

**A JOINT PRICING AND REPLENISHMENT
POLICY FOR PERISHABLE PRODUCTS
WITH FIXED SHELF LIFE AND POSITIVE
LEAD TIMES**

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By

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ABSTRACT

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Most of the existing inventory models in the literature are based on the assumption that the items have infinite shelf life and do not deteriorate no matter how long they stay on the shelf. However this assumption may not be applicable in many situations since there are also many types of products with limited shelf lives. In the inventory literature stored items with fixed finite lifetimes are usually referred to as perishable items. Examples of perishable products include fresh foods, medical products, whole-blood units, packaged chemical products and photographic films.

In this study, we consider the joint pricing and ordering policy, (Q, r, P_1, P_2) , for an inventory model with perishable items, with constant shelf lives and positive lead times. The demand process is assumed to be Poisson. If there is a single batch on hand, the items in a batch are sold at price P_1 . If there are two batches in stock, the items in the older batch are sold at price P_2 , where $P_1 > P_2$. The younger batch is not sold until the older one is totally depleted. Although the shelf lives are constant, the sequence of remaining shelf lives of the items at the instances where stock level hits Q , is a random sequence. The limiting distribution of this sequence is obtained and the analytical derivations of the operating characteristics of the model is based on this limiting distribution. Numerical results are also presented.

Keywords: inventory, perishables, effective shelf life, pricing and ordering policy.

ÖZET

SABİT RAF ÖMÜRLÜ VE POZİTİF TEDARİK SÜRELİ ÜRÜNLER İÇİN ORTAK FİYATLANDIRMA VE TEDARİK POLİTİKASI

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Literatürde yer alan envanter modellerinin birçoğu, ürünlerinin sınırsız raf ömrü olduğunu ve rafta ne kadar kalırlarsa kalsınlar bozulmaya uğramadıklarını varsaymaktadır. Ancak, birçok ürün çeşidinin sınırlı raf ömrü olduğu için, bu varsayım pekçok durumda uygulanabilir olmamaktadır. Envanter literatüründe, depolanan sınırlı ömürlü ürünlere bozulabilir ürünler denilmektedir. Taze besinler, tıbbi ürünler, kan üniteleri, paketlenmiş kimyasal ürünler ve fotoğraf filmleri bozulabilir ürünlere örnek verilebilir.

Bu çalışmada, sabit raf ömürlü ve pozitif tedarik süreli bozulabilir ürünler için envanter modeli olarak ortak fiyatlandırma ve tedarik politikası ele alınmıştır. Talep süreci Poisson dağılımına sahip varsayılmıştır. Yeni öbekteki ürünler P_1 fiyatıyla satılmakta, önceki öbekteki ürünlerin tamamı satılmadan önce yeni bir öbek gelirse fiyat indirimi yapılmakta ve önceki öbekteki ürünler P_1 fiyatından daha küçük olan P_2 fiyatıyla satılmaktadır. Raf ömürleri sabit olmasına rağmen stok seviyesinin Q ' ya geldiği anlarda ürünlerin kalan raf ömrü dizisi bir rasgele dizidir. Bu dizinin limit dağılımı elde edilmekte ve modelin çalışma davranışının türetimi bu limit dağılıma dayanmaktadır. Deneysel çalışma sonuçları da ayrıca sunulmuştur.

Anahtar sözcükler: envanter, bozulabilir ürünler, etkin raf ömrü, fiyatlandırma ve ısmarlama politikası.

To my parents and Gülnar

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NOTATION

Q = Order Quantity

r = Inventory threshold for reorder

P_1 = Regular price

P_2 = Reduced price

λ_1 = Demand arrival rate when price is P_1

λ_2 = Demand arrival rate when price is P_2

L = Lead time

τ = Constant lifetime for a batch of Q

h = Holding cost per unit time

K = Fixed ordering cost

ϕ = Discount rate

X_i = Random variable representing the arrival time of the i^{th} consecutive demand at regular demand (λ_1)

Y_i = Random variable representing the arrival time of the i^{th} consecutive demand at regular demand (λ_2)

$N(t)$ = Counting process associated with demand process at λ_1 in $(0, t)$

$N'(t)$ = Counting process associated with demand process at λ_2 in $(0, t)$

z_n = R.V. representing the effective lifetime in the n^{th} cycle

$H_i(t)$ = Value of the pdf of the gamma r.v. with parameters i and λ_1 at t

$H_{i,\lambda_2}(t)$ = Value of the pdf of the gamma r.v. with parameters i and λ_2 at t

$E(CL)$ = Expected cycle length

$E(SP_1)$ = Expected number of items sold at P_1

$E(SP_2)$ = Expected number of items sold at P_2

$E(OH)$ = Expected on hand inventory per cycle

$E(P)$ = Expected number of units that perish in a cycle

Chapter 1

Introduction and Literature Review

Most of the existing inventory models in the literature are based on the assumption that the items have infinite shelf life and do not deteriorate no matter how long they stay on the shelf. However, this assumption may not be realistic in many situations since there are many types of products with limited shelf lives. In the inventory literature stored items with fixed finite lifetimes are referred to as perishable items. Examples of perishable products include fresh foods, medical products, whole-blood units, packaged chemical products and photographic films.

In this study, we introduce a unified model for the inventory replenishment and pricing of perishable items. To maximize the expected reward per unit time in infinite horizon, price promotions can be used to clear off the items having less remaining useful life. In a price sensitive market, price promotions can be reasonable options to manage the demand rate.

Consider the following example: Strawberries are one of the most perishable fruit crops and are essentially fully ripe at harvest. They have a high rate metabolism and will destroy themselves in a relatively short time. Optimum storage conditions for strawberries are $0^{\circ}C$ and 90 – 95% relative humidity. However, storage-life is dependent on the handling of berries during and after harvest. In

ideal conditions, strawberries can have 7-10 days of storage-life. Due to its high level perishability, strawberries are usually packaged and stored in such conditions and distributed to the retailers in special packages. Thus, all of the strawberries within replenishment order come from the same vintage and have the same shelflife at the store level. Since the shelflives of the strawberries are very short, when the packages are opened the greengrocer would be interested in optimizing the prices he should charge for the fresh and older strawberries as well as how many new packages of strawberries should be ordered.

Controlling inventories of perishable items poses a significant challenge due to limited useful life of items and necessity to monitor the age of the goods in the inventory. If these items are not used before the expiry date, they would be outdated and there would be an additional cost of out dating of perished items.

Although perishable inventories are commonly encountered in real life, a great amount of the existing literature deals with durable goods due to mathematical difficulties in modeling perishable inventories. Main complications in modeling perishable inventories arise from the finiteness of the shelf life and the lead time structures. There are several approaches in the literature for modeling these quantities.

Regarding the lead time, most of early studies consider models with zero lead time in order to avoid the difficulty caused by the need to track of the ages of items in transit. Using results of such models some studies also proposed heuristic methods for positive lead times [23]. The positive, fixed or random lead time models are studied as discussed below (see e.g. Liu et al [1999], Kalpakam and Sapna [1995], Schmidt and Nahmias [1985], etc.).

In some models it is assumed that each item in the same batch have the same shelf life whereas, in some others, items are assumed to have independent random shelf lives with identical distributions. The assumption regarding the structure of the shelf lives and lead times change the structure and the method to arrange the model.

In our study, an inventory model with unit Poisson demand arrivals and fixed

positive lead time is considered. We assume that all the items in a batch have the same fixed shelf life and there is at most one order outstanding at any time. These assumptions are also made in the work of Berk and Gürler [3]. However, in this work we also consider the pricing for perishable items. If there is a single batch on hand, the selling price is P_1 , yet when the items from the previous batch is not depleted by demand or by perishing, the items in an older batch are sold at a discounted price P_2 whenever a new batch joins the stock. The main motivation behind this price reduction is to stimulate a higher demand rate for the older batch in order to sell it out quickly so as to avoid both the perishing cost of the older batch items and aging of the new batch items. The model assumes a higher demand rate (λ_2) at a lower price, P_2 . After the older batch is completely sold or perishes, the new batch is started to be sold at the original price, P_1 , with the original demand rate (λ_1).

To the best of our knowledge this is a novel work that incorporates the perishability structure explicitly in the pricing policy together with inventory replenishment.

We derive the operating characteristics of the model and the long run expected profit rate using an embedded Markov Process approach. We present numerical results to gain insight about how the optimal discount levels are realised with different system parameters. We also compare the proposed policy with fixed price and second market policy which will be discussed in detail next.

The literature on perishable inventory to determine optimal ordering/pricing policies considered different scenarios related to demand patterns, issuing policies, review periods of inventory, *etc.* and analyze how they affect profits. Instead of going over all of them, in the interest of brevity, we will cite only those that are closely related to our work.

The first analysis for fixed shelf life begin with Van Zyl [34] who considers a periodic review inventory problem and computes the optimal ordering policies assuming that the lifetime of items is exactly two periods. The study is generalized to m-period by Fries [13]. He studies a general m-period model with a zero

replenishment lead time. He extends the classical single item multiperiod inventory model where a good in storage perishes exactly l periods after its receipt. Units are followed from the time they are purchased and enter the inventory until they are either used to satisfy demand or perish. For general l the paper obtains the optimal policy recursively and derives several properties of the solution.

Nandakumar and Morton [26] detail the application of a class of heuristics to the fixed-life perishability problem formulated by Fries [13]. Their approach involves viewing periodic inventory problems in the framework of the classic 'news-boy' model. Computational studies reveal that the heuristic policies are near optimal, and are easy to compute.

The first study with a positive lead time for perishable items belongs to Schmidt and Nahmias [29] who considered the $(S - 1, S)$ policy with a fixed shelf life. They assume that demand follows a Poisson process and unmet demands are lost. They also comment that finding an optimal policy for a continuous review perishable inventory system with a positive lead time is extremely complex and it seems unlikely for anyone to be able to find and use it.

Kalpakam and Sapna [18] analyze the $(S - 1, S)$ perishable inventory model with renewal demands and exponential lifetimes. They identify the inventory level as a semi-regenerative process and obtain the steady state operating characteristics of the model.

The continuous review approach has received less attention in the perishable inventory literature. Weiss [32] studied a continuous review perishable inventory model with a Poisson demand process and zero lead time. He pointed out that the (s, S) policy is optimal when the demand process is compound Poisson.

Liu and Lian [24] extends Weiss' work and analyze a continuous review perishable inventory system with a general renewal demand process and instantaneous replenishments. Using a Markov renewal approach, they obtain closed-form solutions for the steady state probability distribution of the inventory level and system performance measures. They develop a closed-form expected cost function.

Chiu [7] proposes an approximate continuous review perishable inventory model under (Q, r) policy. His study provides an approximate solution by assuming positive lead time, fixed shelf life and full backordering. He assumes that no undershoot occurs at the reorder point r . The approximation method is simple and verified by a comparison with the Weiss model. This paper also compares the proposed model to the conventional model with no perishability.

Ravichandran [28] studies a continuous review perishable inventory system of (S, s) type with Poisson demand and positive lead time. An explicit expression for the stationary distribution of the stochastic process, representing the level of inventory is derived for the lost sales case, under a specified aging phenomena of a batch of items. A simple numerical example is provided.

Tekin, Gürler and Berk [30] study a time based control policy for continuous review inventory systems with constant shelf life, Poisson demand and lost sales. They assume a specific aging pattern similar to Ravichandran and analyze the problem under service level criterion.

Lian and Lui [23] consider a continuous review perishable inventory model with renewal batch demands. They assume that lead time is zero and construct an embedded Markov chain. With a probabilistic approach, they derive a closed-form long run average cost function. Numerical analysis is then used to identify the properties of the cost function and to demonstrate the impacts of changing batch sizes and other system parameters. Gürler and Özkaya [15] generalize Lian and Lui's work for the case where the arrivals follow an arbitrary renewal process with batch demands.

Ordering and pricing decisions in perishable inventories received a great interest in the literature. Studies of pricing strategies in revenue management were motivated by research on production-pricing problems. Whitin [33] was the first author who studied a newsvendor model with price effects. In his model, selling price and stocking quantity are set simultaneously. Whitin [33] adapted the newsvendor model to include a probability distribution of demand that depends on the unit selling price, where price is a decision variable. In his study, he determined the optimal stocking quantity as a function of price and corresponding

optimal price.

Mills [25] concentrates on showing the effect of uncertainty on a monopolist's short-run pricing policy under the assumption of additive demand. In his model, the demand is specified as a decreasing function of price. In particular, he shows that the optimal price under stochastic demand is no greater than the optimal price with deterministic demand. Then Karlin and Carr [20] show that the optimal price under stochastic demand is no smaller than the optimal price under the deterministic demand which is opposite of the corresponding relationship found to be true by Mills for the additive demand case under the assumption of random demand which depends on a price as a parameter.

Dana and Petruzzi [17] propose a model with uncertain demand which depends on both price and inventory level. One of the main assumption of their model is that consumer behavior is specified exogenously and price is fixed. They show that it is optimal to carry more inventory and provide a higher service level than the models that ignore this effect. They also suggest that in the endogenous price case the firm's two-dimensional decision problem can be reduced to two sequential, single-variable optimization problems. As a result, the endogenous-price case is as easy to solve as the exogenous-price case.

Agrawal and Seshadri [2] consider a single-period inventory model where retailer faces uncertain demand and makes a purchasing order quantity and selling price decision with the objective of maximizing expected utility. Distribution of demand is a function of the selling price. Their results suggest that in small independent stores prices of new low demand or low brand recognition products will be higher while the prices of mature, high demand or branded products will be lower.

Bisi and Dada [4] formulate a newsvendor model of a single product whose demand distribution is price-dependent and involves unknown parameter(s). Demand in excess of the order quantity is lost and unobserved. The objective is to determine joint ordering and pricing policies in each period over a finite-horizon of N periods that maximize the total expected profit.

Gurnani and Tang [16] determine the optimal ordering decisions for a retailer when facing uncertain cost and random demand for the product in the selling season, and characterize the conditions under which the retailer would defer the ordering decision.

Khouja [21] studies a single period model in which multiple discounts are used to sell excess inventory. In his model, discount unit is increased progressively until all excess inventory is sold. He deals with maximizing the expected profit. He shows that expected profit is concave and derived the optimality condition for the order quantity. Later Khouja proposes an algorithm for identifying the optimal order quantity for the multi-discount single period problem. He develops algorithms for determining the optimal number of discounts under fixed discounting cost. He identifies the optimal order quantity before any demand is realized for Normal and Uniform demand distributions. He also shows how to get the optimal order quantity and price for the Uniform demand case when the initial price is also a decision variable.

Federgruen and Heching [11] analyze a single item periodic review model, where demands in consecutive periods are independent, but their distributions depend on the item's price in accordance with general stochastic demand functions. A replenishment order may be placed at the beginning of some or all of the periods.

Datta and Paul [10] analyze an inventory system where the demand rate is influenced by both displayed stock level and selling price. A finite period system has been considered under multi-replenishment scenario. Optimal selling price and the optimal order quantities have been treated as decision variables.

Burnetas and Smith [5] consider the combined problem of pricing and ordering for a perishable product with unknown demand distribution. No inventory is carried from one period to the next. The demand distribution in any period depends on the price level in the same period and it is unknown. The retailer must decide on the price and lot size in every period based on the previous prices, lot sizes and sales levels.

As an early work, Cohen [9] considers the problem of simultaneously setting selling price and order quantity for an exponentially decaying product under known demand which is a function of unit selling price. He derives an optimal decision quantity and investigates its sensitivity to changes in perishability. Aggarwal and Jaggi [1] extends this model by developing expressions for the optimal pricing and ordering policy for a three-parameter Weibull deterioration distribution for cases of both no-shortages and backlogging. Kang and Kim [19] extended the model of Cohen [9] by considering a finite production rate. The maximum profit-price decisions were computed with changes in product deterioration. The result of their analysis indicates that the trade-off between revenue and loss due to deterioration leads to unexpected patterns of pricing and production decisions. Lazear [22] develops a model including differences in prices based on the type of good, the volume of sales in a particular industry, or the time a good has been on a shelf. Using Bayesian probability methods, Lazear's model predicts decreasing prices based on time on a shelf. He demonstrates that the uniqueness of the good does not affect the price path, and that perishable goods are more likely to have an initial lower price than non-perishable goods.

Wee and Yu [31] considered the effects of the temporary discount sale when the items deteriorate exponentially with time. This study enables to decide on how much to order when there is a temporary price discount and to understand the relationship between the cost savings and the rate of deterioration when temporary price discount purchase occurs at the regular and non-regular replenishment time. Gallego and van Ryzin [14] focus on dynamic pricing where demand is random, price sensitive and function of price. One of their major assumption is that selling must stop after a deadline and restocking may not be allowed. The authors assume revenue is concave and increasing in the demand intensity. For the specific case where demand is generated by a Poisson process, Gallego and van Ryzin [14] find the optimal solution to the pricing problem, where price changes continuously over time. For the general case, they formulate and solve a deterministic problem, which provides bounds on the expected profit. The strongest conclusion they have is that using simple fixed price policies seems to work well in many real instances rather than the optimal dynamic policies. They conclude

that this is very encouraging conclusion since the optimal dynamic policies can require undesirable characteristics in practical applications.

Petruzzi and Dada [27] study the problem of determining inventory and pricing decisions in a two-period retail setting when an opportunity to refine information about uncertain demand is available. The model extends the newsvendor problem with pricing by allowing for multiple suppliers.

Feng and Gallego [12] investigate the optimal time to switch between two pre-determined prices in a fixed selling season. They assume that demand is a Poisson process and function of price. Feng and Gallego propose that the optimal policy for this problem is a threshold policy, whereby price is changed (decreased or increased) when the time left in the horizon passes a threshold (resp., below or above) that depends on the unsold inventory.

One of the recent works in pricing policies of perishables belongs to Chun [8] who considers the demand function as a negative binomial distribution. He proves that the product price should decrease as the supply level increases.

In a periodic setting, Chew et. al. [6] consider jointly determine the price and the inventory allocation for a perishable product with a predetermined lifetime. They assume that demand for the product is price sensitive. A discrete time dynamic programming model is considered to obtain the optimal prices and the optimal inventory allocations for the product with a two period lifetime to maximize the expected revenue. Later they suggest three heuristics when the lifetime is longer than two periods. They extend their results to the case where the price for the product consistently decreases; and the case where the price for the product first increases and later decreases.

In this study, a continuous review perishable inventory system is designed for the items that have fixed lifetimes and constant lead times. The demand process is assumed to be Poisson and depends on price. There is also a positive lead time between the placement and arrival of the orders.

In our model the system includes a pricing and replenishment policy. When

there is only one batch on hand the items in a batch are sold at price P_1 . If a new batch arrives before the items from the previous batch have not sold out, a price reduction is given and the items in the older batch are sold at price P_2 , where $P_1 > P_2$. The younger batch is not used to satisfy the demand until the older batch is totally depleted. In this study we mainly consider the formal description of a pricing and replenishment policy, referred as (Q, r, P_1, P_2) , which is given in detail in Chapter 2.

As a second policy we also consider a model where some aged items have an opportunity to be sold at a secondary market. In this policy, the primary market always starts with fresh single batch and all the items in a fresh batch are sold at price P_{1s} . If a new batch arrives before the older one is depleted, the remaining items from the older batch are sold at a reduced price P_{2s} in a secondary market. We refer to this policy as $(Q_s, r_s, P_{1s}, P_{2s})$.

We provide numerical results that establish the performance of the above policies and their sensitivity behavior. We find that significant saving are obtained by price reduction if the shelf lives are quite short for (Q, r, P_1, P_2) policy model. We also observe that $(Q_s, r_s, P_{1s}, P_{2s})$ model performs better than (Q, r, P_1, P_2) policy since aging of a new batch is not allowed while the older batch is sold. However, this advantage holds when a positive transformation cost or any cost regarding the operation of a secondary market is not considered. The details are provided by the numerical study in Chapter 4.

The rest of the thesis is organized as follows. In Chapter 2 we introduce our models. In Chapter 3, we derive the key operating characteristics of the models. We state the optimization problem that we consider explicitly. In Chapter 4, we present numerical results on a range of parameter settings to explore the performance of the systems. The thesis ends with concluding remarks and comments on possible future work.

Chapter 2

Description of the Model

In this chapter we consider the joint replenishment and pricing issues for a perishable inventory system under continuous review. We introduce a replenishment and a pricing policy which we refer to as "Single Market Two Price Policy". We derive the operating characteristics of the inventory system that operates under this policy. The detailed discussion for this policy is presented in Section 2.1.

As an alternative operating environment, we also consider a setting where the items on hand at the instance when a new batch arrives are sold at a secondary market with independent demands and reduced price. This policy is referred to as "Secondary Market Policy". The corresponding operating characteristics are derived in Section 2.4.

2.1 Single Market Two Price Policy

In the single market two price policy, we consider a single item, single location, continuous review inventory system with positive lead time where the products have fixed lifetimes. We assume that external demands are generated according to a stationary Poisson process with rate $\lambda > 0$ and arrive to the inventory system one at a time. Replenishment is done in batches and a batch has a fixed, finite

shelf life of τ time units. There exists at most one order outstanding at any time. Various costs related to the system are the linear holding cost, h , per unit held in stock per unit time and linear perishing cost, p , for each unit that perishes per unit. There is a nonzero fixed ordering cost, K . The products in the inventory have a constant lifetime and aging of a younger batch does not begin until all units of the previous batch are exhausted either by demand or by perish. Since units are assumed to be equally useful throughout their lifetimes, we assume that a FIFO (first-in-first-out) issuing policy is used.

We assume that the demand rate, λ , is a decreasing function of price. Our model allows a general structure for this function but in our numerical studies we consider a linear demand rate that decreases in price. We join the ordering policy with a pricing policy. The reasoning behind this integration of ordering and pricing policy is to use price changes as a tool for managing the demand rate.

Under these assumptions we propose a modified (Q, r) policy as follows:

(Q, r, P_1, P_2) **Policy:** A replenishment order of Q units is placed when the inventory level hits r by demand or zero by perishing. If there is a single batch on hand, the items are sold at price P_1 . If there are two batches in stock at any time, the items in the older batch are sold at price P_2 where $P_1 > P_2$, while the younger batch is not sold until the older one is totally depleted.

Since we assumed that the rate of the unit Poisson demand arrivals are price sensitive, we let λ_1 be the arrival rate when items are sold at price P_1 and λ_2 be the corresponding rate when price P_2 is used.

According to the commonly used (Q, r) replenishment policy with unit demand, an order is placed when the inventory level hits r units. Since in our model items are subject to perishing after a constant time, items in a batch on hand can deteriorate before the inventory position hits exactly r at a demand occurrence. As a result, we modify the reordering decision slightly and allow for a reorder to be placed at the perishing instances as well.

According to our policy, if the inventory level drops to zero either by demand

arrivals or by perishing the inventory level is increased to Q units by the arrival of a new batch L time units after the order placement. At this instance there is only one batch in stock and the remaining lifetime of all items on hand is exactly τ and the items are sold at price P_1 . When inventory is not depleted during the lead time by demand or by perishing then there will still be some items in the inventory. By the arrival of the orders the inventory level increases above Q units. At this instance there are two batches on hand and the older batch is sold at price P_2 . The new batch is not started to be sold before all items of the previous batch are totally depleted either through demand or by perishing. As the last item in the older batch is sold or the older batch reaches its expiry age, the inventory level drops to Q . At this instance, the items are again sold at price P_1 . Since the items in the new coming batch are aged until the older batch is depleted, their remaining shelf lives are random variables less than τ .

When we consider the consecutive instances where the inventory level hits Q , we observe that the remaining shelf lives of the items constitute a sequence of random variables. We call the distribution of these variables as effective shelf life distribution. Next, we will discuss the properties of this sequence.

2.2 Effective Shelf life Distribution

Consider the inventory system operating under the proposed (Q, r, P_1, P_2) , replenishment and pricing policy at time $t = 0$ with Q fresh items on hand. Let $\{T_n, n \geq 1\}$ be the sequence of time epochs at which the inventory level hits Q for the n 'th time, with $T_1 = 0$. Then, for all $n \geq 1$, $I(T_n) = Q$, where $I(t)$ is the inventory level at time t . Now suppose that $\{Z_n, n \geq 1\}$ is the sequence of effective shelf lives of the items at T_n where, $Z_1 = \tau$. We develop the expressions for the limiting probability distribution of the effective shelf life sequence $\{Z_n, n \geq 1\}$ by considering the system between two consecutive instances at which the inventory level hits Q .

We first demonstrate the effective shelf life process. Let $\{X_j, j \geq 1\}$ be the

sequences of Poisson demand arrival times when there is only one batch in stock and sold at price P_1 with rate λ_1 . Also let $N(t)$ be the counting process associated with demand process with rate λ_1 in $(0, t]$. Similarly, let $\{Y_j, j \geq 1\}$ be the sequences of Poisson demand arrival times when the items are sold at price P_2 with rate λ_2 and let $N'(t)$ be the counting process associated with demand process with rate λ_2 in $(0, t]$ again. We refer to the time between $T_n + 1$ and T_n as the n 'th embedded cycle for $n \geq 1$.

Referring to Figure 2.2, we illustrate a possible realization of the system dynamics.

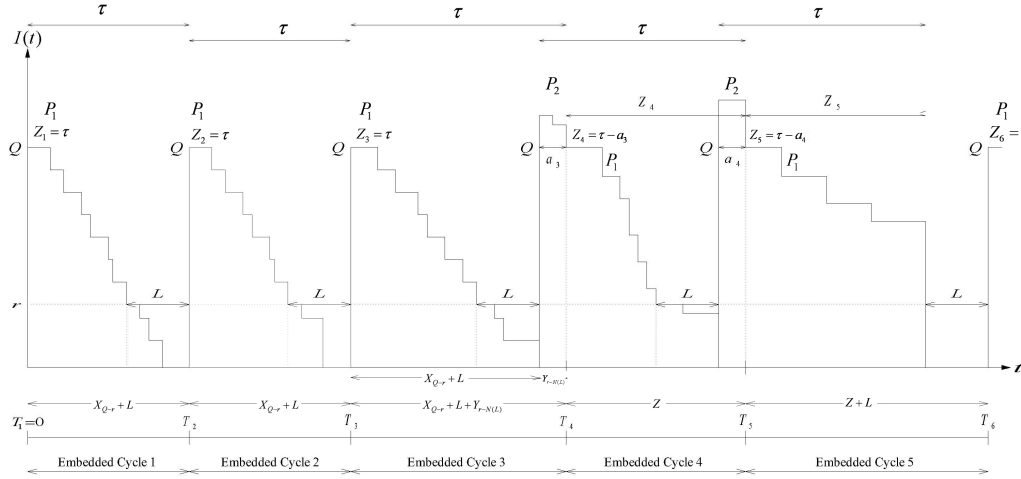


Figure 2.1: Possible cycle realizations

We start with Embedded Cycle 1 at T_1 as the time origin, $t = 0$. In Embedded Cycle 1, a replenishment order is given when the inventory level drops to r after $Q - r$ demands have arrived. During the lead-time period of length L , the remaining r units drop to zero by demand arrivals. Embedded Cycle 1 is completed at the end of lead time and the inventory level is increased to Q units by the arrival of a new batch and all the items in a new batch are sold at price P_1 and Embedded Cycle 2 starts at time $X_{Q-r} + L$. Embedded Cycle 2 is similar to Embedded Cycle 1 with a small difference. A replenishment order of Q units was placed when the inventory level drops to r as in Embedded Cycle 1, however the items reach their effective shelf lives during the lead time and remaining r units drop to zero by perishing and the selling price of all the items

in a new batch is P_1 . Embedded Cycle 3 starts at the end of the lead time and continues during $X_{Q-r} + L + Y_{r-N(L)}$. At this cycle the inventory level drops to r by demand arrivals and a replenishment order is given X_{Q-r} time units after the inventory level hits Q . Outstanding order arrives when there are still some items in the inventory and inventory level increases above Q units. At this time a price reduction is given and all the items in an older batch are sold at price P_2 . These items are depleted by demand during the period $Y_{r-N(L)}$ and Embedded Cycle 4 starts at the end of this period and lasts during the period z . All the items are sold at price P_1 during this period. In Embedded Cycle 4, at the end of the lead time there are still leftover items and inventory level increases above Q units by the arrival of a new batch. At this point a price reduction is given and the items from the older batch are sold at price P_2 . However some of these items perish before they are depleted by demand and this completes the fourth embedded cycle. Then fifth Embedded Cycle starts when inventory level drops to Q . In this cycle the items perish before the replenishment order is placed and cycle continues during the time $z + L$. The process continues in this fashion.

Note that if there are some leftover items at the end of the previous lead time an embedded cycle starts with an effective shelf life strictly smaller than τ but in all other cases the effective shelf life at the beginning of an embedded cycle is exactly τ .

According to the above discussion we can express the effective shelf life sequence as below. The effective shelf life at the beginning of the $n + 1$ 'th period can be written as ;

$$Z_{n+1} = \begin{cases} \tau - Y_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z_n, X_r > L \\ \tau - z_n + X_{Q-r} + L & \text{if } z_n < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z_n, X_r > L \\ \tau, & o.w. \end{cases}$$

Let $F_{n+1}(z|t)$ be the distribution function of the sequence Z_{n+1} given Z_n , then,

$$F_{n+1}(z|t) = \begin{cases} 0 & \text{if } z < \tau + L - t \\ H_{Q-r}(t + z - \tau - L, \lambda) \left[\sum_{i=0}^{r-1} P(N(L) = i) \overline{H}_{r-i}(\tau - z, \lambda_2) \right] & \text{if } \tau + L - t < z < \tau \\ 1 & \text{if } z \geq \tau. \end{cases}$$

Berk and Gürler [3] prove that for a single price policy the limiting distribution for the remaining shelf life process exists.

$$\lim_{n \rightarrow \infty} F_{n+1}(z) = F(z) \quad (2.1)$$

In our study we assume that the limiting distribution also exists. Letting $F(z)$ denote the limiting distribution function we obtain;

$$F(z) = \begin{cases} 0 & \text{if } z < \tau + L - t \\ H_{Q-r}(t + z - \tau - L, \lambda) \left[\sum_{i=0}^{r-1} P(N(L) = i) \overline{H}_{r-i}(\tau - z, \lambda_2) \right] & \text{if } \tau + L - t < z < \tau \\ 1 & \text{if } z \geq \tau. \end{cases}$$

As in the results of Berk and Gürler [3] the equation given in ?? is an integral equation for F which is solved by numerical methods.

2.3 Basic Characteristics

In this section, we present preliminary analysis of the model under consideration. In particular, typical behavior of the inventory process is displayed in detail.

As we mention above we define an embedded cycle as the time between two consecutive instances at which inventory level hits Q . Now, we can list all possible realizations of the inventory level process.

Realization 1: The cycle begins with Q items on hand at $t = 0$ and the items in a batch are sold at price P_1 . After X_{Q-r} units of time, the inventory level drops to r . At this point a replenishment order of Q units is placed and the order arrives after L units of time. However, the inventory level drops to zero before the order arrives. This event occurs at time X_Q and after a certain period of stock out state, the order of Q units arrives at time $X_{Q-r} + L$ and inventory level jumps to Q again. At this point a new embedded cycle begins.

It is important to note that all the items of a new batch start with an age of τ because no time is lost between the arrival of the items and to start selling them.

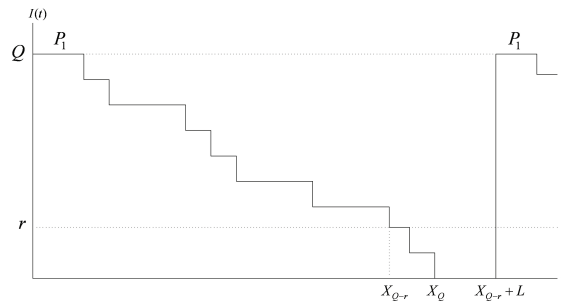


Figure 2.2: Realization 1

Realization 2: This realization is similar to Realization 1 with a small difference. In Realization 1, an order of Q units was placed and all items are sold at price P_1 before the end of the lead time, however in this realization the items reach their effective lifetimes during the lead time and the units that have not been sold, perish at time z . Again a stock out period is encountered and the

order of Q units arrives at time $X_{Q-r} + L$. The new batch starts with an effective lifetime of τ .

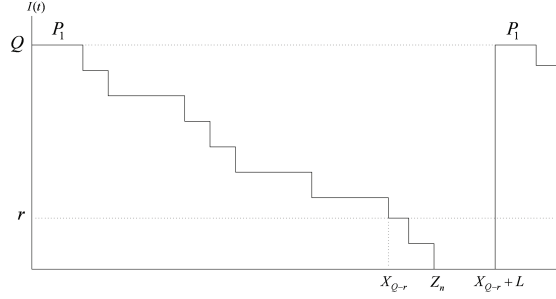


Figure 2.3: Realization 2

Realization 3: In Realization 3 when the inventory level drops to r a replenishment order of quantity Q is placed. The distinguishing characteristics of this realization from the previous realizations is that inventory is not depleted during the lead time, neither by demand nor by perishing. Therefore, when the ordered quantity of Q arrives at point $X_{Q-r} + L$, the inventory level jumps to a level higher than Q . After this point the pricing policy gets involved. Since there are items from both the previous batch and the current batch, we try to sell the items from the older batch more quickly in order to avoid the aging of the items from the new batch. For this reason a price reduction is made for the older batch and all the items in the older batch are sold at price P_2 . The demand rate at this new price is higher than the regular demand rate. Therefore, the process from this point to the time that inventory level drops back to Q , is another process. This difference in process is denoted by the random variable Y_i . This difference causes significant complications in the derivation of operating characteristics for this realization.

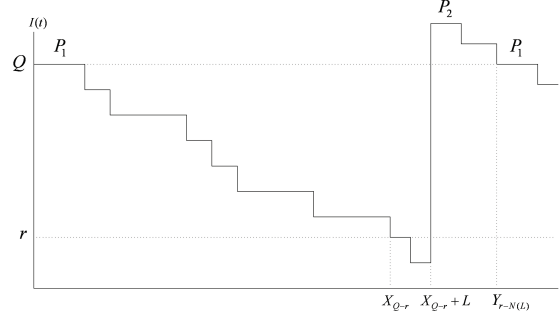


Figure 2.4: Realization 3

Realization 4: The 4th Realization is similar to Realization 3 in the sense that order arrival occurs before the inventory is totally depleted. Thus, again a new process starts after $X_{Q-r} + L$ until the inventory drops to Q . The difference from Realization 3 is that the inventory drops to Q by perishing, not by demand. Therefore the cycle ends at z .

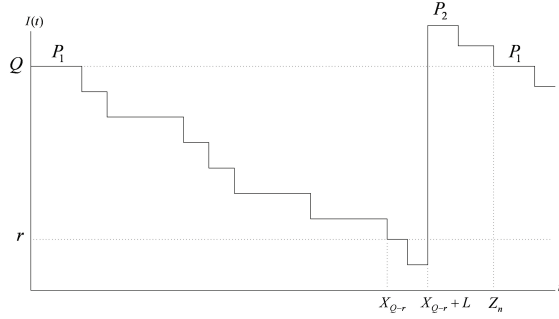


Figure 2.5: Realization 4

Realization 5: Realization 5 can be thought as a special case of Realization 2. In Realization 2, the items perish after the order is placed whereas in Realization 5 the items perish before the order is placed. Therefore, during the whole lead time the demands are lost because of stock out. All the items in a batch are sold at price P_1 .

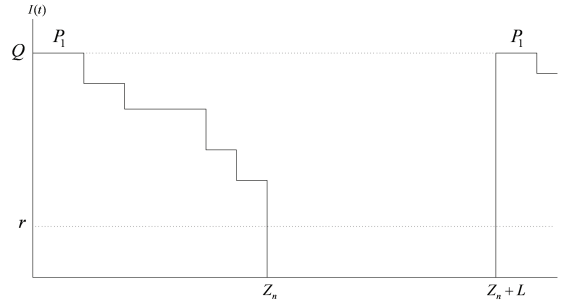


Figure 2.6: Realization 5

2.4 Secondary Market Policy

In this section, we will consider an inventory control policy for a single item, single location, continuous review inventory problem which will be referred as Secondary Market Policy. Similar to what we have covered in Section 2.1, we assume that demand is price sensitive and follows a stationary Poisson process with rate $\lambda > 0$ and arrives at the inventory system one at a time. Replenishment is done in batches and a batch has a fixed, finite shelf life of τ time units and there exists at most one order outstanding at any time.

With these assumptions we propose the following policy;

$(Q_s, r_s, P_{1s}, P_{2s})$ **Policy:** A replenishment order of Q_s units is placed when the inventory level hits r_s by demand or drops to zero by perishing. When there is a single batch in stock, items are sold for price P_{1s} . When a new batch joins the stock after L units time, the older batch is sold at price P_{2s} at a secondary market with an independent arrival process. The items in the new batch are sold at price P_{1s} at a primary market.

According to the above policy, when there are two batches at a time, the items from the older batch are sent to the secondary market. Hence, in the primary market the aging of the items in a new batch is not allowed. Therefore, in the primary market, when the inventory level hits Q , the remaining lifetime of all items in the new batch is exactly τ . Thus, unlike the single market two price policy, we have regenerative cycles starts with an effective lifetime of τ and

during this cycle time, all the items of the older batch are simultaneously sold at the secondary market.

Figure 2.7, illustrate possible realizations of the system dynamics for the primary market.

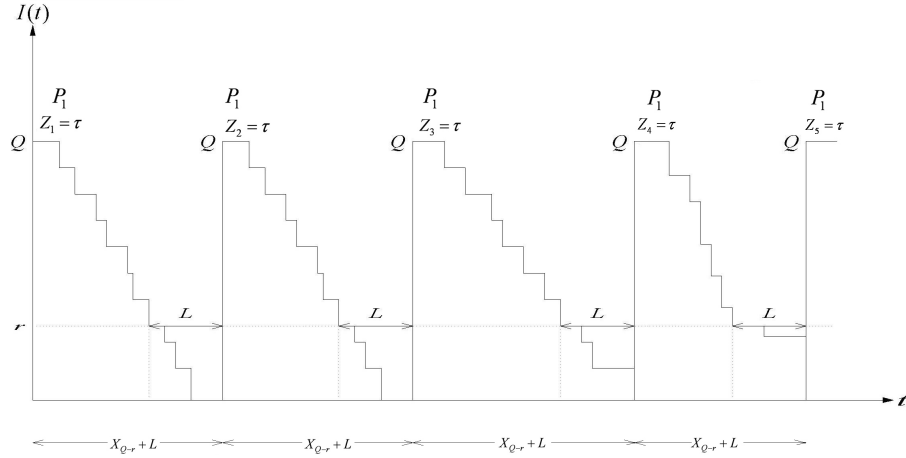


Figure 2.7: Possible cycle realizations for the primary market

We develop the expressions for the operating characteristics of $(Q_s, r_s, P_{1s}, P_{2s})$ policy with respect to the stochastic processes associated with each of possible realizations.

The following section presents the expressions for the expected cycle length and the operating characteristics of proposed single market two price and secondary market policies.

Chapter 3

Operating Characteristics and the Objective Function

In this chapter we obtain the operating characteristics of the inventory system for the single market two price policy and the secondary market policy and construct the objective functions of the corresponding decision models.

3.1 Operating Characteristics of the Single Market Two Price Policy

We will derive the expressions for the operating characteristics of single market two price policy, namely the expected cycle length, the expected holding cost, the expected perishing cost and the expected revenue for a given value of the effective shelf life at the steady state.

We obtain the expressions for the expected values of cycle length, on hand inventory, number of lost sales, number of items that perish and number of items sold at price P_1 or P_2 as a function of the decision variables Q , r , P_1 and P_2 . These expressions are then used to construct the average profit function.

When an embedded cycle begins there are five possible events. The first and second events correspond to fresh batch arrivals when the inventory is depleted during the lead time by demand and perishing, respectively. The third one indicates that all Q units are depleted by demand after the lead time and before they perish and the next one corresponds to the event that the batch perishes after the lead time. Finally, the last one indicates that all items in a batch perish before the reorder point r is reached.

We begin with the cycle length, CL , to derive the operating characteristics of the model of an embedded cycle at steady state. We define a cycle as the time between two consecutive instances at which inventory level hits Q . For a given value $Z = z$ of the effective shelf life,

$$CL = \begin{cases} X_{Q-r} + L & \text{if } X_{Q-r} + X_r < z, X_r < L \\ X_{Q-r} + L & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_{Q-r} + X_r \\ X_{Q-r} + L + Y_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ z & \text{if } z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, X_r > L \\ z + L & \text{if } z < X_{Q-r} \end{cases}$$

where the corresponding events are discussed above.

We also divide CL into categories such as cycle length until reorder point r is reached, cycle length during lead time and cycle length after lead time. Such division can help us to see whether demand or profit is lead time sensitive or not in our future research.

For this case for a given value of $Z = z$ of the effective shelf life, the length of cycle length until reorder point is reached by demand or by perishing is;

$$CL_R = \begin{cases} X_{Q-r} & \text{if } X_{Q-r} < z \\ z & \text{if } X_{Q-r} > z \end{cases}$$

where the first event corresponds to fresh batch arrivals and the replenishment point is reached by demand arrivals and the second one indicates that replenishment point is reached by perishing.

Taking the expectation we obtain;

$$\begin{aligned} E(CL_R|z) &= \frac{(Q-r)}{\lambda_1} L H_{Q-r}(z-L) \\ &+ z \bar{H}_{Q-r}(z). \end{aligned} \quad (3.1)$$

Cycle length during lead time can be calculated as;

$$CL_L = \begin{cases} L & \text{if } X_{Q-r} < z, X_Q < X_{Q-r} + L, X_Q < z \\ L & \text{if } X_{Q-r} < z < X_{Q-r} + L, X_Q > z \\ L & \text{if } X_{Q-r} + L < z, X_r > L \end{cases}$$

where the first event indicates that during lead time all the items on hand are totally depleted by demand and the second one corresponds to the event that items on hand reach their effective lifetimes during lead time and perish. The last one indicates that the inventory level during lead time is not depleted neither by demand nor by perishing.

The expected cycle length during lead time, $E(CL_L|z)$, is equal to L .

Cycle length after lead time is given as;

$$CL_A = \begin{cases} Y_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ z - X_{Q-r} - L & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L. \end{cases}$$

The first event corresponds to the event that items from the previous batch can not be depleted neither by demand nor by perishing and by the arrival of the fresh batch the inventory level increases above Q units and the younger batch is not sold before the older one is depleted by demand arrivals. The second one indicates that the younger batch is started to be sold after the older batch completes its usable lifetime and perishes.

Expression for the expected value of CL_A for a given effective shelf life $Z = z$ is equal to;

$$\begin{aligned} E(CL_A|z) &= \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{(r-j)}{\lambda_2} H_{r-j+1}(z-L-u) dH_{Q-r}(u) \\ &+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} (z-u-L) \bar{H}_{r-j}(z-L-u) dH_{Q-r}(u) \quad (3.2) \end{aligned}$$

Then, we obtain the expression for the expected value of the cycle length for a given effective shelf life $Z = z$ as follows;

$$\begin{aligned} E(CL|Z = z) &= E(CL_R|Z = z) + E(CL_L|Z = z) + E(C_A|Z = z) \\ &= \frac{(Q-r)}{\lambda} L H_{Q-r}(z-L) + z \bar{H}_{Q-r}(z) + L \\ &+ \sum_{i=0}^{r-1} P(N(L) = i) \left[\int_0^{z-L} \frac{r-i}{\lambda_2} H_{r-i+1}(z-L-u) dH_{Q-r}(u) \right. \\ &+ \left. \int_0^{z-L} (z-u-L) \bar{H}_{r-i}(z-L-u) dH_{Q-r}(u) \right] \end{aligned} \quad (3.4)$$

In order to find the holding cost we need the total stock time in a cycle. In other words we need the area under the inventory curve within an embedded cycle which is denoted by OH . Recall that $N(t), t \geq 0$ and $N'(t), t \geq 0$ are the number of demand arrivals with rate λ_1 and λ_2 respectively in an interval of length t . Considering the possible realizations, for a given value of the effective shelf life $Z = z$ we write,

$$OH = \left\{ \begin{array}{ll} \sum_{i=1}^Q X_i & \text{if } X_{Q-r} + X_r < z, X_r < L \\ \sum_{i=1}^{N(z)} X_i + z[Q - N(z)] & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_{Q-r} + X_r \\ \sum_{i=1}^{Q-r+N(L)} X_i + (r - N(L))(X_{Q-r} + L) + \sum_{i=1}^{r-N(L)} Y_i + QY_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ \sum_{i=1}^{Q-r+N(L)} X_i + (r - N(L))(X_{Q-r} + L) + \sum_{i=1}^{N'(z-X_{Q-r}-L)} Y_i \\ \quad + [Q + r - N(L) - N'(z - X_{Q-r} - L)](z - X_{Q-r} - L) & \text{if } z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, \\ & X_r > L \\ \sum_{i=1}^{N(z)} X_i + z[Q - N(z)] & \text{if } z < X_{Q-r} \end{array} \right.$$

To see the changes explicitly in on hand inventory until the reorder point is reached, during the lead time and after the lead time we divide on hand inventory into 3 parts.

Thus, on hand inventory until reorder point is reached either by demand or by perishing is;

$$OH_R = \left\{ \begin{array}{ll} \sum_{i=1}^{Q-r} X_i + rX_{Q-r} & \text{if } X_{Q-r} < z, X_r < L \\ \sum_{i=1}^{N_z} X_i + z[Q - N(z)] & \text{if } X_{Q-r} > z \end{array} \right.$$

where the first event corresponds to the case that after the arrival of a fresh batch the reorder level drops to r by demand arrivals and the second one indicates that the reorder level is reached by perishing.

The expected value is,

$$\begin{aligned} E(OH_R|z) &= \frac{(Q-r)(Q+r+1)}{2\lambda} H_{Q-r+1}(z) \\ &+ zQ\bar{H}_{Q-r}(z) - \frac{\lambda}{2} z^2 \bar{H}_{Q-r-1}(z) \end{aligned} \quad (3.5)$$

Then, on hand inventory during lead time can be calculated as ;

$$OH_L = \begin{cases} \sum_{i=1}^r X_i & \text{if } X_{Q-r} < z, X_Q < X_{Q-r} + L, \\ & X_Q < z \\ \sum_{i=1}^{N(z-X_{Q-r})} X_i \\ + [z - X_{Q-r}][r - N(z - X_{Q-r})] & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_Q \\ \sum_{i=1}^{N(L)} X_i + L[r - N(L)] & \text{if } X_r > L, z > X_{Q-r} + L \end{cases}$$

The first event indicates that during lead time all the items on hand are totally depleted by demand and the second one corresponds to the event that items on hand reach their expiry age during lead time and perish. The third one indicates that the inventory level during lead time is not depleted during by demand nor by perishing.

Taking the expectation, we obtain,

$$\begin{aligned}
E(OH_L|z) &= \frac{r(r+1)}{2\lambda} H_{r+1}(L) H_{Q-r}(z-L) \\
&+ \frac{r(r+1)}{2\lambda} \int_{z-L}^z H_{r+1}(z-u) dH_{Q-r}(u) \\
&+ \int_{z-L}^z (z-u) \left[r\overline{H}_r(z-u) - \lambda(z-u)\overline{H}_{r-1}(z-u) \right] dH_{Q-r}(u) \\
&+ H_{Q-r}(z-L) \frac{\lambda}{2} L^2 \overline{H}_{r-1}(L) \\
&+ LH_{Q-r}(z-L) \left[r\overline{H}_r(L) - \lambda L\overline{H}_{r-1}(L) \right] dH_{Q-r}(u). \tag{3.6}
\end{aligned}$$

Finally, on hand inventory after lead time is;

$$OH_A = \begin{cases} \sum_{i=1}^{r-N(L)} Y_i + QY_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, \\ & X_r > L \\ \sum_{i=1}^{N(z-X_{Q-r})} Y_i + [z - X_{Q-r} - L][Q + r - N(L)] & \\ -N(z - X_{Q-r} - L) & \text{if } z < X_{Q-r} + L + Y_{r-N(L)}, \\ & X_r > L \end{cases}$$

where the first event corresponds to the event that items from the previous batch can not be depleted neither by demand nor by perishing and by the arrival of the fresh batch the inventory level increases above Q units and the younger batch is not sold before the old one is depleted by demand arrivals. The second one indicates that the younger batch is started to sell after the old batch completes its usable lifetime and perish.

Expected value for a given effective shelf life $Z = z$ for the on hand inventory after lead time is,

$$\begin{aligned}
E(OH_A|z) &= \sum_{j=0}^{r-1} P(N(L) = j) \cdot \\
&\int_0^{z-L} (z-L-u) \sum_{i=0}^{r-j-1} P(N'(z-L-u) = i)(Q+r-j-i)dH_{Q-r}(u) \Big] + \\
&\sum_{j=0}^{r-1} \left[P(N(L) = j) \int_0^{z-L} \frac{(r-j)(r-j+1+2Q)}{2\lambda^*} H_{r-j+1}(z-L-u) dH_{Q-r}(u) \right] + \\
&\sum_{j=0}^{r-1} \left[P(N(L) = j) \int_0^{z-L} \frac{\lambda_2}{2} (z-u-L)^2 \bar{H}_{r-j-1}(z-L-u) dH_{Q-r}(u) \right] + \\
&\sum_{j=0}^{r-1} \left[P(N(L) = j) \int_0^{z-L} (z-u-L)[(Q+r-j)\bar{H}_{r-j}(z-L-u)] - \right. \\
&\left. \lambda_2(z-L-u)\bar{H}_{r-j-1}(z-L-u)] dH_{Q-r}(u) \right] \tag{3.7}
\end{aligned}$$

Taking the expectation we obtain the conditional expected on hand inventory in a cycle for given z is;

$$\begin{aligned}
E(OH|Z = z) &= E(OH_R|Z = z) + E(OH_L|Z = z) + E(OH_A|Z = z) \\
&+ \frac{r(r+1)}{2\lambda_1} \int_{z-L}^z H_{r+1}(z-u) dH_{Q-r}(u) \\
&+ \int_{z-L}^z (z-u) \left[r\bar{H}_r(z-u) - \lambda_1(z-u)\bar{H}_{r-1}(z-u) \right] dH_{Q-r}(u) \\
&+ H_{Q-r}(z-L) \frac{\lambda_1}{2} L^2 \bar{H}_{r-1}(L) \\
&+ LH_{Q-r}(z-L) \left[r\bar{H}_r(L) - \lambda_1 L \bar{H}_{r-1}(L) \right] \\
&+ zQ\bar{H}_{Q-r}(z) - \frac{\lambda_1}{2} z^2 \bar{H}_{Q-r-1}(z) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{(r-j)(r-j+1+2Q)}{2\lambda_2} \\
&\cdot H_{r-j+1}(z-L-u) dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{\lambda_2}{2} (z-u-L)^2 \bar{H}_{r-j-1}(z-L-u) dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} (z-u-L) \left[(Q+r-j)\bar{H}_{r-j}(z-L-u) \right. \\
&- \left. \lambda_2(z-L-u)\bar{H}_{r-j-1}(z-L-u) \right] dH_{Q-r}(u)
\end{aligned} \tag{3.8}$$

The following expressions are for the expected number of items sold, $E(SP_1)$, to find the revenue obtained by selling items at price P_1

$$SP_1 = \begin{cases} Q & \text{if } X_{Q-r} + X_r < z, X_r < L \\ N(z) & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_{Q-r} + X_r \\ Q - r + N(L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ Q - r + N(L) & \text{if } z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, X_r > L \\ N(z) & \text{if } X_{Q-r} > z. \end{cases}$$

The first line corresponds to the event that inventory is depleted by demand during the lead time and after an order is placed where no discount is placed and all Q items are sold at price P_1 . The second one corresponds to the events that all Q units perish during the lead time and only items that have not reached their expiry age are sold. The third one indicates that all Q units are depleted after the lead time and before they perish and the items at time $X_{Q-r} + L$ are sold at regular price and the fourth one indicates that the batch perishes after the lead time and only the items at time $X_{Q-r} + L$ are sold. The last one indicates that the batch perishes before the reorder point is reached and items that have not completed their lifetimes are sold.

Then, the conditional expected number of items sold at P_1 price in a cycle for given z is

$$\begin{aligned} E(SP_1|Z = z) &= Q \int_0^L H_{Q-r}(z-) dH_r(v) \\ &+ \sum_{i=0}^{Q-r-1} \sum_{j=0}^{Q-i-1} (i+j) P(N(z-L) = i) P(N(L) = j) \\ &+ 2 \left[\sum_{i=0}^{r-1} P(N(L) = i) (Q-r+i) \int_0^{z-L} H_{r-i, \lambda_2}(z-L-u) dH_{Q-r}(u) \right]. \quad (3.9) \end{aligned}$$

The number of items sold at price P_2 , $E(SP_2)$ in order to find the revenue is

as follows,

$$SP_2 = \begin{cases} r - N(L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ N'(z - X_{Q-r} - L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L. \end{cases}$$

As we have mentioned in Chapter 2 items are sold at price P_2 if the outstanding order arrives when there are still some items in the inventory and the inventory position increases above Q units. At this instance inventory on hand is composed of a number of items from the previous batch and batch of Q units. In order to manage demand a discount policy is applied and the items from the previous batch are sold at price P_2 where $P_2 < P_1$. That is the case which corresponds to the third and fourth events where the items at time $X_r + L$ and $z - X_{Q-r} - L$ have a discount.

Then the expected number of items sold at price P_2 is equal to,

$$\begin{aligned} E(SP_2|z) &= \sum_{i=0}^{r-1} P(N(L) = i)(r - i) \int_0^{z-L} H_{r-i, \lambda_2}(z - L - u) dH_{Q-r}(u) \\ &+ \sum_{i=0}^{r-1} P(N(L) = i) \int_0^{z-L} \sum_{j=0}^{r-i-1} j P(N'(z - u - L) = j) dH_{Q-r}(u). \end{aligned} \quad (3.10)$$

Finally we have the number of perishing items in an embedded cycle, P such that,

$$P = \begin{cases} Q - N(z) & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_r \\ r - N(L) - N'(z - X_{Q-r} - L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_r > L \\ Q - N(z) & \text{if } X_{Q-r} > z. \end{cases}$$

Items that have not exhausted by demand in their lifetimes perish and this corresponds to the second, fourth and last events. Taking the expectation we have,

$$\begin{aligned}
 E(P|z) = & \sum_{i=0}^{Q-r-1} \sum_{j=0}^{Q-i-j} (Q-i-j)P(N(z-L)=i)P(N(L)=j) \\
 & + \sum_{i=0}^{r-1} P(N(L)=i) \sum_{j=0}^{r-i-1} (r-i-j) \int_0^{z-L} P(N'(z-L-u)=j) dH_{Q-r}(u). \quad (3.11)
 \end{aligned}$$

3.2 Operating Characteristics of the Secondary Market Policy

In this section, we explain the expressions for the expected value of the cycle length, on hand inventory, number of items sold at price P_1 and number of items sold at price P_2 and the number of items that perish in an embedded cycle for the secondary market policy.

Let us denote the expected cycle length, expected on hand inventory, expected number of items sold at price P_1 , expected number of items sold at price P_2 and the expected number of items that perish per cycle by, $E(CL_s)$, $E(OH_s)$, $E(SP_{1s})$, $E(SP_{2s})$ and $E(P_s)$ respectively.

In secondary market policy, the items from the previous batch are sold at a secondary market at price P_2 and the new cycle starts with a lifetime of exactly τ .

In Figures 3.1 and 3.2 we illustrate a possible cycle realizations of the system.

The operating characteristics of the secondary market policy, $(Q_s, r_s, P_{1s}, P_{2s})$, can easily be obtained from the derivations we have obtained for (Q, r, P_1, P_2) policy.

We now describe a regenerative cycle for the primary market as the time

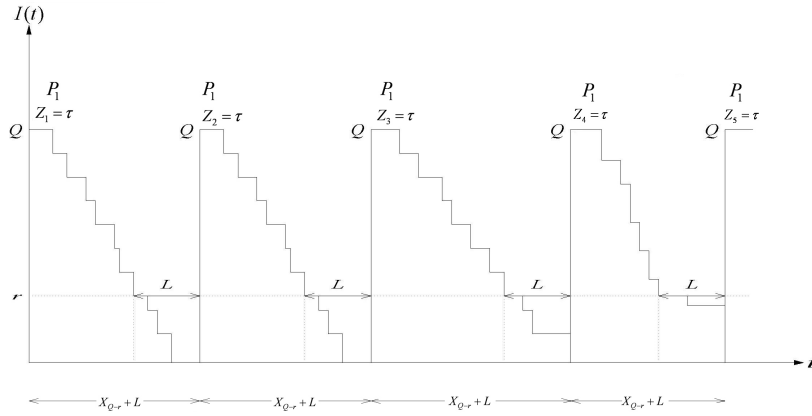


Figure 3.1: Possible cycle realizations for the secondary market policy

between two consecutive instances where the inventory level hits Q . These instances, either follow a stock-out period, so that the items of a new batch has a shelf life of τ , or at the arrival of a new batch there are still items from the previous batch with remaining shelf life z . In the latter case, since the remaining items are sent to a secondary market, the batch in the primary market again starts with fresh life. Hence in the primary market, when the inventory level hits Q , the items always have a shelf life of τ and therefore we have regenerative cycles.

When a new cycle starts for the primary market, the items of the older batch are also started to be sold at price P_2 at the secondary market simultaneously. These items are depleted either by demand or by perishing. We assume that the time until the items are depleted at the secondary market is at most the cycle time of the primary market. Hence we can find the approximate expected profit rate at this policy under the above assumption.

For the expected cycle length, we consider only the cycle length until the end of the lead time in the primary market. This can be obtained by adding a lead time unit to expression 3.1. Hence, the expected value of the cycle length, $E(CL_s)$, is given by the following expression;

$$E(CL_s|z = \tau) = E(CL_R|z = \tau) + L. \quad (3.12)$$

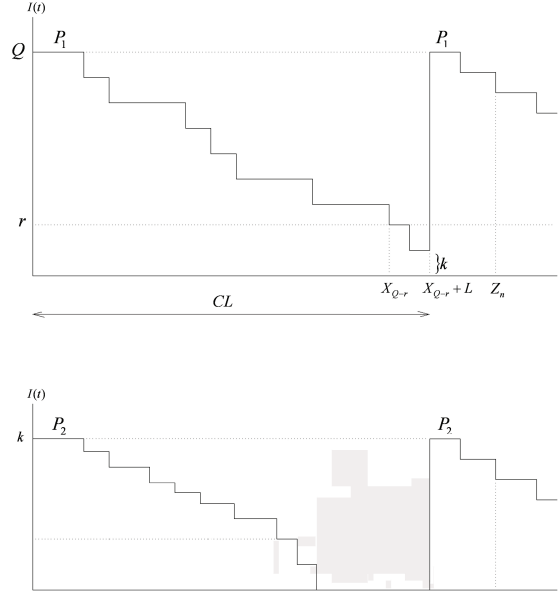


Figure 3.2: Possible cycle realization for the secondary market policy

The expected costs of the secondary market policy is simply obtained as the corresponding expected costs of the single market two price policy with remaining shelf life z is set to $z = \tau$, due to the assumption we made above .

Then, the expected value of the on hand inventory, $E(OH_s)$, and the expected value of the number of items perish $E(P_s)$, are given by the following expressions;

$$E(OH_s|z = \tau) = E(OH_R|z = \tau) + E(OH_L|z = \tau) + E(OH_A|z = \tau), \quad (3.13)$$

$$E(P_s|z = \tau) = E(P|z = \tau). \quad (3.14)$$

The expected revenue of the secondary market policy is simply obtained corresponding expected revenue of the single market two price policy with remaining shelf life z is set to $z = \tau$, due to the assumption we made above.

Then, the expected value of the number of items sold at price P_1 , and the

expected value of the number of items sold at price P_2 , at secondary market are given by the following expressions;

$$E(SP_{1s}|z = \tau) = E(SP_1|z = \tau), \quad (3.15)$$

$$E(SP_{2s}|z = \tau) = E(SP_2|z = \tau). \quad (3.16)$$

3.3 Objective Function

In this section we propose the objective function. Our work addresses the problem of maximizing the expected profit rate. To solve the problem we construct the expected profit rate by using Theorem 1 below [3]. Let $\Phi(t)$ be the profit incurred over the interval $(0, t]$, Φ_i and L_i be the cycle profit and the cycle length of the i th ($i = 1, 2, 3, \dots$) embedded cycle respectively. Since the embedded cycles with the same remaining effective shelf life behave identically, we define for $i \geq 1$,

$$\begin{aligned} \Phi(z) &= E[\Phi_i | Z_i = z] \\ L(z) &= E[L_i | Z_i = z]. \end{aligned} \quad (3.17)$$

Theorem 1. Let F be the limiting distribution function of the sequence of effective shelf lives. Then,

$$\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \frac{\int_L^\tau \Phi(z) dF(z)}{\int_L^\tau L(z) dF(z)}. \quad (3.18)$$

Proof. See [3]

Considering the above result, the expected profit rate, TP for our problem can be stated as follows;

$$TP = \frac{K + \int_0^\tau \alpha(z) dF(z)}{\int_0^\tau E[CL | Z = z] dF(z)}$$

where,

$$\alpha(z) = P_1 \cdot E[SP_1 | Z = z] + P_2 \cdot E[SP_2 | Z = z] - h \cdot E[OH | Z = z] - p \cdot E[Pe | Z = z].$$

Next we will present our numerical study in order to see the general behavior of the optimal policy parameters and the expected profit rate with respect to different cost and system parameters.

Chapter 4

Numerical Analysis

In this chapter we present the results of a numerical study for the performance of the (Q, r, P_1, P_2) and $(Q_s, r_s, P_{1s}, P_{2s})$ policies developed in the previous chapters.

In our model, we assume that external demands are generated according to a stationary Poisson process with rate $\lambda > 0$ and it is a decreasing function of price. In the literature, linear and exponential demand rates are often used by researchers in dealing with pricing. Our model allows a general structure for this function but, in our numerical studies we consider a linear demand rate, λ_i that decreases in price and it is denoted by $\lambda(p_i)$ such that $\lambda_i = \lambda(p_i)$. We consider $\lambda_i = b - ap_i$ where slope a measures the change in demand per unit change in price and b corresponds to the demand rate if the items are given away. We assume that $a > 0$ and $b > 0$ are constants with $0 < p_i < b/a$.

Matlab is one of the commonly accepted software to code mathematical models since it has built-in functions for probability distributions and allows probabilistic and mathematical operations. To find the optimal value of the corresponding discounted expected profit in the acceptable price range we use Matlab. The computational complexity of our model is governed by solving a system of linear equations. Various grid sizes have been tried and we observed that the overall behavior of the shelf life distribution function and the resulting profit rate are not too sensitive to the grid size beyond a certain value. We used increment

size of 0.01 to have precise numerical solutions. We obtained the expected value of the operating characteristics and for optimization we used exhaustive search over a range of policy parameters.

In the following parts we examine the behavior of the effective shelf life under (Q, r, P_1, P_2) policy and search for its impact. We present the numerical results on the sensitivity analysis of the optimal parameters with respect to the environmental parameters.

For (Q, r, P_1, P_2) policy we compare the performance of the inventory system with the secondary market policy and constant pricing policy. In constant pricing policy we do not allow for the pricing promotion and the new and old batches are sold for the same price, P_1 . This comparison allow us to discuss the performance quality of our proposed policy.

4.1 Behavior of the Effective Shelf life Distribution

In this section we investigate the steady state behavior of the effective shelf life distribution and its sensitivity with respect to the control policy parameters where $Q = 17, L = 1, h = 1, p = 5$ and $P_1 = 50, P_2 = 55P_1 = 27.5$. In Figures 4.1, 4.2 and 4.3, we present a sample of the sequence of effective shelflife distributions for consecutive cycles starting with a fresh batch for different τ values.

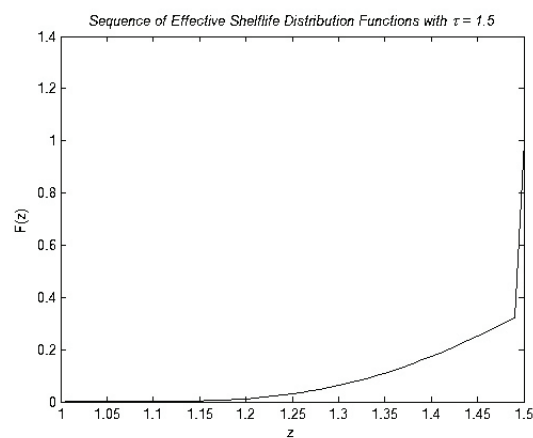


Figure 4.1: Sequence of Effective Shelf life Distribution Functions with $\tau = 1.5$, $a = 0.06$, $b = 13$

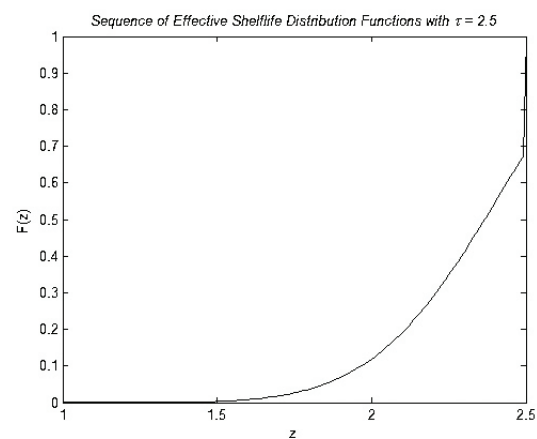


Figure 4.2: Sequence of Effective Shelf life Distribution Functions with $\tau = 2.5$, $a = 0.06$, $b = 13$

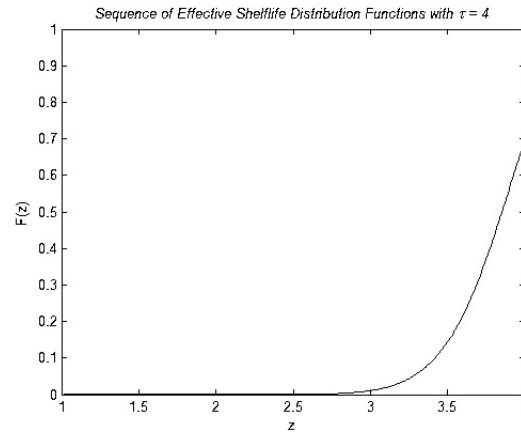


Figure 4.3: Sequence of Effective Shelf life Distribution Functions with $\tau = 4, a = 0.06, b = 13$

We observe that as τ increases the point mass of the effective shelf life at τ that is $P(Z = z)$, decreases probably with smaller perishing probability.

We analyze the sensitivity of the effective shelf life distribution function to the reorder point, r . In Figures 4.4, 4.5 and 4.6 we present a sample of effective shelf life distributions for different values of r . The point mass of the effective shelf life gets smaller as the reorder point, in other words, desired service level r increases.

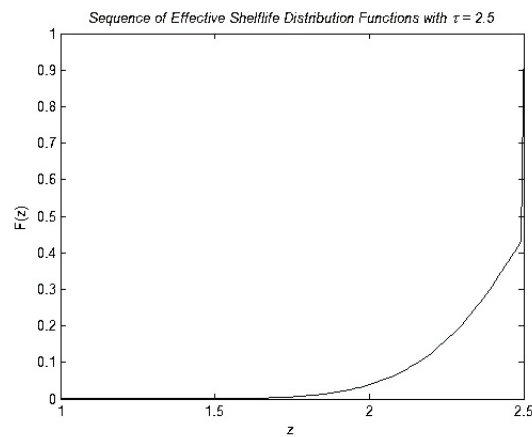


Figure 4.4: Sequence of Effective Shelf life Distribution Functions with $\tau = 2.5, r = 10, a = 0.06, b = 13$

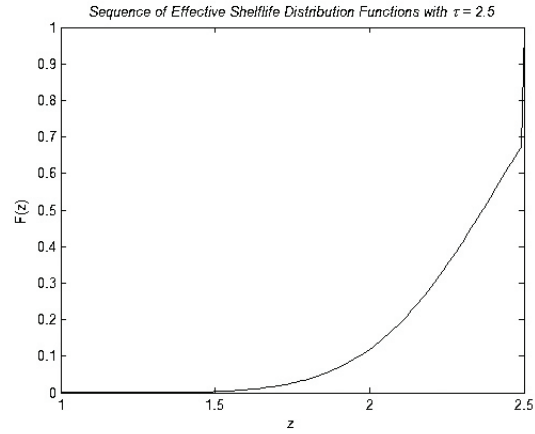


Figure 4.5: Sequence of Effective Shelf life Distribution Functions with $\tau = 2.5, r = 12, a = 0.06, b = 13$

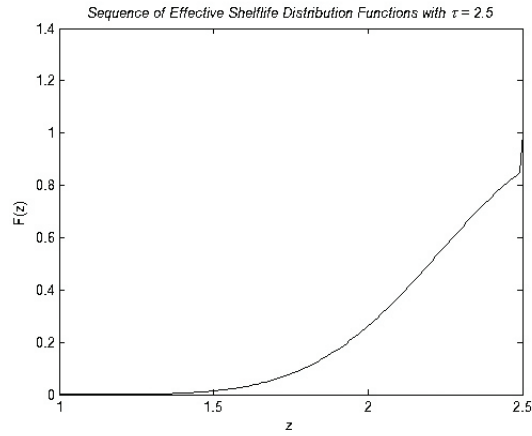


Figure 4.6: Sequence of Effective Shelf life Distribution Functions with $\tau = 2.5, r = 14, a = 0.06, b = 13$

Then, we search for the sensitivity of the effective shelf life distribution to the demand rate and as presented in Figures 4.7 and 4.8 when demand rate λ decreases the effective shelf life distribution gets smaller. Also we observe that the cumulative distribution function of the effective shelf life is increasing more steeply for small values of the effective shelf life with increasing service levels.

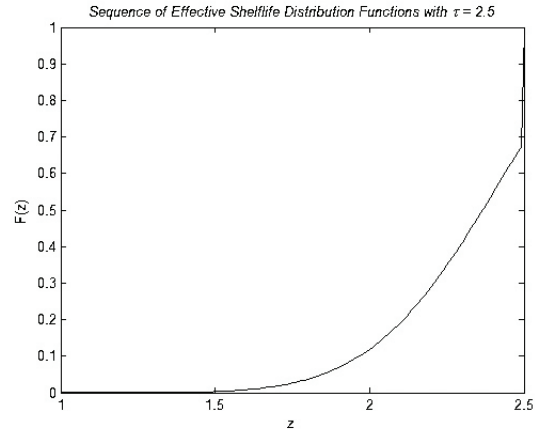


Figure 4.7: Sequence of Effective Shelf life Distribution Functions with $\tau = 2.5$, $Q = 17$, $r = 12$, $a = 0.06$, $b = 13$

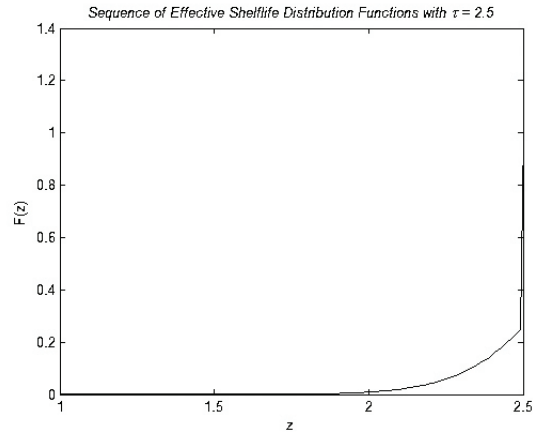


Figure 4.8: Sequence of Effective Shelf life Distribution Functions with $\tau = 2.5$, $Q = 17$, $r = 12$, $a = 0.11$, $b = 19$

4.1.1 Impact of Effective Shelf life

We want to observe the impact of the effective shelf life on the expected profit. To see this effect we present below the values of $E[\text{Profit}|z]$ with respect to different z and τ values. From Table 4.1 one can see how the conditional expected profit changes with τ and price jointly for $L = 1, Q = 17, r = 12, h = 1, K = 50$ and $p = 5$ where $P_1 = 50$ with different choices for P_2 which take on values of 40, 27.5 and 15. The numerical results for $L = 1, Q = 24, r = 15, h = 1, K = 50$ and $p = 5$ where $P_1 = 50$ and P_2 is 40, 27.5 and 15 to see the changes in expected profit for a given $Z = z$ are in Table 4.3.

Table 4.2 and 4.4 present some of the results of the expected profit for different values of τ under the same environmental conditions. When we compare these two cases we observe a high expected profit rate for a given effective shelf life. We can also state when τ gets smaller impact of effective shelf life becomes stronger.

	Price	$E(\text{Profit} z=1.25)$	$E(\text{Profit} z=1.5)$	$E(\text{Profit} z=2)$	$E(\text{Profit} z=2.25)$	$E(\text{Profit} z=2.5)$
$\tau = 1.5$	P2=40	73.6386	275.9821	-	-	-
	P2=27.5	73.3488	272.5052	-	-	-
	P2=15	73.0137	268.5597	-	-	-
$\tau = 2.5$	P2=40	73.6386	275.9821	424.8684	432.6039	434.1031
	P2=27.5	73.3488	272.5052	413.1986	419.3192	420.1897
	P2=15	73.0137	268.5597	400.4965	405.0922	405.4356

Table 4.1: Impact of Effective Shelf life

	$\tau = 1.5$	$\tau = 2.5$
Price	$E(\text{Profit})$	$E(\text{Profit})$
P2=40	245.5979	428.2255
P2=27.5	243.1173	416.0456
P2=15	240.2257	402.6667

Table 4.2: Expected Profit

Price	E(Profit/z=1.25)	E(Profit/z=1.5)	E(Profit/z=2)	E(Profit/z=2.25)	E(Profit/z=2.5)	E(Profit/z=3)	E(Profit/z=3.5)	E(Profit/z=4)
P2=40	-58.0329	25.3805	389.4329	466.3813	490.5534	497.9577	498.2824	498.2948
P2=27.5	-58.0359	25.0048	382.4676	456.0756	478.3554	484.5441	484.7241	484.7265
P2=15	-58.0404	24.4575	372.9845	442.4434	462.6505	467.7959	467.8994	467.8989

Table 4.3: Impact of Effective Shelf life, $\tau = 4$

Price	E(Profit)
P2=40	498.2399
P2=27.5	484.7151
P2=15	467.8969

Table 4.4: Expected Profit, $\tau = 4$

4.2 Sensitivity Analysis - Single Market Two Price Policy

In this section we present experimental setup and numerical results for proposed (Q, r, P_1, P_2) policy.

We use different experimental setups to analyze the sensitivity of the model to various parameters of the system under consideration in order to see how the policy parameters change with respect to different system parameters.

In the analysis, per unit holding cost per unit time is fixed at $h = 1$ and lead time is assumed to be $L = 1$. The increments are taken as 0.01 which enabled us to get more accurate results for the integral approximations. For the inventory system under consideration, inventory level is depleted by perishing of the products in the inventory as well as by demand arrivals. Hence the ordering decision depends not only on the demand rate but also on the lifetime of items. Therefore the relevant performance measures are displayed at different lifetimes. We use the following ranges for the rest of the parameters.

Parameter	Symbol	Values Tested
Ordering cost	K	5, 10, 50
Unit perishing cost	p	5, 10, 40
Shelf life	τ	1.5, 2.5
Slope of demand rate	a	0.11, 0.13, 1.2, 1.75
Intercept of demand rate	b	10.2, 19

Table 4.5: Experimental Setup

The experimental setup defined Table 4.5 is used to display several results. The optimal regular and discounted prices, P_1^* and P_2^* are displayed at the selected remaining lifetimes for their respective values of optimal order quantity Q^* and reorder point r^* .

Parameters															
$p = 5$															
K	τ	a	b	Q^*	r^*	P_1^*	P_2^*	ϕ	TP	Q^*	r^*	P_1^*	P_2^*	ϕ	TP
$K = 5$	$\tau=2.5$	0.11	19	16	15	86	84	3.48	795.83	16	15	85	81	4.70	793.12
		0.13	19	16	15	76	71	6.87	670.415	16	15	75	69	8	667.40
$K = 10$	$\tau=1.5$	1.2	19	11	10	8	5.7	28.75	49.81	11	10	7.6	5.1	32.89	48.09
		1.75	19	10	9	5.7	3.8	33.33	29.17	10	9	5.5	3.4	38.18	28.08
		1.2	10.2	5	4	4.5	2.9	35.55	4.74	5	4	4.3	2.6	39.53	3.85
		1.75	10.2	5	2	3	2	33.33	-0.16	5	2	2.9	1.7	41.37	-0.19
	$\tau=2.5$	0.11	19	16	15	87	84	3.44	793.71	16	15	86	82	4.65	790.20
		0.13	19	15	14	74	69	6.75	666.79	15	14	72	68	6.94	664.58
$K = 50$	$\tau=2.5$	0.11	19	16	15	88	84	3.40	767.70	16	15	87	83	4.60	765.13
		0.13	19	15	14	75	70	6.66	641.16	15	14	74	69	6.80	639.24

$p = 10$															
K	τ	a	b	Q^*	r^*	P_1^*	P_2^*	ϕ	TP	Q^*	r^*	P_1^*	P_2^*	ϕ	TP
$K = 5$	$\tau=2.5$	0.11	19	16	15	86	84	3.48	795.83	16	15	85	81	4.70	793.12
		0.13	19	16	15	76	71	6.87	670.415	16	15	75	69	8	667.40
$K = 10$	$\tau=1.5$	1.2	19	11	10	8	5.7	28.75	49.81	11	10	7.6	5.1	32.89	48.09
		1.75	19	10	9	5.7	3.8	33.33	29.17	10	9	5.5	3.4	38.18	28.08
		1.2	10.2	5	4	4.5	2.9	35.55	4.74	5	4	4.3	2.6	39.53	3.85
		1.75	10.2	5	2	3	2	33.33	-0.16	5	2	2.9	1.7	41.37	-0.19
	$\tau=2.5$	0.11	19	16	15	87	84	3.44	793.71	16	15	86	82	4.65	790.20
		0.13	19	15	14	74	69	6.75	666.79	15	14	72	68	6.94	664.58
$K = 50$	$\tau=2.5$	0.11	19	16	15	88	84	3.40	767.70	16	15	87	83	4.60	765.13
		0.13	19	15	14	75	70	6.66	641.16	15	14	74	69	6.80	639.24

$p = 40$															
K	τ	a	b	Q^*	r^*	P_1^*	P_2^*	ϕ	TP	Q^*	r^*	P_1^*	P_2^*	ϕ	TP
$K = 5$	$\tau=2.5$	0.11	19	16	15	86	84	3.48	795.83	16	15	85	81	4.70	793.12
		0.13	19	16	15	76	71	6.87	670.415	16	15	75	67	10.71	661.76
$K = 10$	$\tau=1.5$	1.2	19	11	10	8	5.7	28.75	49.81	11	10	7.4	4.8	35.13	43.56
		1.75	19	10	9	5.7	3.8	33.33	29.17	10	9	5.3	3.1	41.50	25.03
		1.2	10.2	5	4	4.5	2.9	35.55	4.74	5	4	4.1	2.3	43.90	1.49
		1.75	10.2	5	2	3	2	33.33	-0.16	5	2	2.7	1.5	44.44	-2.09
	$\tau=2.5$	0.11	19	16	15	87	84	3.44	793.71	16	13	85	79	7.05	780.97
		0.13	19	15	14	74	69	6.75	666.79	14	13	70	67	10.66	655.34
$K = 50$	$\tau=2.5$	0.11	19	16	15	88	84	3.40	767.70	15	14	86	80	6.97	770.96
		0.13	19	15	14	75	70	6.66	641.16	14	13	75	68	9.33	653.98

Table 4.6: Sensitivity Analysis with respect to $K, \tau, \lambda, L = 1$ and $h = 1$

Table 4.2 presents our results for the first experimental setup. The findings are intuitive and consistent with our expectations. As the shelf life τ increases the optimal reorder point and the optimal order size are monotonically increasing. The optimal value of Q^* increases as the ordering cost K increases as we expected. A high perishing cost results in lower expected profit.

We observed that a change in the slope of the demand rate a changes the expected profits obtained. The results can be seen in Table 4.2. When the slope of the demand rate a increases, the optimal starting price and the corresponding discounted expected profit decrease. A change in the intercept of the demand rate also has an effect on the optimal prices and corresponding discounted expected profit. An increase in b leads to an increase in optimal regular and discounted prices and results in higher profit rate. When intercept of demand decreases the discount rate increases since in that case demand rate decreases and it is a reasonable option to stimulate the demand rate by price promotion.

We observe that when the arrival of demand gets smaller and remaining shelf life is short the relative decrease in price and the corresponding profits is more noticeable since the price is lowered to increase the demand rate and maximize the expected discounted profit.

The optimal order quantity level Q^* , reorder level r^* and corresponding discounted expected profit are found to be decreasing when perishing cost increases. The unit perishing cost further affects the optimal price decision. When the unit perishing cost is low the optimal price is higher compared to the case with higher perishing cost.

In the next section, we present the results of the numerical study for Secondary Market policy.

4.3 Sensitivity Analysis - Secondary Market Policy

In this section we present the numerical study for the sensitivity analysis of Secondary Market Policy under the same experimental setup as used as in the case of Single Market Two Price Policy.

Table 4.3 displays the optimal values of fixed price P_{1s}^* and P_{2s}^* , optimal order quantity Q_s^* , optimal reorder level r_s^* and optimal discounted expected profit and their sensitivity to different parameter values.

Both single item two price policy and secondary market policy behave in a similar fashion as expected.

We observe that the optimal value for Q_s^* increases as fixed setup cost K increases. When we increase p we observe that optimal value of Q_s^* and P_{2s}^* decreases in order to decrease the number of units that perish and avoid from the perishing cost.

Another observation is that for a given remaining lifetime and unit perishing cost of items in the inventory the optimal prices P_{1s}^* and P_{2s}^* are decreasing in a and they are increasing in b . An increase in a and a decrease in b both means a decrease in the demand rate. Hence a decrease in the demand rate decreases the optimal corresponding profit.

<i>Parameters</i>				<i>p = 5</i>					<i>p = 10</i>				
<i>K</i>	τ	<i>a</i>	<i>b</i>	Q_s^*	r_s^*	P_{1s}^*	P_{2s}^*	TP_s	Q_s^*	r_s^*	P_{1s}^*	P_{2s}^*	TP_s
<i>K</i> = 10	$\tau=1.5$	1.2	19	12	11	8.1	6.9	53.01	12	10	7.9	6.4	52.90
		1.75	19	11	10	5.7	4.5	31.02	11	10	5.3	4.1	30.56
		1.2	10.2	6	5	4.6	3.1	5.06	6	5	4.3	2.8	4.81
		1.75	10.2	5	4	3.2	2.2	-0.07	5	4	3	1.9	-0.17
	$\tau=2.5$												
		0.11	19	17	16	88	87	797.57	17	16	87	86	797.31
		0.13	19	16	15	74	73	671.75	16	15	73	72	671.31
<i>K</i> = 50	$\tau=2.5$	0.11	19	17	16	89	88	775.47	17	16	88	87	775.13
		0.13	19	16	15	75	74	648.35	16	15	74	73	647.88

Table 4.7: Sensitivity Analysis with respect to K, τ, λ , $L = 1$ and $h = 1$

4.4 Sensitivity Analysis - Constant Pricing Policy

As we discussed in previous sections we will compare performance of the (Q, r, P_1, P_2) policy with a constant pricing policy for effective shelf life and Poisson demand rates in the next section. A constant pricing policy is the one that sets a fixed price over the sales period. In this section same experimental setup is used as in the case of single market two price policy in order to see the sensitivity of the constant pricing policy to system parameters. Similar tables are presented for this policy.

Table 4.4 presents the values of optimal fixed price P^* and optimal order quantity Q^* , optimal reorder level r^* and optimal discounted expected profit.

We observe that as the demand rate decreases, the optimal replenishment quantity Q^* decreases if the remaining lifetime is long enough to realize the demand.

When the perishing cost increases, the optimal replenishment quantity for a given remaining shelf life is non increasing and the optimal price is decreasing in order to manage demand and avoid from the perishing cost. A high perishing cost results in lower expected profit.

The optimal replenishment quantity Q^* increases as the ordering cost K increases as we expected.

We will compare and discuss the performance of the single market two price policy with the secondary market and constant pricing policies in the following section.

4.5 Comparison of (Q, r, P_1, P_2) Policy with Constant Pricing and Secondary Market Policy

In this section we compare the performance of single market two price policy with the constant pricing and secondary market policies for Poisson demands rate. The numerical study is based on the same experimental setup as discussed above. The aim of computational study is to give an idea about which policy works better and is more usable than the other under the same environmental conditions.

The parameter values are selected in a way that we are able to observe the effects of perishability for each policy we consider. The fixed parameters are $L = 1$ and $h = 1$. We vary other parameters as presented in Table 4.5.

Our aim is to provide a general idea on the parameter range where single market two price policy outperforms the constant pricing and secondary market policies.

We observe that optimal replenishment quantities are similar in most of the cases for the three policies under the same environmental conditions, this gives us an opportunity to compare the regular prices and the corresponding profits as well.

If we compare the (Q, r, P_1, P_2) policy with the constant pricing policy it is observed that our proposed single market two price policy outperforms the constant pricing policy in all cases. We define a relative improvement in the average profit obtained by (Q, r, P_1, P_2) policy as follows;

$$\Delta\% = \frac{EP(Q^*, r^*, P_1^*, P_2^*) - EP(Q^*, r^*, P^*)}{EP(Q^*, r^*, P^*)} \times 100 \quad (4.1)$$

Table 4.5 displays the relative improvement obtained by applying (Q, r, P_1, P_2) policy compared to using a constant pricing policy.

<i>Parameters</i>		$p = 5$			$p = 10$			$p = 40$		
K	τ	a	b	$\Delta\%$	a	b	$\Delta\%$	a	b	$\Delta\%$
$K = 5$	$\tau=2.5$	0.11	19	0.21	0.11	19	0.185	0.11	19	0.21
		0.13	19	4.67	0.13	19	4.52	0.13	19	0.29
		1.2	19	0.95	1.2	19	1.61	1.2	19	3.88
		1.75	19	1.46	1.75	19	2.45	1.75	19	6.51
$K = 10$	$\tau=1.5$	1.2	10.2	9.97	1.2	10.2	6.94	1.2	10.2	47.52
		1.75	10.2	15.78	1.75	10.2	17.39	1.75	10.2	5
	$\tau=2.5$	0.11	19	0.15	0.11	19	0.09	0.11	19	1.29
		0.13	19	0.24	0.13	19	0.20	0.13	19	0.20
$K = 50$	$\tau=2.5$	0.11	19	0.08	0.11	19	0.10	0.11	19	2.04
		0.13	19	0.09	0.13	19	0.11	0.13	19	3.61

Table 4.9: Relative improvement w.r.t K, τ, λ , $L = 1$ and $h = 1$

For fixed perishing cost the relative profit improvement is higher in cases where the shelf life is shorter. This means that for the systems in which the products have shorter shelf lives that is for highly perishable goods the (Q, r, P_1, P_2) policy results in considerable saving. For the items with longer shelf lives ($\tau = 2.5$) the relative improvement is less than the previous case but still the (Q, r, P_1, P_2) policy performs better than the constant pricing policy.

When b is fixed, an increase in a , in other words a decrease in demand rate results in higher improvement at fixed τ when fixed ordering cost is high.

Unit perishing cost has also an effect on the relative improvement obtained. When perishing cost increases the (Q, r, P_1, P_2) policy yields higher profits at a given shelf life. This effect is noticeable especially when the shelf life is longer.

When we compare the (Q, r, P_1, P_2) policy with $(Q_s, r_s, P_{1s}, P_{2s})$, as one would expect, $(Q_s, r_s, P_{1s}, P_{2s})$ results in notably higher profit in all parameter settings. It is very reasonable since the $(Q_s, r_s, P_{1s}, P_{2s})$ policy has the primary advantage

where the aging of the new batch is not allowed by selling the items at a secondary market.

The value of P_{2s}^* is less than the P_2^* since the aging of the new batch is not allowed, the high level discount rate is not considered.

Chapter 5

Conclusion

In this study, a continuous review perishable inventory system is designed for items that have fixed lifetimes and constant lead time. The demand process is assumed to be Poisson and specified as a decreasing function of price. The system jointly considers a pricing policy with the inventory replenishment policy. The system considers two process. The first process is an ordinary Poisson process with rate, λ_1 , observed when there is only one batch at a time. The second process is observed when there are some leftover items from the previous batch by the arrival of the outstanding order. The second process is also an ordinary Poisson process with a different parameter however, λ_2 , where $\lambda_2 > \lambda_1$.

In this study we mainly consider the description of the single market two price policy, referred as (Q, r, P_1, P_2) policy. According to this policy, when there are two batches at a time, a price reduction is made for the older batch and the younger batch is not used to satisfy the demand until the older one is totally depleted. The model assumes a higher demand rate at a lower price. After the older batch is completely sold or perish, the new batch started to be sold at regular price. Although the shelf lives are constant, the sequence of remaining shelf lives of the items at the instances where stock level is Q is a random sequence.

As a second policy we consider a model that some aged items have an opportunity to be sold at the secondary market, referred as $(Q_s, r_s, P_{1s}, P_{2s})$ policy. In

this policy, primary market always starts with a fresh single batch and if a new batch arrives before the older one is depleted, remaining items from the older batch are sold at reduced price in the secondary market.

Our objective is to determine the optimal pricing and replenishment policy to maximize the expected profit rate. We explain the policy for the (Q, r, P_1, P_2) and $(Q_s, r_s, P_{1s}, P_{2s})$ models in detail to construct a profit maximization problem and analytically derive expressions for the operating characteristics, which are necessary for solving the maximization problem.

A numerical study has been conducted to highlight the basic features of the proposed (Q, r, P_1, P_2) and $(Q_s, r_s, P_{1s}, P_{2s})$ policies. Based on a range of parameter settings, we first analyze the sensitivity of the models to various parameters of the inventory system under consideration.

Then, we compare the (Q, r, P_1, P_2) policy both with the secondary market policy and more general one which is called as constant pricing policy in order to measure the real contribution of (Q, r, P_1, P_2) model. In all cases, we observe that the (Q, r, P_1, P_2) policy outperforms the constant pricing case. However, when we compare the secondary market policy with the (Q, r, P_1, P_2) policy we observe that the secondary market policy performs better than the (Q, r, P_1, P_2) policy. It is a very reasonable result since the aging of the new batch is not allowed in the secondary market policy and we do not consider the transformation cost or any other related cost in order to transform the older batch to the secondary market.

For future research, this study can be extended to the cases in which different demand rate structures can be considered. We can allow batch demands and multiple outstanding orders. Also a transformation cost can be included into the secondary market policy.

Bibliography

- [1] S. P. Aggarwal and C. K. Jaggi. Ordering policy for decaying inventory. *International Journal of Systems Science*, 20:151–155, 1989.
- [2] V. Agrawal and S. Seshadri. Impact of uncertainty and risk aversion on price and quantity in the newsvendor problem. *Manufacturing and Service Operations Management*, 2:410–423, 2000.
- [3] E. Berk and U. Gürlér. Analysis of the (q,r) inventory model for perishables with positive lead times and lost sales. *Operations Research*, 56:1238–1246, 2008.
- [4] A. Bisi and M. Dada. Optimal ordering and pricing by a newsvendor with unobservable lost sales. *Naval Research Logistics*, 143:115–124, 2007.
- [5] A. N. Burnetas and C. E. Smith. Adaptive ordering and pricing for perishable products. *Operations Research*, 48:436–443, 2000.
- [6] E. P. Chew, C. Lee, and R. Liu. Joint onventory allocation and pricing decisions for perishable products. *To appear in International Journal of Production Research*.
- [7] H. N. Chiu. An approximation to the continuous review inventory model with perishable items and lead times. *European Journal of Operational Research*, 87:93–108, 1995.
- [8] Y. H. Chun. Optimal pricing and ordering policies for perishable commodities. *European Journal of Operational Research*, 144:68–82, 2003.

- [9] M. A. Cohen. Joint pricing and ordering policy for exponentially decaying inventory with known demand. *Naval Research Logistics Quarterly*, 24:257–268, 1977.
- [10] T. K. Datta and K. Paul. An inventory system with stock dependent, price sensitive demand rate. *Production Planning and Control*, 12:13–20, 2001.
- [11] A. Federgruen and A. Heching. Combined pricing and inventory control under uncertainty. *Operations Research*, 47:454–475, 1999.
- [12] Y. Feng and G. Gallego. Perishable asset revenue management with markovian time dependent demand intensities. *Department of Industrial Engineering and Operational Research, Columbia University, New York*, 1996.
- [13] B. E. Fries. Optimal ordering policy for a perishable commodity with fixed lifetime. *Operations Research*, 23:46–61, 1975.
- [14] G. Gallego and G. van Ryzin. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40:999–1020, 1994.
- [15] U. Gürler and B. Y. Özkaya. A note on 'continuous review perishable inventory systems: Models and heuristics. *IIE Transactions*, 35:321–323, 2003.
- [16] H. Gurnani and C. S. Tang. Note. optimal ordering decisions with uncertain cost and demand forecast updating. *Management Science*, 45:1456–1462, 1999.
- [17] J. James D. Dana and N. C. Petruzzi. Note: The newsvendor model with endogenous demand. *Management Science*, 47:1488–1497, 2001.
- [18] S. Kalpakam and K. P. Sapna. A lost sales (s-1,s) perishable inventory system with renewal demand. *Naval Research Logistics*, 43:129–142, 1996.
- [19] S. Kang and I. Kim. A study on the price and production level of the deteriorating inventory system. *International Journal of Production Research*, 21:449–460, 1983.

- [20] S. Karlin and C. Carr. Prices and optimal inventory policy. *Studies in Applied Probability and Management Science*, 1962.
- [21] M. Khouja. Optimal ordering, discounting and pricing in the single-period problem. *International Journal of Production Economics*, 65:201–216, 2000.
- [22] E. P. Lazear. Retail pricing and clearance sales. *The American Economic Review*, 76:14–32, 1986.
- [23] Z. Lian and L. Liu. Continuous review perishable inventory systems: Models and heuristics. *IIE Transactions*, 33:809–822, 2001.
- [24] L. Liu and Z. Lian. (s,s) continuous review models for products with fixed lifetimes. *Operations Research*, 47:150–158, 1999.
- [25] E. S. Mills. Uncertainty and price theory. *The Quarterly Journal of Economics*, 73:116–130, 1959.
- [26] P. Nandakumar and T. E. Morton. Near myopic heuristic for the fixed-life perishability problem. *Management Science*, 39:1490–1498, 1993.
- [27] N. C. Petruzzi and M. Dada. Pricing and the newsvendor problem: A review with extensions. *Operations Research*, 47:183–194, 1999.
- [28] N. Ravichandran. Stochastic analysis of a continuous review perishable inventory system with positive lead time and poisson demands. *European Journal of Operational Research*, 84:444–457, 1995.
- [29] C. P. Schmidt and S. Nahmias. (s-1,s) policies for perishable inventory. *Management Science*, 31:719–728, 1985.
- [30] E. Tekin, U. Gürlér, and E. Berk. Age based vs stock level control policies for a perishable inventory system. *European Journal of Operational Research*, 134:309–332, 2001.
- [31] H. Wee and J. Yu. A deteriorating inventory model with a temporary price discount. *International Journal of Production Research*, 53:81–90, 1997.

- [32] H. Weiss. Optimal ordering policies for continuous review perishable inventory models. *Operations Research*, 28:365–374, 1980.
- [33] T. M. Whitin. Inventory control and price theory. *Management Science*, 2:61–68, 1955.
- [34] G. J. J. V. Zyl. Inventory control for perishable commodities. *Unpublished Ph. D. Dissertation, University of North Carolina*, 1964.

Appendix A

Operating Characteristics

The following expressions are for the expected cycle length, $E(CL)$

Cycle Length until reorder point is reached either by demand or by perishing;

$$CL_R = \begin{cases} X_{Q-r} & \text{if } X_{Q-r} < z \\ z & \text{if } X_{Q-r} > z \end{cases}$$

$$\begin{aligned} E(CL_R|z) &= E\left[X_{Q-r}I(X_{Q-r} < z)\right] \\ &+ E\left[zI(X_{Q-r} > z)\right] \\ &= \int_0^z u dH_{Q-r}(u) \\ &+ \int_z^\infty z dH_{Q-r}(u) \\ &= \frac{(Q-r)}{\lambda_1}(L)H_{Q-r}(z-L) \\ &+ z\bar{H}_{Q-r}(z) \end{aligned}$$

Cycle Length during lead time;

$$CL_L = \begin{cases} L & \text{if } X_{Q-r} < z, X_Q < X_{Q-r} + L, X_Q < z \\ L & \text{if } X_{Q-r} < z < X_{Q-r} + L, X_Q > z \\ L & \text{if } X_{Q-r} + L < z, X_r > L \end{cases}$$

$$E(CL_L|z) = L$$

Cycle Length after lead time;

$$CL_A = \begin{cases} Y_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ z - X_{Q-r} - L & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L \end{cases}$$

$$\begin{aligned} E(CL_A|z) &= E \left[Y_{r-N(L)} I(X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L) \right] \\ &+ E \left[[z - X_{Q-r} - L] I(z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, X_r > L) \right] \\ &= \int_0^{z-L} E \left[\sum_{j=1}^{r-1} Y_{r-j} I(Y_{r-j} < z - u - L) \right] dH_{Q-r}(u) \\ &+ \int_0^{z-L} \left[\sum_{j=0}^{r-1} P(N(L) = j) (z - L - u) I(Y_{r-j} < z - u - L) \right] dH_{Q-r}(u) \\ &= \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{(r-j)}{\lambda_2} H_{r-j+1, \lambda_2}(z - L - u) dH_{Q-r}(u) \\ &+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} (z - u - L) \bar{H}_{r-j, \lambda_2}(z - L - u) dH_{Q-r}(u) \end{aligned}$$

The following expressions are for the expected On Hand Inventory per cycle, $E(OH)$:

On Hand Inventory until reorder point is reached either by demand or by perishing;

$$OH_R = \begin{cases} \sum_{i=1}^{Q-r} X_i + rXQ - r & \text{if } X_{Q-r} < z, X_r < L \\ \sum_{i=1}^{N(z)} X_i + z[Q - N(z)] & \text{if } X_{Q-r} > z \end{cases}$$

$$\begin{aligned} E(OH_R|z) &= E\left[\sum_{i=1}^{Q-r} X_i + rX_{Q-r}I(X_{Q-r} < z)\right] \\ &+ E\left[\sum_{i=1}^{N(z)} X_i + z[Q - N(z)]I(X_{Q-r} > z)\right] \\ &= E\left[\sum_{i=1}^{Q-r} X_i I(X_{Q-r} < z)\right] + rE\left[X_{Q-r}I(X_{Q-r} < z)\right] \\ &+ E\left[\sum_{i=1}^{N(z)} X_i I(X_{Q-r} > z)\right] + E\left[z[Q - N(z)]I(X_{Q-r} > z)\right] \\ &= \frac{(Q-r)(Q-r+1)}{2\lambda_1} H_{Q-r+1}(z) + \frac{r(Q-r)}{\lambda_1} H_{Q-r+1}(z) \\ &+ E\left[\sum_{i=1}^{N(z)} X_i I(N_z < Q-r)\right] + E\left[z[Q - N(z)]I(N(z) < Q-r)\right] \\ &= \frac{(Q-r)(Q+r+1)}{2\lambda_1} H_{Q-r+1}(z) \\ &+ \frac{\lambda_1}{2} z^2 \bar{H}_{Q-r-1}(z) + z \sum_{j=0}^{Q-r-1} (Q-j) P(N(z) = j) \end{aligned}$$

$$\begin{aligned}
E(OH_R|z) &= \frac{(Q-r)(Q+r+1)}{2\lambda_1} H_{Q-r+1}(z) \\
&+ \frac{\lambda_1}{2} z^2 \bar{H}_{Q-r-1}(z) + zQ \bar{H}_{Q-r}(z) - \lambda_1 z^2 \bar{H}_{Q-r-1}(z) \\
&= \frac{(Q-r)(Q+r+1)}{2\lambda_1} H_{Q-r+1}(z) \\
&+ zQ \bar{H}_{Q-r}(z) - \frac{\lambda_1}{2} z^2 \bar{H}_{Q-r-1}(z)
\end{aligned}$$

On Hand Inventory during lead time;

$$OH_L = \begin{cases} \sum_{i=1}^r X_i & \text{if } X_{Q-r} < z, X_Q < X_{Q-r} + L, \\ & X_Q < z \\ \sum_{i=1}^{N(z-X_{Q-r})} X_i & \\ + [z - X_{Q-r}][r - N(z - X_{Q-r})] & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_Q \\ \sum_{i=1}^{N(L)} X_i + L[r - N(L)] & \text{if } X_r > L, z > X_{Q-r} + L \end{cases}$$

$$\begin{aligned}
E(OH_L|z) &= E \left[\sum_{i=1}^r X_i I(X_{Q-r} < z, X_Q < X_{Q-r} + L, X_Q < z) \right] \\
&+ E \left[\sum_{i=1}^{N(z-X_{Q-r})} X_i + [z - X_{Q-r}][r - N(z - X_{Q-r})] \right. \\
&\quad \cdot I(X_{Q-r} < z < X_{Q-r} + L, X_Q > z) \left. \right] \\
&+ E \left[\sum_{i=1}^{N(L)} X_i + L[r - N(L)] I(X_r > L, z > X_{Q-r} + L) \right]
\end{aligned}$$

$$\begin{aligned}
E(OH_L|z) &= E\left[\sum_{i=1}^r X_i I(X_r < L, X_r < z - X_{Q-r}, X_{Q-r} < z)\right] \\
&+ E\left[\sum_{i=1}^{N(z-X_{Q-r})} X_i I(X_{Q-r} < z < X_{Q-r} + L, X_Q > z)\right] \\
&+ E\left[(z - X_{Q-r})(r - N(z - X_Q - r))\right. \\
&\quad \cdot \left. I(X_{Q-r} < z < X_{Q-r} + L, X_Q > z)\right] \\
&+ E\left[\sum_{i=1}^{N(L)} X_i I(X_r > L, z > X_{Q-r} + L)\right] \\
&+ E\left[L[r - N(L)]I(X_r > L, z > X_{Q-r} + L)\right] \\
&= \int_0^z E\left[\sum_{i=1}^r X_i I(X_r < \min(L, z - u))\right] dH_{Q-r}(u) \\
&+ E\left[\sum_{i=1}^{N(z-X_{Q-r})} X_i I(X_{Q-r} < z < X_{Q-r} + L, N(z - X_{Q-r}) < r)\right] \\
&+ E\left[(z - X_{Q-r})(r - N(z - X_Q - r))\right. \\
&\quad \cdot \left. I(X_{Q-r} < z < X_{Q-r} + L, N(z - X_{Q-r}) < r)\right] \\
&+ \int_0^{z-L} E\left[\sum_{i=1}^{N(L)} X_i I(N(L) < r)\right] dH_{Q-r}(u) \\
&+ L \int_0^{z-L} E\left[[r - N(L)]I(N(L) < r)\right] dH_{Q-r}(u) \\
&= \int_0^{z-L} E\left[\sum_{i=1}^r X_i I(X_r < L)\right] dH_{Q-r}(u) \\
&+ \int_{z-L}^z E\left[\sum_{i=1}^r X_i I(X_r < (z - u))\right] dH_{Q-r}(u) \\
&+ \int_{z-L}^z E\left[\sum_{i=1}^{N(z-u)} X_i I(N(z - u) < r)\right] dH_{Q-r}(u) \\
&+ \int_{z-L}^z E\left[(z - u)(r - N(z - u))I(N(z - u) < r)\right] dH_{Q-r}(u) \\
&+ \int_0^{z-L} E\left[\frac{\lambda_1}{2} L^2 \overline{H}_{r-1}(L)\right] dH_{Q-r}(u) \\
&+ L \int_0^{z-L} E\left[\sum_{i=1}^{r-1} (r - i) P(N(L) = i)\right] dH_{Q-r}(u)
\end{aligned}$$

$$\begin{aligned}
E(OH_L|z) &= \frac{r(r+1)}{2\lambda_1} \int_0^{z-L} H_{r+1}(L) dH_{Q-r}(u) \\
&+ \frac{r(r+1)}{2\lambda_1} \int_{z-L}^z H_{r+1}(z-u) dH_{Q-r}(u) \\
&+ \int_{z-L}^z E \left[(z-u) \sum_{i=1}^{r-1} (r-i) P(N(z-u)=i) \right] dH_{Q-r}(u) \\
&+ H_{Q-r}(z-L) \frac{\lambda_1}{2} (L^2) \bar{H}_{r-1}(L) \\
&+ LH_{Q-r}(z-L) \sum_{i=1}^{r-1} (r-i) P(N(L)=i) \\
&= \frac{r(r+1)}{2\lambda_1} H_{r+1}(L) H_{Q-r}(z-L) \\
&+ \frac{r(r+1)}{2\lambda_1} \int_{z-L}^z H_{r+1}(z-u) dH_{Q-r}(u) \\
&+ \int_{z-L}^z (z-u) \left[r \bar{H}_r(z-u) - \lambda_1 (z-u) \bar{H}_{r-1}(z-u) \right] dH_{Q-r}(u) \\
&+ H_{Q-r}(z-L) \frac{\lambda_1}{2} L^2 \bar{H}_{r-1}(L) \\
&+ LH_{Q-r}(z-L) \left[r \bar{H}_r(L) - \lambda_1 L \bar{H}_{r-1}(L) \right]
\end{aligned}$$

On Hand Inventory after lead time;

$$OH_A = \begin{cases} \sum_{i=1}^{r-N(L)} Y_i + QY_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, \\ & X_r > L \\ \sum_{i=1}^{N(z-X_{Q-r})} Y_i + [z - X_{Q-r} - L][Q + r - N(L) \\ - N(z - X_{Q-r} - L)] & \text{if } z < X_{Q-r} + L + Y_{r-N(L)}, \\ & X_r > L \end{cases}$$

$$\begin{aligned}
E(OH_A|z) &= E \left[\sum_{i=1}^{r-N(L)} Y_i + QY_{r-N(L)} I(X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L) \right] \\
&+ E \left[\sum_{i=1}^{N(z-X_{Q-r})} Y_i + [z - X_{Q-r} - L] \right. \\
&\cdot [Q + r - N(L) - N(z - X_{Q-r} - L)] \\
&\cdot I(X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L) \left. \right] \\
&+ E \left[\sum_{i=1}^{r-N(L)} Y_i I(X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L) \right] \\
&+ E \left[QY_{r-N(L)} I(Y_{r-N(L)} < z - X_{Q-r} - L, X_r > L) \right] \\
&+ E \left[\sum_{i=1}^{N(z-X_{Q-r})} Y_i I(X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L) \right] \\
&+ E \left[[z - X_{Q-r} - L][Q + r - N(L) - N(z - X_{Q-r} - L)] \right. \\
&\cdot I(X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L) \left. \right] \\
&= \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} E \left[\sum_{i=1}^{r-j} Y_i I(Y_{r-j} < z - u - L) \right] dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \left[QY_{r-j} I(Y_{r-j} < z - u - L) \right] dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \left[\sum_{i=1}^{N'(z-u-L)} Y_i \right. \\
&\cdot I(N(z - u - L) < r - j, \lambda_2) \left. \right] dH_{Q-r}(u)
\end{aligned}$$

$$\begin{aligned}
E(OH_A|z) &= \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{(r-j)(r-j+1)}{\lambda_2} \\
&\quad \cdot H_{r-j+1,\lambda_2}(z-L-u) dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \left[Q \frac{(r-j)}{\lambda_2} H_{r-j+1,\lambda_2}(z-L-u) \right] dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{\lambda_2}{2} (z-u-L)^2 \bar{H}_{r-j-1,\lambda_2}(z-L-u) dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} (z-L-u) \sum_{i=0}^{r-j-1} (Q+r-j-i) \\
&\quad \cdot P(N'(z-L-u) = i) dH_{Q-r}(u) \\
&= \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{(r-j)(r-j+1+2Q)}{2\lambda_2} \\
&\quad \cdot H_{r-j+1,\lambda_2}(z-L-u) dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} \frac{\lambda_2}{2} (z-u-L)^2 \bar{H}_{r-j-1,\lambda_2}(z-L-u) dH_{Q-r}(u) \\
&+ \sum_{j=0}^{r-1} P(N(L) = j) \int_0^{z-L} (z-u-L) [(Q+r-j) \bar{H}_{r-j,\lambda_2}(z-L-u) \\
&\quad - \lambda_2(z-L-u) \bar{H}_{r-j-1,\lambda_2}(z-L-u)] dH_{Q-r}(u)
\end{aligned}$$

The following expressions are for the expected number of items sold at P_1 price, $E(S, P_1)$

$$SP_1 = \begin{cases} Q & \text{if } X_{Q-r} + X_r < z, X_r < L \\ N(z) & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_{Q-r} + X_r \\ Q - r + N(L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ Q - r + N(L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L \\ N(z) & \text{if } X_{Q-r} > z \end{cases}$$

$$\begin{aligned}
E(SP_1|z) &= E\left[QI(X_{Q-r} + X_r < z, X_r < L)\right] \\
&+ E\left[(N(z-L) + N(L))\right. \\
&\quad \cdot \left.I(N(z-L) < Q-r, N(z-L) + N(L) < Q)\right] \\
&+ E\left[(Q-r + N(L))I(X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L)\right] \\
&+ E\left[(Q-r + N(L))\right. \\
&\quad \cdot \left.I(z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, X_r > L)\right] \\
&= \int_0^L \int_0^{z-u} Q dH_{Q-r}(u) dH_r(v) \\
&+ \sum_{i=0}^{Q-r-1} \sum_{j=0}^{Q-i-1} (i+j) P(N(z-L) = i) P(N(L) = j) \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) E\left[(Q-r+i)I(X_{Q-r} + L + Y_{r-i} < z)\right] \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) E\left[(Q-r+i)I(z < X_{Q-r} + L + Y_{r-i}, X_{Q-r} + L < z)\right] \\
&= Q \int_0^L H_{Q-r}(z-) dH_r(v) \\
&+ \sum_{i=0}^{Q-r-1} \sum_{j=0}^{Q-i-1} (i+j) P(N(z-L) = i) P(N(L) = j) \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) (Q-r+i) \int_0^{z-L} H_{r-i, \lambda_2}(z-L-a) dH_{Q-r}(u) \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) (Q-r+i) \int_0^{z-L} \bar{H}_{r-i, \lambda_2}(z-L-a) dH_{Q-r}(u)
\end{aligned}$$

The following expressions are for the expected number of items sold at P_2 price, $E(S, P_2)$

$$SP_2 = \begin{cases} r - N(L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L \\ N'(z - X_{Q-r} - L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L \end{cases}$$

$$\begin{aligned} E(SP_2|z) &= E \left[(r - N(L)) \right. \\ &\quad \cdot \left. I(X_{Q-r} + L + Y_{r-N(L)} < z, X_r > L) \right] \\ &+ E \left[N(z - X_{Q-r} - L) I(z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, X_r > L) \right] \\ &= \sum_{i=0}^{r-1} P(N(L) = i) E \left[(r - i) I(X_{Q-r} + L + Y_{r-i} < z) \right] \\ &+ \sum_{i=0}^{r-1} P(N(L) = i) \\ &\quad \cdot E \left[N(z - X_{Q-r} - L) I(z < X_{Q-r} + L + Y_{r-i}, X_{Q-r} + L < z) \right] \\ &= \sum_{i=0}^{r-1} P(N(L) = i) (r - i) \int_0^{z-L} H_{r-i, \lambda_2}(z - L - u) dH_{Q-r}(u) \\ &+ \sum_{i=0}^{r-1} P(N(L) = i) \int_0^{z-L} \sum_{j=0}^{r-i-1} j P(N'(z - u - L) = j) dH_{Q-r}(u) \end{aligned}$$

The following expressions are for the expected number of units that perish,
 $E(P)$

$$P = \begin{cases} Q - N(z) & \text{if } X_{Q-r} < z < X_{Q-r} + L, z < X_{Q-r} + X_r \\ r - N(L) - N'(z - X_{Q-r} - L) & \text{if } X_{Q-r} + L + Y_{r-N(L)} > z, X_{Q-r} + L < z, X_r > L \\ Q - N(z) & \text{if } X_{Q-r} > z \end{cases}$$

$$\begin{aligned}
E(P|z) &= E\left[(Q - N(z))I(z < X_{Q-r} + L, z < X_{Q-r} + X_r)\right] \\
&+ E\left[(r - N(L) - N'(z - X_{Q-r} - L))\right. \\
&\quad \cdot \left.I(z < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z, X_r > L)\right] \\
&= E\left[(Q - N(z))I(z < X_{Q-r} + L, z < X_Q)\right] \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) E\left[r - i - N'(z - X_{Q-r} - L)\right. \\
&\quad \cdot \left.I(z < X_{Q-r} + L + Y_{r-i}, X_{Q-r} + L < z)\right] \\
&= E\left[(Q - N(z))I(N(z - L) < Q - r, N(z - L) + N(L) < Q)\right] \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) \int_0^{z-L} E\left[r - i - N'(z - X_{Q-r} - L)I(Y_{r-i} > z - L - u)\right] \\
&= \sum_{i=0}^{Q-r-1} \sum_{j=0}^{Q-i-j} (Q - i - j) P(N(z - L) = i) P(N(L) = j) \\
&+ \sum_{i=0}^{r-1} P(N(L) = i) \sum_{j=0}^{r-i-1} (r - i - j) \int_0^{z-L} P(N'(z - L - u) = j) dH_{Q-r}(u)
\end{aligned}$$

Appendix B

Effective Shelflife Distribution

$$Z_{n+1} = \begin{cases} \tau - Y_{r-N(L)} & \text{if } X_{Q-r} + L + Y_{r-N(L)} < z_n, X_r > L \\ \tau - z_n + X_{Q-r} + L & \text{if } z_n < X_{Q-r} + L + Y_{r-N(L)}, X_{Q-r} + L < z_n, X_r > L \\ \tau, & \text{o.w} \end{cases}$$

Suppose that $Z_n = t$ and let $Z_{n+1} = f(x)$. Then, we can write $f(x)$ as follows;

$f(x) = \tau - (Y_{r-N(L)})I(X_{Q-r} + L + Y_{r-N(L)} < t, X_r > L) - (t - X_{Q-r} - L)I(X_{Q-r} + L < t < X_{Q-r} + L + Y_{r-N(L)}, X_r > L)$, where $I(\cdot)$ is the indicator function of its argument.

Let $F_{n+1}(z|t)$ be the conditional cdf of Z_{n+1} given Z_n . Then, by Theorem 1 from Berk and Gürlü [3] we can write $F_{n+1}(z|t)$ as;

$$F_{n+1}(z|t) = \begin{cases} 0 & \text{if } z < \tau + L - t \\ H_{Q-r}(t + z - \tau - L, \lambda) [\sum_{i=0}^{r-1} P(N(L) = i) \overline{H}_{r-i}(\tau - z, \lambda_2)] & \text{if } \tau + L - t < z < \tau \\ 1 & \text{if } z \geq \tau. \end{cases}$$

Appendix C

Possible Realizations

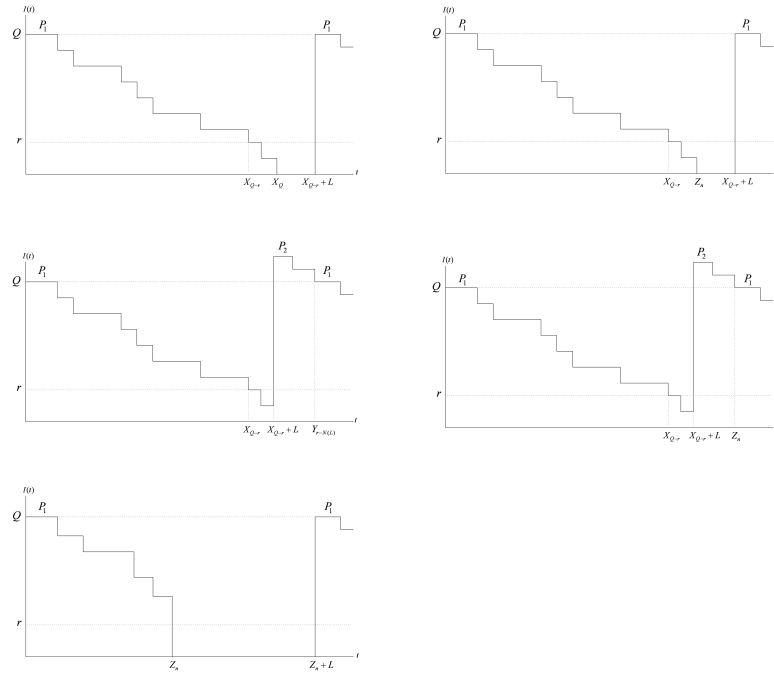


Figure C.1: Possible realizations of the single market two price policy

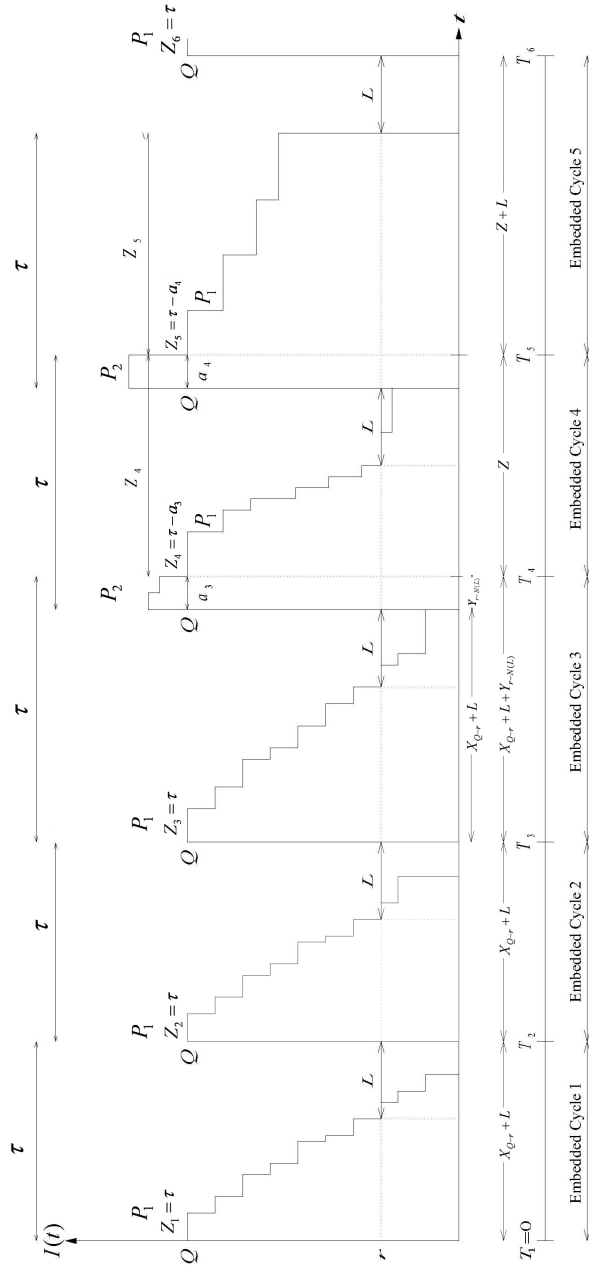


Figure C.2: Possible realizations of the single market two price policy

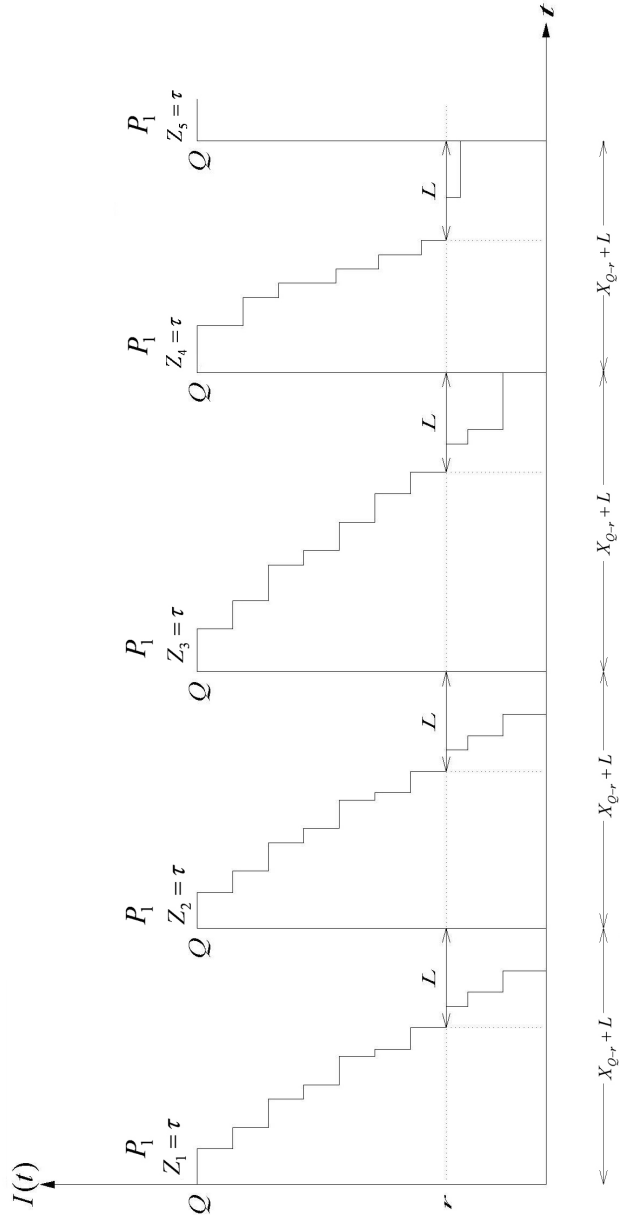


Figure C.3: Possible realizations at the primary market

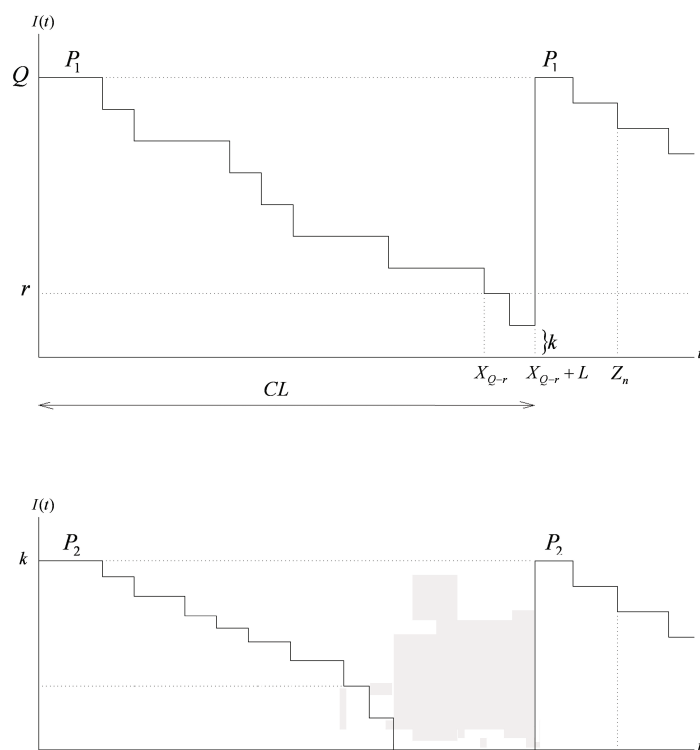


Figure C.4: Possible realization at the secondary market

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%
%           KÖNÜL BAYRAMOĞLU
%
%           BILKENT UNIVERSITY
%           DEPARTMENT OF INDUSTRIAL ENGINEERING
%           2009
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%   In this part of the code we calculate the expected profit value
%   and OHzz, ECLzz, Pzz, SP1zz, SP2zz.
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function[profit,Q, r, OHzz,ECLzz,Pzz,SP1zz,SP2zz]=konul_opt( from_opt_Q,
from_opt_r, from_opt_P1, from_opt_P2 )

```

```

Q = from_opt_Q;
r = from_opt_r;
P1 = from_opt_P1;
P2 = from_opt_P2;

```

```

tao =1.5;
L = 1;
z = L:0.01:tao;
t = L:0.01:tao;
m = length(z);
inc=0.01;

```

```

a=1.2;
b=19;

```

```

lamda_1=b-a*P1;
lamda_2=b-a*P2;
lamdaL = lamda_1 * L;

```

```

kkk=10;
ppp=8;
hhh=1;
ddd=10;

```

```

A = zeros(m,m);

```

```

for i = 1:m,

    sum_P_N_L = 0;

    for k = 0:r-1,

```

```

        sum_P_N_L = sum_P_N_L + poisspdf(k,lamdaL).*(1-gamcdf(tao-t(i),r-
k,1/lamda_2));

    end;

    tstartind = 0;

    for j = 1:m,

        if ((tao+L-t(i) <= t(j)) & (t(j) <= tao))

            HQR = gamcdf(t(j)+t(i)-tao-L,Q-r,1/lamda_1);
            A(i,j)=HQR*sum_P_N_L;

        else

            A(i,j)=0;
        end

    end;

    for j=1:i,

        A(i,j) = A(i,j) - 1;

    end

end;

for i=1:m

    A(m,i)=1;

end

for i=1:m-1,

    b(i)=0;

end

b(m)=1;


f_z=inv(A)*transpose(b);
F_z(1)=f_z(1);
Expz=0;

for i=2:m,

    F_z(i)=F_z(i-1)+f_z(i);
    Expz = Expz + f_z(i-1)*z(i-1);
end
Expz = Expz + f_z(m)*z(m);

```

```

E_CL_z = zeros(size(z));
OH_z = zeros(size(z));
LS_z = zeros(size(z));
P_z = zeros(size(z));
SP1_z = zeros(size(z));
SP2_z = zeros(size(z));

for i = m,

    intpart1=0;
    intpart2=0;
    intpart3=0;
    intpart4=0;
    intpart5=0;

    if(z(i)-L>0)
        x1 = 0:inc:L;

        y1 = gamcdf(z(i)-x1,Q-r+1,1/lamda_1).*gampdf(x1,r,1/lamda_1);
        intpart111 = trapz(x1,y1)*(Q-r)/lamda_1;

        y1 = gamcdf(z(i)-x1,Q-r,1/lamda_1).*gampdf(x1,r,1/lamda_1);
        intpart122 = trapz(x1,y1)*L;

        x2 = z(i)-L:inc:z(i);

        y2 = (1-gamcdf(z(i)-x2,r,1/lamda_1)).*gampdf(x2,Q-r+1,1/lamda_1);
        intpart211 = trapz(x2,y2)*(Q-r)/lamda_1;

        y2 = (1-gamcdf(z(i)-x2,r,1/lamda_1)).*gampdf(x2,Q-r,1/lamda_1);
        intpart222 = trapz(x2,y2)*L;

        intpart5=(z(i)+L)*(1-gamcdf(z(i),Q-r,1/lamda_1));

    else
        x1 = 0:inc:z(i);

        y1 = gamcdf(z(i)-x1,Q-r+1,1/lamda_1).*gampdf(x1,r,1/lamda_1);
        intpart111 = trapz(x1,y1)*(Q-r)/lamda_1;

        y1 = gamcdf(z(i)-x1,Q-r,1/lamda_1).*gampdf(x1,r,1/lamda_1);
        intpart122 = trapz(x1,y1)*L;

        y2 = (1-gamcdf(z(i)-x1,r,1/lamda_1)).*gampdf(x1,Q-r+1,1/lamda_1);
        intpart211 = trapz(x1,y2)*(Q-r)/lamda_1;

        y2 = (1-gamcdf(z(i)-x1,r,1/lamda_1)).*gampdf(x1,Q-r,1/lamda_1);
        intpart222 = trapz(x1,y2)*L;

        intpart5=(z(i)+L)*(1-gamcdf(z(i),Q-r,1/lamda_1));

    end

    intpart1=intpart111+intpart122;

```



```

intpart2=intpart211+intpart222;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% PSZ
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sumPNL=0;

for k=0:Q-r-1
    for l=0:Q-k-1,

        sumPNL=sumPNL+(Q-k-1)*poisspdf(k,lamda_1*(z(i)-
L))*poisspdf(l,lamdaL);

    end;
end;

intpart1111=sumPNL;

intpartSp3 = 0;
intpartSp4 = 0;
intpartSp5 = 0;
intpartSp6 = 0;

%%intpart31
% tüm k?s?mlar buraya al?n?yor
if(z(i)-L>=inc)

    x2 = 0:inc:z(i)-L;

    sumPNL = 0;
    sumPNL2 = 0;

    sumPNL3 = 0;
    sumPNL4 = 0;

    sumPNL5 = 0;
    sumPNL6 = 0;

    sumPNL7 = 0;
    sumPNL8 = 0;

    sumPNL9 = 0;

    sumPNL11 = 0;
    sumPNL12 = 0;

    sumPNL13 = 0;
    sumPNL14 = 0;

    x1 = 0:inc:(r/lamda_2)*5;

    for k = 0:r-1,
        y2 = gamcdf(z(i)-L-x2,Q-r+1,1/lamda_1).*gampdf(x2,r-k,1/lamda_2);
        sumPNL = sumPNL + poisspdf(k,lamdaL)*trapz(x2,y2);
    end;
end;

```

```

y22 = gamcdf(z(i)-L-x2,Q-r,1/lamda_1).*gampdf(x2,r-k,1/lamda_2);
sumPNL2 = sumPNL2 + poisspdf(k,lamdaL)*trapz(x2,y22);

y23 = x2.*gamcdf(z(i)-L-x2,Q-r,1/lamda_1).*gampdf(x2,r-
k,1/lamda_2);
sumPNL3 = sumPNL3 + poisspdf(k,lamdaL)*trapz(x2,y23);

y1 = (gamcdf(z(i)-L,Q-r,1/lamda_1)-gamcdf(z(i)-L-x1,Q-
r,1/lamda_1)).*gampdf(x1,r-k,1/lamda_2);
sumPNL4 = sumPNL4 + poisspdf(k,lamdaL)*trapz(x1,y1);

y222 = gamcdf(z(i)-L-x2,r-k+1,1/lamda_2).*gampdf(x2,Q-
r,1/lamda_1);
sumPNL5 = sumPNL5 + poisspdf(k,lamdaL)*(r-k)*(r-
k+1+2*Q)/2/lamda_2*trapz(x2,y222);

y2222 = (Q+r-k)*(z(i)-L-x2).*(1-gamcdf(z(i)-L-x2,r-
k,1/lamda_2)).*gampdf(x2,Q-r,1/lamda_1);
sumPNL6 = sumPNL6 + poisspdf(k,lamdaL)*trapz(x2,y2222);

if ( k == r - 1)

    sumPNL7 = sumPNL7;
    sumPNL8 = sumPNL8;

else

    y22222 = ((z(i)-L-x2).^2).*(1-gamcdf(z(i)-L-x2,r-k-
1,1/lamda_2)).*gampdf(x2,Q-r,1/lamda_1);
    sumPNL7 = sumPNL7 +
poisspdf(k,lamdaL)*lamda_2*trapz(x2,y22222);

    y222222 = ((z(i)-L-x2).^2).*(1-gamcdf(z(i)-L-x2,r-k-
1,1/lamda_2)).*gampdf(x2,Q-r,1/lamda_1);
    sumPNL8 = sumPNL8 +
poisspdf(k,lamdaL)*(lamda_2/2)*trapz(x2,y222222);
end

sumPNL10 = 0;

% pz22
gu2=gamcdf(z(i)-x2-L,r-k,1/lamda_2).*gampdf(x2,Q-r,1/lamda_1);
sumPNL13=sumPNL13+poisspdf(k,lamdaL)*(r-k)*trapz(x2,gu2);

gu22 = zeros(size(x2));

for l = 0:r-k-1,
    y2222222 = gampdf(x2,Q-r,1/lamda_1).*poisspdf(l,(z(i)-x2-
L)*(lamda_2));
    sumPNL10 = sumPNL10 + (r-k-1)*trapz(x2,y2222222);

```

```

        gu22 = gu22 + poisspdf(1, (z(i)-x2-L)*lamda_2)*1;
    end;

    % sp22
    gu22 = gu22.*gampdf(x2,Q-r,1/lamda_1);
    sumPNL14 = sumPNL14 + poisspdf(k,lamdaL)*trapz(x2,gu22);

    sumPNL9 = sumPNL9 + sumPNL10*poisspdf(k,lamdaL);
    intpart2222=sumPNL9;

    ko2=gamcdf(z(i)-L-x2,r-k,1/lamda_2).*gampdf(x2,Q-r,1/lamda_1);
    sumPNL11 = sumPNL11+poisspdf(k,lamdaL)*(Q-r+k)*trapz(x2,ko2);

    ko22 = (1-gamcdf(z(i)-L-x2,r-k,1/lamda_2)).*gampdf(x2,Q-
r,1/lamda_1);
    sumPNL12 = sumPNL12 + poisspdf(k,lamdaL)*(Q-r+k)*trapz(x2,ko22);

    Pz(i) = intpart1111 + intpart2222;

end;

sumPNL = sumPNL*((Q-r)/(lamda_1));
intpart31 = sumPNL;

sumPNL2 = sumPNL2;
intpart32 = sumPNL2*L;

sumPNL3 = sumPNL3;
intpart33 = sumPNL3;

sumPNL4 = sumPNL4.*z(i);
intpart4 = sumPNL4;

intpart41 = sumPNL5;
intpart421 = sumPNL6;

intpart422 = sumPNL7;
intpart423 = sumPNL8;

intpartSp3 = sumPNL11;
intpartSp4 = sumPNL12;

intpartSp5 = sumPNL13;
intpartSp6 = sumPNL14;

else;

    intpart31 = 0;
    intpart32 = 0;
    intpart33 = 0;
    intpart4 = 0;

    intpart41=0;
    intpart421=0;

    intpart422=0;

```

```

    intpart423=0;

end

intpart3=intpart31+intpart32+intpart33;

E_CL_z(i) = intpart1+intpart2+intpart3+intpart4+intpart5;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% OHZ
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    intpart1 =(Q-r)*(Q+r+1)/2/lamda_1*(gamcdf(z(i),Q-r+1,1/lamda_1));
    intpart21 = z(i)*Q*(1-gamcdf(z(i),Q-r,1/lamda_1));

    if(Q-r>1)
        intpart22 = (((z(i).^2)*lamda_1)/2)*(1-gamcdf(z(i),Q-r-1,1/lamda_1));
    else
        intpart22 = 0;
    end

    if (z(i)-L>=inc)

        x2 = z(i)-L:inc:z(i);

        y2=(gamcdf(z(i)-x2,r+1,1/lamda_1)).*gampdf(x2,Q-r,1/lamda_1);
        intpart31 = trapz(x2,y2)*((r+1)*r)/2/(lamda_1);

        y2=((z(i)-x2)).*(1-gamcdf(z(i)-x2,r,1/lamda_1)).*gampdf(x2,Q-
r,1/lamda_1);
        intpart341 = trapz(x2,y2)*r;

        if (r>1)

            y22=((z(i)-x2).^2).*(1-gamcdf(z(i)-x2,r-
1,1/lamda_1)).*gampdf(x2,Q-r,1/lamda_1);
            intpart33 = trapz(x2,y22)*(lamda_1/2);

            y22=((z(i)-x2)).*((z(i)-x2)).*(1-gamcdf(z(i)-x2,r-
1,1/lamda_1)).*gampdf(x2,Q-r,1/lamda_1);
            intpart342 = trapz(x2,y22)*lamda_1;

        else

            intpart33 = 0;
            intpart342 = 0;

        end

    else

        intpart31=0;
        intpart341=0;

        intpart33 = 0;

```

```

        intpart342 = 0;
    end

    intpart32 = ((r+1)*r)/2/(lamda_1)*gamcdf(L,r+1,1/lamda_1).*gamcdf(z(i)-
L,Q-r,1/lamda_1);

    intpart352 = L*(gamcdf(z(i)-L,Q-r,1/lamda_1));
    intpart353 = r*(1-gamcdf(L,r,1/lamda_1));

    if( r > 1 )
        intpart351 = lamda_1*(L^2)/2*(gamcdf(z(i)-L,Q-r,1/lamda_1))*(1-
gamcdf(L,r-1,1/lamda_1));
        intpart353 = intpart353 - lamda_1*L*(1-gamcdf(L,r-1,1/lamda_1));
    else
        intpart351 = 0;
    end

    intpart35 = intpart351+intpart352*intpart353;

%         intpart4=intpart41+intpart423+intpart421-intpart422;
%         else
%         intpart1=0; intpart21=0; intpart22=0; intpart31=0; intpart32=0;
intpart33=0; intpart34=0; intpart351=0; intpart352=0; intpart353=0;
intpart41=0; intpart421=0; intpart422=0;

    AA(i) = intpart1 + intpart21 - intpart22;
    BB(i) = intpart31 + intpart32 +intpart33 +intpart341-intpart342 +
intpart35;
    CC(i) = intpart41 + intpart421-intpart422 + intpart423;
    OHz(i)=AA(i)+BB(i)+CC(i);

%
%         AA(i)=intpart1+intpart2+intpart3+intpart4;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% LSZ
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    intpart1=0;

    if (L>0)

        x1 = 0:inc:L;

        y1 =lamda_1*(L-x1).*gamcdf(z(i)-x1,Q-
r,1/lamda_1).*gampdf(x1,r,1/lamda_1);
        intpart1=trapz(x1,y1);

        y2=gamcdf(z(i)-x1,Q-r,1/lamda_1).*gampdf(x1,r,1/lamda_1);
        intpartSp1=Q*trapz(x1,y2);

    end

    intpart2=0;

```

```

    if (z(i)-L>=inc)

        x2 = z(i)-L:inc:z(i);

        y2 =lamda_1*(x2+L-z(i)).*(1-gamcdf(z(i)-x2,r,1/lamda_1)).*gampdf(x2,Q-
r,1/lamda_1);
        intpart2=trapz(x2,y2);

    end

    intpart3=lamda_1*L*(1-gamcdf(z(i),Q-r,1/lamda_1));

    LSz(i) =intpart1+intpart2+intpart3;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% sp1z
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    sumPNL=0;

    for k=0:Q-r-1,
        for l=Q-k-1,

            sumPNL=sumPNL+(k+1)*poisspdf(k,lamda_1*(z(i)-
L))*poisspdf(l,lamdaL);
            end;
        end;

    intpart2 = sumPNL;

    SP1z(i) = intpartSp1 + intpart2 + intpartSp3 + intpartSp4;
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% sp2z
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    SP2z(i) = intpartSp5 + intpartSp6;

end

OHzz = OHz*f_z;
ECLzz=E_CL_z*f_z;
LSzz = LSz * f_z;
Pzz = Pz*f_z;
SP1zz = SP1z * f_z;
SP2zz = SP2z *f_z;

% SP1zz = SP1z(m);
% SP2zz = SP2z(m);
% OHzz = OHz(m);
% ECLzz = E_CL_z(m);
% Pzz = Pz(m);

```

```

profit = P1 * SP1zz + P2 * SP2zz - kkk - hhh * OHzz - ppp*Pzz;
profit = profit / ECLzz;

% profit = P1 * SP1zz + P2 * SP2zz - kkk - hhh * OHzz - ppp*Pzz;
% profit = profit / ECLzz;

% ECOST= kkk + ppp*Pzz + ddd*LSzz + hhh * OHzz;
% COST= ECOST /ECLzz

% ECOST= kkk + ppp*Pzz + hhh * OHzz;
% COST= ECOST /ECLzz;

%profit_z(1:251)=P1 * SP1z + P2 * SP2z - kkk - hhh * OHz - ppp*Pz;
%profit_z = profit_z ./ E_CL_z;

%z1 = profit_z(26);
%z2 = profit_z(51);
%z3 = profit_z(101);
%z4 = profit_z(126);
%z5 = profit_z(151);
%z6 = profit_z(251);

```

```
bests = [];
```



```

for i=0:1:1

    best = -inf;

    for P_two=2:0.1:(P_one);

        [a,b,c,d,e,f,g,h] = konul_opt( Q, r, P_one, P_two );

        if ( best < a )

            best = a;
            P_two_opt = P_two;
            OHzz_opt = d;
            ECLzz_opt = e;
            Pzz_opt = f;
            SP1zz_opt = g;
            SP2zz_opt = h;

        end
    end

    fprintf(fid,'X :  %8i %10.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f
%11.4f \n\n', r, Q, P_one, P_two_opt, a, d, e, f, g, h );
    fprintf(fid,'-----
\n\n');

    best = -inf;

    for P_one=(P_two_opt+1):0.1:2*(P_one);

        [a,b,c,d,e,f,g,h] = konul_opt( Q, r, P_one, P_two_opt );

        if ( best < a )

            best = a;
            P_one_opt = P_one;
            OHzz_opt = d;
            ECLzz_opt = e;
            Pzz_opt = f;
            SP1zz_opt = g;
            SP2zz_opt = h;

```

```

        end
    end

fprintf(fid,'X :  %8i %10.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f
%11.4f \n\n', r, Q, P_one_opt, P_two_opt, a, d, e, f, g, h );
        fprintf(fid,'-----
\n\n');

        P_one = P_one_opt;

        bests(end+1) = best;

    end

    while (( bests(end) - bests(end - 1)) > 0.005 * abs(bests(end -
1)) )

        best = -inf;
        P_two_opt = 0;

        for P_two=2:0.1:(P_one);

            [a,b,c,d,e,f,g,h] = konul_opt( Q, r, P_one, P_two );

            if ( best < a )

                best = a;
                P_two_opt = P_two;
                OHzz_opt = d;
                ECLzz_opt = e;
                Pzz_opt = f;
                SP1zz_opt = g;
                SP2zz_opt = h;

            end
        end

        fprintf(fid,'X :  %8i %10.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f
%11.4f \n\n', r, Q, P_one, P_two_opt, a, d, e, f, g, h );
        fprintf(fid,'-----
\n\n');

```

```

best = -inf;

for P_one=(P_two_opt+1):0.1:2*(P_one);

    [a,b,c,d,e,f,g,h] = konul_opt( Q, r, P_one, P_two_opt );

    if ( best < a )

        best = a;
        P_one_opt = P_one;
        OHzz_opt = d;
        ECLzz_opt = e;
        Pzz_opt = f;
        SP1zz_opt = g;
        SP2zz_opt = h;

    end
end

fprintf(fid,'X :  %8i %10.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f %11.4f
%11.4f \n\n', r, Q, P_one_opt, P_two_opt, a, d, e, f, g, h );
    fprintf(fid,'-----
-----
\n\n');

    P_one = P_one_opt;

    bests(end+1) = best;

end

best_Array(end+1) = best;
r_opt_Array(end+1) = r;
q_opt_Array(end+1)= Q;
p_One_Array(end+1) = P_one_opt;
p_Two_Array(end+1) = P_two_opt;
OHzz_Array(end+1) = OHzz_opt;
ECLzz_Array(end+1) = ECLzz_opt;
Pzz_Array(end+1) = Pzz_opt;
SP1zz_Array(end+1) = SP1zz_opt;
SP2zz_Array(end+1) = SP2zz_opt;

end
end

```

```

fclose(fid);

time = cputime - time;

fid1=fopen('konul_opt.txt', 'a');

Q_opt = 0;
r_opt = 0;
best = 0;
OHzz_opt = 0;
ECLzz_opt = 0;
Pzz_opt = 0;
SP1zz_opt = 0;
SP2zz_opt = 0;
P_one_opt = 0;
P_two_opt = 0;

ind = 0;
ind = find( best_Array == ( max( max ( best_Array ) ) ) );

Q_opt = q_opt_Array(ind);
r_opt = r_opt_Array(ind);
best = best_Array(ind);
OHzz_opt = OHzz_Array(ind);
ECLzz_opt = ECLzz_Array(ind);
Pzz_opt = Pzz_Array(ind);
SP1zz_opt = SP1zz_Array(ind);
SP2zz_opt = SP2zz_Array(ind);
P_one_opt = p_One_Array(ind);
P_two_opt = p_Two_Array(ind);

Q_opt
r_opt
best
OHzz_opt
ECLzz_opt
Pzz_opt
SP1zz_opt
SP2zz_opt
P_one_opt
P_two_opt

fprintf(fid1,'-----\n\n');
fprintf(fid1,'Your Optimal Profit Is : %5d\n\n',best);
fprintf(fid1,'-----\n\n');

```

```

fprintf(fid1, 'Your Optimal Q Is : %5d\n\n',Q_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal r Is: %5d\n\n',r_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal P1 Is: %5d\n\n',P_one_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal P2 Is: %5d\n\n',P_two_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimization Time Is: %5d\n\n',time);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal OHzz Is: %5d\n\n',OHzz_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal ECLzz Is: %5d\n\n',ECLzz_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal Pzz Is: %5d\n\n',Pzz_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal SP1zz Is: %5d\n\n',SP1zz_opt);
fprintf(fid1,'-----\n\n');

fprintf(fid1, 'Your Optimal SP2zz Is: %5d\n\n',SP2zz_opt);
fprintf(fid1,'-----\n\n');


fclose(fid1);

```