = \mathcal{P}$. We only need to prove R is a discrete valuation ring. As dim $R \geq 1$, $\mathcal{M} \neq \mathcal{P}$.

Then we can choose $a \in \mathcal{M} - (\mathcal{M}^2 + \mathcal{P})$. Suppose that $\mathcal{M} \neq Ra$. Then there exists $b \in \mathcal{M} - Ra$. Let $I = \mathcal{P}$. Since $\mathcal{P}$ is the set of all zero divisors of R, we get

$$b \notin Ra + Ann_R(x^n) \cap (Ra + Rb)$$

for any $x \in \sqrt{Ra} - \mathcal{P}$ and $n \in \mathbb{N}$. By Theorem 3.24, $a \in \mathcal{M}^2 + \mathcal{P}$, which is a contradiction. Hence $\mathcal{M} = Ra$ and hence R is a discrete valuation ring.

In general case let J be a $\mathcal{P}$-primary ideal of R. By Proposition 3.11, R^2/JR^2 s.t.r.f. as an R/J-module. Hence $R/J \oplus R/J$ s.t.r.f. as an R/J-module. By Proposition 3.20, $(R/J \oplus R/J)_{\mathcal{M}/J}$ s.t.r.f. as an $(R/J)_{\mathcal{M}/J}$-module. As $(R/J)_{\mathcal{M}/J}$ is a discrete valuation ring, R/J is a Dedekind domain and J is a prime ideal with $J = \sqrt{J} = \mathcal{P}$. Hence $\mathcal{P}$ is the only $\mathcal{P}$-primary ideal. Then $R/\mathcal{P}$ is a Dedekind domain and thus $\dim(R/\mathcal{P}) = 1 = \dim R$.

If R is a domain, then $\mathcal{P} = 0$. Hence $R/\mathcal{P} = R$ is a Dedekind domain. □

This theorem shows that Dedekind domains are the only Noetherian domains which s.t.r.f..

The next result is immediate from Theorem 3.25.

Corollary 3.26 *[[8], Corollary 2.4] Suppose that R is a Noetherian ring and R^2 s.t.r.f. as an R-module. Then $\dim R \leq 1$.*

Proof. Assume that $\dim R \geq 1$. Then Theorem 3.25 shows that $\dim R = 1$. □

Proposition 3.27 *[[8], Proposition 2.5] Let R be a Noetherian ring. Suppose that*

(i) $R/\sqrt{0}$ s.t.r.f. as a ring,

(ii) there exist maximal ideals $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ and natural numbers $k_1, k_2, \ldots, k_n$ with

$$\sqrt{0} \cap \mathcal{M}_1^{k_1} \cap \mathcal{M}_2^{k_2} \cap \cdots \cap \mathcal{M}_n^{k_n} = 0.$$

Then R s.t.r.f.

Proof. We may assume that all the k_i's are equal to a common value k, say $k = max\{k_1, \ldots, k_n\}$. Let M be an R-module. To show that R s.t.r.f. it suffices to prove $rad_M(0) \subseteq \langle E_M(0)\rangle$. Clearly $\sqrt{0}M \subseteq \langle E_M(0)\rangle$. Let $m \in rad_M(0)$. Then $m \in K$ for all prime submodules of M. Since $\sqrt{0}M \subseteq K$, we have $m + \sqrt{0}M \in K/\sqrt{0}M$. Hence $m + \sqrt{0}M \in rad_{M/\sqrt{0}M}(0)$ over the ring $R/\sqrt{0}$. Since $R/\sqrt{0}$ s.t.r.f., we have $m + \sqrt{0}M \in \langle E_{M/\sqrt{0}M}(0)\rangle$. Then

$$m + \sqrt{0}M = \sum_i r_i m_i + \sqrt{0}M,$$

where $r_i \in R$, $m_i \in M$ and $r_i^{n_i} m_i \in \sqrt{0}M$, for some natural number n_i. Hence $m = y + \sum_i r_i m_i$ for some $y \in \sqrt{0}M$. If we can show that each $r_i m_i \in \langle E_M(0)\rangle$, then proof is completed.

If $r \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_n$ and $m \in M$ with $r^t m \in \sqrt{0}M$ for some natural number t, then $r^{t+k} m = 0$ by condition (ii). Hence $rm \in E_M(0)$.

Thus if $r_i \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_n$, then $r_i m_i \in E_M(0)$. Suppose that $r_i \notin \mathcal{M}_j$ for some $1 \leq j \leq n$. Without loss of generality, we may assume $r_i \in \mathcal{M}_1 \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_l$ and $r_i \notin \mathcal{M}_{l+1} \cup \cdots \cup \mathcal{M}_n$. Then $R = Rr_i^{n_i} + \mathcal{M}_{l+1} \cap \cdots \cap \mathcal{M}_n$. Hence there exist $s \in R$ and $x \in \mathcal{M}_{l+1} \cap \cdots \cap \mathcal{M}_n$ such that $1 = sr_i^{n_i} + x$. Then $r_i m_i = sr_i^{n_i+1} m_i + r_i x m_i$. $sr_i^{n_i+1} m_i$ and $(r_i x)^{n_i} m_i$ are in $\sqrt{0}M$ since $r_i^{n_i} m_i \in \sqrt{0}M$. Hence $sr_i^{n_i+1} m_i \in \langle E_M(0)\rangle$ and since $r_i x \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_n$, we have $r_i x m_i \in E_M(0)$ by the earlier observation. Therefore $r_i m_i \in \langle E_M(0)\rangle$ for each i. □

Proposition 3.28 *[[8], Proposition 2.6] Suppose that R is Noetherian, $dimR = 1$ and every minimal prime ideal $\mathcal{P}$ of R is the only $\mathcal{P}$-primary ideal in R. Then condition (ii) of Proposition 3.27 is satisfied.*

Proof. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$ be all the minimal prime ideals of R. Since R is Noetherian, a reduced primary decomposition of the zero ideal can be written as follows.

$$0 = J_1 \cap J_2 \cap \cdots \cap J_r \cap I_1 \cap I_2 \cap \cdots \cap I_n,$$

where each J_i is a $\mathcal{P}_i$-primary ideal and I_i is an $\mathcal{M}_i$-primary ideal for some maximal ideal $\mathcal{M}_i$. By assumption, $J_i = \mathcal{P}_i$. Then

$$0 = \sqrt{0} \cap I_1 \cap I_2 \cap \cdots \cap I_n$$

since $\sqrt{0} = J_1 \cap J_2 \cap \cdots \cap J_r$. As I_i is $\mathcal{M}_i$-primary, we have $\mathcal{M}_i^{k_i} \subseteq I_i$ for some large enough natural number k_i. Therefore

$$\sqrt{0} \cap \mathcal{M}_1^{k_1} \cap \mathcal{M}_2^{k_2} \cap \cdots \cap \mathcal{M}_n^{k_n} = 0.$$

□

The following corollary generalize the last statement of Corollary 3.25.

Corollary 3.29 *[[8], Corollary 2.7] Suppose that R is Noetherian, $dim R = 1$ and there exists a unique minimal prime ideal $\mathcal{P}$ in R. Then R s.t.r.f. if and only if $R/\mathcal{P}$ is a Dedekind domain and $\mathcal{P}$ is the only $\mathcal{P}$-primary ideal in R.*

Proof. Necessity is clear by Corollary 3.25. Conversely suppose that $R/\mathcal{P}$ is a Dedekind domain and $\mathcal{P}$ is the only $\mathcal{P}$-primary ideal in R. Then condition (ii) of Proposition 3.27 is satisfied by Proposition 3.28. If we can show $R/\sqrt{0}$ s.t.r.f., then the proof is completed. Since $\mathcal{P}$ is the unique minimal prime ideal of R, $\sqrt{0} = \mathcal{P}$. Hence $R/\mathcal{P} = R/\sqrt{0}$ is a Dedekind domain and s.t.r.f.. Therefore R s.t.r.f. by Proposition 3.27. □

Let $\mathcal{P}$ be a prime ideal of R, M be an R-module, and $M(\mathcal{P}) = \{m \in M : sm \in \mathcal{P}M \text{ for some } s \in R - \mathcal{P}\}$. Then clearly $M(\mathcal{P})$ is a submodule of M containing $\mathcal{P}M$ and $\mathcal{P} \subseteq (\mathcal{P}M : M) \subseteq (M(\mathcal{P}) : M)$.

Proposition 3.30 *[[9], Proposition 1.1] Let M be an R-module and $\mathcal{P}$ be a prime ideal of R such that $M(\mathcal{P}) \neq M$. Then $M(\mathcal{P})$ is a $\mathcal{P}$-prime submodule of M and $M(\mathcal{P})$ is the intersection of all $\mathcal{P}$-prime submodules of M.*

Proof. Let $r \in R$, $m \in M$ with $rm \in M(\mathcal{P})$ and $rM \nsubseteq M(\mathcal{P})$. Then $s(rm) \in \mathcal{P}M$ for some $s \in R - \mathcal{P}$. Since $sr \in R - \mathcal{P}$, we have $m \in M(\mathcal{P})$. Hence $M(\mathcal{P})$ is a prime submodule of M. Let $x \in (M(\mathcal{P}) : M)$ and $x \notin \mathcal{P}$. Then $xy \in M(\mathcal{P})$ for all $y \in M$. Hence $t(xy) \in \mathcal{P}M$ for some $t \in R - \mathcal{P}$. This gives the contradiction that $M = M(\mathcal{P})$. Thus $(M(\mathcal{P}) : M) = \mathcal{P}$.

Let N be a $\mathcal{P}$-prime submodule of M and $n \in M(\mathcal{P})$. Then $kn \in \mathcal{P}M$ for some $k \in R - \mathcal{P}$. Since $(N : M) = \mathcal{P}$ we have $kn \in N$. Thus $n \in N$ as N is prime submodule. Hence $M(\mathcal{P}) \subseteq N$. The reverse inclusion always holds. □

Proposition 3.31 *Let M be a finitely generated R-module. Then $M(\mathcal{P})$ is a prime submodule of M for every prime ideal $\mathcal{P} \supseteq Ann(M)$. If $\mathcal{P}$ is a prime ideal such that $Ann(M)$ is not contained in $\mathcal{P}$, then $M = M(\mathcal{P})$.*

Proof. By Proposition 3.30, it sufficies to show that $M \neq M(\mathcal{P})$ in the case M is finitely generated. Suppose that $M = M(\mathcal{P})$. Then there exist $x_1, x_2 \dots, x_n \in M(\mathcal{P})$ and $s_1, s_2, \dots, s_n \in R - \mathcal{P}$ such that $M = Rx_1 + \dots + Rx_n$ and $s_i x_i \in \mathcal{P}M$ for each i. By the usual determinant argument, we get a contradiction that $s_j \in \mathcal{P}$ for some j. Hence $M \neq M(\mathcal{P})$.

Assume $\mathcal{P}$ is a prime ideal of R such that $\mathcal{P} \nsupseteq Ann(M)$. Then there exists an element $x \in Ann(M) - \mathcal{P}$ and hence $xM = 0 \in \mathcal{P}M$. Thus $M = M(\mathcal{P})$. □

Lemma 3.32 *[[8], Lemma 3.1] Let M be an R-module. Then $rad_M(0) = \bigcap M(\mathcal{P})$, where the intersection is taken over all the prime ideals of R. Furthermore, if M is finitely generated,then the above intersection can be taken over $\{\mathcal{P} : \mathcal{P}$ is a prime ideal of R and $AnnM \subseteq \mathcal{P}\}$.*

Proof. By Proposition 3.30 and 3.31. □

Corollary 3.33 *[[8], Corollary 3.2] Suppose R is Noetherian, $dimR \leq 1$ and M is an R-module. Let $\mathcal{P}_i$ and $\mathcal{M}_j$ be minimal prime and maximal ideals of R respectively. Then $rad_M(0) = (\cap M(\mathcal{P}_i)) \bigcap (\cap \mathcal{M}_j M)$.*

Proof. It is clear that $M(\mathcal{M}_j) = \mathcal{M}_j M$ for every maximal ideal $\mathcal{M}_j$ of R. The result follows from Lemma 3.32. □

The next lemma is a generalization of Lemma 2.20.

Lemma 3.34 *[[8], Lemma 3.3] Suppose R is Noetherian, $dimR \leq 1$, M is an R-module and N is a submodule of M such that $N \neq M$. Then $rad_M(N) = \bigcup rad_L(N)$ where the union is taken over all submodules of L of M which contains N and L/N is finitely generated.*

Proof. Without loss of generality we may assume that $N = 0$. By Lemma 2.16, $rad_L(0) \subseteq rad_M(0)$ for any finitely generated submodule L of M. Let $m \in rad_M(0)$ and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be all the minimal prime ideals of R. By Corollary 3.33, there exist $s_i \in R - \mathcal{P}_i$ with $s_i m \in \mathcal{P}_i M$ for each $1 \leq i \leq n$. Since R is Noetherian, for each $i = 1, 2, \ldots, n$ there are only finitely many maximal ideals of R which contains both s_i and $\mathcal{P}_i$, say $\mathcal{M}_{i1}, \mathcal{M}_{i2}, \ldots, \mathcal{M}_{in_i}$. By Corollary 3.33, $m \in \mathcal{M}_{ij} M$ for each $1 \leq i \leq n$ and for $1 \leq j \leq n_i$. Since $s_i m \in \mathcal{P}_i M$ and $m \in \mathcal{M}_{ij} M$, there exist $p_{ij} \in \mathcal{P}_i$, $a_i \in \mathcal{M}_{ij}$ and $m_{ij}, m_i \in M$ such that

$$s_i m = p_{i1} m_{i1} + \cdots + p_{ik_i} m_{ik_i}$$

and

$$m = a_1m_1 + \cdots + a_tm_t$$

for each i, j, $(1 \leq i \leq n,\ 1 \leq j \leq n_i)$. Hence there exists a finitely generated submodule $L = \sum_{v=1}^{k_i} Rm_{iv} + \sum_{w=1}^{t} Rm_w$ of M such that

(i) $s_im \in \mathcal{P}_iL$ for $i = 1, 2, \ldots, n$,

(ii) $m \in \mathcal{M}_{ij}L$ for each $1 \leq i \leq n$ and $j = 1, 2, \ldots, n_i$.

Let $\mathcal{Q}$ be a maximal ideal of R such that $\mathcal{Q} \notin \{\mathcal{M}_{i1}, \ldots, \mathcal{M}_{in_i}\}$ for all $i = 1, 2, \ldots, n$. We may assume $\mathcal{P}_1 \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is maximal, $Rs_1 + \mathcal{Q} = R$. Then $Rm = Rs_1m + \mathcal{Q}m$. We have $m \in \mathcal{Q}L$ since $s_1m \in \mathcal{P}_1L \subseteq \mathcal{Q}L$ and $m \in L$. By Corollary 3.33, $m \in rad_L(0)$. Hence $rad_M(0) \subseteq \bigcup rad_L(0)$. □

Theorem 3.35 *[[8], Theorem 3.4] Suppose that R is a Noetherian ring. Then the following statements are equivalent.*

(i) R s.t.r.f.,

(ii) $R_{\mathcal{M}}$ s.t.r.f. for any maximal ideal $\mathcal{M}$ of R,

(iii) every finitely generated $R_{\mathcal{M}}$-module s.t.r.f., for any maximal ideal $\mathcal{M}$ of R,

(iv) every finitely generated R-module s.t.r.f..

Proof. $(i) \Rightarrow (iv)$ It is clear.

$(iv) \Rightarrow (i)$ Suppose that every finitely generated R-module s.t.r.f.. Hence R^2 s.t.r.f. and $\dim R \leq 1$ by Corollary 3.26. By Lemma 3.34, $rad_M(0) = \bigcup rad_L(0)$ for every finitely generated submodule L of M. Hence $rad_M(0) = \bigcup \langle E_L(0) \rangle \subseteq \langle E_M(0) \rangle$. Therefore R s.t.r.f..

$(ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (ii)$ The proof is similar with $(iv) \Rightarrow (i)$.

$(iii) \Leftrightarrow (iv)$ By Proposition 3.20. □

Theorem 3.36 *[[8], Theorem 3.5] Any Artinian ring s.t.r.f..*

Proof. Let us first assume that R is a local Artinian ring with maximal ideal $\mathcal{M}$. Then $\mathcal{M}^n = 0$ and $\mathcal{M}^n M = 0 \subseteq N$ for every submodule N of M. Hence $\mathcal{M} = (\langle E_M(N)\rangle : M)$. Then $\langle E_M(N)\rangle$ is a prime submodule of M. Thus R s.t.r.f..

Now let R be an Artinian ring. Then $R_{\mathcal{M}}$ is a local Artinian ring for any maximal ideal $\mathcal{M}$ of R. Hence by the first part of the proof, $R_{\mathcal{M}}$ s.t.r.f. and by Theorem 3.35, R s.t.r.f.. □

By Corollary 3.26, we know that if R is a Noetherian ring and s.t.r.f., then $\dim R \leq 1$. We shall assume that R is one dimensional, Noetherian, local ring by Theorem 3.35 and 3.36. In this case we shall give a necessary condition for R to s.t.r.f..

We shall fix the following notation. Let $\mathcal{M}$ be the maximal ideal of R and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be all the minimal prime ideals of R. We have already dealt with the case $n = 1$ in Corollary 3.29. We define $I_i = \bigcap\limits_{\substack{k=1 \\ k \neq i}}^{n} \mathcal{P}_k$, for $n \geq 2$, $i = 1, 2, \ldots, n$. If $n \geq 3$, we also define $I_{ij} = \bigcap\limits_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{n} \mathcal{P}_k$, for all $1 \leq i, j \leq n$ with $i \neq j$.

Lemma 3.37 *[[8], Lemma 4.1] Suppose that $n \geq 3$ and $\mathcal{M} = I_{ij} + \mathcal{P}_i$, for all $1 \leq i, j \leq n$ with $i \neq j$. Then the following conditions are equivalent.*

(i) $\mathcal{M} = I_i + \mathcal{P}_i$, for some $1 \leq i \leq n$,

(ii) $\mathcal{M} = I_i + \mathcal{P}_i$, for each $i = 1, 2, \ldots, n$,

(iii) $I_{ij} = I_i + I_j$, for all $1 \leq i, j \leq n$ with $i \neq j$,

(iv) $I_{ij} = I_i + I_j$, for some $1 \leq i, j \leq n$ with $i \neq j$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $\mathcal{M} = I_{ij} + \mathcal{P}_i$, for all $1 \leq i, j \leq n$ with $i \neq j$ and $\mathcal{M} = I_i + \mathcal{P}_i$, for some $1 \leq i \leq n$. Then $I_{ij} + \mathcal{P}_i = I_i + \mathcal{P}_i$. Let $a_{ij} \in I_{ij}$. Then $a_{ij} = a_i + p_i$ for some $a_i \in I_i$ and $p_i \in \mathcal{P}_i$. Hence $p_i = a_{ij} - a_i \in \mathcal{P}_i \cap I_{ij} = I_j$ and $a_{ij} \in I_i + I_j$. Therefore $I_{ij} = I_i + I_j$. By assumption, $\mathcal{M} = I_{ij} + \mathcal{P}_i = I_i + I_j + \mathcal{P}_i$. Since $I_j \subseteq \mathcal{P}_i$ we have $\mathcal{M} = I_i + \mathcal{P}_i$ for all i.

$(ii) \Rightarrow (iii)$ It was shown in the proof of $(i) \Rightarrow (ii)$.

$(iii) \Rightarrow (iv)$ Clear.

$(iv) \Rightarrow (i)$ Suppose that $\mathcal{M} = I_{ij} + \mathcal{P}_i$ and $I_{ij} = I_i + I_j$, for some $1 \leq i, j \leq n$ with $i \neq j$. Then $\mathcal{M} = I_i + I_j + \mathcal{P}_i \subseteq I_i + \mathcal{P}_i$ since $I_j \subseteq \mathcal{P}_i$ and $\mathcal{M} = I_i + \mathcal{P}_i$ for some $1 \leq i \leq n$. □

Theorem 3.38 *[[8], Theorem 4.2] Suppose $(R, \mathcal{M})$ is a one dimensional local Noetherian ring and $n \geq 2$. If R^2 s.t.r.f. as an R-module, then there exist $x_1, \ldots, x_n \in R$ such that for $i = 1, 2, \ldots, n$, $I_i = Rx_i + \sqrt{0}$, $\mathcal{P}_i = \sum\limits_{\substack{k=1 \\ k \neq i}}^{n} I_k$ and*

$\mathcal{M} = \mathcal{P}_i + I_i = \sum\limits_{k=1}^{n} I_k$.

Proof. Clearly we may assume $\sqrt{0} = 0$. Hence R is reduced and $\bigcap\limits_{i=1}^{n} \mathcal{P}_i = 0$. By Theorem 3.25, $R/\mathcal{P}_i$ is a Dedekind domain for $i = 1, 2, \ldots, n$. Hence $R/\mathcal{P}_i$ is a discrete valuation ring.

Let $1 \leq i \leq n$ be given. Since $R/\mathcal{P}_i$ is a discrete valuation ring, $\mathcal{M}/\mathcal{P}_i$ is principal in $R/\mathcal{P}_i$. Hence $\mathcal{M} = Ry + \mathcal{P}_i$ for some $y \in \mathcal{M} - \mathcal{P}_i$. Since $\mathcal{P}_i$ is a minimal prime ideal, $I_i \neq 0$ and $I_i \not\subseteq \mathcal{P}_i$. As $R/\mathcal{P}_i$ is a discrete valuation ring, $(I_i + \mathcal{P}_i)/\mathcal{P}_i = (\mathcal{M}/\mathcal{P}_i)^l$ for some $l \geq 1$. Hence $I_i + \mathcal{P}_i = Ry^l + \mathcal{P}_i$. Then there exist $x_i \in I_i$, $p_i \in \mathcal{P}_i$ with $x_i = y^l + p_i$. We must show that x_i generates I_i. Let $a \in I_i$. Then $a = ry^l + q_i$ for some $r \in R$ and $q_i \in \mathcal{P}_i$. It follows that $rx_i - a = rp_i - q_i \in I_i \cap \mathcal{P}_i = 0$. Hence $a = rx_i$ and therefore $I_i = Rx_i$.

Suppose $n = 2$. Then $I_1 = \mathcal{P}_2$ and $I_2 = \mathcal{P}_1$. It remains to show that $\mathcal{M} = \mathcal{P}_1 + \mathcal{P}_2$. Since $R/\mathcal{P}_2$ is a discrete valuation ring, $\mathcal{M} = Ra + \mathcal{P}_2$ where $a \in \mathcal{M} - \mathcal{P}_2$. As $n \geq 2$, R is not a discrete valuation ring and $\mathcal{M} \neq Ra$. Let us choose $b \in \mathcal{M} - Ra$ and let $x \in \sqrt{Ra} - (\mathcal{P}_1 + \mathcal{P}_2)$. Since R is reduced, $\mathcal{P}_1 \cup \mathcal{P}_2$ contains all zero divisors of R. Hence $Ann_R x^n = 0$ for all $n \in \mathbb{N}$. By Theorem 3.24, $a \in \mathcal{M}^2 + \mathcal{P}_1 + \mathcal{P}_2$. Thus $\mathcal{M} = \mathcal{M}^2 + \mathcal{P}_1 + \mathcal{P}_2$ and by Nakayama's Lemma, $\mathcal{M} = \mathcal{P}_1 + \mathcal{P}_2$.

Now suppose that $n \geq 3$. For each $i = 1, 2, \ldots, n$, R/I_i is a one dimensional reduced local ring. By Proposition 3.11, $R/I_i \oplus R/I_i$ s.t.r.f. as an R/I_i-module for each i. By applying induction hypothesis to each $R/I_i \oplus R/I_i$, we get

(i) $\mathcal{M} = \sum\limits_{\substack{k=1 \\ k \neq i}}^{n} I_{ik}$ for $i = 1, 2, \ldots, n$,

(ii) $\mathcal{P}_i = I_j + \sum\limits_{\substack{k=1 \\ k \neq i}}^{n} I_{jk}$ and $I_{ij} = Rx_{ij} + I_j$ for some $x_{ij} \in \mathcal{M}$ and for all $1 \leq i, j \leq n$ with $i \neq j$.

Clearly, $I_{ik} \subseteq \mathcal{P}_j$ if i, j, k are all distinct. Then (i) gives

$$\mathcal{M} = I_{ij} + \mathcal{P}_j = Rx_{ij} + I_j + \mathcal{P}_j, \quad \text{for all i} \neq j. \tag{3.1}$$

Suppose that $\mathcal{M} \neq I_n + \mathcal{P}_n$. By Theorem 3.37,

$$\mathcal{M} \neq I_i + \mathcal{P}_i, \quad \text{for all i}, \tag{3.2}$$

$$I_{ij} \neq I_i + I_j, \quad \text{for all i} \neq \text{j}. \tag{3.3}$$

By (ii) and (3.3), $x_{12} \in I_{12} - (I_1 + I_2)$. Let $x \in \sqrt{Rx_{12}} - (I_1 + I_2)$. Since R is reduced, $Ann_R(x^n) = Ann_R(x)$ for any $n \in \mathbb{N}$. Let $r \in \sqrt{Rx_{12}}$. Then

$r^k \in Rx_{12}$ for some $k \in \mathbb{Z}^+$. Hence $r \in I_{12}$. As $\sqrt{Rx_{12}} \subseteq I_{12}$, $I_{12} \cap \mathcal{P}_1 = I_2$ and $I_{12} \cap \mathcal{P}_2 = I_1$, $x \in I_{12} - (\mathcal{P}_1 \cup \mathcal{P}_2)$. Therefore $Ann_R(x) \subseteq \mathcal{P}_1 \cap \mathcal{P}_2$. Clearly $Rx_{12} + Rx_{23} \subseteq \mathcal{P}_4 \cap \mathcal{P}_5 \cap \ldots \cap \mathcal{P}_n$. Hence

$$Rx_{12} + Ann_R(x) \cap (Rx_{12} + Rx_{23}) \subseteq Rx_{12} + I_3$$

.

Suppose $x_{23} \in Rx_{12}+Ann_R(x)\cap(Rx_{12}+Rx_{23})$. Then $x_{23} \in Rx_{12}+I_3$. By (3.1), $\mathcal{M} = Rx_{23}+I_3+\mathcal{P}_3 \subseteq Rx_{12}+I_3+\mathcal{P}_3 \subseteq I_{12}+I_3+\mathcal{P}_3 \subseteq I_3+\mathcal{P}_3$. Hence $\mathcal{M} = I_3+\mathcal{P}_3$ which contradicts with (3.2). Therefore $x_{23} \notin Rx_{12} + Ann_R(x) \cap (Rx_{12} + Rx_{23})$. Then $x_{12} \in \mathcal{M}^2 + I_1 + I_2$ by Theorem 3.24. By (3.1),

$$\mathcal{M} = Rx_{12} + I_2 + \mathcal{P}_2 \subseteq \mathcal{M}^2 + I_1 + I_2 + \mathcal{P}_2 \subseteq \mathcal{M}^2 + I_2 + \mathcal{P}_2.$$

Hence $\mathcal{M} = \mathcal{M}^2 + I_2 + \mathcal{P}_2$. By Nakayama's lemma, $\mathcal{M} = I_2 + \mathcal{P}_2$ which is a contradiction. Therefore $\mathcal{M} = I_n + \mathcal{P}_n$ and $I_{ij} = I_i + I_j$ for all $i \neq j$ by Lemma 3.37. Then by (i) and (ii),

$$\mathcal{M} = \sum_{\substack{k=1 \\ k \neq i}}^{n} I_{ik} = \sum_{\substack{k=1 \\ k \neq i}}^{n} (I_i + I_k) = \sum_{k=1}^{n} I_k$$

and

$$\mathcal{P}_i = I_j + \sum_{\substack{k=1 \\ k \neq i}}^{n} I_{jk} = I_j + \sum_{\substack{k=1 \\ k \neq i}}^{n} (I_j + I_k) = \sum_{\substack{k=1 \\ k \neq i}}^{n} I_k.$$

□

Theorem 3.39 *[[19], Theorem 5.1] Let $n \geq 2$, $J_1, \ldots, J_n$ be prime ideals in a Noetherian ring R and R is not neseccarily local. Suppose that*

(i) $J_1 \cap J_2 \cap \ldots \cap J_n = 0$,

(ii) R/J_i is a Dedekind domain for all i,

(iii) $J_{k+1} + \bigcap_{i=1}^{k} J_i = \bigcap_{i=1}^{k} (J_{k+1} + J_i)$,

(iv) $R = J_i + J_j$ *or* $R/(J_i + J_j)$ *is semi-simple Artinian, for any* $1 \leq i < j \leq n$.

Then R *s.t.r.f..*

By using the above results we can prove the following:

Theorem 3.40 *[[8], Theorem 1.1] Let* R *be a commutative Noetherian ring and* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *be all the minimal prime ideals of* R*. Then* R *s.t.r.f. if and only if* R *is Artinian or the following conditions are satisfied.*

(i) $\dim R = 1$ *and* $R/\mathcal{P}_i$ *is a Dedekind domain and* $\mathcal{P}_i$ *is the only* $\mathcal{P}_i$*-primary ideal, for each* $i = 1, 2, \ldots, n$.

(ii) for $k = 1, 2, \ldots, n-1$, $(\bigcap_{i=1}^{k} \mathcal{P}_i) + \mathcal{P}_{k+1} = \bigcap_{i=1}^{k} (\mathcal{P}_i + \mathcal{P}_{k+1})$ *if* $n \geq 2$,

(iii) for all $1 \leq i < j \leq n$, $R = \mathcal{P}_i + \mathcal{P}_j$ *or* $R/(\mathcal{P}_i + \mathcal{P}_j)$ *is semi-simple Artinian, if* $n \geq 2$.

Proof. Suppose that R^2 s.t.r.f. as an R-module and R is not Artinian. Hence we may assume $\dim R = 1$, by Corollary 3.26. In view of Corollary 3.29, we may also assume $n \geq 2$. By Corollary 3.25, (i) is satisfied. Under localization at any maximal ideal $\mathcal{M}$ of R, $\mathcal{P}_i R_{\mathcal{M}} = R_{\mathcal{M}}$ if $\mathcal{P}_i \nsubseteq \mathcal{M}$ and $\mathcal{P}_i R_{\mathcal{M}}$ remains prime otherwise. We can apply Theorem 3.38 to $R_{\mathcal{M}}$. Hence (iii) holds in $R_{\mathcal{M}}$ and both sides of the condition (ii) become $\mathcal{M}R_{\mathcal{M}}$ if $\mathcal{M}$ contains $\mathcal{P}_{k+1}$ and $\mathcal{P}_i$ for some $1 \leq i \leq k$. Otherwise, both sides will be equal to $R_{\mathcal{M}}$. Hence (ii) and (iii) holds globally.

Conversely, suppose that (i), (ii) and (iii) hold. By Proposition 3.27 and 3.28, we may assume $\sqrt{0} = 0$. Hence all the conditions required in Theorem 3.39 are satisfied. Therefore R s.t.r.f.. $\square$

Corollary 3.41 *[[8], Corollary 5.2] A Noetherian ring* R *s.t.r.f. if and only if* R^2 *s.t.r.f. as an* R*-module.*

Proof. Suppose that R^2 s.t.r.f.. By Corollary 3.26, $\dim R \leq 1$. If $\dim R = 0$, then R is Artinian and by Theorem 3.36 R s.t.r.f.. If $\dim R = 1$, then by Theorem 3.40 we get R s.t.r.f.. □

Definition 3.42 *Let M be an R-module. Then a proper submodule N of M is called semiprime if whenever $r^k m \in N$ for some $r \in R$, $m \in M$ and positive integer k, then $rm \in N$.*

If N is a submodule of M such that N is an intersection of prime submodules of M, then N is semiprime. The converse of the statement is not true in general. In particular the following theorem shows that if a ring R s.t.r.f., then this property is satisfied for any semiprime submodule of the R-module M.

Theorem 3.43 *[[1], Lemma 10] Let R be any ring. Then the following statements are equivalent for an R-module M.*

(i) M s.t.r.f.

(ii) Every semiprime submodule of M is an intersection of prime submodules of M, and $\langle E_M(\langle E_M(0)\rangle)\rangle = \langle E_M(0)\rangle$.

Proof. $(i) \Rightarrow (ii)$ Let N be a semiprime submodule of M. Then $\langle E_M(N)\rangle = N$. By (i), $rad_M(N) = N$. Hence N is an intersection of prime submodules of M. On the other hand $\langle E_M(0)\rangle = rad_M(0)$ means that $\langle E_M(0)\rangle$ is semiprime and hence $\langle E_M(\langle E_M(0)\rangle)\rangle = \langle E_M(0)\rangle$.

$(ii) \Rightarrow (i)$ Note that $\langle E_M(\langle E_M(0)\rangle)\rangle = \langle E_M(0)\rangle$ implies that $\langle E_M(0)\rangle$ is semiprime. Thus $rad_M(0) \subseteq \langle E_M(0)\rangle$. Hence M s.t.r.f.. □

Definition 3.44 *Let R be a domain. R is called a Bezout domain if every finitely generated ideal is principal.*

Theorem 3.45 *[[1], Theorem 11] Let R be a UFD such that every semiprime submodule of the R-module $R \oplus R$ is an intersection of prime submodules. Then R is a Bezout domain.*

Proof. Suppose that R is not a Bezout domain. Then there exist non-zero elements $a, b \in R$ such that the ideal $Ra + Rb$ is not principal. Let c denote the greatest common divisor of a and b. There exist elements $a_1, b_1 \in R$ such that $a = ca_1$, $b = cb_1$, a_1 and b_1 are coprime, and $I = Ra_1 + Rb_1 \neq R$. For if $Ra_1 + Rb_1 = R$, then $Ra + Rb$ would be principal.

Let $J = \sqrt{I}$. Note that since $I \neq R$ we have $J \neq R$. Let $f = (a_1, b_1) \in F = R \oplus R$. Let $N = J(a_1, b_1)$. We claim that N is a semiprime submodule of $R \oplus R$ which is not an intersection of prime submodules of $R \oplus R$.

Let $K = R(a_1, b_1)$. Then K is a prime submodule of $R \oplus R$ and $N \subseteq K$. For if $K = R \oplus R$, then $1_F = (1, 1) \in K$, that is, $(1, 1) = r(a_1, b_1)$ for some $r \in R$. Then $ra_1 = 1 = rb_1$ and $a_1 = b_1$. This gives us a contradiction. Hence $K \neq R \oplus R$. Let $(x, y) \in R \oplus R$ and $r(x, y) \in K$ for all $r \in R$. Then $(rx, ry) = z(a_1, b_1)$ for some $z \in R$. We have $rx = za_1$ and $ry = zb_1$. Suppose that $r \neq 0$. Then any prime p which divides r must divide z since a_1 and b_1 are coprime. Since R is a UFD, $r = p_1^{k_1} \ldots p_n^{k_n}$ where $p_1, \ldots, p_n$ are irreducible. Since every prime which divides r must divide z, $p_1^{k_1} \ldots p_n^{k_n}$ divides z. Hence r divides z. Then $x = (z/r)a_1$ and $y = (z/r)b_1$. So that $(x, y) = ((z/r)a_1, (z/r)b_1) = (z/r)(a_1, b_1) = (z/r)f$. Hence $(x, y) \in Rf = K$. Thus K is a prime submodule of F. Since $N = Jf$ and $K = Rf$ we have $N \subseteq K$.

Let L be any prime submodule of $R \oplus R$ such that $N = Jf \subseteq L$. Since L is prime $f \in L$ or $J(R \oplus R) \subseteq L$. If $J(R \oplus R) \subseteq L$, then $f = (a_1, b_1) = a_1(1, 0) + b_1(0, 1) \in J(R \oplus R) \subseteq L$. Hence in any case $f = (a_1, b_1) \in L$ and $K = Rf \subseteq L$. Thus K is the intersection of all prime submodules of $R \oplus R$

containing N. Let us show that N is semiprime. Let $s \in R$, $g \in R \oplus R$ and $s^k g \in N$ for some positive integer k. Then $s^k g \in N \subseteq K$ and $s^k(R \oplus R) \subseteq K$ or $g \in K$. If $s^k(R \oplus R) \subseteq K$, then $s = 0$. Suppose $s \neq 0$. Then $g \in K$. Since $K = Rf$ we have $g = tf$ for some $t \in R$. Then $s^k g \in N$ gives $s^k tf \in Jf$, and hence $s^k t \in J$. So $st \in J$. This implies that $sg = stf \in N$. It follows that N is semiprime. Since $J \neq R$ we have $N \neq K$ and hence N is not an intersection of prime submodules. □

Theorem 3.46 *[[1], Corollary 12] Let R be a Noetherian UFD. Then the following statements are equivalent.*

(i) R is a principal ideal domain,

(ii) R s.t.r.f.,

(iii) every finitely generated R-module s.t.r.f.,

(iv) the free R-module $R \oplus R$ s.t.r.f.,

(v) for every R-module M, every semiprime submodule is an intersection of prime submodules of M,

(vi) for every finitely generated R-module M, every semiprime submodule is an intersection of prime submodules of M,

(vii) every semiprime submodule of the free R-module $F = R \oplus R$ is an intersection of prime submodules of F

Proof. $(i) \Rightarrow (ii)$ Since every principal ideal domain is a Dedekind domain, it follows from Theorem 3.19.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ and $(v) \Rightarrow (vi) \Rightarrow (vii)$ are clear.

$(ii) \Rightarrow (v)$, $(iii) \Rightarrow (vi)$, $(iv) \Rightarrow (vii)$ by Lemma 3.43.

$(vii) \Rightarrow (i)$ Since R is Noetherian, every ideal of R is finitely generated. By Lemma 3.45, R is a Bezout domain. Hence R is a principal ideal domain. □

The following corollary gives a partial characterization of Noetherian domains which s.t.r.f..

Corollary 3.47 *[[1], Corollary 13] Let R be a Noetherian domain of finite global dimension. Then the following statements are equivalent.*

(i) R is a Dedekind domain,

(ii) R s.t.r.f.,

(iii) for every R-module M, every semiprime submodule is an intersection of prime submodules of M,

(iv) every semiprime submodule of the free R-module $F = R \oplus R$ is an intersection of prime submodules of F.

Proof. $(i) \Rightarrow (ii)$ By Theorem 3.19.

$(ii) \Rightarrow (iii)$ By Lemma 3.43.

$(iii) \Rightarrow (iv)$ Clear.

$(iv) \Rightarrow (i)$ Suppose that every semiprime submodule of F is an intersection of prime submodules of F. Let $\mathcal{M}$ be any maximal ideal of R. Then the local ring $R_{\mathcal{M}}$ is a unique factorization domain by Auslander-Buchsbaum-Serre Theorem (see, for example, [[20],Theorems 45 and 48]).

Now we check that every semiprime submodule of $F_{\mathcal{M}} = R_{\mathcal{M}} \oplus R_{\mathcal{M}}$ is an intersection of prime submodules of $F_{\mathcal{M}}$. Let N be semiprime submodule of $F_{\mathcal{M}}$. Then $N \cap F$ is a semiprime submodule of F. By [[21], Theorem 4.2], there exist a positive integer n and prime submodules K_i $(1 \leq i \leq n)$ of F such that

$$N \cap F = K_1 \cap \cdots \cap K_n.$$

Suppose that $(K_1 : F) \nsubseteq \mathcal{M}$. Then $(K_1 : F)(K_2 \cap \cdots \cap K_n) \subseteq N \cap F$ and this gives that $N \cap F = K_2 \cap \cdots \cap K_n$. Hence without loss of generality we can assume $(K_i : F) \subseteq \mathcal{M}$ $(1 \leq i \leq n)$. Now it is easy to check that $R_{\mathcal{M}}K_i$ is a prime submodule of $F_{\mathcal{M}}$ and $N = R_{\mathcal{M}}K_1 \cap \cdots \cap R_{\mathcal{M}}K_n$ for each $1 \leq i \leq n$.

By Theorem 3.45, $R_{\mathcal{M}}$ is a Bezout domain and so $R_{\mathcal{M}}$ is a principal ideal domain since $R_{\mathcal{M}}$ is Noetherian. Hence $R_{\mathcal{M}}$ is a discrete valuation ring and thus R is a Dedekind domain. $\square$

CHAPTER 4

GENERALIZATION OF COHEN'S THEOREM

4.1 Cohen's Theorem

Throughout R will denote a commutative ring and M be a unitary module. In this section we will give a generalization of Cohen's Theorem to modules.

Proposition 4.1 *(Cohen's Theorem) R is a Noetherian ring if and only if every prime ideal of R is finitely generated.*

Proof. Suppose every prime ideal of R is finitely generated and R is not Noetherian. Let $\mathcal{A}$ be the set of all ideals I of R which is not finitely generated. Since R is not Noetherian, there exists an ideal of R such that it is not finitely generated. Hence $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is partially ordered by inclusion. Let $\mathcal{B}$ be a non-empty totally ordered subset of $\mathcal{A}$. Define $J = \bigcup_{I_i \in \mathcal{B}} I_i$. Suppose that J is finitely generated, say

$$J = Ra_1 + Ra_2 + \cdots + Ra_t, \quad a_i \in R, \quad 1 \leq i \leq t.$$

Since $J = \bigcup_{I_i \in \mathcal{B}} I_i$, there exists $I_k \in \mathcal{B}$ such that $a_i \in I_k$ for some i. Then there exists $h \in \mathbb{N}$ such that $a_i \in I_h$ for all i since $\mathcal{B}$ is totally ordered. Then

$$J = Ra_1 + Ra_2 + \cdots + Ra_t \subseteq I_h \subseteq J.$$

Thus I_h is finitely generated, a contradiction. Hence J is not finitely generated and is an upper bound for $\mathcal{B}$. By Zorn's Lemma, there exists a maximal element $\mathcal{P} \in \mathcal{A}$.

Our aim is to show that $\mathcal{P}$ is a prime ideal. Let $a, b \in R - \mathcal{P}$ and suppose that $ab \in \mathcal{P}$. Since $\mathcal{P}$ is maximal, $\mathcal{P} + Ra$ is finitely generated. Say $\mathcal{P} + Ra$ is generated by $p_1 + r_1 a, p_2 + r_2 a, \ldots, p_n + r_n a$, where $p_1, p_2, \ldots, p_n \in \mathcal{P}$ and $r_1, r_2, \ldots, r_n \in R$. Let $K = \{r \in R : ra \in \mathcal{P}\}$. Since $K \supseteq \mathcal{P} + Rb \supsetneqq \mathcal{P}$, K is finitely generated. Therefore the ideal aK is also finitely generated. We claim now, $\mathcal{P} = Rp_1 + Rp_2 + \cdots + Rp_n + aK$. Let $r \in \mathcal{P} \subset \mathcal{P} + Ra$. Then there exist $c_1, c_2, \ldots, c_n \in R$ such that

$$r = c_1(p_1 + r_1 a) + \cdots + c_n(p_n + r_n a).$$

Hence $(\sum_{i=1}^{n} c_i r_i)a = r - \sum_{i=1}^{n} c_i p_i \in \mathcal{P}$, so that $\sum_{i=1}^{n} c_i r_i \in K$. Then $r = \sum_{i=1}^{n} c_i p_i + (\sum_{i=1}^{n} c_i r_i)a \in \sum_{i=1}^{n} Rp_i + aK$. Thus $\mathcal{P} \subseteq Rp_1 + Rp_2 + \cdots + Rp_n + aK$ and $\mathcal{P} = Rp_1 + Rp_2 + \cdots + Rp_n + aK$, as claimed. Hence $\mathcal{P}$ is finitely generated, a contradiction. Therefore $ab \notin \mathcal{P}$ and so $\mathcal{P}$ is prime. Thus we have found a prime ideal of R which is not finitely generated. This contradiction shows that R must be Noetherian. □

Proposition 4.2 *Let M and K be two modules. If K is a homomorphic image of M, then $K = M/N$ for any submodule N of M or there exists an epimorphism from M to K.*

Lemma 4.3 *[[4], Lemma] Let N be a submodule of an R-module M. Then N is finitely generated if any of the following two conditions are satisfied.*

(i) If there exists an element $r \in R$ such that both $N + rM$ and $(N :_M \langle r\rangle)$ are finitely generated,

(ii) If there exists an element $m \in M$ such that both $N + Rm$ and $(N : (Rm))$ are finitely generated.

Proof. Let $\psi : (N :_M \langle r\rangle) \to N \cap rM$ and $\varphi : (N : Rm) \to N \cap Rm$. Since ψ and φ are epimorphisms, $N \cap rM$ and $N \cap Rm$ are homomorphic images of

$(N :_M \langle r\rangle)$ and $(N : Rm)$, respectively. Then by the second isomorphism theorem, $(N + rM)/rM \cong N/(N \cap rM)$. Since $N + rM$ is finitely generated, $(N + rM)/rM$ and hence $N/(N \cap rM)$ is finitely generated. Thus N is finitely generated since $N \cap rM$ is finitely generated. Similarly,

$$(N + Rm)/Rm \cong N/(N \cap Rm).$$

Then N is finitely generated. □

Proposition 4.4 *[[4], Proposition 9, [12], Theorem 1] Let N be a proper submodule of an R-module M. Suppose that N is not finitely generated and is maximal among all submodules that are not finitely generated. Then N is a prime submodule.*

Proof. If $r \in R - (N : M)$, then $N \subseteq (N :_M \langle r\rangle)$ and $N \subsetneq N + rM$. Assume that $N \neq (N :_M \langle r\rangle)$. Then both $(N :_M \langle r\rangle)$ and $N + rM$ are finitely generated by the maximality of N. By Lemma 4.3, N is finitely generated which is a contradiction. Therefore $N = (N :_M \langle r\rangle)$ for every $r \in R - (N : M)$ and by Theorem 1.18, N is a prime submodule. □

The following theorem gives a generalization of Cohen's Theorem.

Theorem 4.5 *[[4], Theorem 6] A finitely generated R-module M is Noetherian if and only if every prime submodule of M is finitely generated.*

Proof. Since M is Noetherian, necessity is obvious. Conversely suppose that every prime submodule of M is finitely generated and M is not Noetherian. Let N be a submodule of M which is not finitely generated. Then there exists a submodule K of M which is maximal with respect to this property. Hence by Proposition 4.4, K is a prime submodule of M and finitely generated by our assumption. This contradicts with the fact that K is not finitely generated. Therefore M is Noetherian. □

Independently, Karakaş [12] and Lu [4] have proved that a finitely generated R-module M is Noetherian if and only if every prime submodule of M is finitely generated. In case $M = R$, this result reduces to Cohen's Theorem. Then Smith [11] gave an alternative generalization of Cohen's Theorem to modules.

Lemma 4.6 *[[11], Lemma 6] Let R be a commutative ring and let $\mathcal{P}$ be a prime ideal of R. Let M be a non - zero finitely generated R-module such that $\mathcal{P}$ contains the annihilator of M. Then $M \neq \mathcal{P}M$ and $M/\mathcal{P}M$ is a faithful $R/\mathcal{P}$-module. In particular, $M/\mathcal{P}M$ is not a torsion $R/\mathcal{P}$-module.*

Proof. Suppose $M = \mathcal{P}M$. By Proposition 1.35, we get a contradiction that $\mathcal{P} = R$. Hence $M \neq \mathcal{P}M$. Since $\mathcal{P} \subseteq Ann(M/\mathcal{P}M)$, $M/\mathcal{P}M$ is an $(R/\mathcal{P})$-module. Let $x + \mathcal{P} \in Ann(M/\mathcal{P}M)$. Then $xM \subseteq \mathcal{P}M$. Proposition 1.35 gives $Ann(M/\mathcal{P}M) = \mathcal{P}$.

Now suppose that $M/\mathcal{P}M$ is a torsion $(R/\mathcal{P})$-module. Let $r+\mathcal{P}$ be a non-zero element of $Ann(m + \mathcal{P}M)$ for some $m + \mathcal{P}M \in M/\mathcal{P}M$. Then $rm \in \mathcal{P}M$ and since $M/\mathcal{P}M$ is faithful $r \in \mathcal{P}$, a contradiction. □

Lemma 4.7 *[[11], Lemma 7] Let R be an infinite commutative domain and let M be an R-module of torsion-free rank at least 2. Then M contains an infinite number of 0-primes.*

Proof. Let $x, y \in M$ such that $Rx + Ry$ is a free submodule of M with basis $\{x, y\}$. For each element $r \in R$, let $L_r = R(x + ry)$ and K_r be the submodule of M containing L_r such that K_r/L_r is the torsion submodule of M/L_r. Let $m \in M$, $r \in R$ with $rm \in K_r$ and $m \notin K_r$. Then there exists a non-zero $t \in R$ such that $t(rm) \in L_r$. Since $m \notin K_r$ and t is non-zero we have $r = 0$. Hence $rM \subseteq K_r$. Thus K_r is a prime submodule of M. Now let $\alpha \in (K_r : M)$. Then $\alpha M \subseteq K_r$

and by the above argument $\alpha = 0$. Therefore K_r is a 0-prime submodule of M for each element $r \in R$.

Suppose that $K_r = K_s$ for some $r, s \in R$. By the choice of K_r and K_s for $kn \in K_r$, $kn = k'n'$ for some $n' \in K_s$, $k' \in R$. Then there exist $a, b \in R$ with $a(x + ry) = b(x + sy)$ and $a \neq 0$. Thus $(a - b)x = (sb - ra)y$ and since $\{x, y\}$ is a basis $a = b$ and $sb = ra$. It follows that $r = s$. Thus $\{K_r : r \in R\}$ is an infinite collection of 0-prime submodules of M. □

If one regards the Karakaş- Lu result as giving a test for a finitely generated module to be Noetherian, it is not very efficient since a module can have a large number of prime submodules. In fact, in most situations for every prime ideal $\mathcal{P}$ of R a finitely generated R-module M will contain an infinite number of $\mathcal{P}$-prime submodules, as the next result shows.

Proposition 4.8 *[[11], Lemma 8] Let R be a commutative ring, $\mathcal{P}$ be a prime ideal of R and let M be a finitely generated R-module such that $\mathcal{P}$ contains the annihilator of M. Then M contains only a finite number of $\mathcal{P}$-prime submodules if and only if the domain $R/\mathcal{P}$ is finite or the $R/\mathcal{P}$-module $M/\mathcal{P}M$ has torsion-free rank at most one.*

Proof. Suppose first that $R/\mathcal{P}$ is finite. Since $M/\mathcal{P}M$ is finitely generated, it is finite. Hence M contains only a finite number of 0-primes. Now suppose that $R/\mathcal{P}$ is infinite, $M \neq 0$ and the torsion-free rank of the $R/\mathcal{P}$-module $M/\mathcal{P}M$ is at most one. Let K denote the submodule of M containing $\mathcal{P}M$ such that $K/\mathcal{P}M$ is the $R/\mathcal{P}$-torsion submodule of $M/\mathcal{P}M$. By Lemma 4.6, $K \neq M$. Let us show that K is a $\mathcal{P}$-prime submodule of M. Let $m + K \in T(M/K)$. Then there exist elements $x \in R - \mathcal{P}$ and $z \in R - \mathcal{P}$ such that $zxm \in \mathcal{P}M$. Since $zx \in R - \mathcal{P}$ and $T(M/\mathcal{P}M) = K/\mathcal{P}M$ we have $m \in K$. Hence $T(M/K) = K$. Let $k \in (K : M)$ and $k \notin \mathcal{P}$. Then $km \in K$ for all $m \in M$. There exists an element $y \in R - \mathcal{P}$ such that $y(km) \in \mathcal{P}M$. Since $yk \notin \mathcal{P}$ we have $m \in K$. This gives a contradiction that $K = M$. Thus $(K : M) = \mathcal{P}$. Let H be any $\mathcal{P}$-prime submodule of M. Let $t \in K = T(M/K)$. Then there exist elements $a, b \in R - \mathcal{P}$ such that $(ab)t \in \mathcal{P}M$. Since $\mathcal{P}M \subseteq H$ we have $(ab)t \in H$. Then $t \in H$ since H is a prime submodule of M and $ab \notin \mathcal{P}$. Hence $K \subseteq H$. Since $M/\mathcal{P}M$ has rank one, $H = K$. Thus K is the only $\mathcal{P}$-prime submodule of M.

Conversely suppose that the $R/\mathcal{P}$-module $M/\mathcal{P}M$ has torsion-free rank at least two. By Lemma 4.7, we get a contradiction that M contains an infinite number of $\mathcal{P}$-prime submodules. □

Lemma 4.9 *[[11], Lemma 9] Let M be a finitely generated module and I be an ideal of R such that IM is finitely generated. Then there exists a finitely generated ideal J of R such that $J \subseteq I$ and $IM = JM$.*

Proof. Suppose that $IM = Rx_1 + Rx_2 + \ldots + Rx_k$ for some positive integer k and $x_i \in IM$ $(1 \leq i \leq k)$. Then there exist $a_{ij} \in I$ $(1 \leq j \leq n)$ such that $x_i = a_{i1}m_1 + a_{i2}m_2 + \cdots + a_{in}m_n$ for each i. Let J denote the ideal of R generated by the finite elements a_{ij} for $1 \leq i \leq k$ and $1 \leq j \leq n$. Then $J \subseteq I$ and $IM = JM$. □

Corollary 4.10 *[[11], Corollary 10] Let M be a finitely generated module, I and J be ideals of R such that IM and JM are both finitely generated. Then IJM is finitely generated.*

Proof. By Lemma 4.9, there exist finitely generated ideals I^* and J^* such that $I^* \subseteq I$, $J^* \subseteq J$ and $I^*M = IM$, $J^*M = JM$. Then

$$IJM = IJ^*M = J^*(IM) = J^*I^*M$$

is finitely generated since M and J^*I^* are both finitely generated. □

We need one final lemma to prove the generalization of Cohen's Theorem to modules. The following lemma is called Eakin- Nagata Theorem([15], [16]), as generalized by Formanek [17].

Lemma 4.11 *[[11], Lemma 11] Let R be a ring and let M be a finitely generated R-module such that M satisfies ACC on submodules of the form IM, where I is an ideal of R. Then M is Noetherian.*

Theorem 4.12 *[[11], Theorem 5] Let R be a ring and M be a finitely generated R-module. Then M is Noetherian if and only if the submodule $\mathcal{P}M$ is finitely generated for every prime ideal $\mathcal{P}$ of R.*

Proof. Let R be a ring and M be a finitely generated R-module such that $M\mathcal{P}$ is finitely generated for every prime ideal $\mathcal{P}$ of R. Suppose that M is not Noetherian. By Lemma 4.11, there exists a proper ascending chain

$$A_1M \subset A_2M \subset A_3M \subset \cdots,$$

where A_i is an ideal of R for every $i \geq 1$. Let $A = \sum\limits_{i\geq 1} A_i$. Then AM is not finitely generated. For if AM is finitely generated, then $AM = Ra_1+Ra_2+\cdots+Ra_n$ such that $a_j \in AM$ for $j = 1, 2, \ldots n$. Then there exist $A_{k_j}M$ such that $a_j \in A_{k_j}M$ for all j. Let $k = \max\{k_1, k_2, \ldots, k_n\}$. Hence $a_j \in A_kM$ for all j, that is, $AM = A_kM$. Then $A_iM \subseteq A_kM$ for all i. This gives the contradiction that the ascending chain $A_1M \subset A_2M \subset A_3M \subset \cdots$ must terminate.

Let Δ denote the collection of all ideals B of R such that BM is not finitely generated. $\Delta \neq \emptyset$ since $A \in \Delta$. By Zorn's Lemma, Δ contains a maximal member, say I. By hypothesis, I is not a prime ideal and hence there exist ideals C and D properly containing I such that $CD \subseteq I$. Since I is maximal, CM and DM are finitely generated and by Lemma 4.11, both M/CM and M/DM are Noetherian. Let $C' = (CM : M)$. Then $CM = C'M$ and M/CM is a faithful

R/C'-module. Hence R/C' is a Noetherian ring. It follows that $DM/CDM = DM/C'DM$ is Noetherian and hence M/CDM is Noetherian. But $CDM \subseteq IM$ and CDM is finitely generated by Corollary 4.10. Thus IM is finitely generated, which is a contradiction. Therefore M is Noetherian. $\square$

Corollary 4.13 *[[11], Corollary 12] Let R be a ring and let M be a projective R-module such that $\mathcal{P}M$ is finitely generated for every prime ideal $\mathcal{P}$ of R. Then M is Noetherian.*

Proof. If R is not local, then it contains distinct maximal ideals $\mathcal{P}$ and $\mathcal{Q}$. Since $\mathcal{P}M$ and $\mathcal{Q}M$ are both finitely generated and $M = \mathcal{P}M + \mathcal{Q}M$, M is finitely generated. Then by Theorem 4.12, M is Noetherian. If R is local, then M is free and it is clear that M is finitely generated. Hence again by Theorem 4.12, M is Noetherian. $\square$

We prove the following stronger version of the Karakaş-Lu Theorem.

Theorem 4.14 *[[11], Theorem 13] Let R be a ring. Then a finitely generated R-module M is Noetherian if and only if the submodule $M(\mathcal{P})$ is finitely generated for every prime ideal $\mathcal{P}$ of R containing $Ann_R(M)$.*

Proof. Let M be a finitely generated R-module such that $M(\mathcal{P})$ is finitely generated for every prime ideal $\mathcal{P}$ containing $Ann_R(M)$. Suppose that M is not Noetherian. We adapt the proof of Theorem 4.12. There exists an ideal I of R such that IM is not finitely generated. We can choose $\mathcal{P}$ to be maximal among such ideals. If $\mathcal{P}$ is not prime, then the proof of Theorem 4.12 gives a contradiction. Suppose that $\mathcal{P}$ is prime. If $\mathcal{P}$ does not contain $Ann_R(M)$, then $M(\mathcal{P}) = M$ by Proposition 3.31. If $\mathcal{P}$ contains $Ann_R(M)$, then by the hypothesis $M(\mathcal{P})$ is finitely generated. Thus in any case $M(\mathcal{P})$ is finitely generated.

Let $K = M(\mathcal{P})$. Since K is finitely generated, $K = Rx_1 + Rx_2 + \cdots + Rx_n$ such that $x_i \in M(\mathcal{P})$ for all $1 \leq i \leq n$. Then there exist $c_i \in R - \mathcal{P}$ such that

$c_i x_i \in \mathcal{P}M$. Define $c = c_1 c_2 \dots c_n$. Hence $cK \subseteq \mathcal{P}M$. By the choice of $\mathcal{P}$, the submodule $(\mathcal{P} + Rc)M$ is finitely generated. Hence $\mathcal{P}M \subseteq N + cM$ for some finitely generated submodule N of $\mathcal{P}M$. Since

$$N + cK \subseteq \mathcal{P}M \subseteq K \cap (N + cM) = N + (K \cap cM) = N + cK,$$

we have $\mathcal{P}M = N + cK$ which is a finitely generated submodule of M, a contradiction. Therefore M is Noetherian. □

By Theorem 4.14, we get the following corollary which is the Karakaş - Lu Theorem.

Corollary 4.15 *[[11], Corollary 14] Let R be a commutative ring. Then a finitely generated R-module M is Noetherian if and only if every prime submodule of M is finitely generated.*

Now suppose that R is not a commutative ring for the rest of this section.

Definition 4.16 *A ring R is said to be prime ring if the zero ideal is a prime ideal.*

Definition 4.17 *A prime ring R is called right bounded if every essential right ideal contains a non-zero (two sided) ideal.*

R is a right fully bounded ring if every prime homomorphic image of R is right bounded.

Now we will show the generalization of Cohen's Theorem for the right fully bounded rings.

Lemma 4.18 *[[11], Lemma 2] Let R be a ring such that every ideal is finitely generated as a right ideal. Let I be any proper ideal of R. Then there exist a positive integer n and prime ideals $\mathcal{P}_i$ $(1 \leq i \leq n)$ of R such that $\mathcal{P}_1 \dots \mathcal{P}_n \subseteq I \subseteq \mathcal{P}_1 \cap \dots \cap \mathcal{P}_n$.*

Proof. Suppose the result is false and let Π denote the set of all proper ideals of R which do not satisfy the result. Let $\{A_\lambda : \lambda \in \Lambda\}$ be any chain in Π and let $A = \bigcup_{\lambda\in\Lambda} A_\lambda$. If $A \notin \Pi$, then A contains a product J of prime ideals. Since J is a finitely generated as a right ideal, $J \subseteq A_\lambda$ for some $\lambda \in \Lambda$, a contradiction. Hence $A \in \Pi$. By Zorn's Lemma, Π contains a maximal member B. If B is prime, then B contains a product of prime ideals. Thus B is not prime and there exist ideals C and D, each properly containing B such that $CD \subseteq B$. Since B is the maximal member, there exist prime ideals $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m$ such that

$$\mathcal{P}_1 \ldots \mathcal{P}_n \subseteq C \subseteq \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_n$$

$$\mathcal{Q}_1 \ldots \mathcal{Q}_m \subseteq D \subseteq \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_m.$$

Then we get a contradiction $(\mathcal{P}_1 \ldots \mathcal{P}_n)(\mathcal{Q}_1 \ldots \mathcal{Q}_m) \subseteq CD \subseteq B$. Hence the result is true for any proper ideal of R. $\square$

Lemma 4.19 *[[11], Lemma 3] Let R be a ring such that every prime ideal is finitely generated as a right ideal. Then R satisfies ACC on prime ideals.*

Proof. Let $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \cdots$ be any ascending chain of prime ideals of R and let $I = \bigcup_{i\geq 1} \mathcal{P}_i$. Clearly I is a proper ideal of R. Hence by Lemma 4.18, there exist a positive integer n and prime ideals $\mathcal{Q}_j$ $(1 \leq j \leq n)$ such that

$$\mathcal{Q}_1 \ldots \mathcal{Q}_n \subseteq I \subseteq \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_n.$$

Since $\mathcal{Q}_1 \ldots \mathcal{Q}_n$ is finitely generated right ideal, $\mathcal{Q}_1 \ldots \mathcal{Q}_n \subseteq \mathcal{P}_k$ for some positive integer k. This gives $\mathcal{Q}_j \subseteq \mathcal{P}_k$ for some $1 \leq j \leq n$ and since $I = \bigcup_{i\geq 1} \mathcal{P}_i$ we have $I = \mathcal{P}_k$. Thus R satisfies ACC on prime ideals. $\square$

Lemma 4.20 *[[11], Lemma 4] Let R be a ring such that every prime ideal is finitely generated as a right ideal. Suppose further that $R/\mathcal{P}$ is right Noetherian for every prime ideal $\mathcal{P}$ of R. Then R is Noetherian.*

Proof. Let $\mathcal{P}$ be a prime ideal of R. Since every ideal of $R/\mathcal{P}$ is finitely generated, every ideal I of R which is containing $\mathcal{P}$ is finitely generated. By Lemma 4.18, there exist a positive integer n and prime ideals $\mathcal{P}_i/\mathcal{P}$ $(1 \leq i \leq n)$ such that $(\mathcal{P}_1/\mathcal{P})\dots(\mathcal{P}_n/\mathcal{P}) = 0$. Consider the chain $\mathcal{P}_1 \supseteq \mathcal{P}_1\mathcal{P}_2 \supseteq \dots \supseteq \mathcal{P}_1 \dots \mathcal{P}_n = 0$. By hypothesis, each factor in this chain is a Noetherian right R-module. This implies that $\mathcal{P}_1$ is right Noetherian. Hence R is right Noetherian. □

Theorem 4.21 *[[11], Theorem 1] A right fully bounded ring is right Noetherian if and only if every prime ideal is finitely generated as a right ideal.*

Proof. Let R be a right fully bounded ring such that every prime ideal is finitely generated as a right ideal. Suppose that R is not right Noetherian. By Lemma 4.20, there exists a prime ideal $\mathcal{P}$ such that $R/\mathcal{P}$ is not right Noetherian and by Lemma 4.19, we can choose $\mathcal{P}$ maximal with respect to this property. Say $R' = R/P$. Since

$$R'/\mathcal{Q}' = (R/\mathcal{P})/(\mathcal{Q}/\mathcal{P}) \cong R/\mathcal{Q},$$

every ideal of $R'/\mathcal{Q}'$ is finitely generated as a right ideal.

Let $I/\mathcal{Q}$ be any non-zero proper ideal of $R/\mathcal{Q}$. By Lemma 4.18, there exist a positive integer n and prime ideals $\mathcal{P}_i/\mathcal{Q}$ $(1 \leq i \leq n)$ such that

$$\mathcal{Q} \subseteq \mathcal{P}_1 \dots \mathcal{P}_n \subseteq I \subseteq \mathcal{P}_1 \cap \dots \cap \mathcal{P}_n.$$

Let us say $J = \mathcal{P}_1 \dots \mathcal{P}_n$ and take $R = R/J$ in Lemma 4.20. Since all axioms of Lemma 4.20 are satisfied, R/J is right Noetherian. Hence R/I is right Noetherian. It follows that I/J and hence I is finitely generated right ideal. Now let $E/\mathcal{Q}$ be any essential right ideal of $R/\mathcal{Q}$. Then there exists a non-zero ideal $K/\mathcal{Q}$ such that $K/\mathcal{Q} \subseteq E/\mathcal{Q}$. By the earlier observation, R/K is right Noetherian and K is finitely generated right ideal. Thus E is finitely generated right ideal. Therefore every essential right ideal of R is finitely generated. Since any right ideal of R is a direct summand of an essential right ideal, it follows that every ideal of R is finitely generated and hence R is right Noetherian, a contradiction. $\square$

4.2 ACC for Radical Submodules of Modules

Definition 4.22 *Let M be an R-module and N its submodule. Then N is a radical submodule if $rad_M(N) = N$.*

In this section we consider the relationships between the following five statements for any R-module M.

(i) M satisfies the ACC for radical submodules,

(ii) M satisfies the ACC for prime submodules and each radical submodule is an intersection of a finite number of prime submodules,

(iii) every radical submodule is the radical of a finitely generated submodule,

(iv) every prime submodule is the radical of a finitely generated submodule,

(v) M satisfies the ACC for prime submodules.

Theorem 4.23 *[[6], Theorem 5] For any R-module M, we have $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii) \Rightarrow (iv)$. If M is finitely generated, then $(iv) \Rightarrow (v)$.*

Proof. $(i) \Rightarrow (ii)$ Since every prime submodule is a radical submodule, it is clear that M satisfies the ACC for prime submodules. Let N be a radical submodule of M and put $N = \bigcap_{i \in I} P_i$, where P_i is a prime submodule for each i and this expression is reduced. Assume that I is an infinite index set. Without loss of generality, we may assume that I is countable. Then

$$N = \bigcap_{i=1}^{\infty} P_i \subseteq \bigcap_{i=2}^{\infty} P_i \subseteq \bigcap_{i=3}^{\infty} P_i \subseteq \cdots$$

is an ascending chain of radical submodules which must terminate by our assumption. Hence there exists $j \in \mathbb{N}$ such that $\bigcap_{i=j}^{\infty} P_i = \bigcap_{i=j+t}^{\infty} P_i$ for all $t = 1, 2, \ldots$. This implies that $\bigcap_{i=j+t}^{\infty} P_i \subseteq P_j$, which contradicts that the expression $N = \bigcap_{i=1}^{\infty} P_i$ is reduced. Therefore I is a finite index set.

$(i) \Rightarrow (iii)$ Assume that there exists a radical submodule N which is not the radical of a finitely generated submodule. Let $x_1 \in N$ and let $N_1 = rad_M(Rx_1)$. Then $N_1 \subsetneqq N$, so that there exists $x_2 \in N - N_1$. Let $N_2 = rad_M(Rx_1 + Rx_2)$. Then $N_1 \subsetneqq N_2 \subsetneqq N$, so that there exists $x_3 \in N - N_2$. Continuing by this way we get an ascending chain of radical submodules $N_1 \subsetneqq N_2 \subsetneqq N_3 \subsetneqq \cdots$ which contradicts with (i).

$(iii) \Rightarrow (iv)$ This is clear.

$(iv) \Rightarrow (v)$ Assume that M is finitely generated. Let $P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$ be an ascending chain of prime submodules of M. Then $P = \bigcup_{i=1}^{\infty} P_i$ is a prime submodule by Lemma 2.6. By our assumption, P is a radical of some finitely generated submodule L, say $L = Re_1 + Re_2 + \cdots + Re_n$. Since $L \subseteq rad_M(L) = P = \bigcup_{i=1}^{\infty} P_i$, there exists P_j for some $j \geq 1$ such that $L = Re_1 + Re_2 + \cdots + Re_n \subseteq P_j$. Then $P = rad_M(L) \subseteq rad_M(P_j) = P_j$ and hence the chain of prime submodules terminates. □

Proposition 4.24 *[[6], Proposition 6] Let M be an R-module such that $R/Ann_R(M)$ satisfies the ACC for prime ideals. Then M satisfies the ACC for prime submodules if and only if it satisfies the ACC for $\mathcal{P}$-prime submodules for every prime ideal $\mathcal{P}$ containing $Ann_R(M)$.*

Proof. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be an ascendig chain of prime submodules of M. Then $(K_1 : M) \subseteq (K_2 : M) \subseteq (K_3 : M) \subseteq \cdots$ and $Ann_R(M) \subseteq (K_i : M)$ for all i. By our assumption, there exist positive integers t, k such that

$$(K_t : M) = (K_{t+k} : M) \quad \text{for all} \quad k \geq 1.$$

Hence the chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ must terminate. $\square$

Lemma 4.25 *Let M be an R-module which satisfies the ACC for prime submodules and suppose that each finitely generated submodule of M has only finitely many minimal prime submodules. Then M satisfies the ACC for radicals $rad_M(N)$ of finitely generated submodules N.*

Proof. Let $rad_M(N_1) \subseteq rad_M(N_2) \subseteq rad_M(N_3) \subseteq \cdots$ be an ascending chain of radicals of finitely generated submodules N_i of M for all $i \geq 1$. Let $H_i = \{P_1^{(i)}, P_2^{(i)}, \ldots, P_{n_i}^{(i)}\}$ be the set of all minimal prime submodules of N_i for each i. Then we have

$$\bigcap_{j=1}^{n_1} P_j^{(1)} \subseteq \bigcap_{j=1}^{n_2} P_j^{(2)} \subseteq \bigcap_{j=1}^{n_3} P_j^{(3)} \subseteq \cdots$$

By Proposition 2.7, each $P_j^{(i+1)} \in H_{i+1}$ must contain some $P_k^{(i)} \in H_i$ for every $i \geq 1$. By applying the same way, we can construct an ascending chain of prime submodules since each n_i is a finite number. If $rad_M(N_1) \subseteq rad_M(N_2) \subseteq rad_M(N_3) \subseteq \cdots$ does not terminate, then we will get a non-terminating ascending chain of prime submodules, which is a contradiction. $\square$

Lemma 4.26 *[[6], Lemma 2] Assume the hypothesis in Lemma 4.25. If P is any prime submodule of M, then $P = rad_M(N)$ for some finitely generated submodule N of M.*

Proof. We can prove the lemma in almost the same way as the proof of $(i) \Rightarrow (iii)$ in Theorem 4.23 and the result follows from Lemma 4.25. $\square$

In Theorem 4.5, we gave the generalization of Cohen's Theorem. Now in the following theorem we see its application to modules satisfying the ACC for prime submodules.

Theorem 4.27 *[[6], Theorem 6] Let M be a finitely generated R-module which satisfies the ACC for prime submodules. Then M is Noetherian if and only if* *every finitely generated submodule N of M has only finitely many minimal prime submodules and $rad_M(N)$ is finitely generated.*

Proof. ($\Leftarrow$) Let P be any prime submodule of M. Then $P = rad_M(N)$ for some finitely generated submodule N of M by Lemma 4.26. Hence M is Noetherian by 4.5.

($\Rightarrow$) Let M be a Noetherian module and let N be a finitely generated submodule of M. Then $rad_M(N) = \bigcap_{i \in I} P_i$, where each P_i is a minimal prime submodules of N by Theorem 2.10. Assume I is an infinite index set. Then we have an ascending chain of minimal prime submodules of N

$$\bigcap_{i=1}^{\infty} P_i \subseteq \bigcap_{i=2}^{\infty} P_i \subseteq \bigcap_{i=3}^{\infty} P_i \subseteq \cdots .$$

Since M satisfies ACC for prime submodules there exists $j \in \mathbb{N}$ such that $\bigcap_{i=j}^{\infty} P_i = \bigcap_{i=j+k}^{\infty} P_i$ for all $k = 1, 2, \ldots$. Hence $\bigcap_{i=j+k}^{\infty} P_i = P_j$. This contradicts with the fact that P_i is a minimal prime submodule of N. Therefore N has finitely many minimal prime submodules. By Lemma 4.26, $rad_M(N) = P$ for some prime submodule of M. Then $rad_M(N)$ is finitely generated since M is Noetherian. $\square$

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