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RESULTS OF VARIATIONAL INEQUALITY INVOLVING MONOTONE OPERATOR

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GHADEER MUNTADHER SAFAA AL-TAMEEMI

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By Ghadeer Muntadher Safaa AL-TAMEEMI

February 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

Advisor : Prof. Dr. Faruk POLAT
Co-Advisor : Assist. Prof. Dr. Ayed E. HASHOOSH

Examining Committee Members:

Chairman : Prof. Dr. Faruk POLAT
Department of Mathematics
Çankırı Karatekin University

Member : Assoc. Prof. Dr. Faruk KARAASLAN
Department of Mathematics
Çankırı Karatekin University

Member : Asst. Prof. Dr. Uğur GÖNÜLLÜ
Department of Mathematics and Computer Sciences
İstanbul Kültür University

Approved for the Graduate School of Natural and Applied Sciences

Prof. Dr. Hamit ALYAR
Director of Graduate School

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Gadeer Muntadher Safaa AL-TAMEEMI

ABSTRACT

RESULTS OF VARIATIONAL INEQUALITY INVOLVING MONOTONE OPERATOR

Gadeer Muntadher Safaa AL-TAMEEMI

Master of Science in Mathematics

Advisor: Prof. Dr. Faruk POLAT

Co-Advisor: Asst. Prof. Dr. Ayed E. HASHOOSH

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This thesis is an attempt to establish existence findings for certain variational inequality problems involving set-valued mappings in reflexive and non-reflexive Banach spaces. More precisely, the thesis presents new types of monotonicity in Banach space, all with the aim of obtaining results on the existence of findings of $VI(\omega\eta\kappa)$ in the case of convex subsets by using a KKM mapping, compactness and convex function. All this is done through three tasks. Task one presents a host of essential definitions along with some Theorems. The aim of this task is to help the reader to get some understanding of main results grounded in this thesis. Task two is designed to establish the existence of the solution of $VI(\omega\eta\kappa)$ by generalizing a new class of function in the setting of Banach space. Task three is intended to prove that the concept of well-posedness for $VI(\omega\eta\kappa)$ is equivalent to the existence of its solution. In so aiming, the results presented in this thesis develop and improve some corresponding solutions of several authors.

2023, 57 pages

Keywords: Inequality prolem, Solution existence, KKM-Theorem, Well-posdness, Monotone operator

ÖZET

MONOTON OPERATÖR İÇEREN VARYASYONEL EŞİTSİZLİĞİN SONUÇLARI

Gadeer Muntadher Safaa AL-TAMEEMI
Matematik, Yüksek Lisans
Tez Danışmanı: Prof. Dr. Faruk POLAT
Eş Danışman: Dr. Öğr. Üyesi Ayed E. HASHOOSH
Şubat 2023

Bu tez, dönüşlü ve dönüşsüz Banach uzaylarında küme değerli eşlemeleri içeren belirli varyasyonel eşitsizlik problemleri için varoluş bulguları oluşturma girişimidir. Daha kesin olarak, tez Banach uzayında yeni tekdüzelik türleri sunuyor, bunların hepsi bir KKM eşleme, kompaktlık ve dışbükey fonksiyon kullanarak dışbükey altkümeler durumunda $VI(\omega\eta\kappa)$ bulguların varlığına ilişkin sonuçlar elde etmeyi amaçlıyor. Bütün bunlar üç görevle yapılır. Birinci görev, bazı Teoremlerle birlikte bir dizi temel tanım sunar. Bu görevin amacı, okuyucunun bu tezde temellenen ana sonuçları biraz anlamasına yardımcı olmaktır. İkinci görev, Banach uzayında yeni bir fonksiyon sınıfını genelleştirerek $VI(\omega\eta\kappa)$ çözümünün varlığını kurmak için tasarlanmıştır. Üçüncü görev, iyi pozlama kavramının $VI(\omega\eta\kappa)$ çözümünün varlığına eşdeğer olduğunu kanıtlamayı amaçlamaktadır. Bu amaçla, bu tezde sunulan sonuçlar, birkaç yazarın karşılık gelen bazı çözümlerini geliştirmekte ve iyileştirmektedir.

2023, 57 sayfa

Anahtar Kelimeler: Eşitsizlik problemi, Çözüm varlığı, KKM-Teoremi, İyi pozluluk, Monoton operatör

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Gadeer Muntadher Safaa AL-TAMEMY
Çankırı-2023

CONTENTS

LIST OF SYMBOLS

$\boldsymbol{R}$	The set of Real numbers
$\boldsymbol{N}$	The set of Natural numbers
$\in$	Belonge to
diam (A)	Diameter of a set A
Conv(A)	Convex hull of set A
$\boldsymbol{2^E}$	The family of all a nonempty sets of E
$\langle\ \ \rangle$	Canonical inner product
$\geq$	The greater than or equal
$\leq$	The less than or equal
$\exists$	There exists

LIST OF ABBREVIATIONS

0-diag	0-diagonally convex
et al.	And others.
App.seq	Approximating sequence.
B.S	Banach space.
H.T.L.S	Hausdorff topological vector spaces.
H.C	Hemicontinuous.
IPS	Inner product space.
lsc	Lower Semi Continuous.
M. bi.f	Monotone bifunction.
OPGPEC	Optimization problem with generalized equilibrium constraint.
resp.	Respectively.
s-w-p	Strongly well-posdness.
s.t	Such that.
usc	Upper Semi Continuous.
V.I	Variational inequality.
w-p	well-posedness
w.r.t	With respect to.

1. INTRODUCTION

This thesis is devoted to the study of V. I issues w.r.t pseudomonotonicity as well as well-posedness solutions. It is the theory of V.I developed by (Stampacchia 1964) and Fichera gives powerful methods for solving a wide variety of problems in mathematics and engineering (Fichera 1964). These techniques were developed by (Stampacchia 1964), and (Fichera 1964). The free boundary value issue, for instance, can be investigated well within the context of V.I, the moving boundary value issue can be characterized by a type of V.I, and the traffic assignment issue is a V.I issue. However, the theory of V.I developed thus far is useful to the investigation of free and moving boundary value issues of even order.

Another important concept in nonlinear analysis is the monotone concept of a defined mapping together with continuity and convexity. This includes issues involving V.I, w-p, complementarity, and optimization as well as game theory and other related topics. Pseudomonotonicity, η - α-monotonicity, $\eta - f -$ monotonicity and quasi-monotonicity and relaxed monotonicity have been proposed by many researchers as major generalizations of monotonicity see (Bai *et al.* 2006), (Chen 1999), (Cottle *et al.* 1992), (Fang *et al.* 2003), (Farouq 2001), (Huang *et al.* 2000), (Karamardian at el. 1990), (Karamardian *et al.* 1993), (Luc 2001), as well as the citations included therein. In the research done by (Verma 2003), a new class of V.I accompanied by relaxed monotone operators was analyzed and developed. Recently, (Fang and Huang 2003), defined a relaxed $\eta - \alpha -$ monotone concept. For single-valued maps, (Bai *et al.* 2006) established a relaxed $\eta - \alpha$ - pseudomonotone notion. $\eta - f - \alpha$ relaxed$-f-$pseudomonotone concept was proposed by (Kang *et al.* 2003) for set-valued mappings, which generalizes the monotone idea from (Fang *et al.* 2002). Regarding the concept of w-p, the first basic is derived from J. Hadamard's early twentieth-century concept. It necessitates the existence and uniqueness of the best solution as well as continual dependence on the problem's information. Several methods have been proposed in the mathematical literature to examine distinct w-p in relation to optimization problems and V.I. see (Bednarczuk 1966), E. S. Levitin and

B.T.Polyak, among other mathematicians in (Levitin and Polyak 1966). Others, including (Tykhonov 1966) and others, have developed and studied well-posed issues.

They were able to provide a mathematically exact description of approximation w-p solutions for optimization issues and V.I. These topics now provide a fairly robust foundation for investigation. Since then, several w-p contexts that originated from issues (such as optimization and V.I) have been extended to scalar issues, fixed-point issues, Nash equilibrium issues, saddle point issues, and other issues as well. For additional details, see (Chen et. al 2014). (Fukushima 1986) introduced a modification of the projection methods to solve the issues of variable inequality. where each iteration of the proposed algorithm from a projection over half a distance contains a certain closed convex set rather than the last set itself. Thus, the affinity of the algorithm with the solution can be established under suitable conditions. As outlined by (Bianchi *et al.* 2014), two kinds of w-p for vector equilibrium issues were presented and defined. As a parametric version of equilibrium issue with a w-p idea, another case of w-p has been discussed. E. Miglierina, E. Molho and M. Rocca have already taken advantage of Salamon's proposed method in (2005).

In addition to that, there is the concept of w-p, which has been used to problems involving vector equilibrium. (Kimura *et al.* 2008) gave us a big step into a newly discovered issue of w-p. In addition, many forms of Levitin-Polyak w-p involving broad applications of vector equilibrium, were studied by J. Peng, Y.Wang, and S.Wu (Peng *et al.* 2011). They did this by taking the situation further. In particular, (Long *et al.* 2008) and (Zaslavski 2008), resp., offer examples for Levitin-Polyak w-p for explicitly limited EPs and generic w-p for EPs. It has been shown that Levitin-Polyak w-p may be used to the parametric quasivariational inclusion and disclusion issues, as well as to quasiequilibrium and a scalar equilibrium issues, by (Boonman and Wangkeeree 2018). While Wehbe *et al.* investigated the complete controllability of the a system of two wave equations coupled via velocities using boundary control acting just on one problem, Wehbe *et al.* examined the effect of boundary control (Wehbe *et al.* 2022).

The aim of this thesis is the generalization and practical realization of the solution method to suitable classes of issues both variational inequalities and w-p resp.

This thesis is composed of four chapters. Chapter one presents a host of essential definitions along with some theorems. The aim of this chapter is to help the reader to get some understanding of main results grounded in this thesis. The chapter starts with a section on primary definitions of normed space before it ends up with a brief introduction w-p. We can state all of the aims of this chapter at once by saying that we are trying to ground the investigation in some concepts of convex analysis required for the upcoming chapters. This being said, the chapter appeals to some concept derived from topological spaces and monotone function.

Chapter two, a more generalized new concept of $VI(\omega\eta\kappa)$ mappings w.r.t hybrid-type mappings is introduced. We establish important existence findings of solutions of $VI(\omega\eta\kappa)$ mappings in reflexive B.S by making use of the KKM approach. Our findings are broad generalizations based on a variety of works.

The third chapter of our thesis is considered to be the concluding inquiry, in which we introduce the essential terminology and refer to certain outcomes from the field. The third section defines and expands the notion of w-p of $VI(\omega\eta\kappa)$ mappings. The fourth section introduces a new idea of w-p of optimization issues by means of constraints described with form of parametric for generalized the issue $VI(\omega\eta\kappa)$ mappings and a pertinent w-p.

Finally, in the fourth chapter, we mention the important conclusions of the thesis and future work.

2. PRELIMINARIES AND BASIC RESULTS

2.1 Introduction

In the following chapter, a family of basic definitions as well as primary theorems is introduced. The essential goal of introducing these definitions and theorems is to help us derive main results obtained in this thesis that would be given in the upcoming chapters. With this goal in mind, the chapter begins with the very basic definitions of normed space before we end with a brief introduction to w-p. By doing so, we base our investigation upon some notions of convex analysis that are necessary for our next chapters. Furthermore, we refer to a number of concepts derived from topological spaces. Additionally, we present basic definitions and notions of monotone function.

2.2 Some Concepts of Normed Spaces

This section discusses a variety of concepts, properties, and conclusions that are fundamental to the study of Normed space and Banach space.

Definition 1.1. (Yosida 1980) Consider that E is a vector space on field F (where F can represent either the real or complex numbers field). The function $\|.\|: E \longrightarrow F$ is called norm on E. In the situation that the following holds true:

i. $\| \sigma \| \geq 0 \; \forall \sigma \in E,$
ii. $\| \sigma \| = 0$ if and only if $\sigma = 0, \forall \sigma \in E,$
iii. $\| \sigma\sigma \| = |\sigma| \, \| \sigma \|, \forall \sigma \in E, \sigma \in F,$
iv. $\| \sigma + \varrho \| \leq \| \sigma \| + \| \varrho \|, \forall \sigma, \varrho \in E.$

Then, the pair $(E, \|.\|)$ is called a normed space.

Definition 1.2. (Yosida 1980) Consider that E is a vector space, then a function $\langle .,. \rangle : E \times E \to R$ is considered an internal product of S, if each of the following requirements is met

i. $\langle u,u \rangle \geq 0, \forall u \in E$;
ii. $\langle u,u \rangle = 0$ if and only if $u = 0$;
iii. $\overline{\langle u,v \rangle} = \langle v,u \rangle$; where$\overline{\langle u,v \rangle}$ is implies conjugate to$\langle u,v \rangle$;
iv. $\langle \alpha u + \beta v, z \rangle = \alpha \langle u,z \rangle + \beta \langle v,z \rangle, \forall u,v \in E; \alpha,\beta \in F$.

Then, the pair $(E, \langle .,. \rangle)$ is an inner product space (for shortly, IPS).

Example 1.3. (Yosida 1980) The space $E = F^n$ with the function $\langle u,v \rangle = \sum_{i=1}^{n} u_i \, \overline{v}_i, \forall u,v \in E$,where $\overline{v}_i$ is implies conjugate v, then S is an IPS**.**

Solution

i. $\langle u,u \rangle = \sum_{i=1}^{n} u_i^2 \geq 0$;
ii. $\langle u,u \rangle = 0 \Leftrightarrow \sum_{i=i}^{n} u_i^2 = 0 \Leftrightarrow u_i = 0, \forall i \Leftrightarrow u = 0$;
iii. $\overline{\langle u,v \rangle}$=$\overline{\sum_{i=1}^{n} u_i \overline{v_i}}$ =$\sum_{i=1}^{n} \overline{u_i} v_i = \langle v,u \rangle$;
iv. $\langle \alpha u + \beta v, z \rangle = \sum_{i=1}^{n} (\alpha u_i + \beta v_i) \, \overline{z}_i$

$$= \alpha \sum_{i=1}^{n} u_i \, \overline{z}_i + \beta \sum_{i=1}^{n} v_i \, \overline{z}_i$$
$$= \alpha \langle u,z \rangle + \beta \langle v,z \rangle.$$

Then, the function $\langle .,. \rangle$ is an inner product on S and the pair $(E, \langle .,. \rangle)$ is an IPS.

Theorem 1.4. (Yosida 1980) An IPS S is a normed space, by norm $\|v\| = \sqrt{\langle v,v \rangle}$. Consedring $(E, \langle .,. \rangle)$ is IPS, then

i.$\langle 0,u \rangle = \langle u,0 \rangle = 0$;
ii.$\langle u, \alpha v + \beta z \rangle = \overline{\alpha} \langle u,v \rangle + \overline{\beta} \langle u,z \rangle, \forall u,v,z \in E; \alpha,\beta \in F$.

Definition 1.5. (Yosida 1980) Let $\{\sigma_n\}$ be a sequence in normed space E, then $\{\sigma_n\}$ is said to be bounded sequence, if there is a real positive number $r > 0$, in which

$$\|\sigma_n\| \leq r, \forall n \in Z^{+}.$$

Definition 1.6. (Yosida 1980) Consider that E is a normed space, $\sigma_0 \in E$ and $\varepsilon \geq 0$, then the set $B = \{\sigma \in E, \|\sigma_0 - \sigma\| < \varepsilon\}$ is said to be an open ball by center σ_0 and radius ε, denoted $B_\varepsilon(\sigma_0)$.

Definition 1.7. (Yosida 1980) Consider that $\{\sigma_n\}$ is a sequence in normed space E, then $\{\sigma_n\}$ is said to be converge sequence, if there exist $\sigma \in E$ such that

$$\forall \delta > 0, \exists r \in Z^{+}, \|\sigma_n - \sigma\| < \delta \ \forall n > r.$$

And, σ the a convergent point to sequence $\{\sigma_n\}$ and form write,

$$\lim_{n\to\infty} \sigma_n = \sigma \text{ or } \sigma_n \to \sigma.$$

Definition 1.8. (Yosida 1980) Cauchy sequence $\{\sigma_n\}$ is defined in a normed space E, if for every $\delta > 0$, there is $r \in Z^{+}$such that $\|\sigma_n - \sigma_m\| < \delta, \forall n, m > r.$

Theorem 1.9. (Yosida 1980) Cauchy sequence is a convergent sequence in a normed space.

Definition 1.10. (Yosida 1980) Weakly convergent sequence $\{\sigma_n\}$in a normed space E holds if

$$\lim_{n\to\infty} f(\sigma_n) = f(\sigma) \text{ for some } \sigma \in \text{E}.$$

Definition 1.11. (Yosida 1980) The normed space E is considered a B.S if each Cauchy sequence in E converges to an element of E.

Definition 1.12. (Yosida 1980) In a B.S E. A mapping $\Gamma: K \longrightarrow R$ is considered to be

i. Lower semicontinuouse (for shortly, lsc) at $\sigma_0 \in E$ if

$$\Gamma(\sigma_0) \leq \liminf \Gamma(\sigma_n).$$

ii. Upper Semi Continuous (for shortly , usc) at $\sigma_0 \in E$ if

$$\Gamma(\sigma_0) \geq \limsup \Gamma(\sigma_n).$$

For any sequence $\{\sigma_n\}$ of E where $\sigma_n \longrightarrow \sigma_0$.

Example 1.13. (Yosida 1980) Consider that K is a nonempty and $\Gamma: K \rightarrow R$ in which for every $\sigma \in K$ then

$$\Gamma(\sigma) = \begin{cases} 0 & \sigma < 1 \\ 2 & \sigma = 1 \\ 1/2 & \sigma > 1 \end{cases}$$

then, Γ is usc at $\sigma = 1$, because its value is higher than its neighborhood value.

Example 1.14. (Yosida 1980) Considering K be a nonempty subset of real number and $\Gamma: K \rightarrow R$ such that for every $\sigma \in K$ then

$$\Gamma(\sigma) = \begin{cases} \sigma^2 & \text{if } \sigma \neq 0 \\ -1 & \text{if } \sigma = 0 \end{cases}$$

Then, Γ is lsc at $\sigma = 0$.

Definition 1.15. (Yosida 1980) Consider that A and B are two normed spaces. A mapping $\beta: A \to B$ is called continuous at $\sigma \in$ A if for any given $r >\ 0$, there is $\delta > 0$ such that $\|\beta(\varrho) - \beta(\sigma\,)\| < r$ for all $\varrho \in$ A satisfying $\|\ \varrho - \sigma\ \| < \delta$. Then β is named continuous if, it continuous at every point in K.

2.3 Convex Analysis

Within this section, we will discuss several fundamental concepts and terminology of convex analysis that are essential to the work that we do.

Definition 1.16. (Zeidler 1986) Consider that the assumption that K is a nonempty subset of B.S. E. If

$$r + (1 - r)\varrho \in A, r \in [0,1], \forall\sigma\,, \varrho \in A.$$

Then, K is a convex set.

Lemma 1.17. (Zeidler 1986) Suppose A and B are convex sets in a vector space E of $\|\,.\,\|$ norm. Then

i. The empty set is convex.
ii. $A + B = \{\sigma\ + \varrho; \sigma\ \in A, \varrho \in B\}$ is convex.
iii. δA is convex for any scaler δ .
iv. For any set of convex sets, the intersection is also convex.

Remark 1.18. (Zeidler 1986) It is not necessary for the union of two convex sets to be convex.

Below is an example that would explain to us why the union need not to be convex.

Example 1.19. (Zeidler 1986) Consider that $E = R^3$ is a space and $M = \{(\sigma, 0,0): \sigma \in R\}$, $N = \{(\sigma, \varrho, 0): \sigma + \varrho = 0: \sigma, \varrho \in R\}$ are convex subset of E. If we take $\frac{1}{2} \in [0,1]$, then $\sigma_1, \sigma_2 \in M$, where $\sigma_1 = (2,0,0) \in M, \sigma_2 = (8, -8,0) \in M \cup N$. But

$$\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2 = \frac{1}{2}(2,0,0) + \frac{1}{2}(8, -8,0)$$

$$= (1,0,0) + (4, -4,0)$$

$$= (5, -4,0).$$

Since $(5, -4,0) \notin M$and N, then $(5, -4,0) \notin M \cup N$. Hence $M \cup N$ is not convex.

Definition 1.20. (Zeidler 1986) Consider K is a subset of B.S E, then the smallest convex set in E contained K is said to be convex hull denoted by conv(K).

Proposition 1.21. (Zeidler 1986) On convexity, the following conditions hold

i. $K \subseteq conv(K)$,
ii. $conv\ (K)$ = intersection of all the convex sets that contain K,
iii. Whenever K is a convex set, then $K = conv(K)$.

Definition 1.22. (Zeidler 1986) Let K be subset of a vector space E, and $\sigma_1, \sigma_2 \ldots \sigma_n \in K$, the combination for all $\sigma_1, \sigma_2 \cdots \sigma_n \in A, r_i \geq 0$ in which $\sum_{i=1}^{n} r_i = 1$ the combination $\sum_{i=1}^{n} r_i \sigma_i \in A$ is said to be convex combination.

Definition 1.23. (Zeidler 1986) A function $\beta: K \rightarrow R$ is said to be convex function if

$$\beta(l\sigma + (1-l)\varrho) \leq l\beta(\sigma) + (1-l)\beta(\varrho), \forall \sigma, \varrho \in K, l \in [0,1].$$

The following example would make it clear the above definition.

Example 1.24. (Zeidler 1986) Let $K = C[a,b]$, where A space of real continuous function on $[a,b]$ and $\beta: K \to R$ such that for every $h \in A$. One has

$$\beta\big(h(\sigma)\big) = \int_a^b |h(\sigma)| d\sigma$$

If we take $h(\sigma), k(\sigma) \in A$ and $r \in [0,1]$ then

$$\beta(rh(\sigma) + (1-r)k(\sigma))$$

$$= \int_a^b |rh(\sigma) + (1-r)k(\sigma)| d\sigma$$

$$\leq r \int_a^b |h(\sigma)| d\sigma + (1-r) \int_0^b |k(\sigma)| d\sigma$$

$$= r\beta\big(h(\sigma)\big) + (1-r)\beta\big(k(\sigma)\big).$$

Definition 1.25. (Chadli 2002) A bifunction $\beta: K \times K \to R$ is said vector 0-diag if, for each finite subset $\{\sigma_1, \sigma_2 \ldots . \sigma_n\}$ of A and $\alpha_i \geq 0, i = 1,2 \ldots n$ by $a = \sum_{i=1}^{n} \alpha_i \sigma_i$ and $\sum_{i=1}^{n} \alpha_i = 1$, one has

$$\sum_{i=1}^{n} \alpha_i \beta(\sigma, \sigma_i) \geq 0.$$

Definition 1.26. (Browder 1963) Hemicontinuous β is the set of all real-valued functions defined on a convex set K of E (for shortly, $H.C$) if

$$\lim_{l \to 0} \beta(l\sigma + (1-l)\varrho) = \beta(\varrho) \forall \sigma, \varrho \in A, l \in [0,1].$$

Definition 1.27. (Lee (2009) Let K be a nonempty convex subset of E. Let $T: K \to 2^{E^*}$ and $\eta: K \times K \to E$ be two mappings. Then, T is named to be $\eta - H.C$ if for every $\sigma, \varrho \in K$, a mapping $g: [0,1] \to 2^R$ defined by

$$g(t) = \cup_{u_t \in T(\sigma_l)} \langle u_t, \eta(\varrho, \sigma) \rangle, \text{ where } \sigma_t = \sigma + t(\varrho - \sigma) \text{ is usc at } 0^+.$$

2.4 Some Concepts From Topological Spaces

The findings presented here are a combination of concepts from topological spaces that will be employed in other chapters of the thesis.

Definition 1.28. (Schaefer 1999) Consider that E be a nonempty set. A subset τ of E is a topology on E if

i. E and the empty set, are members of τ.
ii. The union of any numbers of sets in T, whether finite or infinite, is also a member of τ.
iii. The intersection of finite number of sets in also belongs to τ.

A topological space comprises the pair (E, τ).

Definition 1.29. (Schaefer 1999) A topological space E is named a Hausdorff space if for any two distinct points x and y in E, there is two open sets $\aleph$ and M in which x belongs to $\aleph$ and y to M. So,

$\aleph \cap M = \emptyset$.

Definition 1.30. (Schaefer 1999) Considering E is a B.S with norm $\|.\|$. The topology on E induces by norm$\|.\|$is said to be norm topology or strong topology.

Definition 1.31. (Schaefer 1999) Suppose that V is a B.S and $M \subseteq$ V. Then, if M is closed in the weak topology of V, then M is considered to be weakly closed.

Definition 1.32. (Liang Chern 2013) If K is a set. Then a collection of open sets $\{K_i\}$, i $\in I$ is called cover of A if

$$\mathrm{K} \subset \bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{K}_{\mathrm{i}}.$$

Definition 1.33. (Liang Chern 2013) If every open covering of K by an open set of E has a finite sub covering, then the subset K of the space E is compact.

Definition 1.34. (Schaefer 1999) In the weak topology, a subset K of a B.S E is compact if every sequence in K contains a subsequence that converges weakly in A.

Definition 1.35. (Knaster 1929) Consider that K is a nonempty subset of the H.T.L.S E. A mapping $\Gamma: K \longrightarrow P(E)$ is called a KKM-mapping if for each finite subset $\{\varrho_1, \varrho, \dots, \varrho_n\}$ of K, we have conv $\{\varrho_1 \dots . \varrho_n\} \subseteq \bigcup_{i=1}^{n} \Gamma(\varrho_i)$.

Example 1.36. (Knaster 1929) Assume that $K = [0,2]$ is a nonempty from real numbers R and the function $\beta: K \to P(R)$ defined as:

$$\beta(z) = \begin{cases} [1,2] & z > 0 \\ 0 & z = 0 \end{cases}$$

Then, if we take $\{0,1\} \subseteq A$ and con $\{0,1\} = [0,1]$ and $\beta(0) = 0, \beta(1) = [1,2]$. Let $z = 1/2 \in [0,1] = con\, v\{0,1\}$. But $z \notin \beta(0) \cup \beta(1) = \{0\} \cup [1,2]$. That means conv $\{0,1\} \not\subset \cup\, \beta(z_i)$

where $z_i = 0,1$, hence β is not KKM.

Lemma 1.37. (Fan 1972) Consider that K be a nonempty subset of H.T.L.S $\boldsymbol{E}$ if $\boldsymbol{\Gamma}: \boldsymbol{K} \longrightarrow \boldsymbol{P}(\boldsymbol{E})$ is a KKM mapping, $\boldsymbol{\Gamma}(\boldsymbol{\varrho})$ is closed in V for every $\boldsymbol{\varrho} \in \boldsymbol{K}$ and compact for some $\boldsymbol{\varrho} \in \boldsymbol{K}$. Consequently,

$$\bigcap_{a \in A} \Gamma(\varrho) \neq \emptyset.$$

Definition 1.38. (Yosida 1980) Consider that A and B are nonempty set of V. For describe the Hausdorff distance metric $L(.,.)$ between two points A and B, we possess

$$L(A,B) = max\,\{e(A,B), e(B,A)\}$$

where $e(A,B) = sup_{\sigma \in A} d(\sigma, B)$ and $d(\sigma, B) = inf_{\ \sigma \in B} ||\sigma - \varrho||$.

Definition 1.39. (Kuratowski 1968) Consider that K is a subset of E that is not empty. Then, The set K's measure of noncompactness μ is defined as

$$\mu(\mathrm{K}) = \inf\left\{\delta > 0: \mathrm{K} \subset \bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{K_i}, \mathrm{diam}\, \mathrm{K_i} < \delta, \mathrm{i} = 1,2 \ldots \mathrm{n}\right\}$$

where diam refers to the diameter of a set K.

Theorem 1.40. (Liang Chern 2013) A set B in $\boldsymbol{R^n}$ is just compact if it is both closed and bounded.

Definition 1.41. (Liang Chern 2013) A set B is supposed to be relative compact if its closure compact.

Definition 1.42. (Liang Chern 2013) A set B is supposed to be sequentially compact if every sequence in B has converge subsequence with limit in B.

Theorem 1.43. (Liang Chern 2013) A set B in $\boldsymbol{R^n}$ is sequentially compact if and only if it closed and bounded.

Theorem 1.44. (Liang Chern 2013) If B is a set in $\boldsymbol{R^n}$ and has one of the following properties. Then it has the other two.

i. B is closed and bounded,
ii. B is compact,
iii. Every infinite subset of B has a limit point in B.

2.5 Monotone Operator

This section, we will review some definitions and concepts pertaining to the monotone operator in both single valued and set valued contexts.

i. For given a B.S E by dual space E^*,a multivalued mapping $T: E \to 2^{E^*}$ is named to be a monotone operator, given that $\langle \sigma^* - \varrho^*, \sigma - \varrho \rangle \geq 0,$ for each x,y $\in dom(T),\ \sigma^* \in T(\sigma)$ and $\varrho^* \in T(\varrho)$ where $dom(T) = \{\sigma \in E: T\sigma \neq \emptyset\}$ its domain. The set G(T)= $\{(\sigma, \sigma^*) \in E \times E^* s.t\, \sigma^* \in T(\sigma)\}$ is the graph of the operator T.
ii. W. Oettli and H. Riahi, (1998) presented a new type of M. bi.f, and called it as $\Psi\text{–}M$. bi.f. According to him, it is defined as the following:

Assume that $\Psi, B: K \times K \to R$ are two bifunctions. Then, B is said to be $\Psi - M$. bi.f if

$$B(\sigma, \varrho) + B(\varrho, \sigma) \leq \Psi(\sigma, \varrho) + \Psi(\varrho, \sigma) \forall \sigma, \varrho \in A.$$

iii. A. E. Hashoosh, M. Alimohammadi and K. Kalleji, (2016) introduced a new type of M. bi.f and called itα-M. bi.f. It can be defined as below: Assume that $\alpha, B: K \times K \to R$ are two bifunction. Then, B is named to be α -M. bi.f if

$$B(\sigma\,,\varrho) + B(\varrho,\sigma\,) + \alpha(\sigma\,,\varrho) \leq 0\; \forall\, \sigma\,,\varrho \in K.$$

On the other hand, some scholars have introduced other types of monotone trifunctions for examples

1) V. Preda., M. Beldiman and A. Batatorescu, (2007) presented monotone trifunctions and it is defined as the following:
   i. A trifunction $\beta: K \times K \times K \to R$ is said to be monotone if and only if for every $a, b \in A$.Then:

$$\beta(\sigma\,,\varrho,\sigma\,) - \beta(\sigma\,,\varrho,\varrho) \geq 0.$$

   ii. A trifunction $\beta: K \times K \times K \to R$ is said to be general relaxed α -monotone if there is $\alpha: K \times K$ such that for every $\sigma\,,\varrho \in K$we have

$$\beta(\sigma\,,\varrho,\sigma\,) - \beta(\sigma\,,\varrho,\varrho) \geq \alpha(\varrho,\sigma\,).$$

2) O. Chadli, G. Pany and R. Mohapatra, (2020), present a new type of monotone trifunction. They are defined as following.
   i. A trifunction $\beta: K \times K \times K \to R$ is said to be monotone trifunction if and only if:

$$\beta(\sigma\,,\varrho,z) + \beta(\sigma\,,z,\varrho) \leq 0\; \forall\, \sigma\,,\varrho,z \in K.$$

   ii. A trifunction $\beta: K \times K \times K \to R$ is said to be relaxed α –monotone if and only if, there is $\alpha: A \to R^{+}$, with $\alpha(rz) = r^{k}\alpha(z)$ for all $r > 0$ where $k > 1$ and $z \in A$ such that for every $\sigma\,,\varrho,z \in A$ then:

$$\beta(\sigma, \varrho, z) + \beta(\sigma, z, \varrho) \leq \alpha(\varrho - z).$$

iii. Let E represent a normed space and K represent a nonempty subset of E, for all $\sigma, \varrho, z \in K$. Then, a trifunction $\beta: K \times K \times K \to R$ is said to be γ -strongly monotone. If there is a positive constant γ, such that

$$\beta(\sigma, \varrho, z) + \beta(\sigma, z, \varrho) \leq -\gamma \parallel \varrho - z \parallel^2.$$

Definition 1.45. (Lee 2009) Consider that $\mathrm{T}: \mathrm{K} \to 2^{\mathrm{E}^*}, \mathrm{N}: \mathrm{K} \times \mathrm{E}^* \to \mathrm{E}^*$ and $\varpi: K \times K \to E$ be three mappings and $L: K \times K \to R \cup \{+\infty\}$ be a proper function. T is named to be ϖ-coercive w.r.t L if there is an $\sigma_0 \in K$ such that for all $u \in T(\sigma)$ and $u_0 \in T(\sigma_0)$

$$\frac{\langle N(\sigma, u) - N(\sigma_0, u_0), \varpi(\sigma, \sigma_0)\rangle + L(\sigma, \sigma_0)}{\|\varpi(\sigma_0, \sigma)\|} \to \infty,$$

whenever $\parallel \sigma \parallel \to \infty$.

2.6 Some Cases of W-p

Numerous publications in the literature deal with a generalization of Tykhonov well-posedness as it relates to optimization issues with multiple solutions. We mention some concepts of w-p by parametric for novel class of equilibrium issues EP_Ψ and for optimization issues with perturbations that include the classical equilibrium issue as a special case obtained by researchers in (Hashoosh and Alimohammady 2016), (Fang *et al.* 2008), (Zhang *et al.* 2009), and (Kimura 2008), some of which we have developed in the third chapter, in accordance with our new definition, which will be mentioned later. In this context, the following is a description of the formulation of optimization issues that include equilibrium constraints:

Let $F: T \times K \times K \to R$ and $h: T \times K \to R$ are two mappings in which $T \subset E$ is a nonempty set, where E is a B.S. The following is the formulation of the optimization issue by generalized equilibrium constraint (abbreviated OIGEC)

$$\min h(t,u) \text{ where } (t,u) \in T \times K \text{ and } u \in S(t),$$

The parametric generalized equilibrium issue (EP_Ψ) has a set of solutions called S(t), which is given by $u \in S(t)$ if and only if

$$F(t,u,v) + \psi(u,v) \geq 0 \text{ for all } v \in K.$$

We recall the following special versions of the issue:

i. If $\Psi \equiv 0$ then the issue is converted to the parametric equilibrium issue (for short, $EP(t)$. Finding $\sigma \in K$ such that $F(t,\sigma,\varrho) \geq 0, \forall \varrho \in K$, (Fang et. al 2008),
ii. If $\Psi \equiv 0$ and $F(t,\sigma,\varrho) = -h(t,\sigma,\sigma - \varrho)$ for all $\varrho \in K$ then, the issue is converted to the quasivariational parametric inequality (for short, QVI (t)) , K. Zhang, Z. He and D. Gao (2009).
iii. If $\Psi(\sigma,\varrho) \equiv \Psi(\sigma) - \Psi(\varrho) \forall \varrho \in K$ and $F(t,\sigma,\varrho) = \langle \beta(\sigma), \varrho - \sigma \rangle$ then, the issue is converted to the mixed variational inequalities (for short, (MVI)) Y. P. Fang, N. J. Huang and J. C. Yao (2008).

In what follows, several ideas of w-p of generalized equilibrium issues (EP_Ψ) are recalled, and some conditions under which equilibrium issues are s-w-p in a generalized sense are studied.

Definition 1.46. **(**Hashoosh and Alimohammady 2016) $\{(t_n, \sigma_n)\} \subset T \times K$ is an App.seq for (EP_ψ) if there is a nonnegative sequence $\{\epsilon_n\}$ by $\epsilon_n \to 0$ as $n \to \infty$ in which

$$F(t,\sigma_n,\varrho) + \psi(\sigma_n,\varrho) \geq -\epsilon_n \|\varrho - \sigma_n\| \ \forall\, n \in N, \varrho \in K .$$

Definition 1.47. (Hashoosh and Alimohammady 2016) The issue (EP_{ψ}) considered to be s-w-p (or strongly (or weakly) w-p in the generalized sense) if (EP_{ψ}) has a single solution σ , and for every sequence $\{\sigma_n\}$ by $\sigma_n \to \sigma$, every App.seq for (EP_{ψ}) converges strongly to the single solution.

If (EP_{ψ}) has a nonempty solution set $S(t)$, then every approximate solution sequence has a subsequence which strongly (or weakly) converges to the point of $S(t)$.

We'll go through some w-p characterizations for (EP_{Ψ}). $\forall \epsilon > 0$, let us define two sets as follows:

$$\Gamma(\epsilon) := \{(t, \sigma) \in T \times K, F(t, \sigma, \varrho) + \Psi(\sigma, \varrho) \geq -\epsilon \parallel \varrho - \sigma \parallel \forall \varrho \in K\}.$$

And

$$\Lambda(\epsilon) := \{(t, \sigma) \in T \times K, F(t, \varrho, \sigma) + \alpha(\sigma, \varrho) \leq \Psi(\sigma, \varrho) + \epsilon \parallel \varrho - \sigma \parallel \forall \varrho \in K\}.$$

Lemma 1.48. (Hashoosh and Alimohammady 2016) Considering K is a nonempty convex, closed subset of the B.S X, and $F: T \times K \times K \to R$ and $\Psi, \alpha: K \times K \to R$ are three functions. The subsequent conditions must hold

i. $F(t, \sigma, \sigma) = 0, \forall t \in T, \sigma \in K$,
ii. $F(t, \cdot, \cdot)$ is α -M. bi.f and $H.C$ $\forall t \in T$,
iii. $F(t, \sigma, \cdot)$ is convex $\forall t \in T, \sigma \in K$,
iv. $\Psi(\sigma, \cdot)$ and $\alpha(\sigma, \cdot)$ are convex $\forall \sigma \in K$.

Consequently, $\Gamma(\epsilon) = \Lambda(\epsilon)$ $\forall \epsilon > 0$.

Lemma1.49. (Hashoosh and Alimohammady 2016) Considering K is a nonempty convex, closed subset of the B.S X, and $F: T \times K \times K \to R$ and $\Psi, \alpha: K \times K \to R$ satisfy in the following conditions:

i. $F(\cdot, \varrho, \cdot), \alpha(\cdot, \varrho)$ are lsc $\forall \varrho \in K$,
ii. $\Psi(\cdot, \varrho)$ is usc $\forall \varrho \in K$.

Consequently, $\Lambda(\epsilon)$ is closed in $T \times K$ $\forall \epsilon > 0$.

Theorem 1.50. (Fang *et al.* 2008) Consider that $T: X \to X$ is $H.C$ and monotone and $\vartheta: X \to R \cup \{+\infty\}$ be proper, lower semi continuous and convex. Then, $MVI(T, \vartheta)$is strongly α-w-p if and only if

$$\Gamma_\alpha(\epsilon) \neq \emptyset \quad \forall \epsilon > 0, \;\; diam\big(\Gamma_\alpha(\epsilon)\big) \to 0 \text{ as } \epsilon \to 0.$$

Where $MVI(T, \vartheta)$: Fınd $\sigma \in X$ $\langle T(\sigma), \sigma - \varrho\rangle + \vartheta(\sigma) - \vartheta(\varrho) \leq 0$ $\forall \varrho \in X$, and

$$\Gamma_\alpha(\epsilon) = \{\sigma \in X: \langle T(\sigma), \sigma - \varrho\rangle + \vartheta(\sigma) - \vartheta(\varrho) \leq \frac{\alpha}{2}\|\sigma - \varrho\| + \epsilon \forall \varrho \in X.$$

Theorem 1.51. (Fang *et al.* 2008) Let $T: X \to X$ is function in which $\sigma \longmapsto \langle T(\sigma), \sigma - \varrho\rangle$ is lower semi continuous for all $\varrho \in X$,and $\vartheta: X \to R \cup \{+\infty\}$ be proper lower semi continuous and convex. Consequently, $MVI(T, \vartheta)$ is strongly α-w-p in generalized sense if and only if

$$\Gamma_\alpha(\epsilon) \neq \emptyset \quad \forall \epsilon > 0, \qquad \mu\big(\Gamma_\alpha(\epsilon)\big) \to 0 \; as \; \epsilon \to 0.$$

Let's have a discussion about how to formulate optimization issues that have an equilibrium constraint. The formulation of the optimization issue with a generalized equilibrium constraint (OPGPEC) is as follows:

$$\min h\,(t,u) \text{ where } (t,u) \in T \times K\,,$$

where $u \in S(t)$ is a collection of solutions to the parametric general equilibrium issue $(EP_{\Psi}(t))$, defined by $u \in S(t)$ iff

$$F(t,u,v) + \Psi(u,v) \geq 0 \text{ for all } v \in K.$$

Definition 1.52. (Hashoosh 2016) A sequence $\{(t_n, u_n)\} \subset T \times K$ is named to be an App. seq for (OPGPEC) if

i. There is a non-negative sequence $\{\epsilon_n\}$ by $\epsilon_n \to 0$ as $n \to \infty$ in which

$$F(t_n, u_n, v) + \Psi(u_n, v) \geq -\epsilon_n \|v - u_n\| \forall n \in N, v \in K,$$

ii. $\lim_{n\to\infty} h(t_n, u_n) \leq \inf_{r\in T, v\in S(r)} h(r, v).$

Definition 1.53. (Hashoosh 2016) (OPGPEC) is said to be s-w-p (or strongly (or weakly) $w-p$ in the generalized sense) if (OPGPEC) has a single solution σ and for every App.seq for (OPGPEC) converges strongly (or weakly) to the single solution. If $S \neq \emptyset$, then every approximate solution sequence has a subsequence which strongly (or weakly) converges to a point of S.

The following is a definition of the set of approximate solutions to the equation (OPGPEC).

$$\eta(\epsilon, \delta) := \begin{cases} (t,u) \in T \times K; h(t,u) \leq \inf_{r\in T, v\in S(r)} h(r,v) + \delta \text{ and} \\ F(t,u,v) + \Psi(u,v) \geq -\epsilon_n \parallel v - u \parallel \forall v \in K. \end{cases}$$

Theorem 1.54. (Hashoosh 2016) Real B.Ss E and X have subsets T and K that are nonempty, closed, and convex resp., If (OPGPEC) is s-w-p in the generalized sense. Then

$$\eta(\epsilon,\delta) \neq \emptyset\ \forall \epsilon,\delta > 0,\ lim_{(\epsilon,\delta)\to(0,0)}\mu\big(\eta(\epsilon,\delta)\big) = 0.$$

Additionally, assuming the following assumptions are true:

i. $F(\cdot,\sigma\,,\cdot)$ is lsc $\forall\ \sigma\ \in K$,
ii. $F(t,\sigma\,,\cdot)$ is convex $\forall (t,\sigma\,) \in T \times K$,
iii. $F(t,\cdot,\cdot)$ is α -M. bi.f, H.C, $\forall t \in T$,
iv. $F(t,\sigma\,,\sigma\,) = 0, \forall t \in T, \sigma\ \in K$,
v. $\alpha(\cdot,\varrho)$ and $\Psi(\cdot,\varrho)$ are $usc \forall \varrho \in K$,
vi. $\Psi(\sigma\,,\cdot)$ and $\alpha(\sigma\,,\cdot)$ are convex $\forall \sigma\ \in K$,
vii. h is lsc.

Then, the converse holds.

Theorem 1.55. (Hashoosh 2016) Let T and K be nonempty, closed and convex subsets of finite dimensional B.Ss E and X resp., Le $F: T \times K \times K \to R, \Psi, \alpha: K \times K \to R$ and $h: T \times K \to R$ are four mappings. Assume that

i. $F(\cdot,\sigma\,,\cdot)$ is lsc and convex $\forall \sigma\ \in K$,
ii. $F(t,\cdot,\cdot)$ is $\alpha - M.bi.f$ and H.C, $\forall t \in T$,
iii. $F(t,\sigma\,,\sigma\,) = 0, \forall t \in T, \sigma\ \in K$,
iv. $\alpha(\cdot,\varrho)$ and $\Psi(\cdot,\varrho)$ are $usc \forall \varrho \in K$,
v. $\Psi(\sigma\,,\cdot)$ and $\alpha(\sigma\,,\cdot)$ are convex $\forall \sigma\ \in K$,
vi. h is lsc and convex function.

Consequently, (OPGPEC) is s-w-p if and only if it has a single solution.

For more, view previous conclusions, we recall further results and theorems as referred by (Kimura 2008) that are useful in our work.

Consider that X is a non-empty subset of V, Z is a real topological vector space by a solid pointed convex cone $C \subset Z$ and θ_Z zero vectors in the Z and a nonempty subset of Hausdorff topological space. Then the issue for any $p \in \Gamma$, find $x \in X$ in which

$$f(p, \sigma, \varrho) \notin -\int C \text{ , for any } \varrho \in X,$$

is said to be parametric vector equilibrium issue and denoted by VEP_p.

Theorem 1.56. (Kimura 2008) Real H.T.L.S V and X are nonempty subsets of V. Given that Z is a real topological vector space and $C \subset Z$ is a solid, pointed convex cone, Γ is a topological space. Assume f is a vector-valued function from $\Gamma \times X \times X$ to Z by the equation $f(p, \sigma, \sigma) = \theta_Z$, for every $p \in \Gamma$ and $\sigma \in X$. In addition, the following factors are taken into account:

i. $f(\cdot,\cdot,\varrho)$ is $(-C)$ -continuous function on $\Gamma \times X$ $\forall \varrho \in X$;
ii. $f(p,\cdot,\varrho)$ is strictly $(-C)$ -properly quasiconvex on X $\forall p \in \Gamma$ and $\varrho \in X$;

Consequently, $\{\boldsymbol{VEP_p} : \boldsymbol{p} \in \boldsymbol{\Gamma}\}$ is parametrically uniquely w-p.

3. NEW RESULTS OF RELAXED HYBRID-TYPE- $VI(\omega\eta\kappa)$

3.1 Introduction

In this chapter, using a new type of monotonous function is called $(\eta\kappa\alpha\beta - p)$-monotone operator w.r.t the second argument of a hybrid-type mapping $\omega: K \times E^* \to E^*$ in which K is a nonempty set of real reflexive B.S E, E^* is a dual of E. Among the results that we obtained in this chapter, Our findings prove the existence of $VI(\omega\eta\kappa)$ by using KKM-mapping in reflexive B.Ss. Further, the solution of $VI(\omega\eta\kappa)$ is equivalent to the solution of $VI(\omega\eta\kappa\alpha\beta)$. Following is a new definition that we have developed for the pseudomonotone operator that contributed significantly to the solutions of our main findings.

Definition 2.1. A set-valued mapping $T: K \to 2^{E^*}$ is named to be $(\eta\kappa\alpha\beta - p)$-monotone operator w.r.t the second argument of a hybrid-type mapping $\omega: K \times E^* \to E^*$ if $\exists$ a mappings $\eta: K \times K \to E$, $\kappa: K \times K \to R$ and $\alpha: E \to R$ with $\alpha(tz) = \Lambda(t)\alpha(z)$ for $z \in E$, where $\Lambda: (0, \infty) \to (0, \infty)$ is a function by $\frac{\lim_{t\to 0} \Lambda(t)}{t} = 0$, and $\beta: K \times K \to R$ with

$$\lim_{t\to 0}\left[\frac{\Lambda(t)\alpha(\varrho - \sigma)}{t} + \frac{\beta(\sigma, t\varrho + (1-t)\sigma)}{t}\right] = 0,$$

where for any $\sigma, \varrho \in K$

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq 0 \text{ for all } u \in T(\sigma),$$

implies

$$\langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho) \text{ for all } v \in T(\varrho).$$

Remark 2.2.

i) If $\omega(\sigma\,,u) = u$, $\omega(\sigma\,,v) = v$ and $\beta(\sigma\,,\varrho) = 0$ in Definition 2.1. Consequently, the concept of pseudomonotone has been defined in (Kang *et al.* 2003), For all $\sigma\,,\varrho \in K$

$$\langle u,\eta(\varrho,\sigma\,)\rangle + \kappa(\varrho,\sigma\,) \geq 0 \text{ for all } u \in T(\sigma\,),$$

implies

$$\langle v,\eta(\varrho,\sigma\,)\rangle + \kappa(\varrho,\sigma\,) \geq \alpha(\varrho - \sigma\,) \text{ for all } v \in T(\varrho).$$

ii) If T is a single-valued mapping and $\Lambda(t) = t^p$ for $p > 1$, and $\beta(\sigma\,,\varrho) = 0$, Consequently, if we assume the following, we get

a) relaxed η- α-monotone concept defined in (Fang and Huang 2003). For each $\sigma,\varrho \in K$

$$\langle T(\sigma\,) - T(\varrho),\eta(\sigma\,,\varrho)\rangle \geq \alpha(\sigma\; - \varrho).$$

b) relaxed η - α-pseudomonotone concept defined in (Bai *et al.* 2006).

$\langle T(\varrho),\eta(\sigma\,,\varrho)\rangle \geq 0$ implies $\langle T(\sigma\,),\eta(\sigma\,,\varrho)\rangle \geq \alpha(\sigma\; - \varrho)$ For all $\sigma\,,\varrho \in K$,

iii) If $\beta \equiv 0$ and $\lim_{t\to 0}\frac{\Lambda(t)}{t} = 0$, then in Definition 2.1 then we have the following pseudomonotone concept defined in (Lee *et al.* 2009).

$$\langle \omega(\sigma\,,u),\eta(\varrho,\sigma\,)\rangle + \kappa(\varrho,\sigma\,) \geq 0 \text{ for all } u \in T(\sigma\,);$$

implies

$$\langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq \alpha(\varrho - \sigma) \text{ for all } v \in T(\varrho).$$

Next, we analyze two distinct kinds of V.I in reflexive B.Ss:

(i) $VI(\omega\eta\kappa)$ Find $\sigma \in K$ satisfying

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq 0 \text{ for } \varrho \in K \text{ and } u \in T(\sigma). \qquad (2.1)$$

(ii) $VI(\omega\eta\kappa\alpha\beta)$ Find $\sigma \in K$ satisfying

$$\langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho) \text{ for } \varrho \in K \text{ and } \in T(\varrho). \qquad (2.2)$$

3.2 Main Results

We discuss existence of findings for the issue $\boldsymbol{VI(\omega\eta\kappa)}$ in a reflexive B.S.

Theorem 2.3. Consider that E is a real reflexive B.S and that K is a non-empty closed convex subset of this space, $T: K \to 2^{E^*}$ is an η-H.C and $(\eta\kappa\alpha\beta - p)$-monotone operator w.r.t the second argument of a hybrid-type mapping $\omega: K \times E^* \to E^*$. Assume that

i. $\eta(\sigma, \sigma) = \kappa(\sigma, \sigma) = 0 \ \forall \sigma \in K$;
ii. $\sigma \mapsto \eta(., \sigma)$ and $\sigma \mapsto \kappa(., \sigma)$ are convex.

Then, $\sigma \in K$ is a solution of $VI(\omega\eta\kappa)$ if, and only if it is a solution of VI(ωηκαβ).

Proof. If σ is a solution of $VI(\omega\eta\kappa)$, by Definition 2.1 it is a solution of VI(ωηκαβ).

On the other hand, consider that $\sigma \in K$ is a solution of $VI(\omega\eta\kappa\alpha\beta)$ and $\varrho \in K$ is any point. Let

$$\sigma_t = t\varrho + (1-t)\sigma, t \in [0,1], \text{ then } \sigma_t \in K$$

It follows from $VI(\omega\eta\kappa\alpha\beta)$ that, for $u_t \in T(\sigma_t)$

$$\begin{aligned} \langle \omega(\sigma, u_t), \eta(\sigma_t, \sigma) \rangle + \kappa(\sigma_t, \sigma) &\geq \alpha(\sigma_t - \sigma) + \beta(\sigma, \sigma_t) \\ &= \alpha\big(t(\varrho - \sigma)\big) + \beta(\sigma, \sigma_t) \\ &= \Lambda(t)\alpha(\varrho - \sigma) + \beta(\sigma, \sigma_t). \end{aligned} \tag{2.3}$$

As a consequence of conditions (i) and (ii), we get

$$\langle \omega(\sigma, u_t), \eta(\sigma_t, \sigma) \rangle + \kappa(\sigma_t, \sigma)$$

$$= \langle \omega(\sigma, u_t), \eta(t\varrho + (1-t)\sigma, \sigma) \rangle + v(t\varrho + (1-t)\sigma, \sigma)$$

$$\leq t\langle \omega(\sigma, u_t), \eta(\varrho, \sigma) \rangle + (1-t)\langle \omega(\sigma, u_t), \eta(\sigma, \sigma) \rangle + t\kappa(\varrho, \sigma) + (1 - t)\kappa(\sigma, \sigma)$$

$$= t[\langle \omega(\sigma, u_t), \eta(\varrho, \sigma) \rangle + \kappa(\varrho, \sigma)]. \tag{2.4}$$

It follows from (2.3) and (2.4), for all $\varrho \in K, u_t \in T(\sigma_t)$.

$$\langle \omega(\sigma, u_t), \eta(\varrho, \sigma) \rangle + f(\varrho, \sigma) \geq \frac{k(t)}{t}\alpha(\varrho - \sigma) + \frac{1}{t}\beta(\sigma, \sigma_t) \tag{2.5}$$

Since T is η-H.C, letting $t \to 0$ in (2.5), we get

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle + f(\varrho, \sigma) \geq 0, \text{ for all } \varrho \in K \text{ and } u \in T(\sigma).$$

Therefore, $\sigma \in K$ is a solution of $VI(\omega\eta\kappa)$.

Theorem 2.5. Assume that K is a bounded closed convex subset of a reflexive B.S E. Let $T: K \to 2^{E^*}$ be an η-H.C and $(\eta\kappa\alpha\beta - p)$-monotone operator w.r.t the second argument of a mapping $\omega: K \times E^* \to E^*$. Consider that

i. $\eta(\sigma, \sigma) = 0$ and $\kappa(\sigma, \sigma) = 0$ for all $\sigma, \in K$;

ii. $\sigma \mapsto \eta(\sigma, \cdot)$ and $\sigma \mapsto \kappa(\sigma, \cdot)$ are usc and convex in first argument;

iii. $\alpha: E \to R$ and $\beta(., \sigma)$ weakly lsc.

Then you'll have to find at least one solution for $VI(\omega\eta\kappa)$.

Proof. A set-valued mappings $F, G: K \to 2^E$ are defined as follows for each $\varrho \in K$

$$F(\varrho) = \{\sigma \in K: \langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq 0, u \in T(\sigma)\},$$
$$G(\varrho) = \{\sigma \in K: \langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho), v \in T(\varrho)\}.$$

We will assert F to be a KKM mapping. If F doesn't correspond to a KKM mapping, then there is $\{\varrho_1, \varrho_2, \dots, \varrho_n\} \subset K$ in which $\text{co}\{\varrho_1, \varrho_2, \dots, \varrho_n\} \not\subset \bigcup_{i=1}^n F(\varrho_i)$. It means that there is a $\varrho_0 \in \text{co}\{\varrho_1, \varrho_2, \dots, \varrho_n\}$, so $\varrho_0 = \sum_{i=1}^n t_i \varrho_i$, where $t_i \geq 0$ $i = 1, \dots, n, \sum_{i=1}^n t_i = 1$, but $\varrho_0 \notin \bigcup_{i=1}^n F(\varrho_i)$. By the definition of F, we have

$$\langle \omega(\varrho_0, u), \eta(\varrho_i, \varrho_0)\rangle + \kappa(\varrho_i, \varrho_0) < 0, \text{ for some } u \in T(\varrho_0).$$

For $i = 1,2, \dots, n$, it follows from i and ii that

$$
\begin{aligned}
0 \quad &= \langle \omega(\varrho_0, u), \eta(\varrho_0, \varrho_0) \rangle + \kappa(\varrho_0, \varrho_0) \\
&= \left\langle \omega(\varrho_0, u), \eta\left(\sum_{i=1}^{n} t_i \varrho_i, \varrho_0\right) \right\rangle + \kappa\left(\sum_{i=1}^{n} t_i \varrho_i, \varrho_0\right) \\
&\leq \sum_{i=1}^{n} t_i \langle \omega(\varrho_0, u), \eta(\varrho_i, \varrho_0) \rangle + \sum_{i=1}^{n} t_i\, \kappa(\varrho_i, \varrho_0) \\
&= \sum_{i=1}^{n} t_i [\langle \omega(\varrho_0, u), \eta(\varrho_i, \varrho_0) \rangle + \kappa(\varrho_i, \varrho_0)] \\
&< 0,
\end{aligned}
$$

for some $u \in T(\varrho_0)$, which contradicts the hypothesis. Consequently, F is a KKM mapping. Now we demonstrate

$$F(\varrho) \subset G(\varrho), \text{ for all } \varrho \in K.$$

For any given $\varrho \in K$, letting $\sigma \in F(\varrho)$, we have

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle + \kappa(\varrho, \sigma) \geq 0 \text{ for } u \in T(\sigma).$$

Since T is $(\eta\kappa\alpha\beta - p)$-monotone operator

$$\langle \omega(\sigma, v), \eta(\varrho, \sigma) \rangle + \kappa(\varrho, \sigma) \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho) \text{ for } v \in T(\varrho).$$

It follows that $\sigma \in G(\varrho)$, so

$$F(\varrho) \subset G(\varrho), \text{ for all } \varrho \in K$$

Consequently, this demonstrates that G is also a KKM mapping.

On the other hand, let $\langle \sigma_n \rangle$ be a net in $G(\varrho)$ converging weakly to σ_0, then for all $v \in T(\varrho)$,

$$\langle \omega(\sigma_n, v), \eta(\varrho, \sigma_n)\rangle + \kappa(\varrho, \sigma_n) \geq \alpha(\varrho - \sigma_n) + \beta(\sigma_n, \varrho).$$

Since $\sigma \mapsto \eta(\sigma, .)$ and $\sigma \mapsto f(\sigma, .)$ are usc, and α, β are weakly lsc, it follows that

$$\begin{aligned}
\langle \omega(\sigma_0, v), \eta(\varrho, \sigma_0)\rangle + \kappa(\varrho, \sigma_0) &\geq \limsup_n \langle \omega(\sigma_k, v), \eta(\varrho, \sigma_k)\rangle + \limsup_n \kappa(\varrho, \sigma_k) \\
&\geq \limsup_n \big(\langle \omega(\sigma_k, v), \eta(\varrho, \sigma_k)\rangle + \kappa(\varrho, \sigma_k)\big) \\
&\geq \liminf_n \big(\langle \omega(\sigma_k, v), \eta(\varrho, \sigma_k)\rangle + \kappa(\varrho, \sigma_k)\big)
\end{aligned}$$

$$\begin{aligned}
&\geq \liminf_n \alpha(\varrho - \sigma_k) + \beta(\sigma_k, \varrho) \\
&\geq \alpha(\varrho - \sigma_0) + \beta(\sigma_0, \varrho).
\end{aligned}$$

As a result, $\sigma_0 \in G(\varrho)$, proving that $G(\varrho)$ is weakly closed for all $\varrho \in K$. Again, for all $\varrho \in K$ $G(\varrho)$ is weakly compact in K because K is weakly compact and it is bounded closed and convex. Theorem 2.1 and the Fan-KKM Theorem imply that

$$\bigcap_{\varrho \in K} F(\varrho) = \bigcap_{\varrho \in K} G(\varrho) \neq \emptyset.$$

Consequently, there must be at least one solution to $VI(\omega\eta\kappa)$.

Theorem 2.6. Consider that K is a nonempty unbounded closed convex subset of a reflexive B.S E, and a set-valued mapping $T: K \to 2^{E^*}$ be η-H.C, $(\eta\kappa\alpha\beta - p)$-monotone operator w.r.t the second argument of a mapping $\omega: K \times E^* \to E^*$ and $\kappa: K \times K \to R \cup \{+\infty\}$ be a proper function. If the conditions below are met,

i. $\eta(\sigma, \sigma) = 0$ and $\kappa(\sigma, \sigma) = 0$ for all $\sigma, \in K$;
ii. $\eta(\sigma, \varrho) + \eta(\varrho, \sigma) = 0$ and $\kappa(\sigma, \varrho) + \kappa(\varrho, \sigma) = 0$ for all $\sigma, \varrho \in K$;
iii. $\sigma \mapsto \eta(\sigma, \cdot)$ and $\sigma \mapsto \kappa(\sigma, \cdot)$ are usc and convex in first argument ;
iv. $\alpha: E \to R, \beta(., \sigma)$, is weakly lsc;

v. T is η-coercive w.r.t κ.

Consequently, at least one solution for $VI(\omega\eta\kappa)$.

Proof. For $r > 0$, let

$$B_r = \{\varrho \in E : \| \varrho \| \leq r\}$$

Take the following issue into account: Find $\sigma_r \in K \cap B_r$ in which

$$\langle \omega(\sigma_r, u_r), \eta(\varrho, \sigma_r)\rangle + \kappa(\varrho, \sigma_r) \geq 0 \text{ for } \varrho \in K \cap B_r \tag{2.6}$$

and $u_r \in T(\sigma_r)$.

Theorem 2.5 proves that there is only one solution $\sigma_r \in K \cap B_r$ to problem (2.6). Take σ_0 whose norm $\|\sigma_0\|$ is less than r in the coercive conditions (iv). Then, $\sigma_0 \in K \cap B_r$ and

$$\langle \omega(\sigma_r, u_r), \eta(\sigma_0, \sigma_r)\rangle + \kappa(\sigma_0, \sigma_r) \geq 0 \text{ for } u_r \in T(\sigma_r)$$
(2.7)

By condition (i) we get

$$\begin{aligned}
& \langle \omega(\sigma_r, u_r), \eta(\sigma_0, \sigma_r)\rangle + \kappa(\sigma_0, \sigma_r) \\
= \; & -\langle \omega(\sigma_r, u_r) - \omega(\sigma_r, u_0), \eta(\sigma_r, \sigma_0)\rangle + \kappa(\sigma_0, \sigma_r) + \langle \omega(\sigma_r, u_0), \eta(\sigma_0, \sigma_r)\rangle \\
\leq \; & -\langle \omega(\sigma_r, u_r) - \omega(\sigma_r, u_0), \eta(\sigma_r, \sigma_0)\rangle + \kappa(\sigma_0, \sigma_r) + \|\omega(\sigma_r, u_0)\| . \|\eta(\sigma_r, \sigma_0)\| \\
\leq \; & \|\eta(\sigma_r, \sigma_0)\| \left(-\frac{\langle \omega(\sigma_r, u_r) - \omega(\sigma_r, u_0), \eta(\sigma_r, \sigma_0)\rangle + \kappa(\sigma_r, \sigma_0)}{\|\eta(\sigma_r, \sigma_0)\|} + \|\omega(\sigma_r, u_0)\| \right)
\end{aligned}$$

for $u_r \in T(\sigma_r)$ and $u_0 \in T(\sigma_0)$.

If $\|\sigma_r\| = r$ for all r, we can choose r large enough such that the above inequality and the η-coercivity of T w.r.t κ, we get

$$\langle \omega(\sigma_r, u_r), \eta(\sigma_0, \sigma_r)\rangle + \kappa(\sigma_0, \sigma_r) < 0.$$

It is in direct opposition to (2.7). Then, there exists r in which $\|\sigma_r\| < r$. For any $\varrho \in K$, one can choose $0 < \epsilon < 1$ small enough in which

$$\sigma_r + \epsilon(\varrho - \sigma_r) \in K \cap B_r$$

Given that (2.6), it follows that

$$\begin{aligned} 0 \leq\ & \langle \omega(\sigma_r, u_r), \eta(\sigma_r + \epsilon(\varrho - \sigma_r), \sigma_r)\rangle + \kappa(\sigma_r + \epsilon(\varrho - \sigma_r), \sigma_r) \\ \leq\ & (1-\epsilon)\langle \omega(\sigma_r, u_r), \eta(\sigma_r, \sigma_r)\rangle + (1-\epsilon) f(\sigma_r, \sigma_r) + \epsilon\langle \omega(\sigma_r, u_r), \eta(\varrho, \sigma_r)\rangle \\ & + \epsilon f(\varrho, \sigma_r) \\ =\ & \epsilon[\langle \omega(\sigma_r, u_r), \eta(\varrho, \sigma_r)\rangle + \kappa(\varrho, \sigma_r)], \end{aligned}$$

which implies that

$$\langle \omega(\sigma_r, u_r), \eta(\varrho, \sigma_r)\rangle + \kappa(\varrho, \sigma_r) \geq 0,$$

for $\varrho \in K$ and- $u_r \in T(\sigma_r)$. This completes the proof.

Remark 2.7

i. Theorems 2.4-2.6 also hold if T is continuous on finite dimensional subspaces.
ii. Theorems 2.5 and 2.6 improve and generalize the known results of P. Hartman and G. Stampacchia (1966).

4. IMPROVEMENT OF W-p INVOLVING $(\boldsymbol{\eta\kappa\alpha\beta - p})$-MONOTONE OPERATOR

4.1 Introduction

This chapter's objective is to establish the existence and uniqueness of w-p solutions for $VI(\omega\eta\kappa)$ utilizing the $(\eta\kappa\alpha\beta - p)$-monotone operator in relation to the second argument of a hybrid-type. To achieve this, we consider a subset K of the B.S E. We derive a helpful characterization of metric projection as well as adequate conditions for these types of w-p. This chapter's explanation can be used to demonstrate that the concept of w-p for new types of variational inequality is equivalent to the solution's existence and uniqueness.

In order to achieve these outcomes, the present chapter is organized as follows. Section 2 begins with a review of the necessary terminology and references some field-specific findings. The third section defines and extends the notion of w-p to new types of variational inequality (Briefly, V.I). In Section 4, we introduce a novel notion of w-p for optimization issues by constraints specified by a parametric form of $VI(\omega\eta\kappa)$.

4.2 Basic Definitions of Well-Posedness

Our working assumption is that E and W are both B.Ss, and that K is a nonempty, convex, and closed subset of E as well as A subset of W, unless something to the contrary is stated.

The following definitions and results are provided For the convenience of the reader and are necessary in order to demonstrate our basic findings. We recollect the next generalized $VI(\omega\eta\kappa)$ mapping w.r.t the second argument of a hybrid-type mapping $\omega: K \times E^* \to E^*$ if there is a mappings $\eta: K \times K \to E, \kappa: K \times K \to R$and $\alpha: E \to R$ with $\alpha(tz) = \Lambda(t)\alpha(z)$ for $z \in E$, where $\Lambda: (0, \infty) \to (0, \infty)$ is a mapping by $\lim_{t\to 0} \frac{\Lambda(t)}{t} = 0$, and $\beta: K \times K \to R$,with

$$\lim_{t \to 0} \left[\frac{\Lambda(t)\alpha(\varrho - \sigma)}{t} + \frac{\beta(\sigma, t\varrho + (1-t)\sigma)}{t} \right] = 0,$$

in which for any $\sigma, \varrho \in K$

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle + \kappa(\varrho, \sigma) \geq 0 \text{ for all } u \in T(\sigma).$$

With a set-valued mapping $T: K \to 2^{E^*}$.

Following is a formulation of optimization issues by parametric. Let $H: A \times K \longrightarrow R$ and $F: A \times K \times K \to R$ are two functions. The following is a formulation of the optimization problem with generalized parametric models, indicated by OPGPM:

$\min H(s, \sigma) \in A \times K$in which $\sigma \in I(s)$ and $I(s)$ is the solution family of the parametric generalized $Ep_S(\omega\eta\kappa)$ defined by $\sigma \in I(s)$ if

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle + \kappa(s, \varrho, \sigma) \geq 0 \text{ for all } \varrho \in K, u \in T(\sigma). \tag{3.1}$$

Instead of writing $\{Ep_S(\omega\eta\kappa), s \in A\}$ family of generalized (parametric issue).

The following outcomes are possible, based on the specific circumstances:

i. If $\omega \equiv 0$ and $\kappa(s, \varrho, \sigma) = -h(s, \varrho, \varrho - \sigma)$ then the issue can be converted to parametric quasivariational inequality QVI(s) (Lee 2009).

ii. If $\omega \equiv 0$, then the issue can be converted to the parametric equilibrium issue EP(s) (Luc 2001).

iii. If $\langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle = \beta(\varrho, \sigma)$ then the issue can be converted to the optimization issue by generalized equilibrium constraint (OPGPEC) (Chen 2014).

4.3 W-p For $VI(\omega\eta\kappa)$ With Metric Characterizations

The principles of being w-p for $VI(\omega\eta\kappa)$ are introduced in this section. Through the section's results, we outline some conditions under which the $VI(\omega\eta\kappa)$ is s-w-p in the generalized sense.

Definition 3.1 A sequence $\{(s_n, \sigma_n)\} \subset A \times K$ is said to be App.seq for $VI(\omega\eta - \kappa)$ if $\exists$ $\{c_n\} \geq 0$ with $c_n \to 0$ as $n \longrightarrow \infty$ such that

$$\langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n) \rangle + \kappa(s_n, \varrho, \sigma_n) + c_n \|\sigma_n - \varrho\| \geq 0, u \in T(\sigma) \tag{3.2}$$

Definition 3.2 The issue $VI(\omega\eta\kappa)$ is named strongly w-p (or strongly w-p in the generalized sense in $\{\sigma_n\}$ with $\sigma_n \longrightarrow \sigma$) if $VI(\omega\eta\kappa)$ has a single solution σ and for every App.seq $\{\sigma_n\}$ for $VI(\omega\eta\kappa)$ converge strongly to the single solution **(**or If $VI(\omega\eta\kappa)$ has a nonempty solution set I(s), So, there exists a subsequence of all approximation solution sequences that strongly converges to a point of I(s)).

Following are several characterizations of w-p for the $VI(\omega\eta\kappa)$. Two sets are defined for any $c > 0$.

$$\Gamma(c) = \{(s, \sigma) \in A \times K; \langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \geq 0, u \in T(\sigma)\}.$$

$$\begin{aligned} N(c) = \{(s, \sigma) \in A \times K; \langle \omega(\sigma, v), \eta(\varrho, \sigma) \rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \\ \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho) \text{ for } v \in T(\varrho)\}. \end{aligned}$$

In the next Lemma we will prove that the above characterizations are equal.

Lemma 3.3. Suppose E is a real reflexive B.S and that K is a non-empty closed convex subset of this space. Assume that

i. T is $(\eta\kappa\alpha\beta - p)$-monotone operator;

ii. $\eta(\sigma,\sigma) = 0$ and $F(s,\sigma,\sigma) = 0 \ \forall\sigma \in K$;

iii. $\sigma \mapsto \eta(.,\sigma)$ and $\sigma \mapsto \kappa(s,.,\sigma)$ are convex;

iv. $T: K \to 2^{E^*}$ be η-H.C.

Then, $\Gamma(c) = N(c)$ for all $c > 0$.

Proof. Assme that $(s,\sigma) \in \Gamma(c)$, so $(s,\sigma) \in A \times K$ such that

$$\{\langle \omega(\sigma,u), \eta(\varrho,\sigma)\rangle + \kappa(s,\varrho,\sigma) + c\|\sigma - \varrho\| \geq 0, u \in T(\sigma)\}.$$

By $(\eta\kappa\alpha\beta - p)$-monotonicity, we have

$$\langle \omega(\sigma,v), \eta(\varrho,\sigma)\rangle + \kappa(s,\varrho,\sigma) + c\|\sigma - \varrho\| \geq \alpha(\varrho - \sigma) + \beta(\sigma,\varrho).$$

Therefore , $(s,\sigma) \in N(c)$. Conversely, assume that $(s,\sigma) \in N(c)$

$$\langle \omega(\sigma,v), \eta(\varrho,\sigma)\rangle + \kappa(s,\varrho,\sigma) + c\|\sigma - \varrho\| \geq \alpha(\varrho - \sigma) + \beta(\sigma,\varrho) v \in T(\varrho).$$

Then, for all $\sigma \in K$, we assume that $\varrho \in K$ is any point such that

$\sigma_t = t\varrho + (1-t)\sigma , t \in [0,1]$, so $\sigma_t \in$ K, because K is a convex set.

For $u_t \in T(\sigma_t)$

$$\begin{aligned}\langle \omega(\sigma,u_t), \eta(\sigma_t,\sigma)\rangle + \kappa(s,\sigma_t,\sigma) + c\|\sigma - \sigma_t\| &\geq \alpha(\sigma_t - \sigma) + \beta(\sigma,\sigma_t) \\ &= \alpha\big(t(\varrho - \sigma)\big) + \beta(\sigma,\sigma_t) \\ &= k(t)\alpha(\varrho - \sigma) + \beta(\sigma,\sigma_t)\end{aligned} \tag{3.3}$$

We have a conclusion based on conditions (ii) and (iii).

$$\langle \omega(\sigma, u_t), \eta(\sigma_t, \sigma)\rangle + \kappa(s, \sigma_t, \sigma) + c\|\sigma - \sigma_t\|$$

$$= \langle \omega(\sigma, u_t), \eta(t\varrho + (1-t)\sigma, \sigma)\rangle + \kappa(s, t\varrho + (1-t)\sigma, \sigma)$$

$$\leq t\langle \omega(\sigma, u_t), \eta(\varrho, \sigma)\rangle + (1-t)\langle \omega(\sigma, u_t), \eta(\sigma, \sigma)\rangle + t\kappa(\mathrm{s}, \varrho, \sigma) + (1 - t)\kappa(\mathrm{s}, \sigma, \sigma) + tc\|\sigma - \mathrm{y}\|$$

$$= t[\langle \omega(\sigma, u_t), \eta(\varrho, \sigma)\rangle + \kappa(\mathrm{s}, \varrho, \sigma) + c\|\sigma - \mathrm{y}\|] \tag{3.4}$$

It follows from (3.3) and (3.4), for all $\varrho \in K, u_t \in T(\sigma_t)$.

$$\langle \omega(\sigma, u_t), \eta(\varrho, \sigma)\rangle + \kappa(\varrho, \sigma) + c\|\sigma - \varrho\| \geq \frac{k(t)}{t}\alpha(\varrho - \sigma) + \frac{1}{t}\beta(\sigma, \sigma_t), \tag{3.5}$$

Since T is η-H.C, letting $t \to 0$ in (3.5), we get

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \geq 0, \text{ for all } \varrho \in K \text{ and } u \in T(\sigma).$$

Therefore, $(s, \sigma) \in \Gamma(c)$.

Lemma 3.4. Suppose E is a real reflexive B.S and that K is a non-empty closed convex subset of this space and

i. $\omega(.,v), \eta(\varrho,.)$ and $\kappa(.,\varrho,.)$are usc ,
ii. $\beta(.,\varrho)$ and $\alpha(\sigma)$ are lsc.

Then, $N(c)$ is closed in $A \times K$ $\forall c > 0$.

Proof. Consider that $\{(s_n, \sigma_n)\} \subseteq N(c)$ is a sequence such that $(s_n, \sigma_n) \longrightarrow (s, \sigma)$. Then

$$\langle \omega(\sigma_n, v), \eta(\varrho, \sigma_n)\rangle + F(s, \varrho, \sigma_n) + c\|\sigma_n - \varrho\| \geq \alpha(\varrho - \sigma_n) + \beta(\sigma_n, \varrho)$$

Since $\omega(.\,, v), \eta(\varrho, .\,)$and $\kappa(s, \varrho, .\,)$are usc , $\beta(.\,, \varrho)$ and $\alpha(\sigma)$ are lsc then

$$\begin{aligned}
&\langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \\
&\qquad \geq \limsup_{n\to\infty} [\langle \omega(\sigma_n, v), \eta(\varrho, \sigma_n)\rangle + \kappa(s_n, \varrho, \sigma_n) + c\|\sigma_n - \varrho\|] \\
&\qquad \geq \liminf_{n\to\infty} [\alpha(\varrho - \sigma_n) + \beta(\sigma_n, \varrho)] \\
&\qquad \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho).
\end{aligned}$$

Then,

$$\langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho).$$

which implies that $(s, \sigma) \in N(c)$, therefore $N(c)$ is closed in $A \times K$ $\forall c > 0$.

Theorem 3.5. Suppose E is a real reflexive B.S and that K is a non-empty closed convex subset of this space. Assume that $VI(\omega\eta\kappa)$ is w-p, then

$$\Gamma(c) \neq \emptyset \,,\ \lim_{c\to 0} diam\, [\Gamma(c)] = 0. \tag{3.6}$$

In addition to this, provided the following assumptions are correct:

i) $T$is$(\eta\kappa\alpha\beta - p)$-monotone operator and η-H.C;
ii) $\eta(\sigma, \sigma) = 0$ and $\kappa(s, \sigma, \sigma) = 0$ for all $\sigma \in K$;

iii) $\sigma \mapsto \kappa(s,.,\sigma)$ and $\sigma \mapsto \eta(.,\sigma)$are convex;

iv) $\omega(.,v), \eta(\varrho,.)$ and $\kappa(.,\varrho,.)$are usc ;

v) $\beta(.,\varrho)$ and $\alpha(\sigma)$ are lsc.

Then, the converse holds.

Proof. Consider that $VI(\omega\eta\kappa)$ is s-w-p, then $VI(\omega\eta\kappa)$ admit a single solution $(s,\sigma) \in A \times K$ in which

$$\langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(s, \varrho, \sigma) \geq 0.$$

It is obvious that $\Gamma(c) \neq \emptyset$,, for every $c > 0$, through contradiction, you consider that

$$\lim_{n\to\infty} diam\,[\Gamma(c_n)] > p > 0,$$

for some sequence $\{c_n\} > 0$. Both sequences $\{(s_n, \sigma_n)\}$and $\{(s_n, e_n)\} \in \Gamma(c_n)$ that we could find such that

$$\|(s_n, \sigma_n) - (s_n, e_n)\| > p \quad \forall\, n \in N. \tag{3.7}$$

Since $\{(s_n, \sigma_n)\}$ and $\{(s_n, e_n)\}$ are App.seq for $VI(\omega\eta\kappa)$. By w-p of $VI(\omega\eta\kappa)$, they must converge to only one solution of $VI(\omega\eta\kappa)$, which is a contradiction (3.7).

Conversely, if the condition (3.6) holds. Let $\{(s_n, \sigma_n)\}$ be an App.seq for $VI(\omega\eta\kappa)$.

Consequently, there is a nonnegative sequence $\{c_n\}$ by $c_n \to 0$ where $n \to \infty$ so that for all $\varrho \in K$ and $u \in T(\sigma)$

$$\langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle + \kappa(s_n, \varrho, \sigma_n) + c_n\|\sigma_n - \varrho\| \geq 0. \tag{3.8}$$

Then, $\{(s_n, \sigma_n)\} \subseteq \Gamma(c_n)$ .From (3.6), $\{(s_n, \sigma_n)\}$ is indeed a Cauchy sequence, it converges to a point (s, σ). As a result of (3.8), $\omega(.,u), \eta(\varrho,.)$and $\kappa(.,\varrho,.)$ are usc, we get

$$0 = \lim_{n\to\infty} \inf c_n \|\sigma_n - \varrho\|$$

$$\geq \lim_{n\to\infty} \inf [-\langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle - \kappa(s_n, \varrho, \sigma_n)] .$$

By virtue of the property $(\eta\kappa\alpha\beta - p)$-monotonicity of T, we can get

$$\lim_{n\to\infty} \inf [-\langle \omega(\sigma_n, v), \eta(\varrho, \sigma_n)\rangle - \kappa(s, \varrho, \sigma_n) \leq \lim_{n\to\infty} \inf[-\alpha(\varrho - \sigma_n) - \beta(\sigma_n, \varrho)].$$

Therefore,

$$\lim_{n\to\infty} \sup [\langle \omega(\sigma_n, v), \eta(\varrho, \sigma_n)\rangle + \kappa(s, \varrho, \sigma_n)$$

$$\geq \lim_{n\to\infty} \inf[\alpha(\varrho - \sigma_n) + \beta(\sigma_n, \varrho)].$$

$$\geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho).$$

But,

$$\langle \omega(\sigma, v), \eta(\varrho, \sigma)\rangle + \kappa(s, \varrho, \sigma) \geq \lim_{n\to\infty} \sup [\langle \omega(\sigma_n, v), \eta(\varrho, \sigma_n)\rangle + \kappa(s, \varrho, \sigma_n)$$

$$\geq \alpha(\varrho - \sigma) + \beta(\sigma, \varrho).$$

Consequently, (s, σ) is a solution to $VI(\omega\eta\kappa)$ according to Theorem (2.4). The uniqueness follows directly from this (3.6).

Remark 3.6. Observe that the diameter of $\Gamma(c)$ does not tend toward zero if $VI(\omega\eta\kappa)$ has several solutions.

As a consequence of this, the Kuratowski of non-compactness measure for approximation solution set is taken into consideration in the following result rather than the diameter.

Theorem 3.7. Consider K and A to be nonempty, closed subsets of the real B.Ss E and W, resp., if, $VI(\omega\eta\kappa)$ is s-w-p in the generalization sense, then:

$$\Gamma(c) \neq \emptyset, \forall c, \lim_{c\to 0} \mu\,[\Gamma(c)] = 0. \tag{3.9}$$

i) $T$is$(\eta\kappa\alpha\beta - p)$-monotone operator and η-H.C;

ii) $\eta(\sigma\,,\sigma\,) = 0$ and $\kappa(s,\sigma\,,\sigma\,) = 0$ for all $\sigma\ \in K$;

iii) $\sigma\ \mapsto \eta(.,\sigma\,)$ and $\sigma\ \mapsto \kappa(s,.,\sigma\,)$ are convex;

iv) $\omega(.,v), \eta(\varrho,.)$ and $\kappa(.,\varrho,.)$ are usc ;

v) $\beta(.,\varrho)$ and $\alpha(\sigma\,)$ are lsc.

So, the converse is true.

Proof. Let $VI(\omega\eta\kappa)$ be s-w-p in a generalized sense. Then the solution set I of $VI(\omega\eta - \kappa)$ is a nonempty. This indicates that for any $c > 0, \Gamma(c) \neq \emptyset$, since $I \subset \Gamma(c)$. Our claim here that the solution set I of $VI(\omega\eta\kappa)$ is compact. Indeed, for any sequence $\{(s_n, \sigma\,_n)\}$ in $I, \{(s_n, \sigma\,_n)\}$ is an App.seq for$VI(\omega\eta\kappa)$.

As a result, there is a converging subsequence to a point of I. It follows that I must be a compact set. Our claim show that $lim_{c\to 0}\mu[(\Gamma(c)] \to 0$. Therefore, given $I \subseteq \Gamma(c)$

$$L(\Gamma(c), I) = max\{e(\Gamma(c), I), e\big(I, \Gamma(c)\big)\}.$$

By compactness of the solution set I, we get

$$\mu\big(\Gamma(c)\big) \leq 2L(\Gamma(c), I) + \mu(I) = 2e(\Gamma(c), I)$$

where $\mu(I) = 0$, since I is compact. To prove $lim_{c\to 0}\mu[(\Gamma(c)] \to 0$. It is sufficient to prove that $e(\Gamma(c), I) \to 0$ as $c \to 0$. If not, there is constant $c > 0$ and $c_n \to 0$, and$\{(s_n, \sigma_n)\} \subset \Gamma(c_n)$ in which

$$(s_n, \sigma_n) \notin I + B_{\frac{c}{2}}(0), \text{ for all } n \in N. \qquad (3.10)$$

where $B_{\frac{c}{2}}(0)$ represents an open ball with a center value of 0 and a radius value of $c/2$.

Since $\{(s_n, \sigma_n)\} \subseteq \Gamma(c_n)$ is an App.seq of $VI(\omega\eta\kappa)$, then, the generalized w-p of $VI(\omega\eta\kappa)$ that there is a subsequence converges to a point of $(s, \sigma) \in I$, it is in direct opposition to (3.10).

On the other hand, suppose that (3.9). Then, Lemmas 3.3 and 3.4 assert that when $c > 0$, $\Gamma(c)$ is a closed non-empty set. Using the Kuratowski Theorem (Kuratowski 1968), one can get,

$$L(\Gamma(c), I) \to 0 \text{ as } c \to 0. \qquad (3.11)$$

where $\mathrm{I} = \bigcap_{c>0} \Gamma(c)$ is a set that is not empty and is compacted.

Let $\{(s_n, \sigma_n)\} \subset A \times K$ be any approximate solution sequence for $VI(\omega\eta\kappa)$. Then, there is a non-negative sequence $\{c_n\}$ by $c_n \to 0$ where $n \to \infty$, in which

$$\langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle + \kappa(s_n, \varrho, \sigma_n) + c_n\|\sigma_n - \varrho\| \geq 0.$$

This means $(s_n, \sigma_n) \in \Gamma(c_n)$ and this together with (3.9) indicate that

$$d[(s_n, \sigma_n), I] \leq e(\Gamma(c), I) \rightarrow 0.$$

Given that I compact, it is logical to conclude that, there is $(\overline{s}_n, \acute{\sigma}_n) \in I$ in which

$$\|(s_n, \sigma_n) - (\acute{s}_n, \acute{\sigma}_n)\| = d[(s_n, \sigma_n), I] \rightarrow 0.$$

Now, due to the compactness of a solution set I, the sequence $(\overline{s}_n, \acute{\sigma}_n)$ and its subsequence $\{(\overline{s}_{n_k}, \acute{\sigma}_{n_k})\}$ strongly converge to I. There for the corresponding $\{(s_{n_k}, \sigma_{n_k})\}$ subsequence of $\{(s_n, \sigma_n)\}$ converge strongly to $\{(\bar{s}_n, \bar{\sigma}_n)\}$. Hence $VI(\omega\eta\kappa)$ is w-p in generalized sense.

4.4 W-p of Optimization Issues By Generalized Parametric Constraints

Let's begin our discussion of the formulation of optimization issues with constraints here in this section. The following is a formulation of the optimization issue with generalized constraint that is indicated by the symbol (OPGPM):

$\min H(s, \sigma)$ where $(s, \sigma) \in A \times K$, and $\sigma \in I(s)$,

A is a nonempty closed subset of a parametric space, $H: A \times K \longrightarrow R, \kappa: A \times K \times K \rightarrow R$, and $I(s)$ is the solution set of the set of the parametric generalized $Ep_S(\omega\eta - \kappa)$ defined by $\sigma \in I(s)$ if:

$\langle \omega(\sigma, u), \eta(\varrho, \sigma) \rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \geq 0$, for all $u \in T(\sigma)$.

Definition 3.8. A sequence $\{(s_n, \sigma_n)\} \subset A \times K$ is named to be App.seq for (OPGPM) if

i. There is a non-negative sequence $\{c_n\}$ with $c_n \rightarrow 0$ as $n \rightarrow \infty$ in which

$$\langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle + \kappa(s_n, \varrho, \sigma_n) + c_n\|\sigma_n - \varrho\| \geq 0, u \in T(\sigma).$$

ii. $lim_{n\to\infty} sup H(s_n, \sigma_n) \leq \inf_{m\in A, e\in I(m)} H(m, e).$

Definition 3.9. (OPGPM) is said to be strongly w-p (resp., s-w-p in the generalized sense) if (OPGPM) has a single solution and for every App.seq for (OPGPM) converges strongly (or weakly) to the single solution. If $\boldsymbol{I} \neq \boldsymbol{\varphi}$ In addition, each and every solution of App.seq has a subsequence that strongly converges to a point of I.

The following statements define the set of approximate solutions for (OPGPM):

$$\psi(c, \epsilon) = \begin{cases} (s, \sigma) \in \mathrm{A} \times \mathrm{K}; \mathrm{H}(s, \sigma) \leq \inf_{m\in A, e\in I(m)} H(m, e) + \epsilon \ \text{ and} \\ \quad \langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(s, \varrho, \sigma) + c\|\sigma - \varrho\| \geq 0. \end{cases}$$

Theorem 3.10. Consider that K is a nonempty convex, closed subset of a B.S E. And $\beta_1: A \times K \times K \to R$ and $H: A \times K \to R$ are two functions, if (OPGPM) is s-w-p. Then

$$\psi(c, \epsilon) \neq \varphi \ \ \forall c, \epsilon > 0 \text{ , diam } \psi(c, \epsilon) \to 0. \tag{3.11}$$

where $(c, \epsilon) \to (0,0)$. Furthermore, assuming the following assumptions are true:

i. $T$is$(\eta\kappa\alpha\beta - p)$-monotone operator and η-H.C;
ii. $\eta(\sigma, \sigma) = 0$ and $\kappa(s, \sigma, \sigma) = 0 \ \forall \sigma \in K$;
iii. $\sigma \mapsto \eta(., \sigma)$ and $\sigma \mapsto \kappa(s, ., \sigma)$ are convex;
iv. $\omega(., v), \eta(\varrho, .)$ and $\kappa(., \varrho, .)$are usc ;
v. $\beta(., \varrho)$ and $\alpha(\sigma)$ are lsc.
vi. H is lsc.

Then the converse hold.

Proof. Assume that $(OPGPM)$ is s-w-p. Then (OPGPM) admits a single solution (s $, \sigma \in A \times K$ in which

$$\begin{cases} H(s,\sigma) = \inf_{m\in A, e\in I(m)} H(m,e) \wedge \langle \omega(\sigma, u), \eta(\varrho, \sigma)\rangle + \kappa(s,\varrho,\sigma) + c\|\sigma - \varrho\| \geq 0. \end{cases}$$

Obviously, $\psi(c,\epsilon) \neq \varphi, \forall c, \epsilon > 0$, because $(s,\sigma) \in \psi(c,\epsilon)$ $\forall c, \epsilon > 0$. If $diam\, \psi(c,\epsilon) \nrightarrow 0$ as $c \to 0, \epsilon \to 0$ then, there is a constant $p > 0$ and $\{c_n\} \geq 0$ and $\{\epsilon_n\} \geq 0$ by $c_n \to 0$, $\epsilon_n \to 0$ as $n \to 0$ and $(s_n, \sigma_n), (s_n, e_n) \in \psi(c,\epsilon)$ in which

$$\|(s_n,\sigma_n) - (s_n, e_n)\| > p \;\forall\, n \in N. \tag{3.12}$$

Since $(s_n,\sigma_n), (s_n, e_n) \in \psi(c,\epsilon)\;\forall\, n \in N$, so both $\{(s_n,\sigma_n)\}$ and $\{(s_n, e_n)\}$ are App.seq for (OPGPM). By the w-p of (OPGPM). They must strongly converge on a single solution of (OPGPM) a contradiction to (3.12).

On the other hand**,** suppose that condition (3.11) holds. Assume that $\{(s_n,\sigma_n)\}$ be an App.seq for (OPGPM). Then, there is a nonnegative sequence $\{c_n\}$ with $c_n \to 0$ as $n \to \infty$ in which

$$\begin{cases} \lim_{n\to\infty} H(s_n, b_n) \leq \inf_{m\in A, e\in I(m)} H(m,e) \text{ and} \\ \langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle + \kappa(s_n, \varrho, \sigma_n) + c_n\|\sigma_n - \varrho\| \geq 0. \end{cases} \tag{3.13}$$

This yields that $(s_n,\sigma_n) \in \psi(c_n, \epsilon_n)$. It follows from (3.11), that $\{(s_n,\sigma_n)\}$ is a Cauchy sequence and so it converge strongly to a point $(s,\sigma) \in A \times K$. Because of (3.13) and the assumptions $\omega(.,u), \eta(\varrho,.)$ and $\kappa(.,\varrho,.)$ are usc, we can conclude.

$$0 = \lim_{n\to\infty} \inf c_n\|\sigma_n - \varrho\|$$

$$\geq \lim_{n\to\infty} \inf[-\langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle - \kappa(s_n, \varrho, \sigma_n)]\,.$$

By virtue of the property $(\eta\kappa\alpha\beta - p)$-monotonicity of T, we can get

$$\lim_{n\to\infty} \inf\,[-\langle\omega(\sigma_{\,n}, v), \eta(\varrho, \sigma_{\,n})\rangle - \kappa(s, \varrho, \sigma_{\,n}) \leq \lim_{n\to\infty} \inf[-\alpha(\varrho - \sigma_{\,n}) - \beta(\sigma_{\,n}, \varrho)].$$

Therefore,

$$\lim_{n\to\infty} \sup\,[\langle\omega(\sigma_{\,n}, v), \eta(\varrho, \sigma_{\,n})\rangle + \kappa(s, \varrho, \sigma_{\,n})$$

$$\geq \lim_{n\to\infty} \inf[\alpha(\varrho - \sigma_{\,n}) + \beta(\sigma_{\,n}, \varrho)].$$

$$\geq \alpha(\varrho - \sigma\,) + \beta(\sigma\,, \varrho).$$

But,

$$\langle\omega(\sigma\,, v), \eta(\varrho, \sigma\,)\rangle + \kappa(s, \varrho, \sigma\,) \geq \lim_{n\to\infty} \sup\,[\langle\omega(\sigma_{\,n}, v), \eta(\varrho, \sigma_{\,n})\rangle + \kappa(s, \varrho, \sigma_{\,n})$$

$$\geq \alpha(\varrho - \sigma\,) + \beta(\sigma\,, \varrho).$$

Moreover, it is obvious from (3.13) and assumption (v) that

$$\begin{aligned} \inf_{m\in A, e\in I(m)} H(m, e) &\geq \lim_{n\to\infty} \sup H(s_n, \sigma_{\,n}) \\ &\geq \lim_{n\to\infty} \inf H(s_n, \sigma_{\,n}) \\ &\geq H(s, \sigma\,). \end{aligned}$$

Therefore, according to Theorem 2.4, (s,x) is the solution of (OPGPM), and the uniqueness automatically follows from this (3.11). As a result, the proof is now complete.

Theorem 3.11. If A and K are a nonempty convex and closed subset of real B.Ss W and E resp., If (OPGPM) is s-w-p in the generalized sense, one has

$$\psi(c,\epsilon) \neq \emptyset \; \forall c,\epsilon > 0, \lim_{(c,\epsilon)} \mu \, [\psi(c,\epsilon)] = 0. \tag{3.14}$$

In addition, assuming the following hypotheses are correct:

i. $T$is$(\eta\kappa\alpha\beta - p)$-monotone operator and η-H.C;
ii. $\eta(\sigma\,,\sigma\,) = 0$ and $\kappa(s,\sigma\,,\sigma\,) = 0$ for all $\sigma\; \in K$;
iii. $\sigma\; \mapsto \eta(.\,,\sigma\,)$ and $\sigma\; \mapsto \kappa(s,.\,,\sigma\,)$ are convex;
iv. $\omega(.\,,v), \eta(\varrho,.)$ and $\kappa(.\,,\varrho,.)$are usc ;
v. $\beta(.\,,\varrho)$ and$\alpha(\sigma\,)$ are lsc.
vi. H is lsc.

The opposite is also true.

Proof. Using a proof similar to that of Theorem 3.7, we can show that is w-p (OPGETC).

To study the uniqueness of solutions to (OPGPM), we demonstrate that given appropriate conditions the w-p of (OPGPM) is equivalent to the existence and uniqueness of solutions in the following result.

Theorem 3.12. Considering K and A are a nonempty convex and closed subsets of finite dimensional B.S E and W resp., Let $\kappa = A \times K \times K \to R$and $H: A \times K \longrightarrow R$. If

i. T is $(\eta\kappa\alpha\beta - p)$-monotone operator and η-H.C;
ii. $\eta(\sigma\,,\sigma\,) = 0$ and $\kappa(s,\sigma\,,\sigma\,) = 0$ for all $\sigma\; \in K$;
iii. $\sigma\; \mapsto \eta(.\,,\sigma\,)$ and $\sigma\; \mapsto \kappa(s,.\,,\sigma\,)$ are convex;
iv. $\omega(.\,,v), \eta(\varrho,.)$ and $\kappa(.\,,\varrho,.)$are usc ;
v. $\beta(.\,,\varrho)$ and $\alpha(\sigma\,)$ are lsc.
vi. H is convex and lsc.

Therefore, (OPGPM) is a s-w-p only if there is exactly one solution to the issue.

Proof. The necessity is auto. Assume, for the sake of sufficiency, that (OPGPM) has a single solution (s^*, σ^*). Consequentially,

$$\begin{cases} H(s^*, \sigma^*) = \inf\limits_{m\in A, e\in I(m)} H(m, e) \\ \langle \omega(\sigma^*, u), \eta(\varrho, \sigma^*)\rangle + \kappa(s^*, \varrho, \sigma^*) \geq 0. \end{cases} \tag{3.15}$$

Let $\{(s_n, \sigma_n)\} \subset A \times K$ be App.seq for $(OPGPM)$. Then $\exists\, c_n > 0$ with $c_n \to 0$ as $n \to \infty$ such that:

$$\begin{cases} \lim\limits_{n\to\infty} H(s_n, \sigma_n) \leq \inf\limits_{m\in A, e\in I(m)} H(m, e) \\ \langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n)\rangle + \kappa(s_n, \varrho, \sigma_n) + c_n\|\sigma_n - \varrho\| \geq 0. \end{cases} \tag{3.16}$$

we claim that $\{(s_n, \sigma_n)\}$ is bounded. Generally, one can assume that $\|(s_n, \sigma_n)\| \to +\infty$.

$r_n = \frac{1}{\|(s_n, \sigma_n) - (s^*, \sigma^*)\|}$, and

$$\begin{aligned} (t_n, z_n) \quad &= r_n(s_n, \sigma_n) + (1 - r_n)(s^*, \sigma^*) \\ &= [r_n s_n + (1 - r_n)s^*, r_n\sigma_n + (1 - r_n)\sigma^*]. \end{aligned}$$

Let $r_n \in (0,1)$ and $(t_n, z_n) \to (t, z)$ with $(t, z) \neq (s^*, \sigma^*)$. Taking into a account $W \times E$ is finite-dimensional and the closedness and convex of K and A, we have $(t_n, z_n) \in A \times K$. Accordingly, from assumptions $(3.15), (3.16)$ and (vi) for any $(t, z) \in K \times A$, one can get

$$H(s^*, \sigma^*) = \lim_{n\to\infty} \sup r_n H(s_n, \sigma_n) + \lim_{n\to\infty} \sup (1 - r_n)H(s^*, \sigma^*)$$

$$\geq \lim_{n\to\infty} \sup [r_n H(s_n, \sigma_n) + (1 - r_n)H(s^*, \sigma^*)]$$

$$\geq \lim_{n\to\infty} sup\, H(t_n, z_n) \tag{3.17}$$

$$\geq \lim_{n\to\infty} inf\, H(t_n, z_n)$$

$$\geq \mathrm{H}(t, z).$$

Moreover, following from conditions (ii), (iii), (3.15) and (3.16), we have

$$0 = \lim_{n\to\infty} inf\, r_n c_n \|\sigma_{\ n} - \varrho\|$$

$$\geq \lim_{n\to\infty} inf\, -r_n[\langle \omega(\sigma_{\ n}, u), \eta(\varrho, \sigma_{\ n})\rangle + \kappa(s_n, \varrho, \sigma_{\ n})]$$

$$-\ (1 - r_n)[\langle \omega(\sigma^{\ *}, u), \eta(\varrho, \sigma^{\ *})\rangle + \kappa(s^*, \varrho, \sigma^{\ *})]$$

$$\geq \lim_{n\to\infty} inf[-r_n\langle \omega(\sigma_{\ n}, u), \eta(\varrho, \sigma_{\ n})\rangle - (1 - r_n)\langle \omega(\sigma^{\ *}, u), \eta(\varrho, \sigma^{\ *})\rangle]$$

$$-r_n F(s_n, \varrho, \sigma_{\ n}) - (1 - r_n)\kappa(s^*, \varrho, \sigma^{\ *})$$

$$\geq \lim_{n\to\infty} inf\, [-\langle \omega(z_n, u), \eta(\varrho, z_n)\rangle - \kappa(t_n, \varrho, z_n)].$$

By virtue of the property $(\eta\kappa\alpha\beta - p)$-monotonicity of T, we can get

$$\lim_{n\to\infty} \mathrm{inf}[-\langle \omega(z_n, v), \eta(\varrho, z_n)\rangle - \kappa(t_n, \varrho, z_n) \leq \lim_{n\to\infty} \mathrm{inf}[-\alpha(\varrho - z_n) - \beta(z_n, \varrho)].$$

Therefore,

$$\lim_{n\to\infty} \mathrm{sup}\, [\langle \omega(z_n, v), \eta(\varrho, z_n)\rangle + \kappa(t_n, \varrho, z_n)$$

$$\geq \lim_{n\to\infty} \inf[\alpha(\varrho - z_n) + \beta(z_n, \varrho)].$$

$$\geq \alpha(\varrho - \sigma\ ) + \beta(\sigma\ , \varrho).$$

But,

$$\langle \omega(z, v), \eta(\varrho, z)\rangle + \kappa(t, \varrho, z) \geq \lim_{n\to\infty} \sup\ [\langle \omega(z_n, v), \eta(\varrho, z_n)\rangle + \kappa(t_n, \varrho, z_n)$$

$$\geq \alpha(\varrho - \sigma\ ) + \beta(\sigma\ , \varrho).$$

Then, from Theorem 2.4, one can have

$$\langle \omega(z, u), \eta(\varrho, z)\rangle + \kappa(t, \varrho, z) \geq 0. \tag{3.18}$$

Hence, from (3.17), (3.18), we have (t, z)solves (OPGPM) which is a contradiction, so the sequence $\{(s_n, \sigma\ _n)\}$ is bounded. Let $\left\{\left(s_{n_i}, \sigma\ _{n_i}\right)\right\}$ be any subsequence of $\{(s_n, \sigma\ _n)\}$ in which$\left(s_{n_i}, \sigma\ _{n_i}\right) \longrightarrow (s_0, \sigma\ _0)$ as i $\longrightarrow \infty.$ It follows that

$$0 \geq \lim_{n\to\infty} \inf \mathrm{c_{n_i}} \|\sigma\ _{\mathrm{n_i}} - \mathrm{y}\|$$

$$\geq \lim_{n\to\infty} \inf\left[-\left\langle \omega\left(\sigma\ _{n_i}, u\right), \eta\left(\varrho, \sigma\ _{n_i}\right)\right\rangle - \kappa\left(s_{n_i}, \varrho, \sigma\ _{n_i}\right).\right.$$

By virtue of the property $(\eta\kappa\alpha\beta - p)$-monotonicity of T, we can get

$$\lim_{n\to\infty} \inf\ [-\left\langle \omega\left(\sigma\ _{n_i}, v\right), \eta\left(\varrho, \sigma\ _{n_i}\right)\right\rangle - \kappa\left(s_{n_i}, \varrho, \sigma\ _{n_i}\right) \leq \lim_{n\to\infty} \inf\left[-\alpha\left(\varrho - \sigma\ _{n_i}\right) - \beta\left(\sigma\ _{n_i}, \varrho\right)\right].$$

Therefore,

$$\lim_{n\to\infty}\sup\,[\langle\omega(\sigma_{n_i}, v), \eta(\varrho, \sigma_{n_i})\rangle + \kappa(s_{n_i}, \varrho, \sigma_{n_i})$$

$$\geq \lim_{n\to\infty}\inf[\alpha(\varrho - \sigma_{n_i}) + \beta(\sigma_{n_i}, \varrho)].$$

$$\geq \alpha(\varrho - \sigma_0) + \beta(\sigma_0, \varrho).$$

But,

$$\langle\omega(\sigma_0, v), \eta(\varrho, \sigma_0)\rangle + \kappa(s_0, \varrho, \sigma_0) \geq \lim_{n\to\infty}\sup\,[\langle\omega(\sigma_{n_i}, v), \eta(\varrho, \sigma_{n_i})\rangle + \kappa(s_{n_i}, \varrho, \sigma_{n_i})\,]$$

$$\geq \alpha(\varrho - \sigma_0) + \beta(\sigma_0, \varrho). \tag{3.19}$$

From Theorem 2.4 and (3.19) one can have

$$\langle\omega(\sigma_0, u), \eta(\varrho, \sigma_0)\rangle + \kappa(s_0, \varrho, \sigma_0) \geq 0. \tag{3.20}$$

Considering (3.16) and the lower semicontinuity of H, it follows that

$$\lim_{m\in K, e\in I(m)} inf\, H(m, e) \geq \lim_{i\to\infty} sup\, H(s_{n_i}, \sigma_{n_i})$$

$$\geq \lim_{i\to\infty} inf\, H(s_{n_i}, \sigma_{n_i})$$

$$\geq H(s_0, \sigma_0). \tag{3.21}$$

From (3.20) and (3.21), (s_0, σ_0) solves (OPGPM). Given the solution's uniqueness, we get $(s_0, \sigma_0) = (s^*, \sigma^*)$ and hence (s_n, σ_n) converges to (s^*, σ^*). Therefore, (OPGPM) is strong w-p.

Theorem 3. 13. Consider that K and A are a nonempty, closed and convex subsets of finite dimensional B.Ss E and W resp., $\kappa = A \times K \times K \to R$ and $H: A \times K \longrightarrow R$. If, there is some $\epsilon > 0$ in which $\psi(\epsilon, \epsilon)$ is a nonempty and bounded and assume the next statements are true:

i. $T$is $(\eta\kappa\alpha\beta - p)$-monotone operator and η-H.C;
ii. $\eta(\sigma, \sigma) = 0$ and $\kappa(s, \sigma, \sigma) = 0$ for all $\sigma \in K$;
iii. $\sigma \mapsto \eta(., \sigma)$ and $\sigma \mapsto \kappa(s, ., \sigma)$ are convex;
iv. $\omega(., v), \eta(\varrho, .)$ and $\kappa(., \varrho, .)$are usc ;
v. $\beta(., \varrho)$ and $\alpha(\sigma)$ are lsc;
vi. H is convex and lsc.

Therefore, (OPGPM) is s-w-p in the generalized sense.

Proof. Let $\{(s_n, \sigma_n)\} \subseteq A \times K$ be an App.seq for $(OPGPM)$. Then, there is a nonnegative sequence $\{c_n\}$ with $c_n \to 0$ as $n \to \infty$ such that

$$\begin{cases} \lim_{n\to\infty} H(s_n, \sigma_n) = \inf_{m\in A, e\in I(m)} H(m, e) \\ \langle \omega(\sigma_n, u), \eta(\varrho, \sigma_n) \rangle + \kappa(s_n, \varrho, \sigma_n) + c_n \|\sigma_n - \varrho\| \geq 0 \end{cases} \tag{3.22}$$

Since $\psi(\epsilon, \epsilon)$ is a nonempty and bounded. Then, there is n_0 such that $(s_n, \sigma_n) \in \psi(\epsilon, \epsilon)$ for all $n \geq n_0$. Considering's boundedness of $\psi(\epsilon, \epsilon)$, there is subsequence $\{(s_{n_i}, \sigma_{n_i})\}$ of $\{(s_n, \sigma_n)\}$ in which $(s_{n_i}, \sigma_{n_i}) \to (s_0, \sigma_0)$ as $i \to \infty$. Consequently, as proved in Theorem 3.12,(s_0, b_0) solves (OPGPM). Then, (OPGPM) is s-w-p in the generalized sense.

5. CONCLUSIONS AND FUTUREWORK

5.1 Conclusions

As mentioned in the introduction, the aim of the study is to presented $VI(\omega\eta\kappa)$ mapping w.r.t the second argument of a hybrid-type mapping involving KKM-mapping. Moreover, we presented new definition of monotonicity is called $(\eta\kappa\alpha\beta - p)$-monotone operator. Finally, the concepts of extended w-p for $VI(\omega\eta\kappa)$ and established some character-izations. Extended well-posedness is synonymous with the existence and uniqueness of solutions under appropriate conditions, which is what we introduced. In addition to this, the related notions of extended w-p in the generalized sense are presented and examined for $VI(\omega\eta\kappa)$ problems that have more than one solution.

5.2 Future Work

1- Finding the new types of variational inequalities and investigate about major operations of these inequalities, in addition to study the generalized implicit V.I in inclusion issues.

2-Studying the existence and uniqueness of the weak solution to the w-p in Sobolev space.

3-Using well-known fixed point thorey, one can applying of variational issues in all areas for social, industrial, pure, physical, regional, and engineering sciences.

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