DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

RESIDUAL METHOD FOR NONLINEAR SYSTEM OF INITIAL VALUE PROBLEMS

by
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İZMİR

RESIDUAL METHOD FOR NONLINEAR SYSTEM OF INITIAL VALUE PROBLEMS

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**RESIDUAL METHOD FOR NONLINEAR SYSTEM OF INITIAL VALUE PROBLEMS**" completed by **BURCU NOYAN** under supervision of **YRD. DOÇ. MELTEM ADIYAMAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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RESIDUAL METHOD FOR NONLINEAR SYSTEM OF INITIAL VALUE PROBLEMS

ABSTRACT

In this thesis, non-linear system of initial value problems are solved numerically using Residual Method which is based on the minimization of a residual function using the Taylor's series expansion. The convergence of the method applied to non-linear system of initial value problems is analysed. The method is illustrated with Lorenz system, which is a chaotic initial value system, and primary HIV-1 infection problem, which is a non-linear dynamical system.

Keywords: Non-linear initial value systems, Residual method, Bernstein polynomials, Lorenz system, primary HIV-1 infection problem.

BAŞLANGIÇ DEĞER PROBLEMLERİNİN DOĞRUSAL OLMAYAN SİSTEMLERİ İÇİN KALINTI METODU

ÖZ

Bu tezde, başlangıç değer problemlerinin doğrusal olmayan sistemleri, Taylor seri açılımı kullanılarak kalıntı fonksiyonunun minimize edilmesine dayanan kalıntı metoduyla sayısal olarak çözüldü. Başlangıç değer problemlerinin doğrusal olmayan sistemlerine uygulanan metodun yakınsaklığı analiz edildi. Metot, kaotik başlangıç değer sistemi olan Lorenz sistemi ve doğrusal olmayan dinamik sistem olan erken dönem HIV-1 enfeksiyon problemi ile örneklendirildi.

Anahtar kelimeler: Başlangıç değer problemlerinin doğrusal olmayan sistemleri, Kalıntı metodu, Bernstein polinomları, Lorenz sistemi, erken dönem HIV-1 enfeksiyon problemi.

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CHAPTER ONE

INTRODUCTION

In this thesis, we propose Residual method to approximate non-linear system of initial value problems of the form

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)),\ \mathbf{Y}(a) = \mathbf{Y}_0, \tag{1.1}$$

where $\mathbf{Y} : [a,b] \to \mathcal{D}$, $\mathbf{F} \in C^n([a,b] \times \mathcal{D})$, $\mathcal{D}$ is a closed region in $\mathbb{R}^m$ and $\mathbf{Y}_0$ is a finite vector in $\mathcal{D}$.

Investigation of the exact and numerical solutions of non-linear system of initial value problems have been focused by some researchers for many years. Most famous type of them are dynamical systems and chaotic problems.

For an example of dynamical systems, we investigate primary HIV-1 infection problem which is a basic mathematical model widely employed to describe the virus dynamics of primary HIV-1 infection (Phillips, 1996) and described by a system of differential equations given in (Burg et al., 2009)

$$\frac{d\mathbf{T}}{dt} = s - d\mathbf{T} - \beta\mathbf{V}\mathbf{T}, \qquad \mathbf{T}(t_0) = \mathbf{T}_0, \tag{1.2}$$

$$\frac{d\mathbf{I}}{dt} = \beta\mathbf{V}\mathbf{T} - \delta\mathbf{I}, \qquad \mathbf{I}(t_0) = \mathbf{I}_0, \tag{1.3}$$

$$\frac{d\mathbf{V}}{dt} = p\mathbf{I} - c\mathbf{V}, \qquad \mathbf{V}(t_0) = \mathbf{V}_0, \tag{1.4}$$

where $\mathbf{T}$ is the concentration of target cells, s represents the constant influx rate of target cells, d is the target cell loss rate, β is the target cells infection rate constant, $\mathbf{I}$ is the concentration of infected cells, δ their loss rate constant, $\mathbf{V}$ is the serum viral concentration, p is the viral production rate constant and c is the virus concentrate rate constant. Equation (1.2) expresses the dynamics of the target cells, Equation (1.3) describes the dynamics of the infected cells and Equation (1.4) expresses the viral dynamics. For this model it is concluded that the control of HIV-1 in the periphery is limited by the availability of susceptible target cells.

For an example of chaotic problems, we consider Lorenz system

$$\begin{aligned}
\frac{dx_1}{dt} &= a(x_2 - x_1), && x_1(t_0) = x_{10}, \\
\frac{dx_2}{dt} &= -x_1x_3 + bx_1 - x_2, && x_2(t_0) = x_{20}, \\
\frac{dx_3}{dt} &= x_1x_2 - cx_3, && x_3(t_0) = x_{30},
\end{aligned} \tag{1.5}$$

where a, b, c are all greater than zero. These equations were derived by Lorenz (Lorenz, 1963) in the modelling of two dimensional fluid cell between two paralel plates at different temperatures (Motsa et al., 2013). Some examples of the numerical treatment of Lorenz system are given in (Motsa et al., 2013, Rodrigues & Matos, 2013, Abdulaziz et al., 2008, Motsa, 2012, Chowdhury et al., 2009).

To solve these non-linear system of initial value problems numerically, we use Residual method given in (Adiyaman & Oger, 2015) which is based on the minimization of a residual function using the Taylor's series expansion. In this method, interval $[a,b]$ is divided into N subintervals and approximate solution is constructed as a linear combination of Bernstein polynomials on each subinterval. Then unknown coefficients of Bernstein polynomials are obtained using residual method. This method reduces the non-linear system of initial value problems to lower triangular system with non-zero diagonals. This is the significant advantage of the Residual method.

In Chapter 2, Residual method is described for non-linear system of initial value problems. Convergence of the method for non-linear system of initial value problems is analysed in Chapter 3. In Chapter 4, numerical solutions of primary HIV-1 infection problem and Lorenz system is given. In the conclusion, we summarize the study and present our suggestions regarding future works.

CHAPTER TWO

RESIDUAL METHOD FOR NON-LINEAR SYSTEM OF INITIAL VALUE PROBLEMS

2.1 Bézier Curves

The polynomial $u(x)$ of any degree n can be defined as follows

$$u(x) = \sum_{j=0}^{n} c_j B_j^n\left(\frac{x-a}{b-a}\right), \tag{2.1}$$

where

$$B_j^n\left(\frac{x-a}{b-a}\right) = \binom{n}{j}\left(\frac{x-a}{b-a}\right)^j\left(\frac{b-x}{b-a}\right)^{n-j}$$

are Bernstein polynomials over the interval $[a,b]$. The definition (2.1) is known as Bernstein-Bézier form. If $u(x)$ is a vector-valued polynomial, then $u(x)$ is a parametric Bézier curve and the Bézier coefficients c_j are the control points. If $u(x)$ is a scalar-valued polynomial, the function $y = u(x)$ is called an explicit Bézier curve given by $(x, u(x))$. The control points of an explicit curve are $(a + \frac{b-a}{n} j, c_j)$ where c_j is called the Bézier control points.

2.1.1 Some properties of Bézier Curves

Bézier curves are parametric curves which are constructed using linear combination of Bernstein polynomials. These curves are widely used in computer graphics and have many special properties. We can summarize these properties as follows:

The polygon formed by connecting the adjacent control points c_j with straight lines, starting with c_0 and finishing with c_n is called the control polygon.

- **Invariant under affine transformation:** We can rotate, translate, scale and reflect the control polygon. Let T be an affine transformation, then

$$T\left(\sum_{j=0}^{n} c_j B_j^n\left(\frac{x-a}{b-a}\right)\right) = \sum_{j=0}^{n} T(c_j) B_j^n\left(\frac{x-a}{b-a}\right).$$

- **End points interpolation:** The first and last control points are the end points of the curve. In other words; a Bézier curve always interpolates the end control points, that is

  $$u(a) = c_0 \text{ and } u(b) = c_n.$$

  The curve is tangent to the control polygon at the end points.

- **Convex hull property:** Every point of Bézier curve lies inside the convex hull of its defining control points.

- **Variation diminishing property:** Number of intersections of a straight line with the Bézier curve less than or equal to the number of intersections of that line with the control polygon.

- **Symmetry property:** If we reorder the control points from $c_0, c_1, \ldots, c_n$ to $c_n, c_{n-1}, \ldots, c_0$ and using the symmetry property of the Bernstein polynomials, we obtain the following identity which shows the symmetry property of Bézier curves;

  $$\sum_{j=0}^{n} c_j B_j^n\left(\frac{x-a}{b-a}\right) = \sum_{j=0}^{n} c_j^* B_j^n\left(\frac{b-x}{b-a}\right).$$

In addition, many operations on the Bézier curves can be performed using control points. For example, the derivatives of the Bézier curves can be written

$$\frac{d^k}{dx^k} u(x) = \frac{1}{(b-a)^k} \frac{n!}{(n-k)!} \sum_{j=0}^{n-k} \Delta^k c_j B_j^{n-k}\left(\frac{x-a}{b-a}\right) \tag{2.2}$$

for $k = 0,...,n$, where Δ is a difference operator.

Substituting the end points into the last equation, we obtain

$$\begin{aligned} \frac{d^k}{dx^k}u(x)\Big|_{x=a} &= \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\Delta^k c_0, \\ \frac{d^k}{dx^k}u(x)\Big|_{x=b} &= \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\Delta^k c_{n-k}. \end{aligned} \tag{2.3}$$

These formulas can be used to set constraints on Bézier segments to make them connect with C^k continuity and to form a Bézier spline. For example, the curve which consists of the following two Bézier segments

$$u_1(x) = \sum_{j=0}^{n} c_j^1 B_j^n\left(\frac{x-a}{b-a}\right), \quad x \in [a,b]$$

and

$$u_2(x) = \sum_{j=0}^{n} c_j^2 B_j^n\left(\frac{x-b}{c-b}\right), \quad x \in [b,c]$$

satisfies C^1 continuity if control points of them satisfy the following equations

$$c_0^2 = c_n^1$$

and

$$c_1^2 - c_0^2 = c_n^1 - c_{n-1}^1 \quad .$$

2.2 Residual Method

Consider the non-linear system of initial value problem (1.1)

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)), \ \mathbf{Y}(a) = \mathbf{Y}_0.$$

We start the process dividing the interval $[a,b]$ into N equally spaced subintervals $[a_{i-1}, a_i]$, where $a_i = a + ih,\ i = 0,...,N,\ h = (b-a)/N$ and N is a positive integer. The initial value problem (1.1) is written piecewisely as follows

$$\mathbf{Y}_i'(x) = \mathbf{F}(x, \mathbf{Y}_i(x)), \ x \in S_i, \tag{2.4}$$

where $S_i = [a_{i-1}, a_i]$ for $i = 1, ..., N$ and

$$\mathbf{Y}_1(a_0) = \mathbf{Y}_0,\ \mathbf{Y}_i(a_{i-1}) = \mathbf{Y}_{i-1}(a_{i-1}) \text{ for } i = 2, .., N. \tag{2.5}$$

To approximate the solution $\mathbf{Y}_i(x)$, we use nth degree Bézier curve over S_i

$$\mathbf{U}_i(x) = \sum_{j=0}^{n} C_j^i \mathbf{B}_j^n\left(\frac{x-a_{i-1}}{h}\right), \tag{2.6}$$

where

$$\mathbf{B}_j^n\left(\frac{x-a_{i-1}}{h}\right) = \binom{n}{j}\frac{1}{h^n}(x-a_{i-1})^j(a_i-x)^{n-j}$$

are the Bernstein polynomials over the interval $[a_{i-1}, a_i]$ and C_j^i are unknown control points. So $(n+1)$ unknown control points must be determined to obtain approximate solution $U_i(x)$ over each subinterval S_i.

When the initial conditions (2.5) are applied to the approximate solution, we obtain following conditions

$$\mathbf{U}_1(a_0) = \mathbf{U}_0, \quad \mathbf{U}_i(a_{i-1}) = \mathbf{U}_{i-1}(a_{i-1}) \tag{2.7}$$

for $i = 2, ..., N$.

Therefore, we guarantee the continuity the approximate solution

$$\mathbf{U}(x) = \begin{cases} \mathbf{U}_1(x) \text{ if } x \in S_1, \\ \mathbf{U}_2(x) \text{ if } x \in S_2, \\ \vdots \\ \mathbf{U}_N(x) \text{ if } x \in S_N, \end{cases} \tag{2.8}$$

that is, $\mathbf{U}(x) \in C[a, b]$.

From the equation (2.7), we get

$$\begin{aligned} C_0^1 &= Y_0, \\ C_0^i &= C_n^{i-1}. \end{aligned} \tag{2.9}$$

So for each subinterval S_i, we have n unknown control points.

Substitution of (2.6) into the differential equation (2.4) for $i = 1, ..., N$ yields the following piecewise residual function

$$\mathbf{R}(x) = \begin{cases} \mathbf{R}_1(x) \text{ if } x \in S_1, \\ \mathbf{R}_2(x) \text{ if } x \in S_2, \\ \vdots \\ \mathbf{R}_N(x) \text{ if } x \in S_N, \end{cases}$$

where

$$\mathbf{R}_i(x) = \mathbf{U}_i'(x) - \mathbf{F}(x, \mathbf{U}_i(x)),\ x \in S_i.$$

Our aim is to minimize residual function and to find unknown control points. So, we determine unknown control points minimizing the residual function. For this minimization, we force the first n terms in Taylor's expansion of $\mathbf{R}_i(x)$ at $x = a_{i-1}$ be zero, that is

$$\begin{aligned} \mathbf{R}_i(a_{i-1}) &= 0, \\ \mathbf{R}_i'(a_{i-1}) &= 0, \\ \mathbf{R}_i''(a_{i-1}) &= 0, \\ &\vdots \\ \mathbf{R}_i^{(n-1)}(a_{i-1}) &= 0. \end{aligned} \tag{2.10}$$

So, from (2.10) we get

$$\begin{aligned} \mathbf{R}_i^{(k)}(a_{i-1}) &= \mathbf{U}_i^{(k+1)}(a_{i-1}) - \mathbf{F}^{(k)}(a_{i-1}, \mathbf{U}_i(a_{i-1})) \\ &= 0 \end{aligned}$$

$k = 0, \ldots, n-1$, which yield the following linear equations

$$\frac{n(n-1)\ldots(n-k)}{h^{k+1}} \Delta^{k+1} C_0^i - \mathbf{F}^k(a_{i-1}, C_0^i) = 0,$$

for $i = 1, \ldots, N$.

These equations are linear, so $C_1^i, \ldots, C_n^i$ can be solved easily from lower triangular system of linear equations. Therefore, the approximate solutions $\mathbf{U}_i(x)$ are obtained and residual functions $\mathbf{R}_i(x)$ are minimized over each subinterval S_i for $i = 1, ..., N$.

2.3 Convergence of the Method for Non-Linear System of Initial Value Problems

Lemma 2.3.1. *The maximum norm of the Residual functions* $\left\|\boldsymbol{R}_i(x)\right\|_\infty$ *are order of* h^n *for* $i = 1, ..., N$.

Proof. The Taylor's expansion of $\mathbf{R}_i(x)$ at $x = a_{i-1}$ is

$$\begin{aligned}\mathbf{R}_i(x) &= \mathbf{R}_i(a_{i-1}) + (x - a_{i-1})\mathbf{R}_i'(a_{i-1}) + \frac{(x-a_{i-1})^2}{2!}\mathbf{R}_i''(a_{i-1}) + \ldots \\ &+ \frac{(x-a_{i-1})^{n-1}}{(n-1)!}\mathbf{R}_i^{(n-1)}(a_{i-1}) + \frac{(x-a_{i-1})^n}{n!}\mathbf{R}_i^{(n)}(\xi_i)\end{aligned}$$

where ξ_i between (a_{i-1}, x) and $x \in [a_{i-1}, a_i]$. Using equations (2.10) and taking maximum norm of both sides, we get

$$\begin{aligned}\left\|\mathbf{R}_i(x)\right\|_\infty &= \left|\frac{(x-a_{i-1})^n}{n!}\right|\left\|\mathbf{R}_i^{(n)}(\xi_i)\right\|_\infty \\ &\leq Ch^n,\end{aligned}$$

where $C = \dfrac{1}{n!}\max\limits_{x\in[a_{i-1},a_i]}\left\|\mathbf{R}_i^{(n)}(x)\right\|_\infty$. Therefore,

$$\left\|\mathbf{R}(x)\right\|_\infty = O(h^n).$$

□

Lemma 2.3.2. *Let* $\tilde{\boldsymbol{U}}_i(x)$ *be the auxiliary approximate solution of piecewise initial value problem* (2.4) *with initial conditions* (2.5) *then*

$$\boldsymbol{Y}_i^{(l)}(a_{i-1}) = \tilde{\boldsymbol{U}}_i^{(l)}(a_{i-1}), \ \textit{for l=0,...,n} \tag{2.11}$$

and

$$\left\|\boldsymbol{Y}_i(x) - \tilde{\boldsymbol{U}}_i(x)\right\|_\infty \leq Kh^{n+1}, \ \forall x \in S_i \tag{2.12}$$

where $\boldsymbol{Y}_i(x)$ *is the corresponding exact solution and*

$$K = \frac{1}{(n+1)!}\max_{i=1,\ldots,N}\left(\max_{x\in[a_{i-1},a_i]}\left\|\boldsymbol{Y}_i^{(n+1)}(x)\right\|_\infty\right).$$

Proof. We define initial value problem (1.1) piecewisely as

$$\mathbf{Y}_i'(x) = \mathbf{F}(x, \mathbf{Y}_i(x)),\ x \in S_i,$$

where $S_i = [a_{i-1}, a_i]$ for $i = 1, ..., N$ and

$$\mathbf{Y}_1(a_0) = \mathbf{Y}_0, \quad \mathbf{Y}_i(a_{i-1}) = \mathbf{Y}_{i-1}(a_{i-1}) \text{ for } i = 2, ..., N.$$

Since $\tilde{\mathbf{U}}_i(x)$ is the auxiliary approximation solution, it must satisfy the following initial conditions

$$\tilde{\mathbf{U}}_1(a_0) = \mathbf{Y}_0 \text{ and } \tilde{\mathbf{U}}_i(a_{i-1}) = \mathbf{Y}_{i-1}(a_{i-1}). \tag{2.13}$$

Let residual function for $\tilde{\mathbf{U}}_i(x)$ be

$$\tilde{\mathbf{R}}_i(x) = \tilde{\mathbf{U}}_i'(x) - \mathbf{F}(x, \tilde{\mathbf{U}}_i(x)),\ x \in S_i \tag{2.14}$$

and satisfies

$$\tilde{\mathbf{R}}_i^{(k)}(a_{i-1}) = 0, \tag{2.15}$$

for $k = 0, \ldots, n-1$ as in (2.10).

Then,

$$\begin{aligned} 0 &= \tilde{\mathbf{R}}_i^{(k)}(a_{i-1}) \\ &= \tilde{\mathbf{U}}_i^{(k+1)}(a_{i-1}) - \frac{\partial^k \mathbf{F}(x, \tilde{\mathbf{U}}_{i-1}(x))}{\partial x^k}\Big|_{x=a_{i-1}} \\ &= \tilde{\mathbf{U}}_i^{(k+1)}(a_{i-1}) - \frac{\partial^k \mathbf{F}(x, \mathbf{Y}_{i-1}(x))}{\partial x^k}\Big|_{x=a_{i-1}} \\ &= \tilde{\mathbf{U}}_i^{(k+1)}(a_{i-1}) - \mathbf{Y}_i^{(k+1)}(a_{i-1}). \end{aligned}$$

So, we have

$$\tilde{\mathbf{U}}_i^{(l)}(a_{i-1}) = \mathbf{Y}_i^{(l)}(a_{i-1}) \tag{2.16}$$

for $l = 0, \ldots, n$.

To prove (2.12), we start writing Taylor's expansion of $\mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x)$ at $x = a_{i-1}$

$$\begin{aligned} \mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x) &= \Big(\mathbf{Y}_i(a_{i-1}) - \tilde{\mathbf{U}}_i(a_{i-1})\Big) + (x - a_{i-1})\Big(\mathbf{Y}_i'(a_{i-1}) - \tilde{\mathbf{U}}_i'(a_{i-1})\Big) \\ &+ \frac{(x-a_{i-1})^2}{2!}\Big(\mathbf{Y}_i''(a_{i-1}) - \tilde{\mathbf{U}}_i''(a_{i-1})\Big) + \cdots + \frac{(x-a_{i-1})^n}{n!} \\ &\times \Big(\mathbf{Y}_i^{(n)}(a_{i-1}) - \tilde{\mathbf{U}}_i^{(n)}(a_{i-1})\Big) + \frac{(x-a_{i-1})^{n+1}}{(n+1)!}\mathbf{Y}_i^{(n+1)}(\xi_{i-1}), \end{aligned}$$

where ξ_{i-1} is between (a_{i-1}, x). From (2.11), we obtain

$$\mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x) = \frac{(x-a_{i-1})^{n+1}}{(n+1)!}\mathbf{Y}_i^{(n+1)}(\xi_{i-1}).$$

Taking maximum norm of both sides, we get

$$\begin{aligned}\left\|\mathbf{Y}_i(x) - \tilde{U}_i(x)\right\|_\infty &= \left|\frac{(x-a_{i-1})^{n+1}}{(n+1)!}\right| \left\|\mathbf{Y}_i^{(n+1)}(\xi_{i-1})\right\|_\infty \\ &\leq Kh^{n+1},\end{aligned}$$

where $K = \dfrac{1}{(n+1)!}\max\limits_{i=1,\ldots,N}\left(\max\limits_{x\in[a_{i-1},a_i]}\left\|\mathbf{Y}_i^{(n+1)}(x)\right\|_\infty\right)$.

□

Lemma 2.3.3. *Let $\tilde{\boldsymbol{U}}_i(x)$ be the auxiliary approximate solution of piecewise initial value problem* (2.4) *with initial conditions* (2.5) *and $\boldsymbol{U}_i(x)$ be the approximate solution of* (2.4) *with initial condition* (2.7)*, then*

$$\left\|\boldsymbol{U}_i(x) - \tilde{\boldsymbol{U}}_i(x)\right\|_\infty \leq \left(1 + C\left(h + \frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right)\right)\left\|\boldsymbol{U}_i(a_{i-1}) - \tilde{\boldsymbol{U}}_i(a_{i-1})\right\|_\infty, \tag{2.17}$$

where

$$C = \max_{k=0\ldots,n-1}\left(\max_{(x,v)\in\Omega}\left\|\frac{\partial^{k+1}\boldsymbol{F}(x,v)}{\partial v\partial x^k}\right\|_\infty\right)$$

for $\Omega = \{(x,v)|\, x \in [a,b], v \in \mathcal{D}\}$.

Proof. We write the Taylor's expansion of $\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x)$ about $x = a_{i-1}$

$$\begin{aligned}\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x) &= (\tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1})) + (x-a_{i-1})\left(\tilde{\mathbf{U}}_i'(a_{i-1}) - \mathbf{U}_i'(a_{i-1})\right) \\ &+ \frac{(x-a_{i-1})^2}{2!}\left(\tilde{\mathbf{U}}_i''(x) - \mathbf{U}_i''(x)\right) + \cdots + \frac{(x-a_{i-1})^n}{n!}\left(\tilde{\mathbf{U}}_i^{(n)}(x) - \mathbf{U}_i^{(n)}(x)\right).\end{aligned}$$

Using (2.14), (2.15) and mean value theorem, we obtain

$$
\begin{aligned}
\tilde{\mathbf{U}}_i(x)-\mathbf{U}_i(x) &= (\tilde{\mathbf{U}}_i(a_{i-1})-\mathbf{U}_i(a_{i-1}))+(x-a_{i-1}) \\
&\times\Big(\mathbf{F}(a_{i-1},\tilde{\mathbf{U}}_i(a_{i-1}))-\mathbf{F}(a_{i-1},\mathbf{U}_i(a_{i-1}))\Big) \\
&+\frac{(x-a_{i-1})^2}{2!}\left(\frac{\partial\mathbf{F}(x,\tilde{\mathbf{U}}_i(x))}{\partial x}-\frac{\partial\mathbf{F}(x,\mathbf{U}_i(x))}{\partial x}\right)\Big|_{x=a_{i-1}}+\dots \\
&+\frac{(x-a_{i-1})^n}{n!}\left(\frac{\partial^{n-1}\mathbf{F}(x,\tilde{\mathbf{U}}_i(x))}{\partial x^{n-1}}-\frac{\partial^{n-1}\mathbf{F}(x,\mathbf{U}_i(x))}{\partial x^{n-1}}\right)\Big|_{x=a_{i-1}} \\
&= (\tilde{\mathbf{U}}_i(a_{i-1})-\mathbf{U}_i(a_{i-1}))+(x-a_{i-1})\mathbf{F}_v(a_{i-1},\xi_{i-1}^{(1)}) \\
&\times(\tilde{\mathbf{U}}_i(a_{i-1})-\mathbf{U}_i(a_{i-1}))+\frac{(x-a_{i-1})^2}{2!}\left(\frac{\partial^2\mathbf{F}(x,v)}{\partial v\partial x}\Big|_{\substack{x=a_{i-1}\\ v=\xi_{i-1}^{(2)}}}\right) \\
&\times(\tilde{\mathbf{U}}_i(a_{i-1})-\mathbf{U}_i(a_{i-1}))+\dots+\frac{(x-a_{i-1})^n}{n!}\left(\frac{\partial^n\mathbf{F}(x,v)}{\partial v\partial x^{n-1}}\Big|_{\substack{x=a_{i-1}\\ v=\xi_{i-1}^{(n)}}}\right),
\end{aligned}
$$

where $\xi_{i-1}^{(k)}$, $k=1,\dots,n$ are between (a_{i-1},x). Taking maximum norm of both sides and using $C=\max_{k=0\dots,n-1}\left(\max_{(x,v)\in\Omega}\left\|\frac{\partial^{k+1}\mathbf{F}(x,v)}{\partial v\partial x}\Big|_{\substack{x=a_{i-1}\\ v=\xi_{i-1}^{(k+1)}}}\right\|_\infty\right)$

$$
\left\|\tilde{\mathbf{U}}_i(x)-\mathbf{U}_i(x)\right\|_\infty\le\left(1+C\left(h+\frac{h^2}{2}+\dots+\frac{h^n}{n!}\right)\right)\left\|\tilde{\mathbf{U}}_i(a_{i-1})-\mathbf{U}_i(a_{i-1})\right\|_\infty,
$$

where $x\in[a_{i-1},a_i]$. □

Theorem 2.3.4. *Let $\boldsymbol{Y}(x)$ be the exact solution of*

$$
\boldsymbol{Y}'=\boldsymbol{F}(x,\boldsymbol{Y}(x)),\quad \boldsymbol{Y}(x_0)=\boldsymbol{Y}_0
$$

and $\boldsymbol{U}(x)$ be the nth degree approximate function (2.8) *then*

$$
\left\|\boldsymbol{Y}(x)-\boldsymbol{U}(x)\right\|_\infty\le Mh^n,\quad x\in[a,b],
$$

where M is a constant which is independent on h and n.

Proof. We will prove this theorem using mathematical induction. Let $\tilde{\mathbf{U}}_i(x)$ be the corresponding auxilary approximate solution for $x \in S_i$, then

$$\left\|\mathbf{Y}_i(x) - \mathbf{U}_i(x)\right\|_\infty \leq \left\|\mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x)\right\|_\infty + \left\|\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x)\right\|_\infty, \text{ for } i = 1, \ldots, N.$$

Using (2.12) and (2.17), we obtain

$$\left\|\mathbf{Y}_i(x) - \mathbf{U}_i(x)\right\|_\infty \leq Kh^{n+1} + \left(1 + C\left(h + \frac{h^2}{2} + \cdots + \frac{h^n}{n!}\right)\right)\left\|\tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1})\right\|_\infty. \quad (2.18)$$

Since for the first interval $\tilde{\mathbf{U}}_1(x) = \mathbf{U}_1(x)$, we get

$$\left\|\mathbf{Y}_1(x) - \mathbf{U}_1(x)\right\|_\infty \leq Kh^{n+1}, \quad \forall x \in S_1,$$

where

$$K = \frac{1}{(n+1)!} \max_{i=1,\ldots,N} \left(\max_{x \in [a_{i-1}, a_i]} \left\|\mathbf{Y}_i^{(n+1)}(x)\right\|_\infty \right).$$

Then,

$$\begin{aligned}
\left\|\mathbf{Y}_2(x) - \mathbf{U}_2(x)\right\|_\infty &\leq Kh^{n+1} + \left(1 + C\left(h + \tfrac{h^2}{2} + \cdots + \tfrac{h^n}{n!}\right)\right)\left\|\tilde{\mathbf{U}}_2(a_1) - \mathbf{U}_2(a_1)\right\|_\infty \\
&\leq Kh^{n+1} + \left(1 + C\left(h + \tfrac{h^2}{2} + \cdots + \tfrac{h^n}{n!}\right)\right)\left\|\mathbf{Y}_1(a_1) - \mathbf{U}_1(a_1)\right\|_\infty \\
&\leq 2Kh^{n+1} + O(h^{n+2}).
\end{aligned}$$

Suppose that

$$\left\|\mathbf{Y}_{N-1}(x) - \mathbf{U}_{N-1}(x)\right\|_\infty \leq (N-1)\, Kh^{n+1} + O(h^{n+2}).$$

Now, we will show that it is true for N. From (2.18), we have

$$\begin{aligned}
\left\|\mathbf{Y}_N(x)-\mathbf{U}_N(x)\right\|_\infty &\leq (N-1)\,Kh^{n+1}+\left(1+C\left(h+\tfrac{h^2}{2}+\cdots+\tfrac{h^n}{n!}\right)\right)\\
&\times\left\|\tilde{\mathbf{U}}_N(a_{N-1})-\mathbf{U}_N(a_{N-1})\right\|_\infty\\
&\leq Kh^{n+1}+\left(1+C\left(h+\tfrac{h^2}{2}+\cdots+\tfrac{h^n}{n!}\right)\right)\\
&\times\left\|\mathbf{Y}_{N-1}(a_{N-1})-\mathbf{U}_{N-1}(a_{N-1})\right\|_\infty\\
&\leq Kh^{n+1}+\left(1+C\left(h+\tfrac{h^2}{2}+\cdots+\tfrac{h^n}{n!}\right)\right)\\
&\times(N-1)\,Kh^{n+1}+\mathcal{O}(h^{n+2})\\
&\leq NKh^{n+1}+\mathcal{O}(h^{n+2}).
\end{aligned}$$

Since $N=(b-a)/h$, we have

$$\left\|\mathbf{Y}(x)-\mathbf{U}(x)\right\|_\infty \leq Mh^n$$

where M is a constant which is independent on h and n. □

CHAPTER THREE
NUMERICAL EXAMPLES

3.1 Numerical Results

We solve the following three examples numerically to illustrate the applicability of the proposed method. The numerical calculations and all figures in this work are performed using Mathematica.

Example 3.1.1. Consider the system

$$\frac{dx}{dt} = x + 2y - z,$$

$$\frac{dy}{dt} = x + z,$$

$$\frac{dz}{dt} = 4x - 4y + 5z,$$

with the initial conditions

$$x(0) = -1,\ y(0) = 0,\ z(0) = 0.$$

The exact solution of the given system is

$$\begin{aligned} x(t) &= -2e^{2t} + e^{3t}, \\ y(t) &= e^{2t} - e^{3t}, \\ z(t) &= 4e^{2t} - 4e^{3t}. \end{aligned}$$

We apply Residual method to above system of initial value problems to confirm the error analysis of the method. In Table 3.1, we give exact and approximate values of the system and also give the obtained error values for $h = 0.1$ and $n = 4$.

Table 3.1 Error values of Example 3.1.1 for $h = 0.1$ and $n = 4$.

t	$x(t)$	$y(t)$	$z(t)$
0.000	0	0	0
0.100	4.90964(-09)	-5.68949(-09)	-2.27579(-08)
0.200	1.33950(-08)	-1.54966(-08)	-6.19866(-08)
0.300	2.75375(-08)	-3.14144(-08)	-1.25658(-07)
0.400	4.73634(-08)	-6.15767(-08)	-2.46307(-07)
0.500	9.47288(-08)	-1.36299(-07)	-5.45196(-07)
0.600	1.47834(-07)	-4.33625(-08)	-1.7345(-07)
0.700	2.37894(-07)	-2.70111(-07)	-1.08044(-06)
0.800	4.0321(-07)	-8.92306(-08)	-3.56923(-07)
0.900	3.79802(-07)	-1.87333(-06)	-7.49334(-06)
1.000	1.25292(-06)	-1.50373(-06)	-6.01492(-06)

In Table 3.2, we give the observed orders which are well confirmed with theoretical results and computed using the following formula

$$ord(h) = \frac{\log(\frac{e_h}{e_{h/2}})}{\log 2}.$$

Table 3.2 Observed orders for Example 3.1.1.

N	$n = 2$	$n = 3$	$n = 4$
Observed orders for x			
10/20	1.67186	2.621231	3.56238
20/40	1.84842	2.83064	3.80865
40/80	1.93098	2.9302	3.95203
Observed orders for y			
10/20	1.69408	2.63226	3.56882
20/40	1.8598	2.83634	3.81389
40/80	1.93674	2.93312	4.03471
Observed orders for z			
10/20	1.75459	2.72627	3.69628
20/40	1.88481	2.88012	3.87734
40/80	1.94136	2.93665	4.04074

Example 3.1.2. Consider the Lorenz system

$$\frac{dx}{dt} = 10(y-x),$$

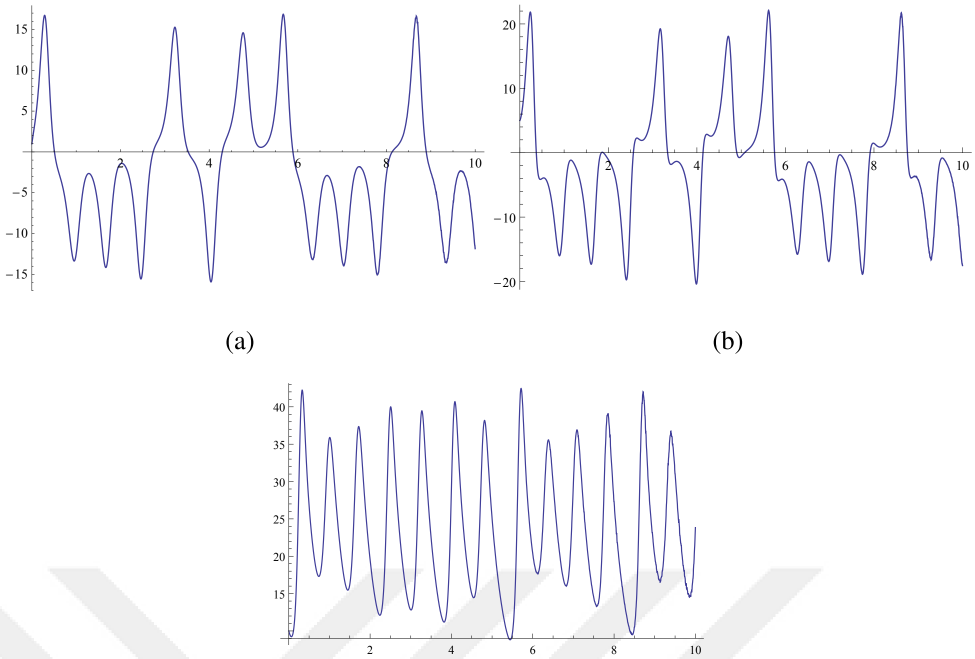
$$\frac{dy}{dt} = -xz+28x-y,$$

$$\frac{dz}{dt} = xy-8/3z,$$

with the initial conditions

$$x(0) = 1,\ y(0) = 5,\ z(0) = 10.$$

We solve Lorenz system using Residual method (RM) and compare our results with the results obtained by piecewise successive linearization method (PSLM) in [6]. Graphs of approximate solutions are given in Figure 3.1 and comparison of the results are given in Table 3.3.



(a) (b)

(c)

Figure 3.1 Graphs of the approximate solutions of Example 3.1.2, (a) Approximate solution of *x(t)* for $n = 4$, (b) Approximate solution of *y(t)* for $n = 4$, (c) Approximate solution of *z(t)* for $n = 4$.

Table 3.3 Numerical solution of the Lorenz system obtained using PSLM [6] and RM.

t	$x_1(t)$		$x_2(t)$		$x_3(t)$	
	RM	REF.[6]	RM	REF.[6]	RM	REF.[6]
2	-1.44432	-1.444359	-1.07476	-1.074977	19.5172	19.517057
4	-14.6729	-14.675080	-20.1954	-20.189107	29.0463	29.063362
6	-2.88223	-2.883028	-4.76728	-4.763557	20.3578	20.355558
8	-2.69165	-2.679252	1.45007	1.429476	27.0352	27.105659
10	-11.847	-12.026645	-17.5237	-17.520281	23.8346	24.300154

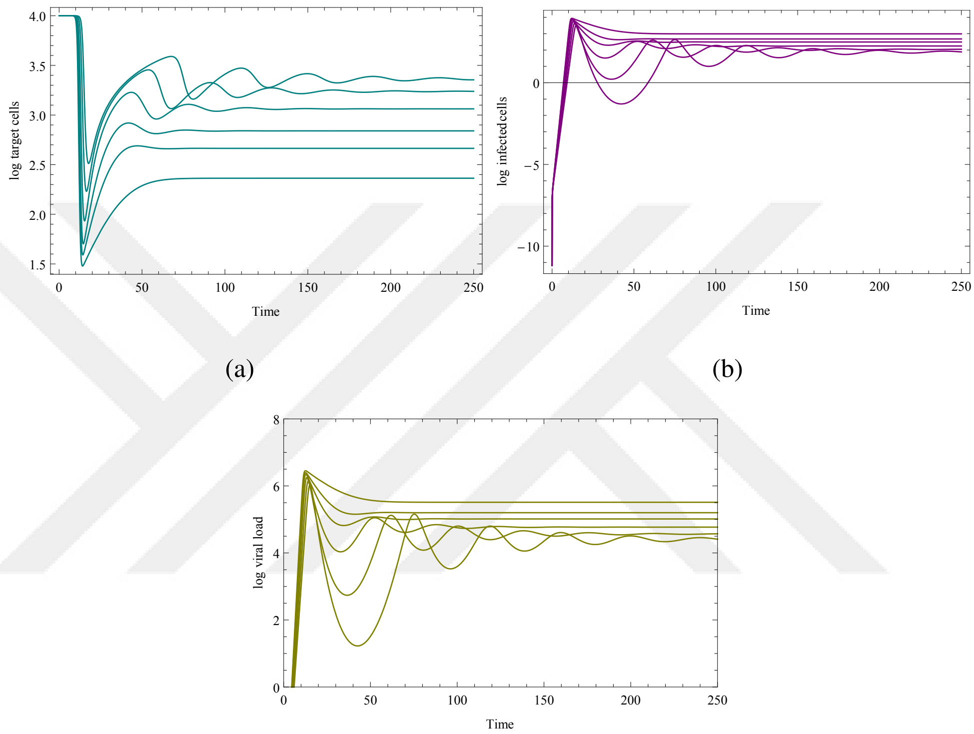
Example 3.1.3. Consider the primary HIV-1 infection problem

$$\frac{dx}{dt} = 10^2 - 10^{-2}x - 1.3x10^{-6}xz,$$
$$\frac{dy}{dt} = 1.3x10^{-6}xz - \delta y,$$
$$\frac{dz}{dt} = 10^3 y - 3z,$$

with the initial conditions

$$x(0) = 10^4,\ y(0) = 0,\ z(0) = 10^{-6}.$$

In the following graphs logarithm of approximate solution of x, y, z with base 10 in 250 days for different infected cell death rates $\delta = 0.1$, $\delta = 0.2$, $\delta = 0.3$, $\delta = 0.5$, $\delta = 0.75$, $\delta = 1.0$ are given.



(a) (b) (c)

Figure 3.2 Graphs of the approximate solutions of Example 3.1.3, (a) Logarithm of approximate number of target cells x with base 10, (b) Logarithm of approximate number of infected cells y with base 10, (c) Logarithm of approximate number of viruses z with base 10.

CHAPTER FOUR
CONCLUSION

We use residual method to approximate non-linear system of initial value problems. As a result, we obtain high order accurate solutions. Observed orders obtained from numerical examples are in good agreement with the predicted ones in the theorem. This method can be extended some special kind of non-linear ordinary differential equations and partial differential equations.

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