

**KARADENİZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ**

MATEMATİK ANABİLİM DALI

REGÜLER SINIR DEĞER PROBLEMİNİN GREEN FONKSİYONU

YÜKSEK LİSANS TEZİ

Çiğdem GEBİÇ

**HAZİRAN 2022
TRABZON**



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Karadeniz Teknik Üniversitesi Fen Bilimleri Enstitüsünde
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ÖNSÖZ

Tez konumun belirlenmesinden, çalışmamın tamamlanıp bu hale getirilmesine kadar geçen süreçte benden desteğini ve yardımını esirgemeyen, lisans ve yüksek lisans öğrenimim boyunca kendisinden, araştırmalarından, çalışmalarından ve kişiliğinden çok şey öğrendiğim sayın Prof. Dr. Haskız COŞKUN hocama emeği için teşekkür eder, saygılarımı sunarım.

Tez çalışmamı hazırlarken bana yardım eden değerli dostlarıma, hayatım boyunca yanımda olan ve her türlü kararımda maddi ve manevi destek sağlayan, bu günlere gelmemde emeği olan anne ve babama, beni her konuda cesaretlendiren, motive eden kardeşlerime teşekkürü bir borç bilirim.

Çiğdem GEBİÇ

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TEZ ETİK BEYANNAMESİ

Yüksek Lisans Tezi olarak sunduğum “REGÜLER SINIR DEĞER PROBLEMİNİN GREEN FONKSİYONU” başlıklı bu çalışmayı baştan sona kadar danışmanım Prof.Dr. Haskız COŞKUN’un sorumluluğunda tamamladığımı, verileri/örnekleri kendim topladığımı, deneyleri/analizleri ilgili laboratuvarlarda yaptığımı/yaptırdığımı, başka kaynaklardan aldığım bilgileri metinde ve kaynakçada eksiksiz olarak gösterdiğimi, çalışma sürecinde bilimsel araştırma ve etik kurallarına uygun olarak davrandığımı ve aksinin ortaya çıkması durumunda her türlü yasal sonucu kabul ettiğimi beyan ederim. 14/06/2022

Çiğdem GEBİÇ

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Yüksek Lisans Tezi

ÖZET

REGÜLER SINIR DEĞER PROBLEMİNİN GREEN FONKSİYONU

Çiğdem GEBİÇ

Karadeniz Teknik Üniversitesi

Fen Bilimleri Enstitüsü

Matematik Anabilim Dalı

Danışman: Prof. Dr. Haskız COŞKUN

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Bu çalışma esas olarak üç bölümden oluşmaktadır.

Birinci bölümde, temel bazı bilgiler yardımıyla sınır değer problemleri için Green fonksiyonlarının elde edilişi incelenmiştir.

İkinci bölümde, $q(x) \in L^1[0, \pi]$ olmak üzere

$$y'' + (\lambda - q)y = 0, 0 \leq x \leq \pi$$

$$\frac{y'(0, \lambda)}{y(0, \lambda)} = -\tan \alpha,$$

$$\cos \beta y'(\pi, \lambda) + \sin \beta y(\pi, \lambda) = 0$$

probleminin Green fonksiyonları asimptotik olarak bulunmuştur. Dirichlet ve Neumann sınır koşullarının sağlanması durumunda da Green fonksiyonu hesaplanmıştır.

Üçüncü bölümde ise potansiyel fonksiyonu

$q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$ olması durumu ele alınarak bazı özel başlangıç değerlerini sağlayan çözümler için asimptotik tahminler elde edilmiş ve bu tahminler kullanılarak Green fonksiyonları hesaplanmıştır. Ayrıca Dirichlet ve Neumann sınır koşulları durumunda Green fonksiyonları asimptotik olarak elde edilmiştir.

Anahtar Kelimeler: Green fonksiyonu, Asimptotik çözümler, Dirichlet ve Neumann sınır koşulları

Master Thesis

SUMMARY

THE GREEN'S FUNCTION OF REGULAR BOUNDARY VALUE PROBLEM

Çiğdem GEBİÇ

Karadeniz Technical University
The Graduate School of Natural and Applied Sciences
Mathematics Graduate Program
Supervisor: Professor. Haskız COŞKUN
2022, 115 Pages

This study basically consists of three sections. In the first section, we formulate Green's functions for boundary value problems based on some main theory. In the second section, we consider the boundary value problem

$$y'' + (\lambda - q)y = 0, 0 \leq x \leq \pi$$

$$\frac{y'(0, \lambda)}{y(0, \lambda)} = -\tan \alpha,$$

$$\cos \beta y'(\pi, \lambda) + \sin \beta y(\pi, \lambda) = 0$$

and compute Green's functions associated with it. In addition to these, we also consider Dirichlet and Neumann boundary conditions.

In the last section, we particularly consider the special integrable potential

$q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$, and find asymptotic estimates for the solutions satisfying certain initial values, and then using these estimates we evaluate Green's functions.

Key Words: Green's functions, Asymptotic solutions, Dirichlet and Neumann boundary conditions

SEMBOLLER DİZİNİ

λ	: özdeğer
$\lambda \rightarrow \infty$	: λ, ∞ 'a yaklaşırken
$W(u, v)(x)$	: Wronskian determinanı
$G(x, y, \lambda)$	: Green fonksiyonu
π	: Pi sayısı, yaklaşık değeri 3,14159265
O, o	: asimptotik terimleri gösteren simgeler
$W(u, v)(x)$	: Wronskian determinanı
$G(x, y, \lambda)$	: Green fonksiyonu

1.GENEL BİLGİLER

1.1.Giriş

İkinci mertebeden lineer bir diferansiyel denklem genel olarak aşağıdaki şekilde ifade edilir.

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = F(x), \quad x_1 < x < x_2 \quad (A_2(x) \neq 0) \quad (1.1)$$

Birçok uygulamada, (1.1) denkleminin bazı yardımcı koşulları sağlaması istenir. Bunların sayısı denklemin mertebesi kadardır. Eğer verilen koşullar tek noktada tanımlı ise bu koşullara başlangıç koşulları, iki noktada tanımlı ise bu koşullara sınır koşulları denir. Sınır koşulları genel olarak

$$\left. \begin{aligned} a_{11}y(x_1) + a_{12}y'(x_1) + b_{11}y(x_2) + b_{12}y'(x_2) &= \alpha \\ a_{21}y(x_2) + a_{22}y'(x_2) + b_{21}y(x_1) + b_{22}y'(x_1) &= \beta \end{aligned} \right\} \quad (1.2)$$

şeklinde ifade edilir. Burada $a_{ij}, b_{ij}, \alpha, \beta$ sabitlerdir. (1.1) denkleminin (1.2) koşullarını sağlayan çözümünü bulma problemine sınır değer problemi denir. Uygulamada kullanılan bazı sınır koşulları (1.2)'nin bir özel durumu olan ayrılmış sınır koşullarıdır. Yani birinci denklem sadece x_1 'e ve ikinci denklem x_2 'ye bağlıdır. Yani

$$\left. \begin{aligned} a_{11}y(x_1) + a_{12}y'(x_1) &= \alpha, & a_{11}^2 + a_{12}^2 &\neq 0 \\ a_{21}y(x_2) + a_{22}y'(x_2) &= \beta, & a_{21}^2 + a_{22}^2 &\neq 0 \end{aligned} \right\} \quad (1.3)$$

(1.2) koşulları bazen ayrılmamış sınır koşulları olarak da adlandırılır. Bunlar uygulamada ayrılmış sınır koşulları kadar yaygın değildir.

Ayrıca $F(x) = 0$ ise (1.1) denkleminde homojendir denir. Eğer $\alpha = \beta = 0$ ise sınır koşullarına da homojen sınır koşulları adı verilir.

Sınır değer problemlerinin özel bir sınıfa ait olan özdeğer problemleri uygulamada özellikle inşaat, uçak, gemi tasarımlarında da kullanılmaktadır.

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = F(x), \quad x_1 < x < x_2 \quad (1.4)$$

$$\left. \begin{aligned} {}_{11}y(x_1) + {}_{12}y'(x_1) &= \alpha, & a_{11}^2 + \frac{2}{12} &\neq 0 \\ {}_{21}y(x_2) + {}_{22}y'(x_2) &= \beta, & a_{21}^2 + \frac{2}{22} &\neq 0 \end{aligned} \right\} \quad (1.5)$$

problemini ele alalım.

$$M := A_2(x)D^2 + A_1(x)D + A_0(x) \quad (1.6)$$

olarak tanımlanırsa (1.4) denklemi

$$M[y] = F(x)$$

olarak yazılabilir. Benzer şekilde sınır operatörleri B_1 ve B_2 aşağıdaki şekilde tanımlanırsa

$$B_1[y] = {}_{11}y(x_1) + {}_{12}y'(x_1) \quad (1.7)$$

$$B_2[y] = {}_{21}y(x_2) + {}_{22}y'(x_2) \quad (1.8)$$

bu durumda (1.5) eşitlikleri

$$B_1[y] = \alpha, \quad B_2[y] = \beta \quad (1.9)$$

olarak kısaca kompakt bir formda ifade edilmiş olur.

$$M[c_1f + c_2g] = c_1M[f] + c_2M[g]$$

(c_1, c_2 sabitler) sağlanırsa operatöre lineerdir denir.

Daha genel olarak

$$L[y] + \lambda r(x)y = 0, \quad x_1 < x < x_2$$

$$\left. \begin{aligned} B_1[y] &= {}_{11}y(x_1) + {}_{12}y'(x_1) = \alpha, & a_{11}^2 + \frac{2}{12} &\neq 0 \\ B_2[y] &= {}_{21}y(x_2) + {}_{22}y'(x_2) = \beta, & a_{21}^2 + \frac{2}{22} &\neq 0 \end{aligned} \right\}$$

$$L = D[P(x)D] + q(x)$$

sınıfına ait problemler özdeğer problemleri olarak adlandırılır. Özel olarak

$$y(x_1) = y(x_2) = 0$$

durumu Dirichlet sınır koşulları,

$$y'(x_1) = y'(x_2) = 0$$

Neumann sınır koşulları olarak bilinmektedir.

Bilindiği üzere homojen olmayan başlangıç değer ve sınır değer problemleri için Green fonksiyonları yardımıyla özel çözümler bulunabilir. Sınır değer problemleri için Green fonksiyonları başlangıç değer problemlerindekine benzer olarak tanımlanmasına rağmen sınır değer problemleri için Green fonksiyonlarının her zaman mevcut olmadığı bilinmektedir. Oysa başlangıç değer problemleri için Green fonksiyonları daima bulunabilir.

1.2. Sınır Değer Problemleri İçin Green Fonksiyonları

Öncelikle

$$M[y] = F(x), \quad x_1 < x < x_2 \quad (1.10)$$

$$B_1[y] = a_{11}y(x_1) + a_{12}y'(x_1) = \alpha$$

$$B_2[y] = a_{21}y(x_2) + a_{22}y'(x_2) = \beta$$

sınır değer problemini ele alalım. Burada M operatörü

$M = A_2(x)D^2 + A_1(x)D + A_0(x)$, $D = \frac{d}{dx}$ şeklindedir. (1.10) problemi için Green fonksiyonlarını elde etmeden önce (1.10) problemi kendine eş forma konulur. Hatırlanacağı üzere $M[y] = F(x)$ kendine eş formda değilse ($A_2'(x) \neq A_1(x)$)

$$p(x) = e^{\int^{A_1(x)} / A_2(x) dx}$$

olmak üzere,

$$\mu(x) = p(x) / A_2(x)$$

ile denklemin her iki tarafı çarpılarak denklem kendine eş forma konur. Bu durumda (1.10) denkleminin operatörü L ile gösterilir ve L kendine eştir denilir. Bu durumda ilgili problem

$$L[y] = f(x), \quad x_1 < x < x_2, \quad B_1[y] = \alpha, \quad B_2[y] = \beta \quad (1.11)$$

$$f(x) = p(x)F(x) / A_2(x) ,$$

$$q(x) = p(x)A_0(x) / A_2(x)$$

şeklinde düzenlenirse

$$L = D[P(x)D] + q(x)$$

olarak ifade edilir. Hatırlanacağı üzere (1.11) denkleminin genel çözümü

$$y = y_H + y_p$$

şeklindedir. Burada y_H

$$L[y] = 0 \quad , \quad B_1[y] = \alpha, \quad B_2[y] = \beta \quad (1.12)$$

ve y_p ise

$$L[y] = f(x) \quad , \quad B_1[y] = 0, \quad B_2[y] = 0 \quad (1.13)$$

denklemin bir özel çözümüdür. (1.12)' nin genel çözümü

$$y_H = c_1 y_1(x) + c_2 y_2(x)$$

şeklindedir. Burada y_1 ve y_2 homojen denklemin lineer bağımsız çözümleri, c_1 ve c_2 homojen olmayan sınır koşullarını sağlayacak şekilde belirlenen sabitlerdir. Şimdi (1.13) problemini ele alalım. Başlangıç değer problemlerindekine benzer olarak

$$y_p = - \int_{x_1}^{x_2} g(x,s) f(s) ds \quad (1.14)$$

şeklinde bir çözümün varlığı kabul edilir. Burada $g(x,s)$ belirlenecek Green fonksiyonunu göstermektedir. ((-) işareti ise $g(x,s)$ nin özel bir fiziksel yoruma sahip olması içindir.) Hesaplamalarda Dirac delta fonksiyonundan yararlanacağız.

1.3. Dirac Delta Fonksiyonu

$g(t)$: $(-\infty, +\infty)$ üzerinde sürekli olmak üzere

$$t \neq a \text{ için } \delta(t - a) = 0,$$

$$\int_{-\infty}^{+\infty} \delta(t - a) dt = 1,$$

$$\int_{-\infty}^{+\infty} g(t) \delta(t - a) dt = g(a),$$

$$\int_{-\infty}^{+\infty} \delta(u - a) du = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases} = H(t - a)$$

özelliklerini sağlar.

(1.14)'ün her iki tarafına L operatörü uygulanırsa ve L operatörünün integral ile yer değiştirebileceği varsayılırsa

$$L[y_p] = L \left[- \int_{x_1}^{x_2} g(x, s) f(s) ds \right] = - \int_{x_1}^{x_2} L[g] f(s) ds = f(x)$$

dir. Son eşitliğin sağ tarafının $f(x)$ 'e eşit olabilmesi için

$$L[g] = -\delta(x - s) \quad , \quad x_1 < x < x_2 \quad (1.15)$$

sağlanmalıdır. (Burada $\delta(x - s)$ dirac delta fonksiyonunun

$\int_{-\infty}^{+\infty} \delta(t - a) f(t) dt = f(a)$ özelliği kullanılmıştır.) $g(x, s)$ ' yi belirlemek için ise (1.15)'den başka bazı koşullar gereklidir. Bunun için (1.13)'teki sınır koşulları (1.14)'e uygulanırsa

$$B_1[y_p] = - \int_{x_1}^{x_2} B_1[g] f(s) ds = 0,$$

$$B_2[y_p] = - \int_{x_1}^{x_2} B_2[g] f(s) ds = 0$$

bulunur. f fonksiyonu herhangi bir fonksiyon olabileceğinden bu son eşitliklerin sağlanabilmesi için

$$B_1[g] = 0, \quad B_2[g] = 0 \quad (1.16)$$

sağlanmalıdır. Sonuç olarak aradığımız Green fonksiyonu

$$L[g] = -\delta(x - s), \quad x_1 < x < x_2, \quad B_1[g] = 0, \quad B_2[g] = 0 \quad (1.17)$$

probleminin çözümüdür. Bu problem (1.13)'e benzerdir. Burada $x_1 < s < x_2$ ve s sabittir. (1.17) problemi (1.13) problemine benzemesine rağmen (1.17)'deki kuvvet fonksiyonu dirac delta fonksiyonudur. Bunun anlamı ise (1.17)'deki g yi bulma problemi (1.13)'deki (y) 'yi bulma probleminden daha basittir. Böylece L operatörü ve özel sınır koşullarıyla ilgili Green fonksiyonu bulunduktan sonra bu Green fonksiyonu (1.13)'ü çözmek için kullanılır.

(1.17)'deki delta fonksiyonu, $g(x, s)$ 'nin $x = s$ civarında detaylı bir biçimde incelenmesi gerekliliğini ortaya koyar. Bunun için de (1.14) göz önüne alınırsa

$$y_p(s^+) - y_p(s^-) = - \int_{x_1}^{x_2} [g(s^+, s) - g(s^-, s)] f(s) ds$$

(Burada $y(s^+) = \lim_{\varepsilon \rightarrow 0^+} y(s + \varepsilon)$, $y(s^-) = \lim_{\varepsilon \rightarrow 0^-} y(s - \varepsilon)$)

bulunur. Fakat bir diferansiyel denklemin çözümü sürekli bir fonksiyon olacağı için son eşitliğin sol tarafı sıfır olur ve f keyfi olduğundan

$$g(s^+, s) = g(s^-, s) \quad (1.18)$$

bulunur. Bu ise g nin $x = s$ de sürekli olması gerektiğini belirtir. Bir adım daha öteye gidilirse, $g(x, s)$ nin türevinin incelenmesi gerekir. Bunun için $L[g] = -\delta(x - s)$, $[s^-, s^+]$ da x 'e göre integre edilirse

$$\left(L = D[P(x)D] + q, \quad L[g] = \int_{s^-}^{s^+} D[P(x)D](g) + \int_{s^-}^{s^+} qg \right) = - \int_{s^-}^{s^+} \delta(x - s) dx$$

$$\left[P(x) \frac{\partial g}{\partial x} \right] \Big|_{x=s^-}^{s^+} + \int_{s^-}^{s^+} q(x)g(x, s) dx = \int_{s^-}^{s^+} \delta(x - s) dx.$$

$q(x)$ ve $g(x, s)$ nin $x = s$ 'de sürekliliğinden sol taraftaki integralin

$$\int_{s^-}^{s^+} q(x)g(x, s)dx = 0$$

olduğu gözlenir. Ayrıca δ fonksiyonları için integral özellikleri kullanılarak ve $p(x)$ 'in sürekli ve $[x_1, x_2]$ üzerinde sıfırdan farklılığı da dikkate alınırsa son eşitlik

$$\left(- \int_{s^-}^{s^+} \delta(x - s)dx \right) = 1,$$

$$\left. \frac{\partial g}{\partial x} \right|_{x=s^-}^{s^+} = -\frac{1}{p(s)}. \quad (1.19)$$

Bu sonuç ise $x = s$ de $g(x, s)$ nin büyüklüğü $\frac{1}{p(s)}$ olan bir sıçrama süreksizliğine sahip olduğunu gösterir. Böylece Green fonksiyonu şöyle tanımlanır.

Tanım 1.1:

$$L[y] = f(x), \quad x_1 < x < x_2, \quad B_1[y] = \alpha, \quad B_2[y] = \beta$$

sınır değer problemi için $g(x, s)$ Green fonksiyonu aşağıdaki özellikleri sahiptir.

$$(L = D[P(x)D] + q(x)) \quad (x_1 < x < x_2)$$

$$a) L[g] = -\delta(x - s), \quad x_1 < x < x_2 \quad (s \text{ sabit})$$

$$b) B_1[g] = 0, \quad B_2[g] = 0$$

$$c) g(s^+, s) = g(s^-, s)$$

$$d) \left. \frac{\partial g}{\partial x} \right|_{x=s^-}^{s^+} = -\frac{1}{p(s)}.$$

Bu tanıma bağlı olarak Green fonksiyonu aşağıdaki şekilde elde edilir. Öncelikle (a)'dan ya $x < s$ veya $x > s$ olduğu görülür. O zaman δ fonksiyonu tanımından $L[g] = 0$ elde edilir. Çünkü her ikisinde $x \neq s$ dir. Sonra, eğer z_1 ve z_2 $L[g] = 0$ homojen denkleminin $B_1[z_1] = 0, B_2[z_2] = 0$ 'ı sağlarsa ve çözümleri ise bu durumda a) ve b) den

$$g(x, s) = \begin{cases} u(s)z_1(x), & x < s \\ v(s)z_2(x), & x > s \end{cases} \quad (1.20)$$

bulunur. Burada u, v belirlenecek fonksiyonlardır. Tanımdaki (c) ve (d) kullanılırsa u ve v fonksiyonları

$$\begin{aligned} v(s)z_2(s) - u(s)z_1(s) &= 0 \\ v(s)z_2'(s) - u(s)z_1'(s) &= -\frac{1}{p(s)} \end{aligned} \quad (1.21)$$

ve (1.21)'i aynı anda sağlayan $u(s)$ ve $v(s)$

$$\begin{aligned} u(s) &= -\frac{z_2(s)}{p(s)W(z_1, z_2)(s)} \\ v(s) &= -\frac{z_1(s)}{p(s)W(z_1, z_2)(s)} \end{aligned} \quad (1.22)$$

bulunur. Burada

$$W(z_1, z_2) = z_1 z_2' - z_1' z_2$$

Wronskian fonksiyonudur.

Bu arada şu önemli noktayı vurgulamak istiyoruz. Eğer y_1 ve y_2 aynı diferansiyel denklemin çözümleri iseler,

$$P(x)W(z_1, z_2)(s) = p(x)W(z_1, z_2)(x) = c$$

ve bu durumda (1.21) deki Green fonksiyonu

$$g(x, s) = \begin{cases} -z_1(x)z_2(s)/c, & x_1 < x < s \\ -z_1(s)z_2(x)/c, & s \leq x < x_2 \end{cases} \quad (x \text{ de\u0131işken ve } s \text{ sabit})$$

veya denk olarak

$$g(x, s) = \begin{cases} -z_1(s)z_2(x)/c, & x_1 < s \leq x \\ -z_1(x)z_2(s)/c, & x_1 < s < x_2 \end{cases} \quad (1.23) \quad (x \text{ sabit ve } s \text{ de\u0131işken})$$

elde edilir. c bir sabit olduğundan (1.23)'den Green fonksiyonunun x ve s 'ye göre simetrik olduğu gözlenir. Yani $g(x, s) = g(s, x)$ 'dir. Bu özellik, Green fonksiyonunun diğer yöntemlerle elde edişinde önemli rol oynar.

1.4. Çözümlü Alıştırımlar

Örnek 1.1:

$$y'' + k^2 y = f(x), \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0, \quad k \neq 0$$

sınır değer problemi için Green fonksiyonunu elde ediniz.

Çözüm:

İlgili homojen denklemin lineer bağımsız çözümleri $y_1 = \cos kx$, $y_2 = \sin kx$. z_1 ve z_2 çözümünü bulmak için önce z_1 ve z_2 nin lineer kombinasyonu olduğu kabul edilir.

$$z_1 = c_{11} \cos kx + c_{12} \sin kx$$

şeklinde kabul edilir. $z_1(0) = 0$ dan $c_{11} = 0$ ve $c_{12} = 1$ alınırsa

$$z_1 = \sin kx,$$

benzer şekilde

$$z_2 = c_{21} \cos kx + c_{22} \sin kx \text{ ve } z_2(1) = 0 \text{ dan}$$

$$c_{21} \cos k + c_{22} \sin k = 0.$$

c_{11} ve c_{22} 'yi öyle seçmeliyiz ki son eşitlik sağlanmalı ve aynı anda c_{21} ve c_{22} ikisi birden sıfır olmamalıdır. Diğer seçimler arasında $c_{21} = \sin k$ ve $c_{22} = -\cos k$,

$$z_2 = \sin k \cos kx - \cos k \sin kx = \sin k(1 - x)$$

bulunur. Son olarak c sabitini belirlemeliyiz.

Öncelikle $L = D^2 + k^2$ kendine eş bir formda ve $P(x) = 1$ olduğunu gözleriz. Buradan

$$c = P(x)W(z_1, z_2)(x) = \begin{vmatrix} \sin kx & \sin k(1-x) \\ k \cos kx & -k \cos k(1-x) \end{vmatrix}$$

$$\text{ve } c = -k \sin k$$

bulunur. Green fonksiyonu c ile bölüneceğinden $k \neq n\pi$ ($n = 1, 2, 3, \dots$) olmalıdır. Bu durumda

$$g(x, s) = \begin{cases} \frac{\sin ks \times \sin k(1-x)}{ks \sin k}, & 0 < s \leq x \\ \frac{\sin kx \times \sin k(1-s)}{ks \sin k}, & x < s < 1 \end{cases}$$

Eğer $k = n\pi$ ($n = 1, 2, 3, \dots$) izin verilirse $c = 0$ bu durumda Green fonksiyonu elde edilemez. Bu durumda

$$z_1 = z_2 = \sin n\pi x$$

ki bunun anlamı

$$y'' + n^2\pi^2 y = 0, \quad y(0) = 0, \quad y(1) = 0$$

probleminin sıfırdan farklı çözümü olduğu anlaşılır.

Bu örnekten anlaşılacağı üzere sınır değer problemi için Green fonksiyonu her zaman mevcut değildir. Yani $c = 0$ ise $g(x, s)$ elde edilemez. Bu noktayı vurgulamak için aşağıdaki daha genel problemi ele alalım.

$$\left. \begin{aligned} L[y] &= f(x) & x_1 < x < x_2, \\ B_1[y] &= \alpha, & B_2[y] &= \beta \end{aligned} \right\} \quad (1.24)$$

Bilindiği üzere (1.24)'ün çözümü için aşağıdaki iki problem oluşturulur:

$$i) \quad L[y] = 0, \quad B_1[y] = \alpha, \quad B_2[y] = \beta,$$

$$ii) \quad L[y] = f(x), \quad B_1[y] = 0, \quad B_2[y] = 0.$$

(1.24)'ün çözümü (i) ve (ii) den elde edilen çözümlerin toplamıdır.

ii) problemini çözmek için Green fonksiyonu araştırılır. Ancak Green fonksiyonun mevcut olabilmesi için

$$L[y] = 0, \quad B_1[y] = 0, \quad B_2[y] = 0$$

sınır değer problemi sadece trivial çözüme sahip olmalıdır. Sıfırdan farklı çözüme sahipse z_1 ve z_2 lineer bağımlıdır ve böylece $W(z_1, z_2) = 0$ veya $c = 0$ dır. Bu durumda Green fonksiyonları elde edilemez. Eğer (2) sadece trivial çözüme sahipse z_1 ve z_2 lineer bağımsız olacağından $c \neq 0$ ve Green fonksiyonu tek türlü bellidir.

Örnek 1.2:

$$y'' + y = \sin x, \quad 0 < x < \frac{\pi}{2},$$

$$y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1$$

problemini Green fonksiyonunu kullanarak çözünüz.

Çözüm:

$$i) \quad y'' + y = 0 \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1$$

$$ii) \quad y'' + y = \sin x, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0$$

Birinci problem $y_H = c_1 \cos x + c_2 \sin x$

$$y(0) = 0 \Rightarrow c_1 = 1$$

$$y\left(\frac{\pi}{2}\right) = -1 \Rightarrow c_2 = -1$$

$$y = \cos x - \sin x.$$

Homojen denklemin lineer bağımsız çözümleri (homojen sınır koşullarını sağlayan)

$$z_1 = \sin x, \quad z_2 = \cos x.$$

alınabilir. $P(x) = 1$ olduğundan

$$P(x)W(z_1, z_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

$$c = -1$$

$$g(x, s) = \begin{cases} \sin s \cos x, & 0 < s \leq x \\ \sin x \cos s, & x < s < \pi/2 \end{cases}.$$

Yukarıdaki ikinci problem kendine eş formdadır. Böylece $f(x) = \sin x$ alınırsa

$$y_p = - \int_0^{\pi/2} g(x, s) \sin s ds = -\cos x \int_0^x \sin^2 s ds - \sin x \int_x^{\pi/2} \cos s \sin s ds$$

$$p = -\frac{1}{2}x \cos x$$

$$= h + p = \left(1 - \frac{x}{2}\right) \cos x - \sin x.$$

Örnek 1.3:

Green fonksiyon yöntemini kullanarak

$$2x^2 y'' + 3x y' - y = 4x^{3/2}, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0$$

problemini çözünüz.

Çözüm:

$$h = 2x^2 y'' + 3x y' - y = 0, \quad y(0) = 0, \quad y(1) = 0$$

probleminin çözümüdür. Bu bir Cauchy Euler denklemdir. Bu denklemin genel çözümü

$$h = c_1 x^{1/2} + c_2 x^{-1}.$$

$y(0) = 0$ sağlanması için $c_2 = 0$ olmalı ve c_1 keyfi olmalıdır.

İkinci sınır koşulundan ise $y(1) = c_1 = 0$ 'dır. Böylece

$$h = 0.$$

Bu ise orijinal problemin bir tek çözümü olabilmesi için gereklidir. y_p 'yi çözmek için önce denklem kendine eş forma konur, $P(x)$ ve $f(x)$ bulunur.

$$P(x) = e^{\int \frac{3x}{2x^2} dx} = x^{3/2}$$

Böylece diferansiyel denklem $2x^2$ ile bölünür ve $P(x) = x^{3/2}$ ile çarpılırsa kendine eş denklem

$$\frac{d}{dx}(x^{3/2} y') - \frac{1}{2} x^{-1/2} y = 2x \quad y(0) = 0, \quad y(1) = 0.$$

elde edilir. Böylece $x^{1/2}$ ve x^{-1} Green fonksiyonlarını oluşturmak için kullanılır. $x^{1/2}$ ve x^{-1} in lineer kombinasyonları oluşturulursa

$$z_1 = c_{11} x^{1/2} + c_{12} x^{-1}$$

$$z_2 = c_{21}x^{1/2} + c_{22}x^{-1}$$

sınır koşulları kullanılırsa

$z_1(0) = 0$ 'dan $c_{12} = 0$, c_{11} keyfidir. Diyelim ki $c_{11} = 1$ olsun ve böylece $z_1 = x^{1/2}$ olur.

$z_2(1) = 0$ 'dan $c_{21} + c_{22} = 0$. $c_{21} = -c_{22} = 1$ ve $z_2 = x^{1/2} - x^{-1}$ olur.

Kolayca gözleneceği üzere

$$c = P(x)W(z_1, z_2)(x) = x^{3/2} \left(\frac{3}{2} x^{-3/2} \right) = \frac{3}{2}$$

ve

$$g(x, s) = \begin{cases} -\frac{2}{3} s^{1/2} (x^{1/2} - x^{-1}), & 0 < s \leq x \\ -\frac{2}{3} x^{1/2} (s^{1/2} - s^{-1}), & x < s < 1 \end{cases}$$

bulunur. Böylece aranan çözüm $y = y_p$, ($y_H = 0$)

$$\begin{aligned} y &= - \int_0^1 g(x, s)(2s)ds \\ &= \frac{2}{3} (x^{1/2} - x^{-1}) \int_0^x s^{1/2} (2s)ds + \frac{2}{3} x^{1/2} \int_x^1 (s^{1/2} - s^{-1})(2s)ds \end{aligned}$$

ve gerekli işlemler yapılırsa

$$y = \frac{4}{5} x^{1/2} (x - 1)$$

elde edilir.

2. YAPILAN ÇALIŞMALAR

Bu bölümde

$$y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$\frac{y'(0, \lambda)}{y(0, \lambda)} = -\tan \alpha \quad (2.2)$$

$$\cos \beta y'(\pi, \lambda) + \sin \beta y(\pi, \lambda) = 0 \quad (2.3)$$

$\alpha, \beta \in [0, \pi]$ sınır koşullarını sağlayan, $q \in L^1[0, \pi]$ ve $\int_0^\pi q(t)dt = 0$ (ortalama değeri sıfır) olan Green fonksiyonu genel olarak

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq y \leq x \leq \pi \end{cases}$$

şeklinde hesaplanacaktır. Bu amaçla ilk olarak φ, ψ ve $w(\lambda)$ için asimptotik çözümler elde edilecektir. Uygulanan yöntem [1] ve [2] çalışmasındaki yöntemlere benzerdir.

Bu tür çalışmalardan bazıları sınır değerlerinde özdeğer parametresi içermektedir. Özdeğer parametresi tek sınır koşulunda olabileceği gibi her iki sınır koşulunda da bulunabilir. [1] çalışmasında λ –bağımlı sol sınır koşulu quadratik, sağ sınır koşulu ise sadece bağımlı ve türevini içermektedir. [2] çalışmasında ise λ parametresi her iki sınır koşullarında da bulunmaktadır. [3] çalışmasında da λ parametresi her iki sınır koşulunda lineer olarak yer almaktadır.

Öncelikle (2.1) denklemini uygun dönüşümle

$$v' = -\lambda + q - v^2 \quad (2.4)$$

dönüştürülür [4].

$S(x, \lambda) := \operatorname{Re} v(x, \lambda)$ ve $T(x, \lambda) := \operatorname{Im} v(x, \lambda)$ olmak üzere

$$F(x, \lambda) := \begin{cases} \left| \int_x^\pi e^{2i\lambda(1/2)t} q(t) dt \right| / \int_x^\pi |q(t) dt|, & \int_x^\pi |q(t)| dt \neq 0 \\ 0, & \int_x^\pi |q(t)| dt = 0 \end{cases}$$

olarak tanımlanırsa

$$\eta(\lambda) := \sup_{0 \leq x \leq \pi} F(x, \lambda),$$

$$\eta(\lambda) \rightarrow 0, \lambda \rightarrow \infty \quad [4].$$

(2.4) için yaklaşık çözümler aşağıdaki gibidir [4] :

$$v(x, \lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(x, \lambda)$$

$$v_1' + 2i\lambda^{1/2}v_1 = q,$$

$$v_2' + 2i\lambda^{1/2}v_2 = -v_1^2,$$

ve $n=3,4,\dots$ için

$$v_n' + 2i\lambda^{1/2}v_n = -\left(v_{n-1}^2 + 2v_{n-1} \sum_{m=1}^{n-2} v_m\right) \quad (2.5).$$

Teorem 2.1:

$$y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$y(0, \lambda) = \cos\alpha, \quad y'(0, \lambda) = -\sin\alpha, \quad \alpha \in [0, \pi] \quad (2.6)$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned} \varphi(x, \lambda) &= \cos\alpha \times \exp\left(\int_0^x S(t, \lambda) dt\right) \\ &\times \left[\cos\left(\int_0^x T(t, \lambda) dt\right) - \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right] \times \sin\left(\int_0^x T(t, \lambda) dt\right) \right] \end{aligned} \quad (2.7)$$

ile verilir. Burada $S(x, \lambda), T(x, \lambda)$ önceden tanımlandığı gibidir.

İspat:

(2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x, \lambda) = c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \quad (2.8)$$

ve

$$\begin{aligned} Z'(x, \lambda) &= c_1 S(x, \lambda) \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^x T(t, \lambda) dt\right) T(x, \lambda) \end{aligned} \quad (2.9)$$

şeklindedir [4]. (2.6) ile verilen başlangıç koşulları sağlanacak şekilde c_1 ve c_2 sabitlerini belirleyelim. (2.6) ile verilen başlangıç koşullarından $Z(0, \lambda) = \cos\alpha$ yerine yazılırsa

$$\begin{aligned} \cos\alpha &= c_1 \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^0 T(t, \lambda) dt\right), \\ \cos\alpha &= c_1 \cos c_2, \quad c_1 = \frac{\cos\alpha}{\cos c_2} \end{aligned} \quad (2.10)$$

$Z'(0, \lambda) = -\sin\alpha$ kullanılırsa

$$\begin{aligned} -\sin\alpha &= c_1 S(0, \lambda) \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^0 T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^0 T(t, \lambda) dt\right) T(0, \lambda), \\ -\sin\alpha &= c_1 S(0, \lambda) \cos c_2 - c_1 \sin c_2 T(0, \lambda) \end{aligned} \quad (2.11)$$

bulunur. (2.10) eşitliğinden bulunan c_1 değeri (2.11)'de yerine yazılırsa;

$$\begin{aligned} -\sin\alpha &= \frac{\cos\alpha}{\cos c_2} S(0, \lambda) \cos c_2 - \frac{\cos\alpha}{\cos c_2} \sin c_2 T(0, \lambda), \\ -\sin\alpha &= \cos\alpha \times S(0, \lambda) - \cos\alpha \times \tan c_2 T(0, \lambda), \end{aligned}$$

$$-\sin\alpha = \cos\alpha[S(0, \lambda) - \tan c_2 T(0, \lambda)]$$

$$\tan c_2 = \frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \Rightarrow c_2 = \tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] \quad (2.12)$$

(2.10) eşitliği kullanılırsa;

$$c_1 = \frac{\cos\alpha}{\cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] \right)} \quad (2.13)$$

bulunur. Son olarak (2.12) ve (2.13) (2.7)'de yerine yazılırsa (2.6) ile verilen başlangıç koşullarını sağlayan çözüme $\varphi(x, \lambda)$ denilirse

$$\begin{aligned} \varphi(x, \lambda) &= \frac{\cos\alpha}{\cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] \right)} \times \exp \left(\int_0^x S(t, \lambda) dt \right) \\ &\times \cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] + \int_0^x T(t, \lambda) dt \right) \end{aligned} \quad (2.14)$$

bulunur.

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ kullanılırsa

$$\begin{aligned} &\cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] + \int_0^x T(t, \lambda) dt \right) \\ &= \cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] \right) \times \cos \left(\int_0^x T(t, \lambda) dt \right) \\ &\quad - \sin \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] \right) \times \sin \left(\int_0^x T(t, \lambda) dt \right) \end{aligned}$$

(2.14)'de yerine yazılırsa ve gerekli işlemler yapılırsa

$$\varphi(x, \lambda) = \cos\alpha \times \exp \left(\int_0^x S(t, \lambda) dt \right)$$

$$\times \left[\frac{\cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan \alpha}{T(0, \lambda)} \right] \right) \times \cos \left(\int_0^x T(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan \alpha}{T(0, \lambda)} \right] \right)} - \frac{\sin \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan \alpha}{T(0, \lambda)} \right] \right) \times \sin \left(\int_0^x T(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(0, \lambda) + \tan \alpha}{T(0, \lambda)} \right] \right)} \right]$$

düzenlenirse;

$$\begin{aligned} \varphi(x, \lambda) &= \cos \alpha \times \exp \left(\int_0^x S(t, \lambda) dt \right) \\ &\times \left[\cos \left(\int_0^x T(t, \lambda) dt \right) - \left[\frac{S(0, \lambda) + \tan \alpha}{T(0, \lambda)} \right] \times \sin \left(\int_0^x T(t, \lambda) dt \right) \right] \end{aligned} \quad (2.7)$$

elde edilir.

Aşağıdaki notasyonları kullanırsak

$$E_0(x, \lambda) := \exp \left(\int_0^x S(t, \lambda) dt \right),$$

$$C_0(x, \lambda) := \cos \left(\int_0^x T(t, \lambda) dt \right),$$

$$S_0(x, \lambda) := \sin \left(\int_0^x T(t, \lambda) dt \right)$$

$\varphi(x, \lambda)$ eşitliğimiz ;

$$\varphi(x, \lambda) = \cos \alpha E_0(x, \lambda) \left[C_0(x, \lambda) - \left[\frac{S(0, \lambda) + \tan \alpha}{T(0, \lambda)} \right] S_0(x, \lambda) \right],$$

$$\varphi(x, \lambda) = \cos \alpha E_0(x, \lambda) C_0(x, \lambda) - \cos \alpha E_0(x, \lambda) S_0(x, \lambda) \frac{S(0, \lambda)}{T(0, \lambda)}$$

$$- \sin \alpha E_0(x, \lambda) S_0(x, \lambda) \frac{1}{T(0, \lambda)} \quad (2.15)$$

şeklinde elde edilir. Son eşitlikteki $E_0(x, \lambda)$, $C_0(x, \lambda)$, $S_0(x, \lambda)$, $S(0, \lambda)$ ve $T(0, \lambda)$ değerleri daha sonra hesaplanacaktır.

Teorem 2.2:

$$y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$y(\pi, \lambda) = \cos\beta, \quad y'(\pi, \lambda) = -\sin\beta, \quad \beta \in [0, \pi] \quad (2.16)$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned} \psi(x, \lambda) = & \cos\beta \exp\left(-\int_x^\pi S(t, \lambda)dt\right) \\ & \times \left[\cos\left(\int_x^\pi T(t, \lambda)dt\right) + \left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right] \times \sin\left(\int_x^\pi T(t, \lambda)dt\right) \right] \end{aligned} \quad (2.17)$$

ile verilir. Burada $S(x, \lambda), T(x, \lambda)$ önceden tanımlandığı gibidir.

İspat:

(2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x, \lambda) = c_1 \exp\left(\int_0^x S(t, \lambda)dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda)dt\right) \quad (2.8)$$

ve

$$\begin{aligned} Z'(x, \lambda) = & c_1 S(x, \lambda) \exp\left(\int_0^x S(t, \lambda)dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda)dt\right) \\ & - c_1 \exp\left(\int_0^x S(t, \lambda)dt\right) \times \sin\left(c_2 + \int_0^x T(t, \lambda)dt\right) T(x, \lambda) \end{aligned} \quad (2.9)$$

şeklindedir [4]. (2.16) ile verilen başlangıç koşulları sağlanacak şekilde c_1 ve c_2 sabitlerini belirleyelim. (2.16) ile verilen başlangıç koşullarından

$Z(\pi, \lambda) = \cos\beta$ ve $Z'(\pi, \lambda) = -\sin\beta$ yazılırsa

$$\cos\beta = c_1 \exp\left(\int_0^\pi S(t, \lambda)dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda)dt\right), \quad (2.18)$$

$$\begin{aligned}
-\sin\beta &= c_1 S(\pi, \lambda) \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \\
&\quad - c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) T(\pi, \lambda)
\end{aligned} \tag{2.19}$$

(2.18) eşitliği (2.19) eşitliğinde kullanılırsa

$$\begin{aligned}
-\sin\beta &= \frac{\cos\beta S(\pi, \lambda) \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)} \\
&\quad - \frac{\cos\beta \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) T(\pi, \lambda)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)}, \\
-\sin\beta &= \cos\beta S(\pi, \lambda) - \cos\beta T(\pi, \lambda) \tan\left(c_2 + \int_0^\pi T(t, \lambda) dt\right), \\
\tan\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) &= \frac{\cos\beta S(\pi, \lambda) + \sin\beta}{\cos\beta T(\pi, \lambda)}, \\
c_2 + \int_0^\pi T(t, \lambda) dt &= \tan^{-1} \left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)} \right], \\
c_2 &= \tan^{-1} \left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt
\end{aligned} \tag{2.20}$$

bulunur. (2.20) eşitliği (2.18)'de yerine yazılırsa

$$\begin{aligned}
\cos\beta &= c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \\
&\quad \times \cos\left(\tan^{-1} \left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^\pi T(t, \lambda) dt\right), \\
c_1 &= \frac{\cos\beta}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) \cos\left(\tan^{-1} \left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)} \right]\right)} \tag{2.21}
\end{aligned}$$

elde edilir. Son olarak (2.20) ve (2.21) eşitlikleri (2.17)'de yerine yazılırsa (2.16) ile verilen başlangıç koşullarını sağlayan çözüme $\psi(x, \lambda)$ denilirse

$$\psi(x, \lambda) = \frac{\cos\beta \exp\left(\int_0^x S(t, \lambda) dt\right)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) \cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right)} \\ \times \cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right] - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt\right)$$

sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\psi(x, \lambda) = \frac{\cos\beta \exp\left(-\int_x^\pi S(t, \lambda) dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right)} \\ \times \cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right] - \int_x^\pi T(t, \lambda) dt\right) \quad (2.22)$$

bulunur.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ kullanılırsa

$$\cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right] - \int_x^\pi T(t, \lambda) dt\right) \\ = \cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right) \times \cos\left(\int_x^\pi T(t, \lambda) dt\right) \\ + \sin\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right) \times \sin\left(\int_x^\pi T(t, \lambda) dt\right)$$

(2.22)'de yerine yazılırsa ve gerekli işlemler yapılırsa

$$\psi(x, \lambda) = \cos\beta \times \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \\ \times \left[\frac{\cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right) \times \cos\left(\int_x^\pi T(t, \lambda) dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right)} \right. \\ \left. + \frac{\sin\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right) \times \sin\left(\int_x^\pi T(t, \lambda) dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right]\right)} \right],$$

$$\begin{aligned}
&= \cos\beta \times \exp\left(-\int_x^\pi S(t, \lambda)dt\right) \\
&\times \left[\cos\left(\int_x^\pi T(t, \lambda)dt\right) + \left[\frac{S(\pi, \lambda) + \tan\beta}{T(\pi, \lambda)}\right] \sin\left(\int_x^\pi T(t, \lambda)dt\right)\right] \quad (2.17)
\end{aligned}$$

elde edilir.

Aşağıdaki notasyonları kullanırsak

$$E_\pi(x, \lambda) := \exp\left(-\int_x^\pi S(t, \lambda)dt\right),$$

$$C_\pi(x, \lambda) := \cos\left(\int_x^\pi T(t, \lambda)dt\right),$$

$$S_\pi(x, \lambda) := \sin\left(\int_x^\pi T(t, \lambda)dt\right)$$

$\psi(x, \lambda)$ eşitliğimiz ;

$$\begin{aligned}
\psi(x, \lambda) &= \cos\beta E_\pi(x, \lambda) C_\pi(x, \lambda) + \cos\beta E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} \\
&\quad + \sin\beta E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} \quad (2.23)
\end{aligned}$$

şeklinde bulunur. Son eşitlikteki $E_\pi(x, \lambda)$, $C_\pi(x, \lambda)$, $S_\pi(x, \lambda)$, $S(\pi, \lambda)$ ve $T(\pi, \lambda)$ değerleri daha sonra hesaplanacaktır.

Öncelikle $S(x, \lambda)$ ve $T(x, \lambda)$ değerlerini hesaplayalım. Bunun için [4][sayfa 870].

Yeterince büyük λ için $|v_2(x, \lambda)| \leq k_2 \eta(\lambda)^2$ olan bir $\{k_2\}$ reel sayı dizisi vardır.

Dolayısıyla

$$v(x, \lambda) = i\lambda^{1/2} + v_1(x, \lambda) + k_2 \eta(\lambda)^2$$

olur. (2.5)'den

$$v_1(x, \lambda) = -e^{-2i\lambda^{1/2}x} \int_x^\pi e^{2i\lambda^{1/2}t} q(t) dt$$

dir. Düzenlemeler yapılırsa

$$\begin{aligned}
v_1(x, \lambda) &= -[\cos(2\lambda^{1/2}x) - i\sin(2\lambda^{1/2}x)] \\
&\quad \times \int_x^\pi [\cos(2\lambda^{1/2}t) + i\sin(2\lambda^{1/2}t)]q(t)dt \\
&= -\cos(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt \\
&\quad -i\cos(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
&\quad +i\sin(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt - \sin(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt
\end{aligned}$$

elde edilir.

$$T(x, \lambda) = \text{Im}v(x, \lambda) = \lambda^{1/2} + \text{Im}v_1(x, \lambda) + O(\eta(\lambda)^2),$$

$$\begin{aligned}
\text{Im}v_1(x, \lambda) &= -\cos(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
&\quad +\sin(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt, \\
T(x, \lambda) &= \lambda^{1/2} - \cos(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
&\quad +\sin(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt + O(\eta(\lambda)^2)
\end{aligned}$$

bulunur.

$$S(x, \lambda) = \text{Re}v(x, \lambda) = \text{Re}v_1(x, \lambda) + O(\eta(\lambda)^2),$$

$$\begin{aligned}
\text{Re}v_1(x, \lambda) &= -\cos(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt \\
&\quad -\sin(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt,
\end{aligned}$$

$$S(x, \lambda) = -\cos(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt \\ - \sin(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\eta(\lambda)^2)$$

elde edilir. Burada

$$\sin \xi_x = \int_x^\pi \cos(2\lambda^{1/2}t)q(t)dt, \quad \cos \xi_x = \int_x^\pi \sin(2\lambda^{1/2}t)q(t)dt \quad (2.24)$$

olarak alınır

$$T(x, \lambda) = \lambda^{1/2} - [\cos(2\lambda^{1/2}x)\cos \xi_x - \sin(2\lambda^{1/2}x)\sin \xi_x] + O(\eta(\lambda)^2)$$

bulunur ve $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ eşitliği kullanılırsa

$$T(x, \lambda) = \lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2) \quad (2.25)$$

olur.

$$S(x, \lambda) = -[\cos(2\lambda^{1/2}x)\sin \xi_x + \sin(2\lambda^{1/2}x)\cos \xi_x] + O(\eta(\lambda)^2)$$

bulunur ve $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ bağıntısı ile

$$S(x, \lambda) = -\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2) \quad (2.26)$$

elde edilir.

$$\begin{aligned} \frac{S(x, \lambda)}{T(x, \lambda)} &= \frac{-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)}{\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)} \\ &= - \left[\frac{\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)}{\lambda^{1/2}} \right] \\ &\quad \times [1 + \lambda^{-1/2}\cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)] \\ &= -\lambda^{-1/2}\sin(2\lambda^{1/2}x + \xi_x) - \lambda^{-1}\sin(2\lambda^{1/2}x + \xi_x)\cos(2\lambda^{1/2}x + \xi_x) \\ &\quad + O(\lambda^{-1/2}\eta(\lambda)^2) \\ &= -\lambda^{-1/2}\sin(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \end{aligned} \quad (2.27)$$

bulunur. Burada

$$\sin(2\lambda^{1/2}x + \xi_x) = O(\eta(\lambda)), \quad \cos(2\lambda^{1/2}x + \xi_x) = O(\eta(\lambda)) \quad (2.28)$$

dir[4]. İlk olarak

$$T(0, \lambda) = T(\pi, \lambda) = \lambda^{1/2} + O(\eta(\lambda)) \quad (2.29)$$

ikinci olarak

$$\begin{aligned} \frac{1}{T(0, \lambda)} &= \frac{1}{T(\pi, \lambda)} = \frac{1}{\lambda^{1/2} + O(\eta(\lambda))} = \frac{1}{\lambda^{1/2}[1 + O(\lambda^{-1/2}\eta(\lambda))]} \\ &= \frac{1}{\lambda^{1/2}} \times [1 + O(\lambda^{-1/2}\eta(\lambda))] = \lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda)), \end{aligned} \quad (2.30)$$

üçüncü olarak

$$S(0, \lambda) = S(\pi, \lambda) = O(\eta(\lambda)) \quad (2.31)$$

ve son olarak

$$\begin{aligned} \frac{S(0, \lambda)}{T(0, \lambda)} &= \frac{S(\pi, \lambda)}{T(\pi, \lambda)} = \frac{O(\eta(\lambda))}{\lambda^{1/2} + O(\eta(\lambda))} = \frac{O(\eta(\lambda))}{\lambda^{1/2}[1 + O(\lambda^{-1/2}\eta(\lambda))]} \\ &= \frac{O(\eta(\lambda))}{\lambda^{1/2}} \times [1 + O(\lambda^{-1/2}\eta(\lambda))] = O(\lambda^{-1/2}\eta(\lambda)) + O(\lambda^{-1}\eta(\lambda)) \\ &= O(\lambda^{-1/2}\eta(\lambda)) \end{aligned} \quad (2.32)$$

elde edilir.

Teorem 2.3: $\lambda \rightarrow \infty$ için

a)

$$\begin{aligned} E_0(x, \lambda) &= \exp\left(\int_0^x S(t, \lambda)dt\right) = 1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\lambda^{-1/2}\eta(\lambda)^2), \end{aligned} \quad (2.33)$$

b)

$$\begin{aligned}
C_0(x, \lambda) &= \cos \left(\int_0^x T(t, \lambda) dt \right) \\
&= \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2),
\end{aligned} \tag{2.34}$$

c)

$$\begin{aligned}
S_0(x, \lambda) &= \sin \left(\int_0^x T(t, \lambda) dt \right) \\
&= \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2),
\end{aligned} \tag{2.35}$$

d)

$$\begin{aligned}
E_\pi(x, \lambda) &= \exp \left(- \int_x^\pi S(t, \lambda) dt \right) \\
&= 1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) + O(\lambda^{-1/2} \eta(\lambda)^2),
\end{aligned} \tag{2.36}$$

e)

$$\begin{aligned}
C_\pi(x, \lambda) &= \cos \left(\int_x^\pi T(t, \lambda) dt \right) \\
&= \cos \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2),
\end{aligned} \tag{2.37}$$

f)

$$\begin{aligned}
S_\pi(x, \lambda) &= \sin \left(\int_x^\pi T(t, \lambda) dt \right) \\
&= \sin \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2),
\end{aligned} \tag{2.38}$$

ile verilir.

İspat:

a) Öncelikle $\int_0^x S(t, \lambda) dt$ integralini hesaplayalım.

$$\begin{aligned} \int_0^x S(t, \lambda) dt &= - \int_0^x \cos(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt \\ &\quad - \int_0^x \sin(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt + O(\lambda^{-1/2}\eta(\lambda)^2) \\ &:= I_1 + I_2 \end{aligned}$$

I_1 için aşağıdaki işlemler yapılırsa

$$\begin{aligned} I_1 &= - \int_0^x \cos(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt \\ dv &= \cos(2\lambda^{1/2}t) dt \quad v = \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t) \\ u &= \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \cos(2\lambda^{1/2}t) q(t) dt \\ I_1 &= - \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t) \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \Big|_0^x \\ &\quad - \frac{1}{2\lambda^{1/2}} \int_0^x \sin(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt. \end{aligned}$$

elde edilir. Benzer şekilde

$$\begin{aligned} I_2 &= - \int_0^x \sin(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt \\ dv &= \sin(2\lambda^{1/2}t) dt \quad v = - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \\ u &= \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \sin(2\lambda^{1/2}t) q(t) dt \\ I_2 &= \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \Big|_0^x \end{aligned}$$

$$+ \frac{1}{2\lambda^{1/2}} \int_0^x \sin(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt$$

bulunur. I_1 ve I_2 'yi $\int_0^x S(t, \lambda) dt$ integralinde yerine yazarsak

$$\begin{aligned} \int_0^x S(t, \lambda) dt &= -\frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t) \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \Big|_0^x \\ &\quad - \frac{1}{2\lambda^{1/2}} \int_0^x \sin(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt \\ &\quad + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \Big|_0^x \\ &\quad + \frac{1}{2\lambda^{1/2}} \int_0^x \sin(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt + O(\lambda^{-1/2} \eta(\lambda)^2) \\ &= \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t) q(t) dt \\ &\quad - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}x) q(x) dx \\ &\quad - \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t) q(t) dt + O(\lambda^{-1/2} \eta(\lambda)^2) \end{aligned}$$

olarak elde edilir ve (2.24) kullanılırsa

$$\begin{aligned} \int_0^x S(t, \lambda) dt &= \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x) \cos \xi_x - \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x) \sin \xi_x \\ &\quad - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}x) q(x) dx + O(\lambda^{-1/2} \eta(\lambda)^2) \\ \int_0^x S(t, \lambda) dt &= \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}x) q(x) dx \\ &\quad + O(\lambda^{-1/2} \eta(\lambda)^2) \end{aligned}$$

bulunur ve Taylor seri açılımı ile

$$\begin{aligned}
E_0(x, \lambda) &= \exp\left(\int_0^x S(t, \lambda) dt\right) = 1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}x) q(x) dt + O(\lambda^{-1/2}\eta(\lambda)^2)
\end{aligned} \tag{2.33}$$

elde edilir.

b) Öncelikle $\int_0^x T(t, \lambda) dt$ 'yi hesaplayalım.

$$\begin{aligned}
\int_0^x T(t, \lambda) dt &= \int_0^x \lambda^{1/2} dt + \int_0^x \sin(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt \\
&\quad - \int_0^x \cos(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt + O(\lambda^{-1/2}\eta(\lambda)^2) \\
&:= I_1 + I_2
\end{aligned}$$

I_1 integrali

$$\begin{aligned}
I_1 &= \int_0^x \sin(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt \\
dv &= \sin(2\lambda^{1/2}t) dt \quad v = -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \\
u &= \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \cos(2\lambda^{1/2}t) q(t) dt \\
I_1 &= -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \Big|_0^x \\
&\quad - \frac{1}{2\lambda^{1/2}} \int_0^x \cos(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt
\end{aligned}$$

şeklinde hesaplanır. Benzer şekilde I_2 integrali de

$$\begin{aligned}
I_2 &= -\int_0^x \cos(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt \\
dv &= \cos(2\lambda^{1/2}t) dt \quad v = \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t)
\end{aligned}$$

$$u = \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \sin(2\lambda^{1/2}t)q(t)dt$$

$$I_2 = -\frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t) \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \Big|_0^x \\ -\frac{1}{2\lambda^{1/2}} \int_0^x \sin(2\lambda^{1/2}t) \sin(2\lambda^{1/2}t) q(t) dt$$

bulunur. Böylece

$$I_1 + I_2 = -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t) q(t) dt \\ +\frac{1}{2\lambda^{1/2}} \int_0^\pi \cos(2\lambda^{1/2}x) q(x) dx \\ -\frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t) q(t) dt \\ -\frac{1}{2\lambda^{1/2}} \int_0^x \cos^2(2\lambda^{1/2}t) q(t) dt \\ -\frac{1}{2\lambda^{1/2}} \int_0^x \sin^2(2\lambda^{1/2}t) q(t) dt$$

elde edilir. (2.24) kullanılırsa ve gerekli düzenlemeler yapılırsa

$$I_1 + I_2 = -\frac{1}{2\lambda^{1/2}} [\cos(2\lambda^{1/2}x) \sin \xi_x + \sin(2\lambda^{1/2}x) \cos \xi_x] \\ +\frac{1}{2\lambda^{1/2}} \int_0^\pi \cos(2\lambda^{1/2}x) q(x) dx \\ -\frac{1}{2\lambda^{1/2}} \left[\int_0^x [\cos^2(2\lambda^{1/2}t) + \sin^2(2\lambda^{1/2}t)] q(t) dt \right] \\ = -\frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x + \xi_x) + \frac{1}{2\lambda^{1/2}} \int_0^\pi \cos(2\lambda^{1/2}x) q(x) dx \\ -\frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt$$

olur, buradan

$$\begin{aligned} \int_0^x T(t, \lambda) dt &= \lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2} x + \xi_x) \\ &+ \frac{1}{2\lambda^{1/2}} \int_0^\pi \cos(2\lambda^{1/2} x) q(x) dx - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \\ &+ O(\lambda^{-1/2} \eta(\lambda)^2) \end{aligned}$$

bulunur ve

$$\begin{aligned} C_0(x, \lambda) &= \cos \left(\int_0^x T(t, \lambda) dt \right) \\ &= \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2) \end{aligned} \quad (2.34)$$

elde edilir.

c) $\int_0^x T(t, \lambda) dt$ b) 'deki gibi elde edilir. Buradan hareketle

$$\begin{aligned} S_0(x, \lambda) &= \sin \left(\int_0^x T(t, \lambda) dt \right) \\ &= \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2) \end{aligned} \quad (2.35)$$

bulunur.

d) Öncelikle $-\int_x^\pi S(t, \lambda) dt$ integralini hesaplayalım.

$$\begin{aligned} -\int_x^\pi S(t, \lambda) dt &= \int_x^\pi \cos(2\lambda^{1/2} t) \left(\int_t^\pi \cos(2\lambda^{1/2} s) q(s) ds \right) dt \\ &+ \int_x^\pi \sin(2\lambda^{1/2} t) \left(\int_t^\pi \sin(2\lambda^{1/2} s) q(s) ds \right) dt + O(\lambda^{-1/2} \eta(\lambda)^2) \\ &:= I_1 + I_2 \end{aligned}$$

I_1 ve I_2 'yı bulalım. İlk olarak

$$I_1 = \int_x^\pi \cos(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt$$

$$dv = \cos(2\lambda^{1/2}t) dt \quad v = \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t)$$

$$u = \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \cos(2\lambda^{1/2}t) q(t) dt$$

$$I_1 = \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t) \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \Big|_x^\pi \\ + \frac{1}{2\lambda^{1/2}} \int_x^\pi \sin(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt.$$

elde edilir. Benzer şekilde

$$I_2 = \int_x^\pi \sin(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt$$

$$dv = \sin(2\lambda^{1/2}t) dt \quad v = -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t)$$

$$u = \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \sin(2\lambda^{1/2}t) q(t) dt$$

$$I_2 = -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \Big|_x^\pi \\ - \frac{1}{2\lambda^{1/2}} \int_x^\pi \sin(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt$$

bulunur. I_1 ve I_2 'yi $\int_x^\pi S(t, \lambda) dt$ integralinde yerine yazarsak

$$-\int_x^\pi S(t, \lambda) dt = -\frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t) q(t) dt \\ + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t) q(t) dt + O(\lambda^{-1/2} \eta(\lambda)^2)$$

ve (2.24) kullanılırsa

$$-\int_x^\pi S(t, \lambda) dt = \frac{1}{2\lambda^{1/2}} [\cos(2\lambda^{1/2}x) \cos \xi_x - \sin(2\lambda^{1/2}x) \sin \xi_x] + O(\lambda^{-1/2} \eta(\lambda)^2)$$

$$-\int_x^\pi S(t, \lambda) dt = \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2} \eta(\lambda)^2)$$

elde edilir. Dolayısıyla

$$\begin{aligned} E_\pi(x, \lambda) &= \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \\ &= 1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2} \eta(\lambda)^2) \end{aligned} \quad (2.36)$$

hesaplanır.

e) Öncelikle $\int_x^\pi T(t, \lambda) dt$ integralini hesaplayalım.

$$\begin{aligned} \int_x^\pi T(t, \lambda) dt &= \int_x^\pi \lambda^{1/2} dt + \int_x^\pi \sin(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt \\ &\quad - \int_x^\pi \cos(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt + O(\lambda^{-1/2} \eta(\lambda)^2) \\ &:= I_1 + I_2 \end{aligned}$$

I_1 integrali için

$$I_1 = \int_x^\pi \sin(2\lambda^{1/2}t) \left(\int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \right) dt$$

kısmi integrasyon yöntemi ile

$$dv = \sin(2\lambda^{1/2}t) dt \quad v = -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t)$$

$$u = \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \cos(2\lambda^{1/2}t) q(t) dt$$

dönüşümleri yerine yazılır, böylece

$$I_1 = -\frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}t) \int_t^\pi \cos(2\lambda^{1/2}s) q(s) ds \Big|_x^\pi$$

$$-\frac{1}{2\lambda^{1/2}} \int_x^\pi \cos(2\lambda^{1/2}t) \cos(2\lambda^{1/2}t) q(t) dt.$$

elde edilir. Benzer şekilde I_2 integralinde de

$$I_2 = - \int_x^\pi \cos(2\lambda^{1/2}t) \left(\int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \right) dt$$

$$dv = \cos(2\lambda^{1/2}t) dt \quad v = \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t)$$

$$u = \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \quad du = 0 - \sin(2\lambda^{1/2}t) q(t) dt$$

dönüşümleri ile

$$I_2 = -\frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}t) \int_t^\pi \sin(2\lambda^{1/2}s) q(s) ds \Big|_x^\pi \\ - \frac{1}{2\lambda^{1/2}} \int_x^\pi \sin(2\lambda^{1/2}t) \sin(2\lambda^{1/2}t) q(t) dt$$

olarak bulunur. Böylece

$$I_1 + I_2 = \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x) \int_x^\pi \cos(2\lambda^{1/2}t) q(t) dt \\ + \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x) \int_x^\pi \sin(2\lambda^{1/2}t) q(t) dt \\ - \frac{1}{2\lambda^{1/2}} \int_x^\pi [\cos^2(2\lambda^{1/2}t) + \sin^2(2\lambda^{1/2}t) q(t) dt]$$

olur. (2.24) kullanılırsa ve gerekli düzenlemeler yapılırsa

$$I_1 + I_2 = \frac{1}{2\lambda^{1/2}} [\cos(2\lambda^{1/2}x) \sin \xi_x + \sin(2\lambda^{1/2}x) \cos \xi_x] - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt,$$

bulduğumuz eşitliği $\int_x^\pi T(t, \lambda) dt$ integralinde yerine yazarsak

$$\int_x^\pi T(t, \lambda) dt = \lambda^{1/2}(\pi - x) + \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt$$

$$+O(\lambda^{-1/2}\eta(\lambda)^2)$$

bulunur ve dolayısıyla

$$\begin{aligned} C_\pi(x, \lambda) &= \cos\left(\int_x^\pi T(t, \lambda)dt\right) \\ &= \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \end{aligned} \quad (2.37)$$

elde edilir.

f) $\int_x^\pi T(t, \lambda)dt$ e) 'deki gibidir. Buradan hareketle

$$\begin{aligned} S_\pi(x, \lambda) &= \sin\left(\int_x^\pi T(t, \lambda)dt\right) \\ &= \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \end{aligned} \quad (2.38)$$

elde edilir.

Teorem 2.4: $\lambda \rightarrow \infty$ için

i)

$$y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$y(0, \lambda) = \cos\alpha, \quad y'(0, \lambda) = -\sin\alpha, \quad \alpha \in [0, \pi] \quad (2.6)$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned} \varphi(x, \lambda) &= \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ &\quad - \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ &\quad - \lambda^{-1/2}\sin\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)) \end{aligned} \quad (2.39),$$

ii)

$$y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$y(\pi, \lambda) = \cos\beta, \quad y'(\pi, \lambda) = -\sin\beta, \quad \beta \in [0, \pi] \quad (2.16)$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned} \psi(x, \lambda) = & \cos\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ & + \cos\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ & + \lambda^{-1/2} \sin\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ & + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \quad (2.40)$$

ile verilir.

İspat:

i) Bulduğumuz (2.30),(2.32),(2.33),(2.34) ve (2.35) eşitliklerini (2.15)'de yerine yazalım.

$$\begin{aligned} \varphi(x, \lambda) = & \cos\alpha E_0(x, \lambda)C_0(x, \lambda) - \cos\alpha E_0(x, \lambda)S_0(x, \lambda) \frac{S(0, \lambda)}{T(0, \lambda)} \\ & - \sin\alpha E_0(x, \lambda)S_0(x, \lambda) \frac{1}{T(0, \lambda)} \\ := & I_1 - I_2 - I_3 \end{aligned} \quad (2.15a)$$

Öncelikle I_1 'i hesaplayalım.

$$\begin{aligned} I_1 = & \cos\alpha E_0(x, \lambda)C_0(x, \lambda) \\ = & \cos\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \right] \end{aligned}$$

$$\begin{aligned}
& +O(\lambda^{-1/2}\eta(\lambda)^2)] \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
& = \cos\alpha \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \right. \\
& \quad \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
& \quad \left. \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right] \\
& +O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

Böylece

$$I_1 = \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)).$$

şeklinde hesaplanır. Benzer şekilde

$$\begin{aligned}
I_2 & = \cos\alpha E_0(x, \lambda) S_0(x, \lambda) \frac{S(0, \lambda)}{T(0, \lambda)} \\
& = \cos\alpha \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
& \quad \times \left[\frac{1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x)}{-\frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\lambda^{-1/2}\eta(\lambda)^2)} \right] \times [O(\lambda^{-1/2}\eta(\lambda))] \\
& = \cos\alpha \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right. \\
& \quad \left. + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right. \\
& \quad \left. - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right] \\
& +O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

işlemleri ile

$$I_2 = \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

bulunur ve son olarak

$$\begin{aligned} I_3 &= \sin\alpha E_0(x, \lambda) S_0(x, \lambda) \frac{1}{T(0, \lambda)} \\ &= \sin\alpha \times \left[\frac{1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x)}{-\frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\lambda^{-1/2}\eta(\lambda)^2)} \right] \\ &\quad \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\ &\quad \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \\ &= \sin\alpha \times \left[\frac{\lambda^{-1/2} \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)}{+\frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)} \right. \\ &\quad \left. -\frac{\lambda^{-1}}{2} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right] \\ &\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned}$$

$$I_3 = \sin\alpha \times \lambda^{-1/2} \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. Bu değerler (2.15a) eşitliğinde yerine yazılırsa

$$\begin{aligned} \varphi(x, \lambda) &= \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad - \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad - \lambda^{-1/2} \sin\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.39) \end{aligned}$$

olarak bulunur.

ii) Bulduğumuz (2.30),(2.32),(2.36),(2.37) ve (2.38) eşitliklerini (2.23)'de yerine yazalım.

$$\begin{aligned}
 \psi(x, \lambda) &= \cos\beta E_\pi(x, \lambda)C_\pi(x, \lambda) + \cos\beta E_\pi(x, \lambda)S_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} \\
 &\quad + \sin\beta E_\pi(x, \lambda)S_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} \\
 &:= I_1 + I_2 + I_3
 \end{aligned} \tag{2.23a}$$

Öncelikle I_1 'i hesaplayalım.

$$\begin{aligned}
 I_1 &= \cos\beta E_\pi(x, \lambda)C_\pi(x, \lambda) \\
 &= \cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
 &\quad \times \left[\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
 &= \cos\beta \times \left[\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt \right) \right. \\
 &\quad \left. + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt \right) \right] \\
 &\quad + O(\lambda^{-1/2}\eta(\lambda)) \\
 I_1 &= \cos\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2).
 \end{aligned}$$

şeklinde hesaplanır. Benzer şekilde

$$\begin{aligned}
 I_2 &= \cos\beta E_\pi(x, \lambda)S_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} \\
 &= \cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times [O(\lambda^{-1/2}\eta(\lambda))] \\
 &\quad \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \cos\beta \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right. \\
&\quad \left. + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right] \\
&\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

$$I_2 = \cos\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

elde edilir ve son olarak

$$\begin{aligned}
I_3 &= \sin\beta E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} \\
&= \sin\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \\
&= \sin\beta \times \left[\lambda^{-1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right. \\
&\quad \left. + \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right] \\
&\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

$$I_3 = \lambda^{-1/2} \sin\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. Bu değerler (2.23a) eşitliğinde yerine yazılırsa

$$\begin{aligned}
\psi(x, \lambda) &= \cos\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
&\quad + \cos\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)
\end{aligned}$$

$$\begin{aligned}
& +\lambda^{-1/2}\sin\beta \times \sin\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned} \tag{2.40}$$

olarak hesaplanır.

Şimdi $G(x, y, \lambda)$ Green fonksiyonunu elde etmek için gerekli olan

$$w(\lambda) := w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

değerini hesaplayalım.

Teorem 2.5: $\lambda \rightarrow \infty$ için (2.6) ve (2.16) koşulları altında

$$\begin{aligned}
w(\lambda) &= w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \\
&= 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\
&\quad + \cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\
&\quad + \sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] + O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{2.41}$$

ile verilir.

İspat:

$\varphi'(x, \lambda)$ 'yı bulmak için (2.12) ve (2.13) değerlerini (2.9) eşitliğinde yerine yazalım.

$$\begin{aligned}
Z'(x, \lambda) &= \frac{\cos\alpha \cdot S(x, \lambda) \cdot \exp\left(\int_0^x S(t, \lambda)dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right)} \\
&\quad \times \cos\left[\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right] + \int_0^x T(t, \lambda)dt\right] \\
&\quad - \frac{\cos\alpha \cdot \exp\left(\int_0^x S(t, \lambda)dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right)} \\
&\quad \times \sin\left[\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right] + \int_0^x T(t, \lambda)dt\right] T(x, \lambda)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\cos\alpha \cdot S(x, \lambda) \cdot E_0(x, \lambda)}{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right)} \\
&\times \left[\cos\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right) \times \cos\left(\int_0^x T(t, \lambda) dt\right) \right. \\
&\quad \left. - \sin\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right) \times \sin\left(\int_0^x T(t, \lambda) dt\right) \right] \\
&- \frac{\cos\alpha E_0(x, \lambda)}{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right)} \\
&\times \left[\sin\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right) \times \cos\left(\int_0^x T(t, \lambda) dt\right) \right. \\
&\quad \left. + \cos\left(\tan^{-1}\left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)}\right]\right) \right. \\
&\quad \left. \times \sin\left(\int_0^x T(t, \lambda) dt\right) \right] \times T(x, \lambda)
\end{aligned}$$

Notasyonlarla ifade edersek

$$Z'(x, \lambda) = \cos\alpha \times S(x, \lambda) E_0(x, \lambda) C_0(x, \lambda)$$

$$- \cos\alpha \times S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda) \times \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right]$$

$$- \cos\alpha \times E_0(x, \lambda) C_0(x, \lambda) \times \left[\frac{S(0, \lambda) + \tan\alpha}{T(0, \lambda)} \right] T(x, \lambda)$$

$$+ \cos\alpha \times E_0(x, \lambda) S_0(x, \lambda) T(x, \lambda)$$

$$Z'(x, \lambda) = \cos\alpha \times S(x, \lambda) E_0(x, \lambda) C_0(x, \lambda) - \cos\alpha \times S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda) \frac{S(0, \lambda)}{T(0, \lambda)}$$

$$- \sin\alpha \times S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda) \frac{1}{T(0, \lambda)}$$

$$- \cos\alpha E_0(x, \lambda) C_0(x, \lambda) \frac{S(0, \lambda)}{T(0, \lambda)} T(x, \lambda)$$

$$-\sin\alpha E_0(x, \lambda) C_0(x, \lambda) \frac{1}{T(0, \lambda)} T(x, \lambda) - \cos\alpha E_0(x, \lambda) S_0(x, \lambda) T(x, \lambda)$$

$$:= I_1 - I_2 - I_3 - I_4 - I_5 - I_6 \quad (2.9a)$$

şeklinde olur. Öncelikle I_1 eşitliğinde $S(x, \lambda)$, $E_0(x, \lambda)$ ve $C_0(x, \lambda)$ eşitlikleri yerine yazılırsa

$$\begin{aligned} I_1 = & \cos\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \right. \\ & \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\ & \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \end{aligned}$$

olur ve gerekli hesaplamalardan sonra

$$\begin{aligned} I_1 = & -\cos\alpha \times \left[\sin(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \right. \\ & + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \\ & \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \\ & \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \times \sin(2\lambda^{1/2}x + \xi_x) \Big] \\ & + O(\lambda^{-1/2}\eta(\lambda)^2) \\ I_1 = & O(\lambda^{-1/2}\eta(\lambda)^2). \end{aligned}$$

olarak elde edilir. I_2 eşitliğinde $S(x, \lambda)$, $E_0(x, \lambda)$, $S_0(x, \lambda)$ ve $\frac{S(0, \lambda)}{T(0, \lambda)}$ yerine yazılır, böylece

$$\begin{aligned} I_2 = & \cos\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \right. \\ & \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \end{aligned}$$

$$\begin{aligned}
& \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [O(\lambda^{-1/2}\eta(\lambda))] \\
& = \cos\alpha \times \left[-\sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right. \\
& \quad - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin(2\lambda^{1/2}x + \xi_x) + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
& \quad \left. \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \sin(2\lambda^{1/2}x + \xi_x) \right] + O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

hesaplamalarından sonra

$$I_2 = O(\lambda^{-1/2}\eta(\lambda))$$

şeklinde hesaplanır. I_3 eşitliği için

$$\begin{aligned}
I_3 &= \sin\alpha S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda) \frac{1}{T(0, \lambda)} \\
&= \sin\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \right. \\
& \quad \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
& \quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))]
\end{aligned}$$

gerekli hesaplamalardan sonra

$$\begin{aligned}
&= \sin\alpha \times \left[-\lambda^{-1/2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right. \\
& \quad - \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \\
& \quad \left. \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{\lambda^{-1}}{2} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \right]
\end{aligned}$$

$$\times \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)\Big] + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_3 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. Benzer şekilde I_4 eşitliği için

$$\begin{aligned} I_4 &= \cos\alpha E_0(x, \lambda) C_0(x, \lambda) \frac{S(0, \lambda)}{T(0, \lambda)} T(x, \lambda) \\ &= \cos\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t)dt \right. \\ &\quad \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\ &\quad \times \left[\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2) \right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right) \right] \\ &= \cos\alpha \times \left[\lambda^{1/2} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right. \\ &\quad \left. + \frac{1}{2} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \cos(2\lambda^{1/2}x + \xi_x) \right. \\ &\quad \left. - \frac{1}{2} \int_0^\pi \sin(2\lambda^{1/2}t) q(t)dt \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right. \\ &\quad \left. - \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \cos(2\lambda^{1/2}x + \xi_x) \right. \\ &\quad \left. - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \cos(2\lambda^{1/2}x + \xi_x) \right. \\ &\quad \left. \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t)dt \right. \\ &\quad \left. \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \cos(2\lambda^{1/2}x + \xi_x) \right] + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned}$$

yapılan işlemlerden sonra

$$I_4 = -\lambda^{1/2} \cos \alpha \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right)$$

şeklinde elde edilir ve I_5 eşitliğinde

$$\begin{aligned} I_5 &= \sin \alpha E_0(x, \lambda) C_0(x, \lambda) \frac{1}{T(0, \lambda)} T(x, \lambda) \\ &= \sin \alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \right. \\ &\quad \left. + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \times \left[\cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \\ &\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2} x + \xi_x) + O(\eta(\lambda)^2)] \times [\lambda^{-1/2} + O(\lambda^{-1} \eta(\lambda))] \\ &= \sin \alpha \times \left[\cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) \right. \\ &\quad \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \\ &\quad \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \lambda^{-1/2} \cos(2\lambda^{1/2} x + \xi_x) \\ &\quad - \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2} x + \xi_x) \times \cos(2\lambda^{1/2} x + \xi_x) \\ &\quad \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2} x + \xi_x) \\ &\quad \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \times \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \left. \right] + O \left(\lambda^{-1/2} \eta(\lambda) \right) \\ I_5 &= -\sin \alpha \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right) \end{aligned}$$

olarak bulunur. Son olarak I_6 hesaplanırsa

$$I_6 = \cos \alpha E_0(x, \lambda) S_0(x, \lambda) T(x, \lambda)$$

$$\begin{aligned}
&= \cos \alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \right. \\
&\quad \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[\sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)]
\end{aligned}$$

gerekli işlemlerden sonra

$$\begin{aligned}
I_6 &= \cos \alpha \times \left[\lambda^{1/2} \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) + \frac{1}{2} \cos(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) - \frac{1}{2} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
&\quad \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) - \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) \\
&\quad \left. + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \times \cos(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \left. \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) + O(\lambda^{-1/2}\eta(\lambda)) \right] \\
I_6 &= \lambda^{1/2} \cos \alpha \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) + O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

şeklinde elde edilir. Bulduğumuz bu değerler (2.9a) eşitliğinde yerine yazılırsa

$$\begin{aligned}
\varphi'(x, \lambda) &= -\lambda^{1/2} \cos \alpha \times \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) \\
&\quad - \sin \alpha \times \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right)
\end{aligned}$$

$$-\lambda^{1/2} \cos \alpha \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right) \quad (2.42)$$

bulunur. Şimdi $\psi'(x, \lambda)$ 'yı bulmak için (2.20) ve (2.21) değerlerini (2.9)'da yerine yazılır.

$$\begin{aligned} Z'(x, \lambda) = & \frac{\cos \beta S(x, \lambda) \exp \left(\int_0^x S(t, \lambda) dt \right)}{\exp \left(\int_0^\pi S(t, \lambda) dt \right) \times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda) + \tan \beta}{T(\pi, \lambda)} \right] \right)} \\ & \times \cos \left[\tan^{-1} \left[\frac{S(\pi, \lambda) + \tan \beta}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt \right] \\ & - \frac{\cos \beta \exp \left(\int_0^x S(t, \lambda) dt \right)}{\exp \left(\int_0^\pi S(t, \lambda) dt \right) \times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda) + \tan \beta}{T(\pi, \lambda)} \right] \right)} \\ & \times \sin \left[\tan^{-1} \left[\frac{S(\pi, \lambda) + \tan \beta}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt \right] T(x, \lambda) \end{aligned}$$

Sınırlar üzerinde düzenlemeler ve gerekli işlemler yapılırsa

$$\begin{aligned} Z'(x, \lambda) = & \cos \beta \times S(x, \lambda) E_\pi(x, \lambda) C_\pi(x, \lambda) \\ & + \cos \beta \times S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} \\ & + \sin \beta \times S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} \\ & - \cos \beta \times E_\pi(x, \lambda) C_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} T(x, \lambda) \\ & - \sin \beta \times E_\pi(x, \lambda) C_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} T(x, \lambda) \\ & + \cos \beta \times E_\pi(x, \lambda) S_\pi(x, \lambda) T(x, \lambda) \\ & := I_1 + I_2 + I_3 - I_4 - I_5 + I_6 \end{aligned} \quad (2.9b)$$

olarak bulunur. I_1 için

$$I_1 = \cos \beta S(x, \lambda) E_\pi(x, \lambda) C_\pi(x, \lambda)$$

$$\begin{aligned}
&= \cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)]
\end{aligned}$$

gerekli hesaplamalardan sonra

$$\begin{aligned}
I_1 &= \cos\beta \times \left[-\sin(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right. \\
&\quad \left. - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \left. \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
I_1 &= O(\lambda^{-1/2}\eta(\lambda)^2).
\end{aligned}$$

şeklinde bulunur. Benzer şekilde

$$\begin{aligned}
I_2 &= \cos\beta S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} \\
&= \cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [O(\lambda^{-1/2}\eta(\lambda))] \\
&= \cos\beta \times \left[-\sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right. \\
&\quad \left. - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \left. \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)) \right]
\end{aligned}$$

$$I_2 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde hesaplanır. I_3 hesaplanırsa

$$\begin{aligned}
I_3 &= \sin\beta S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} \\
&= \sin\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \\
&= \sin\beta \times \left[-\lambda^{-1/2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right. \\
&\quad \left. - \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \left. \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)) \right] \\
I_3 &= O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

olur. I_4 için

$$\begin{aligned}
I_4 &= \cos\beta E_\pi(x, \lambda) C_\pi(x, \lambda) \frac{S(\pi, \lambda)}{T(\pi, \lambda)} T(x, \lambda) \\
&= \cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [O(\lambda^{-1/2}\eta(\lambda))]
\end{aligned}$$

işlemler yapılırsa

$$\begin{aligned}
&= \cos\beta \times \left[\lambda^{1/2} \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + \frac{1}{2} \cos(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) - \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
&\quad \left. + O(\lambda^{-1/2}\eta(\lambda)) \right]
\end{aligned}$$

$$I_4 = \lambda^{1/2} \cos\beta \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda))$$

bulunur. I_5 eşitliğinde

$$\begin{aligned}
I_5 &= \sin\beta E_\pi(x, \lambda) C_\pi(x, \lambda) \frac{1}{T(\pi, \lambda)} T(x, \lambda) \\
&= \sin\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \\
&= \sin\beta \times \left[\cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \right. \\
&\quad \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) - \lambda^{-1/2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) - \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \left. \times \cos(2\lambda^{1/2}x + \xi_x) \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \right]
\end{aligned}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)]$$

$$I_5 = \sin\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

olarak bulunur ve son olarak

$$\begin{aligned} I_6 &= \cos\beta E_\pi(x, \lambda) S_\pi(x, \lambda) T(x, \lambda) \\ &= \cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)\right] \\ &\quad \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2)\right] \\ &\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \\ &= \cos\beta \times \left[\lambda^{1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + \frac{1}{2} \cos(2\lambda^{1/2}x + \xi_x)\right. \\ &\quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) - \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad \times \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ &\quad \left.+ O\left(\lambda^{-1/2}\eta(\lambda)\right)\right] \end{aligned}$$

$$I_6 = \lambda^{1/2} \cos\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. Bulunan eşitlikler (2.9b)'de yerine yazılırsa

$$\begin{aligned} \psi'(x, \lambda) &= -\lambda^{1/2} \cos\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ &\quad - \sin\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \end{aligned}$$

$$\begin{aligned}
& +\lambda^{1/2}\cos\beta \times \sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned} \tag{2.43}$$

olarak bulunur.

Bulunan (2.39),(2.40) ve (2.42),(2.43) eşitlikleri $w(\lambda)$ 'da yerine yazılırsa

$$w(\lambda) = w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

$$\begin{aligned}
& = \begin{bmatrix} \cos\alpha \\ \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\cos\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} -\lambda^{1/2}\cos\beta \\ \times \cos\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ -\sin\beta \\ \times \cos\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\lambda^{1/2}\cos\beta \\ \times \sin\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \\
& - \begin{bmatrix} -\lambda^{1/2}\cos\alpha \\ \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\sin\alpha \\ \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{1/2}\cos\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\lambda^{-1/2}\sin\beta \\ \sin\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}
\end{aligned}$$

olur, çarpma işlemleri yapılırsa

$$\begin{aligned}
w(\lambda) & = -\lambda^{1/2}\cos\beta \times \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\
& \times \cos\left(\lambda^{1/2}(\pi-x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)
\end{aligned}$$

$$\begin{aligned}
& -\cos\alpha \times \sin\beta \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\alpha \times \cos\beta \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\beta \times \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\cos\alpha \times \sin\beta \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& -\lambda^{1/2}\cos\beta \times \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\sin\alpha \times \cos\beta \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)
\end{aligned}$$

$$\begin{aligned}
& -\sin\alpha \times \cos\beta \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\beta \times \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\beta \times \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\cos\alpha \times \sin\beta \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\sin\alpha \times \cos\beta \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\sin\alpha \times \cos\beta \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
& +\lambda^{-1/2}\sin\alpha \times \sin\beta \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)
\end{aligned}$$

$$\begin{aligned}
& +\lambda^{1/2}\cos\beta \times \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\
& \quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\beta \times \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +\cos\alpha \times \sin\beta \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\
& \quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +O(\lambda^{-1}\eta(\lambda))
\end{aligned}$$

şeklinde elde edilir. Son eşitlikteki terimleri sırası ile 1,2, ...,18 olarak isimlendirerek

$$(1 - 17), (2 - 18), (3 - 4), (5 - 12), (6 - 10), (7 - 14), (8 - 15), (9 - 13), (11 - 16)$$

eşleştirirsek kosinüs ve sinüs açılımlarını kullanırsak

$$\begin{aligned}
w(\lambda) &= -\lambda^{1/2}\cos\beta \times \cos\alpha \\
& \quad \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\alpha \times \cos\beta \\
& \quad \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +\cos\alpha \times \sin\beta \\
& \quad \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& -\cos\alpha \times \sin\beta
\end{aligned}$$

$$\begin{aligned}
& \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& + \sin \alpha \times \cos \beta \\
& \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& + \lambda^{-1/2} \sin \alpha \times \sin \beta \\
& \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& + \lambda^{1/2} \cos \beta \times \cos \alpha \\
& \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& + \lambda^{1/2} \cos \beta \times \cos \alpha \\
& \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& + \sin \alpha \times \cos \beta \\
& \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& + O(\lambda^{-1} \eta(\lambda))
\end{aligned}$$

elde edilir, integral sınırları üzerinde de düzenlemeler yapılırsa

$$\begin{aligned}
w(\lambda) &= \lambda^{1/2} \cos \alpha \times \cos \beta \times \sin \left(\lambda^{1/2} \pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t) dt \right) \\
&+ \cos \alpha \times \sin \beta \times \sin \left(\lambda^{1/2} \pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t) dt \right) \\
&- \cos \alpha \times \sin \beta \times \cos \left(\lambda^{1/2} \pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t) dt \right)
\end{aligned}$$

$$\begin{aligned}
& +\sin\alpha \times \cos\beta \times \sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t)dt\right) \\
& +\lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t)dt\right) \\
& +\lambda^{1/2}\cos\beta \times \cos\alpha \times \sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t)dt\right) \\
& +\sin\alpha \times \cos\beta \times \cos\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}} \int_0^\pi q(t)dt\right) + O(\lambda^{-1}\eta(\lambda))
\end{aligned}$$

olarak bulunur. $\int_0^\pi q(t)dt = 0$ olduğu kullanılırsa

$$\begin{aligned}
w(\lambda) &= 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\
& +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\
& +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] + O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{2.41}$$

şeklinde elde edilir.

Teorem 2. 6: $\lambda \rightarrow \infty$ ve $0 \leq x \leq y \leq \pi$ için

$$\begin{aligned}
G(x, y, \lambda) &= \\
& \frac{\begin{bmatrix} \cos\alpha \\ \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ -\cos\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ +\lambda^{-1/2}\sin\beta \\ \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\ +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\ +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] \end{bmatrix}} \\
& +O\left(\lambda^{-1/2}\eta(\lambda)\right).
\end{aligned} \tag{2.44}$$

İspat:

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq y \leq x \leq \pi \end{cases}$$

şeklinde idi. Bulduğumuz (2.39), (2.40) ve (2.41) eşitliklerini $G(x, y, \lambda)$ 'da yerine yazarsak

$0 \leq x \leq y \leq \pi$ için

$$\begin{aligned} G(x, y, \lambda) &= \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} \\ &= \frac{\begin{bmatrix} \cos\alpha \\ \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ -\cos\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ +O(\lambda^{-1/2}\eta(\lambda)) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ +\lambda^{-1/2}\sin\beta \\ \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ +O(\lambda^{-1/2}\eta(\lambda)) \end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\ +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\ +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] + O(\lambda^{-1}\eta(\lambda)) \end{bmatrix}} \end{aligned}$$

(2.45) elde edilir.

$$\begin{aligned} F_1(x, \lambda) &:= \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad - \cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad - \lambda^{-1/2}\sin\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \end{aligned}$$

ve

$$F_2(y, \lambda) := \cos\beta \times \cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right)$$

$$\begin{aligned}
& +\cos\beta \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\
& +\lambda^{-1/2}\sin\beta \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right)
\end{aligned}$$

ve son olarak

$$\begin{aligned}
F_3(\alpha, \beta) &:= 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\
& +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\
& +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)]
\end{aligned}$$

olarak tanımlarsak

$$\begin{aligned}
G(x, y, \lambda) &= \frac{\left(F_1(x, \lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right)\left(F_2(y, \lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right)}{F_3(\alpha, \beta) + O\left(\lambda^{-1}\eta(\lambda)\right)} \\
&= \frac{\left[\begin{aligned} & F_1(x, \lambda) \times F_2(y, \lambda) + F_1(x, \lambda) \times O\left(\lambda^{-1/2}\eta(\lambda)\right) \\ & + F_2(y, \lambda) \times O\left(\lambda^{-1/2}\eta(\lambda)\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \times O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned}\right]}{F_3(\alpha, \beta) + O\left(\lambda^{-1}\eta(\lambda)\right)} \\
&= \frac{F_1(x, \lambda) \times F_2(y, \lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)}{F_3(\alpha, \beta) \times [1 + O\left(\lambda^{-1}\eta(\lambda)\right)]} \\
&= \frac{F_1(x, \lambda) \times F_2(y, \lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)}{F_3(\alpha, \beta)} \times [1 + O\left(\lambda^{-3/2}\eta(\lambda)\right)] \\
&= \frac{F_1(x, \lambda) \times F_2(y, \lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)}{F_3(\alpha, \beta)} \\
&= \frac{F_1(x, \lambda) \times F_2(y, \lambda)}{F_3(\alpha, \beta)} + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

elde ederiz ve (2.45) eşitliği

$$G(x, y, \lambda) =$$

$$\begin{aligned}
& \begin{bmatrix} \cos \alpha \\ \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\ -\cos \alpha \\ \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\ -\lambda^{-1/2} \sin \alpha \\ \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \end{bmatrix} \times \begin{bmatrix} \cos \beta \\ \times \cos \left(\lambda^{1/2} (\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t) dt \right) \\ +\cos \beta \\ \times \sin \left(\lambda^{1/2} (\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t) dt \right) \\ +\lambda^{-1/2} \sin \beta \\ \times \sin \left(\lambda^{1/2} (\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t) dt \right) \end{bmatrix} \\
& = \frac{\begin{bmatrix} 2\lambda^{1/2} \cos \alpha \times \cos \beta \times \sin(\lambda^{1/2} \pi) + \lambda^{-1/2} \sin \alpha \times \sin \beta \times \sin(\lambda^{1/2} \pi) \\ +\cos \alpha \times \sin \beta \times [\sin(\lambda^{1/2} \pi) - \cos(\lambda^{1/2} \pi)] \\ +\sin \alpha \times \cos \beta \times [\sin(\lambda^{1/2} \pi) + \cos(\lambda^{1/2} \pi)] \end{bmatrix}}{+O\left(\lambda^{-1/2} \eta(\lambda)\right)} \quad (2.44)
\end{aligned}$$

bulunur. Benzer şekilde $0 \leq y \leq x \leq \pi$ içinde

$$G(x, y, \lambda) = \frac{\varphi(y, \lambda) \psi(x, \lambda)}{w(\lambda)}$$

hesaplanır.

Aşağıdaki teoremlerde Dirichlet koşulları altında Green fonksiyonu elde edilecektir.

$$\textbf{Teorem 2.7: } y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$y(0, \lambda) = \cos \alpha, \quad y'(0, \lambda) = -\sin \alpha, \quad \alpha = \frac{\pi}{2} \quad (2.46)$$

başlangıç koşullarını sağlayan çözümü

a)

$$\varphi(x, \lambda) = -\lambda^{-1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O\left(\lambda^{-1/2} \eta(\lambda)\right) \quad (2.47)$$

b)

$$\varphi(x, n) = -\frac{1}{(n+1)} \times \sin \left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t) dt \right)$$

$$+O(n^{-2}\eta(n)) \quad (2.48)$$

olarak hesaplanır.

İspat:

a) (2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x, \lambda) = c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \quad (2.8)$$

ve

$$\begin{aligned} Z'(x, \lambda) &= c_1 S(x, \lambda) \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^x T(t, \lambda) dt\right) T(x, \lambda) \end{aligned} \quad (2.9)$$

şeklindedir [4]. (2.46) ile verilen başlangıç koşulları sağlanacak şekilde c_1 ve c_2 sabitlerini belirleyelim. (2.46) ile verilen başlangıç koşullarından $Z(0, \lambda) = \cos\alpha$ yerine yazılırsa

$$\begin{aligned} \cos\alpha &= c_1 \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^0 T(t, \lambda) dt\right) \\ \cos\alpha &= c_1 \cos c_2 \end{aligned} \quad (2.49)$$

$\alpha = \frac{\pi}{2}$, $\cos\alpha = 0$ ve $\cos c_2 \neq 0$ ise bu durumda

$$0 = c_1 \cos c_2, \quad c_1 = 0$$

olur ki bu aşıkâr çözümdür.

$\alpha = \frac{\pi}{2}$, $\cos\alpha = 0$ ve $c_1 \neq 0$ ise bu durumda c_2 'yi bulalım.

$$0 = c_1 \cos c_2, \quad \cos c_2 = 0 \Rightarrow c_2 = \frac{\pi}{2} \quad (2.50)$$

olur.

$Z'(0, \lambda) = -\sin\alpha$ kullanılırsa

$$\begin{aligned}
-\sin\alpha &= c_1 S(0, \lambda) \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^0 T(t, \lambda) dt\right) \\
&\quad - c_1 \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^0 T(t, \lambda) dt\right) T(0, \lambda), \\
-\sin\alpha &= c_1 S(0, \lambda) \cos c_2 - c_1 \sin c_2 T(0, \lambda)
\end{aligned}$$

olur. (2.50) kullanılırsa

$$1 = c_1 T(0, \lambda) \Rightarrow c_1 = \frac{1}{T(0, \lambda)} \quad (2.51)$$

elde edilir. Son olarak (2.50) ve (2.51) değerleri (2.8)'de yerine yazılırsa ve (2.46) ile verilen başlangıç koşullarını sağlayan çözüme $\varphi(x, \lambda)$ denilirse

$$\varphi(x, \lambda) = \frac{1}{T(0, \lambda)} \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(\frac{\pi}{2} + \int_0^x T(t, \lambda) dt\right),$$

elde edilir ve $\cos\left(\frac{\pi}{2} + \beta\right) = -\sin\beta$ kullanılırsa

$$\varphi(x, \lambda) = -\frac{1}{T(0, \lambda)} \exp\left(\int_0^x S(t, \lambda) dt\right) \times \sin\left(\int_0^x T(t, \lambda) dt\right) \quad (2.52)$$

bulunur.

Notasyonlarla ifade edecek olursak (2.52) eşitliği

$$\varphi(x, \lambda) = -\frac{1}{T(0, \lambda)} E_0(x, \lambda) S_0(x, \lambda)$$

şeklinde düzenlenir ve bulduğumuz (2.30), (2.33) ve (2.35) eşitliklerini $\varphi(x, \lambda)$ 'da yerine yazarsak

$$\begin{aligned}
\varphi(x, \lambda) &= -\left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \right. \\
&\quad \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \\
&\quad \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right]
\end{aligned}$$

ve gerekli işlemler yapılırsa

$$\begin{aligned}
\varphi(x, \lambda) &= -\lambda^{-1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\
&\quad - \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2} x + \xi_x) \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\
&\quad + \frac{\lambda^{-1}}{2} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\
&\quad + O(\lambda^{-1/2} \eta(\lambda)) \\
\varphi(x, \lambda) &= -\lambda^{-1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)) \tag{2.47}
\end{aligned}$$

elde edilir.

b) Burada Dirichlet sınır koşulları altında (2.1) – (2.3) özdeğerleri

$$\lambda_n^{1/2} = (n+1) - \frac{1}{(n+1)} A_{2(n+1)} + O(n^{-1} \eta(n)^2) + O(n^{-2} \eta(n)) \tag{2.53}$$

kullanılacaktır [4]. Böylece

$$\begin{aligned}
\lambda_n^{1/2} &= (n+1) + O(\eta(n)), \\
\lambda_n^{-1/2} &= \frac{1}{(n+1) + O(\eta(n))} = \frac{1}{(n+1)[1 + O(n^{-1} \eta(n))]} \\
&= \frac{1}{(n+1)} \times [1 + O(n^{-1} \eta(n))] = \frac{1}{(n+1)} + O(n^{-2} \eta(n)) \tag{2.54}
\end{aligned}$$

elde edilir. O halde

$$\begin{aligned}
\varphi(x, n) &= -\frac{1}{(n+1)} \times \sin \left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t) dt \right) \\
&\quad + O(n^{-2} \eta(n)) \tag{2.48}
\end{aligned}$$

bulunur.

Teorem 2. 8: $y'' + (\lambda - q)y = 0$ (2.1)

denkleminin

$$y(\pi, \lambda) = \cos \beta, \quad y'(\pi, \lambda) = -\sin \beta, \quad \beta = \frac{\pi}{2} \quad (2.55)$$

başlangıç koşullarını sağlayan çözümü

a)

$$\psi(x, \lambda) = \lambda^{-1/2} \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right) \quad (2.56)$$

b)

$$\begin{aligned} \psi(x, n) = & \frac{1}{(n+1)} \sin \left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t) dt \right) \\ & + O(n^{-2} \eta(n)) \end{aligned} \quad (2.57)$$

ile verilir.

İspat:

a) (2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x, \lambda) = c_1 \exp \left(\int_0^x S(t, \lambda) dt \right) \times \cos \left(c_2 + \int_0^x T(t, \lambda) dt \right) \quad (2.8)$$

ve

$$\begin{aligned} Z'(x, \lambda) = & c_1 S(x, \lambda) \exp \left(\int_0^x S(t, \lambda) dt \right) \times \cos \left(c_2 + \int_0^x T(t, \lambda) dt \right) \\ & - c_1 \exp \left(\int_0^x S(t, \lambda) dt \right) \times \sin \left(c_2 + \int_0^x T(t, \lambda) dt \right) T(x, \lambda) \end{aligned} \quad (2.9)$$

şeklindedir [4]. (2.55) ile verilen başlangıç koşulları sağlanacak şekilde c_1 ve c_2 sabitlerini belirleyelim. (2.55) ile verilen başlangıç koşullarından

$Z(\pi, \lambda) = \cos \beta$ ve $Z'(\pi, \lambda) = -\sin \beta$ yazılırsa

$$\cos\beta = c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \quad (2.58)$$

$$\beta = \frac{\pi}{2}, \quad \cos\beta = 0, \quad \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \neq 0$$

ise (2.58) eşitliği

$$0 = c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \Rightarrow c_1 = 0$$

olur ki bu aşıkâr çözümdür.

$$\beta = \frac{\pi}{2}, \quad \cos\beta = 0, \quad c_1 \neq 0 \Rightarrow \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) = 0$$

olur buradan

$$c_2 + \int_0^\pi T(t, \lambda) dt = \frac{\pi}{2} \Rightarrow c_2 = \frac{\pi}{2} - \int_0^\pi T(t, \lambda) dt \quad (2.59)$$

elde edilir.

$$\begin{aligned} -\sin\beta &= c_1 S(\pi, \lambda) \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) T(\pi, \lambda) \end{aligned}$$

(2.59) kullanılırsa

$$\begin{aligned} -1 &= c_1 S(\pi, \lambda) \exp\left(\int_0^\pi S(t, \lambda) dt\right) \cos\left(\frac{\pi}{2}\right) - c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \sin\left(\frac{\pi}{2}\right) T(\pi, \lambda) \\ 1 &= c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) T(\pi, \lambda) \Rightarrow c_1 = \frac{1}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) T(\pi, \lambda)} \end{aligned} \quad (2.60)$$

elde edilir. Son olarak (2.59) ve (2.60) değerleri (2.8)'de yerine yazılırsa ve (2.55) ile verilen başlangıç koşullarını sağlayan çözüme $\psi(x, \lambda)$ denilirse

$$\psi(x, \lambda) = \frac{\exp(\int_0^x S(t, \lambda) dt)}{\exp(\int_0^\pi S(t, \lambda) dt) T(\pi, \lambda)} \cos\left(\frac{\pi}{2} - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt\right)$$

şeklinde bulunur. Sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\psi(x, \lambda) = \frac{1}{T(\pi, \lambda)} \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \cos\left(\frac{\pi}{2} - \int_x^\pi T(t, \lambda) dt\right)$$

olur ve $\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$ kullanılırsa

$$\psi(x, \lambda) = \frac{1}{T(\pi, \lambda)} \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \times \sin\left(\int_x^\pi T(t, \lambda) dt\right) \quad (2.61)$$

elde edilir.

Notasyonlarla ifade edecek olursak

$$\psi(x, \lambda) = \frac{1}{T(\pi, \lambda)} E_\pi(x, \lambda) S_\pi(x, \lambda)$$

şeklindedir. Bulduğumuz (2.30), (2.36) ve (2.38) değerleri $\psi(x, \lambda)$ 'da yerine yazılırsa

$$\begin{aligned} \psi(x, \lambda) &= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)\right] \\ &\times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2)\right] \\ &\times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \end{aligned}$$

gerekli hesaplamalardan sonra

$$\begin{aligned} \psi(x, \lambda) &= \lambda^{-1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt\right) \\ &+ \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt\right) \\ &+ O(\lambda^{-1/2}\eta(\lambda)) \end{aligned}$$

$$\psi(x, \lambda) = \lambda^{-1/2} \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right) \quad (2.56)$$

bulunur.

b) (2.54)'ten

$$\begin{aligned} \psi(x, n) &= \frac{1}{(n+1)} \sin \left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t) dt \right) \\ &\quad + O(n^{-2} \eta(n)) \end{aligned} \quad (2.57)$$

elde edilir.

Teorem 2.9: Dirichlet sınır şartları altında $\lambda \rightarrow \infty$ için

$$\begin{aligned} w(\lambda) &= w_x(\varphi, \psi) = \varphi(x, \lambda) \psi'(x, \lambda) - \varphi'(x, \lambda) \psi(x, \lambda) \\ &= \lambda^{-1/2} \sin(\lambda^{1/2} \pi) + O(\lambda^{-1} \eta(\lambda)) \end{aligned} \quad (2.62)$$

ile verilir.

İspat:

$\varphi'(x, \lambda)$ 'yı bulmak için (2.50) ve (2.51)'i (2.9)'da yerine yazalım. Böylece

$$\begin{aligned} Z'(x, \lambda) &= \frac{S(x, \lambda)}{T(0, \lambda)} \exp \left(\int_0^x S(t, \lambda) dt \right) \times \cos \left(\frac{\pi}{2} + \int_0^x T(t, \lambda) dt \right) \\ &\quad - \frac{1}{T(0, \lambda)} \exp \left(\int_0^x S(t, \lambda) dt \right) \times \sin \left(\frac{\pi}{2} + \int_0^x T(t, \lambda) dt \right) T(x, \lambda) \end{aligned}$$

olur. Ayrıca

$$\cos \left(\frac{\pi}{2} + \int_0^x T(t, \lambda) dt \right) = -\sin \left(\int_0^x T(t, \lambda) dt \right)$$

ve

$$\sin \left(\frac{\pi}{2} + \int_0^x T(t, \lambda) dt \right) = \cos \left(\int_0^x T(t, \lambda) dt \right)$$

olduğu kullanılırsa

$$Z'(x, \lambda) = -\frac{1}{T(0, \lambda)} S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda) - \frac{1}{T(0, \lambda)} E_0(x, \lambda) C_0(x, \lambda) T(x, \lambda)$$

$$:= I_1 + I_2$$

şeklinde olur.

$$I_1 = -\frac{1}{T(0, \lambda)} S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda)$$

$$= -\left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt + O(\lambda^{-1/2}\eta(\lambda)^2)\right] \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))]$$

$$\times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)]$$

gerekli hesaplamalardan sonra

$$= \lambda^{-1/2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right)$$

$$+ \frac{\lambda^{-1}}{2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \times \cos(2\lambda^{1/2}x + \xi_x)$$

$$- \frac{\lambda^{-1}}{2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right)$$

$$\times \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt + O(\lambda^{-1/2}\eta(\lambda))$$

$$I_1 = O(\lambda^{-1/2}\eta(\lambda))$$

bulunur.

$$I_2 = -\frac{1}{T(0, \lambda)} E_0(x, \lambda) C_0(x, \lambda) T(x, \lambda)$$

$$\begin{aligned}
&= - \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \right. \\
&\quad \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times [\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))] \\
&\quad \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)]
\end{aligned}$$

gerekli hesaplamalardan yapılırsa

$$\begin{aligned}
I_2 &= -\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + \lambda^{-1/2} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) - \frac{\lambda^{-1/2}}{2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + \frac{\lambda^{-1}}{2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \\
&\quad + \frac{\lambda^{-1/2}}{2} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \\
&\quad - \frac{\lambda^{-1}}{2} \int_0^\pi \sin(2\lambda^{1/2}t) q(t) dt \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)) \\
I_2 &= -\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) + O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

elde edilir. I_1 ve I_2 'yi $\varphi'(x, \lambda)$ 'da yerine yazarsak

$$\begin{aligned}
\varphi'(x, \lambda) &= -\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt\right) \\
&\quad + O(\lambda^{-1/2}\eta(\lambda)) \tag{2.63}
\end{aligned}$$

şeklinde olur.

Şimdi $\psi'(x, \lambda)$ 'yı bulmak için (2.59) ve (2.60)'ı (2.9)'da yerine yazalım.

$$\begin{aligned} Z'(x, \lambda) &= \frac{S(x, \lambda) \exp\left(\int_0^x S(t, \lambda) dt\right)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) T(\pi, \lambda)} \times \cos\left(\frac{\pi}{2} - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt\right) \\ &\quad - \frac{\exp\left(\int_0^x S(t, \lambda) dt\right)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) T(\pi, \lambda)} \\ &\quad \times \sin\left(\frac{\pi}{2} - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt\right) T(x, \lambda) \end{aligned}$$

sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\begin{aligned} Z'(x, \lambda) &= \frac{S(x, \lambda)}{T(\pi, \lambda)} \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \times \cos\left(\frac{\pi}{2} - \int_x^\pi T(t, \lambda) dt\right) \\ &\quad - \frac{1}{T(\pi, \lambda)} \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \times \sin\left(\frac{\pi}{2} - \int_x^\pi T(t, \lambda) dt\right) T(x, \lambda), \end{aligned}$$

olur.

$$\cos\left(\frac{\pi}{2} - \int_x^\pi T(t, \lambda) dt\right) = \sin\left(\int_x^\pi T(t, \lambda) dt\right),$$

ve

$$\sin\left(\frac{\pi}{2} - \int_x^\pi T(t, \lambda) dt\right) = \cos\left(\int_x^\pi T(t, \lambda) dt\right),$$

olduğu kullanılırsa

$$\begin{aligned} Z'(x, \lambda) &= \frac{1}{T(\pi, \lambda)} S(x, \lambda) \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \sin\left(\int_x^\pi T(t, \lambda) dt\right) \\ &\quad - \frac{1}{T(\pi, \lambda)} \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \cos\left(\int_x^\pi T(t, \lambda) dt\right) \end{aligned}$$

elde edilir ve notasyonlarla ifade edersek

$$Z'(x, \lambda) = \frac{1}{T(\pi, \lambda)} S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) - \frac{1}{T(\pi, \lambda)} E_\pi(x, \lambda) C_\pi(x, \lambda) T(x, \lambda)$$

$$:= I_1 - I_2$$

şeklinde olur.

$$I_1 = \frac{1}{T(\pi, \lambda)} S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda)$$

$$= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right]$$

$$\times \left[-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2) \right] \times \left[\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda)) \right]$$

$$\times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right]$$

$$= -\lambda^{-1/2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)$$

$$- \frac{\lambda^{-1}}{2} \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)$$

$$\times \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda))$$

$$I_1 = O(\lambda^{-1/2}\eta(\lambda))$$

şeklinde hesaplanır.

$$I_2 = -\frac{1}{T(\pi, \lambda)} \exp\left(-\int_x^\pi S(t, \lambda)dt\right) \cos\left(\int_x^\pi T(t, \lambda)dt\right)$$

$$= -\left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right]$$

$$\times \left[\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right]$$

$$\times \left[\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda)) \right] \times \left[\lambda^{1/2} + \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2) \right]$$

gerekli hesaplamalardan sonra

$$\begin{aligned}
&= -\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + \frac{\lambda^{-1/2}}{2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + \frac{\lambda^{-1/2}}{2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
&\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

$$I_2 = -\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. I_1 ve I_2 'yi $\psi'(x, \lambda)$ 'da yerine yazarsak

$$\psi'(x, \lambda) = -\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.64)$$

elde edilir.

Bulduğumuz (2.47), (2.56), (2.63) ve (2.64) eşitlikleri $w_x(\varphi, \psi)$ 'da yerine yazılırsa

$$\begin{aligned}
&w_x(\varphi, \psi) = \\
&= \left[\begin{aligned} &\lambda^{-1/2} \\ &\times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right] \times \left[\begin{aligned} &\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ &+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right] \\
&+ \left[\begin{aligned} &\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right] \times \left[\begin{aligned} &\lambda^{-1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ &+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right]
\end{aligned}$$

şeklinde olur. Böylece

$$\begin{aligned}
w(\lambda) &= \lambda^{-1/2} \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
&\quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)
\end{aligned}$$

$$\begin{aligned}
& +\lambda^{-1/2}\cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\
& \times \sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O(\lambda^{-1}\eta(\lambda)) \\
& =\lambda^{-1/2}\left[\sin\left(\lambda^{1/2}\pi-\frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)\right]+O(\lambda^{-1}\eta(\lambda))
\end{aligned}$$

bulunur. $\int_0^\pi q(t)dt = 0$ (ortalama değeri) olduğundan

$$w(\lambda) = \lambda^{-1/2}\sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda)) \quad (2.62)$$

elde edilir.

Teorem 2.10:

i) Dirichlet sınır şartları altında $\lambda \rightarrow \infty$ ve $0 \leq x \leq y \leq \pi$ için

$$\begin{aligned}
G(x, y, \lambda) &= \\
&= \frac{\left[\sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right] \times \left[\sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\right]}{\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\
&+O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.65),
\end{aligned}$$

ii)

$$\begin{aligned}
G(x, y, n) &= \frac{1}{(n+1)}\sin\left((n+1)x-\frac{1}{2(n+1)}\int_0^x q(t)dt\right) \\
&\times \sin\left((n+1)(\pi-y)-\frac{1}{2(n+1)}\int_y^\pi q(t)dt\right) \\
&+O(n^{-3}\eta(n)) \quad (2.66)
\end{aligned}$$

ile verilir.

İspat:

i)

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq y \leq x \leq \pi \end{cases}$$

şeklinde idi. Bulduğumuz (2.62), (2.63) ve (2.64)'ü $G(x, y, \lambda)$ 'da yerine yazarsak

$0 \leq x \leq y \leq \pi$ için

$$\begin{aligned} G(x, y, \lambda) &= \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} \\ &= \frac{\left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right] \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \right]}{\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\ &\quad + O(\lambda^{-1/2}\eta(\lambda)) \end{aligned} \tag{2.65}$$

bulunur. Benzer şekilde $0 \leq y \leq x \leq \pi$ için de

$$G(x, y, \lambda) = \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)}$$

elde edilir.

ii) Bulduğumuz $G(x, y, \lambda)$ eşitliğinde reversion uygularsak

$$\begin{aligned} G(x, y, \lambda) &= \\ &= \frac{\left[\lambda^{-1/2}\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right] \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)) \right]}{\sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda))} \\ &= \frac{\left[\lambda^{-1/2}\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \right] \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)) \right]}{\sin(\lambda^{1/2}\pi) \times [1 + O(\lambda^{-1}\eta(\lambda))]} \end{aligned}$$

$$\begin{aligned}
&= \frac{\left[\lambda^{-1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \right. \\
&\quad \times \sin \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
&\quad \left. + O \left(\lambda^{-1/2} \eta(\lambda) \right) \right]}{\sin(\lambda^{1/2} \pi)} \times [1 + O(\lambda^{-1} \eta(\lambda))] \\
&= \frac{\left[\lambda^{-1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \right. \\
&\quad \times \sin \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \left. \right]}{\sin(\lambda^{1/2} \pi)} + O \left(\lambda^{-1/2} \eta(\lambda) \right)
\end{aligned}$$

elde edilir. (2.54)'ten

$$\begin{aligned}
G(x, y, n) = & \frac{\left[\frac{1}{(n+1) + o(\eta(n))} \right. \\
& \times \sin \left((n+1)x - \frac{1}{2((n+1) + o(\eta(n)))} \int_0^x q(t) dt \right) \\
& \times \\
& \left. \sin \left((n+1)(\pi - y) - \frac{1}{2((n+1) + o(\eta(n)))} \int_y^\pi q(t) dt \right) \right]}{+O(n^{-3}\eta(n))} \\
& \frac{}{\sin((n+1)\pi + O(\eta(n)))}
\end{aligned}$$

dir, burada $\sin((n+1)\pi + O(\eta(n)))$ değerini hesaplayalım.

$$\begin{aligned}
\sin((n+1)\pi + O(\eta(n))) &= \\
&= \cos((n+1)\pi) \sin(O(\eta(n))) \\
&+ \sin((n+1)\pi) \cos(O(\eta(n))) = O(\eta(n))
\end{aligned}$$

Bulduğumuz değeri $G(x, y, n)$ 'de yerine yazarsak

$$\begin{aligned}
G(x, y, n) = & \frac{1}{(n+1)} \sin \left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t) dt \right) \\
& \times \sin \left((n+1)(\pi - y) - \frac{1}{2(n+1)} \int_y^\pi q(t) dt \right) \\
& + O(n^{-3} \eta(n))
\end{aligned} \tag{2.66}$$

olarak bulunur.

Aşağıdaki teoremlerde Neumann koşulları altında Green fonksiyonu elde edilecektir.

$$\textbf{Teorem 2.11:} y'' + (\lambda - q)y = 0 \tag{2.1}$$

denkleminin

$$y(0, \lambda) = \cos \alpha, \quad y'(0, \lambda) = -\sin \alpha, \quad \alpha = 0 \tag{2.67}$$

başlangıç koşullarını sağlayan çözümü

i)

$$\begin{aligned}
\varphi(x, \lambda) = & \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\
& + O(\lambda^{-1/2} \eta(\lambda))
\end{aligned} \tag{2.68}$$

ii)

$$\begin{aligned}
\varphi(x, n) = & \cos \left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t) dt \right) \\
& - \sin \left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t) dt \right) + O(n^{-2} \eta(n))
\end{aligned} \tag{2.69}$$

şeklindedir.

İspat:

i) (1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x, \lambda) = c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \quad (2.8)$$

ve

$$\begin{aligned} Z'(x, \lambda) &= c_1 S(x, \lambda) \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^x T(t, \lambda) dt\right) T(x, \lambda) \end{aligned} \quad (2.9)$$

şeklindedir [4]. (2.67) ile verilen başlangıç koşulları sağlanacak şekilde c_1 ve c_2 sabitlerini belirleyelim. (2.67) ile verilen başlangıç koşullarından

$$\begin{aligned} Z(0, \lambda) &= \cos\alpha \Rightarrow \cos\alpha = c_1 \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^0 T(t, \lambda) dt\right) \\ \cos\alpha &= c_1 \cos c_2 \Rightarrow 1 = c_1 \cos c_2 \\ &\Rightarrow c_1 = \frac{1}{\cos c_2} \end{aligned} \quad (2.70)$$

$Z'(0, \lambda) = -\sin\alpha$ kullanılırsa

$$\begin{aligned} -\sin\alpha &= c_1 S(0, \lambda) \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^0 T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^0 S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^0 T(t, \lambda) dt\right) T(0, \lambda), \\ -\sin\alpha &= c_1 S(0, \lambda) \cos c_2 - c_1 \sin c_2 T(0, \lambda), \\ 0 &= \frac{1}{\cos c_2} S(0, \lambda) \cos c_2 - \frac{1}{\cos c_2} \sin c_2 T(0, \lambda), \\ S(0, \lambda) &= \tan c_2 T(0, \lambda), \\ c_2 &= \tan^{-1} \left[\frac{S(0, \lambda)}{T(0, \lambda)} \right] \end{aligned} \quad (2.71)$$

olur. (2.71) eşitliği (2.70)'de yerine yazılırsa

$$c_1 = \frac{1}{\cos\left(\tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)} \quad (2.72)$$

bulunur. Son olarak (2.71) ve (2.72) değerleri $Z(x, \lambda)$ 'da yerine yazılırsa ve (2.67) ile verilen başlangıç koşullarını sağlayan çözüme $\varphi(x, \lambda)$ denilirse

$$\begin{aligned} \varphi(x, \lambda) &= \frac{1}{\cos\left(\tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)} \exp\left(\int_0^x S(t, \lambda) dt\right) \\ &\quad \times \cos\left(\tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right] + \int_0^x T(t, \lambda) dt\right) \\ &= \frac{\exp\left(\int_0^x S(t, \lambda) dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)} \left[\cos\left(\tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right) \cos\left(\int_0^x T(t, \lambda) dt\right) \right. \\ &\quad \left. - \sin\left(\tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right) \sin\left(\int_0^x T(t, \lambda) dt\right) \right] \end{aligned}$$

şeklinde elde edilir ve düzenlemeler yapılırsa

$$\begin{aligned} \varphi(x, \lambda) &= \exp\left(\int_0^x S(t, \lambda) dt\right) \cos\left(\int_0^x T(t, \lambda) dt\right) \\ &\quad - \exp\left(\int_0^x S(t, \lambda) dt\right) \left[\frac{S(0,\lambda)}{T(0,\lambda)}\right] \sin\left(\int_0^x T(t, \lambda) dt\right) \end{aligned} \quad (2.73)$$

olarak bulunur.

Natasyonlarla ifade edecek olursak

$$\begin{aligned} \varphi(x, \lambda) &= E_0(x, \lambda)C_0(x, \lambda) - E_0(x, \lambda)S_0(x, \lambda) \left[\frac{S(0,\lambda)}{T(0,\lambda)}\right] \\ &:= I_1 - I_2 \end{aligned}$$

şeklinde olur.

$$\begin{aligned} I_1 &= E_0(x, \lambda)C_0(x, \lambda) \\ &= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \\
& = \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) \\
& \quad \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \\
& \quad \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda))
\end{aligned}$$

$$I_1 = \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda))$$

ve

$$\begin{aligned}
I_2 &= E_0(x, \lambda) S_0(x, \lambda) \left[\frac{S(0, \lambda)}{T(0, \lambda)} \right] \\
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \right. \\
&\quad \left. + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \times [O(\lambda^{-1/2} \eta(\lambda))] \\
&\quad \times \left[\sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \\
&= \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) \\
&\quad \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \\
&\quad \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)) \\
I_2 &= \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda))
\end{aligned}$$

elde edilir. I_1 ve I_2 eşitliklerini $\varphi(x, \lambda)$ 'da yerine yazarsak

$$\begin{aligned}\varphi(x, \lambda) &= \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned}\quad (2.68)$$

bulunur.

ii) (2.54)'ten

$$\begin{aligned}\varphi(x, n) &= \cos\left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t)dt\right) \\ &\quad - \sin\left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t)dt\right) + O(n^{-2}\eta(n))\end{aligned}\quad (2.66)$$

elde edilir.

$$\textbf{Teorem 2.12: } y'' + (\lambda - q)y = 0 \quad (2.1)$$

denkleminin

$$y(\pi, \lambda) = \cos\beta, \quad y'(\pi, \lambda) = -\sin\beta, \quad \beta = 0 \quad (2.74)$$

başlangıç koşullarını sağlayan çözümü

i)

$$\begin{aligned}\psi(x, \lambda) &= \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ &\quad + \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned}\quad (2.75),$$

ii)

$$\begin{aligned}\psi(x, n) &= \cos\left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t)dt\right) \\ &\quad + \sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t)dt\right) + O(n^{-2}\eta(n))\end{aligned}\quad (2.76)$$

ile verilir.

İspat:

i) (2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x, \lambda) = c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \quad (2.8)$$

ve

$$\begin{aligned} Z'(x, \lambda) &= c_1 S(x, \lambda) \exp\left(\int_0^x S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^x T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^x S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^x T(t, \lambda) dt\right) T(x, \lambda) \end{aligned} \quad (2.9)$$

şeklindedir [4]. (2.74) ile verilen başlangıç koşulları sağlanacak şekilde c_1 ve c_2 sabitlerini belirleyelim. (2.74) ile verilen başlangıç koşullarından

$$\begin{aligned} Z(\pi, \lambda) &= \cos\beta \\ 1 &= c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \end{aligned} \quad (2.77)$$

$Z'(\pi, \lambda) = -\sin\beta$ kullanılırsa

$$\begin{aligned} 0 &= c_1 S(\pi, \lambda) \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \\ &\quad - c_1 \exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \sin\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) T(\pi, \lambda) \end{aligned} \quad (2.78)$$

(2.77) eşitliğini (2.78)'de yerine yazarsak

$$\begin{aligned} 0 &= \frac{S(\pi, \lambda) \exp\left(\int_0^\pi S(t, \lambda) dt\right)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)} \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) \\ &\quad - \frac{\exp\left(\int_0^\pi S(t, \lambda) dt\right)}{\exp\left(\int_0^\pi S(t, \lambda) dt\right) \times \cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)} \times \sin\left(c_2 + \int_0^\pi T(t, \lambda) dt\right) T(\pi, \lambda) \\ &= S(\pi, \lambda) - \frac{\sin\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)}{\cos\left(c_2 + \int_0^\pi T(t, \lambda) dt\right)} T(\pi, \lambda), \end{aligned}$$

$$\frac{S(\pi, \lambda)}{T(\pi, \lambda)} = \tan \left(c_2 + \int_0^\pi T(t, \lambda) dt \right)$$

bulunur ve

$$c_2 = \tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt \quad (2.79)$$

olarak elde edilir. (2.79) eşitliğini (2.77)'de yerine yazarsak

$$1 = c_1 \exp \left(\int_0^\pi S(t, \lambda) dt \right) \times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^\pi T(t, \lambda) dt \right)$$

$$c_1 = \frac{1}{\exp \left(\int_0^\pi S(t, \lambda) dt \right) \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \quad (2.80)$$

bulunur. Son olarak (2.79) ve (2.80) değerleri $Z(x, \lambda)$ 'da yerine yazılırsa ve (2.74) ile verilen başlangıç koşullarını sağlayan çözüme $\psi(x, \lambda)$ denilirse

$$\psi(x, \lambda) = \frac{\exp \left(\int_0^x S(t, \lambda) dt \right)}{\exp \left(\int_0^\pi S(t, \lambda) dt \right) \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)}$$

$$\times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt \right)$$

olur, sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\psi(x, \lambda) = \frac{\exp \left(- \int_x^\pi S(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_x^\pi T(t, \lambda) dt \right)$$

$$= \frac{\exp \left(- \int_x^\pi S(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \times \left[\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right) \cos \left(\int_x^\pi T(t, \lambda) dt \right) \right.$$

$$\left. + \sin \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right) \sin \left(\int_x^\pi T(t, \lambda) dt \right) \right]$$

$$\psi(x, \lambda) = \exp \left(- \int_x^\pi S(t, \lambda) dt \right)$$

$$\times \left[\cos \left(\int_x^\pi T(t, \lambda) dt \right) + \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \sin \left(\int_x^\pi T(t, \lambda) dt \right) \right] \quad (2.81)$$

elde ederiz.

Natasyonlarla ifade edecek olursak

$$\psi(x, \lambda) = E_\pi(x, \lambda)C_\pi(x, \lambda) + E_\pi(x, \lambda)S_\pi(x, \lambda) \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right]$$

$$:= I_1 - I_2$$

şeklinde olur. Buradan

$$\begin{aligned} I_1 &= E_\pi(x, \lambda)C_\pi(x, \lambda) \\ &= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\ &\quad \times \left[\cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\ &= \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-3/2}\eta(\lambda)) \\ I_1 &= \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)) \end{aligned}$$

şeklindedir ve

$$\begin{aligned} I_2 &= E_\pi(x, \lambda)S_\pi(x, \lambda) \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \\ &= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[O(\lambda^{-1/2}\eta(\lambda)) \right] \\ &\quad \times \left[\sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \end{aligned}$$

$$\begin{aligned}
&= \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-3/2}\eta(\lambda)\right) \\
I_2 &= \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

şeklinde elde edilir. I_1 ve I_2 eşitlikleri $\psi(x, \lambda)$ 'da yerine yazılırsa

$$\begin{aligned}
\psi(x, \lambda) &= \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
&\quad + \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned} \tag{2.75}$$

bulunur.

ii) (2.54)'ten

$$\begin{aligned}
\psi(x, n) &= \cos\left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t)dt\right) \\
&\quad + \sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t)dt\right) + O(n^{-2}\eta(n))
\end{aligned} \tag{2.76}$$

elde edilir.

Teorem 2.13: Neumann sınır şartları altında $\lambda \rightarrow \infty$ için

$$\begin{aligned}
w(\lambda) &= w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \\
&= 2\lambda^{1/2}\sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{2.82}$$

ile verilir.

İspat: $\varphi'(x, \lambda)$ 'yı bulmak için (2.71) ve (2.72)'yi $Z'(x, \lambda)$ 'da yerine yazalım.

$$Z'(x, \lambda) = \frac{S(x, \lambda)\exp\left(\int_0^x S(t, \lambda)dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]\right)} \times \cos\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right] + \int_0^x T(t, \lambda)dt\right)$$

$$\begin{aligned}
& -\frac{\exp(\int_0^x S(t, \lambda) dt)}{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]\right)} \times \sin\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right] + \int_0^x T(t, \lambda) dt\right) T(x, \lambda) \\
& = S(x, \lambda) E_0(x, \lambda) \left[\frac{\cos\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]\right) \times \cos(\int_0^x T(t, \lambda) dt)}{-\sin\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]\right) \times \sin(\int_0^x T(t, \lambda) dt)} \right] \\
& \quad - E_0(x, \lambda) \left[\frac{\sin\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]\right) \times \cos(\int_0^x T(t, \lambda) dt)}{+\cos\left(\tan^{-1}\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]\right) \times \sin(\int_0^x T(t, \lambda) dt)} \right] T(x, \lambda)
\end{aligned}$$

şeklinde olur ve notasyonlar yardımıyla

$$\begin{aligned}
& = S(x, \lambda) E_0(x, \lambda) \left[\cos\left(\int_0^x T(t, \lambda) dt\right) - \left[\frac{S(0, \lambda)}{T(0, \lambda)}\right] \sin\left(\int_0^x T(t, \lambda) dt\right) \right] \\
& \quad - E_0(x, \lambda) \left[\left[\frac{S(0, \lambda)}{T(0, \lambda)}\right] \cos\left(\int_0^x T(t, \lambda) dt\right) + \sin\left(\int_0^x T(t, \lambda) dt\right) \right] T(x, \lambda)
\end{aligned}$$

$$Z'(x, \lambda) = S(x, \lambda) E_0(x, \lambda) C_0(x, \lambda) - S(x, \lambda) E_0(x, \lambda) S_0(x, \lambda) \left[\frac{S(0, \lambda)}{T(0, \lambda)}\right]$$

$$-E_0(x, \lambda) C_0(x, \lambda) T(x, \lambda) \left[\frac{S(0, \lambda)}{T(0, \lambda)}\right] - E_0(x, \lambda) S_0(x, \lambda) T(x, \lambda)$$

gösterilir.

$$Z'(x, \lambda) := I_1 - I_2 - I_3 - I_4$$

olarak tanımlarsak

$$I_1 = S(x, \lambda) E_0(x, \lambda) C_0(x, \lambda)$$

$$\begin{aligned}
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \right. \\
&\quad \left. + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \\
&\quad \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&= -\sin(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
&\quad - \frac{1}{2\lambda^{1/2}} \sin(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
&\quad \times \cos(2\lambda^{1/2}x + \xi_x) + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \\
&\quad \times \sin(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)) \\
I_1 &= O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

olarak bulunur. İkinci olarak

$$\begin{aligned}
I_2 &= S(x, \lambda)E_0(x, \lambda)S_0(x, \lambda) \left[\frac{S(0, \lambda)}{T(0, \lambda)} \right] \\
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [O(\lambda^{-1/2}\eta(\lambda))] \\
&= -\sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
&\quad - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x)
\end{aligned}$$

$$\begin{aligned} & \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \times \\ & \times \sin(2\lambda^{1/2}x + \xi_x) \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned}$$

$$I_2 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. Üçüncü olarak

$$\begin{aligned} I_3 &= E_0(x, \lambda) C_0(x, \lambda) T(x, \lambda) \left[\frac{S(0, \lambda)}{T(0, \lambda)} \right] \\ &= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\ &\quad \times \left[\cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right) \right] \\ &\quad \times \left[\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2) \right] \\ &= \lambda^{1/2} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{1}{2} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad \times \cos(2\lambda^{1/2}x + \xi_x) - \frac{1}{2} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \\ &\quad \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad \times \cos(2\lambda^{1/2}x + \xi_x) \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ &\quad + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2}t)q(t)dt \times \cos(2\lambda^{1/2}x + \xi_x) \\ &\quad \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned}$$

$$I_3 = \lambda^{1/2} \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right)$$

hesaplanır ve son olarak

$$\begin{aligned}
I_4 &= E_0(x, \lambda) S_0(x, \lambda) T(x, \lambda) \\
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) - \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \\
&\quad \times \left[\sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2} x + \xi_x) + O(\eta(\lambda)^2)] \\
&= \lambda^{1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + \frac{1}{2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\
&\quad \times \cos(2\lambda^{1/2} x + \xi_x) - \frac{1}{2} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \\
&\quad \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \cos(2\lambda^{1/2} x + \xi_x) \\
&\quad \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2} x + \xi_x) \\
&\quad \times \cos(2\lambda^{1/2} x + \xi_x) \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\
&\quad + \frac{1}{2\lambda^{1/2}} \int_0^\pi \sin(2\lambda^{1/2} t) q(t) dt \times \cos(2\lambda^{1/2} x + \xi_x) \\
&\quad \times \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right) \\
I_4 &= \lambda^{1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-1/2} \eta(\lambda) \right)
\end{aligned}$$

elde edilir. I_1, I_2, I_3 ve I_4 değerleri $\varphi'(x, \lambda)$ 'da yerine yazılırsa

$$\begin{aligned}\varphi'(x, \lambda) = & -\lambda^{1/2} \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \\ & -\lambda^{1/2} \sin \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) + O \left(\lambda^{-3/2} \eta(\lambda) \right)\end{aligned}\quad (2.83)$$

bulunur.

$\psi'(x, \lambda)$ 'yı elde etmek için (2.79) ve (2.80)'i $Z'(x, \lambda)$ 'da yerine yazalım.

$$\begin{aligned}Z'(x, \lambda) = & \frac{S(x, \lambda) \exp \left(\int_0^x S(t, \lambda) dt \right)}{\exp \left(\int_0^\pi S(t, \lambda) dt \right) \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \\ & \times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt \right) \\ & - \frac{\exp \left(\int_0^x S(t, \lambda) dt \right)}{\exp \left(\int_0^\pi S(t, \lambda) dt \right) \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \\ & \times \sin \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_0^\pi T(t, \lambda) dt + \int_0^x T(t, \lambda) dt \right) T(x, \lambda)\end{aligned}$$

olur ve sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\begin{aligned}Z'(x, \lambda) = & \frac{S(x, \lambda) \exp \left(- \int_x^\pi S(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \times \cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_x^\pi T(t, \lambda) dt \right) \\ & - \frac{\exp \left(- \int_x^\pi S(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \times \sin \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] - \int_x^\pi T(t, \lambda) dt \right) T(x, \lambda) \\ = & S(x, \lambda) \exp \left(- \int_x^\pi S(t, \lambda) dt \right) \left[\frac{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right) \cos \left(\int_x^\pi T(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \right. \\ & \left. + \frac{\sin \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right) \sin \left(\int_x^\pi T(t, \lambda) dt \right)}{\cos \left(\tan^{-1} \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \right)} \right]\end{aligned}$$

$$\begin{aligned}
& -\exp\left(-\int_x^\pi S(t, \lambda) dt\right) \left[\frac{\sin\left(\tan^{-1}\left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)}\right]\right) \cos\left(\int_x^\pi T(t, \lambda) dt\right)}{\cos\left(\tan^{-1}\left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)}\right]\right)} \right] T(x, \lambda) \\
& = S(x, \lambda) \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \\
& \quad \times \left[\cos\left(\int_x^\pi T(t, \lambda) dt\right) + \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)}\right] \sin\left(\int_x^\pi T(t, \lambda) dt\right) \right] \\
& \quad - \exp\left(-\int_x^\pi S(t, \lambda) dt\right) \\
& \quad \times \left[\left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)}\right] \cos\left(\int_x^\pi T(t, \lambda) dt\right) - \sin\left(\int_x^\pi T(t, \lambda) dt\right) \right] T(x, \lambda)
\end{aligned}$$

elde edilir. Notasyonlarla ifade edecek olursak

$$\begin{aligned}
Z'(x, \lambda) &= S(x, \lambda) E_\pi(x, \lambda) C_\pi(x, \lambda) + S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)}\right] \\
&\quad - E_\pi(x, \lambda) C_\pi(x, \lambda) T(x, \lambda) \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)}\right] + E_\pi(x, \lambda) S_\pi(x, \lambda) T(x, \lambda)
\end{aligned}$$

şeklinde buluruz.

$$Z'(x, \lambda) := I_1 - I_2 - I_3 - I_4$$

olarak tanımlarsak, I_1 değerimiz

$$\begin{aligned}
I_1 &= S(x, \lambda) E_\pi(x, \lambda) C_\pi(x, \lambda) \\
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)]
\end{aligned}$$

$$\begin{aligned}
& \times \left[\cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
& = -\sin(2\lambda^{1/2}x + \xi_x) \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& \quad - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \\
& \quad \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)) \\
I_1 & = O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

şeklinde hesaplanır. İkinci olarak I_2 değerimiz

$$\begin{aligned}
I_2 & = S(x, \lambda) E_\pi(x, \lambda) S_\pi(x, \lambda) \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \\
& = \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
& \quad \times [-\sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \times [O(\lambda^{-1/2}\eta(\lambda))] \\
& \quad \times \left[\sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
& = -\sin(2\lambda^{1/2}x + \xi_x) \times \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\
& \quad - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \times \sin(2\lambda^{1/2}x + \xi_x) \\
& \quad \times \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda)) \\
I_2 & = O(\lambda^{-1/2}\eta(\lambda)),
\end{aligned}$$

üçüncü olarak I_3 değerimiz

$$\begin{aligned}
I_3 &= E_\pi(x, \lambda) C_\pi(x, \lambda) T(x, \lambda) \left[\frac{S(\pi, \lambda)}{T(\pi, \lambda)} \right] \\
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \times \left[O(\lambda^{-1/2}\eta(\lambda)) \right] \\
&\quad \times \left[\cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \\
&= \lambda^{1/2} \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + \frac{1}{2} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) - \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \times \cos(2\lambda^{1/2}x + \xi_x) \\
&\quad + O(\lambda^{-1/2}\eta(\lambda)) \\
I_3 &= \lambda^{1/2} \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda))
\end{aligned}$$

ve son olarak I_4 eşitliğimiz

$$\begin{aligned}
I_4 &= E_\pi(x, \lambda) S_\pi(x, \lambda) T(x, \lambda) \\
&= \left[1 + \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times \left[\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \right] \\
&\quad \times [\lambda^{1/2} - \cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)] \\
&= \lambda^{1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + \frac{1}{2} \cos(2\lambda^{1/2}x + \xi_x)
\end{aligned}$$

$$\begin{aligned}
& \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) - \cos(2\lambda^{1/2}x + \xi_x) \\
& \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}} \cos(2\lambda^{1/2}x + \xi_x) \\
& \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \times \cos(2\lambda^{1/2}x + \xi_x) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \\
I_4 &= \lambda^{1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

şeklinde hesaplanmış olur. I_1, I_2, I_3 ve I_4 değerleri $\psi'(x, \lambda)$ 'da yerine yazılırsa

$$\begin{aligned}
\psi'(x, \lambda) &= -\lambda^{1/2} \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\
&+ \lambda^{1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-3/2}\eta(\lambda)\right) \quad (2.84)
\end{aligned}$$

elde edilir.

Bulduğumuz (2.68),(2.75),(2.80) ve (2.84) eşitlikleri $w_x(\varphi, \psi)$ 'da yerine yazılırsa

$$\begin{aligned}
w_x(\varphi, \psi) &= \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \\
&= \begin{bmatrix} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ -\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} -\lambda^{1/2} \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ +\lambda^{1/2} \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \\
&- \begin{bmatrix} -\lambda^{1/2} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ -\lambda^{1/2} \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ +\sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}
\end{aligned}$$

gerekli hesaplamalardan sonra

$$\begin{aligned}
&= \left[\begin{aligned} &-\lambda^{1/2} \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \times \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \quad (1a) \\ &+\lambda^{1/2} \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \times \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \quad (1b) \\ &+\lambda^{1/2} \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \quad (1b) \\ &-\lambda^{1/2} \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \times \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \quad (1a) \end{aligned} \right] \\
&+ \left[\begin{aligned} &+\lambda^{1/2} \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \quad (1c) \\ &+\lambda^{1/2} \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \times \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \quad (1d) \\ &+\lambda^{1/2} \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \times \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \quad (1d) \\ &+\lambda^{1/2} \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \times \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \quad (1c) \end{aligned} \right] \\
&+O(\lambda^{-1}\eta(\lambda))
\end{aligned}$$

elde edilir ve kosinüs ve sinüs toplam fark formülleri kullanırsa $w(\lambda)$ eşitliği

$$\begin{aligned}
w(\lambda) = &-\lambda^{1/2} \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt + \lambda^{1/2}x \right. \\
&\left. - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \quad (1a)
\end{aligned}$$

$$\begin{aligned}
&+\lambda^{1/2} \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt + \lambda^{1/2}x \right. \\
&\left. - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right) \quad (1b)
\end{aligned}$$

$$\begin{aligned}
&+\lambda^{1/2} \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt + \lambda^{1/2}(\pi - x) \right. \\
&\left. - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \quad (1c)
\end{aligned}$$

$$\begin{aligned}
& +\lambda^{1/2}\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) \right. \\
& \quad \left. - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\
& +O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{1d}$$

ve gerekli düzenlemeler yapılırsa $w(\lambda)$ eşitliğimiz

$$\begin{aligned}
w(\lambda) &= -\lambda^{1/2}\cos\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right) \\
& +\lambda^{1/2}\sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right) + \lambda^{1/2}\cos\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right) \\
& +\lambda^{1/2}\sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right) + O(\lambda^{-1}\eta(\lambda)) \\
w(\lambda) &= 2\lambda^{1/2}\sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{2.82}$$

elde edilir.

Teorem 2.14:

i) Neumann sınır şartları altında $\lambda \rightarrow \infty$ ve $0 \leq x \leq y \leq \pi$ için

$$\begin{aligned}
G(x, y, \lambda) &= \\
&= \frac{\begin{pmatrix} \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \end{pmatrix} \times \begin{pmatrix} \cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +\sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \end{pmatrix}}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\
& +O(\lambda^{-1/2}\eta(\lambda))
\end{aligned} \tag{2.85}$$

ii)

$$\begin{aligned}
G(x, y, n) &= \\
&= \cos \left(-\frac{(n+1)\pi + (n+1)(x-y)}{2((n+1) + O(\eta(n)))} \left(\int_0^x q(t)dt + \int_y^\pi q(t)dt \right) \right) \\
&\quad + \sin \left(+\frac{(n+1)(\pi-y) - (n+1)x}{2((n+1) + O(\eta(n)))} \left(\int_0^x q(t)dt - \int_y^\pi q(t)dt \right) \right) \\
&\quad + O(n^{-3}\eta(n)) \quad (2.86)
\end{aligned}$$

ile verilir.

İspat:

i)

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O(\lambda^{-k}\eta(\lambda)), & 0 \leq y \leq x \leq \pi \end{cases}$$

şeklinde idi. Bulduğumuz (2.82), (2.83) ve (2.84)'ü Green fonksiyonunda yerine yazarsak

 $0 \leq x \leq y \leq \pi$ için

$$\begin{aligned}
G(x, y, \lambda) &= \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} \\
&= \frac{\begin{pmatrix} \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) \\ -\sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) \end{pmatrix} \times \begin{pmatrix} \cos \left(\lambda^{1/2}(\pi-y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt \right) \\ +\sin \left(\lambda^{1/2}(\pi-y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt \right) \end{pmatrix}}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\
&\quad + O(\lambda^{-1/2}\eta(\lambda)) \quad (2.85)
\end{aligned}$$

buluruz. Benzer şekilde $0 \leq y \leq x \leq \pi$ içinde

$$G(x, y, \lambda) = \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)}$$

elde edilir.

ii) Bulduğumuz $G(x, y, \lambda)$ 'ya reversion uygularsak

$$G(x, y, \lambda) =$$

$$\begin{aligned}
& \left[\begin{aligned} & \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ & + \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ & - \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \\ & - \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \times \sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}} \int_y^\pi q(t)dt\right) \end{aligned} \right] \\
& = \frac{\left[\begin{aligned} & + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right]}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi) \times [1 + O(\lambda^{-1}\eta(\lambda))]} \\
& \left[\begin{aligned} & \cos\left(\lambda^{1/2}\pi + \lambda^{1/2}(x - y) - \frac{1}{2\lambda^{1/2}} \left(\int_0^x q(t)dt + \int_y^\pi q(t)dt\right)\right) \\ & + \sin\left(\lambda^{1/2}(\pi - y) - \lambda^{1/2}x + \frac{1}{2\lambda^{1/2}} \left(\int_0^x q(t)dt - \int_y^\pi q(t)dt\right)\right) \end{aligned} \right] \\
& = \frac{\left[\begin{aligned} & + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right]}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\
& \times [1 + O(\lambda^{-1}\eta(\lambda))] \\
& \left[\begin{aligned} & \cos\left(\lambda^{1/2}\pi + \lambda^{1/2}(x - y) - \frac{1}{2\lambda^{1/2}} \left(\int_0^x q(t)dt + \int_y^\pi q(t)dt\right)\right) \\ & + \sin\left(\lambda^{1/2}(\pi - y) - \lambda^{1/2}x + \frac{1}{2\lambda^{1/2}} \left(\int_0^x q(t)dt - \int_y^\pi q(t)dt\right)\right) \end{aligned} \right] \\
& = \frac{\left[\begin{aligned} & + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \right]}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\
& \times O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

elde edilir ve (2.54)'ten

$$G(x, y, n) =$$

$$= \frac{\left[\begin{aligned} &\cos \left(-\frac{(n+1)\pi + (n+1)(x-y)}{2((n+1) + O(\eta(n)))} \left(\int_0^x q(t)dt + \int_y^\pi q(t)dt \right) \right) \\ &+ \sin \left(+\frac{(n+1)(\pi - y) - (n+1)x}{2((n+1) + O(\eta(n)))} \left(\int_0^x q(t)dt - \int_y^\pi q(t)dt \right) \right) \\ &+ O(n^{-3}\eta(n)) \end{aligned} \right]}{2(n+1) \times \sin((n+1)\pi)}$$

bulunur. Burada $\sin((n+1)\pi + O(\eta(n)))$ değerini hesaplayıp $G(x, y, n)$ 'de yerine yazalım.

$$\begin{aligned} \sin((n+1)\pi + O(\eta(n))) &= \\ &= \cos((n+1)\pi) \sin(O(\eta(n))) \\ &\quad + \sin((n+1)\pi) \cos(o(\eta(n))) \\ &= O(\eta(n)) \end{aligned}$$

olarak elde edilir.

$$\begin{aligned} G(x, y, n) &= \\ &= \cos \left(-\frac{(n+1)\pi + (n+1)(x-y)}{2((n+1) + O(\eta(n)))} \left(\int_0^x q(t)dt + \int_y^\pi q(t)dt \right) \right) \\ &\quad + \sin \left(+\frac{(n+1)(\pi - y) - (n+1)x}{2((n+1) + O(\eta(n)))} \left(\int_0^x q(t)dt - \int_y^\pi q(t)dt \right) \right) \\ &\quad + O(n^{-3}\eta(n)) \end{aligned} \tag{2.86}$$

bulunur.

3. POTANSİYEL FONKSİYONUN $q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$ OLMASI DURUMU

Teorem 3.1: $\lambda \rightarrow \infty$

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \quad (3.1)$$

denkleminin

$$y(0, \lambda) = \cos \alpha, \quad y'(0, \lambda) = -\sin \alpha \quad \alpha \in [0, \pi] \quad (3.2)$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned} \varphi(x, \lambda) = & \cos \alpha \times \cos \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \\ & - \cos \alpha \times \sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \\ & - \lambda^{-1/2} \sin \alpha \times \sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) + O(\lambda^{-1/2} \eta(\lambda)) \end{aligned} \quad (3.3)$$

ile verilir.

İspat: Bir önceki bölümde,

$$\int_0^{\pi} q(t) dt = 0$$

olmak koşuluyla

$$y'' + (\lambda - q)y = 0, \quad x \in [0, \pi]$$

$$y(0, \lambda) = \cos \alpha, \quad y'(0, \lambda) = -\sin \alpha, \quad \alpha \in [0, \pi]$$

probleminin çözümünü

$$\varphi(x, \lambda) = \cos \alpha \times \cos \left(\lambda^{1/2} x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t) dt \right)$$

$$\begin{aligned}
& -\cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) \\
& -\lambda^{-1/2}\sin\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.39)
\end{aligned}$$

olarak bulmuştuk. Öncelikle

$$\int_0^\pi \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right) dx = 0$$

sağlanır. $\int_0^x q(t)dt$ değeri hesaplanırsa

$$\int_0^x q(t)dt = \int_0^x \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right) dx = \left(2x^{1/2} - \frac{2x}{\sqrt{\pi}}\right) \Big|_0^x = 2\sqrt{x} - \frac{2x}{\sqrt{\pi}}$$

elde ederiz. Bulduğumuz değeri (2.39)'da yerine yazarsak

$$\begin{aligned}
\varphi(x, \lambda) &= \cos\alpha \times \cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right)\right) \\
& -\cos\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right)\right) \\
& -\lambda^{-1/2}\sin\alpha \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right)\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)
\end{aligned}$$

gerekli düzenlemeler yapılsa

$$\begin{aligned}
\varphi(x, \lambda) &= \cos\alpha \times \cos\left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\
& -\cos\alpha \times \sin\left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\
& -\lambda^{-1/2}\sin\alpha \times \sin\left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (3.3)
\end{aligned}$$

olarak bulunur.

Teorem 3.3: $\lambda \rightarrow \infty$

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \quad (3.1)$$

denkleminin

$$y(\pi, \lambda) = \cos \beta, \quad y'(\pi, \lambda) = -\sin \beta \quad \beta \in [0, \pi] \quad (3.4)$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned} \psi(x, \lambda) = & \cos \beta \times \cos \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ & + \cos \beta \times \sin \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ & + \lambda^{-1/2} \sin \beta \times \sin \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ & + O \left(\lambda^{-1/2} \eta(\lambda) \right) \end{aligned} \quad (3.5)$$

ile hesaplanır.

İspat: Bir önceki bölümde

$$\begin{aligned} \psi(x, \lambda) = & \cos \beta \times \cos \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\ & + \cos \beta \times \sin \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\ & + \lambda^{-1/2} \sin \beta \times \sin \left(\lambda^{1/2} (\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\ & + O \left(\lambda^{-1/2} \eta(\lambda) \right) \end{aligned} \quad (2.40)$$

olarak bulmuştuk. Buradaki $\int_x^\pi q(t)dt$ değerini hesaplayalım.

$$\begin{aligned}\int_x^\pi q(t)dt &= \int_x^\pi \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right) dx = \left(2x^{1/2} - \frac{2x}{\sqrt{\pi}}\right) \Big|_x^\pi \\ &= \left(2\sqrt{\pi} - \frac{2\pi}{\sqrt{\pi}}\right) - \left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right) = \frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\end{aligned}$$

elde ederiz. Bulduğumuz değeri (2.40)'da yerine yazarsak

$$\begin{aligned}\psi(x, \lambda) &= \cos\beta \times \cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\left(\frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\right)\right) \\ &\quad + \cos\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\left(\frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\right)\right) \\ &\quad + \lambda^{-1/2}\sin\beta \times \sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\left(\frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\right)\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned}$$

gerekli düzenlemeler yapılırsa

$$\begin{aligned}\psi(x, \lambda) &= \cos\beta \times \cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}}\right]\right) \\ &\quad + \cos\beta \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}}\right]\right) \\ &\quad + \lambda^{-1/2}\sin\beta \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}}\right]\right) \\ &\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned}\tag{3.5}$$

elde ederiz.

Teorem 3.4: $\lambda \rightarrow \infty$ için, (3.2) ve (3.4) koşulları altında

$$\begin{aligned}w(\lambda) &= w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \\ &= 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi)\end{aligned}$$

$$\begin{aligned}
& +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\
& +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] + O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{3.6}$$

ile verilir.

İspat: (3.3) ve (3.5) de bulunan $\varphi(x, \lambda)$ ve $\psi(x, \lambda)$ kullanılırsa

$$w(\lambda) = w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

olduğundan

$$\begin{aligned}
w(\lambda) &= 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\
& +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\
& +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] + O(\lambda^{-1}\eta(\lambda))
\end{aligned} \tag{3.6}$$

elde edilir. Dikkat edilirse $w(\lambda)$ eşitliğimiz $q(t)$ fonksiyonuna bağlı değildir.

Teorem 3.5:

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \tag{3.1}$$

denkleminin

$$y(0, \lambda) = \cos\alpha, \quad y'(0, \lambda) = -\sin\alpha \quad \alpha \in [0, \pi]$$

ve

$$y(\pi, \lambda) = \cos\beta, \quad y'(\pi, \lambda) = -\sin\beta \quad \beta \in [0, \pi]$$

için Green fonksiyonu ($\lambda \rightarrow \infty$)

$$G(x, y, \lambda) =$$

$$\begin{aligned}
& \begin{bmatrix} \cos\alpha \\ \times \cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\cos\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\lambda^{-1/2}\sin\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \end{bmatrix} \\
&= \frac{\begin{bmatrix} 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\ +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\ +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] \end{bmatrix}}{1} \\
&\quad + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.7}
\end{aligned}$$

olarak bulunur.

İspat:

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0 \leq y \leq x \leq \pi \end{cases}$$

dir. $0 \leq x \leq y \leq \pi$ için bulduğumuz $\varphi(x, \lambda), \psi(x, \lambda)$ ve $w(\lambda)$ 'yı $G(x, y, \lambda)$ 'da yerine yazarsak

$$G(x, y, \lambda) =$$

$$\begin{aligned}
& \begin{bmatrix} \cos\alpha \\ \times \cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\cos\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\lambda^{-1/2}\sin\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \end{bmatrix} \\
&= \frac{\begin{bmatrix} 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\ +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\ +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] \end{bmatrix}}{1}
\end{aligned}$$

$$+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (3.7)$$

elde ederiz. $0 \leq y \leq x \leq \pi$ için de aynı işlemler yapılır.

Aşağıdaki teoremlerde Dirichlet koşulları altında Green fonksiyonu elde edilecektir.

Teorem 3.6:

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \quad (3.1)$$

denkleminin

$$y(0, \lambda) = \cos\alpha, \quad y'(0, \lambda) = -\sin\alpha \quad \alpha = \frac{\pi}{2} \quad (3.8)$$

başlangıç koşullarını sağlayan çözümü ($\lambda \rightarrow \infty$)

i)

$$\varphi(x, \lambda) = -\lambda^{-1/2} \times \sin\left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (3.9)$$

ii)

$$\begin{aligned} \varphi(x, n) = & -\frac{1}{(n+1)} \times \sin\left(\frac{1}{(n+1)} \left[\frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ & + O(n^{-2}\eta(n)) \end{aligned} \quad (3.10)$$

ile hesaplanır.

İspat:

i) Daha önce

$$\varphi(x, \lambda) = -\lambda^{-1/2} \times \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.49)$$

olarak bulmuştuk.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x, \lambda)$ 'da yerine yazılırsa

$$\varphi(x, \lambda) = -\lambda^{-1/2} \sin \alpha \times \sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) + O(\lambda^{-1/2} \eta(\lambda)) \quad (3.9)$$

elde ederiz.

ii) Önceki bölümde

$$\begin{aligned} \varphi(x, n) = & -\frac{1}{(n+1)} \times \sin \left((n+1)x - \frac{1}{2(n+1)} \int_0^x q(t) dt \right) \\ & + O(n^{-2} \eta(n)) \end{aligned} \quad (2.46)$$

olarak hesaplamıştık.

$\int_0^x q(t) dt$ hesaplanır ve $\varphi(x, n)$ 'de yerine yazılırsa

$$\begin{aligned} \varphi(x, n) = & -\frac{1}{(n+1)} \times \sin \left(\frac{1}{(n+1)} \left[\frac{((n+1)^2 \sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \\ & + O(n^{-2} \eta(n)) \end{aligned} \quad (3.10)$$

elde ederiz.

Teorem 3.7:

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \quad (3.1)$$

denkleminin

$$y(\pi, \lambda) = \cos \beta, \quad y'(\pi, \lambda) = -\sin \beta \quad \beta = \frac{\pi}{2} \quad (3.11)$$

başlangıç koşullarını sağlayan çözümü ($\lambda \rightarrow \infty$)

i)

$$\psi(x, \lambda) = \lambda^{-1/2} \sin \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) + O(\lambda^{-1/2} \eta(\lambda)) \quad (3.12)$$

ii)

$$\begin{aligned}\psi(x, n) &= \frac{1}{(n+1)} \times \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{x} - \frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ &\quad + O(n^{-2}\eta(n))\end{aligned}\tag{3.13}$$

ile bulunur.

İspat:

i) Bir önceki bölümde

$$\psi(x, \lambda) = \lambda^{-1/2} \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2}\eta(\lambda))\tag{2.53}$$

olarak bulmuştuk. $\int_x^\pi q(t) dt$ hesaplanır ve $\psi(x, \lambda)$ 'da yerine yazılırsa

$$\psi(x, \lambda) = \lambda^{-1/2} \sin \left(\frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) + O(\lambda^{-1/2}\eta(\lambda))\tag{3.12}$$

elde ederiz.

ii) Daha önce

$$\begin{aligned}\psi(x, n) &= \frac{1}{(n+1)} \times \sin \left((n+1)(\pi - x) - \frac{1}{2(n+1)} \int_x^\pi q(t) dt \right) \\ &\quad + O(n^{-2}\eta(n))\end{aligned}\tag{2.54}$$

olarak hesaplamıştık.

$\int_x^\pi q(t) dt$ hesaplanır ve $\psi(x, n)$ 'de yerine yazılırsa

$$\begin{aligned}\psi(x, n) &= \frac{1}{(n+1)} \times \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{x} - \frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ &\quad + O(n^{-2}\eta(n))\end{aligned}\tag{3.13}$$

elde ederiz.

Teorem 3.8: $\lambda \rightarrow \infty$, (3.8) ve (3.11) koşulları altında

$$\begin{aligned} w(\lambda) &= w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \\ &= \lambda^{-1/2} \sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda)) \end{aligned} \quad (3.14)$$

olarak bulunur.

İspat: (3.9) ve (3.12)'de bulunan $\varphi(x, \lambda)$ ve $\psi(x, \lambda)$ kullanılırsa

$$w(\lambda) = w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

olduğundan

$$w(\lambda) = \lambda^{-1/2} \sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda)) \quad (3.14)$$

elde edilir. Dikkat edilirse $w(\lambda)$ eşitliğimiz $q(t)$ fonksiyonuna bağlı değildir.

Teorem 3.9:

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \quad (3.1)$$

denkleminin

$$y(0, \lambda) = 0, \quad y(\pi, \lambda) = 0$$

sınır değer problemi için Green fonksiyonu ($\lambda \rightarrow \infty$)

i)

$$G(x, y, \lambda) = \frac{\left[\sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \times \sin \left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{y} - \frac{(\lambda\sqrt{\pi} + 1)y}{\sqrt{\pi}} \right] \right) \right]}{\lambda^{1/2} \sin(\lambda^{1/2}\pi)} + O(\lambda^{-1/2}\eta(\lambda)) \quad (3.15)$$

ii)

$$\begin{aligned}
G(x, y, n) &= \frac{1}{(n+1)} \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{x} - \frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\
&\quad \times \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{y} - \frac{((n+1)^2\sqrt{\pi} + 1)y}{\sqrt{\pi}} \right] \right) \\
&\quad + O(n^{-3}\eta(n))
\end{aligned} \tag{3.16}$$

ile verilir.

İspat:

i)

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O(\lambda^{-1/2}\eta(\lambda)), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O(\lambda^{-1/2}\eta(\lambda)), & 0 \leq y \leq x \leq \pi \end{cases}$$

dir. $0 \leq x \leq y \leq \pi$ için bulduğumuz $\varphi(x, \lambda), \psi(x, \lambda)$ ve $w(\lambda)$ 'yi $G(x, y, \lambda)$ 'da yerine yazarsak

$$G(x, y, \lambda) = \frac{\left[\sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \times \sin \left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{y} - \frac{(\lambda\sqrt{\pi} + 1)y}{\sqrt{\pi}} \right] \right) \right]}{\lambda^{1/2} \sin(\lambda^{1/2}\pi)} + O(\lambda^{-1/2}\eta(\lambda)) \tag{3.15}$$

elde ederiz. $0 \leq y \leq x \leq \pi$ için de aynı işlemler yapılır.

ii) Bulduğumuz $G(x, y, \lambda)$ 'ya reversion uygularsak

$$G(x, y, n) = \frac{1}{(n+1)} \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{x} - \frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right)$$

$$\begin{aligned} & \times \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{y} - \frac{((n+1)^2\sqrt{\pi}+1)y}{\sqrt{\pi}} \right] \right) \\ & + O(n^{-3}\eta(n)) \end{aligned} \quad (3.16)$$

elde ederiz.

Aşağıdaki teoremlerde Neumann koşulları altında Green fonksiyonu elde edilecektir.

Teorem 3.10:

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \quad (3.1)$$

denkleminin

$$y(0, \lambda) = \cos \alpha, \quad y'(0, \lambda) = -\sin \alpha \quad \alpha = 0 \quad (3.17)$$

başlangıç koşullarını sağlayan çözümü

i)

$$\begin{aligned} \varphi(x, \lambda) &= \cos \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) - \sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \\ &+ O(\lambda^{-1/2}\eta(\lambda)) \end{aligned} \quad (3.18)$$

ii)

$$\begin{aligned} \varphi(x, n) &= \cos \left(\frac{1}{(n+1)} \left[\frac{((n+1)^2\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \\ &- \sin \left(\frac{1}{(n+1)} \left[\frac{((n+1)^2\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) + O(n^{-2}\eta(n)). \end{aligned} \quad (3.19)$$

İspat:

i) Daha önce

$$\varphi(x, \lambda) = \cos \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right) - \sin \left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt \right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.65)$$

olarak bulmuştuk.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x, \lambda)$ 'da yerine yazılırsa

$$\begin{aligned} \varphi(x, \lambda) &= \cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) - \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ &+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \quad (3.18)$$

elde ederiz.

ii) Önceki bölümde

$$\begin{aligned} \varphi(x, n) &= \cos\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right) \\ &- \sin\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right) + O(n^{-2}\eta(n)) \end{aligned} \quad (2.66)$$

olarak hesaplamıştık.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x, n)$ 'de yerine yazılırsa

$$\begin{aligned} \varphi(x, n) &= \cos\left(\frac{1}{(n+1)}\left[\frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ &- \sin\left(\frac{1}{(n+1)}\left[\frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O(n^{-2}\eta(n)) \end{aligned} \quad (3.19)$$

elde ederiz.

Teorem 3.11: $\lambda \rightarrow \infty$

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \quad (3.1)$$

denkleminin

$$y(\pi, \lambda) = \cos \beta, \quad y'(\pi, \lambda) = -\sin \beta, \quad \beta = 0 \quad (3.20)$$

başlangıç koşullarını sağlayan çözümü

i)

$$\begin{aligned} \psi(x, \lambda) = & \cos \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ & + \sin \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) + O(\lambda^{-1/2} \eta(\lambda)) \end{aligned} \quad (3.21)$$

ii)

$$\begin{aligned} \psi(x, n) = & \cos \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{x} - \frac{((n+1)^2 \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ & + \sin \left((n+1)\pi + \frac{1}{(n+1)} \left[\sqrt{x} - \frac{((n+1)^2 \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right) \\ & + O(n^{-2} \eta(n)) \end{aligned} \quad (3.22)$$

ile bulunur.

İspat:

i) Bir önceki bölümde

$$\begin{aligned} \psi(x, \lambda) = & \cos \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) \\ & + \sin \left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t) dt \right) + O(\lambda^{-1/2} \eta(\lambda)) \end{aligned} \quad (2.72)$$

olarak bulmuştuk. $\int_x^\pi q(t) dt$ hesaplanır ve $\psi(x, \lambda)$ 'da yerine yazılırsa

$$\psi(x, \lambda) = \cos \left(\lambda^{1/2} \pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{x} - \frac{(\lambda \sqrt{\pi} + 1)x}{\sqrt{\pi}} \right] \right)$$

$$+\sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (3.21)$$

elde ederiz.

ii) Daha önce

$$\begin{aligned} \psi(x, n) &= \cos\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right) \\ &+ \sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right) + O(n^{-2}\eta(n)) \end{aligned} \quad (2.73)$$

olarak hesaplamıştık. $\int_x^\pi q(t)dt$ hesaplanır ve $\psi(x, n)$ 'de yerine yazılırsa

$$\begin{aligned} \psi(x, n) &= \cos\left((n+1)\pi + \frac{1}{(n+1)}\left[\sqrt{x} - \frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}}\right]\right) \\ &+ \sin\left((n+1)\pi + \frac{1}{(n+1)}\left[\sqrt{x} - \frac{((n+1)^2\sqrt{\pi} + 1)x}{\sqrt{\pi}}\right]\right) \\ &+ O(n^{-2}\eta(n)) \end{aligned} \quad (3.22)$$

elde ederiz.

Teorem 3.12: $\lambda \rightarrow \infty$, (3.17) ve (3.20) koşulları altında

$$\begin{aligned} w(\lambda) &= w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \\ &= 2\lambda^{1/2} \sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda)) \end{aligned} \quad (3.23)$$

olarak bulunur.

İspat: (3.18) ve (3.21)'de bulunan $\varphi(x, \lambda)$ ve $\psi(x, \lambda)$ kullanılırsa

$$w(\lambda) = w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

olduğundan

$$w(\lambda) = 2\lambda^{1/2} \sin(\lambda^{1/2}\pi) + O(\lambda^{-1}\eta(\lambda)) \quad (2.79)$$

şeklinde hesaplarız. Dikkat edilirse $w(\lambda)$ eşitliğimiz $q(t)$ fonksiyonuna bağlı değildir.

Teorem 3.13:

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}} \right) \right) y = 0 \quad (3.1)$$

denkleminin

$$y'(0, \lambda) = 0, \quad y'(\pi, \lambda) = 0$$

sınır değer problemi için Green fonksiyonu ($\lambda \rightarrow \infty$)

i)

$$G(x, y, \lambda) =$$

$$\frac{\begin{bmatrix} \cos \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \\ -\sin \left(\frac{1}{\lambda^{1/2}} \left[\frac{(\lambda\sqrt{\pi} + 1)x}{\sqrt{\pi}} - \sqrt{x} \right] \right) \end{bmatrix} \times \begin{bmatrix} \cos \left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{y} - \frac{(\lambda\sqrt{\pi} + 1)y}{\sqrt{\pi}} \right] \right) \\ +\sin \left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}} \left[\sqrt{y} - \frac{(\lambda\sqrt{\pi} + 1)y}{\sqrt{\pi}} \right] \right) \end{bmatrix}}{2\lambda^{1/2} \sin(\lambda^{1/2}\pi)}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (3.24)$$

ii)

$$\begin{aligned} G(x, y, n) &= \cos \left((n+1)\pi + (n+1)(x-y) - \frac{1}{(n+1)} \left[\sqrt{x} - \sqrt{y} + \frac{y-x}{\sqrt{\pi}} \right] \right) \\ &\quad \times \sin \left((n+1)(\pi-y) - (n+1)x + \frac{1}{(n+1)} \left[\sqrt{x} + \sqrt{y} - \frac{y+x}{\sqrt{\pi}} \right] \right) \\ &\quad +O(n^{-3}\eta(n)) \end{aligned} \quad (3.25)$$

olarak hesaplanır.

İspat:

i)

$$G(x, y, \lambda) = \begin{cases} \frac{\varphi(x, \lambda)\psi(y, \lambda)}{w(\lambda)} + O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0 \leq x \leq y \leq \pi \\ \frac{\varphi(y, \lambda)\psi(x, \lambda)}{w(\lambda)} + O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0 \leq y \leq x \leq \pi \end{cases}$$

dir. $0 \leq x \leq y \leq \pi$ için bulduğumuz $\varphi(x, \lambda), \psi(x, \lambda)$ ve $w(\lambda)$ 'yı $G(x, y, \lambda)$ 'da yerine yazarsak

$$\begin{aligned} G(x, y, \lambda) = & \frac{\begin{bmatrix} \cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \end{bmatrix} \times \begin{bmatrix} \cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \end{bmatrix}}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)} \\ & + O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{aligned} \quad (3.24)$$

elde ederiz. $0 \leq y \leq x \leq \pi$ için de aynı işlemler yapılır.

ii) Bulduğumuz $G(x, y, \lambda)$ 'ya reversion uygularsak

$$\begin{aligned} G(x, y, n) = & \cos\left((n+1)\pi + (n+1)(x-y) - \frac{1}{(n+1)}\left[\sqrt{x} - \sqrt{y} + \frac{y-x}{\sqrt{\pi}}\right]\right) \\ & \times \sin\left((n+1)(\pi-y) - (n+1)x + \frac{1}{(n+1)}\left[\sqrt{x} + \sqrt{y} - \frac{y+x}{\sqrt{\pi}}\right]\right) \\ & + O(n^{-3}\eta(n)) \end{aligned} \quad (3.26)$$

elde ederiz.

4.SONUÇLAR

1) $y'' + (\lambda - q)y = 0$ (1) probleminin $y(0, \lambda) = \cos\alpha$, $y'(0, \lambda) = -\sin\alpha$,

$\alpha \in [0, \pi]$ başlangıç değer koşullarını sağlayan çözümü Teorem 2.1 ile verildi. Uygun notasyonlar yardımıyla

$$\varphi(x, \lambda) = G_0(E_0, C_0, S_0, T(0, \lambda), S(0, \lambda))$$

olarak ifade edildi.

2) (1) probleminin $y(\pi, \lambda) = \cos\beta$, $y'(\pi, \lambda) = -\sin\beta$, $\beta \in [0, \pi]$ başlangıç değer koşullarını sağlayan çözümü Teorem 2.2 ile verildi. Notasyonlar yardımıyla

$$\psi(x, \lambda) = F_\pi(E_\pi, C_\pi, S_\pi, T(\pi, \lambda), S(\pi, \lambda))$$

ifade edildi.

3) 1) ve 2)'de gerekli E_0, C_0, S_0 ve E_π, C_π, S_π değerleri Teorem 2.3'te hesaplandı.

4) 3)'de hesaplanan değerler yardımıyla $\varphi(x, \lambda)$ ve $\psi(x, \lambda)$ çözümleri λ 'ya bağlı asimptotik olarak Teorem 2.4'de elde edildi.

5) Green fonksiyonu için gerekli olan Wronskian $w(\lambda)$ Teorem 2.5'te hesaplandı ve $G(x, y, \lambda)$ değeri Teorem 2.6'a verildi.

6) Özel olarak Dirichlet koşullarının sağlanması durumu $\left(\alpha = \frac{\pi}{2}\right)$ durumu ele alınarak $\varphi(x, \lambda)$ için asimptotik çözüm bulundu, daha sonra λ_n için asimptotik tahminler kullanılarak $\varphi(x, n)$ çözümü Teorem 2.7'de elde edildi.

7) Benzer şekilde $\beta = \frac{\pi}{2}$ için $\psi(x, \lambda)$ ve daha sonra $\psi(x, n)$ çözümleri Teorem 2.8'de verildi.

8) Dirichlet sınır koşulları için Wronskian $w(\lambda)$ Teorem 2.9'da hesaplandı.

9) Wronskian kullanılarak Dirichlet sınır koşulları durumunda $G(x, y, \lambda)$ asimptotik olarak Teorem 2.10'da verildi.

10) (6-9)'da elde edilen sonuçlara benzer olarak $\alpha = 0$ ve $\beta = 0$ için Neumann sınır koşulları durumu incelenerek $G(x, y, \lambda)$ Green fonksiyonu Teorem 2.14'de ifade edildi.

11) Son olarak yukarıdaki sonuçlar $q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$ potansiyel fonksiyonuna uygulandı.



5.ÖNERİLER

- 1) Elde edilen asimptotik sonuçlarda $q(x)$ potansiyel fonksiyonu integrallenebilen bir fonksiyon olarak kabul edildi. Potansiyel fonksiyonunun singüler nokta içermesi durumu için de asimptotik çözümler araştırılabilir.
- 2) Sınır koşulları her iki uç noktada ya da bir uçta λ 'ya bağlı farklı formlarda kabul edilerek çözümler araştırılabilir.
- 3) Potansiyel fonksiyonunun sınırlı salınımlı bir fonksiyon olması durumu incelenebilir.
- 4) Asimptotik çözümler için hata terimi geliştirilebilir.

6.KAYNAKLAR

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ÖZGEÇMİŞ

İlk ve orta öğrenimini Fatih İlköğretim Okulu'nda, lise öğrenimini Fatih Sultan Mehmet Anadolu Lisesi'nde tamamladı.

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