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RESULTS OF WELL-POSEDNESS FOR MIXED EQUILIBRIUM PROBLEMS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

RESULTS OF WELL-POSEDNESS FOR MIXED EQUILIBRIUM PROBLEMS

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February 2023

This thesis is an attempt to study in a rigorous manner some specific cases of well posedness (w-p) in applied sciences. More precisely, the thesis looks at mixed hemiequilibrium in Banach space, all with the aim of obtaining results on the well posedness of equilibrium problems in the case of convex subsets in B.S, compactness, and convex function. All this is done through three tasks. Task one presents a host of essential definitions along with some theorems. The aim of this task is to help the reader to get some understanding of main results grounded in this thesis. Task two is designed to establish a new kind of Mixed-hemi-equilibrium problem (abbreviation for $(\boldsymbol{MHEP})$) involving a novel type of monotonicity called $\boldsymbol{\psi}$- monotone in the setting of B.S. This will demonstrate that the idea of w-p for $(\boldsymbol{MHEP})$ is equivalent to well posedness for $\boldsymbol{\psi}$- monotone. Task three deals with the optimization of solution for Mixed-hemiequilibrium with parametric characters. In so aiming, the results in this thesis help us develop and improve some corresponding solutions of several authors.

2023, 55 pages

Keywords: Hemiequilibrium problems, Approximating sequence, Optimization problems, Metric characterizations, Well-posed, Monotone bifunction

ÖZET

KARIŞIK DENGE PROBLEMLERİ İÇİN İYİ POZLULUK SONUÇLARI

Ayat Montadhar Safaa AL-TAMEMY
Matematik, Yüksek Lisans
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Şubat 2023

Bu tez, uygulamalı bilimlerdeki bazı özel iyi konumluluk durumlarını titiz bir şekilde inceleme girişimidir. Daha kesin olarak, tez, Banach uzayındaki dışbükey altkümeler, kompaktlık ve dışbükey fonksiyon durumundaki denge problemlerinin iyi konumluluğu hakkında sonuçlar elde etmek amacıyla Banach uzayındaki karma hemi-dengeye bakar. Bütün bunlar üç görevle yapılır. Birinci görev, bazı teoremlerle birlikte bir dizi temel tanım sunar. Bu görevin amacı, okuyucunun bu tezde temellenen ana sonuçları biraz anlamasına yardımcı olmaktır. İkinci görev, yeni bir tür Karma-hemi-denge problemi oluşturmak için tasarlanmıştır (kısaca KHDP) Banach uzayı ortamında ψ-tekdüzelik adı verilen yeni bir monotonluk türünü içerir. Bu, iyi pozlama fikrinin KHDP-iyi pozlamaya eşdeğer olduğunu gösterecektir. Görev üç, parametrik karakterli Karma-hemiequilibrium için çözümün optimizasyonu ile ilgilidir. Böyle bir amaçla, bu tezdeki sonuçlar, birkaç yazarın karşılık gelen bazı çözümlerini geliştirmemize ve iyileştirmemize yardımcı olur.

2022, 55 sayfa

Anahtar Kelimeler: Hemidenge problemleri, Yakınsama dizisi, Optimizasyon problemleri, Metrik karakterizasyonları, İyi pozluluk, Monoton bifunction

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God be praised, for he has bestowed upon us the treasure of wisdom; may prayers and peace be upon our master Muhammad and his pure and pure family, his selected companions, and those who followed them in kindness until the Day of Judgment. As I put the final touches on this thesis, I am grateful to extend my thanks and appreciation to the supervising professor (Prof. Dr. Faruk POLAT) for giving me his time and his suggestion of the topic of the thesis and for his valuable guidance and sound opinions, which had the great merit of enriching the research and showing it as it is. To make my thanks and gratitude to my virtuous supervisor (Co-Advisor: Asst. Prof. Dr. Ayed Eleiwis HASHOOSH), because he has a distinguished role, good opinions and good touches on every page of my thesis. I extend my sincere thanks to the staff of the Mathematics Department, especially the Head of the Mathematics Department. We do not forget everyone who contributed with me to the success of the message, especially my mother. In conclusion, I thank everyone who helped me and helped me complete this research.

Ayat montadhar safaa AL-TAMEMY
Çankırı-2023

CONTENTS

LIST OF SYMBOLS

$\boldsymbol{R}$	The set of Real numbers
$\boldsymbol{N}$	The set of Natural numbers
$\in$	Belong to
diam (A)	Diameter of a set A
Conv(A)	Convex hull of set A
$\mathbf{2}^{E}$	The family of all a nonempty sets of E
$\langle \quad \rangle$	Canonical inner product
$\geq$	The greater than or equal
$\leq$	The less than or equal
$\exists$	There exists

LIST OF ABBREVIATIONS

et al.	And others.
App.seq	Approximating sequence.
B.S	Banach space.
EP	Equilibrium Problem
H.T.L.S	Hausdorff topological vector spaces.
H.C	Hemicontinuous.
IPS	Inner product space.
Lsc	Lower Semi Continuous.
MEP	Mixed Equilibrium Problem
M. bi.f	Monotone bifunction.
OPGPEC	Optimization problem with generalized equilibrium constraint.
resp.	Respectively.
s-w-p	Strongly well-posdness.
s.t	Such that.
usc	Upper Semi Continuous.
V.I	Variational inequality.
w-p	well-posedness
w.r.t	With respect to.

1. INTRODUCTION

Bla Within the realms of optimization and nonlinear operator problems, the idea of w-p holds a significant place in the hierarchy of priorities. There is no doubt that the presence of a solution to a w-p problem and the fact that this solution is the only one possible have proven to be any valuable of optimization theory and non-linear operator theory, which form related particular examples of (EP). It is common knowledge that the French mathematician (Hadamard 1923) of the 20th century was the one who originally conceived of and made widespread use of the mathematical term "well-posedness".

According to Hadamard, the primary concept of w-p is intended to meet three essential features, and the implementations of these conditions inspired a substantial chunk of the succeeding decades' worth of effort. These conditions include the existence of the solution, the solution's uniqueness, and the solution's stability. It is claimed that a issue is w-p if it falls within the parameters of the definition that came before it. In the body of published work in mathematics, a number of approaches have been suggested for discussing varying degrees of w-p in relation to optimization problems and variational inequalities; one such approach is attributed to Bednarczuk in (1994). Mathematicians such as (Levitin and Polyak 1966), as well as (Tykhonov 1966), and others have labored to generate and examine problems that are considered to be w-p. They were able to provide a precise mathematical formulation of an approximate solution of w-p for optimization issues and variational inequalities, which was a successful accomplishment.

These issues currently offer a very fruitful foundation upon which to build study. Several examples of w-p that originated from issues (such as optimization and variational inequality) have since been extended to scalar issues, saddle point issues, fixed-point issues, Nash equilibrium issues, and other types of issues. For more details on dealing with the issues surrounding w-p, (Chen *et al.* 2014) and (Huang 2001) extended w-p of set-valued optimization issues. S. Reich and A.J. Zaslavski (2005), studied w-p and fixed point issues.

(Xiao *et al.* 2011) extended w-p of hemi-variational inequalities and inclusion issues. (Zhang *et al.* 2009) extended w-p of quasi-variational inequalities. (Zhanga and Chen 2016) tended to generalize w-p of Nash-type games issues with set payoff.

One kind of w-p has originated in (Bianchi *et al.* 2009), who presented and formulated two kinds of w-p for vector equilibrium issues. (Chifu and Petru 2008) presented w-p of fixed point. (Lignola and Morgan 2000), tended to generalize w-p by focusing on generalizations for variational inequality issues. With a new w-p concept, a parametric form of an equilibrium issue has been used to show another case of w-p. Salamon suggested this method which has already been used by (Miglierina *et al.* 2005). A new w-p formulation has been used to vector equilibrium issues, (Kimura *et al.* 2008) gave us a way to start thinking about a new w-p issue. Also, J. Peng, Y. Wang, and S. Wu (2011) used the case to study several kinds of Levitin-Polyak w-p in generalized vector equilibrium issues. (Long *et al.* 2008) and (Zaslavski 2008) give cases for Levitin-Polyak w-p for explicitly constrained EPs and generic w-p of EPs. J. P. Revalski looked at w-p in terms of optimization and optimal control issues in 1997. We advise the reader to read the research (Fang *et al.* 2008), (Fang *et al.* 2008) and (Zhang *et al.* 2009).

This thesis is composed of four chapters. Chapter one presents a host of essential definitions along with some Theorems. The aim of this chapter is to help the reader to get some understanding of main results grounded in this thesis. The chapter starts with a section on primary definitions of normed space before it ends up with a brief introduction to w-p. We can state all of the aims of this chapter at once by saying that we are trying to ground the investigation in some concepts of convex analysis required for the upcoming chapters. This being said, the chapter appeals to some concept derived from topological spaces and M. bi.f.

Chapter two is divided into five sections including the introduction. Section two recalls relevant definitions that are important to our investigations. Section three introduce w-p of (MHEP) and ψ – monotonicity.

W-p is discussed in this section for a type of mixed-hemiequilibrium issues involving ψ -M. bi.fs. Section 4 provides metric characterizations of (MHEP) for w-p. This section idealizes various characterizations for the issue's w-p (MHEP). Finally, section five discusses Characterizations of the w-p for (MHEP) in generalized sense.

The purpose of this section is to develop w-p and its characterizations and offer some conceptions under which (MHEP) is strongly w-p or strongly $\psi -$ w-p in the generalized sense.

Chapter three is taken to be the conclusive investigation in our thesis in which the essential definitions are presented, and some recent findings in the subject are referred to. In the third section, we provide a novel concept of w-p for optimization issues that have constraints described by a parametric form of Mixed Hemiequilibrium issue constraint and a relevant w-p. This idea applies to issues that have limits on how they can be optimized. Finally, in the fourth chapter, we mention the important conclusions of the thesis and future work.

2. PRELIMINARIES

2.1 Introduction

In the following chapter, we introduce a family of basic definitions as well as primary theorems. The essential goal of introducing these definitions and theorems is to help us derive main results obtained in this work that would be given in the upcoming chapters. With this goal in mind, the chapter begins with the very basic definitions of normed space before we end with a brief introduction to well posedness.

By doing so, we base our investigation upon some notions of convex analysis that are necessary for our next chapters. Furthermore, we refer to a number of concepts derived from topological spaces. Additionally, we present basic definitions and notions of M. bi.f.

2.2 Some Concepts Relating to Metric and Normed Spaces

Definition 1.1. (Fréchet 1906) Consider that a set $X \neq \emptyset$. A Function $d: X \times X \to \mathbb{R}$ is called metric on$X$if for every $x, y, z \in X$ satisfy this following conditions:

$(D_1) d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.

$(D_2) d(x, y) = d(y, x)$.

$(D_3) d(x, y) \leq d(x, z) + d(z, y)$.

The set$X$which equipped by metric is called metric space, and denoted by (X, d).

Definition 1.2. (Fréchet 1906) Consider that X is metric space and point $x_0 \in X$and a real number $r > 0$, types sets as defined by:

$(a) B(x_0; r) = \{x \in X: d(x, x_0) < r\}$ is called an open ball.

$(b) \overline{B}(x_0; r) = \{x \in X: d(x, x_0) \leq r\}$is called a closed ball.

In both instances, x_0 is referred to as the center and r as the radius. An open ball of radius r is defined as the set of all points in X whose distance from the ball's center is less than r.

Definition 1.3. (Fréchet 1906) Consider that X be a metric space and $S \subseteq X$. S is bounded if there is $x_o \in S$and $r \in \mathbb{R}$ such that $d(x, x_0) < r$, for all $x \in S$.

Definition 1.4. (Fréchet 1906) A sequence $\{x_n\}, n \geq 1$of real numbers is named to be bounded if, there is a real number $M > 0$ in which

$$| x_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Proposition 1.5. (Fréchet 1906) A monotone sequence of a real numbers is convergent if, it is bounded.

Definition 1.6. (Yosida 1980) Consider that S be a vector space on field F (where F is either the field of real or complex number). The function $\|.\|: S \longrightarrow F$ is called norm on V, if the following conditions are hold

i. $\| u \| \geq 0 \ \forall u \in S,$
ii. $\| u \| = 0$ if and only if $u = 0, \forall u \in S,$
iii. $\| ru \| = |r| \| u \|, \forall u \in S, r \in F,$
iv. $\| u + v \| \leq \| u \| + \| v \|, \forall u, v \in S.$

Then the pair $(S, \|.\|)$ is called a normed space.

Example 1.7. (Yosida 1980) Consider that $M = C[0,1]$ where M is a space of real continuous on $[0,1]$ and $\|\cdot\|: M \longrightarrow \mathbb{R}$ defined as follow:

$$\| h \| = \int_0^1 \vee h(t) \vee dt.$$

It is easy to prove $\|\cdot\|$ norm on M, then $M$is a normed space.

Definition 1.8. (Yosida 1980) Consider that $\{u_n\}$ be a sequence in normed space S, then $\{u_n\}$ is told to be bounded sequence, if, there is a real positive number $r > 0$, in which

$$\|u_n\| \leq r, \forall n \in Z^{+.}$$

Definition 1.9. (Yosida 1980) Consider that S be a normed space, $u_0 \in S$ and r is real positive number, then the set $K = \{u \in V, \|u_0 - u\| < r\}$ is said to be open ball by center a_0 and radius r, denoted $B_r(u_0)$.

Definition 1.10. (Yosida 1980) Consider that $\{u_n\}$ be a sequence in normed space S, then $\{u_n\}$ is named to be converge sequence, if, there is $a \in V$ in which

$$\forall \delta > 0, \exists r \in Z^+, \|u_n - u\| < \delta, \text{ for all } n > r.$$

And u the a convergent point to sequence $\{u_n\}$ and form write,

$$\lim_{n \to \infty} u_n = u, \vee u_n \to a.$$

Definition 1.11. (Yosida 1980) Consider that $\{u_n\}$ be a sequence in a normed space$V$, then $\{u_n\}$ is said to be Cauchy sequence in S, if for every $\delta > 0, \exists r \in Z^+$ such that

$$\|u_n - u_m\| < \delta, \forall n, m > r.$$

Theorem 1.12. (Yosida 1980) Any convergent sequence in a normed space is Cauchy sequence.

Definition 1.13. (Yosida 1980) The sequence $\{u_n\}$in a normed space S is said to be weakly converge if $\exists$ $u \in$ S, such that

$$\lim_{n\to\infty} \vartheta(u_n) = \vartheta(u).$$

Definition 1.14. (Yosida 1980) The normed space S is said to be Banach space (B.S) if every Cauchy sequence in $\boldsymbol{S}$ convergent sequence to an element of $\boldsymbol{S}$.

Definition 1.15. (Yosida (1980) Consider that S be a B.S. A mapping $\Gamma: K \longrightarrow R$ is said to be

i. Lower semicontinuouse (abbreviation for, lsc) at $u_0 \in K$ if

$$\Gamma(\mathrm{u}_0) \leq \lim\inf\, \Gamma(\mathrm{u}_\mathrm{n}).$$

ii. Upper Semi Continuous (abbreviation for, usc) at $\mathrm{u}_0 \in$ K if

$$\Gamma(\mathrm{u}_0) \geq \lim\sup \Gamma(\mathrm{u}_\mathrm{n}).$$

For any sequence $\{ u_n\}$ of K such that $u_n \longrightarrow u_0$.

Example 1.16. (Yosida (1980) Consider that A be a nonempty and $\Gamma: K \to R$ such that for every $u \in K$ then

$$\Gamma(u) = \begin{cases} 0 & u < 1 \\ 2 & u = 1 \\ 1/2 & u > 1 \end{cases}$$

Then, Γ is upper semicontinuous at $u = 1$,because its value is higher than its neighborhood value.

Definition 1.17. (Yosida (1980) Consider that K and $T$are two normed spaces. A mapping $\vartheta: K \to T$ is named continuous at $u \in K$if for all $r > 0$, there is $\delta > 0$ in which

$$\|\vartheta(u) - \vartheta(v)\| < r \text{ for all } v \in K \text{ satisfying } \| u - v \| < \delta .$$

Moreover, ϑ is called continuous if it continuous at every point in K.

Definition 1.18. (Clarke 1983) A function $A: S \to R$ is termed locally Lipschitz if for every $u \in S$, there is a neighborhood U of u and a constant $L_u > 0$ in which

$$|A(w) - A(v)| \le L_u \|w - v\|_S; \text{ for each } v, w \in U.$$

Definition 1.19. (Clarke 1983) The functional locally Lipschitz $J: S \to R$ is assumed. Then the generalized derivative of J at $u \in S$ in the direction of $v \in S$, indicate by $J^0(u; v)$, and as expressed by

$$J^0(u; v) = \limsup_{w \to u, \lambda \downarrow 0} \frac{J(w + \lambda v) - J(w)}{\lambda}$$

Lemma 1.20 (Clarke 1983). Consider the function $J: S \to \mathbb{R}$ to be a locally Lipschitz function with rank L_u near to the point $u \in S$.Then

i. The function $v \mapsto J^0(u; v)$ has the properties of being positively, finite, homogeneous, subadditive, and satisfying the condition that

$$|J^0(u; v)| \leq L_u \parallel v \parallel_S;$$

ii. $J^0(u; v)$ is u.s.c;

iii. $J^0(u; -v) = (-J)^0(u; v)$.

2.3 Some Concepts Relating to Convex Analysis

In this section, we discuss fundamental notions and terminology of convex analysis that are relevant to our work.

Definition 1.21. (Zeidler 1986) Consider that A be a nonempty subset of a vector space S. Then, A is called termed set if

$$ru + (1 - r)v \in A, r \in [0,1], \forall u, v \in A.$$

Lemma 1.22. (Zeidler 1986) Assume that K and T are two convex sets in vector space $S$with a norm $\|.\|$.Then

i. i. Both the empty sets and the singleton sets are convex sets.

ii. $\varrho + \sigma = \{u + w; u \in \varrho, w \in \sigma\}$ is convex.

iii. δK is convex for any scaler δ .

iv. The intersection of any collection of convex sets is convex.

v. Every convex set's closure is also a convex set.

Definition 1.23. (Zeidler 1986) Consider that K be a subset of B.S S, then the smallest convex set in S contained K is said to be convex hull denoted by conv(K).

Proposition 1.24. (Zeidler 1986) On convexity, the following facts are true:

i. $K \subseteq conv(K)$,

ii. $conv\ (K) =$ the intersection of all convex sets that contain K,

iii. If K is convex set then $K = conv(K)$.

Definition 1.25. (Zeidler 1986) Consider that K be subset of a vector space S, and $u_1, u_2 \dots u_n \in K$, the combination for all $u_1, u_2 \cdots u_n \in K, r_i \geq 0$ in which $\sum_{i=1}^{n} r_i$=1 the combination $\sum_{i=1}^{n} r_i u_i \in K$ is said to be convex combination.

Definition 1.26 (Zeidler 1986). A function $\vartheta: K \to R$ is said to be convex function if

$$\vartheta(ru + (1-r)v) \leq r\vartheta(u) + (1-r)\vartheta(v), \forall u, v \in K, r \in [0,1].$$

The following illustration should clarify the previous definition.

Example 1.27. (Zeidler 1986) Consider that $K = C[u, v]$, where A space of real continuous function on $[u, v]$ and $\vartheta: K \to R$ such that for every $h \in K$ we have

$$\vartheta\big(h(x)\big) = \int_u^v |h(x)| dx$$

If we take $h(x), k(x) \in A$ and $r \in [0,1]$ then

$$\vartheta(rh(x) + (1-r)k(x))$$

$$= \int_u^v |rh(x) + (1-r)k(x)| dx$$

$$\leq r \int_u^v |h(x)| dx + (1-r) \int_u^v |k(x)| dx$$

$$= r\vartheta\big(h(x)\big) + (1-r)\vartheta\big(k(x)\big).$$

Definition 1.28. (Browder 1963) Consider that ϑ be a real-valued function defined on subset K of S that is convex. If

$$\lim_{t \to 0^+} \vartheta(tx + (1-t)y) = \vartheta(y) \forall x, y \in K, t \in [0,1],$$

Then, A is said to be hemicontinuous (H.C).

2.4 Some Concepts Relating to Topological Spaces

The results presented in this section are collection of the related ideas from topological spaces that will be used in subsequent chapters.

Definition 1.29. (Schaefer 1999) A topological space S is called Hausdorff space, if for any two disjoint points u and v in K, there are two disjoint open sets, K and T in S such that $u \in K, v \in T$. Then

$$K \cap T = \emptyset.$$

Definition 1.30. (Schaefer 1999) Consider that S be a B.S with norm $\|.\|$. The topology on S induces by norm $\|.\|$ is said be norm topology or strong topology.

Definition 1.31. (Schaefer 1999) Suppose that S is a B.S and K ⊆ S. Then, we say that K is weakly closed if it is closed in the weak topology of S.

Definition 1.32 (I-Liang Chern 2013). A subset K of a space S, is said to be compact if every open covering of K by open set of S has a finite subcovering.

Definition 1.33 (I-Liang Chern 2013). If K is a set in $\mathbb{R}^n$ and has one of the following properties. Then it has the other two.

i. K is closed and bounded,
ii. K is compact,
iii. Every infinite subset of K has a limit point in K.

Definition 1.34 (Kuratowski 1968) Assume that N, M are two nonempty subsets of S. The Hausdorff metric $H(\cdot,\cdot)$ between N, M expressed by

$$H(N, M) = max\{e(N, M), e(M, N)\} \text{ where}$$

$$e(N, M) = \sup_{a\in N} d(a, M) \text{ with } d(a, M) = \inf_{b\in M} \| a - b \|.$$

Note that $\{N_n\}$ is a sequence of nonempty subset of S.The sequence N_n is convergent to N by Hausdorff metric if $H(N_n, N)$ converges to zero. It is easy to see that $e(N_n, N) \to 0$, if and only if $d(a_n, N) \to 0, \forall a_n \in N$.

Definition 1.35. (Kuratowski 1968) The measure to which a nonempty subset A can be described as being non-compact, as defined by

$$\mho(A) = inf\{\epsilon > 0: A \subseteq\cup_{j=1}^{n} A_j, diam(A_j) < \epsilon, j = \overline{1,n}\},$$

where the diameter of a set A_j is denoted by the notation diam A_j.

Proposition 1.36. (Liu *et al.* 2016). Consider the set-valued mapping $G: S \longrightarrow 2^S$. If the following statements are interchangeable:

i. G is l.s.c.

ii. If $a \in X$; $\{a_\lambda\}_{\lambda\in J} \subset S$ is a net in S in which $a_\lambda \to a$ and $a^* \in G(a)$, then for each $\lambda \in J$ we can find $a_\lambda^* \in G(u_\lambda)$ in which $a_\lambda^* \to a^*$ in E.

Let us recall the a Mosco convergence presented in (Mosco 1969).

Definition 1.37. (Mosco 1969) Consider that $\{N_n\}$ be a sequence of nonempty subset of S. The sequence N_n is said to be Mosco convergent to N, if $N = \lim\inf_{n\to\infty} N_n$ such that

$\lim\inf_{n\to\infty} N_n$ means the Painleve-Kuratowski stronge limit and define as

$$\liminf_{n\to\infty} N_n = \{a \in S; \text{for all } n \in N, \exists\, a_\lambda \in N_n \text{such that } a_\lambda \to a\}.$$

So, the sequence $\{N_n\}$ is lower semi$-$Mosco converges to N, if $N = \liminf_{n\to\infty} N_n$. Also, if lower semi$-$Mosco converges $\{N_n\}$to N, implies that the closedness of a set N.

Remark 1.38 (Mosco 1969) A subset sequence$\{N_n\}$ of S Mosco converges to N implies that the sequence $\{N_n\}$ also lower semi-Mosco converges to N, On the other hand, the opposite is not always the case.

2.5 Some Concepts Relating to Monotone function

In this section, we recall some definition and notions about monotone operator.

1. E. Blum, W. Oettli (1994), presented M. bi.f and defined it as the following: A bifunction$\vartheta: K \times K \to R$ is called M. bi.f if

$$\vartheta(u,v) + \vartheta(v,u) \leq 0 \ \ \forall u, v \in K.$$

2. W. Oettli and H. Riahi (1998), presented a new type of M. bi.f and calledit as Ψ–M. bi.f. According to him, it is defined as the following:

Assume that $\Psi, \vartheta: K \times K \to R$ are two bifunctions. Then ϑ is said to be Ψ-M. bi.f if

$$\vartheta(u,v) + \vartheta(v,u) \leq \Psi(u,v) + \Psi(v,u) \forall u, v \in K.$$

1. A. E. Hashoosh, M. Alimohammadi and K. Kalleji (2016), introduced a new type of M. bi.f and called it α-M. bi.f. It can be defined as below:

Assume that $\alpha, \vartheta: K \times K \to R$ are two bifunctions. Then, ϑ is said to be α -M. bi.f, if

$$\vartheta(u,v) + \vartheta(v,u) + \alpha(u,v) \leq 0 \ \forall u, v \in K.$$

Example 1.39 (Oettli and Riahi 1998) Consider that $X = l^p$ be a reflexive B.S, $1 < p < \infty$, where

$$X = \left\{ x = \{x_n\} \subseteq R \colon \| \ x \ \| = \left(\sum_{n \geq 1} |x_n|^p \right)^{\frac{1}{p}} \right\}$$

Define a set called $\{K = x \in l^p \colon \| \ x \ \| \leq 1\}$ that is nonempty closed and convex subset of l^p. $\vartheta \colon K \times K \to R$ in which $\vartheta(x;y) = \| \ y - x \ \|^2$. Thus,

$$\vartheta(x;y) + \vartheta(y;x) = 2 \ \| \ y - x \ \|^2 > 0, \text{ for } x \neq y,$$

in which ϑ is not $M.bi.f$. However, one can use $\psi \colon K \times K \to \mathbb{R}$, in which

$$\psi(x;y) = 2 \ \| \ y - x \ \|^2.$$

Therefore, ϑ is ψ - M. bi.f.

Here, we will recall some cases for definition of well posedness and some Theorems presented by previous researchers in this field.

2.6 Some Concepts Relating to W-p

In this section we recalled some concepts of w-p by parametric for novel class of equilibrium issues EP_ψ and for optimization issues with perturbations which includes the classical equilibrium issue as a special case. The application of these ideas demonstrates that the w-p of the generalized equilibrium issue is equivalent to the existence and uniqueness of the results of the issue in (Hashoosh and Alimohammady 2016). Related to this, An explanation of how optimization problems can be formulated using equilibrium constraints is provided below.

Considering $h: T \times K \to R$ and $F: T \times K \times K \to R$ are two functions in which $T \subset E$ is a nonempty set, where E is a B.S. The optimization issue abreast of generalized equilibrium constraint, the organization indicated by (OPGPEC) is structured as follows:

$$min\, h(t, u) s.t, (t, u) \in T \times K \text{ and } u \in S(t),$$

where S(t) represents the solution set of the parametric generalized equilibrium issue $(EP_{(t)})$ and is defined by $u \in S(t)$ if and only if the following condition is satisfied:

$$F(t, u, v) + \psi(u, v) \geq 0, \forall v \in K$$

In what follows, several ideas of w-p of generalized equilibrium issues (EP_{Ψ}) and some circumstances under which equilibrium issues are substantially w-p in a generalized sense are recalled.

Definition 1.40 (Hashoosh 2016) It is claimed that a sequence $\{(t_n, x_n)\} \subset T \times K$ is an approximation sequence for $\left(EP_{\psi}\right)$ if, there is a nonnegative sequence $\{\epsilon_n\}$ abreast of $\epsilon_n \to 0$ as $n \to \infty$ such that it satisfies the following condition:

$$F(t, x_n, y) + \psi(x_n, y) \geq -\epsilon_n \|y - x_n\| \ \forall\, n \in N, y \in K\ .$$

Definition 1.41 (Hashoosh 2016) The issue $\left(EP_{\psi}\right)$ is expressed to be strongly w-p (or strongly (or weakly) w-p in the generalized sense) if $\left(EP_{\psi}\right)$ has only one solution x, and for every sequence $\{x_n\}$ with $x_n \to x$, every App.seq for $\left(EP_{\psi}\right)$ converges strongly to only one solution. If $\left(EP_{\psi}\right)$ has a nonempty solution set $S(t)$, then every approximate solution sequence has a subsequence which strongly (or weakly) converges to the point of $S(t)$.

In what comes next, we will review some descriptions of w-p that apply to (EP_{Ψ}). For any $\epsilon > 0$.

The author identify two sets:

$$\Gamma(\epsilon) := \{(t, x) \in T \times K, F(t, x, y) + \Psi(x, y) \geq -\epsilon \parallel y - x \parallel \forall y \in K\}.$$

And

$$\Lambda(\epsilon) := \{(t, x) \in T \times K, F(t, y, x) + \alpha(x, y) \leq \Psi(x, y) + \epsilon \parallel y - x \parallel \forall y \in K\}.$$

Lemma 1.42. (Hashoosh 2016) Considering K be a nonempty convex, closed subset of a B.S X. Suppose that $F: T \times K \times K \to \mathbb{R}$ and $\Psi, \alpha: K \times K \to \mathbb{R}$are three functions satisfy under these circumstances:

i. $F(t, x, x) = 0, \forall t \in T, x \in K$,
ii. $F(t,\cdot,\cdot)$ is α -M. bi.f and H.C , for all $t \in T$,
iii. $F(t, x,\cdot)$ is convex for all $t \in T, x \in K$,
iv. $\Psi(x,\cdot)$ and $\alpha(x,\cdot)$ are convex for all $x \in K$.

Then, $\Gamma(\epsilon) = \Lambda(\epsilon)$ for all $\epsilon > 0$.

Lemma1.43. (Hashoosh 2016) Consider that K be a nonempty convex and closed subset of a B.S X . Consider that $F: T \times K \times K \to \mathbb{R}$ and $\Psi, \alpha: K \times K \to \mathbb{R}$ satisfy under these circumstances:

i. $F(\cdot, y,\cdot), \alpha(\cdot, y)$ are lsc for all $y \in K$,

ii. $\Psi(\cdot, y)$ is usc for all $y \in K$.

Then, $\Lambda(\epsilon)$ is closed in $T \times K$ for any $\epsilon > 0$.

Theorem 1.44. (Hashoosh 2016) Consider that K is a nonempty convex, closed subset of a B.S X. Consider that $F: T \times K \times K \to \mathbb{R}$ and $\Psi, \alpha: K \times K \to \mathbb{R}$ are three functions. If (EP_Ψ) is strongly w-p, then

$$\Gamma(\epsilon) \neq \emptyset \ \forall \epsilon > 0, lim_{\epsilon\to 0}\ diam\big(\Gamma(\epsilon)\big) = 0.$$

In addition to this, if provided the following conditions are correct:

i. $F(\cdot, x, \cdot)$ is $lsc \forall x \in K$,
ii. $F(t, x, \cdot)$ is convex $\forall (t, x) \in T \times K$,
iii. $F(t, \cdot, \cdot)$ is α -M. bi.f, hemicontinuous, $\forall t \in T$,
iv. $F(t, x, x) = 0, \forall t \in T, x \in K$,
v. $\alpha(\cdot, y)$ and $\Psi(\cdot, y)$ are $usc \forall y \in K$,
vi. $\Psi(x, \cdot)$ and $\alpha(x, \cdot)$ are convex $\forall x \in K$.

Then the converse holds.

Theorem 1.45 (Hashoosh 2016) Consider that T and K are nonempty, closed and convex subsets of real B.S E and X resp., If (EP_Ψ) is strongly w-p in the generalized sense, then

$$\Gamma(\epsilon) \neq \emptyset \forall \epsilon > 0, \lim_{\epsilon\to 0} \mu\left(\Gamma(\epsilon)\right) = 0.$$

In addition to this, if provided the following conditions are correct:

i. $F(\cdot, x, \cdot)$ is lsc for all $x \in K$,
ii. $F(t, x, \cdot)$ is convex for all $(t, x) \in T \times K$,

iii. $F(t,\cdot,\cdot)$ is α -M. bi.f, and H.C for all $t \in T$,

iv. $F(t,x,x) = 0$,for all $t \in T, x \in K$,

v. $\alpha(\cdot, y)$ and $\Psi(\cdot, y)$ are usc for all $y \in K$,

vi. $\Psi(x,\cdot)$ and $\alpha(x,\cdot)$ are convex for all $x \in K$.

Then, the converse holds.

Theorem 1.46 (Hashoosh 2016) Consider that T and K be nonempty, closed and convex subsets of finite dimensional B.Ss E and X resp., If $F: T \times K \times K \to \mathbb{R}$, $\Psi, \alpha: K \times K \to \mathbb{R}$ and $h: T \times K \to \mathbb{R}$ are four functions under these circumstances:

i. $F(\cdot, x,\cdot)$ is lsc and convex for all $x \in K$,

ii. $F(t,\cdot,\cdot)$ is α -M. bi.f and H.C for all $t \in T$,

iii. $F(t,x,x) = 0$, for all $t \in T, x \in K$,

iv. $\alpha(\cdot, y)$ and $\Psi(\cdot, y)$ are usc for all $y \in K$,

v. $\Psi(x,\cdot)$ and $\alpha(x,\cdot)$ are convex for all $x \in K$,

vi. h is convex and lsc.

Therefore, (OPGPEC) is strongly w-p if, and only if, it possesses only one solution to the issue.

Now, we recall further results and theorems as referred by K. Kimura, Y. Liou, S. Wu and J. Yao (2008), that are useful in our work.

Consider that V be a real H.T.L.S and X a nonempty subset of V, let Z be a real topological vector space abreast of a solid pointed convex cone $\boldsymbol{C} \subset \boldsymbol{Z}$ and $\boldsymbol{\theta_Z}$ zero vector in Z. Additionally, let $\boldsymbol{\Gamma}$ be a nonempty subset of H.T.L.S. Then the issue for any $\in \boldsymbol{\Gamma}$, find $x \in \boldsymbol{X}$ in which

$$f(p,x,y) \notin -\int C \text{ , for any } y \in X,$$

is said to be parametric vector equilibrium issue and denoted by VEP_p.

Theorem 1.47. (2008) Consider that X be a nonempty compact convex subset of a real H.T.L.S V, and Z a real topological vector space abreast of a solid pointed convex cone $C \subset Z$. Assume that Γ be a topological space. Consider that f is a vector-valued function from $\Gamma \times X \times X$ to Z abreast of $f(p,x,x) = \theta_Z$ for all $p \in \Gamma$ and $x \in X$. In addition to this, the following presumptions are constructed:

i. $f(\cdot,\cdot,y)$ is $(-C)$ -continuous on $\Gamma \times X$ for each $y \in X$;
ii. $f(p,\cdot,y)$ is strictly $(-C)$ -properly quasi-convex on X for each $p \in \Gamma$ and $y \in X$;
iii. $f(p,x,\cdot)$ is C -continuous and C-quasi-convex on X for each $p \in \Gamma$ and $x \in X$.

Therefore, $\{VEP_p : p \in \Gamma\}$ is parametrically uniquely w-p.

Theorem 1.48. (Kimura 2008) Consider that X be a nonempty compact convex subset of a real H.T.L.S V, and Z a real topological vector space abreast of a solid pointed convex cone $C \subset Z$. Assume that Γ be a topological space. Consider that f is a vector-valued function from $\Gamma \times X \times X$ to Z abreast of $f(p,x,x) \notin -\int C$ for all $p \in \Gamma$ and $x \in X$. . In addition to this, the following presumptions are constructed:

i. $f(\cdot,\cdot,y)$ is $(-C)$ -continuous on $\Gamma \times X$ for each $y \in X$;
ii. $S(p) \neq \emptyset$ for each $p \in \Gamma$. Then $\{VEP_p : p \in \Gamma\}$ is parametrically w-p .

Then, $\{VEP_p : p \in \Gamma\}$ is parametrically w-p.

3. WELL POSED-NESS FOR MIXED HEMIEQUILIBRIUM ISSUES

3.1 Introduction

Nowadays, w-p is worth noting that plays important role in the field of optimization and nonlinear operator issues.

The aim of this chapter is to establish a new class of mixed hemi-equilibrium issues denoted by (MHEP) with ψ- monotone. In order to accomplish this goal, we will be focusing on a subset, K, of B.S. S. We derive a relevant characterisation of metric projection as well as sufficient requirements for these different sorts of w-p. In conclusion, the discussion that has been presented throughout this chapter may be utilized to demonstrate that the idea of w-p for (MHEP) is equal to the concept of w-p for ψ-monotonous.

3.2 Preliminaries

In this chapter, unless stated otherwise assume that two B.Ss E and S and K is a nonempty subset of S. Let S^* be a topological dual space and its norm is denoted by $\|.\|$. In order to make things easier for the reader, we are going to go over a a few prerequisites, such as definitions and outcomes, that have to be imposed before we can show our fundamental results.

Now, we consider the Mixed-hemiequilibrium issue (abbreviation for (MHEP)): Find $u_0 \in \Pi(u_0), \forall v \in \Pi(u_0)$ such that

$$\vartheta(u_0, v) + \rho(v) - \rho(u_0) + J^0\big(\Omega u_0; \Omega\delta(u_0, v)\big) \geq 0 \tag{3.1}$$

such that ϑ is a bifunction from $K \times K$ in to R, in which $\vartheta(u,u) = 0\ \forall u \in K$, and $\Pi: K \multimap K$ set valued. $\Omega: S \to S$ is a linear compact operator, $\delta: S \times S \to S$ single-valued function $\rho: S \to \mathbb{R} \cup \{+\infty\}$and $J: S \to \mathbb{R}$ locally Lipschitz functional.

Remark 2.1. As a result of the convexity of $J^0(u,v)$ and the linearity of $\delta(u,\cdot)$ for every $u \in S$. One can determine that the function $v \mapsto J^0\big(u, \delta(u,v)\big)$ is convex.

Remark 2.2. Because Ω is a linear compact operator, it is obvious that Ωu_n strongly converges to some $\Omega_u \in K$. Therefore, , $\Omega\delta(u_n, v)$ converges strongly to Ωu when $v \in K$. By applying this observation and Lemma (1.16), one can conclude that

$$\lim_n \sup J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \leq J^0(\Omega u; \Omega\delta(u,v)). \tag{3.2}$$

In the sequel, we give the formulation of optimization issue with Mixed-Hemiequilibrium constraint which is denoted shortly by $(OPMHEPC)$. Let $Q: T \times K \to R$ and $\vartheta: T \times K \times K \to R$ be two functions in which T is a nonempty subset of E and $\Pi: K \times K \multimap K$ be set valued Then $(OPMHEPC)$ is formulated as follows:

$$\min Q(t,u) \text{ such that } (t,u) \in T \times K \text{ and } u \in \pi(t),$$

where $\pi(t)$ is solution set of the parametric Mixed hemiequilibrium issue $\big(MHEP(t)\big)$ which is satisfies $(t,u) \in \Pi(t,u)$ iff

$$\vartheta(t,u,v) + \rho(v) - \rho(u) + J^0\big(\Omega u; \Omega\delta(u,v)\big) \geq 0, \forall (t,v) \in \Pi(t,u) \tag{3.3}$$

Instead of writing $\{MHEP(t); t \in T\}$ for the family of Mixed-Hemiequilibrium issue i.e., the parametric issue, we will simply write $(MHEP)$ in this work. The Mixed hemiequilibrium issue $(MHEP)$ which containing many issues as special case been studied intensively. Some special cases of $(MHEP)$ are as below:

(i) If $J = 0, \Pi(t,u) \equiv T \times K$ and $\rho = 0$, then issue $(MHEP)$ is shrink to the parametric equilibrium issue (abbreviation for, $EP(t)$). The following form: Find $u \in k$ such that $\vartheta(t,u,v) \geq 0, \forall v \in K$ (Fang *et al.* 2008).

(ii) If $J = 0, \rho = 0, \Pi(t,u) \equiv T \times K$ and $\vartheta(t,u,v) = -h(t,u,u-v) \forall v \in K$, then issue $MHEP(t)$ is shrink to the parametric quasivariational inequality (abbreviation for $QVI(t)$) (Zhang, *et al.* 2009).

(iii) If $\delta(u;v) = u - v, \rho = 0, \vartheta(t,u,v) = \langle A_u - f, v - u \rangle, \Pi(t,u) \equiv T \times K$ and $\Omega = id$, then issue MHEP is shrink to the parametric hemivariational inequality (Verma 1997).

If $\rho = 0$ and $\Pi(t,u) \equiv T \times K$, then issue $(MHEP)$ is shrink to the parametric Hemi-equilibrium issue (abbreviation for, HEP (t)).(Hashoosh and Jebur 2021). The following form: Find $u \in K$ such that

$\vartheta(t,u,v) + J^0\big(\Omega u; \Omega\delta(u,v)\big) \geq 0, \forall (t,v) \in \Pi(t,u)., \forall v \in K$

Remark 2.3. Throughout this work, in addition to investigate some characterizations of wellposedness for (MHEP) we consider, $\forall \lambda \in (0,1)$

$$\lim_{\lambda \to 0} \frac{\psi(x, x_\lambda)}{\lambda} + \psi(x,y) = 0. \tag{3.4}$$

3.3 W-p of (MHEP) and ψ – monotonicity

In this section, we give ideas of w-p for a class of Mixed hemiequilibrium issues involving of ψ - M. bi.f. For this reason, the following definitions will be given.

Definition 2.4. A sequence$\{(t_n, u_n)\} \subseteq T \times K$ is an approximating sequence for Mixed hemiequilibrium issue $(MHEP)$ if there is $\{\epsilon_n\}$ is positive sequence which $\{\epsilon_n\} \to 0$ as $n \downarrow \infty$ in which $d\big((t_n, u_n), \Pi((t_n, u_n)\big) \leq \epsilon_n$ and for all $(t, v) \in \Pi(t_n, u_n)$ for all $n \in N$,

$$\vartheta(t, u_n, v) + \rho(v) - \rho(u_n) + J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \geq -\epsilon_n \|v - u_n\|. \tag{3.5}$$

Definition 2.5. The issue (MHEP) is called strongly (resp., weakly) w-p if (MHEP) has only one solution u, and for every App.seq for (MHEP) convergent strongly (resp., weakly) to only one solution u.

Definition 2.6. The issue $(MHEP)$ is said to be strongly (resp., weakly) w-p in the generalized sense if the solution set $\pi(t)$ for $(MHEP)$ is a nonempty set and every App.seq $\{t_n, u_n\}$ has subsequence that strongly (resp., weakly) converges to some point of $\pi(t)$.

It is known that the strongly w-p (in the generalized sense) implies the weak w-p (in the generalized sense), but the converse is not generally true.

Next, we give the concepts of ψ-App.seq, ψ -w-p and ψ-w-p in the generalized sense.

Definition 2.7. Consider that $\{(t_n, u_n)\} \subseteq T \times K$ is an ψ -App. seq for Mixed hemiequilibrium issue (MHEP) if there is $\{\epsilon_n\}$ is positive sequence which $\{\epsilon_n\} \to 0$ as $n \downarrow \infty$ in which $d\big((t_n, u_n), \Pi((t_n, u_n)\big) \leq \epsilon_n$ and for all $(\mathrm{t}, \mathrm{v}) \in \Pi(\mathrm{t_n}, \mathrm{u_n})$ for all $n \in N$,

$$\vartheta(t, v, u) - \psi(u, v) \leq \epsilon \parallel v - u \parallel -J^0\big(\Omega u; \Omega\delta(u, v)\big) - \rho(v) + \rho(u_n) + \psi(v, u).$$

Definition 2.8. The Mixed hemiequilibrium issue $(MHEP)$ is called strongly (resp., weakly) ψ - well posedness if $(MHEP)$ has only one solution u, and for every ψ-App.seq for $(MHEP)$ convergent strongly (resp., weakly) to only one solution u.

Definition 2.9. $(MHEP)$ is said to be strongly (resp. , weakly) ψ-w-p in the generalized sense if the solution set $\pi(t)$ for $(MHEP)$ is a nonempty set $\pi(t)$ and every ψ App.seq $\{u_n\}$ has subsequence that strongly (resp., weakly) converges to some point of $\pi(t)$.

Remark 2.10. If ϑ is ψ - M. bi.f, one can get

i. An App.seq for $(MHEP)$ is an ψ-App.seq for $(MHEP)$;
ii. If $(MHEP)$ is strongly (resp., weakly) ψ-w-p, then it is strongly (resp., weakly) w-p ;
iii. $(MHEP)$ is strongly (resp., weakly) ψ-w-p in the generalized sense, then it is strongly (resp., weakly) w-p in the generalized sense.

In order to solve issue (2.3), we impose the following hypotheses:

H1: The mapping $\delta(\cdot;\cdot): S \times S \to S$, it satisfies the following hypotheses:

i. $\delta(u;u) = 0$ for all $u \in S$;
ii. $\delta(u;\cdot)$ is linear operator for all $u \in S$;
iii. for any $v \in S, \delta(u^m;v) \to \delta(u;v)$, whenever $u^m \to u$.

H2: $\vartheta: T \times K \times K \to \mathbb{R}$ is a function, it satisfies the following hypotheses:

i. $\vartheta(t,u,u) = 0$ for all $t \in T, u \in K$;
ii. $\vartheta(t,\cdot,\cdot)$ is ψ - M. bi.f;
iii. $\vartheta(t,\cdot,v)$ is H.C for all $t \in T$;
iv. $\vartheta(t,u,\cdot)$ is convex for all $t \in T, u \in K$;
v. $\vartheta(\cdot,u,\cdot)$ is lsc for all $u \in K$.

H3: $\psi: K \times K \to \mathbb{R}$ is a function, satisfies the following hypotheses

i. $\psi(\cdot, v)$ are H.C for all $t \in T$;

ii. $\psi(u, \cdot)$ is convex for all $t \in T, u \in K$;

iii. $\psi(\cdot,\cdot)$ is usc.

H4: $\rho: S \to \mathbb{R} \cup \{+\infty\}$ is a function on K, satisfies the following hypotheses, where $K \cap dom\ H \neq H$, dom $\rho = \{x \in S: \rho(x) < +\infty\}$ is the effective domain of ρ.

i. ρ is convex;

ii. ρ is l.s.c.

H5: $Q: T \times K \to \mathbb{R}$ is a function, satisfies the following hypotheses

i. $Q(\cdot,\cdot)$ is convex ;

ii. $\psi(\cdot,\cdot)$ is l.s.c.

3.4 W-p of (MHEP) with metric characterizations

This section ideals with some characterizations for w-p of the issue (MHEP). The analysis of our results cannot be obtained straight forwardly from the previeos papers. For each $\epsilon > 0$, we consider that the next sets

$$\Gamma(\epsilon) = \{(t,u) \in T \times K: d\big((t,u), \Pi(t,u)\big) \leq \epsilon \text{ such that}$$
$$\vartheta(t,u,v) + \rho(v) - \rho(u) + J^0\big(\Omega u; \Omega\delta(u,v)\big) \geq -\epsilon \parallel v - u \parallel, \forall (t,v) \in \Pi(t,u)\big\}.$$

and

$$\Lambda(\epsilon) = \{(t,u) \in T \times K: d\big((t,u), \Pi(t,u)\big) \leq \epsilon \text{ such that}$$
$$\vartheta(t,v,u) - \psi(u,v) \leq \epsilon \parallel v - u \parallel + J^0\big(\Omega u; \Omega\delta(u,v)\big) + \rho(v) - \rho(u) + \psi(v,u), \forall (t,v) \in \Pi(t,u)\big\}.$$

The following Lemma plays an important role in the sequel.

Lemma 2.11. Consider that $K_0 \subset K$ is a nonempty convex subset of a real B.S S, if hypotheses $H_1, H_2(i-iv), H_3(i-ii)$ and $H_4(i)$ be satisfied. Then, for each $u_0 \in K_0$ the statements (i) and ii are equivalent:

(i) Find $(t, u_0) \in T \times K_0$ such that for all $(t, v) \in T \times K_0$

$$\vartheta(t, u_0, v) + \rho(v) - \rho(u_0) + J^0\big(\Omega u; \Omega\delta(u_0, v)\big) \geq -\epsilon\|v - u_0\|.$$

(ii) Find $(t, u_0) \in T \times K_0$ such that for all $(t, v) \in T \times K_0$

$$\vartheta(t, v, u_0) - \psi(u_0, v)$$
$$\leq \epsilon\|v - u_0\| + J^0\big(\Omega u_0; \Omega\delta(u_0, v)\big) + \rho(v) - \rho(u_0) + \psi(v, u_0).$$

Proof. Let the inequality (i) holds. Then, $(t, u_0) \in T \times K_0$, $\forall (t, v) \in T \times K_0$,such that

$$\vartheta(t, u_0, v,) + \rho(v) - \rho(u_0) + J^0\big(\Omega u_0; \Omega\delta(u_0, v)\big) \geq -\epsilon\|v - u_0\|. \tag{3.6}$$

According to $H_2(ii)$ and (2.6) we have

$$\vartheta(t, v, u_0) - \psi(u_0, v) - \psi(v, u_0) \leq -\vartheta(t, u_0, v)$$

$$\leq \epsilon\|v - u_0\| + \rho(v) - \rho(u_0) + J^0\big(\Omega u_0; \Omega\delta(u_0, v)\big). \tag{3.7}$$

It means that, inequality (ii) holds. Conversely, suppose that inequality (ii) holds. Let $u_\lambda = u_0 - \lambda(u_0 - v), \lambda \in (0,1)$, and since K_0 is convex. Then $u_\lambda \in K_0$. Now, from Remark 2.1 and $H_4(i)$we get

$$\vartheta(t, u_\lambda, u_0) \leq \epsilon\|u_\lambda - u_0\| + \rho(u_\lambda) - \rho(u_0) + J^0\big(\Omega u_0; \Omega\delta(u_0, u_\lambda)\big) + \psi(u_0, u_\lambda)$$

$$+\psi(u_\lambda, u_0)$$

$$\lambda\left[\epsilon\|v-u_0\|+\rho(v)-\rho(u_0)+J^0\big(\Omega u_0;\Omega\delta(u_0,v)\big)\right]+\psi(u_0,u_\lambda)+\psi(u_\lambda,u_0) \tag{3.8}$$

By virtue convexity of $\vartheta(t,u_0,.)$ and $H_2(i)$ one can obtain

$$0=\vartheta(t,u_\lambda,u_\lambda)\leq\vartheta(t,u_\lambda,u_0)-\lambda[\vartheta(t,u_\lambda,u_0)-\vartheta(t,u_\lambda,v)].$$

Then,

$$\lambda[\vartheta(t,u_\lambda,u_0)-\vartheta(t,u_\lambda,v)]\leq\vartheta(t,u_\lambda,u_0) \tag{3.9}$$

At similarity from convexity of $\psi(u_0,.)$ all $u_0\in K_0$, we have

$$\lambda[\psi(u_\lambda,u_0)-\psi(u_\lambda,v)]\leq\psi(u_\lambda,u_0). \tag{3.10}$$

Thus, it follows from (2.7) - (2.10), one can get

$$\begin{gathered}\lambda[\vartheta(t,u_\lambda,u_0)-\vartheta(t,u_\lambda,v)+\psi(u_\lambda,u_0)-\psi(u_\lambda,v)]\\ \leq\vartheta(t,u_\lambda,u_0)+\psi(u_\lambda,u_0)\\ \leq\lambda\left[\epsilon\|v-u_0\|+\rho(v)-\rho(u_0)+J^0\big(\Omega u_0;\Omega\delta(u_0,v)\big)\right]+\psi(u_0,u_\lambda)+2\psi(u_\lambda,u_0)\end{gathered}$$

The hemicontinuously of $\vartheta(t,.,v)$ and $H_3(i)$implies that

$$\lambda\left[-\vartheta(t,u_0,v)-\psi(u_0,v)-\epsilon\|v-u_0\|-\rho(v)+\rho(u_0)-J^0\big(\Omega u_0;\Omega\delta(u_0,v)\big)\right]$$

$\leq\psi(u_0,u_\lambda)$.

Then,

$$\vartheta(t,u_0,v)+\rho(v)-\rho(u_0)+J^0\big(\Omega u_0;\Omega\delta(u_0,v)\big)+\epsilon\|v-u_0\|$$

$$\geq -\frac{\psi(u_0,u_\lambda)}{\lambda}-\psi(u_0,v)(2.11)$$

Now, from (2.4) and (2.11), we get for all $(t,v)\in T\times K$.

$$\vartheta(t,u_0,v)+\rho(v)-\rho(u_0)+J^0\big(\Omega u_0;\Omega\delta(u_0,v)\big)\geq -\epsilon\|v-u_0\|.$$

Therefore, inequality (i) holds. This completes the proof.

Lemma 2.12. Consider that $K_0\subset K$ is a nonempty convex subset of a real B.S S, if we suppose hypotheses $H_1, H_2(v), H_3$ iii and H_4 ii. Then, the set of all $(t,u_0)\in T\times K_0$ and satisfied inequality ii is closed.

Proof. Suppose that $\Lambda(\epsilon)=$ the set of all $(t,u_0)\in T\times K_0$ and satisfied inequality (ii), $\{(t_n,u_{n_0})\}\subseteq\Lambda(\epsilon)$ is a sequence in which $(t_n,u_{n_0})\to(t,u_0)\in T\times K_0, \forall(t,v)\in T\times K_0$,

$$\begin{aligned}&\vartheta(t_n,v,u_{n_0})-\psi(u_{n_0},v)\\&\leq \epsilon\|v-u_{n_0}\|+\rho(v)-\rho(u)+J^0\left(\Omega u_{n_0};\Omega\delta(u_{n_0},v)\right)\\&+\psi(v,u_{n_0})\end{aligned}\qquad(2.12)$$

By Remark 2.2 and assumptions $H_2(v)$, $H_3(iii)$ and $H_4(ii)$ we have

$$\begin{aligned}\vartheta(t,v,u_0)\quad&\leq \liminf_n \vartheta(t_n,v,u_{n_0})\\&\leq \limsup_n\left[\epsilon\|v-u_{n_0}\|+\rho(v)-\rho(u_{n_0})+J^0\left(\Omega u_{n_0};\Omega\delta(u_{n_0},v)\right)+\psi(v,u_{n_0})+\psi(u_{n_0},v)\right]\\&\leq \epsilon\|v-u_0\|+\rho(v)-\rho(u_0)+J^0\big(\Omega u_0;\Omega\delta(u_0,v)\big)+\psi(v,u_0)+\psi(u_0,v).\end{aligned}$$

Therefore, $\Lambda(\epsilon)$is closed in $T \times K_0$ for $\epsilon > 0$. This completes the proof.

Next, the result is a consequence of Lemmas 2.11 and 2.12.

Corollary 2.13. Consider that $\Pi: K \times K \multimap K$ be a closed, convex valued mapping. If hypotheses $H_1 - H_4$ be satisfied. Then, $(t, u) \in T \times K$ is a solution of $(MHEP)$ if and only if it solves the next inequality:

Find $(t, u) \in \Pi(t, u)$ for all $(t, v) \in \Pi(t, u)$ such that

$$\vartheta(t, v, u) - \psi(u, v) \leq \epsilon\|v - u\| + \rho(v) - \rho(u) + J^0\big(\Omega u; \Omega\delta(u, v)\big) + \psi(v, u)\}.$$

Moreover, $\Gamma(\epsilon) = \Lambda(\epsilon)$.

Here, the fundamental result in this work is given.

Theorem 2.14 Consider that $\Pi: K \times K \multimap K$ be set-valued mapping, and let Π be a closed convex valued-mapping. So, $(MHEP)$ is s-w-p iff the solution set $\pi \neq \emptyset$ of $(MHEP)$ and

$$\lim_{\epsilon \to 0} dim\, \Gamma(\epsilon) = 0. \tag{3.13}$$

Proof. Consider that $(MHEP)$ be s-w-p, then from it is definition $(MHEP)$ admit a unique solution $(t_0, u_0) \in T \times K$. Therefore $\pi \neq \emptyset$. Use the contradiction, assume that $diam\ \Gamma(\epsilon)$ does not tend to 0 as $\epsilon \to 0$. Then $\exists$ a constant $p > 0$ and $\epsilon_n > 0$ as $n \to 0$ such that $\forall n \in N$ $\exists$ two sequences $(t_n, u_n)$and $(t_n, v_n) \in \Gamma(\epsilon)$ satisfing

$$\|(t_n, u_n) - (t_n, v_n)\| > p, \forall n \in N. \tag{3.14}$$

But $\{(t_n, u_n)\}$ and $\{(t_n, v_n)\}$ are both App.seq for $(MHEP)$. Hence,

$$\lim_{n\to\infty}(t_n, u_n) = \lim_{n\to\infty}(t_n, v_n) = (t_0, u_0) \tag{3.15}$$

Using the two equations (2.14)and (2.15), we can get

$$0 < p < \|(t_n, u_n) - (t_n, v_n)\|$$

$$\leq \|(t_n, u_n) - (t_0, u_0)\| + \|(t_n, v_n) - (t_0, u_0)\| \quad \to 0. \tag{3.16}$$

we reach a contradiction.

Conversely, let us the equation (2.14) and $\pi \neq \emptyset$ are hold. It means that π is singltion point set. Set $\pi(t) = u_0$. Suppose that $\{(t_n, u_n)\}$ is App.seq for $(MHEP)$. Then, there is a sequence $\{\epsilon_n\}$ is positive with $\{\epsilon_n\} \to 0$ as $n \downarrow \infty$ in which $d\big((t_n, u_n), \Pi(t_n, u_n)\big) \leq \epsilon_n$ and

$$\vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \geq -\epsilon_n\|v - u_n\| \tag{3.17}$$

For all $n \in N, (t, v) \in \Pi(t_n, u_n)$.

Therefore, $\{(t_n, u_n)\} \in \Gamma(\epsilon_n) \forall n \in N$. Since $u_0 \in \pi(t)$, then $u_0 \in \Gamma(\epsilon_n)$ and

$$\lim_{n\to\infty}\|(t_n, u_n) - (t_0, u_0)\| \leq \lim_{n\to\infty} \Gamma\,(\epsilon_n)$$

$$= 0.$$

Which implies that (t_n, u_n) strongly converges to (t_0, u_0). Therefore $(MHEP)$ is strongly w-p . This completes the proof.

Remark 2.15. If $H_2(ii)$ satisfies, one can have that $(MHEP)$ strongly ψ-w-p .

Theorem 2.16. Assume that $\Pi: K \times K \multimap K$ be set-valued mapping. If ϑ is ψ - M. bi.f, then $(MHEP)$ is strongly ψ-w-p iff the solution set $\pi \neq \emptyset$ of $(MHEP)$ and

$$\lim_{\epsilon \to 0} dim\, \Lambda(\epsilon) = 0. \tag{3.18}$$

Proof the Theorems 2.14 and 2.16 are similar, so we omit it.

Remark 2.17. One can note, the solution set $\pi \neq \emptyset$ of $(MHEP)$ in the Theorems 2.14 and 2.16 are assuming. In the following the authors remove this assumption.

Theorem 2.18. Consider that $\Pi: K \times K \multimap K$ be a nonempty convex closed valued mapping, and let $\Pi(t_n, u_n)$ be lower semi-Mosco convergent to $\Pi(\mathrm{t}_0, \mathrm{u}_0)$ whenever $(t_n, u_n) \to (t_0, u_0)$ such that $(t_0, u_n) \in T \times K$ and $n \to \infty$. If the conditions $(H_1 - H_4)$ hold. Then $(MHEP)$ is s-w-p iff

$$\lim_{\epsilon \to 0} dim\, \Gamma(\epsilon) = 0 \text{ and } \Gamma(\epsilon) \neq \emptyset \forall \epsilon > 0 \tag{3.19}$$

Proof. It is enough to proof the sufficiency part. Let the condition (2.19) holds and $\{(t_n, u_n)\}$ in $T \times K$ is an App.seq for (MHEP). Then $\exists\ \epsilon_n > 0$, $\epsilon_n \to 0$ as $n \to \infty$ such that $d\big((t_n, u_n), \Pi(t_n, u_n)\big) \leq \epsilon_n$ and

$$\vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \geq -\epsilon_n \|v - u_n\|,$$

$\forall(t_n, v) \in \Pi(t_n, u_n)$ and $n \in N$.

It means that $(t_n, u_n) \in \Gamma(\epsilon_n) \forall n \in N$ from condition (2.19). We have that $\{t_n, u_n\}$ Cauchy sequence. So it converges strongly to some point $(t_0, u_0) \in T \times K$. To proof $(t_0, u_0) \in T \times K$ is a solution of $(MHEP)$, we choose $(t_0', u_0') \in \pi(t_n, u_n))$ such that

$$\|(t_n, u_n) - (t_0', u_0')\| \leq \epsilon_n + d\big((t_n, u_n), \Pi(t_n, u_n)\big) \leq 2\epsilon_n.$$

Since $(t_n, u_n) \to (t_0, u_0)$ and $\epsilon_n \to 0$ as $n \to \infty$ we get that $(t_0', u_0') \to (t_0, u_0)$ as $n \to \infty$. Then, by our assumptions $\Pi(\mathrm{t_n}, \mathrm{u_n})$ be lower semi-Mosco convergent to $\Pi(\mathrm{t_0}, \mathrm{u_0})$. That is

$$\liminf_{\mathrm{n}\to\infty} \Pi(\mathrm{t_n}, \mathrm{u_n}) = \Pi(\mathrm{t_0}, \mathrm{u_0}).$$

So, $(t_0, u_0) \in \Pi(t_0, u_0)$. Then, there exists $\{v_n\}$ be a sequence in K with $(t_n, v_n) \in \Pi(t_n, u_n)$ for all $(t_0, v) \in \Pi(t_0, u_0)$ in which $v_n \to v$ as $n \to \infty$. According to $\psi -$ monotonicity of θ, we have

$$\vartheta(t_n, v, u_n) - \psi(u_n, v) - \psi(v, u_n) - \rho(v) + \rho(u_n) - J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big)$$

$$\leq -\vartheta(t_n, u_n, v) - \rho(v) + \rho(u_n) - J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big)$$

$\leq \epsilon_n\|v - u_n\|$ for all $(t, v) \in \Pi(t, u)$ and $n \in N$.

We have

$$\vartheta(t_n, v, u_n) - \psi(u_n, v) - \psi(v, u_n) - \rho(v) + \rho(u_n) - J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \leq \epsilon_n\|v - u_n\|.$$

Since,

$$\vartheta(t_0, v, u_0) - \psi(u_0, v) - \psi(v, u_0) - \rho(v) + \rho(u_0) - J^0\big(\Omega u_0; \Omega\delta(u_0, v)\big)$$

$$\leq \lim \inf_n \left\{-\vartheta(t_n, u_n, v) - \rho(v) + \rho(u_n) - \psi(u_n, v) - \psi(v, u_n) - J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big)\right\}$$

$$\leq \lim \inf_n \epsilon_n \|v - u_n\|$$

$$\leq \epsilon \|v - u_0\|.$$

Hence, $\forall (t_0, v) \in \Pi(t, u_0)$. This fact together with Lemma 2.11, then

$$\vartheta(t_0, u_0, v) + \rho(v) - \rho(u_0) + J^0\big(\Omega u; \Omega\delta(u_0, v)\big) \geq -\epsilon \|v - u_0\|.$$

$\forall (t_0, v) \in \Pi(t, u_0)$.Then, $(t_0, u_0) \in T \times K$ is solve $(MHEP)$.

To proof the uniqueness of the solution. We suppose that $\exists$ another solution $(t^{\#}, u^{\#}) \in T \times K$ in which (t_0, u_0) and $(t^{\#}, u^{\#}) \in \Gamma(\epsilon)$ $\forall \epsilon > 0$, and

$$0 \leq \| (t_0, u_0) - (t^{\#}, u^{\#})\| \leq diam \Gamma(\epsilon) \ \rightarrow 0 \ as \ \epsilon \rightarrow 0.$$

Which is a contraction. This completes the proof.

Corollary 2.19. Suppose that the same assumptions as Theorem 2.18. Then $(MHEP)$ is ψ -s-w-p iff

$$\lim_{\epsilon \to 0} dim\, \Lambda(\epsilon) = 0 \text{ and } \Lambda(\epsilon) \neq \emptyset \forall \epsilon > 0. \tag{3.20}$$

3.5 Characterizations of the w-p for $(MHEP)$ in the generalized sense

The goal of this section is to establish the characterizations of the w-p, and present some notions under which $(MHEP)$ is strongly w-p or strongly ψ - w-p in the generalized sense.

Theorem 2.20. Consider that the hypothesis $H_1 - H_4$ hold. Then $(MHEP)$ is strongly w-p in the generalized sense iff the solution set $\pi \neq \emptyset$ compact of $(MHEP)$ and

$$\lim_{\epsilon \to 0^+} e(\Gamma(\epsilon), \pi) = 0. \tag{3.21}$$

Proof. Consider that $(MHEP)$ be strongly w-p in the generalized sense, then π is nonempty set and $\pi \subset \Gamma(\epsilon) \neq \emptyset, \forall \epsilon > 0$.

Claim 1. Proof that π is compact.

Let $\{t_n, u_n\} \subset \pi \subset \Gamma(\epsilon)$ be a sequence $\forall \epsilon > 0$, then $\{t_n, u_n\}$ is an App.seq for $(MHEP)$. Thus, it has a subsequence converging to some point of π. It means that π is compact.

Claim 2. We show that (2.21) holds.

According by contradiction, we suppose that $e(\Gamma(\epsilon), \pi)$ does not tend to 0 as $\epsilon \to 0$. Then, $\exists\ \epsilon_n > 0$ with $\epsilon_n \to 0$ as $n \to \infty$ so, there is a constant $p > 0$ and $\{t_n^0, u_n^0\} \in \Gamma(\epsilon)$ satisfing

$$p < d(\{t_n^0, u_n^0\}, \pi), \forall n \in N.$$

Since $\{t_n^0, u_n^0\}$ is an App.seq of $(MHEP)$. Thus, there is a subsequence $\{ t_{nk}^0, u_{nk}^0\}$ of $\{t_n^0, u_n^0\}$ converging to some point of π. Then, we get

$$0 < p < d(\{t_{nk}^0, u_{nk}^0\}, \pi), \text{ as } n_k \to \infty,$$

we reach to contradiction.

Conversely, if (2.21) hold, and let (t_n, u_n) be an App.seq for $(MHEP)$ in $T \times K$. Then, $\exists$ $\epsilon_n > 0$ with $\epsilon_n \to 0$ as $n \to \infty$ and $d\big((t_n, u_n), \Pi(t_n, u_n) \leq \ \epsilon_n$ such that $\forall (t_n, v) \in \Pi(t_n, u_n)$

$$\vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \geq -\epsilon \parallel v - u_n \parallel.$$

Therefore, $(t_n, u_n) \in \Gamma(\epsilon_n)$ for all $n \in N$. So, by (2.21) $\exists$ a sequence $\{l_n\}$ in π, in which

$$\|(t_n, u_n) - (t_n, l_n)\| \to 0 \text{ as } n \to \infty.$$

By compactness of π, $\exists$ a subsequence (t_{nk}^0, l_{nk}^0) of $\{t_n^0, l_n^0\}$ strongly converging to some point of $(t_0, u_0) \in \pi$. Therefore, (MHEP)

$$\|(t_0, u_0) - (t_n, u_n)\| \leqslant \|(t_n, u_n) - (t_{nk}, l_{nk})\| + \|(t_{nk}, l_{nk}) - (t_0, u_0)\| \to 0 \text{ as } n_k \to \infty.$$

Therefore, $(MHEP)$ is strongly ψ – w-p in the generalized sense.

Remark 2.21. If $H_2(ii)$ satisfies, one can have that $(MHEP)$ strongly ψ-w-p in generalized sense.

Theorem 2.22. Consider that the hypothesis $H_1 - H_4$ hold. If ϑ is ψ - monotone, then $(MHEP)$ is strongly ψ-w-p in the generalized sense if and only, if the solution set $\pi \neq \emptyset$ compact of $(MHEP)$ and

$$\lim_{\epsilon \to 0^+} e(\Lambda(\epsilon), \pi) = 0. \tag{3.22}$$

Proof the Theorems 2.20 and 2.22 are similar, so we omit it. In above Theorems, the compactness of π plays a importante role, in future work one can remove this condition and get the following results.

Theorem 2.23. Consider that $\Pi: K \times K \multimap K$ be a nonempty convex closed valued mapping, and let $\Pi(t_n, u_n)$ be lower semi-Mosco convergent to $\Pi(t_0, u_0)$ whenever $(t_n, u_n) \to (t_0, u_0)$ such that $(t_n, u_n) \in T \times K$ and $n \to \infty$. If the conditions $H_1 - H_4$ hold. Then $(MHEP)$ is strongly w-p iff

$$\Gamma(\epsilon) \neq 0, \text{ for all } \epsilon > 0 \text{ and } \lim_{\epsilon \to 0} \Omega\left(\Gamma(\epsilon)\right) = 0. \tag{3.23}$$

Proof. Consider that $(MHEP)$ be strongly w-p in the generalized sense. From Theorem 2.20 then π is nonempty compact set and $\pi \subset \Gamma(\epsilon) \neq \emptyset, \forall \epsilon > 0$. Thus, one can get

$$\Omega(\Gamma) = 0, \tag{3.24}$$

$$H(\Gamma(\epsilon), \pi) = max\{e(\Gamma(\epsilon), \pi), e(\pi, \Gamma(\epsilon))\} = e(\Gamma(\epsilon), \pi) \forall \epsilon > 0. \tag{3.25}$$

From (2.24) and (2.25), we have

$$\begin{aligned}\Omega(\Gamma(\epsilon)) &\leq H(\Gamma(\epsilon),\pi)+\Omega(\pi)\\ &= H(\Gamma(\epsilon),\pi)\\ &= e(\Gamma(\epsilon),\pi).\end{aligned}$$

By using condition (2.21), we have

$$\lim_{\epsilon\to 0}\Omega(\Gamma(\epsilon)) = 0.$$

Conversely, suppose that the issue (2.23) holds $\forall\epsilon > 0, cl\big(\mathit{\Gamma}(\epsilon)\big)$ is nonemty closed, non-decreasing and

$$\lim_{\epsilon\to 0}\mathit{\Omega}\left(cl\big(\mathit{\Gamma}(\epsilon)\big)\right) = \lim_{\epsilon\to 0}\mathit{\Omega}\left(\mathit{\Gamma}(\epsilon)\right)$$

$$= 0.$$

It follows that from generalized cantor theorem (see (Kuratowski 1966)) we obtain

$$\lim_{\epsilon\to 0} H\left(cl\big(\mathit{\Gamma}(\epsilon)\big),\mathit{\Gamma}\right) = 0.$$

and $\emptyset \neq \mathit{\Gamma}$ is compact, in which

$$\cap_{\epsilon>0}\, cl\big(\mathit{\Gamma}(\epsilon)\big) = \mathit{\Gamma}.$$

Our claim $\mathit{\Gamma} = \pi$. It is clear that $\pi \subset \mathit{\Gamma}$. To prove $\mathit{\Gamma} \subset \pi$, for any $(t_0, u_0) \in \mathit{\Gamma}$, we have $d\left((t_0,u_0), cl\big(\mathit{\Gamma}(\epsilon_n)\big)\right) = 0, \forall\epsilon_n > 0.$

Therefore, $\exists\ (t_n, u_n) \in \Gamma(\epsilon_n) \forall n \in N$ such that $\|(t_n, u_n) - (t_0, u_0)\| \leq \epsilon_n$ It means that $(t_n, u_n) \to (t_0, u_0)$, and $\Pi(t_n, u_n)$ be lower semi-Mosco convergent sequence to $\Pi(t_0, u_0)$. Therefore, $(t_0, u_0) \in \Pi(t_0, u_0)$ and $d\big((t_n, u_n), \Pi(t_n, u_n)\big) \leq 0$, such that

$$\vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \geq -\epsilon\|v - u_n\|, \forall (t_n, v) \in \Pi(t_n, u_n)$$

and $n \in N$

Therefore, $(t_0, u_0) \in \Pi(t_0, u_0)$. Evidently, according to $\Pi(t_n, u_n)$ is lower semi-Mosco convergent sequence to $\Pi(t_0, u_0)$ for all $(t, v) \in \Pi(t_0, u_0)$, we perhaps choose a sequence $(t_n, v_n) \in T \times K$ with $(t_n, v_n) \in \Pi(t_n, u_n)$ in which $(t_n, v_n) \to (t_0, v_0)$ as $n \to \infty$. According to $\psi -$ monotonicity of ϑ we have

$$\begin{aligned}
0 \quad &= \lim_n \inf \epsilon_n \|v - u_n\| \\
&\geq \lim_n \inf\big\{-\vartheta(t_n, u_n, \mathrm{v}) - \rho(v) + \rho(u_n) - J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big)\big\} \\
&\geq \lim_n \mathrm{i}\ \ \{\vartheta(t_n, v, u_n) - \psi(u_n, \mathrm{v}) - \psi(\mathrm{v}, u_n) - \rho(v) + \rho(u_n) - J^0\big(\Omega u_n; \Omega\delta(u_n, v)\big) \\
&\geq \vartheta(t, v, u) - \psi(u, v) - \psi(v, u) - \rho(v) + \rho(u_n) - J^0\big(\Omega u; \Omega\delta(u, v)\big).
\end{aligned}$$

This fact together with Corollary 2.13 then (t, u) is solve $(MHEP)$. Therefore, $\Gamma = \pi$ and

$$\lim_{\epsilon \to 0} H(cl(\Gamma(\epsilon)), \pi) = \lim_{\epsilon \to 0} e((\Gamma(\epsilon)), \pi) = 0.$$

Due to Theorem 2.20 and compactness of π, then $(MHEP)$ is w-p in the generalized sense.

Theorem 2.24. Consider that the same assumptions as in Theorem 2.23 hold. Then $(MHEP)$ is strongly ψ-w-p in the generalized sense iff

$$\mathit{\Lambda}(\epsilon) \neq 0, \text{ for all } \epsilon > 0 \text{ and } \lim_{\epsilon \to 0} \mathit{\Omega}\left(\mathit{\Lambda}(\epsilon)\right) = 0. \tag{3.26}$$

Proof. Consider that $(MHEP)$ be strongly ψ-w-p in the generalized sense. From Theorem 2.22 and due to θ being $\psi -$ monotonicity, then π is nonempty compact set, $\pi \subset \mathit{\Lambda}(\epsilon) \neq \emptyset$, $\forall \epsilon > 0$. Thus, one can get

$$\Omega(\pi) = 0, \tag{3.27}$$

$$H(\Lambda(\epsilon), \pi) = \max\left\{e(\Lambda(\epsilon), \pi), e\left(\pi, \Lambda(\epsilon)\right)\right\} = e(\Lambda(\epsilon), \pi) \quad \forall > 0. \tag{3.28}$$

From (2.27) and (2.28), we have

$$\begin{aligned} \Omega(\Lambda(\epsilon)) \quad & \leq H(\Lambda(\epsilon), \pi) + \Omega(\pi) \\ & = H(\Lambda(\epsilon), \pi) \\ & = e(\Lambda(\epsilon), \pi). \end{aligned}$$

By using condition (2.21), we have $\lim_{\epsilon \to 0} \Omega(\Gamma(\epsilon)) = 0$. Conversely, suppose that the problem (2.26) holds $\forall \epsilon > 0, \Gamma(\epsilon)$ is nonemty closed set and

$$\lim_{\epsilon \to 0} \Omega(\Lambda(\epsilon)) = 0.$$

It follows that from generalized cantor theorem (see (Kuratowski 1966)), we obtain

$$\lim_{\epsilon \to 0} H((\Lambda(\epsilon), \Lambda) = 0.$$

and $\emptyset \neq \Lambda$ is compact, in which

$$\bigcap_{\epsilon>0} (\Lambda(\epsilon)) = \Lambda.$$

The other evidence is similar to those found of Theorem 2.23, so we omit it here.

4. OPTIMIZATION ISSUES WITH MIXED ISSUE CONSTRAINT

4.1 Introduction

Various approaches to discussing various w-p in the context of optimization issues have been offered in the mathematical literature. Using a parametric form of generalized Mixed Hemi-equilibrium issues, this chapter introduces the idea of w-p for optimization issues with constraints.

4.2 W-p for Optimization issues with Mixed Hemiequilibrium issues Constraints

Let's dive into how optimization issues by a Mixed Hemiequilibrium issue constraint are formulated. Denoted by the acronym OPMHEPC, the following constitutes a formulation of the optimization issue including the Mixed Hemiequilibrium issue constraint:

$minQ(t,u)$ such that $(t,u) \in T \times K \wedge u \in \pi(t)$,

where $u \in \pi(t), T$ is a nonempty closed subset of a parametric spaces, $Q: T \times K \longrightarrow \mathbb{R}$, and $\pi(t)$ is the solution set of the parametric Mixed Hemiequilibrium issue (MHEP(t)) defined$by, u \in \pi(t)$ if and only if:

$$\vartheta(t,u,v) + \rho(v) - \rho(u) + J^0\big(\Omega u; \Omega\delta(u,v)\big) \geq 0, \forall (t,v) \in T \times K$$

Definition 3.1. A sequence $\{(t_n, u_n)\} \subset T \times K$ is said to be App.seq for (OPMHEPC) if $\forall (t,v) \in T \times K, n \in N$.

i. There is a nonnegative sequence $\{\epsilon_n\}$ by $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$,

$$\vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\left(\Omega u_n;\ \Omega\delta(u_n, v)\right) \geq -\epsilon_n\|v - u_n\|.$$

ii. $lim_{n\to\infty} sup\, Q(t_n, u_n) \leq \inf_{m\in T, v\in\pi(m)} Q(m, v).$

Definition 3.2. (OPMHEPC) is called to be strongly w-p (resp., strongly (or weakly) w-p in the generalized sense) if (OPMHEPC) has only one solution and for each App.seq of (OPMHEPC) converges strongly (or weakly) to only one solution. If $\pi \neq \varphi$ and Every approximation solution sequence has a subsequence that strongly (or weakly) converges to π.

An approximation solution set for (OPMHEPC) is defined as follows:

$$\psi(\epsilon, c) = \begin{cases} (t, u) \in \mathrm{T} \times \mathrm{K};\ \mathrm{Q}(t, u) \leq \inf_{m\in T, v\in\pi(m)} Q(m, v) + \mathrm{c} \quad \text{and} \\ \vartheta(t, u, v) + \rho(v) - \rho(u) + J^0\left(\Omega u;\ \Omega\delta(u, v)\right) \geq -\epsilon\|v - u\| \end{cases}$$

$$\forall (t, v) \in T \times K.$$

Theorem 3.3. Assume that K is a nonempty convex, closed subset of a B.S S. İf (OPMHEPC) is strongly w-p. Then

$$\psi(\epsilon, c) \neq \varphi\ \ \forall\, c, \epsilon > 0\ , diam\, \psi(\epsilon, c) \to 0. \tag{4.1}$$

where $(\epsilon, \mathrm{c}) \to (0,0)$. Moreover, if the assumptions $(\mathrm{H}_1 - \mathrm{H}_4)$ and $\mathrm{H}_5(ii)$ hold. Then the converse hold.

Proof. Consider that (OPMHEPC) is strongly w-p. Then (OPMHEPC) admits only one solution $(t, u) \in \mathrm{T} \times \mathrm{K}$ such that

$$\begin{cases} \mathrm{Q}(t,u) \leq \underset{m\in T, v\in\pi(m)}{inf} Q(m,v) \ and \\ \vartheta(t,u,v) + \rho(v) - \rho(u) + J^0\left(\Omega u;\ \Omega\delta(u,v)\right) \geq 0. \end{cases}$$

Obviously, $\psi(\epsilon, c) \neq \varphi$ for any $c, \epsilon > 0$, since $(t,u) \in \psi(\epsilon, c)$ for each $c, \epsilon > 0$. If diam $\psi(c,\epsilon) \nrightarrow 0$ as $c \to 0, \epsilon \to 0$ then, there is a constant $p > 0$ and a nonnegative sequences $\{c_n\}$ and $\{\epsilon_n\}$ accompanied by $c_n \to 0$ and $\epsilon_n \to 0$ where $n \to 0$ and $(t_n, u_n), (t_n, v_n) \in \psi(\epsilon, c)$ in which

$$\|(t_n, u_n) - (t_n, v_n)\| > p \ \ \forall\, n \in \mathbb{N}. \tag{4.2}$$

Since $(t_n, u_n), (t_n, v_n) \in \psi(\epsilon, c)$ $\forall\, n \in \mathbb{N}$. Then, both $\{(t_n, u_n)\}$ and $\{(t_n, e_n)\}$ are approximating sequence for (OPMHEPC). By the w-p of (OPMHEPC), they have to converge strongly to the unique solution of (OPMHEPC) a contradiction to (3.2).

On the other hand, let's assume that (3.1) is correct. Consider that $\{(t_n, u_n)\}$ be an approximating sequence for (OPMHEPC). Then, there is a nonnegative sequence $\{\epsilon_n\}$ accompanied by $\epsilon_n \to 0$ as $n \to \infty$ where

$$\begin{cases} \underset{n\to\infty}{lim}\, sup\, \mathrm{Q}(t_n, u_n) \leq \underset{m\in T, v\in\pi(m)}{inf} Q(m,v) \text{ and} \\ \vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\left(\Omega u_n;\ \Omega\delta(u_n, v)\right) \geq -\epsilon_n\|v - u_n\|. \end{cases} \tag{4.3}$$

This yields that $(t_n, u_n) \in \psi(\epsilon_n, c_n)$. It follows from (3.1), that $\{(t_n, u_n)\}$ is a Cauchy sequence and so it converge strongly to a point $(t,u) \in \mathrm{T} \times \mathrm{K}$. It follows from (3.3) and assumptions $\mathrm{H}_2(ii - v), \mathrm{H}_3(iii)$ and $\mathrm{H}_4(ii)$ that

$$\begin{aligned}
0 &= \lim_n \inf \epsilon_n \|v - u_n\| \\
&\geq \lim_n \inf\{-\vartheta(t_n, u_n, \mathrm{v}) - \rho(v) + \rho(u_n) - J^0(\Omega u_n; \Omega\delta(u_n, v))\} \\
&\geq \lim_n \inf\{\vartheta(t_n, v, u_n) - \psi(u_n, \mathrm{v}) - \psi(\mathrm{v}, u_n) - \rho(v) + \rho(u_n) - J^0(\Omega u_n; \Omega\delta(u_n, v)) \\
&\geq \vartheta(\mathrm{t}, \mathrm{v}, \mathrm{u}) - \psi(\mathrm{u}, \mathrm{v}) - \psi(\mathrm{v}, \mathrm{u}) - \rho(v) + \rho(u) - J^0(\Omega u; \Omega\delta(u, v))
\end{aligned}$$

Furthermore, it is clear from (3.3) and the underlying assumption $\mathrm{H}_5(ii)$ that

$$\begin{aligned}
\inf_{m \in T, v \in \pi(m)} Q(m, v) &\geq \lim_{n \to \infty} \sup Q(t_n, u_n) \\
&\geq \lim_{n \to \infty} \inf Q(t_n, u_n) \\
&\geq Q(t, u).
\end{aligned}$$

So, by using Corollary 2.13, then (t, u) solves (OPMHEPC), the uniqueness is a direct result of the fact that (3.1). As a result, the argument is now complete.

Theorem 3.4. Let T and K be two non-empty, closed subsets of real B.S. E and S, resp., and let us assume that they are convex. If (OPMHEPC) is strongly w-p in the generalized sense, then:

$$\psi(\epsilon, c) \neq \emptyset \quad \forall c, \epsilon > 0, \qquad \lim_{(c,\epsilon) \to (0,0)} \mu[\psi(\epsilon, c)] = 0. \tag{4.4}$$

Moreover, if the assumptions $\mathrm{H}_1 - \mathrm{H}_4$ and $\mathrm{H}_5(ii)$ hold. Then the converse holds.

By the same method and technique with which we proved the Theorem (2.21), it is possible to prove the validity of the Theorem (3.4).

We will prove that under appropriate conditions, in the next finding, the w-p of (OPMHEPC) is equivalent to the existence and uniqueness of solution. This is something that we need to do in order to study the uniqueness of solutions to (OPMHEPC).

Theorem 3.5. Consider that T and K be a nonempty, closed and convex subsets of finte dimensional B.Ss E and S resp., Suppose that $\mathrm{H}_1 - \mathrm{H}_4$ are hold. Then (OPMHEPC) is a strongly w-p if and only if it has only one solution.

Proof. The necessity is very evident. Assume for the purpose of the sufficiency that the issue (OPMHEPC) has a single solution (t^*, u^*). That being the case,

$$\begin{cases} \mathrm{Q}(t^*, u^*) = \inf_{m \in T, v \in \pi(m)} Q(m, v) \\ \vartheta(t^*, u^*, v) + \rho(v) - \rho(u^*) + J^0\left(\Omega u^*; \Omega\delta(u^*, v)\right) \geq 0. \quad \forall\, (t^*, u^*) \in \mathrm{T} \times \mathrm{K} \end{cases} \tag{4.5}$$

Let $\{(t_n, u_n)\} \subset \mathrm{T} \times \mathrm{K}$ be approximating sequence for (OPMHEPC). Then, there is $\epsilon_n > 0$ accompanied by $\epsilon_n \to 0$ as $\mathrm{n} \to \infty$ such that:

$$\begin{cases} lim_{n \to \infty} sup\, Q(t_n, u_n) = \inf_{m \in T, e \in \pi(m)} Q(m, e) \\ \vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\left(\Omega u_n; \Omega\delta(u_n, v)\right) \geq -\epsilon_n \|v - u_n\|. \end{cases} \tag{4.6}$$

Our claim that $\{(t_n, u_n)\}$ is bounded. One can assume that, if not without losing generality, $\|(t_n, u_n)\| \to +\infty$. Let $r_n = \frac{1}{\|(t_n, u_n) - (t^*, u^*)\|}$, and

$$\begin{aligned} (s_n, z_n) \quad &= r_n(t_n, u_n) + (1 - r_n)(t^*, u^*) \\ &= [r_n t_n + (1 - r_n)t^*, r_n u_n + (1 - r_n)u^*]. \end{aligned}$$

Without loss of generality, we assume that $r_n \in (0,1)$ and $(s_n, z_n) \to (s, z)$ accompanied by $(s, z) \neq (t^*, u^*)$ since $\mathrm{E} \times \mathrm{S}$ is finite-dimensional, when the closedness and convexity of T and K are taken into consideration, one has $(s_n, z_n) \in \mathrm{T} \times \mathrm{K}$.

Consequently, assuming $\mathrm{H}_5(i)$ and (3.5), (3.6) for any $(s, z) \in \mathrm{T} \times \mathrm{K}$, we have

$$\mathrm{Q}(t^*, u^*) = \lim_{n\to\infty} sup\, r_n \mathrm{Q}(t_n, u_n) + \lim_{n\to\infty} sup(1 - r_n)\mathrm{Q}(t^*, u^*)$$

$$\geq \lim_{n\to\infty} sup[r_n \mathrm{Q}(t_n, u_n) + (1 - r_n)\mathrm{Q}(t^*, u^*)\,]$$

$$\geq \lim_{n\to\infty} sup\, \mathrm{Q}(s_n, z_n)$$

$$\geq \lim_{n\to\infty} inf\, \mathrm{Q}(s_n, z_n)$$

$$\geq \mathrm{Q}(s, z). \tag{4.7}$$

Moreover, following from conditions $\mathrm{H}_2(i, iv\ and\ v)$, $\mathrm{H}_4(ii)$ (3.5) and (3.6), we have

$$0 = \lim_{n\to\infty} inf\, r_n \epsilon_n \|v - u_n\|$$

$$\geq \lim_{n\to\infty} inf\, -r_n\big[\vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\left(\Omega u_n;\ \Omega\delta(u_n, v)\right)\big]$$

$$-\ (1 - r_n)\big[\vartheta(t^*, u^*, v) + \rho(v) - \rho(u^*) + J^0\left(\Omega u^*;\ \Omega\delta(u^*, v)\right)\big]$$

$$= \lim_{n\to\infty} inf[-r_n\vartheta(t_n, u_n, v) - (1 - r_n)\vartheta(t^*, u^*, v)] - r_n\, \rho(v) + r_n\rho(u_n)$$

$$-(1 - r_n)\rho(v) + (1 - r_n)\rho(u^*) - r_n\, J^0\left(\Omega u_n;\ \Omega\delta(u_n, v)\right) - (1 - r_n)$$

$$J^0\left(\Omega u^*;\ \Omega\delta(u^*, v)\right)$$

$$\geq \lim_{n\to\infty} inf \left[-\vartheta(t_n, z_n, v) - \rho(v) + \rho(z_n) - J^0 \left(\Omega z_n;\ \Omega\delta(z_n, v)\right)\right]$$

$$\geq \lim_{n\to\infty} inf\ [\vartheta(t_n, v, z_n) - \psi(z_n, v) - \psi(v, z_n) - \rho(v) + \rho(z_n)]$$

$$-J^0 \left(\Omega z_n;\ \Omega\delta(z_n, v)\right)$$

$$\geq \vartheta(t, v, z) - \psi(z, v) - \psi(v, z) - \rho(v) + \rho(z) - J^0 \left(\Omega z;\ \Omega\delta(z, v)\right).$$

Then,

$$\vartheta(t, v, z) - \psi(z, v) \quad \leq \psi(v, z) + \rho(v) - \rho(z) + J^0 \left(\Omega z;\ \Omega\delta(z, v)\right) \tag{4.8}$$

Now by Corollary (2.13), one can have

$$\vartheta(t, u, v) + \rho(v) - \rho(u) + J^0 \left(\Omega u;\ \Omega\delta(u, v)\right) \geq 0. \tag{4.9}$$

Hence, from (3.8), (3.9), that (s, z) solves (OPMHEPC) which is a contradiction, so $\{(t_n, u_n)\}$ is bounded. Let $\left\{\left(t_{n_i}, u_{n_i}\right)\right\}$ be any subsequence of $\{(t_n, u_n)\}$ in which $\left(t_{n_i}, u_{n_i}\right) \longrightarrow (t_0, u_0)$ as i $\longrightarrow \infty.$ It follows that

$$0 = \lim_{i\to\infty} \inf \epsilon_{n_i} \left\|v - u_{n_i}\right\|$$

$$\geq \lim_{i\to\infty} \inf\ [-\vartheta\left(t_{n_i}, u_{n_i}, v\right) - \rho(v) + \rho\left(u_{n_i}\right) - J^0 \left(\Omega u_{n_i};\ \Omega\delta\left(u_{n_i}, v\right)\right)]$$

$$\geq \lim_{i\to\infty} \inf [\vartheta\left(t_{n_i}, v, u_{n_i}\right) - \psi\left(u_{n_i}, v\right) - \psi\left(v, u_{n_i}\right) - \rho(v) + \rho\left(u_{n_i}\right) \tag{4.10}$$

$$- J^0 \left(\Omega u_{n_i};\ \Omega\delta\left(u_{n_i}, v\right)\right)]$$

$$\geq \vartheta(t_0, v, u_0) - \psi(u_0, v) - \psi(v, u_0) - \rho(v) + \rho(u_0)$$

$$- J^0 \left(\Omega u_0;\ \Omega\delta(u_0, v)\right).$$

Again by Corollary (2.15) we have $\forall(t, v) \in T \times K$

$$\vartheta(t_0, u_0, v) + \rho(v) - \rho(u_0) + J^0 \left(\Omega u_0;\ \Omega\delta(u_0, v)\right) \geq 0. \qquad (4.11)$$

It follows from (3.6) and $\mathrm{H}_5(ii)$ that

$$\lim_{m\in K, e\in I(m)} inf Q(m, e) \geq \lim_{i\to\infty} sup\, Q\left(t_{n_i}, u_{n_i}\right)$$

$$\geq \lim_{i\to\infty} inf\, Q\left(t_{n_i}, u_{n_i}\right)$$

$$\geq Q(t_0, u_0). \qquad (4.12)$$

From (3.11) and (3.12), (t_0, u_0) solves (OPMHEPC). Given the solution's uniqueness, $(t_0, u_0) = (t^*, u^*)$ and hence (t_n, u_n) converges to (t^*, u^*).Therefore, (OPMHEPC) is strong w-p.

Theorem 3.6. Consider that T and K are a nonempty, closed and convex subsets of finite dimensional Banach space E and K, respectively. If $\exists$ some $c > 0$ in which $\psi(c, c)$ is a nonempty and bounded and assuming $\mathrm{H}_1 - \mathrm{H}_5$ hold. Then (OPMHEPC) is strongly w-p in the generalized sense.

Proof. Consider that $\{(t_n, u_n)\} \subseteq \mathrm{T} \times \mathrm{K}$ be an approximating sequence of (OPMHEPC). Therefore, there is a nonnegative sequence $\{\epsilon_{\mathrm{n}}\}$ by $\epsilon_{\mathrm{n}} \to 0$ where $\mathrm{n} \to \infty$ in which

$$\begin{cases} \lim_{n\to\infty} \sup Q(t_n, u_n) \leq \inf_{m\in T, e\in\pi(m)} Q(m, e) \\ \vartheta(t_n, u_n, v) + \rho(v) - \rho(u_n) + J^0\left(\Omega u_n;\ \Omega\delta(u_n, v)\right) \geq -\epsilon_n\|v - u_n\|. \end{cases} \quad (4.13)$$

Since $\psi(c, c)$ is a nonempty and bounded. So, there is n_0 such that $(t_n, u_n) \in \psi(c, c)$ for each $\mathrm{n} \geq \mathrm{n}_0$. Consider that the boundedness of $\psi(c, c)$, there is a subsequence $\left\{\left(t_{n_i}, u_{n_i}\right)\right\}$ of $\{(t_n, u_n)\}$ in which $\left(t_{n_i}, u_{n_i}\right) \to (t_0, u_0)$ as $i \to \infty$. As a consequence of this, as demonstrated in Theorem 3.5,(t_0, u_0) solves (OPMHEPC). Then (OPMHEPC) is strongly w-p in the generalized sense.

5. CONCLUSION AND FUTURE WORKS

5.1 Concluding Remarks

In this thesis, inspired by the previous works, we investigate the w-p for $MHEP$. Under relatively weak conditions for the data ϑ, Ω and δ involving class of ψ-monotonicity (see Theorems 2.14 and 2.21), some metric characterizations of w-p of (OPMHEPC) are given. Our results generalize and improve many known results and can be applied to many other issues.

5.2 Future Works

There is no inclusive study whatever. Each study left out something and need further elaboration. Throughout our study in this thesis, we come up with the fact that there is much to say about well posedness in B.S and, particularly, monotonicity of bifunction. Throughout our study above, we have noticed that two we would like to further study in the future and these are as follows:

Establish the existence of results w-p of a system of (MHEP) and ψ – monotonicity.

It is possible to study applications of the equation in differential equations or in the fixed point theory.

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