RIEMANN BOUNDARY VALUE PROBLEM

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ABSTRACT

RIEMANN BOUNDARY VALUE PROBLEM

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We study the Riemann boundary value problem with finite and infinite index.

The basis of the solution of the problem with the finite index is the Plemelj-Sokhotski formula. The solution of the problem with infinite index depends not only on the Plemelj-Sokhotski formula but also on some results of the entire functions theory.

Keywords: Cauchy type integral, Hölder condition, Singular integral, Index, Verticity, Homogeneous Riemann boundary value problem, Non-homogeneous Riemann boundary value problem.

ÖZET

RİEMANN'IN SINIR DEĞER PROBLEMİ

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Sonlu ve sonsuz indeksli Riemann sınır değer problemini çalışıyoruz.

Sonlu indeksli problemin çözümünün esası Plemelj-Sokhotski formülüne bağlıdır. Sonsuz indeksli problemin çözümü sadece Plemelj-Sokhotski formülüne değil ayrıca tüm fonksiyonlar teorisinden bazı sonuçlara da bağlıdır.

Anahtar kelimeler: Cauchy tipi integral, Hölder koşulu. Tekil integral, İndeks, Vertisiti, Homojen Riemann sınır değer problemi, Homojen olmayan Riemann sınır değer problemi.

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Chapter 1

Introduction

Theory of boundary value problems for analytic functions is one of the most important branches of complex analysis. It has wide applications, because many practical problems in mechanics, physics and engineering may be transformed to such problems or singular integral equations which are closely related to the boundary value problems.

The classic statement of the Riemann boundary value problem consists of finding a function $\Phi(z)$ which is analytic and bounded in two domains D^+, D^-, with a common boundary - a smooth closed contour L. The function $\Phi(z)$ must admit continuous extension $\Phi^+(t), \Phi^-(t)$ onto L both from D^+ and D^- satisfying on L the boundary condition,

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),$$

where $G(t)$ and $g(t)$ are considered to be known functions satisfying the Hölder condition on L, $G(t) \neq 0$ for $t \in L$. When $g(t) \equiv 0$, the problem is called *homogeneous* and *non-homogeneous*, otherwise.

The main characteristic of the Riemann boundary value problem is the *index*,

$$æ = \mathrm{Ind}_L G(t),$$

defined as the increment of $\arg G(t)$ along L divided by 2π. In the classic statement of the Riemann boundary value problem the index æ is always finite. In the 1960's N.V. Govorov investigated the Riemann boundary value problem with infinite index. He obtained rather complete results under some special assumptions.

The content of this work includes both the classic Riemann boundary value problem (with finite index) and Riemann boundary value problem with infinite index. The work also includes some basic and necessary material for the problem.

The presented work is organized in the following way. In Chapter 2 we recall the necessary definitions and state (some of them with proofs) known results which we will use in the sequel. More specifically, we first explain the definition and some properties of Cauchy type integral. Then we give the definition of the Hölder condition and study some properties of functions satisfying the condition.

In Chapter 3, we prove the existence of boundary values of Cauchy type integrals under some suitable conditions and prove the Plemelj-Sokhotski formula. In this chapter, we also prove an important property of the limiting values, namely, if L is a smooth closed contour and the density of Cauchy type integral satisfies the Hölder condition, then its limiting values also satisfy Hölder condition.

Chapter 4 is devoted to the formulation and solution of the classic Riemann boundary value problem. After introducing the definition of index, we give solutions to the homogeneous and the non-homogeneous problems.

In Chapter 5, we deal with the Riemann problem with an infinite index when the verticity index is less than $1/2$. For the solution of the problem we first give the definition of canonical function and prove some properties of the canonical function. Then using these properties we solve the homogeneous problem. Finally, to solve the non-homogeneous Riemann boundary value problem with an infinite index when the verticity index is less than $1/2$, we first give an approach to the solution where we see the necessary conditions on $g(t)$ and $G(t)$. Then in the last section, we solve the non-homogeneous problem with infinite index.

Chapter 2

Integrals of Cauchy type

2.1 Definition of Cauchy type integral

Let L be a smooth [1] closed contour in the plane of complex variable z. Denote the domain within the contour L, i.e., the interior domain, with D^+ and the domain outside the contour L, i.e., exterior domain with D^-.

DEFINITION: *Let L be a smooth closed (or non-closed) contour and let $f(t)$ be a complex continuous function defined on L, then*

$$F(z) = \frac{1}{2\pi i} \int_L \frac{f(t)}{t - z} dt, \quad z \notin L, \tag{2.1}$$

is called the Cauchy type integral. The function $f(t)$ is called its density and $\frac{1}{t-z}$ its kernel.

The Cauchy type integral is analytic in the variable z in $D^+ \cup D^- = \mathbb{C} \backslash L$. The proof consists in establishing the possibility of differentiation w.r.t. the

[1] By a smooth contour we understand here and hereafter a simple (i.e., without points of intersection with itself), closed (or non-closed) contour with continuously varying tangent, having no cusps.

variable z in the integrand. We shall prove the corresponding result in a more general case and this implies the analyticity of the Cauchy type integral as a particular case.

THEOREM 2.1 *Let L be a smooth contour (closed or non-closed) and let G be a domain. Let $f(\tau, z)$ be a function continuous w.r.t. $(\tau, z) \in L \times G$ and analytic in $z \in G$ for all values τ and the derivative of $f(\tau, z)$ w.r.t. z, i.e., $f_z(\tau, z)$ is continuous w.r.t. (τ, z). Then the function*

$$F(z) = \int_L f(\tau, z) d\tau, \quad z \in G,$$

is analytic in z and moreover

$$F'(z) = \int_L f_z(\tau, z) d\tau, \quad z \in G.$$

Proof: Observe that since $f_z(\tau, z)$ is continuous w.r.t. τ,

$$\begin{aligned} \frac{F(z+\Delta z) - F(z)}{\Delta z} &- \int_L f_z(\tau, z) d\tau \\ &= \int_L \left(\frac{f(\tau, z+\Delta z) - f(\tau, z)}{\Delta z} - f_z(\tau, z) \right) d\tau \quad (2.2) \\ &= \int_L \left(\frac{1}{\Delta z} \int_z^{z+\Delta z} (f_z(\tau, \xi) - f_z(\tau, z)) d\xi \right) d\tau \quad (2.3) \end{aligned}$$

Let

$$M := \max_{z \in L} \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f_z(\tau, \xi) - f_z(\tau, z)) d\xi \right|. \quad (2.4)$$

Then

$$\left| \int_L \left(\frac{1}{\Delta z} \int_z^{z+\Delta z} (f_z(\tau, \xi) - f_z(\tau, z)) d\xi \right) d\tau \right| \leq M \cdot \textit{length of } L. \quad (2.5)$$

Since $f_z(\tau, z)$ is continuous w.r.t. (τ, z) , i.e., $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|f_z(\tau, \xi) - f_z(\tau, z)| < \epsilon$ whenever $|\xi - z| < \delta$ $\forall \tau \in L$. For $|\Delta z| < \delta$, since

$|\xi - z| \leq |\Delta z| < \delta$, we have

$$\left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f_z(\tau, \xi) - f_z(\tau, z)) d\xi \right| \leq \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon.$$

Turning back to (2.5), we get

$$\left| \int_L \frac{1}{\Delta z} \int_z^{z+\Delta z} (f_z(\tau, \xi) - f_z(\tau, z)) d\xi d\tau \right| \leq \epsilon \cdot \textit{length of } L. \tag{2.6}$$

So as $|\Delta z| \to 0$; LHS of (2.6) goes to 0 as well. But then LHS of (2.2) goes to $F'(z) - \int_L f_z(\tau, z) d\tau$ as $\Delta z \to 0$, which gives us that $F'(z) = \int_L f_z(\tau, z) d\tau$. Hence $F(z)$ is analytic w.r.t. z. $\square$

Thus, $F(z)$ is a function analytic in z everywhere in the domain of analyticity of $f(\tau, z)$.

If L is a non-closed contour, then the Cauchy type integral, $F(z)$, is a function analytic in the whole plane, whose possible singularities may lie on L. If L is a closed contour, then by definition $F(z)$ splits into two independent functions; $F^+(z)$ defined in D^+ and $F^-(z)$ defined in D^-. These functions are not generally analytic continuation of each other.

An analytic function $F(z)$ defined in two disjoint domains D^+, D^-, by two expressions $F^+(z)$, $F^-(z)$, is generally called a *sectionally analytic function.*

We now note an important property of the Cauchy type integral: the function $F(z)$ represented by the Cauchy type integral vanishes at infinity.

In fact, let us expand $F(z)$ in the neighborhood of infinity into series w.r.t. the powers of $1/z$. Noting that

$$\frac{1}{\tau - z} = -\frac{1}{z} - \frac{\tau}{z^2} - \dots - \frac{\tau^{n-1}}{z^n} + \dots,$$

and multiplying the series by $\frac{1}{2\pi i} f(\tau)$ and integrating term by term, we get

$$F^-(z) = \sum_{k=1}^{\infty} \frac{c_k}{z^k}, \quad \text{for} \quad |z| > \max_{\tau \in L} |\tau|,$$

where

$$c_k = -\frac{1}{2\pi i}\int_L \tau^{k-1}F(\tau)d\tau.$$

Since in the series the zero power is absent, $F^-(\infty) = 0$.

2.2 Hölder condition

Before dealing with the problem of the behavior of the Cauchy type integral, we consider a very important class of functions for our problem: functions satisfying Hölder condition.

DEFINITION: *Let $f(t)$ be defined on a smooth curve L. If*

$$|f(t_2) - f(t_1)| \leq A|t_2 - t_1|^\lambda \quad (0 < \lambda \leq 1),$$

for arbitrary points t_1, t_2 on L, where A and λ are positive constants, then $f(t)$ is said to satisfy Hölder condition with the Hölder index λ and the Hölder constant A. The class of all such functions is denoted by $H_\lambda(L)$. (Sometimes we will denote it just with H_λ or with $H(L)$ or with H.).

The reason for the restriction $0 < \lambda \leq 1$ in the above definition is: if $\lambda = 0$, then $f(t)$ is a bounded function, which is too general for us; if $\lambda > 1$, then $f(t)$ is constant.

Indeed, if $\lambda > 1$, then

$$\left|\frac{f(t_2) - f(t_1)}{t_2 - t_1}\right| \leq \mathrm{A}|t_2 - t_1|^{\lambda-1},$$

where A is a constant, $(\lambda - 1) > 0$. As $t_2 \to t_1$ in both sides, we obtain $f'(t_1) = 0, \quad \forall t_1 \in L$, which implies that $f(t)$ is constant.

Also, specifically, if $f(t) \in H_1$, we say that $f(t)$ satisfy *Lipschitz condition.*

Now let us list some properties of functions in class H:

1) *If $f(t) \in H$, then $f(t)$ is continuous.*

2) *If $f(t) \in H_\alpha(L)$ and $0 \leq \beta \leq \alpha \leq 1$, then $f(t) \in H_\beta(L)$, i.e., $H_\alpha \subset H_\beta$ if $\beta \leq \alpha$.*

3) *If $f(t)$ is differentiable and has a finite derivative, i.e., $f'(t)$ is bounded, then it satisfies Lipschitz condition.*

4) *If $f_1(t)$ and $f_2(t)$ satisfy Hölder condition in L with indices λ_1, λ_2 respectively, then their sum, product and quotient (under the condition that the denominator does not vanish) satisfy the Hölder condition with the index $\lambda = \min(\lambda_1, \lambda_2)$.*

5) *If $f_1(t)$ and $f_2(t)$ satisfy Hölder condition with indices λ_1, λ_2 respectively, then their composition satisfies Hölder condition with the index $\lambda = \lambda_1 \cdot \lambda_2$.*

6) *If $L = \sum_{j=1}^{n} L_j$ with L_j's (finite in number) non-intersecting to each other and $f(t) \in H_\lambda(L_j)$, then $f(t) \in H_\lambda(L)$.*

7) *If $f(t)$ is defined on L which may be separated into a finite number of arcs L_j such that $L = \sum_{j=1}^{n} L_j$ and if $f(t) \in H_\lambda(L)$, then $f(t) \in H_\lambda(L_j)$ for all $j = 1, ..., n$.*

We will just give the proofs of the second, fourth and fifth properties and the rest are easy to derive from the definition.

For the proof of property 2), assume $f(t) \in H_\alpha(L)$ and $0 < \beta \leq \alpha \leq 1$. Then for any $t_1, t_2 \in L$, $|t_1 - t_2| < 1$, we have

$$|f(t_1) - f(t_2)| \leq A|t_1 - t_2|^\alpha \leq A|t_1 - t_2|^\beta,$$

where A is the Hölder constant. If $|t_1 - t_2| \geq 1$, then

$$|f(t_1) - f(t_2)| \leq (|f(t_1)| + |f(t_2)|)|t_1 - t_2|^\beta \leq 2K|t_1 - t_2|^\beta,$$

where $K = \sup |f(t)|$. So $f(t) \in H_\beta(L)$.

To prove property 4), first observe that if $f_1(t) \in H_{\lambda_1}(L)$ and $f_2(t) \in H_{\lambda_2}(L)$, then by property 2) both $f_1(t)$ and $f_2(t)$ satisfy Hölder condition with the index $\lambda = \min(\lambda_1, \lambda_2)$. Then for any $t_1, t_2 \in L$

$$\begin{aligned} |f_1(t_1) + f_2(t_1) - f_1(t_2) + f_2(t_2)| &\leq |f_1(t_1) - f_1(t_2)| + |f_2(t_1) - f_2(t_2)| \\ &\leq A|t_1 - t_2|^\lambda + B|t_1 - t_2|^\lambda \\ &\leq (A + B)|t_1 - t_2|^\lambda, \end{aligned}$$

where A, B are Hölder constants. Hence we proved the property for the sum. For the product, observe again since $f_1(t), f_2(t) \in H_\lambda(L)$, for any $t_1, t_2 \in L$,

$$\begin{aligned} |f_1(t_1)f_2(t_1) - f_1(t_2)f_2(t_2)| &\leq |f_1(t_1)f_2(t_1) - f_1(t_1)f_2(t_2)| \\ &\quad + |f_1(t_1)f_2(t_2) - f_1(t_2)f_2(t_2)| \\ &\leq |f_1(t_1)||f_2(t_1) - f_2(t_2)| \\ &\quad + |f_2(t_2)||f_1(t_1) - f_1(t_2)| \\ &\leq A_1|f_2(t_1) - f_2(t_2)| + A_2|f_1(t_1) - f_1(t_2)| \\ &\leq A_1 B|t_1 - t_2|^\lambda + A_2 B|t_1 - t_2|^\lambda \leq C|t_1 - t_2|^\lambda, \end{aligned}$$

where A_1 and A_2 are upper bounds and A and B are Hölder constants of $|f_1(t)|$ and $|f_2(t)|$ on L respectively and $C = A_1 B + A_2 A$. Hence we proved the property for the product as well.

The proof of the property for the quotient is similar to the proof for the product.

For the proof of 5), for any $t_1, t_2 \in L$, we have

$$\begin{aligned} |f_1(f_2(t_1)) - f_1(f_2(t_2))| &\leq A|f_2(t_1) - f_2(t_2)|^{\lambda_1} \\ &\leq AB^{\lambda_1}(|t_1 - t_2|^{\lambda_2})^{\lambda_1} = AB^{\lambda_1}|t_1 - t_2|^{\lambda} \end{aligned}$$

where $\lambda = \lambda_1 \cdot \lambda_2$ and A and B are Hölder constants of f_1 and f_2 respectively. Hence we proved that $f_1 \circ f_2(t) \in H_\lambda$. Similarly, $f_2 \circ f_1(t) \in H_\lambda$ as well.

2.3 Principal value of the Cauchy type integral

2.3.1 Principal value of singular integral on straight line

The integral

$$\int_a^b \frac{f(x)}{x-c}dx \qquad (a < c < b),$$

is called a *singular integral.*

It can be defined in the following way:

$$\int_a^b \frac{f(x)}{x-c}dx = \lim_{\epsilon_1, \epsilon_2 \to 0} \left[-\int_a^{c-\epsilon_1} \frac{f(x)}{c-x}dx + \int_{c+\epsilon_2}^b \frac{f(x)}{c-x}dx \right]. \tag{2.7}$$

It is evident (e.g. for $f(x) = 1$) that the limit of the expression may depend on the manner in which ϵ_1 and ϵ_2 tend to zero. Consequently, the integral as improper integral does not exist. It can be given a meaning if a dependence between ϵ_1 and ϵ_2 be established. Assuming that the internal cut out is situated symmetrically w.r.t. the point c, i.e.,

$$\epsilon_1 = \epsilon_2 = \epsilon, \tag{2.8}$$

we are led to the concept of the Cauchy principal value of the improper integral.

DEFINITION: *The Cauchy principal value of the singular integral*

$$\int_a^b \frac{f(x)}{x-c}dx \quad (a<c<b),$$

is the expression

$$\lim_{\epsilon\to 0}\left[\int_a^{c-\epsilon}\frac{f(x)}{x-c}dx+\int_{c+\epsilon}^b\frac{f(x)}{x-c}dx\right].$$

For example, taking into account formula (2.7) and the condition (2.8) we have

$$\int_a^b \frac{dx}{x-c}=\ln\frac{b-c}{c-a}. \tag{2.9}$$

Now consider the integral

$$\int_a^b \frac{f(x)}{x-c}dx,$$

where $f(x)\in H((a,b))$. By representing the integral in the form

$$\int_a^b \frac{f(x)}{x-c}dx=\int_a^b \frac{f(x)-f(c)}{x-c}dx+f(c)\int_a^b\frac{dx}{x-c},$$

we see that in the view of Hölder condition

$$\left|\frac{f(x)-f(c)}{x-c}\right|<\frac{A}{|x-c|^{1-\lambda}},$$

the first integral exists as improper and the second is same with (2.9). So, in fact what we have is the following:

The singular integral

$$\int_a^b \frac{f(x)}{x-c}dx,$$

where $f(x) \in H((a,b))$, exists in the sense of the Cauchy principal value, and it is equal to

$$\int_a^b \frac{f(x)}{x-c}dx = \int_a^b \frac{f(x)-f(c)}{x-c}dx + f(c)\ln\frac{b-c}{c-a}.$$

2.3.2 Principal value of singular curvilinear integral

Let L be a smooth contour and τ, t be complex coordinates of its points. We would like to define the singular integral

$$\int_L \frac{\varphi(\tau)}{\tau - t}d\tau.$$

If σ, s are the lengths of the arcs from a fixed origin of integration to the points τ, t respectively and $\tau = \tau(\sigma)$ is the equation of the contour in the complex form, then substituting in the integral $\tau = \tau(\sigma)$, $t = t(s)$, $d\tau = \tau'(\sigma)d\sigma$ we would reduce it to singular integrals along straight line. Subsequently, we could make full use of the results of the preceding section, without resorting to new considerations. However, it is useful to consider the curvilinear singular integral independently.

Let us describe a circle around the point t of L, t as center and ϱ as the radius of the circle. Let the radius be sufficiently small, so that the circle has only two points of intersection with the curve L, say t_1 and t_2, i.e.,

$$|t_1 - t| = |t_2 - t| = \varrho. \tag{2.10}$$

Denote by l_ϱ the part of the contour L cut out by the circle. Then we have the following definition.

DEFINITION: *The limit of the integral*

$$\int_{L-l_\varrho} \frac{\varphi(\tau)}{\tau - t}d\tau,$$

as $\varrho \to 0$ is called the principal value of the singular integral

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau,$$

if it exists.

For the existence we first consider the simplest case

$$\int_L \frac{d\tau}{\tau - t}.$$

Define

$$\ln(\tau - t) = \ln|\tau - t| + i\arg(\tau - t),$$

where the plane cut is made on the right of the curve, i.e., taken along the straight line parallel to the x-axis and connecting the point t and ∞. Then together with (2.10),

$$\begin{aligned}\int_{L-l_\varrho} \frac{d\tau}{\tau - t} &= \ln(\tau - t)\Big|_a^{t_1} + \ln(\tau - t)\Big|_{t_2}^{b} \\ &= \ln\left(\frac{b-t}{a-t}\right) + i\{\arg(t_1 - t) - \arg(t_2 - t)\}.\end{aligned}$$

Since $\{\arg(t_1 - t) - \arg(t_2 - t)\} \to \pi$ as $\varrho \to 0$, then we have

$$\int_L \frac{d\tau}{\tau - t} = \lim_{l\to 0} \int_{L-l} \frac{d\tau}{\tau - t} = \ln\left(\frac{b-t}{a-t}\right) + i\pi.$$

If the contour is closed, in this case $a = b$, and we have

$$\int_L \frac{d\tau}{\tau - t} = i\pi.$$

Now let us examine the singular integral

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau,$$

where $\varphi(\tau)$ satisfies the Hölder condition on L. Representing it in the form

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + \varphi(t) \int_L \frac{d\tau}{\tau - t}, \qquad (2.11)$$

and repeating the reasoning of the preceding section we find that *the singular integral*

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau,$$

where the function $\varphi(\tau)$ satisfies the Hölder condition, exists in the sense of Cauchy principal value.

The integral can be represented in the following form:

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + \varphi(t) \left[\ln \frac{b-t}{a-t} + i\pi \right].$$

When we have a closed contour, then setting $a = b$ in the above equation we get

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + i\pi\varphi(t).$$

From now on by a singular integral, we will understand its principal value.

2.3.3 Properties of the singular integral

It is quite obvious that a singular integral possesses some of the properties of the ordinary integral, namely integral of a sum is equal to the sum of integrals and a constant factor may be taken outside integral sign. Moreover we have the following important remarks.

In introducing the concept of the principal value it was emphasized as the most significient fact the vicinity cut out is taken symmetrically w.r.t. the point under investigation. In fact, in Section (2.3.1) the significant requirement was not that $\epsilon_1 = \epsilon_2$ (2.8), but that

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{\epsilon_1}{\epsilon_2} = 1,$$

exactly as in section(2.3.2) where the significant assumption is not that the points t_1, t_2 lie on the same circle centered at t, but that the following condition is satisfied:

$$\lim_{t_1, t_2 \to t} \left| \frac{t_2 - t}{t_1 - t} \right| = 1.$$

We also have the following results for the singular integral.

THEOREM:(The rule of change of variable) *If the function* $\tau = \alpha(\zeta)$ *has a continuous first derivative* $\alpha'(\zeta)$ *which does not vanish anywhere, and constitutes a one to one mapping of the contour* L *onto a contour* L', *then*

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \int_{L'} \frac{\varphi[\alpha(\zeta)]\alpha'(\zeta)}{\alpha(\zeta) - \alpha(\xi)} d\zeta,$$

where $t = \alpha(\xi)$.

THEOREM:(Integration by parts) *If* $\varphi(\tau)$ *is a continuously differentiable function and the point* t *does not coincide with an end of the contour* L *(*a *or* b*), then the following formula of the integration by parts is valid:*

$$\int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \pm i\pi\varphi(t) + \varphi(b)\ln(b - t) - \varphi(a)\ln(a - t) - \int_L \varphi'(\tau)\ln(\tau - t) d\tau.$$

For the proofs, we refer Gakhov's book [1], chapter 3, section 4.

Chapter 3

Plemelj-Sokhotski Formula

3.1 Preliminary facts

In this chapter we will consider the existence problem of the boundary values for the Cauchy type integral. We shall prove that they actually exist when $f(t) \in H$ and are closely related to the Cauchy principal value integral. In detail, we will have the following theorem, which is one of the basic.

THEOREM:(Plemelj-Sokhotski Formula) *If L is an arcwise smooth curve and $f(t) \in H(L)$, then, for any point $t_0 \in L$ (t_0 is not an end when L is non-closed), the boundary values of the Cauchy type integral (2.1) exist and the following formula holds:*

$$F^+(t_0) = (1 - \frac{\theta_0}{2\pi})f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)}{t - t_0}dt, \quad t_0 \in L \tag{3.1}$$

$$F^-(t_0) = -\frac{\theta_0}{2\pi}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)}{t - t_0}dt, \quad t_0 \in L, \tag{3.2}$$

where $F^+(t_0)$ and $F^-(t_0)$ represent the limit values of $F(t)$ in (2.1) when $z \to t_0$ from the positive and the negative sides of L respectively, while θ_0

is the angle spanned by the two one-sided tangents of L at t_0 towards the positive sides of L $(0 \leq \theta_0 \leq 2\pi)$.

Observe that by adding and subtracting (3.1) and (3.2), we have the following two formulae:

$$\begin{aligned} F^+(t_0) - F^-(t_0) &= f(t_0), \\ F^+(t_0) + F^-(t_0) &= (1 - \frac{\theta_0}{\pi}) f(t_0) + \frac{1}{\pi i} \int_L \frac{f(t)}{t - t_0} dt. \end{aligned}$$

In particular, if t_0 is a smooth point on L, or L is smooth everywhere, then (3.1) and (3.2) are simplified to

$$F^\pm(t_0) = \pm\frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int_L \frac{f(t)}{t - t_0} dt, \quad t_0 \in L \ . \tag{3.3}$$

From (3.3) we have the following formulae :

$$F^+(t_0) - F^-(t_0) = f(t_0), \tag{3.4}$$

$$F^+(t_0) + F^-(t_0) = \frac{1}{\pi i} \int_L \frac{f(t)}{t - t_0} dt. \tag{3.5}$$

Formula (3.3) is the basic tool for studying Riemann boundary value problems.

The classical proof of this theorem is rather complicated. The simpler proof given below has been taken from J.Lu's book [4], chapter 1, section 2.

We introduce a symbol beforehand. Let L be a continuous curve on the complex plane. For an arbitrary point z in the plane, denote the point on L nearest to z by z_L (if such points are more then one in number, z_L may be any one of them). In particular, if $z = t \in L$, then $t_L = t$. Obviously, $z_L \to t_0$ as $z \to t_0 \in L$. Moreover, for an arbitrary point $t \in L$, we always have

$$|t - z_L| \leq |t - z| + |z - z_L| \leq 2|t - z|. \tag{3.6}$$

If $f(t) \in H_\alpha(L)$, then it follows at once from (3.6)

$$|f(t) - f(z_L)| \leq A|t - z_L|^\alpha \leq 2^\alpha A|t - z|^\alpha. \tag{3.7}$$

3.2 Proof of Plemelj-Sokhotski formula

For the proof of the Plemelj-Sokhotski formula, we need the following lemmas.

LEMMA 3.1 *(see, e.g.,[4], p.11)Assume L is a smooth curve. For two arbitrary points*
$t = t(s)$ and $t_0 = t(s_0)$, we have

$$|t - t_0|/|\widehat{tt_0}| = |t - t_0|/|s - s_0| \to 1 \tag{3.8}$$

uniformly as $s \to s_0$ ($|\widehat{tt_0}|$ denotes the length of arc $\widehat{tt_0}$ of L).

This lemma means that the length of the arc and the chord between two neighboring points on a smooth curve are uniformly equivalent infinitesimals.

With the help of this lemma, we can get the following important fact.

LEMMA 3.2 *Assume L is a smooth curve. There exists a positive constant C $(0 < C < 1)$ such that the following inequalities are valid for any two points t_0 and t on L:*

$$C|\widehat{t_0t}| \leq |t - t_0| \leq |\widehat{t_0t}|. \tag{3.9}$$

Proof: The inequality on the RHS of (3.9) is trivial. Let us prove the LHS of it.

First assume L is a non-closed arc. Then $|\widehat{tt_0}| = |s - s_0|$. By the previous lemma there exists $\delta > 0$ such that

$$|t - t_0|/|\widehat{tt_0}| = |t - t_0|/|s - s_0| \geq 1/2, \tag{3.10}$$

when $|t - t_0| \leq \delta$. On the other hand,

$$|t - t_0|/|\widehat{tt_0}| = |t - t_0|/|s - s_0| \geq \delta/L, \tag{3.11}$$

when $|t - t_0| \geq \delta$. Therefore for $C = \min(1/2, \delta/L)$, we have RHS of (3.9).

Next assume that L is closed. Remembering that $\widehat{tt_0}$ in (3.9) is always regarded as the shortest arc on L, we have $|\widehat{t_0 t}| = |s - s_0|$ when $|t - t_0| \leq \delta$ for sufficiently small δ so that (3.10) is valid for such δ. Moreover, (3.11) is also valid when $|t - t_0| \geq \delta$ provided that the denominator L in it is replaced by $L/2$. $\square$

The following lemma is the main lemma for the proof of Plemelj-Sokhotski formula.

LEMMA 3.3 *Assume L is an arc-wise smooth curve and $f(t) \in H_\alpha(L)$ $(0 < \alpha \leq 1)$. If l is a subarc of L (its length is denoted by l also), then, for any point z in $\mathbb{C}\backslash l$, we have*

$$\left| \int_l \frac{f(t) - f(z_L)}{t - z} dt \right| \leq M l^\alpha, \tag{3.12}$$

where M is a constant independent of L and z.

Proof: Let $l = \widehat{ab}$. Denote the arc length parameter of any point t on L by s_t. We may assume $s_a < s_b$.

First assume l is a smooth arc. As L consists of a finite number of smooth arcs, (3.12) is valid on all such arcs and C, that will be used now, may be taken as the same, i.e., C is independent of L. By (3.6) and (3.7),

$$\begin{aligned}
\left|\int_l \frac{f(t)-f(z_L)}{t-z}dt\right| &\leq 2^\alpha A\int_l \frac{|dt|}{|t-z|^{1-\alpha}} \\
&\leq 2A\int_l \frac{|dt|}{|t-z_L|^{1-\alpha}} \\
&\leq \frac{2A}{C^{1-\alpha}}\int_l \frac{ds}{|s-s_{z_L}|^{1-\alpha}} \\
&= \frac{2A}{C^{1-\alpha}}\left\{\int_{s_a}^{s_{z_L}} \frac{ds}{(s_{z_L}-s)^{1-\alpha}} \int_{s_{z_L}}^{s_b} \frac{ds}{(s-s_{z_L})^{1-\alpha}}\right\} \\
&= \frac{2A}{\alpha C^{1-\alpha}}\{(s_{z_L}-s_a)^\alpha+(s_b-s_{z_L})^\alpha\} \leq Ml^\alpha,
\end{aligned}$$

where $|t-z_L| \geq C|s-s_{z_L}|$ follows from lemma 3.2. Here, $M = \frac{2A}{\alpha C^{1-\alpha}}$ and $l^\alpha = (s_{z_L}-s_a)^\alpha+(s_b-s_{z_L})^\alpha$. Hence, we have the lemma. □

Proof of Plemelj-Sokhotski Formula: First assume L is closed, oriented counter clockwisely. Denote the interior and the exterior regions by D^+ and D^- respectively.

Rewrite (2.1) as

$$\begin{aligned}
F(z) &= \frac{1}{2\pi i}\int_L \frac{f(t)-f(z_L)}{t-z}dt + \frac{f(z_L)}{2\pi i}\int_L \frac{dt}{t-z} \\
&=: I_1(z)+I_2(z), \quad z \notin L.
\end{aligned}$$

Since $f(t) \in H(L)$, so

$$\lim_{z\to t_0} I_2(z) = f(t_0)\lim_{z\to t_0}\frac{1}{2\pi i}\int_L \frac{dt}{t-z}.$$

By the Residue theorem,

$$\frac{1}{2\pi i}\int_L \frac{dt}{t-z} = \begin{cases} 1, & z \in D^+, \\ 0, & z \in D^-, \end{cases}$$

from which it follows at once

$$I_2^+(t_0) = f(t_0), \quad I_2^-(t_0) = 0.$$

Now let us prove

$$\lim_{z\to t_o} I_1(z) = \frac{1}{2\pi i}\int_L \frac{f(t)-f(t_0)}{t-t_0}dt. \tag{3.13}$$

Note that

$$\begin{aligned}
&\left|I_1(z) - \int_L \frac{f(t)-f(t_0)}{t-t_0}dt\right| \\
&\le \frac{1}{2\pi}\left|\int_{L-L_\eta} \frac{f(t)-f(z_L)}{t-t_0}dt - \int_{L-L_\eta}\frac{f(t)-f(t_0)}{t-t_0}dt\right| \\
&\quad + \frac{1}{2\pi}\left|\int_{L_\eta} \frac{f(t)-f(z_L)}{t-t_0}dt\right| + \left|\int_{L_\eta}\frac{f(t)-f(t_0)}{t-t_0}dt\right| \\
&=: \frac{1}{2\pi}(\triangle_1+\triangle_2+\triangle_3),
\end{aligned}$$

where η is a sufficiently small positive number. Assume $f(t)\in H_\alpha(L)$. By the lemma (3.1),

$$\triangle_2 \le M\eta^\alpha, \quad \triangle_3 \le M_1\eta^\alpha.$$

Hence for arbitrary given $\epsilon > 0$, η may be taken so small that $\triangle_2, \triangle_3 < \epsilon/3$. Fix such an η and restrict $|z-t_0| < \eta/2$.

When $t \in L\backslash L_\eta$ we have $|t-t_0| > \eta$ and hence $|t-z| > \eta/2$. Therefore

$$\begin{aligned}
\triangle_1 &= \\
&= \left|\int_{L-L_\eta}\frac{f(t)-f(t_0)}{t-z}dt + \int_{L-L_\eta}\frac{f(t)-f(z_L)}{t-z}dt - \int_{L-L_\eta}\frac{f(t)-f(t_0)}{t-t_0}dt\right| \\
&\le \int_{L-L_\eta}\frac{|f(t)-f(t_0)||z-t_0|}{|t-z||t-t_0|}|dt| + \int_{L-L_\eta}\frac{|f(t_0)-f(z_L)|}{|t-z|}|dt| \\
&\le \frac{8M_1L}{\eta^2}|z-t_0| + \frac{2L}{\eta}|f(t_0)-f(z_L)|,
\end{aligned}$$

where $M_1 = \max|f(t)|$. Since $f(z_L)\to f(t_0)$ as $z\to t_0$, RHS of the above inequality is less then $\epsilon/3$, when z is sufficiently close to t_0. Thus (3.13) is proved. Therefore,

$$I_1^+(t_0) = \frac{1}{2\pi i}\int_L \frac{f(t)}{t-t_0}dt - \frac{f(t_0)}{2\pi i}\int_L \frac{dt}{t-t_0}.$$

Formulas (3.1) and (3.2) are obtained by using the result about $I_2^+(t_0)$.

Next, assume $L = \widehat{ab}$ is an arcwise smooth non-closed arc. We may add a smooth arc L' to L such that $L \cup L'$ becomes an arcwise smooth closed contour. At the same time, we extend $f(t)$ on L' s.t. $f(t) \in H(L \cup L')$. Hence, by the result already proved, we have the theorem also for the case when L is non-closed. $\square$

THEOREM 3.1 *Assume L is an arcwise smooth curve and $f(t) \in H(L)$. Then for any $t_0 \in L$ (including its ends),*

$$\lim_{z\to t_0} \frac{1}{2\pi i}\int_L \frac{f(t)-f(t_0)}{t-z}dt = \frac{1}{2\pi i}\int_L \frac{f(t)-f(t_0)}{t-t_0}dt, \quad z \notin L. \tag{3.14}$$

Proof: Since

$$\frac{1}{2\pi i}\int_L \frac{f(t)-f(t_0)}{t-z}dt = \frac{1}{2\pi i}\int_L \frac{f(t)-f(z_L)}{t-z}dt + \left(\frac{f(z_L)-f(t_0)}{2\pi i}\right)\int_L \frac{dt}{t-z},$$

the second term on the RHS tends to zero as $z \to t_0$ and the first one tends to the RHS of (3.14), by (3.13) (the proof of which is effective also for non-closed L), so (3.14) follows. $\square$

3.3 An important property of the boundary values of Cauchy type Integral

Having proved the existence of the limiting values, now we can give an important property of them which will be very useful later on.

THEOREM 3.2 *If L is a smooth closed contour and $f(t) \in H_\lambda(L)$, then $F^+(t), F^-(t) \in H_\lambda(L)$ when $\lambda < 1$ and $F^+(t), F^-(t) \in \bigcup_{\alpha<1} H_\alpha(L)$ when $\lambda = 1$.*

Proof: It follows from the relation (2.11) that it is sufficient to prove the theorem stated above for the function

$$F_1(t) := \frac{1}{2\pi i}\int_L \frac{f(\tau)-f(t)}{\tau - t}d\tau.$$

For this let us estimate the expression

$$|F_1(t_2) - F_1(t_2)| = \left|\frac{1}{2\pi i}\int_L \left(\frac{f(\tau)-f(t_2)}{\tau - t} - \frac{f(\tau)-f(t_1)}{\tau - t_1}\right) d\tau\right|, \qquad (3.15)$$

for two arbitrary, sufficiently close points t_1 and t_2.

We describe a circle of radius δ centered at t_1 such that it intersects L at two points a and b. The part of the contour L lying within this circle is denoted by l. Let t_2 be an arbitrary fixed point on the arc l other then points a and b. We set $\delta = k|t_2 - t_1|$ where $k > 1$.

Denote by $s = s(t,\tau)$ the length of the smallest of the two arcs of the contour L, with the ends t and τ.

From the smoothness of the contour L, for any two points t_1 and t_2 we can write

$$s(t_1,t_2) \leq m|t_1 - t_2|, \qquad (3.16)$$

where m is a positive constant.

Let us cut off an arc l of the contour L, taking on both sides of the point t_1 equal arcs of lengths $2s(t_1,t_2)$. The ends of this arc will be denoted by a and b. The integral (3.15), that we are interested in, can be represented as follows:

$$\begin{aligned} F_1(t_2) - F_1(t_2) &= \frac{1}{2\pi i}\int_l \frac{f(\tau)-f(t_2)}{\tau - t_2}d\tau - \frac{1}{2\pi i}\int_l \frac{f(\tau)-f(t_1)}{\tau - t_1}d\tau \\ &+\frac{1}{2\pi i}\int_{L\setminus l}\left\{\frac{f(\tau)-f(t_2)}{\tau - t_2} - \frac{f(\tau)-f(t_1)}{\tau - t_1}\right\} d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_l \frac{f(\tau)-f(t_2)}{\tau-t_2}d\tau - \frac{1}{2\pi i}\int_l \frac{f(\tau)-f(t_1)}{\tau-t_1}d\tau \\
&\quad + \int_{L\setminus l} \frac{f(t_1)-f(t_2)}{\tau-t_1}d\tau \\
&\quad + \int_{L\setminus l} \frac{[f(\tau)-f(t_2)](t_2-t_1)}{(t_2-t_1)(\tau-t_2)}d\tau \\
&=: I_1+I_2+I_3+I_4.
\end{aligned}
$$

We can estimate I_2 in the following way:

$$
\begin{aligned}
|I_2| &\leq \frac{1}{2\pi}\int_l \left|\frac{f(\tau)-f(t_1)}{\tau-t_1}\right| |d\tau| \leq C_1 \int_l \frac{|d\tau|}{|\tau-t_1|^{1-\lambda}} \\
&\leq C_2 \int_0^{C|t_2-t_1|} \frac{ds}{s^{1-\lambda}} \leq C_3 s^\lambda(t_1,t_2) \leq A_1|t_1-t_2|^\lambda,
\end{aligned}
$$

where A_1 and C_i, $i=1,2,3$, are positive constants.

In an similar way

$$|I_1| \leq C_5|t_2-t_1|^\lambda,$$

where C_5 is a positive constant.

For the integral I_3 we have

$$|I_3| \leq \frac{|f(t_1)-f(t_2)|}{2\pi}\left|\int_{L\setminus l}\frac{d\tau}{\tau-t_1}\right| \leq \frac{A|t_2-t_1|^\lambda}{2\pi}\left|\int_{L\setminus l}\frac{d\tau}{\tau-t_1}\right|,$$

where A is the Hölder constant.

The last integral can be directly computed, namely

$$\int_{L\setminus l}\frac{d\tau}{\tau-t_1} = \ln\frac{a-t_1}{b-t1}.$$

Consequently, it is bounded for any t_1 on L. Therefore, we have the estimate

$$|I_3| \leq A_3|t_2-t_1|^\lambda, \quad A_3 = constant.$$

Let us now estimate the most complicated integral I_4. Again, making use of the Hölder condition and the smoothness condition, we obtain

$$\begin{aligned}|I_4| &\leq A\frac{|t_2-t_1|}{2\pi}\int_{L\setminus l}\frac{|d\tau|}{|\tau-t_1||\tau-t_2|^{1-\lambda}}\\ &\leq B_1|t_2-t_1|\int_{L\setminus l}|\tau-t_1|^{\lambda-2}\left|\frac{\tau-t_1}{\tau-t_2}\right|^{1-\lambda}|d\tau|.\end{aligned}$$

But $|\tau-t_1|-|t_1-t_2|\leq|\tau-t_2|$ and $|\tau-t_2|\geq\delta=k|t_2-t_1|$, so that

$$|\tau-t_1|\leq\frac{k-1}{k}|\tau-t_2|.$$

It follows that

$$I_4\leq B_2\left(\frac{k-1}{k}\right)^{1-\lambda}|t_1-t_2|\int_R^\delta r^{\lambda-2}dr,$$

where

$$R=\max_{\tau\in L\setminus l}|\tau-t_1|.$$

If $\lambda<1$, computing the latter integral we find

$$|I_4|\leq A_4|t_2-t_1|^\lambda.$$

If $\lambda=1$, in an analogous way we obtain the estimate

$$|I_4|\leq A_4'|t_2-t_1|\,|\log|t_2-t_1||\,.$$

We have B_1, B_2, A_4, A_4' positive constants again. Since, $\log x$ increases slower as $x\to 0$ than any negative power $|x|^{-\epsilon}$ ($\epsilon>0$), we obtain

$$|I_4|\leq A_4'|t_2-t_1|^{1-\epsilon}.$$

Putting together all the estimates, we see that the theorem is valid. □

Chapter 4

Riemann Boundary Value Problem with finite index

4.1 Formulation of the problem, homogeneous and non-homogeneous problems

Let L be a simple smooth positively oriented closed contour. Let L divide $\mathbb{C}$ into three regions: interior domain D^+, exterior domain D^- and L. Assume $G(t)$, $g(t)$ be two functions defined on L satisfying the Hölder condition. Let $G(t) \neq 0, \forall t \in L$. It is required to find two functions $(\Phi^+(z), \Phi^-(z))$ satisfying the following conditions :

1) $\Phi^+(z)$ analytic in D^+, continuous in $D^+ \cup L$,

2) $\Phi^-(z)$ analytic and bounded in D^- and continuous in $D^- \cup L$,

3) On the contour, the relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L, \tag{4.1}$$

is satisfied.

The problems of existence, uniqueness and finding the explicit expressions of such two functions $(\Phi^+(z), \Phi^-(z))$ are the subject of *Riemann boundary value problem* .

When $g(t) \equiv 0$, the problem is called the *homogeneous Riemann boundary value problem* . So for the homogeneous Riemann boundary value problem, the boundary condition (4.1) becomes

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in L. \tag{4.2}$$

The function $G(t)$ is called the *coefficient of the Riemann boundary value problem*, $g(t)$ is called the *free term of the Riemann boundary value problem.*

4.2 The Index

DEFINITION: *By the index of the function $G(t)$ w.r.t. the contour L we call the dived by 2π increment of its argument, in traversing the curve in the positive direction.*

If the increment of the quantity ω in traversing the contour L be denoted by $[\omega]_L$, the index of $G(t)$ can be written in the form

$$æ = \operatorname{Ind} G(t) = \frac{1}{2\pi}[\arg G(t)]_L. \tag{4.3}$$

The index can easily be expressed in terms of logarithm of the function: Indeed we have

$$\log G(t) = \log |G(t)| + i \arg G(t).$$

After having traversed the contour L, $|G(t)|$ returns to its original value, so that

$$\text{æ} = \frac{1}{2\pi i}[\log G(t)]_L. \tag{4.4}$$

We have the following evident assertions:

1) *For a function which is continuous on a closed contour and does not vanish anywhere on it, the index is either an integer or zero.*

2) *The index of a product of functions is equal to the sum of the indices of the factors. The index of a quotient is equal to the difference those of the dividend and the divisor.*

3) *If $G(t)$ is the boundary value of a function analytic inside (or outside) the contour, then its index is equal to the number of zeros inside (or to minus the number of zeros outside) the contour.*

4) *If the function $G(z)$ is analytic inside the contour except at a finite number of points where it may have poles, then the number of zeros is to be replaced by the difference between the number of zeros and the number of poles* [1].

4.3 Solution of the homogeneous problem

Assume that the homogeneous Riemann boundary value problem is solvable, and let $(\Phi^+(z), \Phi^-(z))$ be its solution. Denote the number of zeros of the functions $(\Phi^+(z), \Phi^-(z))$ in the domains of definition D^+, D^-, by N^+, N^-,

[1]The number of poles should be counted the number of times equal to their multiplicity.

respectively. Computing the index of both sides of the relation

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in L,$$

and using the properties 2) and 3) of index in section 2, we get

$$N^+ + N^- = \operatorname{Ind} G(t) = æ.$$

The index æ of the coefficient of the Riemann problem will be called the *index of the problem.*

A very important property of the homogeneous problem is that it is *linear*. Let $(\Phi_1^+(z), \Phi_1^-(z))$ and $(\Phi_2^+(z), \Phi_2^-(z))$ be two different solutions,then both of them satisfy (4.2), which implies $(\alpha\Phi_1^+(z) + \beta\Phi_2^+(z), \alpha\Phi_1^-(z) + \beta\Phi_2^-(z))$ is a solution as well, for any $\alpha, \beta \in \mathbb{C}$. This property shows us that if a problem has a solution, different from the trivial one, then its *not unique.*

Without loss of generality, from now on we will assume $0 \in D^+$.

MAIN THEOREM: *Let $G(t)$ satisfy Hölder condition on L. Then*

i) *If $æ < 0$, then the homogeneous problem has only the trivial solution.*

ii) *If $æ \geq 0$, then the homogeneous problem has $æ + 1$ linearly independent solutions.*

We will give a useful definition and prove a theorem before the proof of the main Theorem.

DEFINITION: *The canonical function of the homogeneous problem is :*

$$X(z) := \begin{cases} \exp\{f(z)\} & z \in D^+ \\ z^{-æ}\exp\{f(z)\} & z \in D^- \end{cases}$$

where

$$f(z) := \frac{1}{2\pi i}\int_L \frac{\log(G(\xi)\xi^{-æ})}{\xi - z}d\xi.$$

THEOREM 4.1 *The function $X(z)$ is analytic in $D^+ \cup D^-$, and moreover for each $\xi_0 \in L$, there exists*

$$X^+(\xi_0) = \lim_{z \to \xi_0} X(z), \quad z \in D^+,$$

and

$$X^-(\xi_0) = \lim_{z \to \xi_0} X(z), \quad z \in D^-,$$

where both X^+ and X^- are continuous on L and satisfy the following condition on the boundary

$$X^+(\xi_0) = G(\xi_0) \cdot X^-(\xi_0), \quad \forall \xi_0 \in L.$$

Observe that it is exactly the same condition with our main functions $(\Phi^+(z), \Phi^-(z))$ just the only boundedness condition for outside (i.e., for $z \in D^-$) is not necessarily satisfied.

Proof: Clearly $\log(G(\xi) \cdot \xi^{-æ})$ is continuous. Moreover it is single-valued. Indeed,

$$\log(G(\xi) \cdot \xi^{-æ}) = \log|G(\xi) \cdot \xi^{-æ}| + i \mathrm{Arg}\,(G(\xi) \cdot \xi^{-æ}),$$

and we have by definition that the increment of $\mathrm{Arg}\, G(\xi)$ is $2\pi æ$. Clearly, increment of $\mathrm{Arg}\,(\xi^{-æ})$ is $-æ 2\pi$. Hence the increment of $\mathrm{Arg}\,(G(\xi) \cdot \xi^{-æ})$ is equal to the sum of the increment of $\mathrm{Arg}\,(G(\xi))$ and the increment of $\mathrm{Arg}\,(\xi^{-æ})$, so its zero. Therefore, $\log(G(\xi) \cdot \xi^{-æ})$ is single-valued.

Since $G(\xi)$ satisfies Hölder condition on L, $\log(G(\xi) \cdot \xi^{-æ})$ satisfies Hölder condition as well. Consider the Cauchy type integral

$$f(z) = \frac{1}{2\pi i} \int_L \frac{\log(G(\xi) \cdot \xi^{-æ})}{\xi - z} d\xi.$$

By Plemelj-Sokhotski formula there exist its continuous boundary values on L, $f^+(\xi_0)$, $f^-(\xi_0)$, such that

$$f^+(\xi_0) - f^-(\xi_0) = \log(G(\xi_0) \cdot \xi_0^{-\text{æ}}), \quad \xi_0 \in L.$$

It follows that both $X^+(\xi_0)$ and $X^-(\xi_0)$ exists and, moreover,

$$X^+(\xi_0) = \exp\{f^+(\xi_0)\} \text{ and } X^-(\xi_0) = \xi_0^{-\text{æ}} \exp\{f^-(\xi_0)\},$$

which gives us that they are continuous on L and satisfy

$$\begin{aligned} \frac{X^+(\xi_0)}{X^-(\xi_0)} &= \xi_0^{\text{æ}} \exp\{f^+(\xi_0) - f^-(\xi_0)\} \\ &= \xi_0^{\text{æ}}[G(\xi_0) \cdot \xi_0^{-\text{æ}}] = G(\xi_0). \end{aligned}$$

Since $f(z)$ is analytic in $D^+ \cup D^-$, the function $X(z)$ is also analytic in $D^+ \cup D^-$ by Theorem 2.1. $\square$

Proof of the main Theorem:

Case 1: Assume æ < 0. Let $(\Phi^+(z), \Phi^-(z))$ be a solution of the homogeneous problem different from the trivial one ($\not\equiv 0$). Set

$$P(z) = \begin{cases} \frac{\Phi^+(z)}{X(z)}, & z \in D^+, \\ \frac{\Phi^-(z)}{X(z)}, & z \in D^-. \end{cases}$$

Then

$$P^+(\xi_0) = \frac{\Phi^+(\xi_0)}{X^+(\xi_0)} = \frac{G(\xi_0)\Phi^-(\xi_0)}{G(\xi_0)X^-(\xi_0)} = \frac{\Phi^-(\xi_0)}{X^-(\xi_0)} = P^-(\xi_0). \tag{4.5}$$

So $P(z)$ is analytic in D^+ and D^-, and by (4.5) its boundary values coincide on L. Then $P(z)$ is continuous on L and by Painlevé theorem $P(z)$ is an entire function.

Since $\Phi^-(z)$ is assumed to be a solution, then it is bounded. By definition $|X(z)| \to \infty$, as $z \to \infty$. Then $P(z)$ is bounded as well. Moreover, $P(z)$ is constant by Liouville theorem, since $P(z)$ is both bounded and entire. But then since $P(\infty) = 0$, $P(z)$ is zero identically, so are $\Phi^+(z)$ and $\Phi^-(z)$, which contradicts with our assumption. Hence when æ < 0 we have only the trivial solution.

Case 2: Assume æ ≥ 0. Let $(\Phi^+(z), \Phi^-(z))$ be a solution of the homogeneous problem different from the trivial one. Set again

$$P(z) = \begin{cases} \frac{\Phi^+(z)}{X(z)}, & z \in D^+, \\ \frac{\Phi^-(z)}{X(z)}, & z \in D^-. \end{cases}$$

In the same way as in Case 1, we conclude that $P(z)$ is an entire function. Since for large enough z,

$$|P(z)| = \left|\frac{\Phi^-(z)}{X(z)}\right| \leq C|z|^{æ},$$

by the well-known generalization of the Liouville theorem $P(z)$ is a polynomial with $\deg P \leq$ æ.

So what we have proved is that if $(\Phi^+(z), \Phi^-(z))$ is a solution then it can be represented in the form

$$\Phi^+(z) = P(z)X^+(z), \quad \Phi^-(z) = P(z)X^-(z),$$

where $P(z)$ is a polynomial with $\deg P \leq$ æ.

But the converse is also true, for any polynomial $P(z)$ with $\deg P \leq$ æ, any function of the form $(P(z)X^+(z), P(z)X^-(z))$ is a solution of the problem.

In fact, letting $\Phi^+(z) = P(z)X^+(z), \quad \Phi^-(z) = P(z)X^-(z)$, where $P(z)$ is any polynomial with $\deg P \leq$ æ, we have by Theorem 4.1,

$$\frac{\Phi^+(\xi_0)}{\Phi^-(\xi_0)} = \frac{X^+(\xi_0)}{X^-(\xi_0)} = G(\xi_0).$$

Moreover, $\Phi^+(z)$ is analytic in D^+ and continuous in $D^+ \cup L$, by the Theorem 4.1 together with the fact that $P(z)$ is a polynomial. Similarly $\Phi^-(z)$ is analytic in D^- and continuous in $D^- \cup L$. Also,

$$|P(z)X^-(z)| \leq C|z|^{æ}|X^-(z)| \leq C|z|^{æ}|z|^{-æ} \leq C,$$

hence $P(z)X^-(z)$ is bounded in D^- as well.

Therefore

$$\begin{aligned}\Phi(z) = P(z)X(z) &= (a_0 + a_1 z + \ldots + a_{æ} z^{æ})X(z) \\ &= a_0 X(z) + a_1(zX(z)) + \ldots + a_{æ}(z^{æ}X(z)),\end{aligned}$$

is a solution of the problem.

Hence for $æ \geq 0$ there exists $æ + 1$ linearly independent solutions where $\{X(z), zX(z), z^2X(z), \ldots, z^{æ}X(z)\}$ is one of the basis for the space of solutions. $\square$

4.4 Solution of the non-homogeneous problem

THEOREM 4.2 *In the case $æ \geq 0$ the non-homogeneous problem has solution and the general solution can be represented in the form*

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(\xi)}{G(\xi)X^-(\xi)(\xi - z)} d\xi + P(z)X(z),$$

where $X(z)$ is the canonical function and $P(z)$ is a polynomial with $degP \leq æ$.

Proof: In addition to the conditions of the homogeneous case, now we are assuming that $g(\xi) \not\equiv 0$ satisfies Hölder condition as well.

First assume there exists a solution of the non-homogeneous problem, denote it by $\Psi(z)$. Consider $\Psi(z)/Q(z)$ where $Q(z)$ is a non trivial solution of the homogeneous problem. Then

$$\frac{\Psi^+(\xi)}{Q^+(\xi)} = \frac{\Psi^-(\xi)G(\xi) + g(\xi)}{Q^-(\xi)G(\xi)} = \frac{\Psi^-(\xi)}{Q^-(\xi)} + \frac{g(\xi)}{Q^-(\xi)G(\xi)}, \quad \xi \in L,$$

which means that

$$\left[\frac{\Psi(\xi)}{Q(\xi)}\right]^+ - \left[\frac{\Psi(\xi)}{Q(\xi)}\right]^- = \frac{g(\xi)}{Q^-(\xi)G(\xi)}, \quad \xi \in L. \tag{4.6}$$

Now remembering Plemelj-Sokhotski formula, if we set

$$\Psi(z) = \frac{Q(z)}{2\pi i}\int_C \frac{g(\xi)}{G(\xi)Q^-(\xi)(\xi - z)}d\xi := Q(z)f_1(z), \tag{4.7}$$

we see that (4.6) is valid.

Moreover, since $Q(z) = X(z)P(z)$, where $P(z)$ is a polynomial, by Theorem 4.1, we see that $Q(z)$ is analytic in $D^+ \cup D^-$ and $Q^+(\xi), Q^-(\xi)$ are continuous on L.

For $f_1(z)$ in (4.7) observe that since $g(z)/(G(z)Q^-(z)) \in H$ by the properties of Hölder condition in Chapter 2, then $\Psi(z)$ satisfies the following three conditions:

$\Psi^+(z)$ is analytic in D^+, continuous in $D^+ \cup L$,

$\Psi^-(z)$ is analytic in D^-, continuous in $D^- \cup L$,

on the contour the relation $Q^+(\xi) = G(\xi)Q^-(\xi) + g(\xi)$, $\xi \in L$, is valid.

Setting $P(z) = 1$ in the equation of $Q(z)$, i.e. setting $Q(z) = X(z)$ and then defining $\Psi_1(z) := \Psi(z) + P(z)X(z)$ the above conditions are still valid, because of the fact that $P(z)X(z)$ is the general solution of the homogeneous problem. Now, to complete the proof of the theorem, all we must prove is that $\Psi^-(z)$ is bounded in D^-, which gives us the boundedness of $\Psi_1^-(z)$ in D^-.

The equality (4.6) with $Q(z) = X(z)$ remains in force also in the case $æ < 0$. Hence the function

$$f_2(z) := \frac{\Psi(z)}{X(z)} - \frac{1}{2\pi i} \int_C \frac{g(\xi)d\xi}{G(\xi)X^-(\xi)(\xi - z)},$$

is analytic both in D^+ and D^- and its boundary values on $\dot{L}$ coincide. Hence $f_2(z)$ is an entire function. Since for $æ < 0$ the function $\Psi(z)/X(z)$ vanishes at infinity, we conclude that $f_2(z) \not\equiv 0$. Hence

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(\xi)d\xi}{G(\xi)X^-(\xi)(\xi - z)} =: X(z)f_3(z). \tag{4.8}$$

In the expression for the function $\Psi(z)$, the first factor has a pole of order $-æ$ at infinity, whereas the second term, as an integral of the Cauchy type has in the general case a zero of order not bigger than one at infinity. Hence $\Psi^-(z)$ has at infinity a pole of order not exceeding $-æ - 1$. Thus, (4.8 is a solution if $æ = -1$, but if $æ < -1$, the non-homogeneous problem in general does not have a solution. It has solutions only when there are some additional conditions. To derive the latter, let us expand the Cauchy type integral $f_3(z)$ into a series in the neighborhood of infinity

$$f_3^-(z) = \sum_{k=1}^{\infty} c_k z^{-k},$$

where

$$c_k = -\frac{1}{2\pi i} \int_L \frac{g(\xi)}{X^+(\xi)} \xi^{k-1} d\xi.$$

For analyticity of $\Psi(z)$ at infinity it is necessary that the first $-æ - 1$ coefficients of the expansion of $f_1(z)$ vanish. This implies that additionally to our theorem to have a solution of the non-homogeneous problem in the case of a negative index ($æ < -1$) it is necessary and sufficient that the following

$-æ-1$ conditions are satisfied:

$$\int_L \frac{g(\xi)}{X^+(\xi)}\xi^{k-1}d\xi = 0 \quad (k = 1, 2, ..., -æ - 1).$$

For the case $æ \geq 0$ since $f(z)$ is bounded and since

$$|X(z)| = \left| z^{-æ}\frac{1}{2\pi i}\int_L \frac{\log(G(\xi)\xi^{æ})}{\xi - z}d\xi \right| \to 0, \quad z \to \infty,$$

then $\Psi^-(z)$ is bounded in D^-. Hence $\Psi_1(z)$ is bounded in D^-, which completes the proof. $\square$

Chapter 5

Riemann Boundary Value Problem with an infinite index when the verticity index is less than 1/2

5.1 Statement of the homogeneous problem

Let D be a domain in the complex z-plane such that the boundary ∂D is a ray $L = \{z : t_0 \leq z < \infty\} =: [t_0, \infty)$ where[1] $0 < t_0 \leq 1$, and $D = \mathbb{C} \backslash [t_0, \infty)$. In what follows the letters t and τ will denote the points of L.

Let $G(t)$ be a continuous function on L non vanishing at all points $t \in L$, $t \neq \infty$. We say that the index of the function $G(t)$ equals to $+\infty$

[1]Choosing $t_0 \in (0, 1]$ does not restrict the generality since by an appropriate rotation and shifting one can get it.

(respectively $-\infty$) if $\lim_{t\to\infty} \arg G(t) = +\infty$. In this case we write

$$\mathrm{Ind}_L G(t) = +\infty\ (-\infty),$$

and say that the point $t = \infty$ is a point of *positive (negative) verticity* of the function $G(t)$.

DEFINITION: *We say that a function $f(t)$ on L satisfies the Hölder condition with the exponent μ, $0 < \mu \leq 1$, if the following inequality holds*

$$|f(t_1) - f(t_2)| \leq C \left| \frac{1}{t_1} - \frac{1}{t_2} \right|^{\mu}, \tag{5.1}$$

for any t_1, $t_2 \in L$, and some positive constant C depending on f. Note that for a bounded part of L this definition is equivalent to the traditional one:

$$|f(t_1) - f(t_2)| \leq C|t_1 - t_2|^{\mu}. \tag{5.2}$$

We will again denote the class of all functions satisfying Hölder condition on L (where L is unbounded now) with $H_\mu(L)$ or with $H(L)$ or with H or with H_μ.

In future we will use the following property of functions of $H(L)$. If $f(t) \in H(L)$, then there exists the limit

$$\lim_{t\to\infty} f(t) = \lambda. \tag{5.3}$$

For the proof of this let first $F_1(1/t) =: f(t)$. Since $f(t) \in H(L)$ where $L = [t_0, \infty)$, then denoting $F_1(1/t) =: F(t)$, we get that $F(t) \in H(l)$ where $l = (0, 1/t_0]$. To show that limit of $f(t)$ exists as $t \to \infty$, we must show that the limit of $F(t)$ exists as $t \to 0$. Firstly observe that by fixing $t_1 \in l$, for any $t \in l$, since $|F(t) - F(t_1)| \leq A|t - t_1|^{\alpha}$ $0 < \alpha \leq 1$, we get

$$|F(t)| \leq |F(t_1)| + A|t - t_1|^{\alpha}.$$

Assume $\lim_{t\to\infty} F(t)$ does not exist. Let $\{t_n\}_{n=1}^{\infty}$ and $\{t_{1_m}\}_{m=1}^{\infty}$ be two sequences both tending to zero, where

$$\liminf_{t\to 0} F(t) = \lim_{n\to\infty} F(t_n) = a \neq b = \lim_{m\to\infty} F(t_{1_m}) = \limsup_{t\to 0} F(t).$$

Let $|a-b| = \rho > 0$.

Then for $n \geq n_0$ and $m \geq m_0$, $\exists t_n, t_{1_m}$ such that

$$|a - F(t_n)| < \frac{\rho}{3}, \quad |b - F(t_{1_m})| < \frac{\rho}{3}.$$

Then

$$\begin{aligned} \rho = |a-b| &= |a - F(t_n) + F(t_n) - F(t_{1_m}) + F(t_{1_m}) - b| \\ &< \frac{2\rho}{3} + |F(t_n) - F(t_{1_m})|, \end{aligned}$$

which implies

$$|F(t_n) - F(t_{1_m})| > \frac{\rho}{3}. \tag{5.4}$$

But since $F(t) \in H(l)$, we have

$$|F(t_n) - F(t_{1_m})| \leq A|t_n - t_{1_m}|^{\alpha}, \quad 0 < \alpha \leq 1. \tag{5.5}$$

Then by (5.4) and (5.5), we have

$$A|t_n - t_{1_m}|^{\alpha} > \frac{\rho}{3}. \tag{5.6}$$

Since both sequences tends to zero, then we can find $|t_n| < \left(\frac{\rho}{6A}\right)^{1/\alpha}$ and $|t_{1_m}| < \left(\frac{\rho}{6A}\right)^{1/\alpha}$. But this gives us

$$A|t_n - t_{1_m}|^{\alpha} < \frac{\rho}{3}, \tag{5.7}$$

for some $n > n_1$ and $m > m_1$. Then choosing $N = \max\{n_0, n_1\}$ and $A = \max\{m_0, m_1\}$, both (5.6) and (5.7) holds for $n > N, m > M$, which is a contradiction to the assumption. Hence $\lim_{t\to 0} F(t) = \lim_{t\to\infty} f(t)$ exists.

Let a function $G(t)$ on the curve $L = [t_0, \infty)$ satisfy the conditions

1)

$$\arg G(t) = 2\pi\psi(t)t^\rho, \tag{5.8}$$

where

$$\rho > 0, \;\; \psi(t) \in H(L), \;\; \psi(\infty) = \lambda > 0, \;\; -1 < \psi(t_0) \leq 0, \;\; |t_0| \leq 1, \tag{5.9}$$

2)

$$\log|G(t)| \in H(L). \tag{5.10}$$

Denote $\tilde{L} = L\backslash\{t_0, \infty\}$. By the *homogeneous Riemann boundary value problem with index $+\infty$, verticity order ρ, and verticity coefficient λ* we mean the following problem:

Find a function $\Psi(z)$ analytic and bounded in D whose boundary values on $\tilde{L}$ satisfy the relation

$$\Psi^+(t) = G(t)\Psi^-(t), \;\; t \in \tilde{L}. \tag{5.11}$$

The verticity order of the problem is ρ in (5.8). In this chapter we consider this problem in the case $0 < \rho < 1/2$.

5.2 Canonical function

Let us study the problem posed in the previous section. Denote

$$X(z) = \exp\left\{\frac{z}{2\pi i}\int_{t_0}^{\infty} \frac{\log G(\tau)}{\tau(\tau - z)}d\tau\right\}. \tag{5.12}$$

Let us show that the integral in the right hand side is convergent.

For the proof of this, observe that $|\log G(t)| = O(t^\rho), t \to \infty$. Indeed since $\log G(t) = \log|G(t)| + i \arg G(t)$, then by (5.8) $\log G(t) = \log|G(t)| + 2\pi i\psi(t)t^\rho$. By (5.3), both $\log|G(t)|$ and $\psi(t)$ are bounded as $t \to \infty$; say $|\log|G(t)|| \le M_1$ and $|\psi(t)| \le M_2$. Then

$$\begin{aligned} |\log G(t)| &= ((\log|G(t)|)^2 + 4\pi^2\psi(t)^2 t^{2\rho})^{1/2} \\ &\le (M_1^2 + 4\pi^2 M_2^2 t^{2\rho})^{1/2}. \end{aligned}$$

So for large enough t, $|\log G(t)| \le M_1 t^\rho$, i.e., $|\log G(t)| = O(t^\rho), t \to \infty$. Using this and the fact that $0 \notin L$, we see that the integral is convergent.

We shall prove that the function $X(z)$ possesses the following properties:

1) $X(z)$ *is analytic in* D, *continuous up to* $\tilde{L}$ *and does not vanish in* $D \cup \tilde{L}$,

2) $X(z)$ *satisfies the boundary condition* (5.11), i.e.,

$$X^+(t) = G(t)X^-(t), \quad t \in \tilde{L},$$

3) $X(z)$ *is normalized by the condition* $X(0) = 1$,

4) *In a sufficiently small neighborhood of the beginning* t_0 *of the curve* L, *the following estimate holds*

$$C_1|z - t_0|^\alpha < |X(z)| < C_2, \quad 0 \le \alpha < 1, \tag{5.13}$$

5) $X(t)$ *is bounded in* D.

Note that [2] for $\forall t_1, t_2 \in L'$ where L' is any finite subarc of L we have

$$||t_1|^\rho - |t_2|^\rho| \le |t_1 - t_2|^\rho, \tag{5.14}$$

[2]Proof follows from the fact that for any $\lambda, 0 \le \lambda < 1$, $\lambda^\rho + (1-\lambda)^\rho \ge 1$ for $0 \le \rho \le 1$.

since $\rho < 1/2$. Then since the sum and the product of functions satisfying Hölder condition also satisfy Hölder condition, by (5.8), (5.10) and (5.14) $\log G(t) \in H_{L'}$, on any finite arc $L' \subset L$.

Now we divide the integral representing $\log X(z)$ into two parts as

$$\begin{aligned}\log X(z) &= \frac{z}{2\pi i}\int_{L'}\frac{\log G(\tau)}{\tau(\tau-z)}d\tau + \frac{z}{2\pi i}\int_{L-L'}\frac{\log G(\tau)}{\tau(\tau-z)}d\tau \\ &=: \log X_1(z) + \log X_2(z).\end{aligned}$$

By Plemelj-Sokhotski formula and by Theorem 3.2 we get that the first integral, $\log X_1(z)$ satisfies Hölder condition and so is analytic in $\mathbb{C}\backslash L'$, continuous up to L'. Moreover the second integral, $\log X_2(z)$ is analytic on L' and property 1) follows.

To prove property 2), observe that $\log X_2^+(t) = \log X_2^-(t)$ and $\log X_1^+(t) = \log X_1^-(t) + \log G(t)$ for $t \in L'$ by (3.5). This gives us $X^+(t) = G(t)X^-(t)$ for $t \in \tilde{L}$.

Property 3) follows directly from the definition of $X(z)$.

To prove property 4), we divide the curve L into two parts as $L = l \cup L^*$ where l is a finite curve as $l = [t_0, t_1)$ and $L^* = L\backslash l = [t_1, \infty)$. We obtain

$$\begin{aligned}\log X(z) &= \frac{1}{2\pi i}\int_{t_0}^{t_1}\frac{\log G(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{t_0}^{t_1}\frac{\log G(\tau)}{\tau}d\tau \\ &\qquad + \frac{z}{2\pi i}\int_{t_1}^{\infty}\frac{\log G(\tau)}{\tau(\tau-z)}d\tau \\ &=: I_1(z) + I_2(z) + I_3(z).\end{aligned}$$

First we will show that $I_3(z) = O(1), \quad z \to t_0$.

Note that since $|\log G(\tau)| = O(\tau^\rho)$ as $\tau \to \infty$, we have

$$\begin{aligned}|I_3(z)| &\leq \frac{|z|}{2\pi}\int_{t_1}^{\infty}\frac{|\log G(\tau)|}{\tau|\tau - t|}d\tau \\ &\leq \frac{|z|}{2\pi}\int_{t_1}^{\infty}\frac{\tau^\rho}{\tau|\tau - z|}d\tau. \qquad (5.15)\end{aligned}$$

Since we can choose z in a sufficiently small neighborhood of t_0 and since $t_0 < t_1$ and $\tau \in [t_1, \infty)$, we can find an $\epsilon > 0$ such that
$|z - t_0| < \epsilon < (1/2)|\tau - t_0|$ holds. Then

$$|\tau - z| \geq |\tau - t_0| - |z - t_0| \geq |\tau - t_0| - \epsilon \geq \frac{|\tau - t_0|}{2}.$$

The inequality (5.15) becomes

$$\begin{aligned}
|I_3(z)| &\leq \frac{|z|}{\pi} \int_{t_1}^{\infty} \frac{\tau^{\rho-1}}{|\tau - t|} d\tau \\
&\leq \frac{|t_0| + \epsilon}{\pi} \int_{t_1}^{\infty} (\tau - t_0)^{\rho-2} d\tau \\
&= \frac{|t_0| + \epsilon}{\pi} \cdot \frac{(t_1 - t_0)^{\rho-1}}{\rho - 1} := \mathit{constant}.
\end{aligned}$$

Here the second inequality follows from $|z| \leq |z - t_0| + |t_0| < |t_0| + \epsilon$ and from $\tau^{1-\rho} \geq (\tau - t_0)^{1-\rho}$, which is valid since $0 < \rho < 1/2$.
Hence, $|I_3(z)| \leq C \Rightarrow I_3 = O(1), \quad z \to t_0$.

Observe that $I_2(z) = O(1), \quad z \to t_0$, as well, because I_2 does not depend on z.

Now, we will examine $I_1(z)$. First of all let z be in the neighborhood of t_0, where $|t_1 - t_0| > |z - t_0|$.

Case 1. Assume Re $z \in [t_0, t_1]$, and denote Re $z = x$. Let

$$\begin{aligned}
I_1(z) &= \frac{1}{2\pi i} \int_{t_0}^{t_1} \frac{\log G(\tau) - \log G(x)}{\tau - z} d\tau + \frac{\log G(x)}{2\pi i} \int_{t_0}^{t_1} \frac{d\tau}{\tau - z} \\
&:= I_{11}(z) + I_{12}(z).
\end{aligned}$$

Let us first estimate $I_{11}(z)$. Observe that

$$\begin{aligned}
|I_{11}(z)| &\leq \frac{1}{2\pi} \int_{t_0}^{t_1} \frac{|\log G(\tau) - \log G(x)|}{|\tau - z|} d\tau \\
&\leq \frac{A}{2\pi} \int_{t_0}^{t_1} \frac{|\tau - x|^{\mu}}{|\tau - z|} d\tau \qquad (5.16)
\end{aligned}$$

$$\leq \frac{A}{2\pi}\int_{t_0}^{t_1}|\tau - x|^{\mu-1}d\tau \tag{5.17}$$

$$= \frac{A}{2\pi}\left[\int_{t_0}^{x}(x-\tau)^{\mu-1}d\tau \int_{x}^{t_1}(\tau - x)^{\mu-1}d\tau\right]$$

$$= \frac{A}{2\pi\mu}\left[(x-t_0)^{\mu} + (t_1 - x)^{\mu}\right]. \tag{5.18}$$

Here, (5.16) follows from the fact that $\log G(\tau) \in H_{\mu}((t_0, t_1))$; (5.17) does from the fact that $|\tau - z| \geq |\tau - x|$. Hence $I_{11}(z) = O(1)$, as $z \to t_0$, since in (5.18) the last term tends to a constant as $z \to t_0$ (i.e., $x \to t_0$).

Observe that,

$$I_{12}(z) = \frac{\log G(x) - \log G(t_0)}{2\pi i}\int_{t_0}^{t_1}\frac{d\tau}{\tau - z} + \frac{\log G(t_0)}{2\pi i}\int_{t_0}^{t_1}\frac{d\tau}{\tau - z}. \tag{5.19}$$

We claim that

$$I_{12}(z) = O(1) - \frac{\log|t_0 - z|}{2\pi i}(\log|G(t_0)| + i \arg G(t_0)) \tag{5.20}$$

$$= O(1) + \frac{\log|t_0 - z|}{2\pi}(i\log|G(t_0)| - \arg G(t_0)),$$

as $z \to t_0$. Let us explain how we get (5.20) from (5.19). Since $\log G(x) \in H_{L'}$ on any finite arc $L' \subset L$, the first integral in (5.19) can be estimated as

$$\left|\frac{\log G(x) - \log G(t_0)}{2\pi i}\int_{t_0}^{t_1}\frac{d\tau}{\tau - z}\right| \leq \frac{A|x - t_0|^{\alpha}}{2\pi i}|\log(t_1 - z) - \log(t_0 - z)|$$

$$= O(1), \quad z \to t_0. \tag{5.21}$$

Moreover the second integral in (5.19) can be evaluated as

$$\frac{\log G(t_0)}{2\pi i}\int_{t_0}^{t_1}\frac{d\tau}{\tau - z} = \frac{\log G(t_0)}{2\pi i}[\log(t_1 - z) - \log(t_0 - z)]$$

$$= \frac{\log G(t_0)}{2\pi i}[\log\frac{|t_1 - z|}{|t_0 - z|} + \arg\{(t_1 - z) - (t_0 - z)\}]$$

$$= O(1) - \frac{\log G(t_0)}{2\pi i}\log|t_0 - z|$$

$$= O(1) - \frac{\log|t_0 - z|}{2\pi i}(\log G(t_0) + i \arg G(t_0)), \tag{5.22}$$

as $z \to t_0$. Hence from (5.21) and (5.22), we have (5.20).

Therefore, in this case we have

$$I_1(z) = O(1) + \frac{\log|t_0 - z|}{2\pi}(i \log|G(t_0)| - \arg G(t_0)), \quad z \to t_0. \qquad (5.23)$$

Case 2. For the case $\operatorname{Re} z \in \tilde{L} \backslash (t_0, t_1)$, we have

$$\begin{aligned} I_1(z) &= \frac{1}{2\pi i}\int_{t_0}^{t_1} \frac{\log G(\tau) - \log G(t_0)}{\tau - z} d\tau + \frac{\log G(\tau)}{2\pi i}\int_{t_0}^{t_1} \frac{d\tau}{\tau - z} \\ &:= I_{11}(z) + I_{12}(z). \end{aligned}$$

Further,

$$\begin{aligned} |I_{11}(z)| &\leq \frac{A}{2\pi}\int_{t_0}^{t_1} \frac{|\tau - t_0|^\mu}{|\tau - z|} d\tau \qquad (5.24) \\ &\leq \frac{A}{2\pi}\int_{t_0}^{t_1} |\tau - t_0|^{\mu-1} d\tau \\ &= \frac{A}{2\pi}\int_{t_0}^{t_1} (\tau - t_0)^{\mu-1} d\tau \\ &= \frac{A}{2\pi} \cdot \frac{(t_1 - t_0)^\mu}{\mu} \end{aligned}$$

We utilized the fact that $\log G(z) \in H_\mu$, and the inequality $|\tau - t_0| < |\tau - z|$, which we have in this case. Hence $I_{11}(z) = O(1)$, as $z \to t_0$.

The integral $I_{12}(z)$ can be estimated as in case 1. So, we have the same result as in (5.23) for $I_1(z)$.

Therefore,

$$\log X(z) = \frac{\log|t_0 - z|}{2\pi}(i \log|G(t_0)| - \arg G(t_0)) + O(1), \quad z \to t_0.$$

Setting

$$\alpha = -\frac{1}{2\pi} \arg G(t_0),$$

and taking into account (5.8), (5.9), we obtain

$$\log|X(z)| = \alpha \log|z - t_0| + O(1), \quad 0 \leq \alpha < 1, \quad z \to t_0,$$

which yields property 4).

We will prove a statement which is stronger than property 5), using the following theorem.

THEOREM 5.1 *Let a function $\psi(t) \in H_\mu(L)$ on the ray $L = [t_0, \infty)$ be such that $\psi(\infty) = \lambda \neq 0$. Let $z = re^{i\theta}$. Consider the function*

$$f(z) = \exp\left\{ z \int_{t_0}^{\infty} \frac{\psi(\tau)|\tau|^\rho}{\tau(\tau - z)} d\tau \right\}, \quad 0 < \rho < 1.$$

This function is of order ρ in the domain D and

$$\begin{aligned} h_f(\theta) &:= \lim_{r\to\infty} r^{-\rho} \log |f(re^{i\theta})| \\ &= -\frac{\pi\lambda}{\sin \pi\rho} \cos \rho(\theta - \pi), \quad 0 \leq \theta \leq 2\pi. \end{aligned}$$

Here the convergence is uniform w.r.t. θ, $0 \leq \theta \leq 2\pi$.

LEMMA 5.1

$$I(z) := z \int_0^{t_0} \frac{\tau^{\rho-1}}{\tau - z} d\tau = O(1) \quad as \quad |z| \to \infty;$$

i.e. I(z) is bounded.

Proof: Let $|z| = r$ and observe that for large enough z, $|\tau - z| \geq |\tau - r| = r - \tau \geq r - t_0$. Then,

$$|I(z)| \leq r \int_0^{t_0} \frac{\tau^{\rho-1}}{|\tau - z|} d\tau \leq \frac{r}{r - t_0} \int_0^{t_0} \tau^{\rho-1} d\tau = \frac{t_0^\rho}{\rho} \cdot \frac{r}{r - t_0},$$

which gives $|I(z)| \leq C = const.$ as $|z| = r \to \infty$, i.e., $I(z) = O(1)$ as $|z| \to \infty$. □

LEMMA 5.2

$$I_1(z) := z \int_0^{\infty} \frac{\tau^{\rho-1}}{\tau - z} d\tau = -\frac{\pi z^\rho}{\sin \rho\pi} e^{-i\rho\pi}.$$

Proof: Consider the following contour integral

$$I(z) := z \int_C \frac{\xi^{\rho-1}}{\xi - z} d\xi$$

where C is the contour consisting of the real axis from $\xi = R_1$ to $\xi = R_2$, say l_1, where $0 < R_1 < R_2$; a circle C_{R_2} of radius R_2 above the real axis, large enough that z is inside the contour; the real axis again from R_2 to R_1, say l_2; and finally a circle C_{R_1} of radius R_1. We take R_1 small and R_2 large. The small circle is necessary to avoid the singularity of the integrand at $\xi = 0$, and the large circle is necessary to close up the contour.

The multivalued function $\xi^{\rho-1}$ is taken to be real, say $\tau^{\rho-1}$ on l_1 and at all other points it has the form $\xi^{\rho-1} = |\xi|^{\rho-1} e^{(\rho-1)\theta_1 i}$, i.e., complex valued so that on l_2, it takes the value $(\tau e^{2\pi i})^{\rho-1}$.Then by the Residue theorem,

$$I(z) = z 2\pi i Res_{(\xi=z)} \frac{\xi^{\rho-1}}{\rho - z} = 2\pi i z^{\rho}. \tag{5.25}$$

$$I(z) = z\left(\int_{l_1} + \int_{C_{R_2}} + \int_{l_2} + \int_{C_{R_1}}\right) \frac{\xi^{\rho-1}}{\xi - z} d\xi. \tag{5.26}$$

Since $|\xi| = R_1$ on C_{R_1} and $|\xi - z| \geq ||\xi| - |z|| = r - R_1$,

$$\left|z \int_{C_{R_1}} \frac{\xi^{\rho-1}}{\xi - z} d\xi\right| \leq 2\pi r R_1 \frac{R_1^{\rho-1}}{|\xi - z|} \leq \frac{2\pi r R_1^{\rho}}{r - R_1} \to 0 \quad \text{as} \quad R_1 \to 0,$$

which is valid since $\rho > 0$. Hence

$$\left|z \int_{C_{R_1}} \frac{\xi^{\rho-1}}{\xi - z} d\xi\right| = O(1) \quad \text{as} \quad R_1 \to 0. \tag{5.27}$$

Similarly, since $|\xi| = R_2$ on C_{R_2} and $|\xi - z| \geq ||\xi| - |z|| = R_2 - r$,

$$\left|z \int_{C_{R_2}} \frac{\xi^{\rho-1}}{\xi - z} d\xi\right| \leq r 2\pi R_2 \frac{R_2^{\rho-1}}{R_2 - r} = r 2\pi \frac{R_2^{\rho}}{R_2 - r} \to 0 \quad \text{as} \quad R_2 \to \infty,$$

since $\rho < 1$. Hence

$$\left| z \int_{C_{R_2}} \frac{\xi^{\rho-1}}{\xi - z} d\xi \right| = o(1) \quad \text{as} \quad R_2 \to \infty. \tag{5.28}$$

Moreover,

$$\begin{aligned}
z\left(\int_{l_1} + \int_{l_2}\right) \frac{\xi^{\rho-1}}{\xi - z} d\xi &= z \int_{R_1}^{R_2} \frac{\tau^{\rho-1}}{\tau - z} d\tau - z \int_{R_1}^{R_2} \frac{(\tau e^{2\pi i})^{\rho-1}}{\tau - z} d\tau \\
&= z(1 - e^{2\pi i(\rho-1)}) \int_{R_1}^{R_2} \frac{\tau^{\rho-1}}{\tau - z} d\tau \\
&= z(1 - e^{2\pi i\rho}) \int_{R_1}^{R_2} \frac{\tau^{\rho-1}}{\tau - z} d\tau \\
&= -z 2\pi i e^{i\pi\rho} \sin \pi\rho \int_{R_1}^{R_2} \frac{\tau^{\rho-1}}{\tau - z} d\tau. \qquad (5.29)
\end{aligned}$$

And (5.29) tends to

$$-z 2\pi i e^{i\pi\rho} \sin \pi\rho \int_0^{\infty} \frac{\tau^{\rho-1}}{\tau - z} d\tau, \quad \text{as} \quad R_1 \to 0, \quad R_2 \to \infty. \tag{5.30}$$

Then, by using (5.27), (5.28) and (5.30) in the formula (5.26) and then taking the two formulas of $I(z)$ in (5.25) and (5.26) as $R_1 \to 0$ and $R_2 \to \infty$, we get

$$-z 2 i e^{\pi i\rho} \sin \pi\rho \int_0^{\infty} \frac{\tau^{\rho-1}}{\tau - z} d\tau = 2\pi i z^{\rho},$$

which means

$$I_1(z) = z \int_0^{\infty} \frac{\tau^{\rho-1}}{\tau - z} d\tau = -\frac{\pi z^{\rho}}{\sin \rho\pi} \exp^{-i\rho\pi}.$$

□

LEMMA 5.3

$$I_2(z) := z \int_{t_0}^{\infty} \frac{\tau^{\rho-1}}{\tau - z} d\tau = -\frac{\pi z^{\rho}}{\sin \rho\pi} e^{-i\rho\pi} + O(1) \quad as \quad |z| \to \infty.$$

Proof: Since $I_2(z) = I_1(z) - I(z)$ where $I(z)$ and $I_1(z)$ are taken exactly as in lemma 5.1 and as in lemma 5.2, respectively; then the proof follows from these two lemmas. $\square$

Proof of Theorem 5.1: Denote

$$\log f(z) := I(z) = \frac{z}{2\pi i}\int_{t_0}^{\infty} \frac{\psi(\tau)\tau^{\rho-1}}{\tau - z} d\tau.$$

Case 1. Assume that $\alpha < \arg z < 2\pi - \alpha$ where $0 < \alpha < \pi/2$, i.e., z is outside of the angle $A_\alpha := \{z : |\arg z| \leq \alpha\}$.

STEP1.1: Note that for $z \notin A_\alpha$, $\tau \geq t_0$, we have $|z - \tau| \geq \tau \sin\alpha$ (because $\tau \sin\alpha$ is the shortest distance from τ to sides of A_α). Moreover, for $z \notin A_\alpha$, $\tau \geq t_0$, we have also

$$|z - \tau| \geq |z| \sin\alpha.$$

Indeed, if $\operatorname{Re} z > 0$, then $|z - \tau| \geq |\operatorname{Im} z| = |z||\sin(\arg z)| \geq |z|\sin\alpha$. If $\operatorname{Re} z \leq 0$, then $|z - \tau| \geq |z| \geq |z|\sin\alpha$.

Hence, for any β, $0 < \beta < 1$, we have

$$|z-\tau| = |z-\tau|^{\beta}|z-\tau|^{1-\beta} \geq (\tau\sin\alpha)^{\beta}(|z|\sin\alpha)^{1-\beta} = \tau^{\beta}|z|^{1-\beta}\sin\alpha. \quad (5.31)$$

Now, we have

$$\begin{aligned} I(z) &= \frac{z}{2\pi i}\int_{t_0}^{\infty} \frac{\psi(\tau) - \psi(\infty)}{\tau - z}\tau^{\rho-1} d\tau + \psi(\infty)\frac{z}{2\pi i}\int_{t_0}^{\infty} \frac{\tau^{\rho-1}}{\tau - z}\tau^{\rho-1} d\tau \\ &=: I_1(z) + I_2(z). \end{aligned}$$

STEP 1.2: We will consider now $I_1(z)$. By the Hölder condition, we have $|\psi(\tau) - \psi(\infty)| \leq C\tau^{-\mu}$, therefore

$$|I_1(z)| \leq \frac{|z|}{2\pi}\int_{t_0}^{\infty} \frac{C\tau^{-\mu+\rho-1}}{|\tau - z|} d\tau \leq \frac{C|z|^{\beta}}{2\pi \sin\alpha}\int_{t_0}^{\infty} \tau^{-\mu+\rho-1-\beta} d\tau;$$

where the last inequality above follows from (5.31).

Choose β such that $\max(-\mu+\rho,0)<\beta<\rho$. Then $-\mu+\rho-1-\beta<-1$ and therefore the last integral converges. Hence

$$|I_1(z)|\le C_1|z|^\beta,\quad \text{i.e.,}\quad |I_1(z)|=o(|z|^\rho),\qquad |z|\to\infty.$$

STEP 1.3: By lemma 5.3 and with the fact that $\psi(\infty)=\lambda$, we have

$$I_2(z)=-\frac{\lambda\pi z^\rho}{\sin\pi\rho}e^{-\pi i\rho}+O(1),\qquad |z|\to\infty.$$

STEP 1.4: By using the results of step 1.2 and step 1.3 in step 1.1, we get

$$I(z)=\log f(z)=-\frac{\lambda\pi z^\rho}{\sin\pi\rho}e^{-\pi i\rho}+o(|z|^\rho),\qquad |z|\to\infty.$$

Since $\log|f(z)|=\mathrm{Re}(\log f(z))$ and $z=re^{i\theta}$, we have

$$\log|f(z)|=-\frac{\lambda\pi r^\lambda}{\sin\pi\lambda}\cos\rho(\theta-\pi)+o(r^\rho),\quad 0\le\theta\le 2\pi,$$

and this implies that, we have

$$\begin{aligned} h_f(\theta) &= \lim_{r\to\infty} r^{-\rho}\log|f(re^{i\theta})| \\ &= -\frac{\pi\lambda}{\sin\pi\rho}\cos\rho(\theta-\pi),\ \ 0\le\theta\le 2\pi. \end{aligned}$$

Case 2. Assume $z\in A_\alpha$, $\mathrm{Re}\,z=x>t_0$.

STEP2.1: Note that in this case

$$|z|\le\frac{\mathrm{Re}\,z}{\cos\alpha}=:\frac{x}{\cos\alpha}. \tag{5.32}$$

(because $|z|=x/\cos(\arg z)\le x/\cos\alpha$).

Write

$$\begin{aligned} I(z) &= \frac{z}{2\pi i}\int_{t_0}^\infty\frac{\psi(\tau)-\psi(\infty)}{\tau-z}\tau^{\rho-1}d\tau+\psi(\infty)\frac{z}{2\pi i}\int_{t_0}^\infty\frac{\tau^{\rho-1}d\tau}{\tau-z} \\ &=: I_3(z)+I_4(z). \end{aligned}$$

STEP 2.2: Now we will consider $I_3(z)$. We have

$$\begin{aligned}
|I_3(z)| &\leq \frac{|z|}{2\pi}\int_{t_0}^{\infty}\frac{|\psi(\tau)-\psi(x)|}{|\tau-z|}\tau^{\rho-1}d\tau \\
&\leq \frac{A|z|}{2\pi}\int_{t_0}^{\infty}\frac{C\left|\frac{1}{\tau}-\frac{1}{x}\right|^{\mu}}{|\tau-z|}\tau^{\rho-1}d\tau \\
&= \frac{A|z|}{2\pi x^{\mu}}\int_{t_0}^{\infty}\frac{C|\tau-x|^{\mu}}{|\tau-z|}\tau^{\rho-\mu-1}d\tau \\
&\leq \frac{AC|z|}{2\pi x^{\mu}}\int_{t_0}^{\infty}|\tau-x|^{\mu-1}\tau^{\rho-\mu-1}d\tau,
\end{aligned}$$

where the last inequality follows from the fact that $|\tau-z|\geq|\tau-x|$. By change of variable $\tau=ux$ first and then by (5.32) , we have

$$\begin{aligned}
|I_3(z)| &\leq \frac{|z|C}{2\pi x^{\mu-\rho+1}}\int_{t_0/x}^{\infty}|1-u|^{\mu-1}u^{\rho-\mu-1}du \\
&\leq \frac{Cx^{\rho-\mu}}{2\pi\cos\alpha}\int_{t_0/x}^{\infty}|1-u|^{\mu-1}u^{\rho-\mu-1}du. \qquad (5.33)
\end{aligned}$$

If $0<\mu<\rho<1$, then

$$C_1 := \int_0^{\infty}|1-u|^{\mu-1}u^{\rho-\mu-1}du<\infty. \qquad (5.34)$$

Indeed,

$$C_1=\int_0^1(1-u)^{\mu-1}u^{\rho-\mu-1}du+\int_1^{\infty}(u-1)^{\mu-1}u^{\rho-\mu-1}du =: I_{31}+I_{32}.$$

For $0<u<1/2$ we have $(1-u)^{\mu-1}<2^{1-\mu}=:A_1$, and for $u>1/2$ we have $u^{\rho-\mu-1}<2^{\mu+1-\rho}=:A_2$. Hence for I_{31}, we have

$$\begin{aligned}
I_{31} &= \int_0^{1/2}(1-u)^{\mu-1}u^{\rho-\mu-1}du+\int_{1/2}^{1}(1-u)^{\mu-1}u^{\rho-\mu-1}du \\
&\leq A_1\int_0^{1/2}u^{\rho-\mu-1}du+A_2\int_{1/2}^{1}(1-u)^{\mu-1}du \\
&= A_1\left.\frac{u^{\rho-\mu}}{\rho-\mu}\right|_0^{1/2}+A_2\left.\frac{(1-u)^{\mu}}{\mu}\right|_{1/2}^{1}<\infty.
\end{aligned}$$

For I_{32} we have the following

$$\begin{aligned} I_{32} &= \int_1^2 (u-1)^{\mu-1} u^{\rho-\mu-1} du + \int_2^\infty (u-1)^{\mu-1} u^{\rho-\mu-1} du \\ &\leq \int_1^2 (u-1)^{\mu-1} du + \int_2^\infty (u-1)^{\rho-2} du \\ &= \left. \frac{(u-1)^\mu}{\mu} \right|_1^2 + \left. \frac{(u-1)^{\rho-1}}{\rho-1} \right|_2^\infty < \infty. \end{aligned}$$

Therefore (5.34) is valid. Hence

$$|I_3(z)| \leq \frac{Cx^{\rho-\mu}}{2\pi\cos\alpha} \cdot C_1 \text{ i.e., } I_3(z) = o(x^\rho) = o(|z|^\rho), \quad |z| \to \infty.$$

If $0 < \rho \leq \mu < 1$, then we have (for x large enough, s.t. $(t_0/x) < 1/2$)

$$\begin{aligned} \int_{t_0/x}^\infty |1-u|^{\mu-1} u^{\rho-\mu-1} du &= \int_{t_0/x}^{1/2} |1-u|^{\mu-1} u^{\rho-\mu-1} du \\ &\qquad + \int_{1/2}^\infty |1-u|^{\mu-1} u^{\rho-\mu-1} du \\ &\leq \int_{t_0/x}^{1/2} 2^{1-\mu} u^{\rho-\mu-1} du + \int_{1/2}^\infty |1-u|^{\mu-1} u^{\rho-\mu-1} du \\ &=: 2^{1-\mu} \int_{t_0/x}^{1/2} u^{\rho-\mu-1} du + C_2. \end{aligned}$$

Substituting this into (5.33), we get

$$|I_3(z)| \leq \frac{Cx^{\rho-\mu}}{2\pi\cos\alpha} \left\{ 2^{1-\mu} \int_{t_0/x}^{1/2} u^{\rho-\mu-1} du + C_1 \right\}. \tag{5.35}$$

Evidently,

$$\int_{t_0/x}^{1/2} u^{\rho-\mu-1} du \leq \begin{cases} \frac{(t_0/x)^{\rho-\mu}}{\mu-\rho}, & \text{for } \rho-\mu < 0 \\ \log \frac{x}{2t_0}, & \text{for } \rho-\mu = 0. \end{cases}$$

Substituting this into (5.35), we get

$$|I_3(z)| \leq \begin{cases} \frac{Ct_0^{\rho-\mu} 2^{1-\mu}}{2\pi\cos\alpha(\mu-\rho)} + \frac{Cx^{\rho-\mu}}{2\pi\cos\alpha} C_2 = o(x^\rho) = o(|z|^\rho), & \text{for } \rho-\mu < 0 \\ \frac{C}{2\pi\cos\alpha} 2^{1-\mu} \log\frac{x}{2t_0} + \frac{CC_2}{2\pi\cos\alpha} = o(x^\rho) = o(|z|^\rho), & \text{for } \rho-\mu = 0. \end{cases}$$

STEP 2.3: Similarly to step 1.3, by lemma 5.3 and with the fact that $\psi(z) = \lambda$,

$$I_4(z) = -\frac{\lambda\pi z^\rho}{\sin\pi\rho}e^{-\pi i\rho} + O(1).$$

STEP 2.4: By using the results of step 2.2 and step 2.3 in step 2.1, we get

$$I(z) = \log f(z) = -\frac{\lambda\pi z^\rho}{\sin\pi\rho}e^{-\pi i\rho} + o(|z|^\rho), \quad |z| \to \infty.$$

Then similarly to step 1.4,

$$\log|f(z)| = -\frac{\lambda\pi r^\lambda}{\sin\pi\lambda}\cos\rho(\theta - \pi) + o(r^\rho) \quad 0 \le \theta \le 2\pi.$$

This implies that,

$$\begin{aligned} h_f(\theta) &= \lim_{r\to\infty} r^{-\rho}\log|f(re^{i\theta})| \\ &= -\frac{\pi\lambda}{\sin\pi\rho}\cos\rho(\theta-\pi), \ \ 0 \le \theta \le 2\pi. \end{aligned}$$

□

THEOREM 5.2 *Let conditions (5.8)-(5.10) be fulfilled and $X(z)$ be the function defined by (5.12). Then $X(z)$ is bounded in D and has the order of decrease ρ and*

$$\begin{aligned} h_X(\theta) &= \lim_{r\to\infty} r^{-\rho}\log|X(re^{i\theta})| \\ &= -\frac{\pi\lambda}{\sin\pi\rho}\cos\rho(\theta-\pi), \ \ 0 \le \theta \le 2\pi. \end{aligned} \tag{5.36}$$

Here the convergence is uniform w.r.t. θ.

Proof: STEP 1: Represent $X(z)$ in the form

$$\begin{aligned} X(z) = \exp\left\{\frac{z}{2\pi i}\int_{t_0}^{\infty}\frac{\log|G(\tau)|}{\tau(\tau-z)}d\tau\right\} \times \\ \times \exp\left\{z\int_{t_0}^{\infty}\frac{\varphi(\tau)|\tau|^\rho}{\tau(\tau-z)}d\tau\right\} =: X_1(z)X_2(z). \end{aligned} \tag{5.37}$$

STEP 2: After changing the variables $\tau = 1/u$, $z = 1/\zeta$ we pass to a finite curve $l = (0, 1/t_0]$. Denote $G^*(u) = G(1/u)$. In the neighborhood of $\zeta = 0$, i.e.; in the neighborhood ∞ of z,

$$\log|X_1(z)| = \mathrm{Re}(\log X_1(z)) = \frac{1}{2\pi}\mathrm{Im}\left(\int_0^{1/t_0} \frac{\log|G^*(u)|}{u-\zeta}du\right). \tag{5.38}$$

STEP 3:

$$\begin{aligned}\int_0^{1/t_0} \frac{\log|G^*(u)|}{u-\zeta}du &\le \int_0^{1/t_0} \frac{\log|G^*(u)| - \log|G^*(\zeta)|}{u-\zeta}du \\ &\quad + \log|G^*(\zeta)| \int_0^{t_0} \frac{du}{u-\zeta} \\ &=: I_1(z) + I_2(z).\end{aligned} \tag{5.39}$$

Since $\log|G^*(u)| \in H_\mu(l)$, we have $|\log|G^*(u)| - \log|G^*(\zeta)|| \le A|u-\zeta|^\mu$. Hence by denoting $|\zeta| = \delta$,

$$\begin{aligned}|I_1(1/\zeta)| &\le A\int_0^{1/t_0} |u-\zeta|^{\mu-1}du \\ &= A\int_0^{\delta}(\delta-u)^{\mu-1}du + A\int_\delta^{1/t_0}(u-\delta)^{\mu-1}du \\ &= \frac{A}{\mu}\left(\delta^\mu + (\frac{1}{t_0}-\delta)^\mu\right) \to \frac{A}{\mu t_0^\mu} \quad \text{as} \quad \delta \to 0,\end{aligned}$$

i.e., $|I_1(z)| = O(1)$ as $|z| \to \infty$.

Hence $\mathrm{Im}(I_1(z)) = O(1)$ as $|z| \to \infty$.

Also since

$$I_2(1/\zeta) = -\log|G^*(\zeta)|\left[\log\left(\zeta - \frac{1}{t_0}\right) - \log\zeta\right],$$

we have $\mathrm{Im}(I_2(z)) = -\log|G^*(\zeta)|[\arg(\zeta - 1/t_0) - \arg\zeta]$.

Hence $\mathrm{Im} I_2(1/\zeta) = O(1)$ as $|\zeta| \to 0$ i.e.,

$$\mathrm{Im}(I_2(z)) = O(1) \quad \text{as} \quad z \to \infty.$$

So by (5.38) and (5.39) we have

$$\log|X_1(z)| = \frac{1}{2\pi}\mathrm{Im}(I_1(z)+I_2(z)) = O(1) \quad \text{as} \quad z \to \infty, \tag{5.40}$$

which implies that $\frac{1}{C} < X_1(z) < C$ for $|z| > r_0$.

STEP 4: Since $\log|X(z)| = \log|X_1(z)| + \log|X_2(z)|$ and $X_2(z)$ is the same as the function $f(z)$ in Theorem 5.1, we see that (5.36) follows from Theorem 5.1 and (5.37).

STEP 5: Since $\lambda > 0$ and $0 < \rho < 1/2$, we have

$$\max_{0\le\theta\le 2\pi} h_X(\theta) = -\pi\lambda\cot\rho\pi < 0.$$

Therefore

$$\max_{0\le\theta\le 2\pi} |X(re^{i\theta})| < \exp\left(-\frac{\pi}{2}\lambda r^\rho \cot\rho\pi\right), \quad r > r_0.$$

This implies the statement of the theorem and in particular, property 5) of the function $X(z)$. $\square$

DEFINITION: *A function $X(z)$ satisfying conditions 1)-5) is called the canonical function of the Riemann boundary value problem (5.11).*

5.3 Solution of the homogeneous problem. Description of solutions of order ρ

DEFINITION: *Denote by $\tilde{B}_L$ the class of functions analytic in D and bounded in any disk $|z| \le R$ and by B_L the class of functions analytic and bounded in D.*

First consider the homogeneous problem in the class $\tilde{B}_L$.

THEOREM 5.3 *General solution of the homogeneous problem in the class $\tilde{B}_L$ can be expressed by the relation*

$$\Phi(z) = F(z)X(z) \tag{5.41}$$

where $X(z)$ is the canonical function and $F(z)$ is an arbitrary entire function.

Proof: Let $\Phi(z) \in \tilde{B}_L$ be a solution. Then the function $F(z) = \Phi(z)/X(z)$ is analytic in D and satisfies the condition $F^+(t) = F^-(t)$ on $\tilde{L}$. It follows from (5.13) that near the point $z = t_0$ this function can be estimated as $F(z) = O(|z - t_0|^{-\alpha})$, $0 \leq \alpha < 1$. By the Cauchy estimation and Painlevé theorem $F(z)$ is an entire function. On the other hand, we get that for any entire function $F(z)$ the relation

$$\Phi(z) = F(z)X(z)$$

defines a solution of the problem (5.11) belonging to the class $\tilde{B}_L$. □

Let us see what additional conditions on the function $F(z)$ provide the inclusion $\Phi(z) \in B_L$, where $\Phi(z)$ is obtained from (5.41).

THEOREM 5.4 *If $\Phi(z) \in B_L$, then the order of growth ρ_F of the corresponding function $F(z)$ does not exceed ρ.*

Proof: Let the contrary be true, i.e., let $\rho_F > \rho$. Then there exists a sequence $z_n \to \infty$ such that

$$\log|F(z_n)| > |z_n|^{\rho_F} > |z_n|^{(\rho+\rho_F)/2}.$$

Also by (5.36) we have

$$\log|X(z_n)| \geq -C|z_n|^{\rho},$$

where C is a positive constant. Then

$$\log|\Phi(z_n)| = \log|X(z_n)| + \log|F(z_n)| > -C|z_n|^\rho + |z_n|^{(\rho+\rho_F)/2},$$

where the last term tends to infinity as $z_n \to \infty$. But this means that $|\Phi(z_n)| \to \infty$ as $z_n \to \infty$ which contradicts to $\Phi(z) \in B_L$. $\square$

THEOREM 5.5 *For a solution $\Phi(z) \in \tilde{B}_L$ to belong to the class B_L it is necessary and sufficient that the order ρ_F of the corresponding entire function $F(z)$ does not exceed ρ and also*

$$\log|F(t)| \leq -Re\left\{t\int_{t_0}^{\infty} \frac{\varphi(\tau)|\tau|^\rho}{\tau(\tau-t)}d\tau\right\} + O(1), \tag{5.42}$$

for all $t \in L$, $t \to \infty$.

Proof: **Necessity.** Let $\Phi(z) \in B_L$, then $\Phi(z)$ is bounded in D. Then

$$\log|F(t)| = \log|\Phi^\pm(t)| - \log|X^\pm(t)| \leq O(1) - \log|X^\pm(t)|. \tag{5.43}$$

Relation (5.37) taking into account the Sokhotski formulas (3.3) yields

$$\log|X^\pm(t)| = \pm\frac{1}{2}\log|G(t)| + \mathrm{Re}\left\{t\int_{t_0}^{\infty} \frac{\varphi(\tau)|\tau|^\rho}{\tau(\tau-t)}d\tau\right\} + \Gamma(t), \tag{5.44}$$

where

$$\Gamma(t) = \mathrm{Im}\left\{\frac{t}{2\pi}\int_{t_0}^{\infty} \frac{\log|G(\tau)|}{\tau(\tau-t)}d\tau\right\}.$$

Since $\log|G(t)| \in H(L)$, we have $|\log|G(t)|| < C$ and, by (5.40) $|\Gamma(t)| < C$. Now (5.42) follows from (5.43) and (5.44). The necessity of $\rho_F \leq \rho$ is proved in the previous theorem.

Sufficiency. Let $\rho_F \leq \rho$ and also relation (5.42) holds. Then

$$\begin{aligned}
\log|\Phi^{\pm}(t)| &= \log|X^{\pm}(t)| + \log|F(t)| \\
&\leq Re\left\{t\int_L \frac{\varphi(\tau)|\tau|^{\rho}}{\tau(\tau-t)}d\tau\right\} + \log|F(t)| + O(1) \\
&\leq Re\left\{t\int_L \frac{\varphi(\tau)|\tau|^{\rho}}{\tau(\tau-t)}d\tau\right\} - Re\left\{t\int_L \frac{\varphi(\tau)|\tau|^{\rho}}{\tau(\tau-t)}d\tau\right\} + O(1) \\
&= O(1) \qquad (5.45)
\end{aligned}$$

We utilized first (5.44) and then (5.42). So by (5.45), we have $|\Phi^{\pm}(t)| < C$. For some ρ_1, $\rho < \rho_1 < 1/2$, we have

$$\begin{aligned}
\max_{|z|=r}|\Phi(z)| &= \max_{|z|=r}|F(z)X(z)| \\
&\leq \max_{|z|=r}|F(z)|\max_{|z|=r}|X(z)| \\
&\leq Ce^{r^{\rho}} \leq e^{r^{\rho_1}} \qquad (5.46)
\end{aligned}$$

for all sufficiently large r, (we used the facts that $|X(z)|$ is bounded and

$$\max_{|z|=r}|F(z)| < C\exp(r^{\rho}) < \exp(r^{\rho_1}), \quad \text{for} \quad \rho_F \leq \rho < \rho_1.$$

)

Then, we have $\Phi(z) \in \tilde{B}_L$, $|\Phi^{\pm}(t)| < C$ for $t \in \tilde{L}$, and

$$\max_{|z|=r}|\Phi(z)| \leq \exp(r^{\rho_1}), \quad r > r_0, \ \ 0 < \rho_1 < 1/2.$$

By Phragmén-Lindelöf principle, we get $\Phi(z) \in B_L$. □

THEOREM 5.6 *Let conditions (5.8)-(5.10) be fulfilled. Then the homogeneous Riemann boundary problem (5.11) has infinitely many solutions and the general form of solution is as follows*

$$\begin{aligned}
\Phi(z) &= F(z)X(z) \\
&= Qz^m \prod_{n=1}^{n_0}\left(1 - \frac{z}{z_n}\right)\cdot\exp\left\{\frac{z}{2\pi i}\int_L \frac{\log G(\tau)}{\tau(\tau-z)}d\tau\right\}, \qquad (5.47)
\end{aligned}$$

$$Q = constant, \quad n_0 \leq \infty, \quad 0 < |z_1| \leq |z_2| \leq ..., \; z_n \to \infty,$$

where $F(z)$ is an entire function satisfying (5.42) and having the order $\rho_F \leq \rho$.

Proof: Due to the inequality $\rho_F < 1/2$ and the Hadamard factorization theorem the function $F(z)$ admits the representation

$$F(z) = Qz^m \prod_{n=1}^{n_0} \left(1 - \frac{z}{z_n}\right), \quad 0 \leq m < \infty, \quad 0 \leq n_0 \leq \infty,$$

(for $n_0 = 0$ we assume $F(z) = Qz^m$). Then assertion of the theorem follows directly from Theorem 5.5. $\square$

COROLLARY: *The homogeneous Riemann boundary value problem has infinitely many linearly independent solutions.*

Proof: In fact, there are infinitely many linearly independent functions of order less than ρ. Every such function $F(z)$ generates a solution because Theorem 5.1 yields,

$$\operatorname{Re}\left\{t \int_{t_0}^{\infty} \frac{\varphi(\tau)|\tau|^\rho}{\tau(\tau - t)} d\tau\right\} = -\pi\lambda|t|^\rho \cot \rho\pi + o(|t|^\rho), \quad t \to \infty, \qquad (5.48)$$

and, hence, $F(z)$ satisfies condition (5.42). $\square$

COROLLARY: *The canonical function is unique.*

Proof: Assume there exist two canonical functions, $X(z), X_1(z)$ say. Then by Theorem 5.6, $\Phi(z) = X(z) = X_1(z)F(z)$ where $F(z)$ is entire. Then

$$F(z) = \frac{X(z)}{X_1(z)},$$

implies that both $X(z)$ and $X_1(z)$ have the same zeros and they can be the zeros of $F(z)$. Since canonical function does not vanish in $D \cup \tilde{L}$, they can only have zeros either at t_0 or at ∞. At infinity $F(z)$ does not have a zero, thus the canonical functions can not have as well. By (5.13) at t_0 they can not have a zero as well. This gives us that $F(z)$ have no zeros, hence it is constant. But we know that by the properties of the canonical function that $X(0) = X_1(0) = 1$. Hence $F(z) = 1$ and $X(z) = X_1(z)$. $\square$

Denote by γ_F the exponent of convergence of the zeros of the function $F(z)$.

THEOREM 5.7 *The general form of a solution $\Phi(z)$ of the homogeneous problem in the class of functions having the order of decrease ρ and $h_\Psi(\theta) < 0$ is given by the relation (5.47) where $F(z)$ is an entire function s.t., either*

$$1)\rho_F < \rho \quad (\gamma_F < \rho), \tag{5.49}$$

or

$$2)\rho_F = \rho \quad (\gamma_F = \rho), \quad \textit{and} \quad h_F(0) < \pi\lambda \cot \rho\pi. \tag{5.50}$$

We have the representation

$$h_\Phi(\theta) = \begin{cases} -\frac{\pi\lambda}{\sin \pi\rho} \cos \rho(\theta - \pi), & \rho_F < \rho, \\ h_F(\theta) - \frac{\pi\lambda}{\sin \pi\rho} \cos \rho(\theta - \pi), & \rho_F = \rho. \end{cases}$$

To prove theorem we need the following lemma, which can be easily obtained from general properties of trigonometrically convex functions.(see, e.g., [3], Ch.8)

LEMMA 5.4 *If a function $h(\theta)$ is trigonometrically ρ-convex on the segment $[\theta_1, \theta_2]$, and $|\theta_1 - \theta_2| < \pi/\rho$ and $h(\theta_k) < 0, k = 1, 2$, then $h(\theta) < 0$ for all $\theta \in [\theta_1, \theta_2]$.*

Proof of Theorem 5.7: Let either (5.49) or (5.50) be fulfilled. By Theorems 5.2 and 5.3 we have

$$h_X(\theta) = -\frac{\pi\lambda}{\sin \pi\rho} \cos \rho(\theta - \pi), \tag{5.51}$$

$$\text{and } \Phi(z) = F(z)X(z). \tag{5.52}$$

In the case $\rho_F < \rho$ we have $h_F(\theta) = 0$ and hence the representation in the theorem follows. Consider the case $\rho_F = \rho$. The function $h_F(\theta)$ is trigonometrically ρ-convex on $[0, 2\pi]$. Since $h_\Phi(\theta)$ is trigonometrically ρ-convex and by (5.50) and (5.51) $h_\Phi(0) = h_\Phi(2\pi) < 0$. Since $\rho < 1/2$, Lemma 5.4 provides $h_\Phi(\theta) < 0, \theta \in [0, 2\pi]$, and so $\Phi(z)$ has the order of decrease ρ. From (5.52) and (5.51) we derive the representation in the theorem.

Conversely, let $\Phi(z)$ have the order of decrease ρ and $h_\Phi(\theta) < 0, \theta \in [0, 2\pi]$. By Theorem 5.4 we have $\rho_F \leq \rho$. If $\rho_F = \rho$, then, setting $\theta = 0$ in to the representation given in the theorem and taking into account that $h_\Phi(\theta) < 0$, we get $h_F(0) < \pi\lambda \cot \rho\pi$. If $\rho_F < \rho$, then $h_F(\theta) = 0$ and theorem follows from the representation given in the theorem. □

5.4 Formulation of the non-homogeneous problem and an approach to its solution

The non-homogeneous problem has the form

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \tilde{L}. \tag{5.53}$$

We will assume that $t_0 = 1$, i.e., $L = \{t : 1 \leq t \leq \infty\}$. We also suppose that the functions $G(t)$ and $g(t)$ satisfy the following conditions

1) $\arg G(t) = 2\pi\varphi(t)t^\rho, \quad 0 < \rho < 1/2,$ (5.54)

where $\varphi(t) \in H_{\mu_0}(L), \quad \rho < \mu_0 \leq 1,$ (5.55)

$\varphi(\infty) = \lambda > 0, \quad -1 < \varphi(1) \leq 0,$ (5.56)

2) $\log |G(t)| \in H_\mu, \quad g(t) \in H_\mu,$ (5.57)

3) $g(\infty) = 0,$ (5.58)

4) either $G(1) = 1$ and $g(1) = 0$, or $G(1) \neq 1$. (5.59)

Denote $D = \mathbb{C} \backslash L$. Let B be the class of functions bounded and analytic in D (This is a special case of the class B_L for $t_0 = 1$). We will study the problem (5.53) in the class B. Similarly to the the problem with finite index case, it suffices to construct a special solution $\Phi_0(z)$, then the general solution has the form $\Phi(z) = \Phi_0(z) + \Psi(z)$, where $\Psi(z)$ is the general solution of the corresponding homogeneous problem $\Psi^+(t) = \Psi^-(t)G(t)$, $1 < t < \infty$.

Let $\Psi_0(z) \in B$ be a special solution of the homogeneous problem, which does not vanish on $\tilde{L}$. Then dividing (5.53) by $\Psi^+(t)$, and taking into account the formula (5.11) we can represent the boundary condition (5.53) in the form

$$\frac{\Phi^+(t)}{\Psi_0^+(t)} - \frac{\Phi^-(t)}{\Psi_0^-(t)} = \frac{g(t)}{\Psi_0^+(t)}.$$

Then a special solution is supposed to be in the form

$$\Phi_0(z) = \frac{\Psi_0(z)}{2\pi i} \int_1^\infty \frac{g(x)}{\Psi_0^+(x)(x-z)} dx, \tag{5.60}$$

and what remains is to choose the function $\Psi_0(z)$ to provide the convergence of the integral in the RHS of (5.60) and also the inclusion of the obtained solution into the class B.

Note that, generally speaking, the canonical function $X(z)$ can not be chosen as the function $\Psi_0(z)$, since (5.36) yields

$$|X^+(t)| < \exp(-at^\rho), \quad 0 < a < \pi\lambda \cot \rho\pi, \quad t > t_0, \tag{5.61}$$

then to make the integral convergent when $\Psi_0(z) = X(z)$ we must add the condition

$$|g(t)| < \exp(-bt^\rho), \quad b > 0, \quad t > t_0. \tag{5.62}$$

According to Theorem 5.5 the function $\Psi_0(z)$ must be constructed in the form $\Psi_0(z) = X(z)F_0(z)$, where $F_0(z)$ is an entire function satisfying some special conditions. These functions have to admit some non uniqueness of the desired function $F_0(z)$. We will show that this non uniqueness may be controlled in such a way that the obtained function $\Psi_0(z)$ will possess the following properties:

$$1)\ \Psi_0(z) \in B, \qquad 2)\ 1/|\Psi_0^+(t)| < C, \quad c \le t < \infty, \tag{5.63}$$

$$3)\ g(t)/\Phi_0^+(t) \in H([c,\infty)), \qquad \lim_{t\to\infty} g(t)/\Psi_0^+(t) = 0. \tag{5.64}$$

(From now on c will be an arbitrary fixed number bigger than 1.)

Observe that $\rho_{F_0} = \rho$ where ρ_{F_0} is the order of the function $F_0(z)$, because we know that by Theorem 5.4 the order $\rho_{F_0} \le \rho$, but if $\rho_{F_0} < \rho$, then by virtue of Theorem 5.7, the function $\Psi^+(t)$ would satisfy an inequality of the form (5.61), which means that we need to introduce condition (5.62) again. Thus $\rho_{F_0} = \rho$.

By Theorem 5.2

$$h_{\Psi_0}(\theta) = -\frac{\pi\lambda}{\sin \pi\rho} \cos \rho(\theta - \pi) + h_{F_0}(\theta), \quad 0 \le \theta \le 2\pi.$$

Since the function $\Psi_0(z)$ satisfies (5.63), $h_{\Psi_0}(0) = 0$ and hence

$$h_{F_0}(0) = \lambda\pi \cot \rho\pi. \tag{5.65}$$

Further 1) in (5.63) yields $h_{\Psi_0}(\theta) \le 0$ and, hence,

$$h_{F_0}(\theta) \le \frac{\pi\lambda}{\sin \pi\rho} \cos \rho(\theta - \pi) = -h_X(\theta), \quad 0 < \theta < 2\pi. \tag{5.66}$$

This means geometrically that the curve $h_{F_0}(\theta)$ lies below the curve $-h_X(\theta)$ and touches it at the points $\theta = 2\pi n$, $n \in \mathbb{Z}$.

There are many functions satisfying (5.65) and (5.66). The simplest ones are those having zeros located on one ray. By virtue of (5.63) the function $F_0(z)$ does not vanish on the ray $\arg z = 0$. Let us choose the ray $\arg z = \pi$ and construct the desired function in the form

$$F_0(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{r_n}\right), \quad 0 < r_1 < r_2 < ...,$$

for some still unspecified values $r_n, n \in \mathbb{Z}^+$.

Let $n(r)$ be the counting function of zeros of the function $F_0(z)$. We demand the existence of the limit

$$\lim_{r\to\infty} r^{\rho} n(r) = \Delta, \qquad 0 < \Delta < \infty.$$

The value of this limit is called the *density* of zeros, and will be determined later.

Taking into account the identity that we proved in Lemma 5.2.

$$z\int_0^{\infty} \frac{dx}{x^{1-\rho}(x-z)} = -\frac{\pi z^{\rho}}{\sin\pi\rho} e^{-i\rho\pi}, \quad 0 < \arg z < 2\pi,$$

together with the Riemann-Stieltjes integral, we obtain

$$\begin{aligned} \log F_0(t) &= \int_0^{\infty} \log\left(1 + \frac{t}{x}\right) dn(x) \\ &= \frac{\pi\Delta}{\sin\pi\rho} t^{\rho} + t\int_0^{\infty} \frac{n(x) - \Delta x^{\rho}}{x(x+t)} dx. \end{aligned} \tag{5.67}$$

LEMMA 5.5 *The function $X^+(t)$ admits the representation*

$$\log X^+(t) = i(\eta\log t + \lambda\pi t^{\rho}) - \lambda\pi\cot(\pi\rho)t^{\rho} + q(t), \tag{5.68}$$

where $q(t) \in H_{\sigma}([c,\infty)), \sigma = \min\{\mu_0 - \rho, \mu\}$ and $\mu < 1$ is the exponent in the Hölder condition in (5.57) and η is a real constant.

Proof: STEP 1: By Plemelj-Sokhotski formula we have,

$$\log X^{+}(t) = \frac{1}{2}\log G(t) + \frac{t}{2\pi i}\int_{1}^{\infty}\frac{\log G(\tau)}{\tau(\tau - t)}d\tau$$

Now by changing the variables $\tau = 1/u$, $t = 1/v$, then by (5.42) and by the fact that $\log G(t) = \log|G(t)| + i\arg G(t) = \log|G(t)| + 2\pi i\varphi(t)t^{\rho}$, we have

$$\begin{aligned}
\log X^{+}(1/v) &= \frac{1}{2}\log G(1/v) - \frac{1}{2\pi i}\int_{0}^{1}\frac{\log G(1/u)}{u-v}d\tau \\
&= \frac{1}{2}\log|G(1/v)| + \pi i\varphi(1/v)\frac{1}{v^{\rho}} \\
&\quad -\frac{1}{2\pi i}\int_{0}^{1}\frac{\log|G(1/u)|}{u-v}du - \int_{0}^{1}\frac{\varphi(1/u)}{u^{\rho}(u-v)}du. \qquad (5.69)
\end{aligned}$$

Letting $\log|G(1/u)| = \psi(u)$ and $\varphi_{*}(u) = \varphi(1/u)$, and then adding and subtracting the term

$$\int_{0}^{1}\frac{\lambda}{u^{\rho}(u-v)}du + \frac{\pi i\lambda}{v^{\rho}}$$

to (5.69), we get

$$\begin{aligned}
\log X^{+}(1/v) &= -\frac{1}{2\pi i}\int_{0}^{1}\frac{\psi(u)}{u-v}dv - \int_{0}^{1}\frac{\varphi_{*}(u)-\lambda}{u^{\rho}(u-v)}du \\
&\quad -\lambda\int_{0}^{1}\frac{du}{u^{\rho}(u-v)} + \frac{\psi(v)}{2} + \frac{\pi i}{v^{\rho}}(\varphi_{*}(v) - \lambda) + \frac{\pi i\lambda}{v^{\rho}}, \qquad (5.70)
\end{aligned}$$

for $0 < v < 1$. Using relations (5.55)-(5.57), we obtain $\psi(u) \in H_{\mu}([c,\infty))$. Also observe that by definition $\varphi_{*}(u) \in H_{\mu}([c,\infty))$ and $\varphi_{*}(0) = \lambda$. Taking into account the properties of a singular integral in a neighborhood of point of discontinuity of the density (see [5], Ch. 3), we find

$$-\frac{1}{2\pi i}\int_{0}^{1}\frac{\psi(u)}{u-v}du = \frac{\psi(0)}{2\pi i}\log v + \psi_{0}(v), \quad \psi_{0}(v) \in H_{\mu}.$$

STEP 2: We claim that

$$(\varphi_{*}(u) - \lambda)u^{-\rho} \in H_{\mu_0-\rho}([1,\infty)). \qquad (5.71)$$

Hence we have $\lim_{u\to\infty}(\varphi_*(u)-\lambda)u^{-\rho}=0$ and by Theorem 3.2 the second integral in (5.70) satisfies Hölder condition as well, i.e.,

$$-\int_0^1 \frac{\varphi_*(u)-\lambda}{u^\rho(u-v)}du \in H. \tag{5.72}$$

Now let us prove the claim. In fact what we have in general is the following claim:

If $\varphi(t)\in H_\mu(L)$ and if $0\le\rho<\mu\le 1$, then the function

$$\Psi(t)=\frac{\varphi(t)-\varphi(t_0)}{|t-t_0|^\rho}$$

where t_0 is a fixed point on L, satisfies Hölder condition on L with the index $\mu-\rho$.

For the proof, without loss in generality let $|t-t_0|=r$. Define $\omega(r)=\varphi(t)-\varphi(t_0)$, i.e., $\omega(r+h)=\varphi(t+h)-\varphi(t_0)$, where $h>0$ (this does not effect the generality as well).Then

$$\begin{aligned}|\Psi(r+h)-\Psi(r)| &= \left|\frac{\omega(r+h)}{(r+h)^\rho}-\frac{\omega(r)}{r^\rho}\right| \\ &= \left|\frac{\omega(r+h)-\omega(r)}{(r+h)^\rho}+\omega(r)\left\{\frac{1}{(r+h)^\rho}-\frac{1}{r^\rho}\right\}\right| \\ &\le \frac{|\omega(r+h)-\omega(r)|}{(r+h)^\rho}+|\omega(r)|\left\{\frac{(r+h)^\rho-r^\rho}{r^\rho(r+h)^\rho}\right\} \\ &=: \Delta_1+\Delta_2.\end{aligned}$$

For estimation of Δ_1, we note

$$|\omega(r+h)-\omega(r)|=|\varphi(t+h)-\varphi(t_0)|\le Ah^\mu,$$

therefore

$$\Delta_1\le\frac{Ah^\mu}{(r+h)^\rho}=A\left(\frac{h}{r+h}\right)^\rho h^{\mu-\rho},$$

which gives us that Δ_1 satisfies the required condition.

For estimation of Δ_2, we observe that $|\omega(r)| = |\varphi(t) - \varphi(t_0)| \leq Ar^\mu$ and so

$$\Delta_2 \leq Ar^{\mu-\rho}\frac{(r+h)^\rho - r^\rho}{(r+h)^\rho}.$$

Now we will consider two possible cases $r \leq h$ and $r > h$.

In the first case $r \leq h$. Using the inequality

$$(r+h)^\rho - r^\rho \leq h^\rho$$

we see that

$$\Delta_2 \leq A\left(\frac{h^\mu}{(r+h)^\rho}\right) \leq A\left(\frac{h}{r+h}\right)^\rho h^{\mu-\rho} \leq Ah^{\mu-\rho}.$$

In the second case $r > h$, using the inequality

$$(r+h)^\rho - r^\rho = r^\rho\left[\left(1+\frac{h}{r}\right)^\rho - 1\right] \leq \rho h r^{\rho-1},$$

we see that

$$\Delta_2 \leq A\rho h r^{\mu-\rho-1} = A\rho\left(\frac{h}{r}\right)^{1+\rho-mu} \cdot h^{\mu-\rho} \leq A\rho h^{\mu-\rho}.$$

Thus the statement is verified.

Therefore we have (5.71) and so (5.72).

STEP 3: For the third integral in (5.70), we claim that

$$-\lambda\int_0^1 \frac{du}{(u-v)u^\rho} = -\frac{\lambda\pi}{v^\rho}\cot\rho\pi + \Omega(v), \quad \Omega(v) \in H_1. \tag{5.73}$$

The proof of this is similar to the proof of the identity in Lemma 5.2. Consider the integral

$$I(t) = t\int_C \frac{d\xi}{\xi^{1-\rho}(\xi - t)},$$

where C is the contour consisting of the real axis from $\xi = r$ to $\xi = t - \epsilon$, say l_1, where $0 < r < t - \epsilon$ and $\epsilon > 0$ being a sufficiently small number; and then a semi circle C_{ϵ_1} of radius ϵ above the real axis around t, joining $\xi = t - \epsilon$ and $\xi = t + \epsilon$; then the real axis from $\xi = t + \epsilon$ to $\xi = R$, say l_2; a circle C_R of radius R above the real axis; then from $\xi = R$ to $\xi = t + \epsilon$, say $l_{2'}$; a semi circle C_{ϵ_2} of radius ϵ under the real axis around t, joining $\xi = t + \epsilon$ and $\xi = t - \epsilon$; then the real axis from $\xi = t - \epsilon$ to $\xi = r$, say $l_{1'}$; and finally a circle C_r of radius r around 0.

The multivalued function $\xi^{1-\rho}$ is taken to be real, say $\tau^{1-\rho}$, on l_1, C_{ϵ_1}, l_2 and at all other points complex valued. Then on $l_{1'}, C_{\epsilon_2}, l_{2'}$ it takes the value $\tau^{1-\rho}e^{2\pi i(1-\rho)}$. By the Residue theorem $I(t) = 0$, and

$$I(t) = t\left(\int_{l_1} + \int_{C_{\epsilon_1}} + \int_{l_2} + \int_{C_R} + \int_{l_{2'}} + \int_{C_{\epsilon_2}} + \int_{l_{1'}} + \int_{C_r}\right) \frac{d\xi}{\xi^{1-\rho}(\xi - t)}.$$

Now observe that

$$\left| t\int_{C_R} \frac{d\xi}{\xi^{1-\rho}(\xi - t)}\right| \leq 2\pi t \frac{R^\rho}{R-t} \to 0 \quad \text{as} \quad R \to \infty,$$

since $\rho < 1/2$ and $|\xi| = R$ here.

Also since $\rho < 0$ and $\xi = r$ on C_r,

$$\left| t\int_{C_r} \frac{d\xi}{\xi^{1-\rho}(\xi - t)}\right| \leq 2\pi t \frac{r^\rho}{r-t} \to 0 \quad \text{as} \quad r \to 0.$$

By the above consideration

$$t\left(\int_{l_1} + \int_{l_2} + \int_{l_{2'}} + \int_{l_{1'}}\right) \frac{d\xi}{\xi^{1-\rho}(\xi - t)} = (1-e^{-2\pi i(1-\rho)})t\left(\int_{l_1} + \int_{l_2}\right) \frac{d\tau}{\tau^{1-\rho}(\tau - t)}.$$

Moreover by using the Laurent series expansion around t, we get

$$\begin{aligned} t\int_{C_{\epsilon_1}} \frac{d\xi}{\xi^{1-\rho}(\xi - t)} &= t\int_{C_{\epsilon_1}} \frac{d\tau}{\tau^{1-\rho}(\tau - t)} \\ &= t^\rho \int_{\epsilon_1} \frac{d\tau}{(\tau - t)} + I_1(t), \end{aligned}$$

where $I_1(t)$ is analytic. Then since $\tau = t + \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$,

$$t \int_{C_{\epsilon_1}} \frac{d\tau}{\xi^{1-\rho}(\tau - t)} = -t^\rho \int_0^\pi i d\theta + I_1(t) = -\pi i t^\rho + I_1(t).$$

Similarly, since $\xi^{1-\rho} = \tau^{1-\rho} e^{2\pi i(1-\rho)}$ on C_{ϵ_2} and using Laurent series expansion around t again, we get

$$\begin{aligned} t \int_{C_{\epsilon_2}} \frac{d\xi}{\xi^{1-\rho}(\xi - t)} &= \frac{t^\rho}{e^{2\pi i(1-\rho)}} \int_{C_{\epsilon_2}} + I_2(t) \frac{\tau}{\tau - t} \\ &= -t^\rho e^{2\pi i(1-\rho)} \int_{-\pi}^0 i d\theta = -\pi i t^\rho e^{2\pi i \rho} + I_2(t), \end{aligned}$$

where $I_2(t)$ is analytic.

Hence using all these calculations and letting $\epsilon \to 0$ where, we get

$$I(t) = (-2i \sin \pi\rho e^{2\pi i\rho}) t \int_0^\infty \frac{d\tau}{\tau^{1-\rho}(\tau - t)} + \pi i t^\rho (1 + e^{2\pi i \rho}),$$

as $r \to 0$ and $R \to \infty$. Moreover, $I(t)$ is equal to 0 by the Residue theorem. Hence we get

$$t \int_0^\infty \frac{d\tau}{\tau^{1-\rho}(\tau - t)} = -\pi t^\rho \cot \rho\pi. \tag{5.74}$$

Therefore

$$\begin{aligned} -\int_0^1 \frac{du}{u^\rho (u - v)} &= t \int_1^\infty \frac{d\tau}{\tau^{1-\rho}(\tau - t)} \\ &= -t \int_0^1 \frac{d\tau}{\tau^{1-\rho}(\tau - t)} + t \int_0^\infty \frac{d\tau}{\tau^{1-\rho}(\tau - t)} \\ &=: I_3(t) + t \int_0^\infty \frac{d\tau}{\tau^{1-\rho}(\tau - t)}. \end{aligned} \tag{5.75}$$

But $I_3(t)$ satisfies Lipschitz condition. For the proof of this, observe for some t, t_1, we have

$$\begin{aligned} |I_3(t) - I_3(t_1)| &= \left| t \int_0^1 \frac{d\tau}{\tau^{1-\rho}(\tau - t)} - t_1 \int_0^1 \frac{d\tau}{\tau^{1-\rho}(\tau - t_1)} \right| \\ &\le \left| \int_0^1 \frac{\tau^\rho (t - t_1)}{(\tau - t)(\tau - t)} d\tau \right| \\ &\le C|t - t_1|, \end{aligned}$$

where C is a constant. So, $I_3(t) \in H_1$. Hence by (5.74) and (5.75) we get (5.73).

STEP 4: Using step 1, step 2 and step 3 in the formula (5.70), we get

$$\log X^+(1/v) = i\left(-\frac{\psi(0)}{2\pi}\log v + \frac{\pi\lambda}{v^\rho}\right) - \frac{\pi\lambda}{v^\rho}\cot\rho\pi + Q(v),$$

where $Q(v) \in H_\sigma$. Turning back to initial variables τ and t, we get the lemma. □

Comparing relations (5.67) and (5.68), define $\Delta = \lambda\cos\rho\pi$. Then we have

$$\log\Psi^+(t) = i(\eta\log t + \lambda\pi t^\rho) + q(t) + t\int_0^\infty \frac{n(x) - \Delta x^\rho}{x(x+t)}dx. \qquad (5.76)$$

Now $n(x)$ must be chosen "close enough" to Δx^ρ in such a way that

$$I(t)\left(= t\int_0^\infty \frac{n(x) - \Delta x^\rho}{x(x+t)}dx\right) \in H_\rho([c,\infty)). \qquad (5.77)$$

Letting

$$n(x) = \left[\Delta x^\rho + \frac{1}{2}\right]$$

we have (5.77), which we will prove later on.And this gives us that

$$r_n = \left(\frac{2n-1}{2\lambda\cos\rho\pi}\right)^{\frac{1}{\rho}}, \quad n = 1, 2, ...$$

So this completes the construction of the functions $F_0(z)$ and $\Psi_0(z)$.

5.5 Solution of the non-homogeneous problem

THEOREM 5.8 *Let*

$$F_0(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{r_n}\right), \quad r_n = \left(\frac{2n-1}{2\lambda\cos\rho\pi}\right)^{\frac{1}{\rho}}.$$

Then both the solutions of the homogeneous problem

$$\Psi_0(z) = X(z)F_0(z),$$

and the solution of the non-homogeneous problem

$$\Phi_0(z) = \frac{\Psi_0(z)}{2\pi i}\int_1^\infty \frac{g(x)}{\Psi_0^+(x)(x-z)}dx, \tag{5.78}$$

belong to the class B.

We need several auxiliary statements.

LEMMA 5.6 *If a function $f(t)$ has continuous derivative on the segment $[c,\infty)$ which satisfies $|f'(t)| < Ct^{-1-\rho}$, for some ρ, $0 < \rho \le 1$, then $f(t) \in H_\rho([c,\infty))$.*

Proof: Since

$$|f(t_1) - f(t_2)| = \left|\int_{t_1}^{t_2} f'(t)dt\right| \le \frac{C}{\rho}\left|\frac{1}{t_1^\rho} - \frac{1}{t_2^\rho}\right| \le \frac{C}{\rho}\left|\frac{1}{t_1} - \frac{1}{t_2}\right|^\rho,$$

it follows that $f \in H_\rho([c,\infty))$. $\square$

LEMMA 5.7 *Let $f(x)$ and $\varphi(x)$ be real-valued functions on the segment $[c,\infty)$. Suppose that the function $f(t)$ is Riemann integrable on every finite segment $[c,t]$ and*

$$\left|\int_c^t f(x)dx\right| < C,$$

and also the function $\varphi(x)$ is continuous, and has at most n extremal points on $[c,\infty)$ and $\varphi(\infty) = 0$. Then

$$\left|\int_c^t f(x)\varphi(x)dx\right| < C(2n+2)\max_{c\le x<\infty}|\varphi(x)|$$

for any $t \in [c,\infty)$.

Proof: Take $b > c$ sufficiently large and find $n+2$ points $a = x_0 < x_1 < \cdots < x_{n+1} = b$ such that the function $\varphi(x)$ is monotonic on each interval $[x_k, x_{k+1}]$. The second mean value theorem gives

$$\begin{aligned}\left|\int_c^b f(x)\varphi(x)dx\right| &= \left|\sum_{k=0}^{n}\left\{\varphi(x_k)\int_{x_k}^{\xi_k} f(x)dx + \varphi(x_{k+1})\int_{\xi_k}^{x_{k+1}} f(x)dx\right\}\right| \\ &\leq (2n+2)C \max_{c\leq x<\infty} |\varphi(x)|.\end{aligned}$$

In the case $b = \infty$ the convergence of the integral follows from the fact that $\varphi(x)$ approaches zero monotonically as $x \to +\infty$, $x \in [x_{n+1}, \infty)$. □

LEMMA 5.8 *If* $n(x) = [\Delta x^\rho + 1/2]$, *then*

$$I(t)\left(= t\int_0^\infty \frac{n(x) - \Delta x^\rho}{x(x+t)}dx\right) \in H_\rho([c, \infty)).$$

Proof: Let $s = t\Delta^{1/\rho}$ and let $u = \Delta x^\rho$. Denote $\tilde{n} = n(\Delta^{-1/\rho}u^{1/\rho})$, we have

$$I(t) = \frac{s}{\rho}\int_0^\infty \frac{\tilde{n}(u) - u}{u(u^{1/\rho} + s)}du =: \frac{1}{\rho}I_0(s).$$

It suffices to prove that $I_0(s) \in H_\rho$. We have

$$I_0'(s) = \int_0^\infty \{\tilde{n}(u) - u\}\frac{u^{1/\rho - 1}}{(u^{1/\rho} + s)^2}du, \quad \tilde{n}(u) = \left[u + \frac{1}{2}\right].$$

For a fixed $\rho > 0$ denote

$$f(u) = \tilde{n}(u) - u, \quad \varphi(u) = u^{1/\rho-1}(u^{1/\rho} + s)^{-2}.$$

The function $\varphi(u)$ has the only extremal point in $[0, \infty)$. This is the point of maximum. We have

$$\max_{0\leq u<\infty} \varphi(u) = \frac{1}{4}s^{-1-\rho}(1-\rho)^{1-\rho}(1+\rho)^{1+\rho} < \frac{1}{2}s^{-1-\rho},$$

$$\varphi(\infty) = 0.$$

Since

$$\left| \int_0^t f(u)du \right| \leq \frac{1}{8}$$

for any $t > 0$, Lemma 5.7 yields $|I_0'(s)| \leq \frac{5}{16}s^{-1-\rho}$ and, by Lemma 5.6, $I_0(s) \in H_\rho([c,\infty))$. $\square$

LEMMA 5.9 *Let $f(t) \in H_\mu([c,\infty))$, $f(\infty) = 0$, and σ be a real number. Then*

$$\begin{aligned} \psi_1(t) &= f(t)e^{i\sigma \log t} \in H_\mu([c,\infty)), \\ \psi_2(t) &= f(t)e^{i\sigma t^\rho} \in H_{\frac{\mu}{\rho+1}}([c,\infty)). \end{aligned}$$

Proof: We restrict ourselves to the consideration of the function $\varphi_2(t)$. The function $\varphi_1(t)$ may be studied in the same way and even a little simpler. To have a finite segment $[0, 1/c]$ let us change the variables by setting $t = 1/u$ and denote $g(u) = \varphi_2(1/u)$, $h(u) = f(1/u)$. For $0 < u_1 < u_2 < 1/c$ we have

$$\begin{aligned} |g(u_1) - g(u_2)| &\leq |h(u_1) - h(u_2)| + |h(u_1)||\exp(-i\sigma u_2^{-\rho}) - \exp(-i\sigma u_1^{-\rho})| \\ &\leq C(u_2 - u_1)^\mu + C_1 u_1^\mu \min\{2, \sigma\rho(u_2 - u_1)u_1^{-1-\rho}\}. \end{aligned}$$

If $2 \leq \sigma\rho(u_2 - u_1)u_1^{-1-\rho}$, then $u_1^\mu \leq (\sigma\rho/2)^{\mu/(\rho+1)}(u_2 - u_1)^{\mu/(\rho+1)}$ and $\min\{2, \sigma\rho(u_2 - u_1)u_1^{-1-\rho}\} = 2$, that which implies $g(u) \in H_{\mu/(\rho+1)}([0, 1/c])$. If $2 \geq \sigma\rho(u_2 - u_1)u_1^{-1-\rho}$, then

$$u_1^{\mu-\rho-1} \leq (\sigma\rho/2)^{(\mu-\rho-1)/(\rho+1)}(u_2 - u_1)^{(\mu-\rho-1)/(\rho+1)}$$

and

$$\min\{2, \sigma\rho(u_2 - u_1)u_1^{-1-\rho}\} = 6\rho(u_2 - u_1)u_1^{-1-\rho},$$

so we again obtain the desired conclusion. $\square$

Proof of Theorem 5.8: First we will prove that $\Psi_0(z) \in B$. If a function is in the class $H([c,\infty))$ then it is bounded; hence $q(t)$ in (5.76) is bounded. Therefore, taking the real part of (5.76) and using the Lemma 5.8, we get $|\Psi_0^+(t)| \le C \le \infty, c \le t < \infty$. Remember that $0 < C_1 \le |G(t)| \le C_2 < \infty, c < t < \infty$, by (5.10), and $|\Psi_0^+(t)| \le C, c \le t < \infty$ by (5.11). Since $\Psi_0(z) = X(z)F_0(z)$ where $F_0(z)$ is an entire function, the functions $\Psi^{\pm}(t)$ are bounded on $[1,c]$ and, hence, on $[1,\infty)$. Since $\rho_{F_0} = \rho < 1/2$, by applying the Phragmén-Lindelõf principle, we obtain $\Psi_0(z) \in B$.

Now, we will prove that $\Phi_0(z) \in B$. Using (5.76) and Lemma 5.8 we see that

$$\frac{1}{\Psi_0^+(t)} = Q(t)\exp\{-i(\eta\log t + \lambda\pi t^{\rho})\}, \quad c \le t \le \infty,$$

where $Q(t) \in H([c,\infty))$. Since $g(t) \in H$ and $g(\infty) = 0$, Lemma 5.9 gives

$$g(t)/\Psi_0^+(t) \in H([c,\infty)), \quad \lim_{t\to\infty} g(t)/\Psi_0^+(t) = 0. \tag{5.79}$$

Now, rewrite equality (5.78) in the form

$$\Phi_0(z) = \frac{\Psi_0(z)}{2\pi i}\left[\int_1^c + \int_c^\infty\right]\frac{g(x)}{\Psi_0^+(x)(x-z)}dx.$$

It follows from (5.79) that the second integral is bounded in any domain $|z-c| > \epsilon$. In the neighborhood of $[1,c]$ the first integral can be estimated as

$$\int_1^c \frac{g(x)}{\Psi_0^+(x)(x-z)}dx = O\left(\frac{1}{|z|}\right), \quad z \to \infty.$$

This is because $g(x)/\Phi_0^+(x)$ is bounded and $|x-z| \ge |z| - |x| \ge |z| - c$ in the neighborhood of $[1,c]$.

Since $\Psi_0(z) \in B$ and c can be chosen arbitrary close to 1, the function $\Phi_0(z)$ is bounded in $|z-1| > \epsilon$.

It remains to study the behavior of $\Phi_0(z)$ near the point $t = 1$. Since $F_0(z) \neq 0$ in the neighborhood of point $z = 1$, we may represent the function $\log \Psi_0(z)$ in the form

$$\begin{aligned} \log \Psi_0(z) &= -\frac{\log G(1)}{2\pi i} \log(z-1) \\ &\quad + \frac{1}{2\pi i} \int_1^c \frac{\log G(x) - \log G(1)}{x - z} dx + f_0(z), \end{aligned} \tag{5.80}$$

for all z from a sufficiently domain $D \cap \{z : |z-1| < c\}$, where $f_0(z)$ is analytic in $|z-1| < c$. Denote

$$-\log G(1)/(2\pi i) = \alpha + \beta i.$$

By (5.56) we have $0 \leq \alpha < 1$. Using the properties of boundary values of the Cauchy-type integrals (see, e.g. [5], Sec.22) we have

$$\log \Psi_0^+(t) = (\alpha + \beta i) \log(t-1) + P(t), \quad 1 \leq t \leq 1 + \delta,$$

for sufficiently small $\delta > 0$, where $P(t) \in H([1, 1+\delta])$. Then

$$\frac{g(t)}{\Psi_0^+(t)} = \psi(t)(t-1)^{-\alpha-\beta i}, \quad 1 \leq t \leq t + \delta,$$

where $\psi(t) \in H$, and also, by (5.59), we have if $\alpha + \beta i = 0$, then $\psi(1) = 0$. Therefore, (see, e.g. [5], Ch. 4), we have

$$\begin{aligned} \int_1^\infty \frac{g(x)}{\Psi_0^+(x)(x-z)} dx &= \int_1^{1+\delta} + \int_{1+\delta}^\infty \frac{g(x)}{\Psi_0^+(x)(x-z)} dx \\ &= \frac{Q}{(z-1)^{\alpha+\beta i}} + \varphi_0(z), \end{aligned} \tag{5.81}$$

for sufficiently small $|z-1|$. Here $Q = 0$ if $\alpha + \beta i = 0$ and the function $\varphi_0(z)$ is either bounded if $\alpha = 0$, or admits the estimate $|\varphi(z)| = O(|z-1|^{-\alpha_0})$, $0 < \alpha_0 < \alpha, z \to 1$ if $\alpha > 0$. Now (5.80) and (5.81) yields the boundedness of the function $\Phi_0(z)$ in the neighborhood of the point $z = 1$. This completes the proof.

THEOREM 5.9 *Let conditions (5.54)-(5.59) be fulfilled. Then the non-homogeneous Riemann boundary value problem (5.53) has infinitely many solutions in the class B. The general formula of the solution is as follows:*

$$\Phi(z) = X(z)\left\{\frac{F_0(z)}{2\pi i}\int_1^\infty \frac{g(x)}{X^+(x)F_0(x)(x-z)}dx + F(x)\right\}, \qquad (5.82)$$

where

$$F_0(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right), \quad r_n = \left(\frac{2n-1}{2\lambda\cos\rho\pi}\right)^{1/\rho},$$

and $F(z)$ is an entire function, whose order does not exceed ρ, and such that

$$\log|F(t)| \leq (\pi\lambda\cot\pi\rho)t^\rho + O(1), \quad t\to\infty. \qquad (5.83)$$

Proof: By Theorem 5.6 we know the explicit expression of the general form of the homogeneous Riemann boundary value problem. Together with the explanation in section 5.4 and the Theorem 5.8 we get that the non-homogeneous Riemann boundary value problem (5.53) has infinitely many solutions in the class B and the general form is as in (5.82), together with the relation (5.42) instead of (5.83). But under the condition (5.55) and (5.56) relations (5.42) and (5.83) are equivalent. Actually the arguments that are used when proving Lemma 5.5 imply that, under the assumption (5.55), we have

$$t\int_1^\infty \frac{\varphi(x)}{x^{1-\rho}(x-t)}dx = -(\lambda\pi\cot\pi\rho)t^\rho + p(t), \quad p(t)\in H([c,\infty)).$$

Hence together with (5.42) we have (5.83). □

Bibliography

[1] Gakhov, F.D., *Boundary Value Problems.* Pergamon Press, New York, 1966.

[2] Govorov, N.V., *Riemann's Boundary Problem with Infinite Index.* Birkhauser Verlag, Basel-Boston, 1994.

[3] Levin, B.Ya., *Lectures on entire functions.* American Mathematical Society, Rhode Island-U.S.A., 1996.

[4] Lu, J.K., *Boundary Value Problems for Analytic Functions*, vol. 16. World Scientific, Singapore, 1993.

[5] Muskhelishvili, N.I., *Singular Integral Equations.* P. Noordhoff N.V., Groningen, Holland, 1953.

[6] Titchmarsh, E.C., *The Theory of Functions*, 2^{nd} ed..Oxford University Press, New York, 1939.