ANALYSIS OF A FULLY DISCRETE FOURIER PSEUDOSPECTRAL METHOD FOR THE ROSENAU EQUATION

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ABSTRACT

In this thesis study, we consider the Rosenau equation with a single power type nonlinear term. The Rosenau equation is proposed as an alternative to the celebrated Korteweg–de Vries (KdV) equation. It describes the propagation of nonlinear waves in anharmonic crystal lattices. This equation possesses solitary wave solutions when the order of nonlinear term and wave speed is greater than 1. However, an exact solution is not available in the literature when the nonlinear term is single power type. Therefore, the numerical investigation of the equation becomes important to understand the dynamics of the waves. There are many numerical studies in the literature mostly using the finite difference and finite element methods. Moreover, most of these studies focus on the numerical analysis of the scheme and do not present numerical results. In this work, we present a numerical scheme combining the Fourier pseudospectral method and second order finite difference method. We also present the truncation error and stability analysis of the proposed scheme. We then introduce some numerical results to verify the theoretical analysis. For this aim, we derive initial solitary wave profiles with the Petviashvili iteration method. Then, we observe the evolution of these solutions using the proposed scheme.

ÖZETÇE

Bu tez çalışmasında Rosenau denklemi tek bir doğrusal olmayan terim içerdiği durumda incelenmektedir. Rosenau denklemi Korteweg–de Vries (KdV) denklemine alternatif olarak önerilmiştir. Harmonik olmayan kristal latislerdeki (kafeslerdeki) dalgaların hareketini betimlemektedir. Bu denklem doğrusal olmayan terimin derecesi ve dalganın hızı birden büyük olduğu durumda soliter (yalnız) dalga çözümlerine sahiptir. Fakat bu soliter dalga çözümlerinin açık formu bilinmemektedir. Bu yüzden sayısal incelemeler dalgaların dinamiğini anlamak için oldukça önemlidir. Literatürde bu denklem ağırlıklı olarak sonlu farklar ve sonlu elemanlar yöntemleriyle çalışılmıştır fakat bu çalışmalar genel olarak önerilen sayısal şemanın yakınsaklık analizi üzerinedir, sayısal sonuçlar içermemektedir. Bu çalışmada Fourier pseudospektral ve sonlu farklar yöntemlerini birleştiren bir sayısal şema önerilmektedir. Ardından, önerilen sayısal şemanın yakınsaklık ve stabilite analizi yapılmaktadır. Teorik analizi doğrulamak için bazı sayısal sonuçlar ayrıca verilmiştir. Bu amaçla denklemin başlangıç soliter (yalnız) dalga profilleri Petviashvili yöntemiyle üretilmiştir. Ardından önerilen sayısal şema kullanılarak bu profillerin zamanda ilerlemesi gözlemlenmiştir.

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CHAPTER I

INTRODUCTION

In this thesis study, we consider the "Rosenau equation"

$$u_t + u_x + u_{xxxxt} + (g(u))_x = 0, \tag{R}$$

where $g(u) = u^p$ and $p \geq 2$ is an integer. The Korteweg-de Vries (KdV) equation models not only the waves at the surface of an inviscid fluid [1] but it also models the evolution of the long waves in anharmonic dense crystal lattices [2]. However, as the KdV equation is derived under the assumption of weak anharmonicity, both the shape and the behavior of high-amplitude waves cannot be well predicted. To overcome such shortcomings Rosenau proposed (R) as an alternative to the KdV equation.

The Cauchy problem associated with the Rosenau equation with the initial condition

$$u(0, x) = u_0(x) \tag{I}$$

has been studied by Park [3]. The author gives the existence of unique global solution with $g \in C^2(\mathbb{R})$ and $u_0 \in H_0^4(\Omega)$. The result has been generalized to the higher dimensional cases in [4].

There are studies in the literature presenting the exact solutions of (R) and its variants for special forms of $g(u)$. Recently, Demirci et al. [5] obtained the periodic wave solutions of (R) in terms of the elliptic functions for $g(u) = \frac{u^{p+1}}{p+1}$ and Alotaibi et al. [6] derived both the periodic and the solitary wave solutions of (R) for $g(u) = -10u^3 + 12u^5$. Even though the solitary wave solutions of (R) exist for $g(u) = u^p$ [7], there are no known exact solitary wave solutions in the literature. As an alternative, Zuo [8] proposed Rosenau-KdV and Rosenau-Kawahara equations by inserting additional dispersive terms to (R) and obtained solitary wave solutions of these equations via sine-cosine and tanh methods. However, these equations are not derived from physical principles. Due to limited knowledge about exact solutions, especially in the case of $g(u) = u^p$, (R) is widely studied from a numerical analysis point of view. For example, Erbay et al. [9] and Demirci et al. [5] generated solitary wave solutions of (R) for $g(u) = u^p$ by using Petviashvili iteration method [10]. Chung [11] proved the existence and uniqueness of the finite difference approximate solutions of (R) when $g(u) = \frac{u^2}{2}$. In [12], Chung et al. also studied finite element solutions of the generalized Rosenau equation and the corresponding error estimates. Lee et al. [13] constructed Galerkin-Crank-Nicolson fully discrete finite element method to approximate the solution of the generalized Rosenau equation. They also demonstrated the boundedness and optimal convergence rate of the approximate solution. Danumjaya et al. [14] developed finite element methods for (R) and they established consistency, a priori bounds and optimal error estimates for both semidiscrete and fully discrete schemes. Omrani [15] proposed a three-level conservative finite difference scheme for (R) when $g(u) = \frac{u^2}{2}$, then derived both the unconditional stability and uniqueness of the scheme. Similar to [15], Atouani et al. [16] also proposed a higher-order conservative finite difference scheme. Manickam et al. [17] applied the second-order splitting method to the Rosenau equation. Then an orthogonal cubic spline collocation method is used to approximate the resulting system. Erbay et al. [18] introduced a discrete convolution operator to approximate the solutions of a nonlocal nonlinear unidirectional wave equation which becomes (R) for some special form of the kernel function and in this case, the proposed semi-discrete scheme has a quadratic convergence rate in space.

Most of the studies in the literature focus on the numerical analysis of the scheme. A few studies present numerical experiments: In [14, 18, 19] the schemes are tested by using the exact solution given for the case where $g(u) = -10u^3 + 12u^5$. Numerical examples are given by using random functions as initial conditions in [11] and [16]. Recently, in [5], the propagation of the solitary wave solutions for (R) with $g(u) = u^p$ is numerically investigated.

As far as we know, there are not many studies in the literature applying the spectral methods to (R). In this study, we generate the solitary wave solutions of (R) for various nonlinearities by using "Petviashvili iteration method". We then combine a Fourier pseudospectral and a second-order finite difference method for space and time discretizations respectively to investigate the evolution of solutions in time. We also present the truncation error and stability analysis of our scheme. To the best of our knowledge, the current study is the first on the analysis of a Fourier pseudospectral method for the Rosenau equation.

CHAPTER II

PRELIMINARIES

2.1 Basic Inequalities and Definitions

Theorem 2.1 (Cauchy's inequality)**.** *For any $a, b \in \mathbb{R}$, we have*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}. \tag{C}$$

Theorem 2.2 (Cauchy-Schwarz inequality)**.** *For any two vector u and v in an inner product space V, we have*

$$\langle u, v\rangle \leq |\langle u, v\rangle| \leq ||u|| \cdot ||v||, \tag{C-S}$$

where $||u|| \coloneqq \langle u, u\rangle^{\frac{1}{2}}$.

Definition 2.1. For final time T and temporal step size Δt, we define the time index set as $\mathcal{T} \coloneqq \{1, 2, \cdots, M-1\}$. Here, $M \coloneqq \frac{T}{\Delta t}$ is the number of grid points in time. We will use the following norms in the analysis of the numerical scheme.

$$\langle f, g\rangle \coloneqq h \sum_{i=0}^{2N} f_i g_i \quad \text{and} \quad ||f||_2 \coloneqq \langle f, f\rangle^{\frac{1}{2}}.$$

$$||f||_{\ell^2(\mathcal{T})} \coloneqq \left(\Delta t \sum_{k=1}^{M-1} |f^k|^2\right)^{\frac{1}{2}}, \quad ||f||_{\ell^2(\mathcal{T};\ell^2)} \coloneqq \left(\Delta t \sum_{k=1}^{M-1} ||f^k||_2^2\right)^{\frac{1}{2}},$$

$$\text{and} \quad ||f||_{\ell^\infty(\mathcal{T};H^m)} \coloneqq \max_{k\in\mathcal{T}} ||f^k||_{H^m}.$$

Here H^m denotes the Sobolev space of order m. For its definition and theory, see Chapter 7 of [20].

The next corollary gives the estimate for the first-order derivative in terms of the second-order derivative. Note that, D_N denotes the derivative operator and acts on grid functions. It is evaluated by multiplying the Fourier coefficients of a grid function by ik.

Corollary 2.3. *For any periodic grid function f, we have*

$$||D_N f||_2^2 \leq \frac{1}{2}\left(||f||_2^2 + ||D_N^2 f||_2^2\right).$$

Proof.

$$||D_N f||_2^2 = \langle D_N f, D_N f\rangle = -\langle f, D_N^2 f\rangle \leq \frac{1}{2}\left(||f||_2^2 + ||D_N^2 f||_2^2\right).$$

In the second equality, we used the summation by parts formula. In the last inequality, we used Cauchy-Schwarz (C-S) and Cauchy (C) inequalities. □

Lemma 2.4 ([21], Lemma 1)**.** *Let f and g be non-negative grid functions and let g be non-decreasing. If*

$$f_l \leq g_l + Ck\sum_{i=1}^{l-1} f_{i+1},$$

for any $C, k \geq 0$, then

$$f_l \leq g_l \exp(Ckl), \quad l = 0, 1, \cdots, M-1.$$

2.2 Sobolev Space Preliminaries

In this section, we list some useful properties of the Sobolev spaces.

Theorem 2.5 ([20], Theorem 7.39)**.** *For $m > 1/2$, $H^m(\Omega)$ is a Banach algebra. So, for any $u, v \in H^m(\Omega)$, we have*

$$||uv||_{H^m(\Omega)} \leq C||u||_{H^m(\Omega)}||v||_{H^m(\Omega)}$$

where $\Omega \subseteq \mathbb{R}$ is an open and bounded interval.

The next theorem plays an important role in truncation error analysis. (2) is given in [22] without its proof. We present the proof of the inequality (1) which is needed for the analysis of the current scheme. In a similar fashion, one can also prove (2).

Theorem 2.6 (Estimates for the central differences)**.** *[22] For any $u \in H^3(0,T)$ and $M = \frac{T}{\Delta t}$, we have the following two estimates:*

$$\left(\Delta t \sum_{k=1}^{M-1} \left| \frac{u^{k+1} - u^{k-1}}{2\Delta t} - \partial_t u^k \right|^2 \right)^{\frac{1}{2}} \leq C\Delta t^2 ||u||_{H^3(0,T)}, \tag{1}$$

$$||D_t^2 u||_{\ell^2(\mathcal{T})} := \left(\Delta t \sum_{k=1}^{M-1} \left| \frac{u^{k+1} - 2u^k + u^{k-1}}{\Delta t^2} \right|^2 \right)^{\frac{1}{2}} \leq C||u||_{H^2(0,T)} \tag{2}$$

where, $u^k := u(\cdot, k\Delta t)$ for any $k \in \mathcal{T}$.

Proof. We will prove (1) only. Proof of (2) is similar to that given for (1). From Taylor's theorem with integral remainder, we have

$$u(\cdot, t + \Delta t) = u(\cdot, t) + \partial_t u(\cdot, t)\Delta t + \partial_t^2 u(\cdot, t)\Delta t^2 + \frac{1}{2}\int_t^{t+\Delta t} \partial_s^3 u(\cdot, s)(t + \Delta t - s)^2 ds, \tag{3}$$

$$u(\cdot, t - \Delta t) = u(\cdot, t) - \partial_t u(\cdot, t)\Delta t + \partial_t^2 u(\cdot, t)\Delta t^2 + \frac{1}{2}\int_t^{t-\Delta t} \partial_s^3 u(\cdot, s)(t - \Delta t - s)^2 ds. \tag{4}$$

Substracting (3) from (4) and taking the absolute value of both sides give the following:

$$\begin{aligned}
&\left| \frac{u(\cdot, t + \Delta t) - u(\cdot, t - \Delta t)}{2\Delta t} - \partial_t u(\cdot, t) \right| \\
&= \frac{1}{4\Delta t} \left| \int_t^{t+\Delta t} \partial_s^3 u(\cdot, s)(t + \Delta t - s)^2 ds - \int_t^{t-\Delta t} \partial_s^3 u(\cdot, s)(t - \Delta t - s)^2 ds \right| \\
&\leq \frac{1}{4\Delta t} \left(\int_t^{t+\Delta t} \left| \partial_s^3 u(\cdot, s)(t + \Delta t - s)^2 \right| ds + \int_{t-\Delta t}^{t} \left| \partial_s^3 u(\cdot, s)(t - \Delta t - s)^2 \right| ds \right) \qquad (5) \\
&\leq \frac{1}{4\Delta t} \Bigg(\left(\int_t^{t+\Delta t} |\partial_s^3 u(\cdot, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_t^{t+\Delta t} (t + \Delta t - s)^4 ds \right)^{\frac{1}{2}} \\
&\qquad + \left(\int_{t-\Delta t}^{t} |\partial_s^3 u(\cdot, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_{t-\Delta t}^{t} (t - \Delta t - s)^4 ds \right)^{\frac{1}{2}} \Bigg) \qquad (6) \\
&= \frac{\Delta t^{\frac{3}{2}}}{4\sqrt{5}} \left(\left(\int_t^{t+\Delta t} |\partial_s^3 u(\cdot, s)|^2 ds \right)^{\frac{1}{2}} + \left(\int_{t-\Delta t}^{t} |\partial_s^3 u(\cdot, s)|^2 ds \right)^{\frac{1}{2}} \right) \qquad (7) \\
&\leq C\Delta t^{\frac{3}{2}} ||u||_{H^3(t-\Delta t, t+\Delta t)} \qquad (8)
\end{aligned}$$

(5) comes from the triangle inequality and the property $\left|\int_a^b f dx\right| \le \int_a^b |f| dx$. For (6), we use Hölder's inequality. In (7), we calculate the integrals. For (8), we use the inclusion $[t, t+\Delta t], [t-\Delta t, t] \subset [t-\Delta t, t+\Delta t]$ together with the definition of Sobolev norm. Taking the square of both sides of (8) and then multiplying both sides by Δt, we obtain;

$$\Delta t \left| \frac{u(\cdot, t+\Delta t) - u(\cdot, t-\Delta t)}{2\Delta t} - \partial_t u(\cdot, t) \right|^2 \le C\Delta t^4 ||u||^2_{H^3(t-\Delta t, t+\Delta t)}. \tag{9}$$

We do summation on both sides of (9) over set $\mathcal{T}$ and we switch to notation of the Definition 2.1.

$$\Delta t \sum_{k=1}^{M-1} \left| \frac{u^{k+1} - u^{k-1}}{2\Delta t} - \partial_t u^k \right|^2 \le C\Delta t^4 \sum_{k=1}^{M-1} ||u||^2_{H^3((k-1)\Delta t, (k+1)\Delta t)}. \tag{10}$$

For the sum in the RHS of (10), first, we use the property $\int_a^b = \int_a^c + \int_c^b$ of the integral. This property unites the integrals. Then, the inclusion $((k-1)\Delta t, (k+1)\Delta t) \subset (0, T)$, $k \in \mathcal{T}$ completes the proof of (1). $\square$

2.3 FFT Preliminaries

In this section, we will give some fundamental definitions and results of the FFT theory.

For $f(x) \in L^2(\Omega)$, $\Omega := (-L, L)$ where $L > 0$, we define its Fourier series as

$$f(x) = \sum_{l=-\infty}^{\infty} \hat{f}_l e^{2\pi i l x/L}, \quad \text{and} \quad \hat{f}_l = \int_\Omega f(x) e^{-2\pi i l x/L} dx.$$

Let S_N be a space of trigonometric polynomials up to degree N in variable x, i.e,

$$S_N := \text{span}\{e^{2\pi i l x/L} | -N \le l \le N\}. \tag{11}$$

Then, we define truncated Fourier series of f onto S_N via projection operator $\mathcal{P}_N$ as:

$$\mathcal{P}_N f(x) = \sum_{l=-N}^{N} \hat{f}_l e^{2\pi i l x/L}.$$

$\mathcal{P}_N f$ gives a good approximation for the Fourier series of the f. But it has a drawback from computational perspective. As we can see from its definition, coefficients of $\mathcal{P}_N f$ are comes from the evaluation of the integral $\hat{f}_l$. In practice, we don't know $f(x)$ for each $x \in \Omega$, we just know for finitely many sample points. For this reason, in general, it is impossible to evaluate coefficient integral $\hat{f}_l$. In order to fix this issue, we introduce a new notion which is called the interpolation operator.

At a set of $2N+1$ spatial grid points $x_j := -L + hj$ with $h = \frac{2L}{2N+1}$, we define the discrete Fourier series of f as:

$$f(x_j) = \sum_{l=-N}^{N} \tilde{f}_l e^{2\pi i l x_j / L}, \quad j = 0, 1, \ldots, 2N$$

with the discrete Fourier coefficients

$$\tilde{f}_l = \sum_{j=0}^{2N} f(x_j) e^{-2\pi i l x_j / L}.$$

Hence

$$\mathcal{I}_N f(x) = \sum_{l=-N}^{N} \tilde{f}_l e^{2\pi i l x / L}.$$

is the Fourier interpolation of f and it satisfies $f(x_j) = \mathcal{I}_N f(x_j)$ at each grid point x_j.

Lemma 2.7. *[23, 24] Let f and all of its derivatives (up to m^*-th order) be are continuous and periodic on Ω. Then, we have the following estimate*

$$||\partial^k f(x) - \partial^k \mathcal{I}_N f(x)||_{L^2} \leq C h^{m^* - k} ||f||_{H^{m^*}}$$

for $0 \leq k \leq m^*$ and $m^* > \frac{1}{2}$. Here, $h = \frac{2L}{2N+1}$ is the spatial step size.

Remark 2.8. *This lemma is a convergence result for the interpolation operator, which shows that, for any fixed h and k, higher regularity i.e. bigger m^* improves the convergence rate.*

Remark 2.9. *In the proof of Theorem 4.1, we will use the Lemma 2.7 with $k = 0$, $k = 1$, and an integer $m^* \geq 2$.*

Lemma 2.10. *[25] For any $\varphi \in S_{pN}$ with positive integer p, we have*

$$||\mathcal{I}_{pN}\varphi||_{H^s} \leq \sqrt{p}||\varphi||_{H^s}.$$

For the definition of S_{pN}, we replace N with pN in (11).

CHAPTER III

CONSERVED QUANTITIES OF THE ROSENAU EQUATION

In this chapter, we obtain the conserved quantities of (R) by using *Noether's theorem*. Conserved quantities can also be derived by applying integration by parts method.

3.1 A Lagrangian Density for the Rosenau Equation

In [26], the authors proposed a Lagrangian density for the Rosenau-RLW-KdV equation. Since the Rosenau equation can be obtained from a Rosenau-KdV-RLW equation, we can use this Lagrangian density in our derivation. The Lagrangian density of the Rosenau equation is:

$$\mathcal{L} = U_t U_x + (U_x)^2 + U_{xxx}U_{xxt} + \frac{2}{p+1}U_x^{p+1}, \tag{12}$$

where $u = U_x$.

Henceforth we use the notation $\xi_1 = t,\ \xi_2 = x$, and $\xi = (\xi_1, \xi_2)$ for the time and space coordinates. We also set $v_1 = U,\ v_2 = U_{xx}$, and $v = (v_1, v_2)$. We interpret ∇v as a *Jacobian*. To be more precise, $(\nabla v)_{i,j} = \frac{\partial v_i}{\partial \xi_j}$. In this notation, (12) can be written as:

$$\mathcal{L} = \frac{\partial v_1}{\partial \xi_1}\frac{\partial v_1}{\partial \xi_2} + \left(\frac{\partial v_1}{\partial \xi_2}\right)^2 + \frac{\partial v_2}{\partial \xi_2}\frac{\partial v_2}{\partial \xi_1} + \frac{2}{p+1}\left(\frac{\partial v_1}{\partial \xi_2}\right)^{p+1}. \tag{13}$$

Theorem 3.1 (Noether's Theorem)**.** *[27, p. 177,178] If the functional*

$$J[v] = \int_{\Omega} \mathcal{L}(\xi, v, \nabla v) d\xi$$

is invariant under the family of transformations

$$\xi_i^* = \Phi_i(\xi, v, \nabla v; \epsilon) \sim \xi_i + \epsilon \varphi_i(\xi, v, \nabla v)$$

$$v_j^* = \Psi_j(\xi, v, \nabla v; \epsilon) \sim v_j + \epsilon \psi_j(\xi, v, \nabla v)$$

$(i, j = 1, 2)$ *for an arbitrary region* Ω*, then*

$$\sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \left(\sum_{j=1}^{2} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial v_j}{\partial \xi_i} \right)} \widetilde{\psi_j} + \mathcal{L} \varphi_i \right) = 0 \tag{14}$$

where,

$$\widetilde{\psi_j} = \psi_j - \sum_{i=1}^{2} \frac{\partial v_j}{\partial \xi_i} \varphi_i. \tag{15}$$

Theorem 3.2. *For the Lagrangian density* (12), (R) *admits the following three conservation laws:*

$$E(t) := \int_{\mathbb{R}} (u^2 + \frac{2}{p+1} u^{p+1}) dx = constant \quad (energy\ conservation) \tag{16}$$

$$P(t) := \int_{\mathbb{R}} (u^2 + u_{xx}^2) dx = constant \quad (momentum\ conservation) \tag{17}$$

$$M(t) := \int_{\mathbb{R}} u dx = constant \quad (mass\ conservation). \tag{18}$$

Proof. Using the Lagrangian density (13) in (14), we have;

$$\frac{\partial}{\partial \xi_1} \left(\frac{\partial v_1}{\partial \xi_2} \widetilde{\psi_1} + \frac{\partial v_2}{\partial \xi_2} \widetilde{\psi_2} + \varphi_1 \mathcal{L} \right) + \frac{\partial}{\partial \xi_2} \left(\left(\frac{\partial v_1}{\partial \xi_1} + 2 \frac{\partial v_1}{\partial \xi_2} + 2 \left(\frac{\partial v_1}{\partial \xi_2} \right)^p \right) \widetilde{\psi_1} + \frac{\partial v_2}{\partial \xi_1} \widetilde{\psi_2} + \varphi_2 \mathcal{L} \right) = 0. \tag{19}$$

Let $\delta t = \epsilon \varphi_1$, $\delta x = \epsilon \varphi_2$, $\delta U = \epsilon \psi_1$, $\delta U_{xx} = \epsilon \psi_2$, and $u = U_x$. From (15), we have the following implications:

$$\widetilde{\psi_1} = \psi_1 - \frac{\partial v_1}{\partial \xi_1} \varphi_1 - \frac{\partial v_1}{\partial \xi_2} \varphi_2 \implies \epsilon \widetilde{\psi_1} = \delta U - U_t \delta t - U_x \delta x, \tag{20}$$

$$\widetilde{\psi_2} = \psi_2 - \frac{\partial v_2}{\partial \xi_1} \varphi_1 - \frac{\partial v_2}{\partial \xi_2} \varphi_2 \implies \epsilon \widetilde{\psi_2} = \delta U_{xx} - U_{xxt} \delta t - U_{xxx} \delta x. \tag{21}$$

By using (20) and (21) in (19), we obtain:

$$R_t := \frac{\partial}{\partial t}(U_x(\delta U - U_t\delta t - U_x\delta x) + U_{xxx}(\delta U_{xx} - U_{xxt}\delta t - U_{xxx}\delta x) + \delta t\mathcal{L})$$

$$Q_x := \frac{\partial}{\partial x}((U_t + 2U_x + 2(U_x)^p)(\delta U - U_t\delta t - U_x\delta x) + U_{xxt}(\delta U_{xx} - U_{xxt}\delta t - U_{xxx}\delta x) + \delta x\mathcal{L})$$

so that, $R_t + Q_x = 0$. Assume that the derivatives of U tend to 0 whenever $|x| \to \infty$. Then integration of the last equation with respect to the variable x on $\mathbb{R}$ gives

$$\frac{d}{dt}\int_{\mathbb{R}} R\ dx = 0 \tag{22}$$

where,

$$R = U_x(\delta U - U_t\delta t - U_x\delta x) + U_{xxx}(\delta U_{xx} - U_{xxt}\delta t - U_{xxx}\delta x) + \delta t\mathcal{L}. \tag{23}$$

Conservation of the energy: If we set $\delta x = \delta U = \delta U_{xx} = 0$ and $\delta t = \epsilon$ in (23), then (22) gives the energy conservation:

$$E(t) := \int_{\mathbb{R}} \left(u^2 + \frac{2}{p+1}u^{p+1}\right) dx = constant.$$

Conservation of the momentum: If we set $\delta t = \delta U = \delta U_{xx} = 0$ and $\delta x = \epsilon$ in (23), then (22) gives the momentum conservation:

$$P(t) := \int_{\mathbb{R}} \left(u^2 + u_{xx}^2\right) dx = constant.$$

Conservation of the mass: If we set $\delta t = \delta x = \delta U_{xx} = 0$ and $\delta U = \epsilon$ in (23), then (22) gives the mass conservation:

$$M(t) := \int_{\mathbb{R}} u dx = constant.$$

□

CHAPTER IV

ANALYSIS OF THE NUMERICAL SCHEME

For the initial value problem (R)-(I), we propose the numerical scheme

$$(1+D_N^4)\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right)+D_N\left(\frac{u^{n+1}+u^{n-1}}{2}\right)+D_N((u^n)^p)=0 \tag{24}$$

which combines Fourier pseudospectral and second order finite-difference methods for space and time discretizations respectively. Here u^n is the value of the approximate solution at time $t_n = n\Delta t$ and $u^0 = u_0(x_i)$.

In order to evaluate u^1, we use the following additional scheme;

$$(1+D_N^4)\left(\frac{u^1-u^0}{\Delta t}\right)+D_N(u^1)+D_N((u^0)^p)=0.$$

4.1 Truncation Error Analysis

Theorem 4.1. *Assume that $u \in L^\infty(0,T;H^{m+2}) \cap H^3(0,T;H^4)$ is the exact solution of (R) and $m \geq 1$ is an integer. Let $U_N(x,t) := \mathcal{P}_N u(x,t)$ and $U^n := \mathcal{I}_N U_N(x_i,t_n)$ be its discrete interpolation. Then*

$$(1+D_N^4)\left(\frac{U^{n+1}-U^{n-1}}{2\Delta t}\right)+D_N\left(\frac{U^{n+1}+U^{n-1}}{2}\right)+D_N((U^n)^p)=\tau^n \tag{25}$$

where $\tau = (\tau^1, \cdots, \tau^{M-1})$ satisfies

$$||\tau||_{\ell^2(\mathcal{T};\ell^2)} \leq C(\Delta t^2 + h^m) \tag{26}$$

and C depends only on the exact solution.

Proof. We first define the quantities;

$$
\begin{aligned}
F_1^n &:= \frac{U^{n+1} - U^{n-1}}{2\Delta t}, & F_{1e}^n &:= (\partial_t U_N)(\cdot, t^n),\\
F_2^n &:= \frac{D_N^4(U^{n+1} - U^{n-1})}{2\Delta t}, & F_{2e}^n &:= (\partial_{xxxxt} U_N)(\cdot, t^n),\\
F_3^n &:= \frac{D_N(U^{n+1} + U^{n-1})}{2}, & F_{3e}^n &:= (\partial_x U_N)(\cdot, t^n),\\
F_4^n &:= D_N((U^n)^p), & F_{4e}^n &:= (\partial_x (U_N)^p)(\cdot, t^n).
\end{aligned}
$$

Using (1) for U_N, we have:

$$||F_1 - \mathcal{I}F_{1e}||_{\ell^2(\mathcal{T};\ell^2)} \leq C\Delta t^2 ||U_N||_{H^3(0,T;L^2)} \leq C\Delta t^2 ||u||_{H^3(0,T;L^2)}, \tag{27}$$

and

$$
\begin{aligned}
||F_2 - \mathcal{I}F_{2e}||_{\ell^2(\mathcal{T};\ell^2)} &\leq C\Delta t^2 ||\partial_{xxxx} U_N||_{H^3(0,T;L^2)}\\
&\leq C\Delta t^2 ||U_N||_{H^3(0,T;H^4)}\\
&\leq C\Delta t^2 ||u||_{H^3(0,T;H^4)}.
\end{aligned} \tag{28}
$$

It is possible to write

$$\frac{1}{2}(U^{n+1} + U^{n-1}) - U_N(\cdot, t^n) = \frac{1}{2}\Delta t^2 D_t^2 U_N.$$

By using above equation in (2), we obtain

$$||D_t^2 \partial_x U_N||_{\ell^2(\mathcal{T})} \leq C||\partial_x U_N||_{H^2(0,T)},$$

which implies

$$||F_3 - \mathcal{I}F_{3e}||_{\ell^2(\mathcal{T};\ell^2)} \leq C\Delta t^2 ||u||_{H^2(0,T;H^1)}. \tag{29}$$

Now, we will estimate the nonlinear terms F_4 and F_{4e} as follows:

$$
\begin{aligned}
&||D_N((U^n)^p) - \mathcal{I}\partial_x((U_N^n)^p)||_2 \\
&\qquad = ||\partial_x \mathcal{I}((U_N^n)^p) - \mathcal{I}\partial_x((U_N^n)^p)||_{L^2} \\
&\qquad \leq ||\partial_x \mathcal{I}((U_N^n)^p) - \partial_x((U_N^n)^p)||_{L^2} + ||\partial_x((U_N^n)^p) - \mathcal{I}\partial_x((U_N^n)^p)||_{L^2} \\
&\qquad \leq C_1 h^{m^*-1}||(U_N^n)^p||_{H^{m^*}} + C_2 h^{m^*}||\partial_x(U_N^n)^p||_{H^{m^*}} && (30) \\
&\qquad \leq C_1 h^{m^*-1}||(U_N^n)^p||_{H^{m^*+1}} + C_2 h^{m^*}||(U_N^n)^p||_{H^{m^*+1}} \\
&\qquad \leq C_1 h^{m^*-1}||(U_N^n)^p||_{H^{m^*+1}} + C_2 h^{m^*-1}||(U_N^n)^p||_{H^{m^*+1}} \\
&\qquad \leq C h^{m^*-1}||(U_N^n)^p||_{H^{m^*+1}} \\
&\qquad = C h^{m}||U_N^n||^p_{H^{m+2}} && (31) \\
&\qquad \leq C h^{m}||U_N||^p_{L^\infty(0,T;H^{m+2})}.
\end{aligned}
$$

In (30), we apply Lemma 2.7 for $k = 0$ and $k = 1$. Since h is the small spatial step size, we have $h^{m^*-1} \geq h^{m^*}$ for any integer $m^* \geq 2$. In (31), we use the algebra property of the Sobolev spaces, Theorem 2.5, and we set $m^* - 1 =: m$. The last estimate leads to the following

$$
||F_4 - \mathcal{I}F_{4e}||_{\ell^2(\mathcal{T};\ell^2)} \leq C h^m ||u||^p_{L^\infty(0,T;H^{m+2})}. \qquad (32)
$$

A combination of (27), (28), (29), and (32) completes the proof of (26). □

4.2 Stability Analysis

In this section, we present the stability analysis of the proposed numerical scheme (24). The next lemma is a consequence of discrete mass conservation and we omit its proof. A similar and detailed analysis can be found in [28, 29].

Let $e_i^n := U_i^n - u_i^n$ and $u_N^n, e_N^n \in S_N$ be interpolations of the u^n and e^n respectively.

Lemma 4.2. *[28, 29] In any time step $t_n = n\Delta t$ where $n \geq 0$ is an integer, we have the estimate*

$$||e_N^n||_{H^2} \leq C(||D_N^2 e^n||_2 + h^m). \tag{33}$$

Theorem 4.3. *Assume that $u \in L^\infty(0,T; H^{m+2}) \cap H^3(0,T; H^4)$ is the exact solution of* (R) *and $m \geq 1$ is an integer. We have the following result for the error function:*

$$||e_N^n||_{\ell^\infty(\mathcal{T};H^2)} \leq C(\Delta t^2 + h^m).$$

Here, constant C depends only on the final time T and exact solution u.

Proof. Subtracting (24) from (25) and multiplying both sides by $2\Delta t$, we obtain;

$$(1+D_N^4)(e^{n+1}-e^{n-1}) = -\Delta t D_N(e^{n+1}+e^{n-1}) + 2\Delta t\tau^n - 2\Delta t D_N((U^n)^p - (u^n)^p). \tag{34}$$

First, we take the inner product of both sides of (34) with $e^{n+1} + e^{n-1}$. For the LHS of (34), we have the following equations:

$$\langle e^{n+1} - e^{n-1}, e^{n+1} + e^{n-1}\rangle = ||e^{n+1}||_2^2 - ||e^{n-1}||_2^2, \tag{35}$$

$$\langle D_N^4(e^{n+1} - e^{n-1}), e^{n+1} + e^{n-1}\rangle = \langle D_N^2(e^{n+1} - e^{n-1}), D_N^2(e^{n+1} + e^{n-1})\rangle \tag{36}$$

$$= ||D_N^2 e^{n+1}||_2^2 - ||D_N^2 e^{n-1}||_2^2. \tag{37}$$

In (35) and (37), we use the bilinearity and symmetry properties of the inner product. Then, we set $||x||_2^2 = \langle x, x\rangle$. In (36), we use the "summation by parts formula".

Now, we estimate the RHS of (34). For the first term of RHS of (34), we have:

$$
\begin{aligned}
-\Delta t\langle D_N(e^{n+1}+e^{n-1}), e^{n+1}+e^{n-1}\rangle & \\
&\leq \Delta t|\langle D_N(e^{n+1}+e^{n-1}), e^{n+1}+e^{n-1}\rangle| \\
&= \Delta t|\langle D_N e^{n+1}, e^{n+1}\rangle + \langle D_N e^{n-1}, e^{n-1}\rangle| \\
&\leq \Delta t\left(|\langle D_N e^{n+1}, e^{n+1}\rangle| + |\langle D_N e^{n-1}, e^{n-1}\rangle|\right) \\
&\leq \frac{\Delta t}{2}\left(||D_N e^{n+1}||_2^2 + ||e^{n+1}||_2^2 + ||D_N e^{n-1}||_2^2 + ||e^{n-1}||_2^2\right) \quad (38) \\
&\leq \Delta t\left(||D_N^2 e^{n+1}||_2^2 + ||e^{n+1}||_2^2 + ||D_N^2 e^{n-1}||_2^2 + ||e^{n-1}||_2^2\right). \quad (39)
\end{aligned}
$$

In (38), we use the Cauchy-Schwarz (C-S) and Cauchy (C) inequalities. In (39), we benefit from Corollary 2.3.

For the second term of RHS of (34), we have:

$$
\begin{aligned}
2\Delta t\langle \tau^n, e^{n+1}+e^{n-1}\rangle &= 2\Delta t\left(\langle \tau^n, e^{n+1}\rangle + \langle \tau^n, e^{n-1}\rangle\right) \\
&\leq 2\Delta t||\tau^n||_2^2 + \Delta t\left(||e^{n+1}||_2^2 + ||e^{n-1}||_2^2\right). \quad (40)
\end{aligned}
$$

In the last inequality, we use the Cauchy-Schwarz (C-S) and Cauchy (C) inequalities.

For the nonlinear term of RHS of (34), we start with the decomposition:

$$
(U^n)^p - (u^n)^p = e^n \sum_{k=0}^{p-1} (U^n)^k (u^n)^{p-1-k}, \quad 0 \leq k \leq p-1. \quad (41)
$$

$$
\begin{aligned}
||D_N\left(e^n (U^n)^k (u^n)^{p-1-k}\right)||_2 &= ||\partial_x \mathcal{I}_{pN}\left(e_N^n (U_N^n)^k (u_N^n)^{p-1-k}\right)||_{L^2} \\
&\leq \sqrt{p}||e_N^n (U_N^n)^k (u_N^n)^{p-1-k}||_{H^2} \quad (42) \\
&\leq C||e_N^n||_{H^2}||U_N^n||_{H^2}^k||u_N^n||_{H^2}^{p-1-k} \quad (43)
\end{aligned}
$$

In (42), we use Lemma 2.10. At this point, we start to estimate $||U_N^n||_{H^2}^k||u_N^n||_{H^2}^{p-1-k}$, afterwards, we will use the result in (43). From regularity of the exact solution, we can write;

$$
||U_N||_{L^\infty(0,T;H^2)} \leq ||U_N||_{L^\infty(0,T;H^{m+2})} \leq ||u||_{L^\infty(0,T;H^{m+2})} = C_1.
$$

Hence, at any time step t_n we have;

$$||U_N^n||_{H^2} \leq C_1. \tag{44}$$

Assume a-priori that, the error function has a H^2-bound:

$$||e_N^n||_{H^2} \leq 1. \tag{45}$$

Together with (44), a-priori assumption (45) gives a H^2-bound for u_N^n:

$$||u_N^n||_{H^2} = ||U_N^n - e_N^n||_{H^2} \leq ||U_N^n||_{H^2} + ||e_N^n||_{H^2} \leq C_1 + 1 =: C_2.$$

By definition, $C_2 > 1$ and $C_2 > C_1$ so, we have

$$||U_N^n||_{H^2} \leq C_2. \tag{46}$$

By using (33),(44),(46) in (43), we get

$$||D_N \left(e^n (U^n)^k (u^n)^{p-1-k}\right)||_2 \leq C_3 \left(||D_N^2 e^n||_2 + h^m\right).$$

and thus

$$\begin{aligned}
-2\Delta t \langle D_N & \left(e^n (U^n)^k (u^n)^{p-1-k}\right), e^{n+1} + e^{n-1}\rangle \\
& \leq 2\Delta t ||D_N \left(e^n (U^n)^k (u^n)^{p-1-k}\right)||_2 \left(||e^{n+1}||_2 + ||e^{n-1}||_2\right) \\
& \leq 2\Delta t C_3 \left(||D_N^2 e^n||_2 + h^m\right) \left(||e^{n+1}||_2 + ||e^{n-1}||_2\right) \\
& \leq \Delta t C_3 \left(||D_N^2 e^n||_2^2 + ||e^{n+1}||_2^2 + ||e^{n-1}||_2^2 + h^{2m}\right)
\end{aligned}$$

Since the last estimate holds for any $0 \leq k \leq p-1$, by decomposition (41) and the bilinearity of the inner product, we have:

$$-2\Delta t \langle D_N \left((U^n)^p - (u^n)^p\right), e^{n+1} + e^{n-1}\rangle \leq \Delta t C_4 \left(||D_N^2 e^n||_2^2 + ||e^{n+1}||_2^2 + ||e^{n-1}||_2^2 + h^{2m}\right). \tag{47}$$

By using (35),(37),(39),(40), and (47) in (34), we obtain the following estimate

$$
\begin{aligned}
||e^{n+1}||_2^2 + ||D_N^2 e^{n+1}||_2^2 - ||e^{n-1}||_2^2 - ||D_N^2 e^{n-1}||_2^2 \\
\leq \Delta t \left(||D_N^2 e^{n+1}||_2^2 + ||e^{n+1}||_2^2 + ||D_N^2 e^{n-1}||_2^2 + ||e^{n-1}||_2^2\right) \\
+ \Delta t C_4 \left(||D_N^2 e^{n}||_2^2 + ||e^{n+1}||_2^2 + ||e^{n-1}||_2^2\right) \\
+ \Delta t \left(||e^{n+1}||_2^2 + ||e^{n-1}||_2^2\right) + 2\Delta t ||\tau^n||_2^2 + \Delta t C_4 h^{2m}. \quad (48)
\end{aligned}
$$

To further estimate (48), we introduce the 2-point momentum-error norm:

$$
M^{n+1} := ||e^{n+1}||_2^2 + ||D_N^2 e^{n+1}||_2^2 + ||e^{n}||_2^2 + ||D_N^2 e^{n}||_2^2.
$$

Clearly, the left hand side of (48) is equal to $M^{n+1} - M^n$ and the first three terms at the right side are less than $M^{n+1} + M^n$. Thus we obtain

$$
M^{n+1} - M^n \leq \Delta t (2 + C_4)\left(M^{n+1} + M^n\right) + \Delta t C_4 h^{2m} + 2\Delta t ||\tau^n||_2^2. \quad (49)
$$

Thanks to the truncation error analysis estimate (26), we can further estimate (49) as:

$$
M^{n+1} \leq M^n + \Delta t C_5 \left(M^{n+1} + M^n\right) + \Delta t C_4 h^{2m} + C\Delta t \left(\Delta t^2 + h^m\right)^2, \quad (50)
$$

where $C_5 := 2 + C_4$.

By doing summation from $i = 1$ to $i = n$ on the both sides of (50), we get:

$$
M^{n+1} \leq M^1 + \Delta t C_5 \left(\sum_{i=1}^{n} M^{i+1} + \sum_{i=1}^{n} M^{i}\right) + \Delta t n C_4 h^{2m} + C\Delta t n \left(\Delta t^2 + h^m\right)^2. \quad (51)
$$

We transfer the last term of the first sum into the LHS of (51). Then, the identity $\sum_{i=1}^{n} M^i = \sum_{i=0}^{n-1} M^{i+1}$ and the bound $\Delta t n \leq T$ gives

$$
(1 - \Delta t C_5) M^{n+1} \leq M^1 + 2\Delta t C_5 \sum_{i=0}^{n-1} M^{i+1} + T C_4 h^{2m} + TC(\Delta t^2 + h^m)^2. \quad (52)
$$

Using the "collocation spectral approximation of the initial data" [25], we have the bound $M^1 \leq C_6 h^{2m}$, and assuming that Δt is sufficiently small to satisfy $\Delta t C_5 \leq \frac{1}{2}$, then (52) becomes;

$$
M^{n+1} \leq C_7 h^{2m} + C_8 (\Delta t^2 + h^m)^2 + 2\Delta t C_5 \sum_{i=0}^{n-1} M^{i+1} \quad (53)
$$

where, $C_7 \coloneqq 2C_6 + 2TC_4$ and $C_8 \coloneqq TC$, so constants are depend on the final time T. Note that, $2C_5\Delta t(n+1) \leq 2C_5 T =: C_9$. Then an application of Lemma 2.4 on (53) gives;

$$M^{n+1} \leq \left(C_7 h^{2m} + C_8(\Delta t^2 + h^m)^2\right) e^{C_9} \leq C_{10}\left(h^{2m} + (\Delta t^2 + h^m)^2\right),$$

where $C_{10} \coloneqq e^{C_9} \max\{C_8, C_7\}$. By the definition of M^{n+1}, we have $||D_N^2 e^n||_2^2 \leq M^{n+1}$. Since both sides are positive, we can take the square root:

$$||D_N^2 e^n||_2 \leq (M^{n+1})^{\frac{1}{2}} \leq \left(C_{10}\left(h^{2m} + (\Delta t^2 + h^m)^2\right)\right)^{\frac{1}{2}} \leq C_{11}\left(\Delta t^2 + h^m\right). \qquad (54)$$

By using (54) in (33), we get

$$||e_N^n||_{H^2} \leq C(||D_N^2 e^n||_2 + h^m) \leq C(\Delta t^2 + h^m). \qquad (55)$$

Finally, we complete the proof by taking the maximum of (55) over the set $\mathcal{T}$. $\square$

CHAPTER V

NUMERICAL RESULTS

This chapter is devoted to numerical results. Our main aim is to verify Theorem 4.3—stability analysis result numerically.

5.1 Petviashvili Iteration Method

A solitary wave solution $u(x,t) = \psi(x - ct)$ with wave speed c of (R) satisfies the ordinary differential equation

$$(1 - c)\psi - c\psi^{(4)} + \psi^p = 0 \tag{56}$$

after integration. Here the derivative is with respect to the variable $\xi := x - ct$ and ψ satisfies the decay property $\lim_{\xi\to\pm\infty} \psi^{(k)}(\xi) = 0$, where $k \geq 0$ is an integer.

In [7], Arnesen extended the results in the literature for the existence and stability of solitary wave solutions of the nonlocal equation,

$$u_t + (f(u))_x + (Lu)_t = 0, \tag{57}$$

by using concentration-compactness method. Here L is a Fourier multiplier operator defined as

$$\widehat{Lu(k)} = m(k)\hat{u}(k)$$

where $\widehat{\ }$ denotes the Fourier transform with respect to the spatial coordinate and k is the wave number. Equation (57) becomes (R) by choosing $L = \partial_x^4$ and $f(u) = u+g(u)$. In case of $g(u) = u^p$, the conditions in [7] are satisfied for (R). Hence, Theorem 2.1 of [7] states that the solitary wave solutions of (R) exists for all $c, p > 1$.

By using the Pohozaev type identities it has been shown in [5] that (56) does not admit any nontrivial solutions if:

i) c is negative and p is odd or *ii)* $c \in (0, 1)$ and p is positive.

Combining the existence results in [7] and nonexistence results in [5], the existence problem of the solitary wave solutions is open when $c < 0$ and p is even.

As exact solitary wave solutions are not available in the literature, we first construct these solutions numerically by the Petviashvili iteration method. The method is well used for the numerical evaluation of both solitary waves and periodic waves. See for example [30, 31, 32, 33]. To find the profile ψ of the solitary wave solution, we apply the Fourier transform to (56) which gives

$$(1 - c - ck^4)\widehat{\psi} + \widehat{\psi^p} = 0,$$

The Petviashvili iteration method is then proposed as;

$$\widehat{\psi}_{n+1}(k) = \frac{M_n^\gamma}{c + ck^4 - 1}\widehat{\psi_n^p}(k)$$

by introducing the stabilizing factor

$$M_n = \frac{\int_{\mathbb{R}}(c + ck^4 - 1)(\widehat{\psi}_n(k))^2 dk}{\int_{\mathbb{R}} \widehat{\psi}_n(k)\widehat{\psi_n^p}(k) dk}$$

where γ is chosen as $p/(p-1)$. As an initial guess, we choose $\psi_0 = \mathrm{sech}(x)$. We control the efficiency of the process by the error between two consecutive iterations, by checking how close the stabilization factor is to 1 and by the residual error. We refer [5] and [9] for the details of application of the method to the Rosenau equation. We also check the Fourier coefficients of the approximate solution. For the application of the method we take the space interval as $[-100, 100]$. The number of the grid points is $N = 2^{10}$ when $p = 2$ and the errors are less than of order 10^{-6} after 15 iterations. The Fourier coefficients decrease to order 10^{-9}. To obtain similar efficiency we increase the number of grid points to $N = 2^{11}$ and $N = 2^{12}$ for $p = 20$ and $p = 80$ respectively. Figure 1 shows the numerically generated solitary wave profiles of (R) for several values of p with wave speed $c = 2$ and for several values of c when $p = 2$.

Remark 5.1. *Here we would like to note that even though there is a nonexistence result when p is odd and there are no results when p is even for $c < 0$, the iteration does not give a result satisfying the equation for any p values when $c < 0$.*

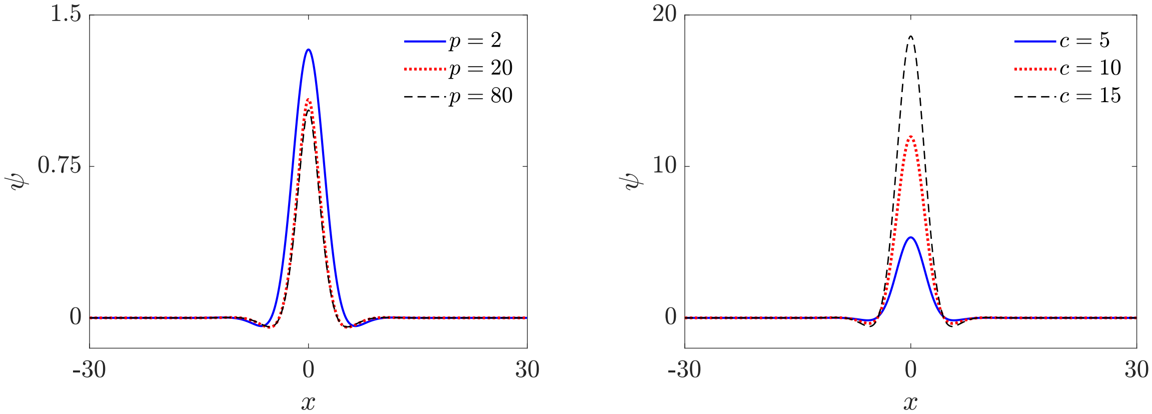

Figure 1: Numerically evaluated solitary waves of the Rosenau equation for: several values of p with $c = 2$ (left), and various values of c with $p = 2$ (right).

5.2 Evolution of Solitary Wave

As far as we know, exact solitary wave solutions of (R) are unknown for $g(u) = u^p$. For this reason, first, we use the Petviashvili iteration method to evaluate an initial profile. Then, we use our numerical scheme (24) to simulate generated initial profile. In Figures 2 and 3 we present propagation of the solitary waves up to time $T = 30$ for $p = 2$ and $p = 20$, respectively. In both experiments, we chose the time step as $\Delta t = 10^{-4}$. The right sides of the figures show the variation of conserved quantities (16),(17), and (18) in time. We observe that the quantities are well preserved by the proposed scheme.

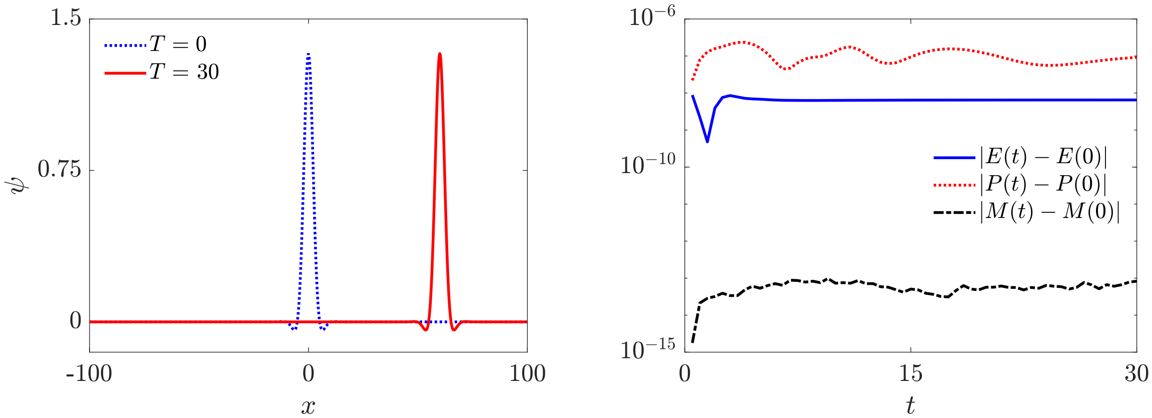

Figure 2: Propagation of the solitary wave up to time $T = 30$ (left) and variation of the conserved quantities in time (right) when $c = p = 2$.

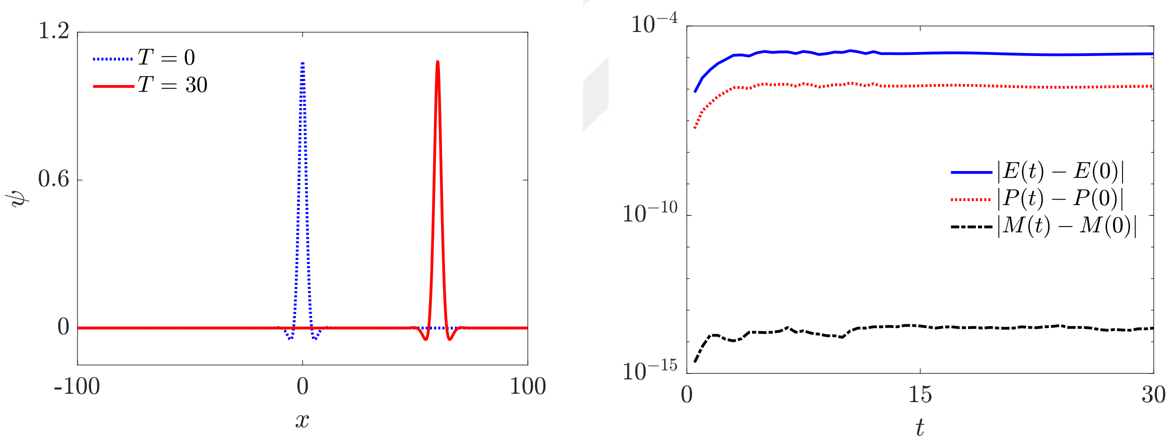

Figure 3: Propagation of the solitary wave up to time $T = 30$ (left) and variation of the conserved quantities in time (right) when $c = 2$ and $p = 20$.

5.3 Convergence Rates of the Numerical Scheme

In this section, we will calculate the convergence rates in spatial and temporal directions by using the formulas:

$$\text{Order in Time} := \log_2\left(\frac{||u_{\Delta t} - u_{ex}||_\infty}{||u_{\Delta t/2} - u_{ex}||_\infty}\right),$$

$$\text{Order in Space} := \log_2\left(\frac{||u_N - u_{ex}||_\infty}{||u_{N/2} - u_{ex}||_\infty}\right).$$

Here, for any grid function u, ℓ^∞-norm is defined as $||u||_\infty := \max_i |u(x_i)|$.

Convergence Rates in Time. In this numerical experiment, we will calculate the convergence rates in time up to $T = 1$. We take $c = 2$, $L = 50$, and $N = 1024$. As the exact solitary wave solutions of (R) are not known, as an exact solution, we consider the numerical solution when $\Delta t = 0.1 \times 2^{-9}$.

Table 1: Converge rates in time for nonlinearities $p = 2$ and $p = 3$.

Δt	ℓ^∞-Error ($p = 2$)	Order	ℓ^∞-Error ($p = 3$)	Order
0.1×2^{-1}	3.8073e-4	-	6.6269e-4	-
0.1×2^{-2}	9.5032e-4	2.0023	1.6525e-4	2.0037
0.1×2^{-3}	2.3745e-5	2.0008	4.1279e-5	2.0012

Table 2: Converge rates in time for nonlinearities $p = 10$ and $p = 11$.

Δt	ℓ^∞-Error ($p = 10$)	Order	ℓ^∞-Error ($p = 11$)	Order
0.1×2^{-1}	0.0024	-	0.0028	-
0.1×2^{-2}	5.8003e-4	2.0256	6.9139e-4	2.0314
0.1×2^{-3}	1.4435e-4	2.0065	1.7189e-4	2.0079

Convergence Rates in Space. Similar to previous numerical experiment, we take $c = 2$, $L = 50$, $T = 1$, and $\Delta t = 10^{-5}$ to calculate convergence rates in space. As an exact solution, we use the numerical solution when $N = 1024$. At this point, it is natural to ask how one can compare two numerical solutions for two different space step sizes N_1 and N_2. Because the corresponding solution vectors u_{N_1} and u_{N_2} will have a different number of components. Hence, we can not directly subtract these solution vectors. We overcome this difficulty via `MATLAB`'s built-in `intersect` command. It matches the space grid points.

The figure below depicts the reference solution and matched solution.

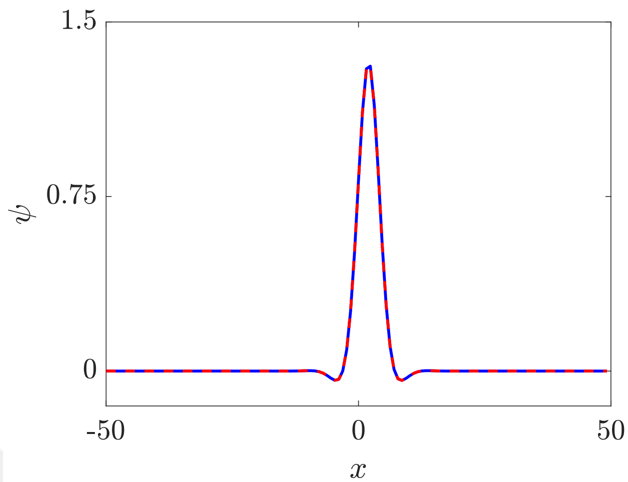

Figure 4: Blue curve represents the restriction of 1024 grid points to the $N = 128$ space grid points, while red dashed curve represents the $N = 128$.

Table 3: Convergence rates in space for nonlinearities $p = 2$ and $p = 3$.

N	ℓ^∞-Error ($p = 2$)	Order	ℓ^∞-Error ($p = 3$)	Order
32	0.2403	-	0.2448	-
64	8.2123e-4	8.1928	0.016	3.9329
128	2.7431e-7	11.5478	1.8492e-6	13.0817

Table 4: Convergence rates in space for nonlinearities $p = 10$ and $p = 11$.

N	ℓ^∞-Error ($p = 10$)	Order	ℓ^∞-Error ($p = 11$)	Order
32	0.368	-	0.3759	-
64	0.1678	1.1329	0.1858	1.0163
128	0.0065	4.6924	0.01	4.2136

5.4 Evolution of Perturbed Solitary Wave

In the last numerical expirement, we look at the evolution of the perturbed solitary wave in time. For this aim, we multiply the initial solitary wave by factor 1.2.

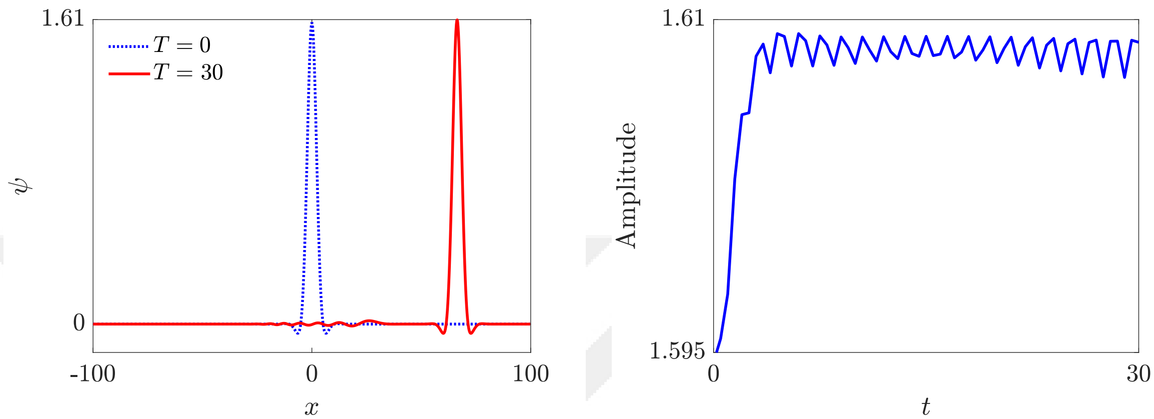

Figure 5: Propagation of the perturbed solution up to $T = 30$ when $c = p = 2$ (left) and variation of its amplitude (right).

In [7], it has been shown that the solitary waves are stable for all $c, p > 1$. Figure 5 agrees with this theoretical result.

CHAPTER VI

CONCLUSIONS

In this thesis study, we suggested a numerical scheme for the Rosenau equation which combines Fourier pseudospectral and second-order finite difference methods for space and time discretizations, respectively. We performed truncation error analysis and stability analysis of the numerical scheme. We also gave several numerical results to check the efficiency of the numerical scheme. We have generated initial solitary wave profiles of the Rosenau equation using the "Petviashvili iteration method" and then observed the propagation of these initial profiles.

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