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RESULTS OF GENERALIZED EQUILIBRIUM PROBLEMS WITH TRI-FUNCTION

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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Ibtisam Abbas Lafta AL-GBURI

ABSTRACT

RESULTS OF GENERALIZED EQUILIBRIUM PROBLEMS WITH TRIFUNCTION

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February 2023

The branch of mathematics that deals with equilibrium issues is just a few decades old, allowing us to investigate and solve problems in a wide variety of scientific domains, including but not limited to pure and applied sciences. The thesis here says first, findings for a novel state of the trifunctional for equilibrium problem $GP(H, L_1, L_2)$ on convex and closed sets that are either bounded or unbounded in Banach spaces are introduced. This is done by employing a generalized form of monotonicity known as ϑL-g - monotone are introduced. Second, it explores certain findings associated with this subject and defines and expands the idea of hemiequilibrium issues.

2023, 49 pages

Keywords: Fixed point theorem, Equilibrium problems, Solution existence, KKM mapping

ÖZET

ÜÇ İŞLEVLİ FONKSİYONLARLA DENGE PROBLEMLERİNİN GENELLEŞTİRİLMESİ İÇİN SONUÇLAR

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Denge konularıyla ilgilenen matematik dalı sadece birkaç yıllıktır ve teorik ve uygulamalı bilimler dahil ancak bunlarla sınırlı olmamak üzere çok çeşitli bilimsel alanlardaki sorunları araştırmamıza ve çözmemize olanak tanır. Buradaki tez öncelikle, Banach uzaylarında hem sınırlı hem de sınırsız olan dışbükey ve kapalı kümeler üzerinde $GP(H, L_1, L_2)$ denge problemi için üç işlevli yeni bir duruma ilişkin bulguların tanıtıldığını söyler. Bu, ϑL-g olarak bilinen genelleştirilmiş bir monotonluk biçimi kullanılarak yapılmaktadır. İkinci olarak, bu konuyla ilişkili belirli bulguları araştırıyor ve hemiequilibrium konularını tanımlıyor ve genişletiyoruz.

2023, 49 sayfa

Anahtar Kelimeler: Sabit nokta teoremi, Denge Problemleri, Çözüm varlığı, KKM-Teoremi

PREFACE AND ACKNOWLEDGEMENTS

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Ibtisam Abbas Lafta AL-GBURI
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CONTENTS

LIST OF SYMBOLS

N	The set of natural numbers
Q	*T*he set of rational numbers
R	The set of real numbers
2^X	The set of all subsets of X
d	Distance function
∩	Intersection
∪	Union

LIST OF ABBREVIATIONS

l.s.c	Lower semi continuous
max	Maximum
min	Minimum
resp.	Respectively
sup	Supremum
u.s.c	Upper semi continuous
w.r.t	With respect to
EP	Equilibrium problem.
0-DC	0-diagonal convex
KKM	Knaster Kuratowski Mazurkiewicz therory
H.C	Hemicontinuous
conv(A)	Convex hull.
B-S	Banach space.
T.H.V.S	Topological Hausdorff vector space

1. INTRODUCTION

The field of mathematics that deals with equilibrium concerns is one that is only a few decades old, which allows us to examine and find solutions to issues originating in a wide variety of scientific domains, For instance, some of the most important applications can be found in fields such as engineering mechanics, biological economics, physics, and other applied sciences. The innovation that (Muu and Oettli 1992) came up with was one of the foundations for its development. In addition, the study of tri-function equilibrium issues has been a stim-ulating area of scientific research because it encompasses and interconnects a number of fundamental areas of variational inequality issues and their generalizations, including minimax issues, game theory, variational-like inequalities, and quasi-variational optim-ization issues. In addition, from its inception, the study of tri-function equilibrium difficulties has been a stimulating area of scientific research.

1.1 Background

Recently, the issues of equilibrium bases put up by two well-known scientists (Blum and Oettli 1994) have received widespread use in a variety of different scientific fields, including both pure and practical scientific fields. The issue of equilibrium is a special case of a large number of other mathematical issues, such as complementary issues, mathematical programming issues, fixed point issues, variational issues, hemivariational issues, and the hemiequilibrium issue see (Plubtieng and Punpaeng 2007), (Alwan *et al.* 2019), (Rădulescu and Repovš 2010), (Matei 2019) and (Klarbring 1986).

Scientists presented the idea of the vector variational inequality issue, but it was changed Giannessi 1980). Here's how they describe the vector equilibrium issue:

Given a bifunction F(x,y) valued in a real ordered locally convex vector space and the closed convex subset K of a real T.H.V.S, find a $\varkappa \in K$ in which $F(\varkappa, y) \geq 0$ for all $y \in K$. Results for the unifying model were found by M. Bianchi and coworkers, who postulated that F is a quasimonotone or pseudomonotone bifunction (Bianchi *et al.*1997).

Konnov, I. V. and Yao, J. C. introduced the concepts of a $F_{(C,D)}$ -pseudomonotone mapping and of a (C, D)-pseudo-monotone pair of mappings by employing Fan’s lemma also thet established several existence findings for generalized vector equilibrium issues see (Konnov and Yao 1999).

(Wang *et al.* 2001) offered an innovative approach to additional gradient computation as a potential solution to the problem of variational inequality. The method employs an improved step-size criterion and a new means of inspecting that is not used in any other projection-type methods. Its global convergent under a new extended monotonicity criterion is demonstrated. (Noor 2004) thought about an unexplored category of equilibrium issues called hemi-equilibrium issues. He proposed and examined a class of iterative algorithms for addressing hemiequilibrium issues using the auxiliary principle technique, wherein the convergence of the algorithms is conditional on either pseudo-monotonicity or partially relaxed strong monotonicity. He also investigated the hemiequilibrium issue class, which consists of equilibrium issues with nondifferentiable Lipschitz continuous functions. Several iterative strategies for addressing these types of equilibrium issues are proposed, and their merits are analyzed using the auxiliary principle technique in (Noor 2006).

In (Sach and Lee 2009) there are presented sufficient conditions for the local closedness and local openness properties as well as the lower (upper) semicontinuity characteristics of the solution sets of a general model that contains as special types a large number of generalized vector quasi-equilibrium issues with set-valued maps. There is also a model application, which can be thought of as a system of generalized vector quasi-equilibrium issues.

(Quoc *et al.* 2012) proposed an equilibrium theory for Hadamard manifolds that proves the presence of equilibrium points for a bifunction. Using the fact that Picard iteration converges for strongly nonexpansive mappings and the notion of resolvents for bifunctions in this context, applications are shown for variational inequality, fixed point, and Nash equilibrium issues. Then, an algorithm is created to approach equilibrium points. (Han and Huang 2017) characterized the generalized $l - B -$ well-posedness and the generalized $u - B -$well-posedness of set optimization issues. In addition, the Hausdorff upper semi-continuity of $l -$minimal solution mapping and $u -$minimal solution mapping is determined under the assumption that the set optimization issue is $l - H -$well-posed and $u - H -$well-posed, respectively.

Using a fixed point theorem involving the statuses of limited and unbounded closed convex subsets in real reflexive B-S, (Hashoosh and Jaber 2020) gave existence results for a general class of systems of nonlinear hemi-equilibrium issues.

Finally, (Bloach *et al.* 2022) established some metric characterizations for the well-posed tri-equilibrium like issues. Moreover, some conditions under which the equilibrium-like s are strongly well-posed is given.

1.2 Overviews

Equilibrium issues is a critical innovative concept and brought about general tools, such unified, rigorous and innovative tools that permitted us to study a wide spectrum of issues. These are issues that have originated from different areas of sciences.

The purpose of this thesis is to present new type of trifunction in equilibrium issues. By using KKM technique and according to relevant suppositions on the study of nonlinear mappings, we will prove the existence of a results for generalized equilibrium $GP(H, L_1, L_2)$ involving class of monotone operator in the setting of real reflexive B-S.

This thesis is composed of five chapters. The first chapter is devoted to the general introduction (background and Organization) to the thesis. This chapter gave a survey of the literature on equilibrium issues. Now we introduce the following chapters:

Chapter two presents a family of essential definitions along with some Theorems. The aim of this chapter is to assist the reader to get acquainted with main results grounded thereof. The chapter starts with a section on primary definitions of normed space before it ends up with a brief introduction to equilibrium issues. We can state all of the aims of this chapter at once by saying that we are trying to ground the investigation in some concepts of convex analysis required for the upcoming chapters. This being said, the chapter appeals to some concept derived from topological spaces and monotone trifunction. Finally, we mention fact concerning of Hemivariational inequality and hemiequilibrium issues.

There are three parts to Chapter 3, beginning with the Introduction. In the second section, we will examine several key terms for our research. In section three, existence results for a novel state of the trifunctional for equilibrium issue $GP(H, L_1, L_2)$ on convex-closed sets that are either limited or unbounded in B-Ss are introduced using a generalized type of monotonicity known as ϑL-g-monotone.

Chapter four is intended to be an conclusive investigation in our thesis in which we provided the necessary definitions on which to base our main findings in this field. As for the third section, it defines and extends the concept of the hemiequilibrium issues.

Finally, chapter five contains some conclusions and Future work. The solutions given in this thesis improve some corresponding results of many scholars.

2. PRELIMINARIES AND BASIC RESULTS

2.1 Introduction

This chapter introduces a number of essential concepts and tentative theories related to our primary findings. The essential goal of introducing these definitions and theorems is to help us derive main results obtained in this thesis that would be given in the upcoming chapters. With this goal in mind, the chapter begins with the very basic definitions of normed space before we end with a brief introduction to equilibrium issues.

By doing so, we base our investigation upon some notions of convex analysis that are necessary for our next chapters. Furthermore, we refer to a number of concepts derived from topological spaces. Additionally, we present basic definitions and notions of monotone trifunction.

2.2 Some Concepts of Normed Spaces

Definition 2.1. (Yosida 1980) Considering V be a nonempty set, and R is a real number, then the real function $d: V \times V \to R$ is called function metric, if the next conditions are hold

i. $d(k,s) \geq 0$, for all $k, s \in V$,
ii. $d(k,s) = 0$ if and only if $k = s$,
iii. $d(k,s) = d(s,k)$, for all $k.s \in V$,
iv. $d(k,s) \leq d(k,\omega) + d(\omega,s)$, for all $k, s, \omega \in V$.

Then the pair (V, d) is called a metric.

Definition 2.2. (Yosida 1980) Let V be a vector space on field F (where F is either the field of real or complex number). The function $||.||: V \longrightarrow F$ is called norm on V, if the following conditions are hold

i. $\parallel s \parallel \geq 0 \; \forall s \in V$,
ii. $\parallel s \parallel = 0$ if and only if $s = 0, \forall s \in V$,
iii. $\parallel rs \parallel = |r| \parallel s \parallel, \forall s \in V, r \in F$,
iv. $\parallel k + s \parallel \leq \parallel k \parallel + \parallel s \parallel, \forall k, s \in V$.

Then, the pair $(V, ||.||)$ is called a normed space.

Example 2.3. (Yosida 1980) Considering $j = C[0,1]$, where M is a space of real continuous on $[0,1]$ and $\|\cdot\|: j \longrightarrow R$defined as follow:

$$\parallel h \parallel = \int_0^1 |h(t)| dt.$$

It is easy to prove $\|\cdot\|$ norm on j, then j is a normed space.

Definition 2.4. (Yosida 1980) Consider $\{s_n\}$ is a sequence in normed space V, then $\{s_n\}$ is named to be bounded sequence, if there is a real positive number $r > 0$, in which

$$||s_n|| \leq r, \forall n \in Z^{+.}$$

Definition 2.5. (Yosida 1980) Consider V is a normed space, $a_0 \in V$ and r is real positive number, then the set $B = \{s \in V, ||s_0 - s|| < r\}$ is said to be an open ball by center s_0 and radius r, denoted $B_r(s_0)$.

Definition 2.6. (Yosida 1980) Consider $\{s_n\}$ is a sequence in normed space V, then $\{s_n\}$ is said to be converge sequence, if there is $a \in V$ in which

$$\forall\delta > 0, \exists r \in Z^{+}, \|s_n - s\| < \delta \ \forall n > r.$$

And, s the a convergent point to sequence $\{s_n\}$ and form write,

$$\lim_{n\to\infty} s_n = s, \vee\ s_n \to s.$$

Definition 2.7. (Yosida 1980) Consider $\{s_n\}$ is a sequence in a normed space$V$, then $\{s_n\}$ is said to be Cauchy sequence in V, if for every $\delta > 0$,there is $r \in Z^{+}$ in which

$$\|s_n - s_m\| < \delta, \forall n, m > r.$$

Theorem 2.8. (Yosida 1980) In a normed space, any convergent sequence is a Cauchy sequence.

Definition 2.9. (Yosida 1980) In a normed space V, the sequence $\{s_n\}$ is named to weakly converge if and only if there is $s \in V$ in which.

$$\lim_{n\to\infty} \beta(s_n) = \beta(s).$$

Definition 2.10. (Yosida 1980) The normed space V is said to be B-S if every Cauchy sequence in V convergent sequence to an element of V.

Definition 2.11. (Yosida 1980) Consider that V is B-S. A mapping $\Gamma: A \longrightarrow R$ is named to be

i. Lower semicontinuouse (briefly, lsc) at $s_0 \in V$ if

$$\Gamma(s_0) \leq \lim\inf\ \Gamma(s_n).$$

ii. Upper Semi Continuous (briefly, usc) at $s_0 \in V$ if

$$\Gamma(s_0) \geq lim\,sup\,\Gamma(s_n).$$

For any sequence $\{ s_n\}$ of V in which $s_n \longrightarrow s$.

Example 2.12. (Yosida 1980) Considering A be a nonempty and $\Gamma: A \to R$ in which for all $s \in A$ then

$$\Gamma(s) = \begin{cases} 0 & s < 1 \\ 2 & s = 1 \\ 1/2 & s > 1. \end{cases}$$

Then Γ is upper semicontinuous at $s = 1$, due to the fact that its value exceeds that of its neighbors.

Example 2.13. (Yosida 1980) Consider that A is a nonempty subset of real number and $\Gamma: A \to R$ in which for every $s \in A$ then

$$\Gamma(s) = \begin{cases} s^2 & \text{if } s \neq 0 \\ -1 & \text{if } s = 0. \end{cases}$$

Then, Γ is lower semicontinuous at $s = 0$.

Definition 2.14. (Yosida 1980) Consider that A and B are two normed spaces. A mapping $\beta: A \to B$ is called continuous at $s \in$ A if for any given $r > 0$ there is $\delta > 0$ in which

$\|\beta(k) - \beta(s)\| < r$ for all k ∈ A satisfying $\| k - s \| < \delta$.

Then, β is called continuous if it continuous at every point in A.

2.3 Convex Analysis

Within this section, we discuss several fundamental concepts and terminology of convex analysis that are essential to the work that we do.

Definition 2.15. (Zeidler 1986) Considering A be a nonempty subset of a vector space V. Then A is called convex set if

$$rs + (1 - r)k \in A, r \in [0,1], \forall s, k \in A.$$

Lemma 2.16. (Zeidler 1986) Assume that A and B are two convex sets in vector space V by a norm $\|.\|$.Then

i. The empty set is convex.
ii. $A + B = \{a + b; a \in A, b \in B\}$ is convex.
iii. $\delta A$is convex for any scaler δ .
iv. Every collection of convex sets will yield a convex result at their intersection.

Remark 2.17. (Zeidler 1986) It is not necessary for the union of two convex sets to be convex.

Below is an example that would explain to us why the union need not to be convex.

Example 2.18. (Zeidler 1986) Considering $V = R^3$ be a space and $M = \{(k, 0,0), a \in R\}, N = \{(a, s, 0); k + s = 0, k, s \in R\}$ are convex subset of V. If we take $\frac{1}{2} \in [0,1]$, then $\varkappa_1, \varkappa_2 \in M \varkappa_1 = (2,0,0) \in M, \varkappa_2 = (8, -8,0) \in M \cup N$. But

$$\frac{1}{2}\varkappa_1 + \frac{1}{2}\varkappa_2 = \frac{1}{2}(2,0,0) + \frac{1}{2}(8, -8,0)$$

$$= (1,0,0) + (4, -4,0)$$

$$= \quad (5, -4, 0).$$

Since $(5, -4, 0) \notin M$and N, then $(5, -4, 0) \notin M \cup N$. Hence $M \cup N$ is not convex.

Definition 2.19. (Zeidler 1986) Consider that A is a subset of B-S V, then the smallest convex set in V contained A is named to be convex hull denoted by conv(A).

Proposition 2.20. (Zeidler 1986) On convexity, the following conditions hold

i. $A \subseteq conv(A)$,
ii. $conv\ (A)$ = the intersection of all convex sets that contain A,
iii. If A is convex set then $A = conv(A)$.

Definition 2.21. (Zeidler 1986) Consider that A be subset of a vector space V, and $s_1, s_2 \ldots s_n \in A$, the combination for all $s_1, s \cdots s \in A, r_i \geq 0$ in which $\sum_{i=1}^{n} r_i = 1$, the combination $\sum_{i=1}^{n} r_i s_i \in A$ is said to be convex combination.

Definition (2.22). (Zeidler 1986) A function $\beta: A \rightarrow R$ is said to be convex function if

$$\beta(rk + (1-r)s) \leq r\beta(k) + (1-r)\beta(s), \forall\ k, s \in A, r \in [0,1].$$

The following example would make it clear the above definition.

Example 2.23. (Zeidler 1986) Considering $A = C[k, s]$, where A space of real continuous function on $[k, s]$ and$\beta: A \rightarrow R$ in which for every $h \in A$, we have

$$\beta(h(\varkappa)) = \int_k^s \ |h(\varkappa)| d\varkappa.$$

If we take $h(\varkappa), k(\varkappa) \in A$ and $r \in [0,1]$ then

$$\beta(rh(\varkappa) + (1-r)k(\varkappa))$$

$$= \int_{k}^{s} |rh(\varkappa) + (1-r)k(\varkappa)|d\varkappa$$

$$\leq r\int_{k}^{0} |h(\varkappa)|d\varkappa + (1-r)\int_{0}^{s} |k(\varkappa)|d\varkappa$$

$$= r\beta\big(h(\varkappa)\big) + (1-r)\beta\big(k(\varkappa)\big).$$

Definition 2.24. (Chadli and Chiang 2002) Considering $\beta: A \times A \to R$ be bifunction, then it is named vector 0-D convexity if, for each finiteset $\{s_1, s_2 \dots . s_n\}$ of A and $\alpha_i \geq 0$, where $i = \overline{1, n}$, by $a = \sum_{i=1}^{n} \alpha_i s_i$ and $\sum_{i=1}^{n} \alpha_i = 1$, one has the following:

$$\sum_{i=1}^{n} \alpha_i \beta(s, s_i) \geq 0.$$

Definition 2.25. (Chadli and Pany 2020) Consider that $\beta: A \times A \times A \longrightarrow R$ be a trifunction, then it is named vector 0-DC, if for every finite subsets $\{s_1, s_2, \dots, s_n\}$ and $\{k_1, k_2, \dots, k_n\}$ of A, $\alpha_i \geq 0, i = \overline{1, n}$ in which $k_0 = \sum_{i=1}^{n} \alpha_i \, k_i$ and $\sum_{i=1}^{n} \alpha_i = 1$ and

$$\sum_{i=1}^{n} \alpha_i \, \beta(s_i, k_0, k_i) \geq 0.$$

Definition 2.26. (Browderm1963) β is argued that a real-valued function defined on a convex subset A of V is H.C, if and only if the following coditions are met:

$$\lim_{r \to 0} \beta(rk + (1-r)s) = \beta(s) \forall k, s \in Ar \in [0,1].$$

Remark 2.27. (Hashoosh and Alimohammadi 2016) Considering V be a B-S and A be a nonempty-convex set of $V. h, \beta: A \times A \longrightarrow R\{+\infty\}$ are real bi-functions and$s_r = rk + (1-r)s$ and$a\,, b \in A, r \in [0,1]$ in which

$$\frac{\lim_{r\to 0} h(s, s_r)}{r} = 0;$$

$$h(k, s) \leqslant \frac{\lim_{r\to 0} r - 1}{r} [\beta(k, k) + h(s, s)].$$

2.4 Some Concepts From Topological Spaces

The results presented in this section are collection of the related ideas from topological spaces that will be used in subsequent chapters.

Definition 2.28. (Schaefer 2013) A topological space V is called Hausdorff space, if for any two disjoint points k and s in V, there is two disjoint open sets, A and B in V in which k ∈A , $s \in B$. Then

A∩ B =∅.

Definition 2.29. (Schaefer 2013) Consider V is a B-S with norm $\|.\|$. The topology on V induces by norm$\|.\|$is said to be norm topology or strong topology.

Remark 2.30. (Schaefer 2013) If V is the topology that is produced by norm, and W is the topology that is induced by functional, then W is weaker than V.

Definition 2.31. (Schaefer 2013) Let's say that V is a B-S, and that A ⊆ V. If set A is closed in the weak topology of space V, then we refer to set A as having a weak closure.

Definition 2.32. (I-Liang 2013) If A is a set. Then a collection of open sets $\{A_i\}$, i ∈ I is called cover of A if

$A \subset \bigcup_{i\in I} A_i$.

Definition 2.33. (I-Liang 2013) A subset A of a space V is compact if every open covering of A by an open set V possesses a finite sub covering.

Definition 2.34. (Schaefer 2013) If every sequence in a subset A of a B-S V contains a subsequence that converges weakly in A, then that subset A is referred to as being compact in the weak topology.

Definition 2.35. (Knaster and Kuratowski 1929) Considering A be a nonempty set of T.H.V.S, a mapping $\Gamma: A \longrightarrow P(V)$ is said to be KKM -mapping if , for any finite set $\{s_1, s_2, \dots, s_n\}$ of A, we ge$t$ conv $\{s_1 \dots . s_n\} \subseteq \cup_{i=1}^{n} \Gamma(s_i)$.

Example 2.36. (Knaster and Kuratowski 1929) Assume that $A = [0,2]$ is a nonempty from real numbers R and the function $\beta: A \rightarrow P(R)$ defined as:

$$\beta(s) = \begin{cases} [1,2] & s > 0 \\ 0 & s = 0. \end{cases}$$

Then, if we take $\{0,1\} \subseteq A$ and con $\{0,1\} = [0,1]$ and $\beta(0) = 0, \beta(1) = [1,2]$. Considering $s = 1/2 \in [0,1] = conv\,\{0,1\}$. But $s \notin \beta(0) \cup \beta(1) = \{0\} \cup [1,2]$.That means conv $\{0,1\} \not\subset \cup\, \beta(s_i) : s_i = 0,1$. Hence β is not KKM.

Lemma 2.37. (Fan 1972) Considering A is a nonempty subset of T.H.V.S V, and a mapping Γ:A⟶P(V) be KKM in addition to $\Gamma(s)$ is closed in V for every instance of $s \in A$ and has a compact representation for $s \in A$. As a result, $\cap_{s\in A} \Gamma(s) \neq \emptyset$, where diam refers to the diameter of the set A.

Theorem 2.38. (I-Liang 2013) A set B in R^n is compact if and only if it closed and bounded.

Theorem 2.39. (I-Liang 2013) If B is a set in $\boldsymbol{R^n}$ and has one of the following properties. Then, it has the other two.

i. B is closed and bounded,
ii. B is compact,
iii. Every infinite subset of B has a limit point in B.

Definition 2.40. (Clarke 1983) A function $A: S \rightarrow R$ is said local Lipschitz if for any $u \in S$ there is a neighborhod U of u and $L_u > 0$ in which

$$|A(w) - A(v)| \leq L_u \|w - v\|_S; \text{ for each } v, w \in U.$$

Definition 2.41. (Clarke 1983) Considering $J: S \rightarrow R$ is a local Lipschitz functional. A generalized derivatived of J at $u \in S$ in the direction of $v \in S$, symbolized by $J^0(u; v)$, is defined as:

$$J^0(u; v) = \limsup_{w \rightarrow u,\ \lambda \downarrow 0} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

Lemma 2.42. (Clarke 1983) Considering $J: S \rightarrow R$ be local Lipschitz function of rank L_u near the point $u \in S$. Then

i. $v \mapsto J^0(u; v)$ is a finite set, positively-homogeneous, sub-additive and fulfils that $|J^0(u; v)| \leq L_u \parallel v \parallel_S$;
ii. $J^0(u; v)$ is usc.
iii. $J^0(u; -v) = (-J)^0(u; v)$.

2.5 Monotone Function

The next part reviews several key concepts and definitions from the study of the monotone operator.

1. (Blum and Oetti 1994), presented monotone bifunction and defined it as the following: A bifunction $\beta: A \times A \rightarrow R$ is termed monotone bifunction if

$$\beta(k, s) + \beta(s, k) \leq 0 \ \ \forall\, k, s \in A.$$

2. (Riahi 1998) presented a new type of monotone bifunction, and calledit as Ψ– Monotone bifunction. According to him, it is defined as the following:

Assume that $\Psi, \beta: A \times A \to R$ are two bifunctions. Then, β is said to be Ψ-Monotone bifunction if

$$\beta(k,s) + \beta(s,k) \leq \Psi(k,s) + \Psi(s,k) \ \forall \ k,s \in A.$$

(Hashoosh 2016) presented a brand new type of monotone bifunction and gave it the name " α -monotone bifunction". It can be summed up as follows:

Assume that $\alpha, \beta: A \times A \to R$ are two bifunction. Then β is said to be α -Monotone bifunction if

$$\beta(k,s) + \beta(s,k) + \alpha(k,s) \leq 0, \forall \ k,s \in A.$$

On the other hand, some scholars have introduced other types of monotone trifunctions.

3. (Preda 2007) presented monotone trifunctions, and it is defined as the following:
   a. A trifunction $\beta: A \times A \times A \to R$ is said to be monotone if and only if for every $k,s \in A$. Then:

$\beta(k,s,k) - \beta(k,s,s) \geq 0.$

   b. A trifunction $\beta: A \times A \times A \to R$ is said to be general relaxed α -monotone if there is $\alpha: A \times A$ such that for every $a,b \in A$, we have

$\beta(k,s,k) - \beta(k,s,s) \geq \alpha(k,s).$

4. (Chadli and Ram 2020) present a new type of monotone trifunction. They are defined as following.

a. A trifunction $\beta: A \times A \times A \to R$ is said to be monotone trifunction if and only if:

$$\beta(k, s, \omega) + \beta(k, \omega, s) \leq 0 \text{ for all } k, s, \omega \in A.$$

b. A tri-function $\beta: A \times A \times A \to R\ is$ said tobe relaxed α –monotone, if and only if there is $\alpha: A \to R^+$, accompanied by $\alpha(rz) = r^k \alpha(z)$ for all $r > 0$ where $k > 1$ and $z \in A$ in which for every $k, s, \omega \in A$ then:

$$\beta(k, s, \omega) + \beta(k, \omega, s) \leq \alpha(s - \omega).$$

c. Considering A is a nonempty subset of a normed space, then a trifunction $\beta: A \times A \times A \to R$ is said to be γ -strongly monotone. If there is a positive constant γ, in which

$$\beta(k, s, \omega) + \beta(k, \omega, s) \leq -\gamma \parallel s - \omega \parallel^2, \text{ for all } k, s, \omega \in A.$$

2.6 Some Cases of Equilibrium Issues

The equilibrium issues provide a framework for plenty consequential issue such as optimization issues, variation inequality issues, complementarity issues, inequality issues and fixed point issues. Furthermore, The problem of equilibrium has been extensively applied to a variety of other fields of study, including economics, mechanics, and engineering science. During the course of the most recent few years, a wealth of findings about the existence of answers to questions pertaining to equilibrium have been established.

In this section, we present this type of issues called Ψ-equilibrium issues to deal with the study of the existence of solution for such a class of equilibrium issues as convex and closed sets in B-S (Hashoosh 2016), The issue is to find $\varkappa \in K$ in which,

$$F(\varkappa, y) + \Psi(\varkappa, y) \geq 0 \quad \text{for all } y \in K, \tag{2.1}$$

where $F, \Psi, \alpha: K \times K \to R$ is real-valued bifunction, K is a nonempty subset of B-SS, and F is an equilibrium function, where $F(\varkappa, \varkappa) = 0$. Let us recall now some special state of issue as below.

i. If $\Psi \equiv 0$ ergo the issue is then minimized to the classic equilibrium issue (briefly, EP), which is to find $\varkappa \in K$ in which $F(\varkappa, y) \geq 0$ for all $y \in K$; (Nikaido 1955).

ii. If $\Psi(\varkappa, y) = \Psi(y) - \Psi(\varkappa)$ for all $y \in K$ ergo, the issue is minimized to the mixed equilibrium issue (briefly, MEP); (Mahato 2013).

iii. If $\Psi(\varkappa, y) = \langle \beta\varkappa, y - \varkappa \rangle$ for all $y \in K$ ergo, the issue is minimized to the generalized equilibrium issue (briefly, GEP); (Kamraksa 2011).

Theorem 2.43. (Hashoosh 2016) Considering $F: K \times K \to R$ is a Ψ-monotone bifunction, with H.C in the 1.term and convex in the 2.term. If $\Psi, \alpha: K \times K \to R$ is convex in the 2.term, then the general equilibrium issue (EP_{Ψ}) is same to the next issue.

Let $\varkappa \in K$ in which

$$F(y, \varkappa) + \alpha(\varkappa, y) \leq \Psi(\varkappa, y) \text{ for all } y \in K. \qquad (2.2)$$

Theorem 2.45. (Hashoosh 2016) Provided that K is a non-empty, bounded-closed convex subset of a real reflexive BS X. Suppose that all of the conditions listed below are satisfied.

i. $F: K \times K \to R$ is α-monotone bifunction, 0-DC, H.C in 1.term and lsc, convex in second term;

ii. $\Psi: K \times K \to R$ is convex in the 2.term, usc in the 1.term and 0 diagonal convex;

iii. $\alpha: K \times K \to R$ is Isc in the 1.term and convex in the 2.term.

Then, (EP_{Ψ}) has a solution.

Corollary 2.45. (Hashoosh 2016) Assume that $0 \in K$ is a nonempty convex subset of B-S X and assumptions (i-iii) in Theorem (2.43) hold. Moreover,

i. $\Psi(\varkappa, 0) = 0$;

ii. $\zeta_1 = \{\varkappa \in K: \alpha(\varkappa, 0) \leq 0\}$ and $\zeta_2 = \{\varkappa \in K: F(0, \varkappa) \leq 0\}$ are relative compact sets.

Then, (EP_{Ψ}) has a solution.

Theorem 2.46. (Hashoosh 2016) Take into account the same assumptions as in Theorem (2.44), but this time without the constraint that K be bounded. Meanwhile, let's assume the following conditions hold:

i. Consider $\Psi(\varkappa, \varkappa) = 0$, for all $\varkappa \in K$,

ii. There is $\varkappa_0 \in K$ in which $F(\varkappa, \varkappa_0) + \Psi(\varkappa, \varkappa_0) \leq 0$, where $\varkappa \in K$ and $\| \varkappa \|$ is large enough.

Then, the issue (EP_{Ψ}) admits a solution.

We recall further results and theorems given in (Preda 2007) that are help in our work.

Theorem 2.47. (Preda 2007) Considering K be a nonempty closed convex subset of reflexive B-S E. Then the following issues are equivalent

P1: Find $x \in K$, in which

$$\Psi(y, x; x) \geq 0, \text{ for any } y \in K. \tag{2.3}$$

P2: Find $x \in K$, in which

$$\Psi(y, \varkappa; y) \geq \alpha(\varkappa, y), \text{ for any } y \in K. \tag{2.4}$$

Where the some assumptions holds.

Theorem 2.48. (Preda 2007) If we suppose that E is a real reflexive B-S and K is a nonempty bounded closed convex subset of E, then we can refer to E's dual space as E*. When all of the following prerequisites are satisfied:

i. $\Psi(y, \varkappa; \cdot)$ is H.C for any fixed $\varkappa, y \in K$;
ii. $\Psi(\cdot, \varkappa; z)$ is a convex and lsc function on K, for any fixed $\varkappa, z \in K$;
iii. $\Psi(\varkappa, y; z) + \Psi(y, \varkappa; z) = 0$ for all $\varkappa, y, z \in K$;
iv. $\Psi(y, \varkappa; \cdot)$ is α-monotone accompanied by $\lim_{t \searrow 0} \frac{\alpha(\varkappa, \varkappa + t(y - \varkappa))}{t} = 0$;
v. $\alpha(\cdot, y)$ is weakly lsc for any fixed $y \in K$, i.e., for any sequence $\{\varkappa_n\}$ that converges to $\varkappa$ in $\sigma(\mathrm{E}, \mathrm{E}^*)$ we have

$$\alpha(\varkappa, y) \leq \lim_{n \to \infty} \inf \alpha(\varkappa_n, y), \text{ for any } y \in K.$$

Then, the issue (2.3) is solvable.

2.7 Hemi-equilibrium Issues

In a nutshell, hemiequilibrium issues are a unifying framework for a wide variety of important issues, including optimization, variation inequality, complemen-tarity, minimax inequality, and fixed point issues. Additionally, hemiequilibrium issues have been widely used to the study and advancement of various sciences, including economics, mechanics, and engineering. Recent years have seen the introduction of numerous conclusions pertaining to the presence of solutions for equilibrium issues and vector equilibrium issues (see(Blum and Oetti 1994), (Ansari 1999) and (Bianchi 2004)).

This section deals with the study of the existence and uniqueness of solutions for the class of hemiequilbrium issues on convex and closed sets in B-S. The issues of this sort are referred to as hemiequilibrium issues (Alwan *et al.* 2019). The following is a definition of the equation of the hemiequilibrium issue:

Find a $u \in K$ in which

$$\vartheta(u, v) + J^0(u, v - u) \geq 0, \text{ for all } v \in K. \tag{2.5}$$

Where $\vartheta: K \times K \to R$ is real-value bifunction, K is a nonempty subset of a B-S S, then ϑ in (2.5) is which $\vartheta(u, u) = 0$. And considering $\forall\lambda \in (0,1)$ then

$$\lim_{\lambda \to 0} \frac{\psi(u, u_\lambda)}{\lambda} + \psi(u, v) = 0. \tag{2.6}$$

Here some existence results and uniqueness of solutions under suitable conditions for issue (2.5) are studied, in which used in chapter three.

Lemma 2.49. (Alwan *et al.* 2019) Assume $\vartheta, \psi: K \times K \to R$ are two bifunctions. If ϑ is ψ-monotone bifunction and the follows are satisfy:

(i) ϑ, ψ are H.C in 1.term and convex in 2.term.

(ii) $J: S \to R$is a locally Lipschitz functional .

Then a hemiequilibrium issue (2.5) is equivalent to the following issue:

Find a $u \in K$ in which

$$\vartheta(v, u) - \psi(u, v) \leq J^o(u, v - u) + \psi(v, u).$$

Theorem 2.50. (Alwan *et al.* 2019) Considering K be a nonempty closed bounded convex subset of reflexive B-S S. Assume $\vartheta, \psi: K \times K \to R$ are two bifunctions. If the following requirements are met, then the issue (2.5) can have at least one solution:

i. ϑ is ψ-monotone bifunction ;

ii. $\vartheta(\cdot, u)$ and $\psi(\cdot, v)$ are H.C ;

iii. $\vartheta(u,\cdot)$and $\psi(u,\cdot)$ are convex;

iv. $\vartheta(u,\cdot)$ is l.s.c and $\psi(\cdot,\cdot)$ is u.s.c.

3. RESULTS OF GENERALIZED EQUILIBRIUM ISSUES WITH TRIFUNCTION

3.1 Introduction

By generalizing a novel kind of trifunction equilibrium issues in a B-S, the purpose of this chapter is to prove some discoveries concerning the existence and uniqueness of the solution to (EP). This will be accomplished by proving that (EP) has a solution. In addition to this, we examine a novel example of the trifunction $GP(H, L_1, L_2)$ so that we can uncover altering contexts which the equilibrium state is maintained through the utilization of ϑL-g- monotone subsets, notably convex and closed sets (bounded or unbounded). We have arrived at this point to vouch for the reality of the solution to (EP), thereby enhancing the most recent properties in its body of published work.

The chapter is divided into three sections in order to achieve the aforementioned purpose. The second section reviews some definitions that are significant to the research. The third section introduces ϑL-g monotone, a novel type of monotonicity. At the conclusion of this chapter, we attempt to demonstrate the existence and uniqueness of $GP(H, L_1, L_2)$.).

3.2 Preliminaries

Considering A be a nonempty set of real reflexive B-S V and $L_1, \vartheta: A \times A \longrightarrow R$ be real bi-functions, $L_2: A \times A \times A \longrightarrow R$ is a real tri function, $H: A \to V^*$ be an arbitrary nonlinear mapping and $g: A \longrightarrow A$ is affine function. By the following generalized equilibrium briefly $GP(H, L_1, L_2)$ we understand the issue of understand the issue of finding $s \in A$ in which

$$\langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) \geq 0 \forall k, g(\omega) \in A. \qquad (3.1)$$

It is interesting to mention below some special cases:

i. If $L_1 = 0, H = 0$ and $s \equiv g(\omega)$ then, issue $GP(H, L_1, L_2)$ is minimizes to the trifunction equilibrium issue posed by scientist Preda et al which is to find $s \in A$ in which $L_2(k, s, s) \geq 0 \; \forall \; k \in A$, see (Preda et al 2007).

ii. If $L_1\big(s, g(\omega)\big) \coloneqq h\big(s, g(\omega)\big) - h(s, s)$ and $H = 0$, then, the issue $GP(H, L_1, L_2)$ is minimizes to the form in general mixed equilibrium issue which is to find $s \in A$ in which $L_2\big(k, s, g(\omega)\big) + h\big(s, g(\omega)\big) - h(s, s) \geq 0$, see (Chadli 2002).

iii. If $L_2 = 0$, $H = 0$, $k \equiv g(\omega)$ issue $GP(H, L_1, L_2)$ is minimizes to the classical equilibrium issue which is to find $s \in A$ in which $L_1(s, k) \geq 0, \forall k \in A$; see Nikaidô and Isoda 1955).

iv. If $L_2\big(k, s, g(\omega)\big) = L_2\big(g(\omega)\big) - L_2(s)$ and $H = 0$, then the issue $GP(H, L_1, L_2)$ is minimizes to the form in mixed equilibrium issue see for instance (Moudafi 2002), which is to find $s \in A$ in which

$$L_1\big(s, g(\omega)\big) + L_2\big(g(\omega)\big) - L_2(s) \geq 0 \;\; \forall \; g(\omega) \in A.$$

Here, we take a look at the equilibrium trifunction$GP(H, L_1, L_2)$ in which for all $k, s \in A$ then $L_2(k, s, s) = 0$.

Remark 3.1. Considering V be a B-S and A be a nonempty-convex set of $V. \vartheta, L_1: A \times A \longrightarrow R\{+\infty\}$ bifunctions, $g: A \longrightarrow A$ is affine function and $s_r = rg(\omega) + (1 - r)s$, and $s, g(\omega) \in A, r \in [0,1]$. Then

$$\lim_{r \to 0} \frac{\vartheta(s,\; s_r)}{r} = 0;$$

$$\vartheta\big(s, g(\omega)\big) \leqslant \lim_{r \to 0} \frac{(r-1)}{r} [L_1(s, s) + \vartheta(s, s)].$$

3.3 Main Results on Tri- Equilibrium Issue

In the first part of this section, the important findings obtained from the Tri-equilibrium function will be discussed, where we aim to synthesize some results about the existence of an equilibrium issue $GP(H, L_1, L_2)$ using the technique of 0-DC. Second, based on the results that we have got, we are able to demonstrate that there is a solution to the problem of $GP(H, L_1, L_2)$ without the hypothesis that set A is bounded. Finally, The solution's uniqueness is shown.

Here, we introduce a new kind of monotonicity is called ϑL-g-monotone and defined as:

A trifunction$L_2: A \times A \times A \to R$ is called ϑL-g-monotone if

$$L_2\big(k, s, g(\omega)\big) + L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \leq 0.$$

where $\vartheta: A \times A \longrightarrow R$ is a $r$eal bifunctions. The researchers present the first result, which center around the issues of equivalence with $GP(H, L_1, L_2)$.

Theorem 3.2. Assume that $L_2: A \times A \times A \to R$ is ϑL-g-monotone trifunction H.C in the 2-term, convex in 3-term,ϑ , $L_1: A \times A \to R$ be convex in the 2-term, $H: A \to V^*$ be arbitrary nonlinear mapping and $g: A \longrightarrow A$is affine function. then the equilibrium issue $GP(H, L_1, L_2)$ is equivalent to the following issue:

$$L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \leq \langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big). \qquad (3.2)$$

Proof. Suppose that $GP(H, L_1, L_2)$has a solution $s \in A$ in which

$$\langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) \geq 0 \forall k, g(\omega) \in A.$$

SinceL_2 is ϑL-g-monotone trifunction, then

$$L_2\big(k,s,g(\omega)\big)+L_2(k,g(\omega),s)+\vartheta\big(s,g(\omega)\big)\le 0.$$

So,

$$L_2(k,g(\omega),s)+\vartheta\big(s,g(\omega)\big)\le -L_2\big(k,s,g(\omega)\big).$$

Notice that, from $GP(H,L_1,L_2)$ it follows that

$$L_2(k,g(\omega),s)+\vartheta\big(s,g(\omega)\big)\le \langle Hs,g(\omega)-s\rangle+L_1\big(s,g(\omega)\big).$$

Therefore, $s\in A$ is a solution of issue (3.2).

Assume that $s\in A$ is a solution of issue (3.2), Considering $k,g(\omega)\in A$, and $s_r=rg(\omega)+(1-r)s, r\in$ [0,1]. Hence, $s_r\in A$. From convexity of A. Therfore,

$$L_2(k,g(\omega),s)+\vartheta\big(s,g(\omega)\big)\le \langle Hs,g(\omega)-s\rangle+L_1\big(s,g(\omega)\big).$$

To conclude, then,

$$L_2(k,s_r,s)+\vartheta(s,s_r)\le \langle Hs,s_r-s\rangle+L_1(s,s_r). \tag{3.3}$$

Since, $0=L_2(k,s_r,s_r)$, then

$$0=L_2(k,s_r,s_r)$$

$$\le rL_2\big(k,s_r,g(\omega)\big)+(1-r)L_2(k,s_r,s),$$

It means that,

$$r\left[L_2(k,s_r,s)-L_2\big(k,s_r,g(\omega)\big)\right] \le L_2(k,s_r,s). \qquad (3.4)$$

By Convexity of ϑ, L_1 in the 2-term, one can get

$$L_1(s,s_r) \le rL_1\big(s,g(\omega)\big)+(1-r)L_1(s,s).$$

Then,

$$r\left[L_1(s,s)-L_1\big(s,g(\omega)\big)\right] \le L_1(s,s)-L_1(s,s_r). \quad (3.5)$$

Also, $\qquad \vartheta(s,s_r) \le r\vartheta\big(s,g(\omega)\big)+(1-r)\vartheta(s,s)$

$$r\left[\vartheta(s,s)-\vartheta\big(s,g(\omega)\big)\right] \le \vartheta(s,s)-\vartheta(s,s_r). \quad (3.6)$$

From (3.4), (3.5) , (3.6) and (3.3) we obtain that

$$r\left[L_2(k,s_r,s)-L_2\big(k,s_r,g(\omega)\big)+L_1(s,s)-L_1\big(s,g(\omega)\big)+\vartheta(s,s)-\vartheta\big(s,g(\omega)\big)\right]$$

$$\le L_2(k,s_r,s)+L_1(s,s)-L_1(s,s_r)+\vartheta(s,s)-\vartheta(s,s_r) \ (3.7)$$

$$\le r\langle Hs,g(\omega)-s\rangle-2\vartheta(s,s_r)+L_1(s,s)+\vartheta(s,s).$$

Since the hemicontinuouity of L_2, we have

$$r\left[-L_2\big(k,s_r,g(\omega)\big)-L_1\big(s,g(\omega)\big)-\langle Hs,g(\omega)-s\rangle\right]$$

$$\le -2\vartheta(s,s_r)+r\vartheta\big(s,g(\omega)\big)+(1-r)[L_1(s,s)+\vartheta(s,s)],$$

divided by (-r), one can obtain

$$\langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big)$$

$$\geq \frac{2\vartheta(s, s_r)}{r} - \vartheta\big(s, g(\omega)\big) - \frac{(1-r)}{r}[L_1(s, s) + \vartheta(s, s)].$$

From Remark (3.1) we observe that

$$\langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) \geq 0.$$

That is means $s$is a solution of issue $GP(H, L_1, L_2)$. Therefore, we are done with the proof.

Next, in second main result we proved the existence of the issue $GP(H, L_1, L_2)$.

Theorem 3.3. Suppose if a nonempty convex-closed and bounded set A of reflexive B-S V, and further assume that the following are true:

i. $L_1: A \times A \to R$is convex in 2- term, u.s.c in 1. term and 0-DC

ii. $L_2: A \times A \times A \to R$is ϑL-g-monotone trifunction, H.C in 2.term, 0-DC and (l.s.c) in 3- term.

iii. $\vartheta: A \times A \to R$ is (l.s.c) in 1. term and convex in 2.term.

iv. $s \mapsto \langle Hs, g(\omega) - s\rangle$ u.s.c on A by respect weak*topology of V*.

v. $g: A \longrightarrow A$ is affine function.

Then, the issue $GP(H, L_1, L_2)$ have one solution.

Proof. If define set-valued mappings $\Gamma, S: A \to 2^V$ as follow, $\forall g(\omega) \in A$

$$\Gamma\big(g(\omega)\big) = k \in A\big\{s \in A; \langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) \geq 0\big\}$$

$$S\big(g(\omega)\big) = k \in A\big\{s \in A; L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big)$$
$$\leq \langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big)\big\}.$$

Then, the issue has a solution :

$$\bigcap_{k\in A} \Gamma(g(\omega)) \neq \emptyset.$$

If $g(\omega) = s$, then $\langle Hs, s - s\rangle + L_1(s, s) + L_2(k, s, s) = 0 \geq 0$. It means that $\big(g(\omega)\big) \neq \emptyset$.

We will now use the opposite to show that $\Gamma\big(g(\omega)\big)$ is KKM-mapping. Considering $\Gamma\big(g(\omega)\big)$ be not KKM, then there is a finite set $\{s_1, s_2, \dots, s_n\}$ of A and $\alpha_i \geq 0$, $i = 1,2, \dots, n$, $\sum_{i=1}^{n} \alpha_i = 1$, in which $s_0 = \sum_{i=1}^{n} \alpha_i s_i \notin \bigcup_{i=1}^{n} \Gamma(s_i)$. From this we conclude, $s_0 \notin \Gamma(s_i)$ $\forall i = 1,2, \dots, n$, it means that,

$$\langle Hs_0, s_i - s_0\rangle + L_1(s_0, s_i) + L_2(k, s_0, s_i)) < 0. \tag{3.8}$$

So,

$\sum_{i=1}^{n} \alpha_i [\, \langle Hs_0, s_i - s_0\rangle + L_1(s_0, s_i) + L_2(k, s_0, s_i)) < 0]$.

From the convexity of L_1 and L_2, this would contradict 0-DC. Then, Γ is KKM-mapping. Next, from Theorem (3.2) we have that$\Gamma\big(g(\omega)\big) \subseteq S\big(g(\omega)\big)$ for all $g(\omega) \in A$. Hence $S\big(g(\omega)\big)$ is KKM. Since $L_2(k, g(\omega), .), \vartheta\big(., g(\omega)\big)$ are (l.s.c) and $L_1(., s)$ and $s \longmapsto \langle Hs, g(\omega) - s\rangle$ is (u.s.c), then

$L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \leq \lim_{n\to\infty} inf\big[L_2(k, g(\omega), s_n) + \vartheta\big(s_n, g(\omega)\big)\big]$

$$\leq \lim_{n\to\infty}\left[\langle Hs_n, g(\omega) - s_n\rangle + L_1\big(s_n, g(\omega)\big)\right]$$

$$\leq \langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big).$$

So,

$$L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \leq \langle Hs, g(\omega) - s\rangle + L_1\big(s, g(\omega)\big).$$

Therefore $S\big(g(\omega)\big)$ is weakly-closed $\forall g(\omega) \in A$, A is weakly compact because it is a nonempty, limited, closed, and convex subset of real reflexive B-S V, as shown by Lemma (2.37) and Theorem (3.2).

$$\bigcap_{k\in A} \Gamma(g(\omega)) = \bigcap_{k\in A} S(g(\omega)) \neq \emptyset.$$

Consequently, the issue $GP(H, L_1, L_2)$ has a solution. This ends the proof.

Here, we would like to point out that through the Theorem (3.3) we obtained this Corollary, but without the condition of compactness. Therefore, we are done with the proof.

Corollary 3.4. Consider that A is a nonempty-convex set of B-S V in which $0 \in A$. If the assumptions (i-v) in Theorem (3.3) and the next assumptions hold

i. $L_1(s, 0) = 0$,

ii. $S_1 = \{s \in A; \vartheta(s, 0) \leq -\langle Hs, s\rangle\}$ and $S_2 = \{s \in A; L_2(k, 0, s) \leq 0\}$ are relativly compact sets.

Then, $GP(H, L_1, L_2)$ has a solution.

Proof. It suffices to prove $S(g(\omega))$ is compact for some $g(\omega) \in A$

$$\mathrm{S}(g(\omega)) = \bigcap_{k\in A} \{s \in A; L_2(k, g(\omega), s) + \vartheta(s, g(\omega)) \leq \langle Hs, g(\omega) - s\rangle + L_1(s, g(\omega))\}.$$

$$= \bigcap_{k\in A} \left\{ s \in A; L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \right.$$

$$\left. \leq \langle Hs, g(\omega) - s\rangle + \frac{L_1\big(s, g(\omega)\big)}{2} + \frac{L_1(s, g(\omega))}{2} \right\}.$$

Then,

$$\mathrm{S}\big(g(\omega)\big) \subseteq \bigcap_{\mathrm{k}\in\mathrm{A}} \left\{ s \in A; L_2(k, g(\omega), s) \leq \frac{L_1\big(s, g(\omega)\big)}{2} \right\}$$

$$\cup \left\{ \mathrm{s} \in \mathrm{A}; \vartheta(s, g(\omega)) \leq \frac{L_1\big(s, g(\omega)\big)}{2} + \langle Hs, g(\omega) - s\rangle \right\}.$$

So,

$$\mathrm{S}(0) \subseteq \bigcap_{\mathrm{k}\in\mathrm{A}} \left\{ \mathrm{s} \in \mathrm{A}; L_2(k, 0, s) \leq \frac{L_1(s, 0)}{2} \right\} \cup \left\{ \mathrm{s} \in \mathrm{A}, \vartheta(s, 0) \leq \frac{L_1(s, 0)}{2} + \langle Hs, 0 - s\rangle \right\}.$$

From conditions (i-ii), we have $\Gamma(0)$ is a subset of relative compact sets S_1 and S_2, then $\Gamma(e)$ is a compact for some $g(\omega) \in A$, closed for every $g(\omega) \in A$ and by Lemma (2.37) we have

$$\bigcap_{g(\omega)\in\mathrm{A}} \Gamma\big(g(\omega)\big) \neq \emptyset.$$

Then, $GP(H, L_1, L_2)$ admits solution. Therefore, we are done with the proof.

We would like to point out that the next result is a development of the Theorem (3.3) after dispensing with the boundedness clause.

Theorem 3.5. Taking into account the fact that the identical hypotheses as in Theorem (3.3) hold without the boundedness of A. Suppose additionally the following assumptions are satisfied:

i. $L_1(s,s) = 0$,
ii. For all $g(\omega)_0 \in A$, in which

$$L_1(s, g(\omega)_0) + L_2(k, s, g(\omega)_0) + \langle Hs, g(\omega)_0 - s\rangle < 0, \text{ for all } k \in A,$$

where $s \in A$ and $\| s \|$ is large encugh in which $\| s \| \geqslant \|g(\omega)_0\|$.

Then, the issue $GP(H, L_1, L_2)$has a solution.

Proof. Take a set $A_n = \{g(\omega) \in A; \| g(\omega) \| \leq n, n > 0\}$and consider the issue is to find $s_n \in A_n$ in which

$$\langle Hs_n, g(\omega) - s_n\rangle + L_1\big(s_n, g(\omega)\big) + L_2\big(k, s_n, g(\omega)\big) \geq 0, \forall k, g(\omega) \in A_n$$

Based on the boundedness of A_n and Theorem (3.3), we conclude that the issue $GP(H, L_1, L_2)$ has at least one solution.

For all $n > 0$ we will choose $\|g(\omega)_0\| < n_0$, then A_{n_0} has one solution s_{n_0} in which

$$\langle Hs_{n_0}, g(\omega)_0 - s_{n_0}\rangle + L_1\big(s_{n_0}, g(\omega)_0\big) + L_2\big(k, s_{n_0}, g(\omega)_0\big) \geq 0. \qquad (3.9)$$

So, $\|s_{n_0}\| \leq n_0$ because $s_{n_0} \in A_{n_0}$ if $\|s_{n_0}\| = n_0$, that is s_{n_0} large enough, From condition (ii), we conclude

$$\left\langle Hs_{n_0}, g(\omega)_0 - s_{n_0}\right\rangle + L_1\left(s_{n_0}, g(\omega)_0\right) + L_2\left(k, s_{n_0}, g(\omega)_0\right) < 0 \text{ for all } k \in A \quad (3.10)$$

Which is contradictions with (3.8). Then, $\left\|s_{n_0}\right\| < n_0$. For each $g(\omega) \in A$, one can take $r \in (0,1)$, small enough in which $z = rg(\omega) + (1-r)s_{no} \in A_{n_0}$. Since A_{n_0} convex, so $z \in A_{n_0}$. From (3.8), for each $z \in A$, one can get

$$0 \leq \left\langle Hs_{n_0}, z - s_{n_0}\right\rangle + L_1\left(s_{n_0}, z\right) + L_2\left(k, s_{n_0}, z\right)$$

$$\leq \left\langle Hs_{n_0}, rg(\omega) + (1-r)s_{no} - s_{n_0}\right\rangle + L_1\left(s_{n_0}, rg(\omega) + (1-r)s_{no}\right) + L_2\left(k, s_{n_0}, rg(\omega) + (1-r)s_{no}\right)$$

$$\leq r\left\langle Hs_{n_0}, g(\omega) - s_{no}\right\rangle + rL_1\left(s_{n_0}, g(\omega)\right) + (1-r)L_1\left(s_{n_0}, s_{no}\right) + rL_2\left(k, s_{n_0}, g(\omega)\right) + (1-r)L_2\left(k, s_{n_0}, s_{n_0}\right).$$

Dividing by r

$$0 \leq \left\langle Hs_{n_0}, g(\omega) - s_{no}\right\rangle + L_1\left(s_{n_0}, g(\omega)\right) + L_2\left(k, s_{n_0}, g(\omega)\right) \text{ for all } k, g(\omega) \in A.$$

Then s_{n_0} is a solution of $GP(H, L_1, L_2)$. Therefore, we are done with the proof.

Let's end this section with an introduction the uniqueness of the solution for the issue $GP(H, L_1, L_2)$.

Theorem 3.6. Consider that the previous hypotheses from Theorem (3.3) are still valid.. By adding:

i. $L_1$is a monotone bifunction,

ii. There is $r > 0$, in which

$L_2\big(k, s, g(\omega)\big) \leq -r\|s - g(\omega)\|^2 \quad \forall k, s, g(\omega) \in A,$

iii. There is $t > 0$, in which

$\langle Hs, g(\omega) - s\rangle \leq -t\|s - g(\omega)\|^2 \quad \forall k, s, g(\omega) \in A.$

Then, $GP(H, L_1, L_2)$ has a unique solution.

Proof. The existence of solutions is deduced by Theorem (3.3), hence the only thing left to do is demonstrate that each solution is unique. Taking it for granted that the problem can be solved in two different ways, say s_1 and s_2 for $GP(H, L_1, L_2)$ so we have,

$$\langle Hs_1, g(\omega) - s_1\rangle + L_1\big(s_1, g(\omega)\big) + L_2\big(k, s_1, g(\omega)\big) \geq 0$$

$$\langle Hs_2, g(\omega) - s_2\rangle + L_1\big(s_2, g(\omega)\big) + L_2\big(k, s_2, g(\omega)\big) \geq 0.$$

Since $s_1, s_2 \in A$, we have,

$$\langle Hs_1, s_2 - s_1\rangle + L_1(s_1, s_2) + L_2(k, s_1, s_2) \geq 0,$$

$$\langle Hs_2, s_1 - s_2\rangle + L_1(s_2, s_1) + L_2(k, s_2, s_1) \geq 0.$$

Then,

$\langle Hs_1, s_2 - s_1\rangle + L_1(s_1, s_2) + L_2(k, s_1, s_2) + \langle Hs_2, s_1 - s_2\rangle + L_1(s_2, s_1) +$
$L_2(k, s_2, s_1) \geq 0.$

This is rewarding,

$$\langle Hs_1, s_2 - s_1\rangle + L_2(k, s_1, s_2) + \langle Hs_2, s_1 - s_2\rangle + L_2(k, s_2, s_1) \geq -L_1(s_1, s_2) - L_1(s_2, s_1) \tag{3.11}$$

Since L_1 is a monotone bifunction, we have

$$L_1(s_1, s_2) + L_1(s_2, s_1) \leq 0 \forall s_1, s_2 \in A, \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\langle Hs_1, s_2 - s_1\rangle + L_2(k, s_1, s_2) + \langle Hs_2, s_1 - s_2\rangle + L_2(k, s_2, s_1) \geq 0. \tag{3.13}$$

From condition (ii), one can get

$$L_2(k, s_1, s_2) \leq -r\|s_1 - s_2\|^2,$$

$$L_2(k, s_2, s_1) \leq -r\|s_2 - s_1\|^2,$$

So,

$$L_2(k, s_1, s_2) + L_2(k, s_2, s_1) \leq -2r\|s_2 - s_1\|^2. \tag{3.14}$$

Also,

$$\langle Hs_1, s_2 - s_1\rangle \leq -t\|s_1 - s_2\|^2.$$

And

$$\langle Hs_2, s_1 - s_2\rangle \leq -t\|s_2 - s_1\|^2$$

$$\langle Hs_1, s_2 - s_1\rangle + \langle Hs_2, s_1 - s_2\rangle \leq -2t\|s_2 - s_1\|^2. \tag{3.15}$$

From (3.14), (3.15) and (3.13), one can obtain

$$0 \leq \langle Hs_1, s_2 - s_1 \rangle + L_2(k, s_1, s_2) + \langle Hs_2, s_1 - s_2 \rangle + L_2(k, s_2, s_1)$$

$$\leq -2(r+t)\|s_2 - s_1\|^2 \leq 0.$$

A contradiction, then $\|s_2 - s_1\|^2 = 0 \Longrightarrow s_2 = s_1$.

Therefore $GP(H, L_1, L_2)$ has a unique solution. Therefore, we are done with the proof.

4. FINDINGS OF HEMI-EQUILIBRIUM PRBLEMS

4.1 Introduction

By generalizing a new type of hemi-equilibrium issues trifunction in B-S, denoted as $HEP(J, L_1, L_2)$ the purpose of this chapter is to prove some findings of the existence of the solution to (HEP). This is done so that novel situations can be discovered. under which the hemi-equilibrium state holds by making use of ϑL-g -monotone trifunction.

In order to establish the above objective, the chapter comprises of three sections. Section two recalls some definitions that are significant to the study. Section three introduces a new kind of hemi-equilibrium issues trifunction. By the end of the chapter, we attempt to establish the existence and for $HEP(J, L_1, L_2)$.

4.2 Preliminaries

Considering A be a nonempty set of real-reflexive B-S V.and $L_1, \vartheta: A \times A \longrightarrow R$ be valued bifunctions, $L_2: A \times A \times A \longrightarrow R\ is$ real-valued tri-function, $J: V \to R$ locally Lipschitz functional. and $g: A \longrightarrow A$ is affine function. By the following generalized hemi-equilibrium briefly $HEP(J, L_1, L_2)$ we understand the issue of finding $s \in A$ in which

$$L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) + J^0(s, g(\omega) - s) \geq 0\ \forall\ k, g(\omega) \in A (4.1)$$

It is interesting to mention below some special cases:

iv. If $L_1 = 0, J^0 = 0$ and $s \equiv g(\omega)$ then, issue $HEP(J, L_1, L_2)$ is minimizes to the trifunction equilibrium issue , which is to find $s \in A$ in which $L_2(k, s, s) \geq 0\ \forall\ k \in A$,see (Preda *et al.* 2007).

v. If $L_1(s, g(\omega)) = h(s, g(\omega)) - h(s, s)$ and $J^0 = 0$, then the issue $HEP(J, L_1, L_2)$ is minimizes to the form in general mixed equilibrium issue which is to find $s \in A$ in which $L_2(k, s, g(\omega)) + h(s, g(\omega)) - h(s, s) \geq 0$, see (Chadli 2002).

vi. If $L_2 = 0$, $J^0 = 0$, $k \equiv g(\omega)$ issue $HEP(J, L_1, L_2)$ is minimizes to the classical equilibrium issue which is to find $s \in A$ in which $L_1(s, k) \geq 0$ for all $k \in A$; see (Nikaidô and Isoda 1955).

vii. If $L_2(k, s, g(\omega)) = L_2(g(\omega)) - L_2(s)$ and $J^0 = 0$, then the issue $HEP(J, L_1, L_2)$ is minimizes to the mixed equilibrium (MEP) see for instance (Moudafi 2002), which is to find $s \in A$ in which

$$L_1(s, g(\omega)) + L_2(g(\omega)) - L_2(s) \geq 0 \forall g(\omega) \in A.$$

4.3 Main Results

In this section, we present some results for the hemiequilibrium issue $HEP(J, L_1, L_2)$ in this section. Notably, the outcomes of this section show that the issue $HEP(J, L_1, L_2)$ is without making the assumption that set A is bounded.

In the next lemma, we are going to make the assumption that A is a nonempty set of areal reflexive B-S V.

Lemma 4.1. Assume that $L_2: A \times A \times A \to R$ is ϑL-g-monotone trifunction H.C in the 2-term, convex in 3-term,ϑ,$L_1: A \times A \to R$be convex in the 2-term, $J: V \to R$ locally Lipschitz functional. and $g: A \longrightarrow A$ is affine function, then the equilibrium issue $HEP(J, L_1, L_2)$ is equivalent to the following issue:

$$L_2(k, g(\omega), s) + \vartheta(s, g(\omega)) \leq J^0(s, g(\omega) - s) + L_1(s, g(\omega)). \quad (4.2)$$

Proof. Suppose that $HEP(J^0, L_1, L_2)$ has a solution $s \in A$ in which

$L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) + J^0\big(s, g(\omega) - s\big) \geq 0 \;\forall\; k, g(\omega) \in A.$

Since L_2 is ϑL-g-monotone trifunction, then

$$L_2\big(k, s, g(\omega)\big) + L_2\big(k, g(\omega), s\big) + \vartheta\big(s, g(\omega)\big) \leq 0.$$

So,

$$L_2\big(k, g(\omega), s\big) + \vartheta\big(s, g(\omega)\big) \leq -L_2\big(k, s, g(\omega)\big).$$

Notice that, from $HEP(J, L_1, L_2)$ it follows that

$$L_2\big(k, g(\omega), s\big) + \vartheta\big(s, g(\omega)\big) \leq J^0\big(s, g(\omega) - s\big) + L_1\big(s, g(\omega)\big).$$

Therefore, $s \in A$ is a solution of issue (4.2).

Assume that $s \in A$ is a solution of issue (4.2), Considering $k, g(\omega) \in A$, and $s_\varepsilon = \varepsilon g(\omega) + (1 - \varepsilon)s, \varepsilon \in [0,1]$ then $s_\varepsilon \in A$ because A is convex set. Thus,

$$L_2\big(k, g(\omega), s\big) + \vartheta\big(s, g(\omega)\big) \leq J^0\big(s, g(\omega) - s\big) + L_1\big(s, g(\omega)\big).$$

Then,

$$L_2(k, s_\varepsilon, s) + \vartheta(s, s_\varepsilon) \leq J^0(s, s_\varepsilon - s) + L_1(s, s_\varepsilon).$$

$$L_2(k, s_\varepsilon, s) + \vartheta(s, s_\varepsilon) - L_1(s, s_\varepsilon) \leq \varepsilon J^0\big(s, g(\omega) - s\big). \qquad (4.3)$$

Since, $0 = L_2(k, s_\varepsilon, s_\varepsilon)$, then

$$0 = L_2(k, s_\varepsilon, s_\varepsilon)$$

$$\leq \varepsilon L_2\big(k, s_\varepsilon, g(\omega)\big) + (1-\varepsilon) L_2(k, s_\varepsilon, s).$$

It means that,

$$\varepsilon\big[L_2(k, s_\varepsilon, s) - L_2\big(k, s_\varepsilon, g(\omega)\big)\big] \leq L_2(k, s_\varepsilon, s). \qquad (4.4)$$

By convexity of ϑ, L_1 in the 2-term, one can get

$$L_1(s, s_\varepsilon) \leq \varepsilon L_1\big(s, g(\omega)\big) + (1-\varepsilon) L_1(s, s).$$

Then, $$\varepsilon\big[L_1(s, s) - L_1\big(s, g(\omega)\big)\big] \leq L_1(s, s) - L_1(s, s_\varepsilon). \quad (4.5)$$

Also, $$\vartheta(s, s_\varepsilon) \leq \varepsilon \vartheta\big(s, g(\omega)\big) + (1-\varepsilon)\vartheta(s, s)$$

$$\varepsilon\big[\vartheta(s, s) - \vartheta\big(s, g(\omega)\big)\big] \leq \vartheta(s, s) - \vartheta(s, s_\varepsilon). \qquad (4.6)$$

From (4.4), (4.5), (4.6) and (4.3), we obtain that

$$\varepsilon\big[L_2(k, s_\varepsilon, s) - L_2\big(k, s_\varepsilon, g(\omega)\big) + L_1(s, s) - L_1\big(s, g(\omega)\big) + \vartheta(s, s) - \vartheta\big(s, g(\omega)\big)\big]$$

$$\leq L_2(k, s_\varepsilon, s) + L_1(s, s) - L_1(s, s_\varepsilon) + \vartheta(s, s) - \vartheta(s, s_\varepsilon) \qquad (4.7)$$

$$\leq \varepsilon J^0(s, g(\omega) - s) - 2\vartheta(s, s_\varepsilon) + L_1(s, s) + \vartheta(s, s).$$

Since the hemicontinuouity of L_2, we have

$$r\big[-L_2\big(k, s, g(\omega)\big) - L_1\big(s, g(\omega)\big) - J^0(s, g(\omega) - s)\big]$$

$$\leq -2\vartheta(s,s_{\varepsilon}) + \varepsilon\vartheta(s,g(\omega)) + (1-\varepsilon)[L_1(s,s) + \vartheta(s,s)]$$

divided by (-r), one can obtain

$L_1(s,g(\omega)) + L_2(k,s,g(\omega)) + J^0(s,g(\omega)-s)$

$$\geq \frac{2\vartheta(s,s_{\varepsilon})}{\varepsilon} - \vartheta\big(s,g(\omega)\big) - \frac{(1-\varepsilon)}{\varepsilon}[L_1(s,s) + \vartheta(s,s)].$$

From Remark (3.1) we observe that

$$L_1(s,g(\omega)) + L_2(k,s,g(\omega)) + J^0(s,g(\omega)-s) \geq 0.$$

That is means s is a solution of issue $HEP(J,L_1,L_2)$. Therefore, we are done with the proof.

Next, in second main result we will prove the existence of the issue $HEP(J,L_1,L_2)$.

Theorem 4.2. Consider that the following conditions are met and that A is a non-empty, closed, finite, and convex subset of the reflexive BS V.

i. $L_1: A \times A \to R$ is convex in 2- term, u.s.c in 1- term and 0-DC

ii. $L_2: A \times A \times A \to R$ is ϑL-g-monotone trifunction, H.C in 2- term, 0-DC and (l.s.c) in 3- term.

$\vartheta: A \times A \to R$is (l.s.c) in 1- term and convex in 2- term.

Then, the issue $HEP(J,L_1,L_2)$admits at least one solution.

Proof. Define two set $-$valued mappings $N,M: A \to 2^V$ as follow, $\forall g(\omega) \in A$

$$\mathrm{N}(g(\omega)) = \bigcap_{k\in A}\{s \in A; L_1\big(s, g(\omega)\big) + L_2\big(k, s, g(\omega)\big) + J^0(s, g(\omega) - s) \geq 0\}$$

$$\mathrm{M}(g(\omega)) = \bigcap_{k\in A}\{s \in A; L_2(k, g(\omega), s) + \vartheta(s, g(\omega))$$
$$\leq J^0(s, g(\omega) - s) + L_1(s, g(\omega))\}.$$

Then, the issue has a solution if:

$$\bigcap_{k\in A}\mathrm{N}(g(\omega)) \neq \emptyset.$$

Since $g(\omega) \in \mathrm{N}(g(\omega)) \Longrightarrow \mathrm{N}(g(\omega)) \neq \emptyset$.

Now, in order to demonstrate that $\Gamma\big(g(\omega)\big)$ is KKM-mapping, we shall prove the opposite. Let $\Gamma\big(g(\omega)\big)$ be not KKM, Hence there must be a set that is finite. $\{s_1, s_2, \dots, s_n\}$ of A and $\alpha_i \geq 0$, $i = 1,2, \dots, n, \sum_{i=1}^{n} \alpha_i = 1$, in which $s_0 = \sum_{i=1}^{n} \alpha_i s_i \notin \bigcup_{i=1}^{n} \mathrm{N}(s_i)$. Then, $s_0 \notin \mathrm{N}(s_i)\ \forall\ i = 1,2, \dots, n$, it means that,

$$L_1(s_0, s_i) + L_2(k, s_0, s_i)) + J^0(s_0, s_i - s_0) < 0. \tag{4.8}$$

So,

$\sum_{i=1}^{n} \alpha_i[\, L_1(s_0, s_i) + L_2(k, s_0, s_i)) + J^0(s_0, s_i - s_0) < 0]$.

From the convexity of L_1 and L_2, this would contradict 0-DC. Then, N is KKM-mapping.

Next, from Lemma (4.1) one can have that $N\big(g(\omega)\big) \subseteq M\big(g(\omega)\big)$ for all $g(\omega) \in A$. Hence $M\big(g(\omega)\big)$ is KKM. Since $L_2(k, g(\omega), .), \vartheta\big(., g(\omega)\big)$ are (l.s.c) and $L_1(., s)$ is (u.s.c), then

$$L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \leq \lim_{n\to\infty} inf\big[L_2(k, g(\omega), s_n) + \vartheta\big(s_n, g(\omega)\big)\big]$$

$$\leq \lim_{n\to\infty}\big[J^0(s_n, g(\omega) - s_n) + L_1\big(s_n, g(\omega)\big)\big]$$

$$\leq J^0(s, g(\omega) - s) + L_1\big(s, g(\omega)\big)$$

So,

$$L_2(k, g(\omega), s) + \vartheta\big(s, g(\omega)\big) \leq J^0(s, g(\omega) - s) + L_1\big(s, g(\omega)\big).$$

Therefore, $M\big(g(\omega)\big)$ is weakly closed $\forall g(\omega) \in A$, Since A is a nonempty set of real reflexive B-S V that is closed, bounded, and convex, we may conclude that A is weakly compact by applying Theorem 2.37 and Lemma 4.1.

$$\bigcap_{k\in A} N(g(\omega)) = \bigcap_{k\in A} M(g(\omega)) \neq \emptyset.$$

Then, the issue $HEP(J, L_1, L_2)$has a solution. And this completes the proof.

Corollary 4.3. Assume that A is a nonempty subset convex of B-S V in which $0 \in A$. If the assumptions (i-iii) in Theorem (4.2) and the following assumptions hold

i. $L_1(s, 0) - J^0(s, s) = 0$,
ii. $K_1 = \{s \in A; \vartheta(s, 0) \leq 0\}$ and $K_2 = \{s \in A; L_2(k, 0, s) \leq 0\}$ are relativly compact sets.

Then, the solution to the equation $HEP(J, L_1, L_2)$ is possible.

Proof. It is sufficient to demonstrate that $M(g(\omega))$ is compact for some $g(\omega) \in A$

$$\mathrm{M}(g(\omega)) = \bigcap_{k \in A} \{s \in A; L_2(k, g(\omega), s) + \vartheta(s, g(\omega))$$
$$\leq J^0(s, g(\omega) - s) + L_1(s, g(\omega))\}.$$

$$= \bigcap_{k \in A} \Big\{s \in A; L_2(k, g(\omega), s) + \vartheta(s, g(\omega))$$
$$\leq \frac{J^0(s, g(\omega) - s) + L_1(s, g(\omega))}{2} + \frac{J^0(s, g(\omega) - s) + L_1(s, g(\omega))}{2}\Big\}.$$

Then,

$$\mathrm{M}(g(\omega)) \subseteq \bigcap_{\mathrm{k} \in \mathrm{A}} \left\{s \in A; L_2(k, g(\omega), s) \leq \frac{J^0(s, g(\omega) - s) + L_1(s, g(\omega))}{2}\right\}$$

$$\cup \left\{\mathrm{s} \in \mathrm{A}; \vartheta(s, g(\omega)) \leq \frac{J^0(s, g(\omega) - s) + L_1(s, g(\omega))}{2}\right\}.$$

$$\mathrm{M}(0) \subseteq \bigcap_{\mathrm{k} \in \mathrm{A}} \left\{\mathrm{s} \in \mathrm{A}; L_2(k, 0, s) \leq \frac{-J^0(s, s) + L_1(s, 0)}{2}\right\}$$

$$\cup \left\{\mathrm{s} \in \mathrm{A}, \vartheta(s, 0) \leq \frac{-J^0(s, s) + L_1(s, 0)}{2}\right\}\Big\}.$$

From conditions (i, ii), $M(0)$ is a subset of relative compact sets K_1 and K_2, then $M(g(\omega))$ is a compact for some $g(\omega) \in A$, closed for every $g(\omega) \in A$ and by Theorem (2.37) we have

$$\bigcap_{g(\omega)\in \mathrm{A}} \mathrm{M}\big(g(\omega)\big) \neq \emptyset.$$

Then, $HEP(J, L_1, L_2)$ admits solution. Therefore, we are done with the proof.

5. CONCLUSIONS AND FUTURE WORK

5.1 Conclusions

The equilibrium branch of mathematics is only a few decades old, allowing us to research and solve issues in a wide range of scientific fields, including but not limited to pure and applied sciences. This thesis asserts Initially, we present findings for a novel state of the trifunctional for equilibrium issue $\boldsymbol{GP}(\boldsymbol{H}, \boldsymbol{L_1}, \boldsymbol{L_2})$ on convex-closed sets that are either limited or unbounded in B-Ss. This is accomplished by introducing a generalized version of monotonicity known as $\boldsymbol{\vartheta}$L-g – monotone operator. It also defines and expands the concept of hemiequilibrium difficulties and examines particular results linked with this topic.

5.2 Future Work

i. Finding the new types of open set for trifunctional for equilibrium problem and investigate about major operations of these inequalities, in addition to study the generalized implicit vector equilibrium problems.

ii. Studying well-posedness of a generalized new type of trifunctional for equilibrium problems.

Finding applications of trifunctional for equilibrium problems in partial differential equations.

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