SOLUTIONS TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS IN TERMS OF SCHWARTZ DISTRIBUTIONS

(SCHWARTZ DAĞILIMLARI VASITASIYLA LİNEER KISMİ TÜREVLİ DENKLEMLERİN ÇÖZÜMLERİ)

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List of Symbols

$\mathbb{N}$	:	Set of natural numbers.
$\mathbb{R}$	:	Set of real numbers.
$\mathbb{Z}$	:	Set of integers.
$\mathbb{C}$	:	Complex number field.
C	:	Space of continuous functions.
C^∞	:	Space of smooth functions.
$\mathbf{L}^1$	:	Space of all integrable functions.
$\mathbf{L}^1$	:	Hilbert space.
$\mathbf{L}^\infty$	:	Space of bounded measurable functions
$\mathscr{S}$	:	Schwartz space.
$\mathscr{D}$	:	Space of test function.
$\mathscr{D}'$	:	Space of Schwartz distributions.
W	:	Sobolev space.
T	:	Torus.
$Dom(f)$	:	Domain of definition of a function f.
$supp(f)$	:	Support of a test function f.
$\mathscr{F}$	:	Fourier transform.
$\overline{\mathscr{F}}$	:	Inverse Fourier transform.
$< .,. >$	:	Distribution operator.
Δ	:	Laplace operator.
$\square$	:	Wave operator.
$*$	:	Convolution operator.
$\otimes$	:	Tensor product.
τ	:	Translation operator.
$\|\|.\|\|$	:	Norm
$\Re$	:	Real part
$\Im$	:	Imaginary part
$N_{m,p}$	:	Semi-norm

ABSTRACT

This thesis explores Schwartz distributions, their properties, and applications to study solution of linear partial differential equations with constant coefficients. Special type of distribution such as Heaviside or Dirac delta functions are also discussed.

Additionally, important topics such as the Hadamard finite part, Cauchy principal values, and the Laplace operator are examined. The thesis also addresses the solution of differential equations using Fourier transforms, providing examples and discussing the elementary solution of the wave equation and the Laplace equation.

Furthermore, the solutions a differential equation in the space of Schwartz distributions and the iterated Laplace operator are discussed.

With the introduction, this thesis consists of ten chapters. First chapter provides an introduction to the topic of the thesis. It discusses the concept of distributions. The chapter also emphasizes different aspects related to distributions, such as compactly supported distributions, differentiation of distributions, and the sum and product of distributions by a smooth function.

In the third chapter , the Fourier transform in the context of the real line, $\mathbb{R}$, is discussed. Firstly, by defining the Fourier transform on $\mathbb{R}$, the aim is to better understand the Fourier transform on $\mathbb{R}$. The space of infinitely differentiable functions with compact support and the space of rapidly decreasing functions are examined.

Next, the Fourier transform is defined on $\mathbb{R}^n$. It is studied in the space $\mathbf{L}^1(\mathbb{R}^n)$, the space $\mathscr{S}(\mathbb{R}^n)$ is examined, and the Fourier transform on the space $\mathbf{L}^2(\mathbb{R}^n)$ is discussed.

In the fourth chapter is devoted to the study of the Fourier transform on the torus.

In the sixth chapter, the Hadamard finite part and its consequences are emphasized. Additionally, the relationship between the Cauchy principal value integral and the Hadamard finite part is discussed. Furthermore, the application of the Hadamard finite part to differential equations is examined.

In the seventh chapter , the Laplace operator is explored, particularly in different coordinate systems.

The eighth chapter focuses on solving partial differential equations using the Fourier transform. This chapter discusses the elementary solution for the wave equation and highlights the application and significance of the Fourier transform in solving differential equations.

In the last chapter, the elementary solution of iterated Laplacian is examined.

Keywords: Schwartz distributions, elementary solution of linear partial differential equation, Hadamard finite part, Cauchy principal values, Heaviside function, Dirac delta function, Fourier transform, Laplace and Wave equation, smooth functions.

ÖZET

Bu tez, Schwartz dağılımlarını, özelliklerini ve sabit katsayılı lineer kısmi diferansiyel denklemlerin çözümlerini incelemektedir. Heaviside veya Dirac delta fonksiyonları gibi özel türdeki dağılımlar da ele alınmaktadır.

Ayrıca, Hadamard sonlu parça , Cauchy temel değerleri ve Laplace operatörü gibi önemli konular da ele alınmaktadır. Tez ayrıca Fourier dönüşümlerini kullanarak kısmi diferansiyel denklemlerin çözümlerini ele almakta, örnekler sunmakta ve dalga denklemi ve Laplace denklemi gibi temel çözümleri tartışmaktadır.

Ayrıca, Schwartz dağılımlarının uzayında bir diferansiyel denklemin çözümü ve iterasyonlu Laplace operatörü de tartışılmaktadır.

Giriş ile birlikte, bu tez on bölümden oluşmaktadır. İlk bölüm, Schwartz dağılımları hakkında bir giriş sunmaktadır.

Bu tez, dağılımların kavramını ele alır. Ayrıca, kompakt destekli dağılımlar, dağılımların türetilmesi ve düzgün bir fonksiyon tarafından dağılımların toplamı ve çarpımı gibi dağılımlarla ilgili farklı yönleri de araştırır.

Üçüncü bölümde, Fourier dönüşümü, reel çizgi $\mathbb{R}$ bağlamında ele alınmaktadır. Öncelikle, $\mathbb{R}$ üzerinde Fourier dönüşümü tanımlanarak, $\mathbb{R}$ üzerindeki Fourier dönüşümünü daha iyi anlamak amaçlanmaktadır. Sonsuz kez türetilebilir ve kompakt destekli fonksiyonların uzayı ile hızla azalan fonksiyonlar uzayı incelenmektedir.

Ardından, Fourier dönüşümü $\mathbb{R}^n$ üzerinde tanımlanmaktadır. $\mathbf{L}^1(\mathbb{R}^n)$ uzayında çalışılmakta, $\mathscr{S}(\mathbb{R}^n)$ uzayı incelenmekte ve $\mathbf{L}^2(\mathbb{R}^n)$ uzayında Fourier dönüşümü üzerinde durulmaktadır.

Dördüncü bölüm, torus üzerinde Fourier dönüşümünün incelenmesine ayrılmıştır.

Altıncı bölümde, Hadamard sonlu parça ve sonuçları üzerinde durulmaktadır. Ayrıca, Cauchy özel değerleri integrali ve Hadamard sonlu parça arasındaki ilişki ele alınmaktadır. Ek olarak, Hadamard sonlu parçanın diferansiyel denklemlere uygulanması incelenmektedir.

Yedinci bölümde, farklı koordinat sistemlerinde Laplace operatörü incelenmektedir.

Sekizinci bölüm Fourier dönüşümünü kullanarak kısmi diferansiyel denklemlerin çözümlerini ele almaktadır. Dalga denklemi için temel çözümün tartışıldığı bu bölüm, Fourier dönüşümünün diferansiyel denklemleri çözmedeki uygulamasını ve önemini vurgulamaktadır.

Son bölümde, tekrarlanan Laplasyen'in temel çözümü incelenmektedir.

Anahtar Sözcükler: Schwartz dağılımlar, lineer kısmi türevli denklemlerin temel çözümü, Hadamard sonlu parça, Cauchy temel değeri, Heaviside fonksiyon, Dirac delta fonksiyonu, Fourier dönüşümleri, Laplace ve Dalga denklemi, düzgün fonksiyonlar,

1 Introduction

Schwartz distributions represent a class of generalized functions that play a significant role in functional analysis and frequently arise in fields such as mathematical physics, wave equations, and quantum mechanics.

In this thesis, we initially address the definition and properties of Schwartz distributions. The fundamental characteristics of these distributions, such as operations of addition and multiplication, their relationship with differential operators.

1.1 Distributions

In this chapter, we will define the Schwartz distributions and their properties.

Proposition 1.1.1. *For any compact $\mathscr{K}$ of $\mho$, there exist numbers $d_k \in \mathbb{N}$ and $c_k > 0$ such that for any function $\varphi \in \mathscr{D}_{\mathscr{K}}(\mho)$, the following inequality holds:*

$$|< T, \varphi >| \leq c_k \sup_{|\alpha| \leq d_k, x \in \mho} |\partial^\alpha \varphi|(x). \tag{1.1.1}$$

The set $\mathscr{D}_{\mathscr{K}}(\mho)$ is defined as the subset of test functions in $\mathscr{D}(\mho)$ whose support is contained within $\mathscr{K}$.

Definition 1.1.1. *We call* ***distribution****, any linear form on$\mathscr{D}(\mho)$ which satisfies the continuity properties, for any compact $\mathscr{K} \subset \mho$. We denote by $\mathscr{D}^{'}(\mho)$,* ***the set of distributions*** *on $\mho$.*

A distribution T is a linear functional defined on $\mathscr{D}$, where its restriction to each compact subset $\mathscr{D}_{\mathscr{K}}$ of $\mathbb{R}^n$ is continuous. In simpler terms, a distribution T is a functional $\varphi \rightarrow < T, \varphi >$, defined for all functions $\varphi \in \mathscr{D}$, and it possesses the following properties:

- T is linear:

$$\begin{cases} < T, \varphi_1 + \varphi_2 > = < T, \varphi_1 > + < T, \varphi_2 > \\ < T, k\varphi > = k < T, \varphi >, \quad k \in \mathbb{C} \end{cases} \tag{1.1.2}$$

- T is continuous.

Definition 1.1.2. *If, for a given distribution T, there exists a number $d \in \mathbb{N}$ such that $d_k \leq d$, for any compact $\mathscr{K} \subset \mho$, then the smallest number d having this property is called the* ***order***.

Example 1.1.1. *Consider a function f that is locally integrable on $\mathbb{R}^n$. The integral*

$$< f, \varphi >= \int f(x)\varphi(x)dx \tag{1.1.3}$$

defines a distribution for all function $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

Recall that the definition of some spaces:

- $supp(f) = \overline{\{x \in Dom(f) : f(x) \neq 0\}} \subset Dom(f)$
- $C(\mho)$: the space of continuous functions on an open set $\mho$
  $C(\mho) = C^0(\mho) = \{f : \mho \to \mathbb{C} \mid \text{f is continuous}\}$.
- $C^\infty(\mho)$: the space of smooth functions on an open set $\mho$.
- $C_c^\infty(\mho)$: the space of smooth functions with compact support in $\mho$, also denoted as $\mathscr{D}(\mho)$, is the space of test function on $\mho$, which consists of smooth functions with compact support. $C_c^\infty(\mho) = \mathscr{D}(\mho) = \{f \in C^\infty(\mho) \mid \text{supp(f) is compact}\}$.
- $\mathscr{D}'(\mho)$: the space of distributions, which consists of linear functional on $\mathscr{D}(\mho)$.

Definition 1.1.3. *A distribution T called* ***positive*** *if $< T, \varphi >\geq 0$ for all $\varphi \geq 0$, $\varphi \in \mathscr{D}$.*

Remark 1.1.1. *The distributions of order zero are said to be* ***measures***. *(chapter II page 25)*

1.1.1 Compactly Supported Distributions

Let $\mathcal{K} = supp(T)$ be the compact support of T, $\mathcal{K} \subset U$, α be a function of $\mathscr{D}(U)$, equals to 1 on an open neighbourhood of $\mathscr{K}$. Then we have:

$$< T, \varphi >=< T, \alpha\varphi >, \quad \forall \varphi \in \mathscr{D}(U), \tag{1.1.4}$$

since $\varphi - \alpha\varphi$ has its support in the complementary open of $\mathscr{K}$
i.e.: $supp(\varphi - \alpha\varphi) \cap supp(f) = \emptyset$.

Theorem 1.1.2. *A distribution is positive if and only if it is a positive measure.*

Proof. Consider a compact set $\mathscr{K}$ and a positive function $\varphi_0 \in \mathscr{D}$ such that $\varphi \equiv 1$ in a neighbourhood of $\mathscr{K}$.
For all φ in $\mathscr{D}$, we know that:

$$-\varphi_0(x) \sup_{x\in\mathscr{K}} |\varphi| \leq \varphi(x) \leq \varphi_0(x) \sup_{x\in\mathscr{K}} |\varphi|, \quad \varphi \geq 0.$$

Therefore, we have the following:

$$\begin{aligned} T \text{is positive} \quad &\Longleftrightarrow < T, \varphi > \geq 0 \\ &\Longleftrightarrow \; | < T, \varphi > | = | < T, \varphi_0 \varphi > | \\ &\Longleftrightarrow \; | < T, \varphi > | \geq < T, \varphi_0 > \sup_{x\in\mathscr{K}} |\varphi|. \end{aligned}$$

□

Definition 1.1.4. *A distribution T is called **harmonic** if it is a solution of the differential equation $\Delta T = 0$.*

Definition 1.1.5. *A distribution T is said to be **sub-harmonic** in a domain $\mho$ if $\Delta T \geq 0$ on $\mho$.*

We notice that the space $\mathbf{L}^1_{\text{loc}}$ consists of functions that are locally integrable. For a function f to be a member of $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$, if the function f is integrable on every compact set $\mathscr{K} \subset \mathbb{R}$.

Theorem 1.1.3. *A sub-harmonic distribution is locally integrable function in $\mho$ (integrable on all compacts of $\mho$), which, in the interior of any closed ball $\mathscr{B}_p$ center 0 and radius p, admits the representation:*

$$T(x) = h(x) + E * \mu, \tag{1.1.5}$$

where h is harmonic and μ is the restriction of ΔT to the ball (μ is a positive Radon measure on $\mathscr{B}_p$) [reference for radon measure], $E = -k_n|x|^{2-n}$ and $k_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}$.

Proof. Let $u(x) = (E * \mu)(x)$.
If μ has compact support, then we have:

$$\begin{aligned} \Delta u &= \Delta(E * \mu) \\ &= \Delta E * \mu \\ &= \delta * \mu \\ &= \mu. \end{aligned}$$

We know also that $\mu = \Delta T$ in the interior of $\mathcal{B}_p$ (since μ is the restriction of ΔT to $\mathcal{B}_p$). Therefore,

$$
\begin{aligned}
\Delta T &= \Delta u \\
\Longrightarrow \Delta(T-u) &= 0 = \Delta h \\
\Longrightarrow T-u &= h
\end{aligned}
$$

is a harmonic function.
So, we finally have;

$$T(x) = h(x) + E * \mu.$$

Since μ has compact support and $E \in \mathbf{L}^1_{\text{loc}}$, $E * \mu \in \mathbf{L}^1_{\text{loc}}$. ((($\{E, E*\mu\} \in \mathbf{L}^1_{\text{loc}}$)))

$\square$

1.2 Differentiation of Distributions

Consider a function f that belongs to the set $C^1(U)$, continuously differentiable on the open $U \subset \mathbb{R}^n$. For all φ in $\mathscr{D}(U)$, we have:

$$\int_U \frac{\partial f}{\partial x^1}\varphi dx = -\int_U f\frac{\partial \varphi}{\partial x^1}dx. \tag{1.2.1}$$

Assuming that f belongs to the space of locally integrable functions $\mathbf{L}_{\text{loc}}(U)$, the existence of the second integral allows us to interpret it as defining the derivative (distribution) of f. In general, the derivative of a distribution is defined as follows:

Definition 1.2.1. *Let T be a distribution on $\mathscr{D}'(U)$, U is an open in $\mathbb{R}^n$. For all φ in $\mathscr{D}(U)$, **the partial differential** $\frac{\partial T}{\partial x^i}$ of T is given by:*

$$< \frac{\partial T}{\partial x^i}, \varphi >= - < T, \frac{\partial \varphi}{\partial x^i} > . \tag{1.2.2}$$

Since the derivative of test function is again a test function, then we can consider successive derivatives of distributions. By applying the derivative operation repeatedly, we can obtain higher-order derivatives of distributions. The following equation represents **the successive differentiation**:

$$< D^j T, \varphi >= (-1)^{|j|} < T, D^j \varphi >, \quad \forall \varphi \in \mathscr{D}(U). \tag{1.2.3}$$

Definition 1.2.2. *Given a test function* $\varphi \in \mathscr{D}(\mathbb{R}^n)$, ***the Dirac delta function*** *often referred to as the Dirac measure, has the following properties:*

- *at the origin*

$$< \delta, \varphi >= \varphi(0) \tag{1.2.4}$$

- *at the point* $x = \alpha$

$$< \delta_\alpha, \varphi >= \varphi(\alpha). \tag{1.2.5}$$

Example 1.2.1. *We define* ***the Heaviside function*** *on* $\mathbb{R}$ *by:*

$$\begin{cases} H(x) = 1 & for\ x \geq 0 \\ H(x) = 0 & for\ x < 0. \end{cases} \tag{1.2.6}$$

The derivative is by definition:

$$< H', \varphi >= - < H, \varphi' >= -\int_0^\infty \varphi'(x)dx = \varphi(0) \tag{1.2.7}$$

So, we have $< H', \varphi >=< \delta, \varphi >$. *This show that the derivative of* H *is* δ*:*

$$H' = \delta. \tag{1.2.8}$$

Example 1.2.2. *For all* $\varphi \in C^\infty$, *the derivative of the Dirac delta function is defined in the sense of distributions as follows:*

on $\mathbb{R}$*:*

$$< \delta', \varphi >= - < \delta, \varphi' >= -\varphi'(0) \tag{1.2.9}$$

$$< \delta^{(p)}, \varphi >= (-1)^{(p)} \varphi^{(p)}(0). \tag{1.2.10}$$

on $\mathbb{R}^n$*:*

$$< D^j\delta, \varphi >= (-1)^{|j|} < \delta, D^j\varphi >= (-1)^{|j|}(D^j\varphi)(0). \tag{1.2.11}$$

We can examine the derivative at the point $\alpha \in \mathbb{R}^n$*:*

$$< D^j\delta_\alpha, \varphi >= (-1)^{|j|}(D^j\varphi)(\alpha). \tag{1.2.12}$$

1.3 Sum and Product of Distribution by a Function C^∞

Let T, T_1, T_2 be in $\mathscr{D}'(U)$. We can define the sum of two distributions as follows:

$$< T_1 + T_2, \varphi > = < T_1, \varphi > + < T_2, \varphi >, \quad \forall \varphi \in \mathscr{D}(U) \tag{1.3.1}$$

and the product of a distribution by a function $f \in C^\infty(U)$:

$$< fT, \varphi > = < T, f\varphi >, \quad \forall \varphi \in \mathscr{D}(U), \quad f\varphi \in \mathscr{D}(U). \tag{1.3.2}$$

As a starting point, we can examine the multiplication on δ:

Example 1.3.1. *Let us consider the multiplication of δ with x. For any φ in $\mathscr{D}(\mathbb{R})$, we have:*

$$\begin{aligned} < x\delta' + \delta, \varphi > &= < (x\delta)', \varphi > \\ &= - < x\delta, \varphi' > \\ &= - < \delta, x\varphi' > \\ &= -x\varphi'(0)|_{x=0} \\ &= 0. \end{aligned}$$

$$\Longrightarrow x\delta' + \delta = 0 \Longrightarrow x\delta' = -\delta \tag{1.3.3}$$

1.3.1 Division Problem

To better understand what we have done and the Dirac delta function, let's solve a few problems. First, let's look at the general results of multiplying distribution by a function.

Exercise 1.3.1. *Show that the equation*

$$xT = 0 \tag{1.3.4}$$

has the solution in $\mathscr{D}'(\mathbb{R}^n)$

$$T = C\delta, \quad C\text{:}constant. \tag{1.3.5}$$

Proof. Let φ be in $\mathscr{D}(\mathbb{R})$. We have:

$$< xT, \varphi > = 0 \iff < T, x\varphi > = 0 \tag{1.3.6}$$

We know also that all φ in $\mathscr{D}(\mathbb{R})$ can be represented in the following form:

$$\varphi(x) = \varphi(0)\theta(x) + x\psi(x) \tag{1.3.7}$$

where $\theta \in \mathscr{D}(\mathbb{R})$, fixed, such that $\theta(0) = 1$; $\psi \in \mathscr{D}(\mathbb{R})$. Then we have:

$$\begin{aligned} < T, \varphi(x) > &= < T, \varphi(0)\theta(x) > + < T, x\psi(x) > \\ &= \varphi(0) < T, \theta(x) > \\ &= C\varphi(0) \end{aligned}$$

with $C =< T, \theta >$.
Thus,

$$T = C\delta.$$

□

Exercise 1.3.2. *Find the general solution of*

$$(x - \alpha)T = 0 \tag{1.3.8}$$

in $\mathscr{D}'(\mathbb{R})$.

Proof. For $\varphi \in \mathscr{D}(\mathbb{R})$, we have:

$$< (x - \alpha)T, \varphi >= 0 \iff < T, (x - \alpha)\varphi >= 0 \tag{1.3.9}$$

We have the knowledge that every function φ can be represented in the following form:

$$\varphi(x) = \varphi(\alpha)\theta(x) + (x - \alpha)\psi(x) \tag{1.3.10}$$

where $\theta \in \mathscr{D}(\mathbb{R})$ such that $\theta(\alpha) = \theta'(\alpha) = 1$; with $\varphi'(\alpha) - \varphi(\alpha) = \psi(\alpha) \in \mathscr{D}(\mathbb{R})$. We have then:

$$\begin{aligned} < T, \varphi(x) > &= < T, \varphi(\alpha)\theta(x) > + < T, (x - \alpha)\psi(x) > \\ &= \varphi(\alpha) < T, \theta(x) > \\ &= C_0\varphi(\alpha) \end{aligned}$$

with $C_0 =< T, \theta >$. Hence,

$$T = C_0 \delta_\alpha. \tag{1.3.11}$$

□

Remark 1.3.1. *We will adopt the following notation when we solve the problems:*

T_p: *particular solution*

T_h: *homogeneous solution*

The general solution deduces by adding the particular solution to the homogeneous solution.

Exercise 1.3.3. *Find the general solution of*

$$(x-\alpha)^2 T = 0 \tag{1.3.12}$$

in $\mathscr{D}'(\mathbb{R})$.

Proof. We know that $C\delta_\alpha$ is the general solution of $(x-\alpha)T = 0$, where C is any constant. By definition, the Dirac delta function has the property that $\delta_\alpha = 0$ for $x \neq \alpha$ and $\int_{-\infty}^{\infty} \delta_\alpha(x), dx = 1$. Therefore, the equation $(x-\alpha)^2 T = 0$ is equivalent to $(x-\alpha)T = C\delta_\alpha$ with constant C. To see the equivalence between these two equations, we can multiply both sides by $(x-\alpha)$ to get $(x-\alpha)^2 T = c\delta_\alpha(x-\alpha)$. The right side simplifies to $c\delta_\alpha(x-\alpha) = 0$. Therefore, we have $(x-\alpha)^2 T = 0$.
By the definition of the derivative of Dirac delta, we have:

$$\begin{aligned}
< (x-\alpha)\delta_a', \varphi(x) > &= - < \delta_\alpha, D((x-\alpha)\varphi(x)) > \\
&= - < \delta_\alpha, \varphi(x) + (x-\alpha)\varphi'(x) > \\
&= - \left(\varphi(x) + (x-\alpha)\varphi'(x)\right|_{x=\alpha} \\
&= -\varphi(\alpha) \\
&=< -\delta_\alpha, \varphi(x) > .
\end{aligned}$$

Therefore,

$$(x-\alpha)\delta_a' = -\delta_\alpha. \tag{1.3.13}$$

Thus, the particular solution is:

$$(x-\alpha)T_p = C\delta_\alpha \Longrightarrow (x-\alpha)T_p = -C(x-\alpha)\delta_a' \Longrightarrow T_p = -C\delta_a' \tag{1.3.14}$$

Finally, we have the general solution as:

$$T = C_1\delta_\alpha + C_2\delta'_a, \quad C_1, C_2 : \text{constants.} \tag{1.3.15}$$

$\square$

Exercise 1.3.4. *Find the general solution of*

$$(x-\alpha)T = \delta_\beta \tag{1.3.16}$$

in the space $\mathscr{D}'(\mathbb{R})$.

Proof. In the context of exercise 1.3.1, the homogeneous solution T_h is represented by $C\delta_\alpha$. We will examine the particular solution in two part:

- If $\alpha \neq \beta$, we can write:

$$< (x-\alpha)T_p, \varphi >=< T_p, (x-\alpha)\varphi >=< \delta_\beta, \varphi > \tag{1.3.17}$$

  Therefore, the particular solution is:

$$T_p = \frac{\delta_\beta}{\beta - \alpha} \tag{1.3.18}$$

  because when $x \neq \beta$, $\delta_\beta = 0$. So, the general solution is:

$$T = \frac{\delta_\beta}{\beta - \alpha} + C\delta_\alpha \tag{1.3.19}$$

  with a constant C.

- If $\alpha = \beta$, can be expressed in the following form:

$$(x-\alpha)T = \delta_\alpha \tag{1.3.20}$$

  We found the homogeneous and particular solution of (1.3.19) in the previous exercises. By the equations (1.3.10) and (1.3.12), the general solution is:

$$T = C\delta_\alpha - \delta'_a \tag{1.3.21}$$

  where C is any constant.

$\square$

Exercise 1.3.5. *Find the general solution of*

$$(x-\alpha)T = \delta_b' \tag{1.3.22}$$

in $\mathscr{D}'(\mathbb{R})$.

Proof. By the exercise 1.3.2, the homogeneous solution is $T_h = C\delta_\alpha$, with any constant C. To calculate T_p, we will evaluate the two conditions on α, β:

- If $\alpha \neq \beta$, we have:

  $$< (x-\alpha)T_p, \varphi > = < \delta_b', \varphi > . \tag{1.3.23}$$

  Therefore, the particular solution is

  $$T_p = \frac{\delta_b'}{\beta - \alpha}, \tag{1.3.24}$$

  because when $x \neq \beta$, $\delta_b' = 0$. So, the general solution is:

  $$T = \frac{\delta_b'}{\beta - \alpha} + C\delta_\alpha, \tag{1.3.25}$$

  with a constant C.

- If $\alpha = \beta$, the equation takes the form:

  $$(x-\alpha)T = \delta_a'. \tag{1.3.26}$$

  As the definition of the derivative of the Dirac delta function, we obtain:

  $$\begin{aligned} < (x-\alpha)\delta_a'', \varphi(x) > &= (-1)^2 < \delta_\alpha, D^2((x-\alpha)\varphi(x)) > \\ &= < \delta_\alpha, 2\varphi'(x) + (x-\alpha)\varphi''(x) > \\ &= \big(2\varphi'(x) + (x-\alpha)\varphi''(x)\big|_{x=\alpha} \\ &= 2\varphi'(\alpha) \\ &= < -2\delta_a', \varphi(x) > . \end{aligned}$$

  So,

  $$(x-\alpha)\delta_a'' = -2\delta_a' \tag{1.3.27}$$

Therefore,

$$(x-\alpha)T_p = \delta'_a \Longrightarrow (x-\alpha)T_p = -\frac{(x-\alpha)}{2}\delta''_a \tag{1.3.28}$$

$$\Longrightarrow T_p = \frac{-\delta''_a}{2}. \tag{1.3.29}$$

In conclusion, the general solution can be expressed as:

$$T = \frac{-\delta''_a}{2} + C\delta_\alpha, \quad C\text{:constant.} \tag{1.3.30}$$

□

Exercise 1.3.6. *Determine the general solution of the equation*

$$(1+x^2)(1-x^2)^2 T = 0 \tag{1.3.31}$$

in the space $\mathscr{D}'(\mathbb{R})$.

Proof. We can divide both sides by $(1+x^2)$ as long as $1+x^2$ is not equal to zero. Since $1+x^2$ is always positive, we can safely divide by it. So $(1+x^2)(1-x^2)^2T = 0$ is equivalent to $(1-x^2)^2T = 0$ on $\mathbb{R}$.
Now, our goal is to determine the general solution for the equation

$$(1-x^2)^2 T = (1-x)^2(1+x)^2 T = 0. \tag{1.3.32}$$

Therefore, using the help of the exercise 1.3.3 we can obtain the general solution with two different equation solutions.
First, let us consider

$$(x-1)^2 T = 0. \tag{1.3.33}$$

The solution can be expressed as follows:

$$T_1 = C_1\delta'_1 + C_2\delta_1, \quad C_1, C_2 : \text{constants.} \tag{1.3.34}$$

The second equation

$$(x+1)^2 T = 0. \tag{1.3.35}$$

has the solution

$$T_{-1} = D_1\delta'_{-1} + D_2\delta_{-1}, \quad D_1, D_2 : \text{constants.} \tag{1.3.36}$$

Consequently, the general solution of (1.3.32) is

$$T == C_1\delta'_1 + C_2\delta_1 + D_1\delta'_{-1} + D_2\delta_{-1} \tag{1.3.37}$$

where C_1, C_2, D_1, D_2 are any constants. □

1.4 Classification of Partial Differential Equations

Definition 1.4.1. *A **partial derivative equation(PDE)** of order d with constant coefficients on $\mathbb{R}$ is written:*

$$\sum_{|a|\leq d} a_\alpha D^\alpha u = f, \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{1.4.1}$$

where a_α given numbers, $f, u \in \mathscr{D}(\mathbb{R}^n)$.
The principal part of the differential operator is:

$$\sum_{|a|=d} a_\alpha D^\alpha.$$

*A sub-variety S of dimension $n-1$, of the equation $s(x) = 0$ is said to be **characteristic** for PDE* (1.4.1)*, if it verifies the first order PDE :*

$$\sum_{|a|=d} a_\alpha p^\alpha = 0 \tag{1.4.2}$$

where, following our usual notations,

$$p^\alpha = p_1^{\alpha_1} \ldots p_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{1.4.3}$$

and we set:

$$p_i = \frac{\partial s}{\partial x^i}. \tag{1.4.4}$$

The properties of the solutions of a PDE depend essentially on the nature of its characteristics.We are thus led to the classification which we state for PDE, where we suppose the coefficients a_α to be real.

- The PDE (1.4.1) is said to be **elliptic** if the polynomial of p^{α}

$$\sum_{|a|=d} a_{\alpha} p^{\alpha} = 0, \quad p \neq 0 \tag{1.4.5}$$

does not vanish for any systems of p^i reals(it has no real characteristics).

- The PDE is said to be x^i -**hyperbolic** if the polynomial at p_1 :

$$\sum_{|a|=d} a_{\alpha} p^{\alpha} = 0, \quad p \neq 0 \tag{1.4.6}$$

has d real and distinct roots for all systems of numbers $p_2, ..., p_n$ reals.

Example 1.4.1. *The Laplace's equation*

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2} u = 0 \tag{1.4.7}$$

is elliptic.
The Wave equation

$$\Box u = \left(\frac{\partial^2}{(\partial x^1)^2} - \sum_{i=2}^{n} \frac{\partial^2}{(\partial x^i)^2} \right) u = 0 \tag{1.4.8}$$

is hyperbolic.

- An important class of equations which are neither elliptic nor hyperbolic are the parabolic equations: these equations are not characterized only by their principal part.The PDE is said to be x^1***-parabolic*** if it can be expressed in the following form:

$$\frac{\partial u}{\partial x_1} - \sum_{\substack{|a| \leq d \\ a_1 = 0}} a_{\alpha} D^{\alpha} u = f \tag{1.4.9}$$

with

$$\sum_{\substack{|a| \leq d \\ a_1 = 0}} a_{\alpha} p^{\alpha}, \quad \forall p \neq 0, \quad \text{real} \tag{1.4.10}$$

which implies that $\sum_{\substack{|a| \leq d \\ a_1 = 0}} a_{\alpha} D^{\alpha}$ is elliptic, in $n-1$ variables.

Example 1.4.2. *The Heat equation*

$$\frac{\partial u}{\partial t} - \Delta u = 0 \tag{1.4.11}$$

is t- parabolic.
The equation

$$\frac{\partial u}{\partial t} + \Delta u = 0 \tag{1.4.12}$$

is not t- parabolic, is (-t)- parabolic.

2 Newtonian Potential

Consider in $\mathbb{R}^2$ the equation

$$-\Delta u = f \tag{2.0.1}$$

where $f \in \varepsilon'$ and $u \in D'(\mho)$. ε' means the dual to ε, in other words it is $C^\infty(\mho)$.
The convolution between E and f is said to be **the Newtonian potential of f**:

$$u = -E * f = k_n |x|^{2-n} * f. \tag{2.0.2}$$

for $n \geq 3$, where $k_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}$. Note that we have for $n = 2$:

$$E = (2\pi)^{-1} log|x|. \tag{2.0.3}$$

2.1 The Volume Potential

Let $\mho$ be a bounded open of $\mathbb{R}^n$ and $supp(f) \subset \overline{\mho}$. Suppose $f \in \mathbf{L}^\infty(\mho)$ (so $\in \mathbf{L}^\infty(\mathbb{R}^n)$).
By definition of convolution, the potential u of f has the form:

$$\begin{aligned} u(y) &= (-E * f)(y) \\ &= -\int_\mho E(y-x) f(x) dx \\ &= k_n \int_\mho \frac{1}{|y-x|^{n-2}} f(x) dx. \end{aligned}$$

We know that $\in \mathbf{L}^\infty(\mathbb{R}^n)$, so it is bounded by a constant M in $\mho$. Thus we have $f(x) \leq M$ $\forall x \in \mho$.
Consider a sequence y_k in $\mathbb{R}^n$ that converges to y_0 as k approaches infinity for a fixed point $y_0 \in \mathbb{R}^n$. We want to show that $\lim_{k\to\infty} u(y_k) = u(y_0)$.

By converging of y_k to y_0, we have $|y_k - x| \to |y_0 - x|$ $\forall x \in \mho$. Furthermore, we obtain:

$$\begin{aligned}
|u(y_k) - u(y_0)| &= \left| k_n \int_\mho \frac{1}{|y_k - x|^{n-2}} f(x)dx - k_n \int_\mho \frac{1}{|y_0 - x|^{n-2}} f(x)dx \right| \\
&\leq k_n \int_\mho \left| \frac{1}{|y_k - x|^{n-2}} - \frac{1}{|y_0 - x|^{n-2}} \right| |f(x)|dx \\
&\leq k_n M \int_\mho \left| \frac{1}{|y_k - x|^{n-2}} - \frac{1}{|y_0 - x|^{n-2}} \right| dx.
\end{aligned}$$

By the dominated convergence theorem, we can interchange the limit and the integral:

$$\lim_{k\to\infty} |u(y_k) - u(y_0)| \leq k_n M \int_\mho \lim_{k\to\infty} \left| \frac{1}{|y_k - x|^{n-2}} - \frac{1}{|y_0 - x|^{n-2}} \right| dx = 0.$$

This shows that $\lim_{k\to\infty} u(y_k) = u(y_0)$.
We know that δ is unitary for the convolution and the convolution is commutative and associative. Then, the differential of u in the sense of distribution is

$$\frac{\partial u}{\partial x^i} = -f * \frac{\partial E}{\partial x^i}. \tag{2.1.1}$$

Outside the origin, we have:

$$E = -k_n |x|^{2-n} \Longrightarrow \frac{\partial E}{\partial x^i} = -(2-n)k_n |x|^{1-n} \frac{x^i}{|x|} = (n-2)k_n \frac{x^i}{|x|^n}$$

the integral of $\left|\frac{\partial E}{\partial x^i}\right|$ over $\mho$ is finite. Because:

$$\begin{aligned}
\int_\mho \left| \frac{\partial E}{\partial x^i} \right| dx &= \int_\mho |(n-2)k_n| \frac{|x^i|}{|x|^n} \\
&\leq \int_\mho |(n-2)k_n| \frac{c}{|x|^n} \\
&< \infty
\end{aligned}$$

for a constant c as $\mho$ is bounded. Thus, we have:

$$\frac{\partial E}{\partial x^i} \in \mathbf{L}^1_{\text{loc}}(\mho). \tag{2.1.2}$$

Consequently, $\frac{\partial u}{\partial x^i}$ is continuous.

Theorem 2.1.1. *Let $f \in \mathbf{L}^\infty(\mho)$ and $supp(f) \subset \overline{\mho}$. Suppose that the potential u of f is in*

$C^1(\mathbb{R}^n)$, bounded and

$$\lim_{x\to\infty} u(x) = 0 \quad in\ \mathbb{R}^n, \quad \lim_{x^i\to\infty} \frac{\partial u}{\partial x^i} = 0. \tag{2.1.3}$$

Then we have:

$$||u||_{C^1(\mathbb{R}^n)} \leq K(\mho)||f||_{\mathbf{L}^\infty(\mho)} \tag{2.1.4}$$

where $K(\mho)$ is a number depending only on $\mho$.

We have more precisely:

$$|u(y)| \leq K(y,\mho)||f||_{\mathbf{L}^\infty(\mho)} \tag{2.1.5}$$

where $K(y,\mho) = k_n \int_\mho \frac{dx}{|y-x|^{n-2}}$ is uniformly bounded for $y \in \mathbb{R}^n$ as $\lim_{|y|\to\infty} K(y,\mho) = 0$. Recall that

$$||f||_{C^1(\mathbb{R}^n)} = \sum_{i=1}^{n} \sup_{x\in\mho} \left|\frac{\partial f}{\partial x^i}\right|, \quad ||f||_{\mathbf{L}^\infty(\mho)} = \sup_{x\in\mho} |f(x)|. \tag{2.1.6}$$

The $j-th$ derivatives of u in $\mathscr{D}'$ are:

$$-D^j u = f * D^j E = D^j f * E. \tag{2.1.7}$$

If $j \geq$, then $D^j E \notin \mathbf{L}^1_{\text{loc}}$: the hypothesis $f \in \mathbf{L}^\infty(\mho)$, $supp(f) \subset \overline{\mho}$ and the same hypothesis $f \in \mathscr{D}^0(\mho)$ are not sufficient to assure that the second derivatives of u are not usual function, i.e.; u is usual solution of $\Delta u = -f$; this result will be ensured by the following theorem:

Theorem 2.1.2. ***(i)*** *If $f \in W^1_\infty(\mathbb{R}^n)$ and $supp(f) \subset \overline{\mho}$, then the potential u of f is in $C^2(\mathbb{R}^n)$.*
(ii) *If $supp(f) \subset \overline{\mho}$, $f \in \mathbf{L}^\infty(\mho)$ and f has the fist derivative of the form;*

$$\frac{\partial f}{\partial x^i} = g_i + h_i \tag{2.1.8}$$

where $g_i \in \mathbf{L}^\infty(\mho)$, $supp(g_i) \subset \overline{\mho}$, $supp(h_i) \subset \partial\mho$, then the potential u of f is in $C^2(\mho)$ and $C^\infty(\complement\overline{\mho})$.

Proof. **(i)** In $\mathscr{D}'$, we have;

$$\begin{aligned} u = -f * E &\Longrightarrow \frac{\partial u}{\partial x^j} = f * \frac{\partial E}{\partial x^j} \\ &\Longrightarrow -\frac{\partial^2 u}{\partial x^i \partial x^j} = \frac{\partial f}{\partial x^i} * \frac{\partial E}{\partial x^j} \\ &\Longrightarrow u \in C^2(\mathbb{R}^n). \end{aligned}$$

(ii) By (i), we have;

$$-\frac{\partial^2 u}{\partial x^i \partial x^j} = \frac{\partial f}{\partial x^i} * \frac{\partial E}{\partial x^j}$$

This implies that

$$\begin{aligned} -\frac{\partial^2 u}{\partial x^i \partial x^j} &= (g_i + h_i) * \frac{\partial E}{\partial x^j} \\ &= g_i * \frac{\partial E}{\partial x^j} + h_i * \frac{\partial E}{\partial x^j}. \end{aligned}$$

□

Remark 2.1.1. *If $\mho$ is a regular open in the sense of Stokes's formula and $f \in C^2(\overline{\mho})$, $suppf \subset \overline{\mho}$, then the under conditions of previous theorem (ii), we have;*

$$u \in C^1(\mathbb{R}^n) \cap C^2(\complement(\partial\mho)). \tag{2.1.9}$$

Remark 2.1.2. *If f is not a compact support but is a measurable function, bounded, tending to zero at infinity faster than $|x|^{-2}$, i.e. such as;*

$$f \in \mathbf{L}^\infty(\mathbb{R}^n), \quad |f(x)| < \frac{A}{|x|^{2+\alpha}} \quad \textit{for } |x| > 1, 0 < \alpha < 1 \tag{2.1.10}$$

then $f * E$ is a function C^1, given by

$$u(y) = k_n \int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^{n-2}} dx \tag{2.1.11}$$

We will show that (2.1.11) *satisfies the equation $\Delta u = f$ using the Fubini's Theorem. Let $\varphi \in \mathscr{D}$. The inner product gives us*

$$
\begin{aligned}
< f,\varphi > &= \int_{\mathbb{R}^n} f(x)\varphi(x)dx \\
&= \int_{\mathbb{R}^n} f(x)(\Delta\varphi * E(x)dx \\
&= \int_{\mathbb{R}^n} f(x)\left(-k_n\int_{\mathbb{R}^n}\frac{\Delta\varphi(y)}{|y-x|^{n-2}}dy\right)dx \\
&= \int_{\mathbb{R}^n}\left(-k_n\int_{\mathbb{R}^n}\frac{f(x)}{|y-x|^{n-2}}dx\right)\Delta\varphi(y)dy \\
&= \int_{\mathbb{R}^n}(f(x)*E)(y)\Delta\varphi(y)dy \\
&=< f*E,\Delta\varphi > \\
&=< f,\Delta\varphi * E >
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow < f,\varphi > &=< f*E,\Delta\varphi > \\
&=< \Delta(f*E),\varphi >
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow f &= \Delta(f*E) \\
&= \Delta u
\end{aligned}
$$

2.1.1 The Single Layer Potential

Let S be differentiable sub-manifold of $\mathbb{R}^n$ of dimension $n-1$ at finite distance. Let f be a measure of support S identifiable with an integrable function f_0 on S. The potential u of f is the given by the usual integral:

$$u(y) = k_n\int_{\mathbb{R}^n}\frac{f_0(x)}{|y-x|^{2-n}}\mho_x \tag{2.1.12}$$

where $\mho_x$ is the form of Leray of S at the point x. Note that $u(y) \in C^\infty(\complement(S))$.

2.1.2 The Double Layer Potential

Let f be a measure of support S identifiable with an integrable function f_0 on S. Let f be a distribution of order 1 of S of the form:

$$f = \sum_{i=1}^{n} \frac{\partial (f_1 n^i)}{\partial x^i} \tag{2.1.13}$$

where n^i are the normal components to S and f_1 is a measure of support S, identifiable with an integrable function on S. The potential u of f is then:

$$\begin{aligned}
u = f * E &= \left(\sum_{i=1}^{n} \frac{\partial (f_1 n^i)}{\partial x^i} \right) * E \\
&= \sum_{i=1}^{n} \left(\frac{\partial (f_1 n^i}{\partial x^i} \right) * E \\
&= \sum_{i=1}^{n} f_1 n^i * \frac{\partial E}{\partial x^i} \\
&= \sum_{i=1}^{n} \left(\int_S f_1 n^i(x) \frac{\partial E}{\partial x^i}(y-x)dx \right) \\
&= \sum_{i=1}^{n} \left(\int_S f_1 n^i(x)(n-2)k_n|y-x|^{1-n} \frac{(y-x)^i}{|y-x|} dx \right) \\
&= \int_S \frac{f_1}{|y-x|^{1-n}} \left((n-2)k_n \sum_{i=1}^{n} n^i(x) \frac{(y-x)^i}{|y-x|} \right) dx \\
&= \int_S \frac{f_1}{|y-x|^{1-n}} a dx
\end{aligned}$$

where $a = (n-2)k_n \sum_{i=1}^{n} n^i(x) \frac{(y-x)^i}{|y-x|}$.

2.1.3 Stokes's Theorem

Let $\mho$ be an open $\partial\mho$ (with its boundary satisfying the Lipschitz condition), $f \in C^1(\mho) \cap C^0(\overline{\mho})$. We have:

$$\int_{\mho} \frac{\partial f}{\partial x^i} dx = \int_{\partial\mho} f n^i \mho \tag{2.1.14}$$

where $\mho$ is $(n-1)$-form on $\partial\mho$ and $n^i = (n^1, \ldots, n^n)$ is normal vector to $\partial\mho$. And we denote:

$$\partial\mho = \{x \in \mathbb{R}^n : s(x) = 0\} \tag{2.1.15}$$

$$\mho \wedge ds(x) = d(x) = dx_1 \wedge \cdots \wedge dx_n \tag{2.1.16}$$

2.2 Energy Integral, Uniqueness Theorem

Let $\mho$ be a bounded open of $\mathbb{R}^n$ and $\overline{\mho}$ be its closure. Let u, φ be two functions of $C^1(\overline{\mho}) \cap C^2(\mho)$. In $\mho$, we have:

$$u\Delta\varphi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(u\frac{\partial\varphi}{\partial x^i}\right) - \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}\frac{\partial\varphi}{\partial x^i} \tag{2.2.1}$$

If we integrate on the both sides on $\mho$, we have then:

$$\begin{aligned}
\int_{\mho}(u\Delta\varphi)(x) &= \int_{\mho}\left[\sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(u\frac{\partial\varphi}{\partial x^i}\right) - \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}\frac{\partial\varphi}{\partial x^i}\right] dx \\
&= \sum_{i=1}^{n}\left[\int_{\mho} \frac{\partial}{\partial x^i}\left(u\frac{\partial\varphi}{\partial x^i}\right) dx\right] - \int_{\mho}\sum_{i=1}^{n} \frac{\partial u}{\partial x^i}\frac{\partial\varphi}{\partial x^i} dx \\
&= \sum_{i=1}^{n}\left(\int_{\partial\mho} u\frac{\partial\varphi}{\partial x^i} n^i \mho\right) - \int_{\mho}\sum_{i=1}^{n} \frac{\partial u}{\partial x^i}\frac{\partial\varphi}{\partial x^i} dx
\end{aligned}$$

where n^i is the normal to $\partial\mho$ and $\mho$ is the Leary's form of $\partial\mho$. Taking $\varphi = u$, we get the energy integral:

$$\int_{\mho}(u\Delta\varphi)(x) = \int_{\partial\mho} u\frac{\partial\varphi}{\partial x^i} n^i \mho - \int_{\mho}\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x^i}\right)^2 dx \tag{2.2.2}$$

2.2.1 Uniqueness Theorem

The equation $\Delta u = T$ has only are solution $u \in C^1(\overline{\mho}) \cap C^2(\mho)$ taken on $\partial\mho$ given values. Notice the definition of u as the following:

$$u(y) = \int_{\mho} u(\Delta E(x-y))dx \tag{2.2.3}$$

Proof. Suppose that u_1 and u_2 are solutions of $\Delta u = T$. We will show that $u_1 = u_2$. The difference function $u = u_1 - u_2$ satisfies this equation. We have then

$$\Delta u = 0 \quad \text{in } \mho \Longrightarrow u = 0 \quad \text{on } \partial\mho \tag{2.2.4}$$

$$
\begin{aligned}
&\Longrightarrow 0 = \int_{\mho} u\Delta u = \int_{\partial\mho} \sum_{i=1}^{n} u \frac{\partial \varphi}{\partial x^i} n^i \mho - \int_{\mho} \sum_{i=1}^{n} (\frac{\partial u}{\partial x^i})^2 dx \\
&\Longrightarrow \int_{\mho} \sum_{i=1}^{n} (\frac{\partial u}{\partial x^i})^2 dx = 0 \\
&\Longrightarrow \frac{\partial u}{\partial x^i} = 0 \quad \text{in } \mho \\
&\Longrightarrow u = 0 \quad \text{in } \mho \\
&\Longrightarrow u_1 = u_2.
\end{aligned}
$$

□

2.3 Green's Formula

Let u, φ be two functions of $C^1(\overline{\mho}) \cap C^2(\mho)$.

$$
u\Delta\varphi - \varphi\Delta u = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(u \frac{\partial \varphi}{\partial x^i} \right) - \sum_{i=1}^{n} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^i} - \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\varphi \frac{\partial u}{\partial x^i} \right) + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x^i} \frac{\partial u}{\partial x^i}
$$

$$
u\Delta\varphi - \varphi\Delta u = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(u \frac{\partial \varphi}{\partial x^i} - \varphi \frac{\partial u}{\partial x^i} \right) \tag{2.3.1}
$$

By the integration on $\mho$, we have:

$$
\begin{aligned}
\int_{\mho} (u\Delta\varphi - \varphi\Delta u) dx &= \int_{\mho} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(u \frac{\partial \varphi}{\partial x^i} - \varphi \frac{\partial u}{\partial x^i} \right) dx \\
&= \int_{\partial\mho} \sum_{i=1}^{n} \left(u \frac{\partial \varphi}{\partial x^i} - \varphi \frac{\partial u}{\partial x^i} \right) n^i \mho \quad \text{by Stokes's Theorem}
\end{aligned}
$$

The general potential formula (the volume potential, the single and layer potential) is

$$
u(y) = -k_n \int_{\mho} (|x-y|^{2-n} \Delta u) dx + k_n \int_{\partial\mho} \left(u \frac{\partial}{\partial n} |x-y|^{2-n} - |x-y|^{2-n} \frac{\partial u}{\partial n} \right) \mho_x \tag{2.3.2}
$$

$\frac{\partial u}{\partial n}$ is the exterior normal derivative on $\partial\mho$ of u.

$$
\begin{aligned}
u(y) &= -k_n \int_{\mho} (|x-y|^{2-n}\Delta u - u\Delta|x-y|^{2-n})dx \\
&= -k_n \int_{\mho} (|x-y|^{2-n}\Delta u dx + k_n \int_{\mho} u\Delta|x-y|^{2-n}dx \\
&= -k_n \int_{\mho} (|x-y|^{2-n}\Delta u dx + k_n \left[\int_{\partial\mho} \sum_{i=1}^{n} u\frac{\partial|x-y|^{2-n}}{\partial x^i} n^i\mho - \int_{\mho} \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}\frac{\partial|x-y|^{2-n}}{\partial x^i} dx\right] \\
&= -k_n \int_{\mho} (|x-y|^{2-n}\Delta u dx + k_n \left[\int_{\partial\mho} \sum_{i=1}^{n} u\frac{\partial|x-y|^{2-n}}{\partial x^i} n^i\mho - \int_{\mho} \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}|x-y|^{2-n} n^i\mho\right] \quad \text{by Stokes's T} \\
&= -k_n \int_{\mho} (|x-y|^{2-n}\Delta u dx + k_n \int_{\partial\mho} \sum_{i=1}^{n} \left(u\frac{\partial|x-y|^{2-n}}{\partial n} - |x-y|^{2-n}\frac{\partial u}{\partial n}\right)\mho_x
\end{aligned}
$$

Let

$$
E(x-y) = \varphi(x) = \begin{cases} |x-y|^{2-n} & \text{for n} \neq 2 \\ \log|x-y| & \text{for n} = 2 \end{cases} \tag{2.3.3}
$$

in $C^\infty(\{(x,y) \in \mathbb{R}^{2n} \, | x \neq y\})$. Let us define an open set $\mho_\varepsilon$ which is $\mho$ deprived of the ball $B_\varepsilon : \{x \in \mathbb{R}^n : |x-y| < \varepsilon\}$.
Notice that $\Delta E = 0$ in $\mho_\varepsilon$.

$$
\partial\mho_\varepsilon = \partial\mho \cup \partial B_\varepsilon.
$$

For $y \in \mho$, the Green's Formula gives

$$
\int_{\mho_\varepsilon} (|x-y|^{2-n}\Delta u - u\Delta|x-y|^{2-n})dx = \int_{\partial\mho_\varepsilon} \sum_{i=1}^{n} \left(u\frac{\partial|x-y|^{2-n}}{\partial x^i} - |x-y|^{2-n}\frac{\partial u}{\partial x^i}\right) n^i\mho
$$

The function $|x-y|^{2-n}\Delta u$ is integrable on $\mho$. We have also:

$$
\lim_{\varepsilon\to 0} \int_{\mho_\varepsilon} (|x-y|^{2-n}\Delta u)dx = \int_{\mho} (|x-y|^{2-n}\Delta u)dx
$$

The integral of $|x-y|^{2-n}\Delta u$ on $\partial B_r (= \{x \in \mathbb{R}^n : |x-y| = r\})$ is :

$$
\begin{aligned}
\int_{\mho_r} (|x-y|^{2-n}\Delta u)dx &= \int_{\partial\mho_r} \sum_{i=1}^{n} \left(u\frac{\partial r^{2-n}}{\partial x^i} - r^{2-n}\frac{\partial u}{\partial x^i}\right) n^i\mho \\
&= \int_{\Sigma} \sum_{i=1}^{n} \left((2-n)ur^{1-n} - r^{2-n}\frac{\partial u}{\partial x^i}\right) n^i\mho
\end{aligned}
$$

with $\mho = rd\Sigma$, $\sum_{i=1}^{n}(n^i)^2 = 1$.
Note that $\sum$ is the unit $(n-1)$-sphere.$(= \{x \in \mathbb{R}^n : \|x\| = 1\})$.

3 Fourier Transform in $\mathbb{R}$

From now on, we discuss the Fourier transform in $\mathbb{R}$. We establish several properties to be able to prove the Inversion Theorem. We study the operation of convolution and the Parseval-Plancherel Theorems.

3.1 Fourier Transform in $\mathscr{D}(\mathbb{R})$

When using the Fourier transform, it is assumed that the functions under consideration are complex-valued.
Let us consider an open set $\mho$ in $\mathbb{R}$. We work in the space $\mathcal{D}(\mho)$

Definition 3.1.1. *We define a function φ that is piece-wise continuous and has compact support.*
Consider the functions $\mathscr{F}\varphi$ and $\overline{\mathscr{F}}\varphi$ as follows:

$$\begin{cases} i)\ \mathscr{F}\varphi(\xi) = \int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(x)dx \\ ii)\ \overline{\mathscr{F}}\varphi(\xi) = \int_{\mathbb{R}} e^{2i\pi x\xi}\varphi(x)dx \end{cases} \tag{3.1.1}$$

$\mathscr{F}\varphi$ and $\overline{\mathscr{F}}\varphi$ are well-defined (and also convergent) because

$$\varphi(x)e^{-2i\pi x\xi} = \varphi(x)\cos(2\pi x\xi) - i\varphi(x)\sin(2\pi x\xi)$$

$$\begin{aligned} \mathscr{F}\varphi(\xi) &= \int_{\mathbb{R}} \varphi(x)\cos(2\pi x\xi) - i\int_{\mathbb{R}} \varphi(x)\sin(2\pi x\xi) \\ &= \Re[\mathscr{F}\varphi(\xi)] + \Im[\mathscr{F}\varphi(\xi)] \end{aligned}$$

$$supp(\Re(e^{-2i\pi x\xi}\varphi(x))) \subset supp(\varphi(x)) \tag{3.1.2}$$

$$supp(\Im(e^{-2i\pi x\xi}\varphi(x))) \subset supp(\varphi(x)) \tag{3.1.3}$$

*$\mathscr{F}\varphi(\xi)$ is referred to as **the Fourier Transform** of $\varphi(x)$.*

Example 3.1.1. *We shall now compute the Fourier transform of the function f defined as follows:*

$$f(x) = \begin{cases} b, & |x| \leq a \\ 0, & |x| > a \end{cases} \tag{3.1.4}$$

$$\begin{aligned}
\mathscr{F} f(\xi) &= \int_{\mathbb{R}} e^{-2i\pi x\xi} f(x)dx \\
&= \int_{-a}^{+a} e^{-2i\pi x\xi} b dx \\
&= \frac{b}{-2i\pi\xi} e^{-2i\pi x\xi}\Big|_{-a}^{+a} \\
&= \frac{-b}{2i\pi\xi}(e^{-2i\pi a\xi} - e^{2i\pi a\xi}) \\
&= \frac{-b}{2i\pi\xi}(-2i)\sin(2\pi a\xi) \\
&= \frac{b}{\pi\xi}\sin(2\pi a\xi).
\end{aligned}$$

Remark 3.1.1. *The Fourier transform is well-defined $\forall \varphi \in \mathcal{D}(\mathbb{R})$ as well.*

Proposition 3.1.1. *Let $\varphi \in \mathscr{D}(\mathbb{R})$.*
i) *The Fourier transform of φ is k-times differentiable for any $k \geq 1$. More specifically, if φ is k times continuously differentiable, then its Fourier transform $\mathscr{F}\varphi$ will also be k times continuously differentiable.*
ii)

$$\mathscr{F}(\varphi^{(k)})(\xi) = \mathscr{F}((\frac{\partial}{\partial x})^k \varphi(\xi)) = (2i\pi\xi)^k \mathscr{F}\varphi(\xi) \tag{3.1.5}$$

Proof. **i)**Since the functions $\xi \mapsto e^{-2i\pi x\xi}$ are infinitely differentiable, and the functions $x^k\varphi(x)$ are continuous and bounded on the compact support of φ, using the Lebesgue's Dominated Convergence Theorem (see [5]), Differentiating $\mathscr{F}\varphi(\xi)$ under the integral sign. This yields:

$$\begin{aligned}
(\frac{\partial}{\partial \xi})^k \int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(x)dx &= \int_{\mathbb{R}} (\frac{\partial}{\partial \xi})^k e^{-2i\pi x\xi}\varphi(x)dx \\
&= \int_{\mathbb{R}} [(-2i\pi x)^k \varphi(x)] e^{-2i\pi x\xi} dx
\end{aligned}$$

which is convergent because $x^k\varphi(x) \in \mathcal{D}(\mathbb{R})$.
ii) Let us calculate $\mathscr{F}(\frac{\partial}{\partial x})^k\varphi$. Firstly we calculate the partial derivative of the product

$\varphi(x)e^{-2i\pi\xi}$:

$$\frac{\partial}{\partial x}(\varphi(x)e^{-2i\pi x\xi}) = \varphi'(x)e^{-2i\pi x\xi} + (-2i\pi\xi)e^{-2i\pi x\xi}\varphi(x).$$

Using integration by parts, if we integrate both sides , we obtain:

$$\begin{aligned}\int_{\mathbb{R}} (\frac{\partial}{\partial x})\varphi(x))e^{-2i\pi x\xi}dx &= (2i\pi\xi)\int_{\mathbb{R}} \varphi(x))e^{-2i\pi x\xi}dx \\ &= (2i\pi\xi)\mathscr{F}\varphi(\xi).\end{aligned}$$

In other words, we can express it as follows:

$$(\mathscr{F}\varphi^{(1)})(\xi) = (2i\pi\xi)(\mathscr{F}\varphi(\xi)), \qquad \forall\varphi \in \mathscr{D}(\mathbb{R}). \tag{3.1.6}$$

If we repeat this process k times, we obtain:

$$\begin{aligned}(\mathscr{F}\varphi^{(2)})(\xi) &= (2i\pi\xi)(\mathscr{F}\varphi^{(1)})(\xi) \\ &= (2i\pi\xi)^2(\mathscr{F}\varphi(\xi)) \\ &\vdots\end{aligned}$$

$$(\mathscr{F}\varphi^{(k)})(\xi) = (2i\pi\xi)^k(\mathscr{F}\varphi(\xi)). \tag{3.1.7}$$

□

Proposition 3.1.2. *For all $k \geq 1$ and for all $\xi \in \mathbb{R}$, we have :*
i) $|2\pi\xi|^k|\mathscr{F}\varphi(\xi)| \leq \|\varphi^k\|_{\mathbf{L}^1} = \int_{\mathbb{R}} |(\frac{\partial}{\partial x})^{(k)}\varphi(x)|dx$
ii) $|(\frac{\partial}{\partial\xi})^k\mathscr{F}\varphi(\xi)| \leq \|(2i\pi x)^k\varphi(x)\|_{\mathbf{L}^1} = \int_{\mathbb{R}} |(2i\pi x)^k\varphi(x)|dx.$

Proof. **i)** We have:

$$\begin{aligned}|2\pi\xi|^k|\mathscr{F}\varphi(\xi)| &= |(2i\pi\xi)^k\int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(x)dx| \\ &= |(2i\pi\xi)^k(\mathscr{F}\varphi(\xi))| \\ &= |\mathscr{F}\varphi^{(k)}(\xi)|.\end{aligned}$$

This implies that:

$$
\begin{aligned}
|(\mathscr{F}\varphi^{(k)}(\xi)| &= \left|\int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi^{(k)}(x)dx\right| \\
&\leq \int_{\mathbb{R}} |e^{-2i\pi x\xi}|.|\varphi^{(k)}(x)|dx \\
&= \int_{\mathbb{R}} |\varphi^{(k)}(x)|dx \\
&= \|\varphi^{(k)}\|_{\mathbf{L}^1}.
\end{aligned}
$$

ii) Since $|e^{-2i\pi x\xi}| = 1$, we can deduce the following:

$$
\begin{aligned}
|(\frac{\partial}{\partial\xi})^{(k)}\mathscr{F}\varphi(\xi)| &= \left|\int_{\mathbb{R}}[(-2i\pi x)^k\varphi(x)]e^{-2i\pi x\xi}dx\right| \quad \text{by proposition 3.1.1} \\
&\leq \int_{\mathbb{R}} |(2i\pi x)^k\varphi(x)|dx \\
&\leq \|(2i\pi x)^k\varphi(x)\|_{\mathbf{L}^1}.
\end{aligned}
$$

□

3.2 Fourier Transform in the Schwartz Space $\mathscr{S}(\mathbb{R})$

Definition 3.2.1. *The space $\mathscr{S}(\mathbb{R})$ ($\subset C^\infty(\mathbb{R})$) consists of functions φ that are infinitely differentiable on $\mathbb{R}$ and satisfy the following conditions:*

$$
\textit{for any } k \geq 0, \qquad \sup_{x\in\mathbb{R}} |x^k\varphi(x)| < +\infty \tag{3.2.1}
$$

*$\mathscr{S}(\mathbb{R})$ is commonly referred to as **the space of rapidly decreasing functions**.*

Example 3.2.1. ***The Gaussian density:***
***The Gaussian density** function g is an element of $\mathscr{S}(\mathbb{R})$, which is defined as follows:*

$$
g(x) = e^{-\pi x^2}. \tag{3.2.2}
$$

This function belongs to the Schwartz space because it is infinitely differentiable and rapidly decreasing as $|x|$ tends to infinity.
Let us recall that:

$$
\int_{\mathbb{R}} e^{-\pi x^2}dx = 1. \tag{3.2.3}
$$

What we are thus proving is:
Let us denote $I = \int_{\mathbb{R}} e^{-\pi x^2} dx$ *then:*

$$\begin{aligned}
I^2 &= \left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right)\left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right)\\
&= \left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right)\left(\int_{\mathbb{R}} e^{-\pi y^2} dy\right)\\
&= \iint_{\mathbb{R}^2} e^{-\pi x^2} e^{-\pi y^2} dxdy\\
&= \iint_{\mathbb{R}^2} e^{-\pi(x^2+y^2)} dxdy\\
&= \int_0^{2\pi}\int_0^{\infty} re^{-\pi r^2} drd\theta.
\end{aligned}$$

By change of variable $u = \pi r^2$*, we obtain:*

$$\begin{aligned}
I^2 &= 2\pi \int_0^{\infty} e^{-u} r \frac{du}{2\pi r}\\
&= \int_0^{\infty} e^{-u} du\\
&= e^{-u}\Big|_0^{\infty}\\
&= 1.
\end{aligned}$$

3.2.1 Elementary Properties of the Fourier Transform

Proposition 3.2.1. *The Gaussian density is a fixed point of the Fourier transform. In other saying:*

$$\mathscr{F}(g(x)) = g(\xi). \tag{3.2.4}$$

Proof.

$$\begin{aligned}
\mathscr{F}(g(\xi)) &= \int_{\mathbb{R}} e^{-2i\pi x\xi} g(x)dx \\
&= \int_{\mathbb{R}} e^{-2i\pi x\xi} e^{-\pi x^2} dx \\
&= \int_{\mathbb{R}} e^{-\pi(2ix\xi + x^2)} dx \\
&= \int_{\mathbb{R}} e^{-\pi(x^2 + 2ix\xi - (i\xi)^2 + (i\xi)^2)} dx \\
&= \int_{\mathbb{R}} e^{-\pi((x+i\xi)^2 - (i\xi)^2)} dx \\
&= \int_{\mathbb{R}} e^{-\pi(x-i\xi)^2 - \pi\xi^2} dx \\
&= e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi(x-i\xi)^2} dx.
\end{aligned}$$

By using the Cauchy's integral theorem, by making the change of variable $y = x - i\xi$, and utilizing the fact that $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$, we obtain:

$$\begin{aligned}
\mathscr{F}(g(\xi)) &= e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi y^2} dy \\
&= e^{-\pi\xi^2}
\end{aligned}$$

□

Proposition 3.2.2. *For a function* φ *belongs to the Schwartz space , we define the translation operator* τ_a *as follows:*

$$\tau_a \varphi(\xi) = \varphi(\xi - a). \tag{3.2.5}$$

Then we have:

$$\mathscr{F}(\tau_a \varphi)(\xi) = e^{-2i\pi\xi a} \mathscr{F} \varphi(\xi). \tag{3.2.6}$$

Proof.

$$\begin{aligned}
\mathscr{F}(\tau_a\varphi)(\xi) &= \int_{\mathbb{R}} e^{-2i\pi x\xi}(\tau_a\varphi)(x)dx \\
&= \int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(x-a)dx \\
&= \int_{\mathbb{R}} e^{-2i\pi(y+a)\xi}\varphi(y)dy \quad \text{where } y = x-a \\
&= e^{-2i\pi a\xi}\int_{\mathbb{R}} e^{-2i\pi y\xi}\varphi(y)dy \\
&= e^{-2i\pi\xi a}\mathscr{F}\varphi(\xi).
\end{aligned}$$

□

Proposition 3.2.3. *For $\varphi \in \mathscr{S}(\mathbb{R})$, we define the dilation operation is defined in the following:*

$$\varphi_h(\xi) = \frac{1}{h}\varphi(\frac{\xi}{h}). \tag{3.2.7}$$

Then, we have:
i) $\mathscr{F}(\varphi_h)(\xi) = \mathscr{F}\varphi(h\xi)$
ii) $\mathscr{F}(\varphi(h\xi)) = (\mathscr{F}\varphi)_h(\xi)$.

Proof. **i)** Since

$$\mathscr{F}(\varphi_h)(\xi) = \int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi_h(x)dx,$$

we have:

$$\begin{aligned}
\mathscr{F}(\varphi_h)(\xi) &= \int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(\frac{x}{h})\frac{dx}{h} \\
&= \int_{\mathbb{R}} e^{-2i\pi y(h\xi)}\varphi(y)dy \quad \text{where } y = \frac{x}{h} \\
&= \mathscr{F}\varphi(h\xi).
\end{aligned}$$

ii) As

$$\mathscr{F}(\varphi(h\xi)) = \int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(hx)dx,$$

we have:

$$\begin{aligned}\mathscr{F}(\varphi(h\xi)) &= \int_{\mathbb{R}} e^{-2i\pi\frac{y}{h}\xi}\varphi(y)\frac{dy}{h} \quad \text{where } y = hx \\ &= \frac{1}{h}\mathscr{F}(\varphi(\frac{\xi}{h})) \\ &= \mathscr{F}(\varphi_h)(\xi).\end{aligned}$$

□

Definition 3.2.2. *Let* $\varphi, \psi \in \mathscr{S}(\mathbb{R})$. ***The convolution operation*** *is defined as follows:*

$$(\varphi * \psi)(x) = \int_{\mathbb{R}} \varphi(x-y)\psi(y)dy. \tag{3.2.8}$$

Lemma 3.2.4. *For two functions* φ *and* ψ *in* $\mathscr{S}(\mathbb{R})$*, we have:*

$$\varphi * \psi = \psi * \varphi. \tag{3.2.9}$$

Proof.

$$\begin{aligned}(\varphi * \psi)(x) &= \int_{-\infty}^{+\infty} \varphi(x-y)\psi(y)dy \\ &= -\int_{+\infty}^{-\infty} \varphi(z)\psi(x-z)dz \quad \text{where } z = x-y\ ,\ dz = -dy \\ &= \int_{-\infty}^{+\infty} \varphi(z)\psi(x-z)dz \\ &= (\psi * \varphi)(x).\end{aligned}$$

□

Theorem 3.2.5. ***(The convolution theorem for the Fourier transform states)***
Consider two functions φ *and* ψ *in* $\mathscr{S}(\mathbb{R})$*. Then the Fourier transform of their convolution is equal to the point-wise product of their Fourier transforms:*

$$\mathscr{F}(\varphi * \psi)(\xi) = \mathscr{F}\varphi(\xi)\,\mathscr{F}\psi(\xi). \tag{3.2.10}$$

Proof.

$$
\begin{aligned}
\mathscr{F}(\varphi * \psi)(\xi) &= \int_{\mathbb{R}} e^{-2i\pi x\xi}(\varphi * \psi)(x)dx \\
&= \int_{\mathbb{R}} e^{-2i\pi x\xi}\left(\int_{\mathbb{R}} \varphi(x-y)\psi(y)dy\right)dx \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2i\pi x\xi}\varphi(x-y)dx\right)\psi(y)dy \quad \text{by Fubini's Theorem} \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2i\pi(y+z)\xi}\varphi(z)dz\right)\psi(y)dy \quad \text{where } z = x-y \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2i\pi z\xi}\varphi(z)dz\right)e^{-2i\pi y\xi}\psi(y)dy \\
&= \left(\int_{\mathbb{R}} e^{-2i\pi z\xi}\varphi(z)dz\right)\left(\int_{\mathbb{R}} e^{-2i\pi y\xi}\varphi(y)dy\right) \quad \text{by Fubini's Theorem} \\
&= \mathscr{F}\varphi(\xi)\mathscr{F}\psi(\xi).
\end{aligned}
$$

□

See [5] for the Fubini's theorem.

3.2.2 The Inversion Theorem

We will now prove the fundamental theorem regarding the Fourier transform, which states that $\mathscr{F}$ is a bijective map from $\mathscr{S}(\mathbb{R})$ to itself, and its inverse is denoted as $\overline{\mathscr{F}}$.

Theorem 3.2.6. *The transformation $\overline{\mathscr{F}}$, which is a linear operator from $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}(\mathbb{R})$, is the inverse of the Fourier transformation $\mathscr{F}$. In other words, for any $\varphi \in \mathscr{S}(\mathbb{R})$, we have:*

$$\overline{\mathscr{F}}(\mathscr{F}\varphi)(x) = \varphi(x). \tag{3.2.11}$$

Proof. Let φ in $\mathscr{S}(\mathbb{R})$.We will show the theorem using the Gaussian density, which is $g(x) = e^{-\pi x^2}$.

$$\begin{aligned}
\int_{\mathbb{R}} e^{2i\pi a\xi}\mathscr{F}\varphi(\xi)g(h\xi)d\xi &= \int_{\mathbb{R}} \mathscr{F}(\tau_{-a}\varphi)(\xi)g(h\xi)d\xi \quad \text{by (3.2.6)}\\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2i\pi x\xi}(\tau_{-a}\varphi)(x)dx\right)g(h\xi)d\xi\\
&= \int_{\mathbb{R}}(\tau_{-a}\varphi)(x)\left(\int_{\mathbb{R}} e^{-2i\pi x\xi}g(h\xi)d\xi\right)dx \quad \text{by Fubini's Theorem}\\
&= \int_{\mathbb{R}}(\tau_{-a}\varphi)(x)\mathscr{F}(g(h\xi))(x)dx\\
&= \int_{\mathbb{R}}(\tau_{-a}\varphi)(x)(\mathscr{F} g)_h(x)dx \quad \text{by Proposition 3.2.3 ii)}\\
&= \int_{\mathbb{R}}(\tau_{-a}\varphi)(x)\frac{1}{h}(\mathscr{F} g)(\frac{x}{h})dx \quad \text{because } \varphi_h(\xi) = \frac{1}{h}\varphi(\frac{\xi}{h})\\
&= \int_{\mathbb{R}}(\tau_{-a}\varphi)(hz)(\mathscr{F} g)(z)dz \quad \text{where } z = \frac{x}{h}\\
&= \int_{\mathbb{R}}\varphi(hz+a)g(z)dz \quad \text{because } \tau_a\varphi(\xi) = \varphi(\xi-a)\text{, by Proposition 3.2.1}
\end{aligned}$$

We have :

$$\int_{\mathbb{R}} e^{2i\pi a\xi}\mathscr{F}\varphi(\xi)g(h\xi)d\xi = \int_{\mathbb{R}}\varphi(hz+a)g(z)dz.$$

By applying the Dominated Convergence Theorem of Lebesgue, as h approaches zero, we have:

$$\begin{aligned}
\int_{\mathbb{R}} e^{2i\pi a\xi}\mathscr{F}\varphi(\xi)g(0)d\xi &= \int_{\mathbb{R}}\varphi(a)g(z)dz\\
\iff \int_{\mathbb{R}} e^{2i\pi a\xi}\mathscr{F}\varphi(\xi)d\xi = \varphi(a) &\quad \text{because } g(0) = 1 \text{ and (3.2.3)}\\
&\Longrightarrow \overline{\mathscr{F}}(\mathscr{F}\varphi)(a) = \varphi(a).
\end{aligned}$$

□

Proposition 3.2.7. ***(Parseval-Plancherel)***.
For φ and ψ in $\mathscr{S}(\mathbb{R})$, where $\mathscr{F}$ denotes the Fourier transform and $(\varphi,\psi) = \int_{\mathbb{R}}\varphi(x)\overline{\psi}(x)dx$ represents the inner product, we have the following properties:

$$(\varphi,\psi) = (\mathscr{F}\varphi,\mathscr{F}\psi). \tag{3.2.12}$$

Furthermore, the Fourier transform defines an isometry with respect to the $\mathbf{L}^2$ norm from the space $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}(\mathbb{R})$.

Proof. The result is evident by the Inversion Theorem because:

$$(\varphi, \psi) = (\varphi, \overline{\mathscr{F}}(\mathscr{F}\,\psi)) = (\mathscr{F}\,\varphi, \mathscr{F}\,\psi). \tag{3.2.13}$$

And we have:

$$\|\varphi\|_{\mathbf{L}^2} = (\varphi, \varphi) = (\mathscr{F}\,\varphi, \mathscr{F}\,\varphi) = \|\mathscr{F}\,\|_{\mathbf{L}^2}. \tag{3.2.14}$$

$\square$

4 Fourier Transform in $\mathbb{R}^n$

In this chapter, we discuss the Fourier transform in $\mathbb{R}^n$. We examine some inequalities using the $\mathbf{L}^1$ norm. We study the isomorphism of the Fourier transform on the space of infinitely differentiable functions with rapid decay.

4.1 Fourier Transform in $\mathbf{L}^1(\mathbb{R}^n)$

Let $\mho$ be an open set in $\mathbb{R}^n$. The space $\mathbf{L}^1(\mho)$ is a vector space defined by:

$$\mathbf{L}^1(\mho) := \{\varphi : \mho \to \mathbb{R} \,\text{such that } |\varphi| \text{ be integrable}\}. \tag{4.1.1}$$

Thus, if $\varphi \in \mathbf{L}^1(\mho)$ then $\varphi_+(x) = max\{\varphi(x), 0\}$ and $\varphi_-(x) = -min\{\varphi(x), 0\}$ are integrable and we have then:

$$\int_\mho \varphi(x)dx = \int_\mho \varphi_+(x)dx - \int_\mho \varphi_-(x)dx. \tag{4.1.2}$$

The vector space $\mathbf{L}^1(\mho)$ is therefore a complete space equipped with the norm

$$|\varphi|_{\mathbf{L}^1} = \int_\mho |\varphi(x)|dx. \tag{4.1.3}$$

Definition 4.1.1. *Let $\varphi \in \mathbf{L}^1(\mathbb{R}^n)$. We define the functions $\mathscr{F}\varphi$ and $\overline{\mathscr{F}}\varphi$ as follows:*

$$\begin{cases} i)\,\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx \\ ii)\,\overline{\mathscr{F}}\varphi(\xi) = \int_{\mathbb{R}^n} e^{2i\pi<x,\xi>}\varphi(x)dx \end{cases} \tag{4.1.4}$$

where $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n)$ and

$$< x, \xi >= \sum_{i=1}^n x_i\xi_i. \tag{4.1.5}$$

*We call $\mathscr{F}\varphi$ and $\overline{\mathscr{F}}\varphi$ **the direct and inverse Fourier transforms**, respectively.*

Definition 4.1.2. *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$. The function*

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{R} \\ (x,y) &\mapsto \varphi(x)\varphi(y) \end{aligned} \tag{4.1.6}$$

is infinitely differentiable and has compact support. This function is ***the tensor product*** *of the functions φ and ψ, denoted by $\varphi \otimes \psi$.*

$$supp(\varphi \otimes \psi) = supp(\varphi) \times supp(\psi) \subset \mathbb{R}^{n+m}. \tag{4.1.7}$$

Proposition 4.1.1. *Suppose that $n = p+q$, where $p,q \in \mathbb{R}$, and that φ is the tensor product of the functions φ_1 and φ_2. Then the Fourier transform of φ is the tensor product of the Fourier transforms of φ_1 and φ_2.*

Proof. Let $x = (x_1, x_2) \in \mathbb{R}^n$ where $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^q$.
Let $\varphi_1 \in \mathcal{D}(\mathbb{R}^p)$ and $\varphi_2 \in \mathcal{D}(\mathbb{R}^q)$. We suppose that :

$$\varphi(x_1,x_2) = \varphi_1(x_1) \otimes \varphi_2(x_2). \tag{4.1.8}$$

Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n$ where $\xi_1 \in \mathbb{R}^p$ and $\xi_2 \in \mathbb{R}^q$. Our aim is to demonstrate that:

$$\mathscr{F}\,\varphi(\xi_1,\xi_2) = \mathscr{F}\,\varphi_1(\xi_1) \otimes \mathscr{F}\,\varphi_2(\xi_2).$$

$$\begin{aligned} \mathscr{F}\,\varphi(\xi) &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx \\ &= \int_{\mathbb{R}^n} e^{-2i\pi\sum_{i=1}^n x_i\xi_i}\varphi(x)dx \\ &= \int_{\mathbb{R}^n} e^{-2i\pi(\sum_{i=1}^p x_i\xi_i + \sum_{j=p+1}^n x_j\xi_j)}\varphi(x)dx \\ &= \int_{\mathbb{R}^n} e^{-2i\pi(<x_1,\xi_1>+<x_2,\xi_2>)}\varphi(x)dx \\ &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^q} e^{-2i\pi<x_1,\xi_1>}e^{-2i\pi<x_2,\xi_2>}\varphi_1(x_1)\varphi_2(x_2)dx_1dx_2 \\ &= \left(\int_{\mathbb{R}^p} e^{-2i\pi<x_1,\xi_1>}\varphi_1(x_1)dx_1\right)\left(\int_{\mathbb{R}^q} e^{-2i\pi<x_2,\xi_2>}\varphi_2(x_2)dx_2\right) \quad \text{by Fubini's Theorem} \\ &= \mathscr{F}\,\varphi_1(\xi_1) \otimes \mathscr{F}\,\varphi_2(\xi_2). \end{aligned}$$

□

Example 4.1.1. ***The Gaussian density***

The Gaussian density *function g is a function in $\mathscr{S}(\mathbb{R}^n)$ and is defined as:*

$$g(x) = e^{-\pi\|x\|^2}. \tag{4.1.9}$$

It has been proven that the Fourier transform of the function $x \mapsto e^{-\pi|x|^2}$ is itself, as stated in Proposition 3.3. Therefore, we have:

$$e^{-\pi\|x\|^2} = \int_{\mathbb{R}} e^{-2i\pi x\xi} e^{-\pi x^2} dx = \mathscr{F}\, g(\xi).$$

Since $g(x) = e^{-\pi\|x\|^2}$, $x \in \mathbb{R}^n$ and

$$\begin{aligned} e^{-\pi\|x\|^2} &= e^{-\pi(x_1^2+\cdots+x_n^2)} \\ &= e^{-\pi x_1^2} \dots e^{-\pi_n^2} \\ &= \prod_{i=1}^{n} e^{-\pi x_i^2}. \end{aligned}$$

By the previous proposition, we have:

$$\begin{aligned} Fg(\xi) &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} g(x) dx \\ &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} e^{-\pi\|x\|^2} dx \\ &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} (\prod_{i=1}^{n} e^{-\pi x_i^2}) dx \\ &= Fg_1(\xi_1) \dots Fg_n(\xi_n). \end{aligned}$$

where $g(\xi) = g_1(\xi_1)\dots g_n(\xi_n)$ and the function $g_i : x_i \longmapsto e^{-\pi x_i^2}$. Therefore, we have:

$$\begin{aligned} \mathscr{F}(e^{-\pi\|x\|^2})(\xi) &= \mathscr{F}(e^{-\pi x_1^2}) \dots \mathscr{F}(e^{-\pi x_n^2}) \\ &= e^{-\pi\xi_1^2} \dots e^{-\pi\xi_n^2} \\ &= e^{-\pi\|\xi\|^2}. \end{aligned}$$

This tells us that the Gaussian density is a fixed point of the Fourier transform in $\mathscr{S}(\mathbb{R}^n)$.

Proposition 4.1.2. *Let φ be in $\mathbf{L}^1(\mathbb{R}^n)$ such that the function $x = (x_1, \dots, x_n) \mapsto x_i\varphi(x)$ is also in $\mathbf{L}^1(\mathbb{R}^n)$. Then $\mathscr{F}\,\varphi$ has a partial derivative with respect to ξ_i that is continuous, bounded, and we have:*

i) $\mathscr{F}(-2\pi x_i\varphi) = \frac{\partial}{\partial\xi_i}\mathscr{F}\,\varphi$.

ii) $\|\frac{\partial}{\partial\xi_i}\mathscr{F}\,\varphi\|_\infty \leq \|2\pi x_i\varphi\|_{\mathbf{L}^1}$.

Proof. **i)** We denote $\tilde{\varphi} := -2\pi x_i\varphi$ and we calculate the Fourier transform of $\tilde{\varphi}$.

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\tilde{\varphi}(x)dx &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}(-2\pi x_i)\varphi(x)dx \\
&= \int_{\mathbb{R}^n} e^{-2i\pi(x_1\xi_1+\cdots+x_i\xi_i+\cdots+x_n\xi_n)}(-2\pi x_i)\varphi(x)dx \\
&= \int_{\mathbb{R}^n} (\frac{\partial}{\partial\xi_i})(e^{-2i\pi(x_1\xi_1+\cdots+x_i\xi_i+\cdots+x_n\xi_n)}\varphi(x))dx \\
&= (\frac{\partial}{\partial\xi_i})\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx \quad \text{by Lebesgue Theorem} \\
&= (\frac{\partial}{\partial\xi_i})\mathscr{F}\,\varphi(\xi).
\end{aligned}$$

ii) As $\|\frac{\partial}{\partial\xi_i}\mathscr{F}\,\varphi(\xi)\|_\infty = \sup_{\xi\in\mathbb{R}^n}|\frac{\partial}{\partial\xi_i}\mathscr{F}\,\varphi(\xi)|$ and $|e^{-2\pi x_i\xi}| = 1$, we have:

$$\begin{aligned}
|\frac{\partial}{\partial\xi_i}\mathscr{F}\,\varphi(\xi)| &= \mathscr{F}(-2\pi x_i\varphi) \\
&= \left|\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}(-2\pi x_i)\varphi(x)dx\right| \\
&\leq \int_{\mathbb{R}^n} |e^{-2i\pi<x,\xi>}||(-2\pi x_i)\varphi(x)|dx \\
&\leq \int_{\mathbb{R}^n} |2\pi x_i\varphi(x)|dx \\
&= \|2\pi x_i\varphi\|_{\mathbf{L}^1}.
\end{aligned}$$

□

Definition 4.1.3. *Let us consider an open set $\mho$ of $\mathbb{R}^n$. D is a linear operator from $\mathcal{D}(\mho)$ to $\mathcal{D}(\mho)$ defined by:*

$$D^\alpha = D_1^{\alpha_1}\ldots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\ldots\partial\xi_n^{\alpha_n}}, \tag{4.1.10}$$

where $|\alpha| = \alpha_1+\cdots+\alpha_n$.

Proposition 4.1.3. *Let $\varphi \in \mathbf{L}^1(\mathbb{R}^n)$. The function $x \mapsto x^\alpha\varphi(x)$ belongs to $\mathbf{L}^1(\mathbb{R}^n)$ such that :*
i) *$D^\alpha\mathscr{F}\,\varphi = \mathscr{F}((-2i\pi x)^\alpha\varphi)$,*
ii) *$|D^\alpha\mathscr{F}\,\varphi(\xi)| \leq \|(2i\pi x)^\alpha\varphi\|_{\mathbf{L}^1}$.*

Proof. **i)** We can differentiate the Fourier transform of φ under the integral sign using the Dominated Convergence Theorem of Lebesgue (See [5]), and we obtain:

$$\begin{aligned}
D^{\alpha}\mathscr{F}\varphi(\xi) &= D^{\alpha}\left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx\right)\\
&= \int_{\mathbb{R}^n}\left(\frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\ldots\partial\xi_n^{\alpha_n}}e^{-2i\pi(x_1\xi_1+\cdots+x_n\xi_n)}\varphi(x)\right)dx\\
&= \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial\xi_1^{\alpha_1}}e^{-2i\pi x_1\xi_1}\ldots\frac{\partial}{\partial\xi_n^{\alpha_n}}e^{-2i\pi x_n\xi_n}\right)\varphi(x)dx\\
&= \int_{\mathbb{R}^n}(-2i\pi x_1)^{\alpha_1}\ldots(-2i\pi x)^{\alpha_n}e^{-2i\pi<x,\xi>}\varphi(x)dx\\
&= \int_{\mathbb{R}^n}(-2i\pi x)^{\alpha}e^{-2i\pi<x,\xi>}\varphi(x)dx\\
&= \mathscr{F}((-2i\pi x)^{\alpha}\varphi).
\end{aligned}$$

ii) As $|e^{-2i\pi<x,\xi>}|=1$, we obtain:

$$\begin{aligned}
|D^{\alpha}\mathscr{F}\varphi(\xi)| &= |\mathscr{F}((-2i\pi x)^{\alpha}\varphi)(\xi)|\\
&= \left|\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}(-2i\pi x)^{\alpha}\varphi(x)dx\right|\\
&= \int_{\mathbb{R}^n}|e^{-2i\pi<x,\xi>}(-2i\pi x)^{\alpha}\varphi(x)|dx\\
&\le \int_{\mathbb{R}^n}|(2i\pi x)^{\alpha}\varphi(x)|dx\\
&= \|(2i\pi x)^{\alpha}\varphi\|_{\mathbf{L}^1}.
\end{aligned}$$

□

Proposition 4.1.4. *Let* φ *a function integrable.If the partial derivative of* φ *with respect to* x_i *belongs to* $\mathbf{L}^1(\mathbb{R}^n)$ *so the function* $\xi_i\mathscr{F}\varphi$ *is continuous, bounded, and we have:*
i) $\mathscr{F}(\frac{\partial\varphi}{\partial x_i}) = (2i\pi\xi_i)\mathscr{F}\varphi$.
ii) $\|2\pi\xi_i\mathscr{F}\varphi\|_{\infty} \le \|\frac{\partial\varphi}{\partial x_i}\|_{\mathbf{L}^1}$.

Proof. **i)** We have:

$$\begin{aligned}
\mathscr{F}(\frac{\partial\varphi}{\partial x_i})(\xi) &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}(\frac{\partial\varphi(x)}{\partial x_i})dx.\\
&= \int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}(e^{-2i\pi(x_1\xi_1+\cdots+x_i\xi_i+\cdots+x_n\xi_n)}\varphi(x))dx
\end{aligned}$$

By applying integration by parts, we obtain:

$$
\begin{aligned}
\mathscr{F}(\frac{\partial\varphi}{\partial x_i})(\xi) &= \left(\int_{\mathbb{R}^{n-1}} \varphi(x)e^{-2i\pi<x,\xi>}\right)\Big|_{x_i=-\infty}^{x_i=+\infty} - \int_{-\infty}^{+\infty}(-2i\pi\xi_i)e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= (2i\pi\xi_i)\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= (2i\pi\xi_i)\mathscr{F}\varphi(\xi).
\end{aligned}
$$

ii) Since $|e^{-2i\pi<x,\xi>}| = 1$, we obtain:

$$
\begin{aligned}
|2\pi\xi_i\mathscr{F}\varphi(\xi)| &= |\mathscr{F}(\frac{\partial\varphi}{\partial x_i})(\xi)| \quad \text{par i)} \\
&= \left|\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}(\frac{\partial\varphi(x)}{\partial x_i})dx\right| \\
&\leq \int_{\mathbb{R}^n}|e^{-2i\pi<x,\xi>}(\frac{\partial\varphi(x)}{\partial x_i})|dx \\
&= \int_{\mathbb{R}^n}|(\frac{\partial\varphi(x)}{\partial x_i})|dx \\
&\leq \|\frac{\partial\varphi}{\partial x_i}\|_{\mathbf{L}^1}.
\end{aligned}
$$

□

Proposition 4.1.5. *Let $\lambda \in \mathbb{R}-\{0\}$, φ and f two functions in $\mathbf{L}^1(\mathbb{R}^n)$ such that $\varphi : x \mapsto f(\lambda x)$. Then the Fourier transform of φ is the function $\xi \mapsto \frac{1}{|\lambda|^n}\mathscr{F} f(\frac{\xi}{\lambda})$.*

Proof. We have:

$$
\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}f(\lambda x)dx. \tag{4.1.11}
$$

By changing variables with $\lambda x = y$, we obtain:

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}f(\lambda x)dx &= \int_{\mathbb{R}^n} e^{-2i\pi<\frac{y}{\lambda},\xi>}f(y)\frac{dy}{|\lambda|^n} \\
&= \frac{1}{|\lambda|^n}\int_{\mathbb{R}^n} e^{-2i\pi<y,\frac{\xi}{\lambda}>}f(y)dy \\
&= \frac{1}{|\lambda|^n}\mathscr{F} f(\frac{\xi}{\lambda}).
\end{aligned}
$$

□

Proposition 4.1.6. *Let $\varphi^* : x \mapsto \overline{\varphi(-x)}$ a function in $\mathbf{L}^1(\mathbb{R}^n)$. Then $\mathscr{F}\varphi^*$ is the complex conjugate of the Fourier transform of $\varphi \in \mathbf{L}^1(\mathbb{R}^n)$. In other words, we have:*

$$
\mathscr{F}\varphi^*(\xi) = \overline{\mathscr{F}\varphi(\xi)}. \tag{4.1.12}
$$

Proof. Since
$$\int_{-\infty}^{+\infty}\varphi(-x)dx = -\int_{+\infty}^{-\infty}\varphi(y)dy = \int_{-\infty}^{+\infty}\varphi(y)dy,$$
we have:

$$\begin{aligned}
\mathscr{F}\,\varphi^*(\xi) &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi^*(x)dx \\
&= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\overline{\varphi(-x)}dx \\
&= \int_{\mathbb{R}^n} \overline{e^{2i\pi<x,\xi>}\varphi(-x)}dx \\
&= \overline{\int_{\mathbb{R}^n} e^{2i\pi<x,\xi>}\varphi(-x)dx} \\
&= \overline{\mathscr{F}\,\varphi(\xi)}.
\end{aligned}$$

□

Proposition 4.1.7. ***i)*** *Let $\varphi \in \mathbf{L}^1(\mathbb{R}^n)$. So we have:*

$$\mathscr{F}(\tau_a\varphi)(\xi) = e^{-2i\pi<a,\xi>}\mathscr{F}\,\varphi(\xi), \tag{4.1.13}$$

where $\tau_a\varphi(\xi) = \varphi(\xi - a)$.
ii) *Let f and $\varphi : x \longmapsto e^{2i\pi<a,x>}f(x)$ in $\mathbf{L}^1(\mathbb{R}^n)$. So we have:*

$$\mathscr{F}\,\varphi(\xi) = \tau_a\mathscr{F}\,f(\xi). \tag{4.1.14}$$

Proof. **i)** Indeed,

$$\begin{aligned}
\mathscr{F}(\tau_a\varphi)(\xi) &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}(\tau_a\varphi)(x)dx \\
&= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x-a)dx \\
&= \int_{\mathbb{R}^n} e^{-2i\pi<y+a,\xi>}\varphi(y)dy \quad \text{where } x-a=y \\
&= e^{-2i\pi<a,\xi>}\int_{\mathbb{R}^n} e^{-2i\pi<y,\xi>}\varphi(y)dy \quad \text{because } <.,.> \text{ is bilinear.} \\
&= e^{-2i\pi<a,\xi>}\mathscr{F}\,\varphi(\xi).
\end{aligned}$$

ii) We have:

$$\begin{aligned}
\mathscr{F}\,\varphi(\xi) &= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}e^{2i\pi<a,\xi>}f(x)dx.
\end{aligned}$$

As $< .,. >$ is bilinear, we have then:

$$\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} e^{-2i\pi<x,-a>} f(x)dx = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi-a>} f(x)dx$$
$$= \mathscr{F} f(\xi - a)$$
$$= \tau_a \mathscr{F} f(\xi).$$

$\square$

Theorem 4.1.8. *The Fourier transform is an algebra homomorphism and a continuous mapping from the space $\mathbf{L}^1(\mathbb{R}^n)$ to the space $C^0(\mathbb{R}^n)$ of continuous functions on $\mathbb{R}^n$ that tend to zero at infinity. For $\varphi, \psi \in \mathbf{L}^1(\mathbb{R}^n)$, we have:*
i) $\mathscr{F}(\varphi * \psi) = \mathscr{F}\varphi F\psi$.
ii) $\|\mathscr{F}\varphi\|_\infty \leq \|\varphi\|_{\mathbf{L}^1}$.
Furthermore, the image of $\mathbf{L}^1(\mathbb{R}^n)$ is dense in $C^0(\mathbb{R}^n)$.

Proof. **i)**First, let's show that the Fourier transform is an algebra homomorphism.
Let φ and ψ be in $\mathbf{L}^1(\mathbb{R}^n)$. We calculate the Fourier transform of $\varphi * \psi$.

$$\mathscr{F}(\varphi * \psi)(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} (\varphi * \psi)(x)dx$$
$$= \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} \left(\int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy \right) dx. \quad \text{by definition 3.2.2}$$

According to Fubini's theorem, we can interchange the order of integration. By changing the variable $x = y + z$, we have:

$$\mathscr{F}(\varphi * \psi)(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi<y+z,\xi>} \left(\int_{\mathbb{R}^n} \varphi(z)\psi(y)dy \right) dx$$
$$= \left(\int_{\mathbb{R}^n} e^{-2i\pi<z,\xi>} \varphi(z)dz \right) \left(\int_{\mathbb{R}^n} e^{-2i\pi<y,\xi>} \psi(y)dy \right)$$
$$= \mathscr{F}\varphi \mathscr{F}\psi.$$

Now, let's verify the continuity of the Fourier transform.

ii) Since $|e^{-2i\pi<x,\xi>}|=1$, on a:

$$
\begin{aligned}
\|\mathscr{F}\varphi(\xi)\|_\infty &= \sup_{\xi\in\mathbb{R}^n}|\mathscr{F}\varphi(\xi)| \\
&= \left|\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx\right| \\
&\le \int_{\mathbb{R}^n} |e^{-2i\pi<x,\xi>}\varphi(x)|dx \\
&= \int_{\mathbb{R}^n} |\varphi(x)|dx \\
&= \|\varphi\|_{\mathbf{L}^1}.
\end{aligned}
$$

□

Lemma 4.1.9. ***(Riemann-Lebesgue)***
Let A be a subset of $\mathbb{R}^n$. If $\varphi\in A$, of complex values, then

$$\lim_{\|\xi\|\to\infty}\int_A e^{-2i\pi<\xi,x>}\varphi(x)dx=0. \tag{4.1.15}$$

Proof. We utilize the theorem for the integrable function $1_A\varphi$. □

4.2 Fourier Transform of an Infinitely Differentiable Function with Rapid Decay

Definition 4.2.1. *$\mathscr{S}(\mathbb{R}^n)$ is the space of infinitely differentiable functions on $\mathbb{R}^n$. A function φ in $\mathscr{S}(\mathbb{R}^n)$ is said to have rapid decay if for any integers m and p, The following condition holds:*

$$N_{m,p}(\varphi)=\sup_{x\in\mathbb{R}^n;|\alpha|\le m}\|x\|^p|D^\alpha\varphi(x)|<+\infty \tag{4.2.1}$$

where

$$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n),\quad |\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n \tag{4.2.2}$$

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n}}. \tag{4.2.3}$$

*$N_{m,p}$ is a **semi-norm** for $\mathscr{S}(\mathbb{R}^n)$. That is, for any $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ and any complex scalar λ, $N_{m,p}$ satisfies the following properties:*
i) $0 \leq N_{m,p} < +\infty$,
ii) $N_{m,p}(\varphi+\psi) \leq N_{m,p}(\varphi) + N_{m,p}(\psi)$,
iii) $N_{m,p}(\lambda\varphi) = |\lambda| N_{m,p}(\varphi)$.

Theorem 4.2.1. *The Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$. The inverse transform is given by:*

$$\varphi(x) = \int_{\mathbb{R}^n} e^{2i\pi<\xi,x>} \mathscr{F}\varphi(\xi) d\xi. \tag{4.2.4}$$

Proof. The Fourier transform being linear, to show that the transform is injective, i.e., $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ such that $\mathscr{F}\varphi = \mathscr{F}\psi$, it is sufficient to demonstrate that $\mathscr{F}\varphi = 0$ implies $\varphi = 0$. Since $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we can apply Theorem 3.8 and obtain $\varphi = 0$.
We can verify the inverse transform using the Gaussian density defined as $g(x) = e^{-\pi x^2}$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $\varepsilon > 0$. We define the following function:

$$\varphi_\varepsilon(x) := \int_{\mathbb{R}^n} g(\varepsilon\xi) e^{2i\pi<x,\xi>} \mathscr{F}\varphi(\xi) d\xi. \tag{4.2.5}$$

Then we have:

$$\begin{aligned}
\varphi_\varepsilon(x) &= \int_{\mathbb{R}^n} g(\varepsilon\xi) e^{2i\pi<x,\xi>} \left(\int_{\mathbb{R}^n} e^{-2i\pi<y,\xi>} \varphi(y) dy \right) d\xi \\
&= \int_{\mathbb{R}^n} \varphi(y) \left(\int_{\mathbb{R}^n} g(\varepsilon\xi) e^{-2i\pi<y-x,\xi>} d\xi \right) dy \quad \text{by the Fubini's Theorem} \\
&= \int_{\mathbb{R}^n} \varphi(y) \varepsilon^{-n} (\mathscr{F} g)(\frac{y-x}{\varepsilon}) dy \quad \text{by Proposition 4.1.5}
\end{aligned}$$

We can make a change of variable using $y - x = z\varepsilon$;

$$\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi(x+z\varepsilon) \mathscr{F} g(z) dz = \int_{\mathbb{R}^n} \varphi(x+z\varepsilon) g(z) dz \quad \text{because } \mathscr{F} g(z) = g(z)$$

When ε tends to zero, the limit of φ_ε is:

$$\begin{aligned}
\lim_{\varepsilon\to 0} \varphi_\varepsilon(x) &= \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^n} \varphi(x+z\varepsilon) g(z) dz \right) \\
&= \int_{\mathbb{R}^n} \left(\lim_{\varepsilon\to 0} [\varphi(x+z\varepsilon) g(z)] \right) dz \quad \text{by Lebesgue Theorem, by examples 3.2.1 and 4.1.1} \\
&= \varphi(x).
\end{aligned}$$

□

In $\mathscr{S}(\mathbb{R}^n)$, the Fourier transform possesses the following properties:

Proposition 4.2.2. *For any $\alpha, \beta \in \mathbb{N}^n$ and any $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, we have:*
i)

$$(-2i\pi\xi)^{\beta}(D^{\alpha}\mathscr{F}\varphi)(\xi) = \mathscr{F}(D^{\beta}((-2i\pi x)^{\alpha}\varphi(x)))(\xi). \tag{4.2.6}$$

Note that D^{α} represents the multi-index notation for partial derivatives, where α is a multi-index consisting of non-negative integers.
ii)

$$\int_{\mathbb{R}^n} \mathscr{F}\varphi(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^n} \mathscr{F}\psi(x)\varphi(x)dx. \tag{4.2.7}$$

iii)

$$\int_{\mathbb{R}^n} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{R}^n} \mathscr{F}\varphi(\xi)\overline{\mathscr{F}\psi(\xi)}d\xi. \tag{4.2.8}$$

iv) (Parseval's Theorem) *The Fourier transform preserves the $\mathbf{L}^2$ norm. In other words*

$$\|\varphi\|_{\mathbf{L}^2} = \|\mathscr{F}\varphi\|_{\mathbf{L}^2}. \tag{4.2.9}$$

Proof. **i)** We can apply the Fourier transform to the partial derivative $D^{\alpha}\varphi$.

$$\begin{aligned}
D^{\alpha}(\mathscr{F}\varphi(\xi)) &= D^{\alpha}\left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx\right) \\
&= \int_{\mathbb{R}^n} \frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\dots\partial\xi_n^{\alpha_n}} e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= \int_{\mathbb{R}^n} (-2i\pi x_1)^{\alpha_1}\dots(-2i\pi x_n)^{\alpha_n} e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= \int_{\mathbb{R}^n} (-2i\pi x)^{\alpha} e^{-2i\pi<x,\xi>}\varphi(x)dx.
\end{aligned}$$

If we multiply by $(2i\pi\xi)^\beta$, we obtain:

$$\begin{aligned}
(2i\pi\xi)^\beta \int_{\mathbb{R}^n} (-2i\pi x)^\alpha e^{-2i\pi<x,\xi>} \varphi(x)dx &= \int_{\mathbb{R}^n} (2i\pi\xi)^\beta (-2i\pi x)^\alpha e^{-2i\pi<x,\xi>} \varphi(x)dx \\
&= \int_{\mathbb{R}^n} (2i\pi\xi_1)^{\beta_1} \dots (2i\pi\xi_n)^{\beta_n} (-2i\pi x)^\alpha e^{-2i\pi<x,\xi>} \varphi(x)dx \\
&= \int_{\mathbb{R}^n} \frac{\partial^{|\beta|}}{\partial\xi_1^{\beta_1} \dots \partial\xi_n^{\beta_n}} (-2i\pi x)^\alpha e^{-2i\pi<x,\xi>} \varphi(x)dx \\
&= \int_{\mathbb{R}^n} D^\beta \left((-2i\pi x)^\alpha e^{-2i\pi<x,\xi>} \varphi(x) \right) dx \\
&= \mathscr{F}(D^\beta (-2i\pi x)^\alpha \varphi(x))(\xi).
\end{aligned}$$

ii) We have:

$$\int_{\mathbb{R}^n} \mathscr{F}\, \varphi(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} \varphi(x)dx \right) \psi(\xi)d\xi. \tag{4.2.10}$$

Since the function $(x,\xi) \mapsto e^{-2i\pi<x,\xi>}\varphi(x)\psi(\xi)$ is integrable with respect to $dxd\xi$, we can interchange the order of integration using Fubini's theorem. Thus, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathscr{F}\, \varphi(\xi)\psi(\xi)d\xi &= \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>} \psi(\xi)d\xi \right) dx \\
&= \int_{\mathbb{R}^n} \varphi(x)\, \mathscr{F}\, \psi(x)dx.
\end{aligned}$$

iii) As we have previously observed, the inner product in the space $\mathscr{S}(\mathbb{R}^n)$ is defined as follows:

$$(\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(x)\overline{\psi}(x)dx. \tag{4.2.11}$$

By writing the form of the inner product, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi(x)\overline{\psi}(x)dx &= (\varphi, \psi) \\
&= (\varphi, \overline{\mathscr{F}}(\mathscr{F}\, \psi)) \\
&= (\mathscr{F}\, \varphi, \mathscr{F}\, \psi) \quad \text{by Inversion Theorem} \\
&= \int_{\mathbb{R}^n} \mathscr{F}\, \varphi(\xi)\overline{\mathscr{F}\, \psi(\xi)}d\xi.
\end{aligned}$$

iv) As $\|\varphi\|_{\mathbf{L}^2}^2 = \int_{\mathbb{R}^n} \varphi(x)\overline{\varphi}(x)dx$, the proposition 4.2.2 ii) implies that:

$$\|\varphi\|_{\mathbf{L}^2}^2 = \int_{\mathbb{R}^n} \mathscr{F}\,\varphi(\xi)\overline{\mathscr{F}\,\varphi}(\xi)d\xi = \|\mathscr{F}\,\varphi\|_{\mathbf{L}^2}^2.$$

□

Theorem 4.2.3. *Let $\varphi * \psi \in \mathscr{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. So we have:*

$$\mathscr{F}(\varphi * \psi)(\xi) = \mathscr{F}\,\varphi(\xi)\,\mathscr{F}\,\psi(\xi) \tag{4.2.12}$$

and

$$\mathscr{F}(\varphi\psi)(\xi) = \mathscr{F}\,\varphi(\xi) * \mathscr{F}\,\psi(\xi). \tag{4.2.13}$$

Proof. We have already shown that the Fourier transform is an algebra homomorphism in Theorem 3.17. Therefore, the first equality is proven. For the second equality, we can utilize certain properties and the Fubini's theorem. By the definition of the convolution operation, we have:

$$\begin{aligned}
(\mathscr{F}\,\varphi * \mathscr{F}\,\psi)(\xi) &= \int_{\mathbb{R}^n} \mathscr{F}\,\varphi(\xi-\eta)\,\mathscr{F}\,\psi(\eta)d\eta \\
&= \int_{\mathbb{R}^n} \left[\left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi-\eta>}\varphi(x)dx\right)\left(\int_{\mathbb{R}^n} e^{-2i\pi<y,\eta>}\psi(y)dy\right)\right]d\eta \\
&= \int_{\mathbb{R}^n} \left[\left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>+<x,-\eta>}\varphi(x)dx\right)\left(\int_{\mathbb{R}^n} e^{-2i\pi<y,\eta>}\psi(y)dy\right)\right]d\eta \\
&= \int_{\mathbb{R}^n} \left[\left(\int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx\right)\left(\int_{\mathbb{R}^n} e^{-2i\pi<y-x,\eta>}\psi(y)dy\right)\right]d\eta.
\end{aligned}$$

Thus;

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2i\pi<y-x,\eta>}\psi(y)dy &= e^{2i\pi<x,\eta>}\int_{\mathbb{R}^n} e^{-2i\pi<y,\eta>}\psi(y)dy \\
&= e^{2i\pi<x,\eta>}\,\mathscr{F}\,\psi(\eta).
\end{aligned}$$

By the inversion theorem, we have the following implication:

$$\int_{\mathbb{R}^n} e^{2i\pi<x,\eta>}\,\mathscr{F}\,\psi(\eta) = \overline{\mathscr{F}}(\mathscr{F}\,\psi)(x) = \psi(x). \tag{4.2.14}$$

Therefore, we have:

$$(\mathscr{F}\,\varphi * \mathscr{F}\,\psi)(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)\psi(x)dx = \mathscr{F}(\varphi\psi)(\xi).$$

$\square$

4.3 The Extension of The Fourier Transform to Space $\mathbf{L}^2(\mathbb{R}^n)$

In this chapter, we will now extend the Fourier transform from the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ to the space $\mathbf{L}^2(\mathbb{R}^n)$.

Theorem 4.3.1. ***(Plancherel-Riesz)***
There exists a unique transformation from $\mathbf{L}^2(\mathbb{R}^n)$ to $\mathbf{L}^2(\mathbb{R}^n)$ that coincides with the Fourier transform on $\mathbf{L}^2(\mathbb{R}^n) \cap \mathbf{L}^1(\mathbb{R}^n)$.
We will still call it ***The Fourier Transform****. This transformation possesses the following properties:*
i) *For any $\varphi \in \mathbf{L}^2(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$,*

$$\mathscr{F}(\mathscr{F}\varphi)(x) = \varphi(-x). \tag{4.3.1}$$

ii) *For any $\varphi \in \mathbf{L}^2(\mathbb{R}^n)$,*

$$\|\mathscr{F}\varphi\|_{\mathbf{L}^2} = \|\varphi\|_{\mathbf{L}^2}. \tag{4.3.2}$$

iii) *For any $\varphi, \psi \in \mathbf{L}^2(\mathbb{R}^n)$, we have:*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} \mathscr{F}\varphi(\xi)\overline{\mathscr{F}\psi}(\xi)d\xi = \int_{\mathbb{R}^n} \varphi(x)\overline{\psi}(x)dx \\ \\ \displaystyle\int_{\mathbb{R}^n} |\mathscr{F}\varphi(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\varphi(x)|^2 dx. \end{cases} \tag{4.3.3}$$

iv) *For $\varphi \in \mathbf{L}^2(\mathbb{R}^n)$ and for every positive real number R, we define:*

$$\mathscr{F}\varphi_R(\xi) = \int_{\|x\| \le R} e^{-2i\pi<x,\xi>}\varphi(x)dx \tag{4.3.4}$$

$$\overline{\mathscr{F}}(\mathscr{F}\varphi_R)(x) = \int_{\|\xi\| \le R} e^{2i\pi<\xi,x>}(\mathscr{F}\varphi_R)(\xi)d\xi \tag{4.3.5}$$

then $\|F\varphi_R - F\varphi\|_{\mathbf{L}^2} \to 0$ and $\|\overline{\mathscr{F}}(\mathscr{F}\varphi_R) - \varphi\|_{\mathbf{L}^2} \to 0$.

Proof. As the space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions is dense in $\mathbf{L}^2(\mathbb{R}^n)$ and considering $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ as a unitary map according to

the extension theorem, there exists a unique continuous linear map
$\mathscr{F} : \mathbf{L}^2(\mathbb{R}^n) \to \mathbf{L}^2(\mathbb{R}^n)$such that $\mathscr{F}$ is the Fourier transformation.
i) We consider a sequence $\varphi_k \in \mathscr{S}(\mathbb{R}^n)$ that converges to φ in the sense of $\mathbf{L}^2(\mathbb{R}^n)$.

$$\begin{aligned}
\mathscr{F}(\mathscr{F}\varphi)(x) &= \mathscr{F}(\mathscr{F}(\lim_{k\to+\infty}\varphi_k))(x) \\
&= \lim_{k\to+\infty}\mathscr{F}(\mathscr{F}\varphi_k)(x) \\
&= \lim_{k\to+\infty}\varphi_k(-x) \\
&= \varphi(-x).
\end{aligned}$$

ii) Since $\mathscr{F}$ and $\|.\|_{\mathbf{L}^2}$ are continuous, we have:

$$\begin{aligned}
\|\mathscr{F}\varphi\|_{\mathbf{L}^2} &= \|\lim_{k\to+\infty}\mathscr{F}\varphi_k\|_{\mathbf{L}^2} \\
&= \lim_{k\to+\infty}\|\mathscr{F}\varphi_k\|_{\mathbf{L}^2} \\
&= \lim_{k\to+\infty}\|\varphi_k\|_{\mathbf{L}^2} \\
&= \|\lim_{k\to+\infty}\varphi_k\|_{\mathbf{L}^2} \\
&= \|\varphi\|_{\mathbf{L}^2}.
\end{aligned}$$

iii) Indeed, we have shown that the Fourier transformation in $\mathscr{S}(\mathbb{R}^n)$ is an onto isometry from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$ with respect to the norm induced by $\mathbf{L}^2(\mathbb{R}^n)$. This isometry can be uniquely extended to an isometry from $\mathbf{L}^2(\mathbb{R}^n)$ to $\mathbf{L}^2(\mathbb{R}^n)$.
iv) According to (iii), the function $\varphi_R = \varphi\,\mathbb{1}_{\|x\|\leq R}$ converges to φ, and the function
$\mathscr{F}\varphi_R = \mathscr{F}\varphi\,\mathbb{1}_{\|\xi\|\leq R}$ converges to $\mathscr{F}\varphi$ when tends to infinity. So, as $\mathscr{F}\varphi_R$ converges to $\mathscr{F}\varphi$, and correspondingly, $\overline{\mathscr{F}}(\mathscr{F}\varphi_R)$converges to $\overline{\mathscr{F}}(\mathscr{F}\varphi) = \varphi$ in $\mathbf{L}^2(\mathbb{R}^n)$. $\square$

5 Fourier Transform on $\mathbf{T}^n$

Definition 5.0.1. *The n-dimensional torus, or n-torus, is the Cartesian product of the unit circle* $\mathbf{S^1}$ *with itself n times in a topological space. It is denoted by* $\mathbf{T}^n$*.*

$$\mathbf{T}^n = \underbrace{\mathbf{T}^1 \times \cdots \times \mathbf{T}^1}_{n\ times} \tag{5.0.1}$$

For $n = 1$*, we obtain the circle* $\mathbf{T}^1 = \mathbf{S}^1$*. It is known that* $\mathbf{T}^1$ *is isomorphic to* $\mathbb{R}/\mathbb{Z}$*, the quotient of the real line by the integers.*

$$\forall x, y \in \mathbf{T}^1, x \sim y \Leftrightarrow x \equiv y(mod\mathbb{Z}). \tag{5.0.2}$$

5.1 Fourier Transform in $\mathbf{L}^1(\mathbf{T}^n)$

Definition 5.1.1. *For any function* f *in* $\mathbf{L}^1(\mathbf{T}^n)$*, it is true that* f *is periodic in each variable. That is, for any* $(x_1, \ldots, x_n) \in \mathbb{R}^n$ *and any* $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$*, we have:*

$$f(x_1 + \mu_1, \ldots, x_n + \mu_n) = f(x_1, \ldots, x_n). \tag{5.1.1}$$

Definition 5.1.2. *Let* φ *be a function belonging to* $\mathbf{L}^1(\mathbf{T})^n$*. The Fourier transform of* φ *is defined as follows:*

$$\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx. \tag{5.1.2}$$

We call the Fourier series of φ *"the formal Fourier series" of* φ*.*

$$\sum_{\xi \in \mathbb{Z}^n} \mathscr{F}\varphi(\xi)e^{2i\pi<x,\xi>}. \tag{5.1.3}$$

Remark 5.1.1. *For* $n = 1$*, when* φ *is a real-valued function, we define:*

$$a_n = \int_0^1 \varphi(t)\cos ntdt \qquad n \leq 0, \tag{5.1.4}$$

$$b_n = \int_0^1 \varphi(t)\sin ntdt \qquad n \leq 1. \tag{5.1.5}$$

and the associated Fourier series, denoted $\mathscr{S}_f$, is given by:

$$\mathscr{S}_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{5.1.6}$$

Proposition 5.1.1. *Let $x \in \mathbf{T}^n$. We have:*

$$\mathscr{F}(\frac{\partial \varphi}{\partial x_i}) = (2i\pi\xi_i)\mathscr{F}\varphi. \tag{5.1.7}$$

Proof. We have:

$$\begin{aligned}
\mathscr{F}(\frac{\partial \varphi}{\partial x_i})(\xi) &= \int_{\mathbf{T}^n} e^{-2i\pi<x,\xi>}(\frac{\partial \varphi(x)}{\partial x_i})dx \\
&= \int_{\mathbf{T}^n} \frac{\partial}{\partial x_i} e^{-2i\pi(x_1\xi_1+\cdots+x_i\xi_i+\cdots+x_n\xi_n)}\varphi(x)dx.
\end{aligned}$$

By integrating by parts, we obtain:

$$\begin{aligned}
\mathscr{F}(\frac{\partial \varphi}{\partial x_i})(\xi) &= \left(\int_{\mathbf{T^{n-1}}} \varphi(x)e^{-2i\pi<x,\xi>}\right)\Big|_{x_i=-\infty}^{x_i=+\infty} - \int_{-\infty}^{+\infty}(-2i\pi\xi_i)e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= (2i\pi\xi_i)\int_{\mathbf{T}^n} e^{-2i\pi<x,\xi>}\varphi(x)dx \\
&= (2i\pi\xi_i)\mathscr{F}\varphi(\xi).
\end{aligned}$$

□

Proposition 5.1.2. *The Fourier transform of a translated function has the following expression. For $a \in \mathbf{T}^n$ and $\xi \in \mathbb{Z}^n$.*

$$\mathscr{F}(\tau_a\varphi)(\xi) = e^{-2i\pi<a,\xi>}\mathscr{F}\varphi(\xi) \quad \textit{where} \quad \tau_a\varphi(\xi) = \varphi(\xi - a). \tag{5.1.8}$$

Proof. We establish the following equality:

$$\begin{aligned}
\mathscr{F}(\tau_a\varphi)(\xi) &= \int_{\mathbf{T}^n} e^{-2i\pi<x,\xi>}(\tau_a\varphi)(x)dx \\
&= \int_{\mathbf{T}^n} e^{-2i\pi<x,\xi>}\varphi(x-a)dx \\
&= \int_{\mathbf{T}^n} e^{-2i\pi<y+a,\xi>}\varphi(y)dy \quad \text{where } x-a=y \\
&= e^{-2i\pi<a,\xi>}\int_{\mathbf{T}^n} e^{-2i\pi<y,\xi>}\varphi(y)dy \quad \text{as } <.,.> \text{ is bilinear} \\
&= e^{-2i\pi<a,\xi>}\mathscr{F}\varphi(\xi).
\end{aligned}$$

□

Proposition 5.1.3. *Let* $\varphi \in \mathbf{L}^1(\mathbf{T}^n)$. *We have:*

$$\mathscr{F}(e^{2i\pi<x,k>}\varphi(x))(\xi) = \mathscr{F}\varphi(\xi - k). \tag{5.1.9}$$

Proof.

$$\begin{aligned}
\mathscr{F}(e^{2i\pi<x,k>}\varphi(x))(\xi) &= \int_{\mathbf{T}^n} e^{-2i\pi<x,\xi>}e^{2i\pi<x,k>}\varphi(x)dx \\
&= \int_{\mathbf{T}^n} e^{-2i\pi<x,\xi-k>}\varphi(x)dx \\
&= \mathscr{F}\varphi(\xi - k).
\end{aligned}$$

□

5.2 Fourier Transform in $\mathbf{L}^2(\mathbf{T}^n)$

Theorem 5.2.1. ***(Riesz)*** *The Fourier transform on the torus* $\mathbf{T}^n$ *is an isomorphism of* $\mathbf{L}^2(\mathbf{T}^n)$ *and* $\mathbf{l}^2(\mathbb{Z}^n)$. *We have:*

$$\begin{cases} \int_{\mathbf{T}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} = \sum_{m\in\mathbb{Z}^n} f(m)\overline{g(m)} & \forall f,g \in \mathbf{l}^2(\mathbb{Z}^n) \\ \int_{\mathbf{T}^n} |\widehat{f}(\xi)|^2 = \sum_{m\in\mathbb{Z}^n} |f(m)|^2 & \forall f \in \mathbf{l}^2(\mathbb{Z}^n) \end{cases} \tag{5.2.1}$$

where

$$\widehat{f}(m) = \int_{\mathbf{T}^n} e^{-2i\pi<m,t>} f(t)dt, \tag{5.2.2}$$

and we have:

$$\widehat{f_N}(t) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m|\leq N}} e^{-2i\pi mt} f(m) \longmapsto f \quad in \quad \mathbf{L}^2(\mathbf{T}^n). \tag{5.2.3}$$

Proof. We will prove it in three steps:
i. Let us consider $\varphi \in \mathbf{L}^2(\mathbf{T}^n)$ such that $\widehat{\varphi}(m) = \int_{\mathbf{T}^n} e^{-2i\pi<m,t>}\varphi(t)dt$ for all m in $\mathbb{Z}^n$. Now, let P be a trigonometric polynomial with finitely many terms. For any trigonometric polynomial P,

$$P_N(t) = \sum_{k\in\mathbb{Z}^n\cap P} P_k(t)e^{2i\pi kt}, \quad \text{we have} \quad \int_{\mathbf{T}^n} \varphi(t)\overline{P_N(t)}dt = 0.$$

This tells us that φ is orthogonal to P_N. The set of trigonometric polynomials $P_N(t)_{N\in\mathbb{N}}$

is dense in $\mathcal{C}^0(\mathbf{T}^n)$. We know that $\|\varphi\|_2 \leq \|\varphi\|_\infty$, then $P_N(t)_N \in \mathbb{N}$ is dense in the space $\mathbf{L}^2(\mathbf{T}^n)$.
for any function $\varphi \in \mathcal{C}^0(\mathbf{T}^n)$, there exists a sequence of trigonometric polynomials $P_N(t)_{N\in\mathbb{N}}$ such that $P_N(t)$ converges uniformly to φ on $\mathbf{T}^n$. Let $P_N(t) \longmapsto \varphi(t)$ in $\mathbf{L}^2(\mathbf{T}^n)$.

$$\int_{\mathbf{T}^n} |\varphi(t)|^2 dt = \lim_{N\mapsto\infty} \int_{\mathbf{T}^n} \varphi(t)(\overline{\varphi}(t) - \overline{P_N}(t))dt = 0 \tag{5.2.4}$$

Because

$$\left| \int_{\mathbf{T}^n} \varphi(t)(\overline{\varphi}(t) - \overline{P_N}(t))dt = 0 \right| \leq \|\varphi\|_{\mathbf{L}^2(\mathbf{T}^n)} \|\varphi - P_N\|_{\mathbf{L}^2(\mathbf{T}^n)} \tag{5.2.5}$$

Therefore $\varphi(t) = 0\ \mathbf{L}^2(\mathbf{T}^n)$.
This shows that if $\widehat{\varphi}(m) = 0$ for all $m \in \mathbb{Z}^n$, then $\varphi(t) = 0$ in $\mathbf{L}^2(\mathbf{T}^n)$.
ii. Let $\varphi \in \mathbf{L}^2(\mathbf{T}^n)$. By using the definition of the norm in $\mathbf{L}^2$ and equation (33), we obtain the following equality:

$$\begin{aligned} \|\varphi_N\|_{\mathbf{L}^2}^2 &= \int_{\mathbf{T}^n} \varphi_N(t)\overline{\varphi_N(t)}dt \\ &= \sum_{|m|\leq N} |\widehat{\varphi}(m)|^2 \int_{\mathbf{T}^n} e^{-2i\pi mt} e^{2i\pi mt} dt \\ &= \sum_{|m|\leq N} |\widehat{\varphi}(m)|^2. \end{aligned}$$

By the triangle inequality, we have:

$$0 \leq \ \|\varphi_N - \varphi\|_{\mathbf{L}^2}^2 \leq \|\varphi_N\|_{\mathbf{L}^2}^2 - \|\varphi\|_{\mathbf{L}^2}^2$$

$$\Rightarrow \|\varphi\|_{\mathbf{L}^2}^2 \leq \|\varphi_N\|_{\mathbf{L}^2}^2$$
$$\Rightarrow \|\varphi\|_{\mathbf{L}^2}^2 \leq \sum_{|m|\leq N} |\widehat{\varphi}(m)|^2.$$

By (i), we know that the system of functions $e^{-2i\pi<m,t>} : m \in \mathbb{Z}^n$ is an orthonormal complete system. Therefore, we obtain the Bessel's inequality:

$$\sum_{|m|\leq N} |\widehat{\varphi}(m)|^2 \leq \|\varphi\|_{\mathbf{L}^2}^2. \tag{5.2.6}$$

We have:

$$\|\varphi\|_{\mathbf{L}^2}^2 = \sum_{|m|\leq N} |\widehat{\psi}(m)|^2. \tag{5.2.7}$$

This shows that the Fourier transform of φ belongs to $\mathbf{l}^2(\mathbb{Z}^n)$.
iii. Let $\psi_N \in \mathbf{L}^2(\mathbf{T}^n)$ be such that for all m in $\mathbb{Z}^n$,

$$\psi_N(m) = \sum_{|m|>N} e^{-2i\pi mt}\widehat{\varphi}(m). \tag{5.2.8}$$

We have then:

$$g(t) := \psi_N + \varphi_N = \sum_{m\in\mathbb{Z}^n} e^{-2i\pi mt}\widehat{\varphi}(m). \tag{5.2.9}$$

Thus

$$\{\widehat{\psi}_N(m)\}_{m\in\mathbb{Z}^n} \in \mathbf{l^2}(\mathbb{Z}^n), \quad \text{i.e.:} \quad \sum_{|m|>N} |\widehat{\psi}(m)|^2 < +\infty$$

We have then:

$$\sum_{|m|\leq N} |\widehat{\varphi}(m)|^2 + \sum_{|m|>N} |\widehat{\psi}(m)|^2 < +\infty \tag{5.2.10}$$

When $N \to 0$,

$$\begin{aligned} \sum_{|m|\leq N} |\widehat{\varphi}(m)|^2 &\to 0 \\ \sum_{|m|>N} |\widehat{\psi}(m)|^2 &\to 0 \\ \Rightarrow \sum_{|m|\leq N} |\widehat{\varphi}(m)|^2 + \sum_{|m|>N} |\widehat{\psi}(m)|^2 &\to 0 \end{aligned}$$

Therefore, the series $g(t)$ converges in $\mathbf{L}^2(\mathbf{T}^n)$. We can thus write the following implications:

$$\begin{aligned} \widehat{g}(m) &= \widehat{\varphi}(m). \\ \Rightarrow \widehat{(g-\varphi)}(m) &= 0 \quad \forall m \in \mathbb{Z}^n. \\ \Rightarrow g - \varphi &= 0 \quad \text{in} \quad \mathbf{L}^2(\mathbf{T}^n). \end{aligned}$$

The relation shows that we indeed have an isometry. $\square$

6 Hadamard Finite Part

6.1 Hadamard Finite Part

The Cauchy Principal Value integral does not work for a higher singularity. For instance, consider $\varphi(x) = 1$ and $y = 0$ in

$$\int_{a_1}^{a_2} \frac{\varphi(x)}{(x-y)^2} \quad a_1 < x < a_2, \tag{6.1.1}$$

that is,

$$\int_{a_1}^{a_2} \frac{dx}{x^2} \quad a_1 < x < a_2.$$

The integral is not convergent, neither does the principal value exist. Since

$$\int_{[a_1,a_2]\setminus(-\varepsilon,\varepsilon)} \frac{dx}{x^2} = \int_{a}^{-\varepsilon} \frac{dx}{x^2} + \int_{\varepsilon}^{a_2} \frac{dx}{x^2} = \frac{1}{a_1} - \frac{1}{a_2} + \frac{2}{\varepsilon},$$

$pv \int_{a_1}^{a_2} \frac{dx}{x^2} = \lim_{\varepsilon\to 0}(\frac{1}{a_1} - \frac{1}{a_2} + \frac{2}{\varepsilon})$ is not finite. Hadamard finite part integral is defined by disregarding the finite part $\frac{2}{\varepsilon}$, and keeping the finite part, i.e.

$$Pf \int_{a_1}^{a_2} \frac{dx}{x^2} = \frac{1}{a_1} - \frac{1}{a_2}.$$

Definition 6.1.1. *Let $\varepsilon > 0$ and denote*

$$F(\varepsilon, a_0) = \int_{[a_1,a_2]\setminus(a_0-\varepsilon,a_0+\varepsilon)} f(x,a_0)dx \quad a_1 < a_0 < a_2, \tag{6.1.2}$$

where the singularity appears at the point $x = a_0$. If $F(\varepsilon, a_0)$ is decomposed into

$$F(\varepsilon, a_0) = F_0(\varepsilon, a_0) + F_1(\varepsilon, a_0)$$

and

$$\lim_{\varepsilon\to 0} F_0(\varepsilon, a_0) < \infty \quad \lim_{\varepsilon\to 0} F_1(\varepsilon, a_0) = \infty,$$

then the finite part integral is defined by keeping "the finite part", i.e.:

$$Pf\int_{a_1}^{a_2} f(x,a_0)dx = \lim_{\varepsilon\to 0} F_0(\varepsilon,a_0) \tag{6.1.3}$$

$$= \lim_{\varepsilon\to 0}[F(\varepsilon,a_0) - F_1(\varepsilon,a_0)]. \tag{6.1.4}$$

Example 6.1.1. *Finite part of* $\frac{1}{x^2}$ *on* $[-1,1]$*:*
We know that $\varphi(x) = \varphi(0) + \varphi'(0) + \varphi''(0) + \dots$*, then we have:*

$$\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x^2}dx = \int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(0)}{x^2}dx + \int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{x\varphi'(0)}{x^2}dx + \dots$$

If we solve in two parts:

$$\begin{aligned}
\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(0)}{x^2}dx &= \int_{-1}^{-\varepsilon} \frac{\varphi(0)}{x^2}dx + \int_{\varepsilon}^{1} \frac{\varphi(0)}{x^2}dx \\
&= -2\varphi(0) + \frac{2\varphi(0)}{\varepsilon} \\
\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{x\varphi'(0)}{x^2}dx &= \int_{-1}^{-\varepsilon} \frac{\varphi'(0)}{x}dx + \int_{\varepsilon}^{1} \frac{\varphi'(0)}{x}dx \\
&= \varphi'(0)\left(\ln|x|\Big|_{-1}^{-\varepsilon} + \ln|x|\Big|_{\varepsilon}^{1}\right) \\
&= 0.
\end{aligned}$$

The divergent part of $< \frac{1}{x^2}, \varphi >$ *is equals to* $\frac{2\varphi(0)}{\varepsilon}$ *and we can define the distribution finite part of* $\frac{1}{x^2}$ *as the following limit:*

$$< Pf\frac{1}{x^2}, \varphi >= \lim_{\varepsilon\to 0}\left(\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^2}dx - \frac{2\varphi(0)}{\varepsilon}\right) = -2\varphi(0). \tag{6.1.5}$$

Note that: $Pf\int_{-1}^{1}\frac{dx}{x^2}$ *is the finite part of* $\lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon}\frac{dx}{x^2} + \int_{\varepsilon}^{1}\frac{dx}{x^2}\right) = \lim_{\varepsilon\to 0}\left(\frac{2}{\varepsilon} - 2\right)$.
Therefore$Pf\frac{1}{x^2}$ *is equals to the distribution* $-2\delta(x)$.

Example 6.1.2. *Finite part of* $\frac{1}{x^3}$ *on* $[-1,1]$*:*
We have:

$$\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x^3}dx = \int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(0)}{x^3}dx + \int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{x\varphi'(0)}{x^3}dx + \int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{x^2\varphi''(0)}{x^3}dx + \dots.$$

If we solve in tree parts:

$$\begin{aligned}
\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(0)}{x^3}dx &= \int_{-1}^{-\varepsilon} \frac{\varphi(0)}{x^3}dx + \int_{\varepsilon}^{1} \frac{\varphi(0)}{x^3}dx \\
&= \left[\frac{-1}{2x^2}\Big|_{-1}^{-\varepsilon} + \frac{-1}{2x^2}\Big|_{\varepsilon}^{1} \right] = 0 \\
\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{x\varphi'(0)}{x^3}dx &= \int_{-1}^{-\varepsilon} \frac{\varphi'(0)}{x^2}dx + \int_{\varepsilon}^{1} \frac{\varphi'(0)}{x^2}dx \\
&= -2\varphi'(0) + \frac{2\varphi'(0)}{\varepsilon} \\
\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} \frac{x^2\varphi''(0)}{x^3}dx &= \int_{-1}^{-\varepsilon} \frac{\varphi''(0)}{x}dx + \int_{\varepsilon}^{1} \frac{\varphi''(0)}{x}dx \\
&= \varphi''(0) \left(\ln|x| \Big|_{-1}^{-\varepsilon} + \ln|x| \Big|_{\varepsilon}^{1} \right) \\
&= 0.
\end{aligned}$$

The divergent part of $< \frac{1}{x^3}, \varphi >$ *is equals to* $\frac{2\varphi'(0)}{\varepsilon}$ *and we can defined the distribution finite part of* $\frac{1}{x^3}$ *as the following limit:*

$$< Pf\frac{1}{x^3}, \varphi >= \lim_{\varepsilon \to 0} \left(\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^3}dx - \frac{2\varphi'(0)}{\varepsilon} \right). \tag{6.1.6}$$

6.2 Cauchy Principal Values Integrals

6.2.1 The Principal value of $\frac{1}{x}$

The function $f : x \longmapsto \frac{1}{x}$ is not locally integrable. It do not define a distribution. However, taking care to avoid the singularity at 0 and by deleting a symmetrical integration with respect to 0, we will still be able to associate a distribution to f.

Let $\varphi \in C_0^\infty(\mathbb{R}) = \mathscr{D}(\mathbb{R})$.

$$
\begin{aligned}
< pv\frac{1}{x}, \varphi > = pv\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}dx &:= \left(\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx\right) \\
&= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx\right) \\
&= \lim_{\varepsilon\to 0}\left(\int_{\varepsilon}^{\infty} \frac{\varphi(-x)}{-x}dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx\right) \\
&= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}dx.
\end{aligned}
$$

As $x \to 0$, the limit of $\frac{\varphi(x)-\varphi(-x)}{x}$ is $2\varphi'(0)$:

$$
\begin{aligned}
\lim_{x\to 0}\frac{\varphi(x)-\varphi(-x)}{x} &= \lim_{x\to 0}\left(\frac{\varphi(x)-\varphi(0)}{x} + \frac{\varphi(-x)-\varphi(0)}{-x}\right) \\
&= \varphi'(0)+\varphi'(0) \\
&= 2\varphi'(0).
\end{aligned}
$$

Remark 6.2.1. *The distribution $pv\frac{1}{x}$ is a distribution of order 1 on $\mathbb{R}$. Since $\lim_{x\to 0}\frac{\varphi(x)-\varphi(-x)}{x} = 2\varphi'(0)$, for any $\varphi \in \mathscr{D}(\mathbb{R})$, we can write:*

$$
\frac{\varphi(x)-\varphi(-x)}{x} \leq 2\sup|\varphi'|. \tag{6.2.1}
$$

This show that $\int_0^\infty \frac{\varphi(x)-\varphi(-x)}{x}dx$ exists. Suppose that $supp\varphi \subset [-a,a]$.

$$
\begin{aligned}
\left|< pv\frac{1}{x}, \varphi >\right| &= \left|\lim_{\varepsilon\to 0}\int_{|x|\leq\varepsilon} \frac{\varphi(x)}{x}dx\right| \\
&= \left|\frac{1}{2}\int_{-a}^{a} \frac{\varphi(x)-\varphi(-x)}{x}dx\right| \\
&\leq \frac{1}{2}\int_{-a}^{a} \left|\frac{\varphi(x)-\varphi(-x)}{x}\right|dx \\
&\leq \frac{1}{2}\int_{-a}^{a} 2\sup|\varphi'|dx \\
&= 2a\sup|\varphi'|
\end{aligned}
$$

$$
\Longrightarrow \left|< pv\frac{1}{x}, \varphi >\right| \leq 2a\sup|\varphi'|.
$$

Thus $pv\frac{1}{x}$ is a distribution of order at most 1. We need to show that $pv\frac{1}{x}$ is not of order 0.

Exercise 6.2.1. *Show that $pv\frac{1}{x}$ is an odd distribution.*

Proof.

$$
\begin{aligned}
< pv\frac{1}{-x},\varphi > &= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{-x}dx+\int_{\varepsilon}^{\infty}\frac{\varphi(x)}{-x}dx\right)\\
&= \lim_{\varepsilon\to 0}\left(\int_{\varepsilon}^{\infty}-\frac{\varphi(-x)}{-x}dx+\int_{\varepsilon}^{\infty}\frac{\varphi(x)}{-x}dx\right)\\
&= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\varphi(x)-\varphi(-x)}{-x}dx\\
&= -< pv\frac{1}{x},\varphi > .
\end{aligned}
$$

$\Longrightarrow pv\frac{1}{x}$ is odd. ☐

Exercise 6.2.2. *Show that $xpv\frac{1}{x}=1$ and so $supp(pv\frac{1}{x})=\mathbb{R}$.*

Proof. Let $\varphi\in\mathscr{D}(\mathbb{R})$, we have:

$$
\begin{aligned}
< xpv\frac{1}{x},\varphi > &=< pv\frac{1}{x},x\varphi >\\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\varphi(x)dx\\
&= \int_{\mathbb{R}}\varphi(x)dx\\
&=< 1,\varphi >\\
\Longrightarrow xpv\frac{1}{x} &= 1.
\end{aligned}
$$

☐

Exercise 6.2.3. *Show that $\frac{\partial}{\partial x}\ln|x| = pv\frac{1}{x}$.*

Proof. Let $\varphi\in\mathscr{D}(\mathbb{R})$ and $\ln|x|\in\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$.

$$
\begin{aligned}
< \frac{\partial}{\partial x}\ln|x|,\varphi > &= -< \ln|x|,\varphi'(x) >\\
&= -\int_{\mathbb{R}}\ln|x|\varphi'(x)dx\\
&= -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\ln|x|\varphi'(x)dx
\end{aligned}
$$

By integration by part, we have:

$$\begin{aligned}
\int_{|x|\geq\varepsilon} \ln|x|\varphi'(x)dx &= \int_{-\infty}^{-\varepsilon} \ln(-x)\varphi'(x)dx + \int_{\varepsilon}^{\infty} \ln(x)\varphi'(x)dx \\
&= (\ln(-x)\varphi(x)|_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} \ln(-x)'\varphi(x)dx + (\ln(x)\varphi(x)|_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty} \ln(x)'\varphi(x)dx \\
&= \ln\varepsilon\varphi(-\varepsilon) - \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}dx - \ln\varepsilon\varphi(\varepsilon) - \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx \\
&= \ln\varepsilon(\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx.
\end{aligned}$$

Then,

$$\begin{aligned}
< \frac{\partial}{\partial x}\ln|x|, \varphi > &= -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \ln|x|\varphi'(x)dx \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \left[-\ln\varepsilon(\varphi(-\varepsilon) - \varphi(\varepsilon)) + \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx\right] \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx \\
&= < pv\frac{1}{x}, \varphi > .
\end{aligned}$$

This is what we want to prove. □

Exercise 6.2.4. *Show that* $xPf\frac{1}{x^2} = pv\frac{1}{x}$ *and* $x^2Pf\frac{1}{x^2} = \{1\}$.

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R})$.

$$\begin{aligned}
< xPf\frac{1}{x^2}, \varphi > &= < Pf\frac{1}{x^2}, x\varphi > \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{x\varphi(x)}{x^2}dx \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx \\
&= < pv\frac{1}{x}, \varphi > . \\
\Longrightarrow xPf\frac{1}{x^2} &= pv\frac{1}{x}.
\end{aligned}$$

$$
\begin{aligned}
< x^2 Pf\frac{1}{x^2}, \varphi > &=< Pf\frac{1}{x^2}, x^2\varphi > \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{x^2\varphi(x)}{x^2}dx \\
&=< 1, \varphi > \\
\implies x^2 Pf\frac{1}{x^2} = \{1\}.
\end{aligned}
$$

□

Exercise 6.2.5. *Find the Fourier transform of the distribution $pv\frac{1}{x}$.*

Proof. Firstly note that $xpv\frac{1}{x} = 1$ and $\mathscr{F}(\delta) = 1$. Let $\varphi \in \mathscr{D}(\mathbb{R})$.

$$
\begin{aligned}
< \delta, \varphi >= \varphi(0) &= \mathscr{F}^{-1}(\mathscr{F}(\varphi(0))) \\
&= \int_{\mathbb{R}} e^{2i\pi 0x}\mathscr{F}(\varphi(x))dx \\
&= \int_{\mathbb{R}} \mathscr{F}(\varphi(x))dx \\
&=< 1, \mathscr{F}(\varphi(x)) > \\
&=< \mathscr{F}(1), \varphi > \\
\implies \delta = \mathscr{F}(1).
\end{aligned}
$$

We want to find $\mathscr{F}(pv\frac{1}{x})$. Beside, we know that:

$$xpv\frac{1}{x} = 1 \implies \mathscr{F}(xpv\frac{1}{x}) = \mathscr{F}(1) = \delta.$$

Then we can conclude distribution of δ,

$$
\begin{aligned}
<\delta,\varphi> = <\mathscr{F}(1),\varphi> &= <\mathscr{F}(xpv\frac{1}{x}),\varphi> \\
&= <xpv\frac{1}{x},\mathscr{F}(\varphi)> \\
&= <pv\frac{1}{x},x\mathscr{F}(\varphi)> \\
&= <pv\frac{1}{x},x\frac{F(\varphi')}{2i\pi x}> \quad \text{since} \quad \mathscr{F}(\varphi(x)) = \frac{1}{(2\pi i x)^k}\mathscr{F}(\varphi^k(x)) \\
&= <\frac{1}{2i\pi}\mathscr{F}(pv\frac{1}{x}),\varphi'> \\
&= <-\frac{1}{2i\pi}\frac{\partial}{\partial x}(\mathscr{F}(pv\frac{1}{x})),\varphi>
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \delta = -\frac{1}{2i\pi}\frac{\partial}{\partial x}(\mathscr{F}(pv\frac{1}{x})) \\
&\Longrightarrow -2i\pi\delta = \frac{\partial}{\partial x}(\mathscr{F}(pv\frac{1}{x})) \\
&\Longrightarrow \mathscr{F}(pv\frac{1}{x}) = -2i\pi H + c_0.
\end{aligned}
$$

where c_0:constant and H is Heaviside function. Note that: $H' = \delta$. □

6.3 Application Hadamard Finite Part to Differential Equation

In the previous sections, we examined test functions and their properties, and performed operations on them. In this section, we will investigate the solutions of differential equations in the distribution space. Here, the finite part and Heaviside functions that we have learned before will be helpful in this regard.

Let us recall that $\mathscr{D}$ is the space of C^∞ functions with compact support on $\mathbb{R}$ and $\mathscr{D}'$ is the space of distribution on $\mathbb{R}$.

For any function $\varphi \in \mathscr{D}$ of support in $[-a,a)$, we define the functional u by:

$$<u,\varphi> = \int_0^a \frac{\varphi(x)-\varphi(0)}{x}dx + \varphi(0)\log a. \tag{6.3.1}$$

We will note

$$u = Pf\frac{H(x)}{x}. \tag{6.3.2}$$

where H is the function Heaviside.

Let us start by verifying the linearity of u. Let $\varphi_1, \varphi_2 \in \mathscr{D}$, and let α be a constant. Then we have:

$$\begin{aligned}
< u, \alpha\varphi_1 + \varphi_2 > &= \int_0^a \frac{(\alpha\varphi_1 + \varphi_2)(x) - (\alpha\varphi_1 + \varphi_2)(0)}{x} dx + (\alpha\varphi_1 + \varphi_2)(0) \log a \\
&= \alpha \int_0^a \frac{\varphi_1(x) - \varphi_1(0)}{x} dx + \int_0^a \frac{\varphi_2(x) - \varphi_2(0)}{x} dx + (\alpha\varphi_1(0) + \varphi_2(0)) \log a \\
&= \alpha < u, \varphi_1 > + < u, \varphi_2 > .
\end{aligned}$$

Thus, u satisfies the linearity property.

Also the absolute value of the functional $< u, \varphi >$ is bounded for any test function $\varphi \in \mathscr{D}$. By the Mean Value Theorem, we have:

$$\int_0^a \frac{\varphi(x) - \varphi(0)}{x} dx = \int_0^a \varphi'(\xi) dx \tag{6.3.3}$$

for some $\xi \in (0, x)$. Taking the absolute value, we have:

$$\begin{aligned}
| < u, \varphi > | &= \left| \int_0^a \frac{\varphi(x) - \varphi(0)}{x} dx + \varphi(0) \log a \right| \\
&\leq \int_0^a |\varphi'(\xi)| dx + |\varphi(0)| |\log a| \quad \text{(using triangle inequality)} \\
&\leq a \sup_{0 \leq x \leq a} |\varphi'(\xi)| + |\varphi(0)| |\log a| \quad (\text{since } 0 \leq \xi \leq a).
\end{aligned}$$

Next, we need to show that u is a continuous functional on $\mathscr{D}$. Let φ_n be a sequence in $\mathscr{D}$ such that $\varphi_n \to 0$ in $\mathscr{D}$, i.e., $\varphi_n(x) \to 0$ and all its derivatives converge uniformly to zero as $n \to \infty$. We want to show that $< u, \varphi_n > \to 0$.

Since φ_n converges to zero in $\mathscr{D}$, we have $\varphi_n(x) \to 0$ uniformly on any compact subset of $(0, a)$. This means that for any $\varepsilon > 0$, there exists N_1 such that for all $n > N_1$, we have $|\varphi_n(x)| < \varepsilon$ for all $x \in [0, a]$.

$$\begin{aligned}
| < u, \varphi_n > | &= \left| \int_0^a \frac{\varphi_n(x) - \varphi_n(0)}{x} dx + \varphi_n(0) \log a \right| \\
&\leq \left| \int_0^a \frac{\varphi_n(x)}{x} \right| + \left| \int_0^a \frac{\varphi_n(0)}{x} dx \right| + |\varphi_n(0) \log a| \\
&\leq \varepsilon \log a + 2|\varphi_n(0)| \log a.
\end{aligned}$$

Since $\varphi_n(0) \to 0$. Thus, for any $\varepsilon > 0$, there exists N_2 such that for all $n > N_2$, we have

$|\varphi_n(0)| < \varepsilon$. Choosing $N = \max N_1, N_2$, we can bound the expression above as:

$$| < u, \varphi_n > | \leq 3\varepsilon \log a. \tag{6.3.4}$$

We see that $< u, \varphi_n >$ converges to zero as $n \to \infty$. Since u is a linear functional and is continuous on $\mathscr{D}$, it is a distribution in $\mathscr{D}'$ of order 1.
To find the support of u, we need to determine the values of x for which $< u, \varphi > \neq 0$ for any $\varphi \in \mathscr{D}$. If φ has support in $(-\infty, 0)$, we know that $\varphi(x) = 0$ for $x \geq 0$. Thus,

$$\begin{aligned} < u, \varphi > &= \int_0^a \frac{\varphi(x) - \varphi(0)}{x} dx + \varphi(0) \log a \\ &= -\varphi(0) \int_0^a \frac{1}{x} dx + \varphi(0) \log a \\ &= 0. \end{aligned}$$

Therefore, the support of u is the complement of the support of φ, which is the interval $[0, +\infty)$.
Let us denote u' as the derivative of u. We will calculate the product xu'. By definition:

$$< u', \varphi >= - < u, \varphi' >= -\left(\int_0^a \frac{\varphi'(x) - \varphi'(0)}{x} dx + \varphi'(0) \log a \right). \tag{6.3.5}$$

Note that $\varphi(a) = 0$ according to the condition of support. Integrating by parts, we have:

$$\begin{aligned} < u', \varphi > &= \varphi'(0) + a^{-1}\varphi(0) - \int_0^a \frac{\varphi(x) - x\varphi'(0) - \varphi(0)}{x^2} dx - \varphi'(0) \log a \\ &= -\left(-\varphi'(0)a^{-1} + \varphi'(0) \log a + \int_0^a \frac{\varphi(x)}{x^2} dx - \int_0^a \frac{x\varphi'(0) + \varphi(0)}{x^2} dx \right) + \varphi'(0) \\ &= -\left(\sum_{k=0}^{1} \frac{\varphi^k(0)}{k!} \frac{a^{k-1}}{k-1} + \int_0^a \frac{\varphi(x)}{x^2} dx + \int_0^a \left(\sum_{k=0}^{1} \frac{\varphi^k(0)}{x^2 k!} \right) dx + \varphi'(0) \right) \quad \text{by the Taylor formula} \\ &= -Pf \int_0^a \frac{\varphi(x)}{x^2} dx + \varphi'(0) \\ &=< -Pf \frac{H(x)}{x^2}, \varphi > + \varphi'(0) \\ &=< -Pf \frac{H(x)}{x^2}, \varphi > + < -\delta', \varphi > \\ &=< -Pf \frac{H(x)}{x^2} - \delta', \varphi > . \end{aligned}$$

$$\Longrightarrow u' = -Pf \frac{H(x)}{x^2} - \delta'. \tag{6.3.6}$$

By the definition of product of a distribution by a C^∞ function,see the section 1.3, we have:

$$< xu', \varphi > = < u', x\varphi > . \tag{6.3.7}$$

If we replace φ with $x\varphi$ in equation (6.3.5):($(x\varphi)' = \varphi + x\varphi'$)

$$\begin{aligned}
< xu', \varphi > &= < u', x\varphi > \\
&= -\int_0^a \frac{(x\varphi)(x) - (x\varphi)(x)|_{x=0} - (x\varphi)'(x)|_{x=0}}{x^2} dx + a^{-1}(x\varphi)(x)|_{x=0} - \log a (x\varphi)'(x)|_{x=0} + (x\varphi(\\
&= -\int_0^a \frac{\varphi(x) - \varphi(0)}{x} dx - \log a \varphi(0) + \varphi(0) \\
&= -Pf \int_0^a \frac{H(x)}{x} dx + \varphi(0) \\
&= < -Pf \frac{H(x)}{x} + \delta, \varphi > .
\end{aligned}$$

$$\Longrightarrow xu' = -Pf \frac{H(x)}{x} + \delta. \tag{6.3.8}$$

Hence,if we combining two equations (6.3.6) and (6.3.8), we obtain:

$$xu' = x\left(-Pf \frac{H(x)}{x^2} - \delta'\right) = -Pf \frac{H(x)}{x} + \delta.$$

$$\Longrightarrow xPf \frac{H(x)}{x^2} = Pf \frac{H(x)}{x}. \tag{6.3.9}$$

Here, we used the equality $x\delta' = -\delta$. (See the example 1.3.1 in chapter 1.1.)
Note that every distribution on an open interval of $\mathbb{R}$ has a primitive. Therefore, we can determine a primitive of u. Let v be a primitive of u such that

$$< v, \varphi' > = - < v', \varphi > = - < u, \varphi > . \tag{6.3.10}$$

Hence, by definition of u we have:

$$< v, \varphi' > = - < u, \varphi > = -\int_0^a \frac{\varphi(x) - \varphi(0)}{x} dx - \varphi(0) \log a. \tag{6.3.11}$$

Integrating by parts, we obtain:

$$
\begin{aligned}
< v,\varphi' > &= -[(\varphi(x)-\varphi(0)\log x|_0^a + \int_0^a \varphi'(x)\log x dx - \varphi(0)\log a \\
&= -(\varphi(a)-\varphi(0))\log a + \int_0^a \varphi'(x)\log x dx - \varphi(0)\log a \\
&= \int_0^a \varphi'(x)\log x dx, \quad \text{because } \varphi(a)=0 \\
&=< H(x)\log x, \varphi' > .
\end{aligned}
$$

Consequently, we find a primitive for u as:

$$v = H(x)\log x + C, \quad C\text{:constant.} \tag{6.3.12}$$

Exercise 6.3.1. *Write a solution in D'^{+}(distribution of support $[0,+\infty]$) of the differential equation of*

$$xy' + y = \delta. \tag{6.3.13}$$

Proof. We know that $(xy(x))' = xy'(x) + y(x)$ and $H' = \delta$, H:function Heaviside. So we have this implication:

$$
\begin{aligned}
xy' + y = \delta &\Longrightarrow (xy)' = \delta \\
&\Longrightarrow xy = H + C, \quad C\text{:constant} \\
&\Longrightarrow y(x) = \frac{H(x)+C}{x}, \quad C\text{:constant.}
\end{aligned}
$$

Here, y is general solution in $\mathscr{D}'$. We can write this solution as the following:

$$y = Pf\frac{H(x)}{x} + Cpv\frac{1}{x}. \tag{6.3.14}$$

The proof of $xPf\frac{H(x)}{x} = H(x)$ can be easily established .For any $\varphi \in \mathscr{D}$:

$$
\begin{aligned}
< xPf\frac{H(x)}{x}, \varphi > &=< Pf\frac{H(x)}{x}, x\varphi > \\
&= \int_0^\infty \frac{x\varphi(x)}{x} dx \\
&=< H(x), \varphi > .
\end{aligned}
$$

Therefore the general solution of (6.3.13) in $\mathscr{D}'$ is

$$y = Pf\frac{H(x)}{x} + Cpv\frac{1}{x} + C_0\delta, \quad C, C_0\text{:constants.} \tag{6.3.15}$$

Recall that $C_0\delta$ is a homogeneous solution of (6.3.13). (See the exercise 1.3.1 in the section 1.3.

Now, we look the general solution for the space D'^{+}. For this, we need to remember our knowledge about the support of Hadamard finite part and Cauchy principal values. In the section 6.2, we shown $supp(pv\frac{1}{x}) = \mathbb{R}$. (See, the exercise 6.2.2). Hence, the solutions in D'^{+} are:

$$Pf\frac{H(x)}{x} + C_0\delta, \quad C_0\text{:constant.} \tag{6.3.16}$$

□

Thanks to the above exercise, we can obtain the general solution in $\mathscr{D}'$ of

$$xy' + y = Pf\frac{H(x)}{x}. \tag{6.3.17}$$

This equation can be written in the following form:

$$(xy)' = Pf\frac{H(x)}{x} \tag{6.3.18}$$

We found a primitive $xy = H(x)\log x + C$, C:constant for previous equation (See paragraph 9). Therefore, we have the general solution of (6.3.18) is:

$$y = Pf\frac{H(x)\log x}{x} + Cpv\frac{1}{x} + C_0\delta \tag{6.3.19}$$

where C, C_0 are constant, $C_0\delta$ is the homogeneous solution of (6.3.18) and

$$< Pf\frac{H(x)\log x}{x}, \varphi >= \int_0^a \frac{\varphi(x) - \varphi(0)}{x}\log x dx + \frac{\log a}{a}\varphi(0). \tag{6.3.20}$$

7 Laplace Operator

7.1 In Polar Coordinates

Recall that **the Laplace's equation** in $\mathbb{R}^2$ in terms of the usual (x,y) coordinates system is defined as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=u_{xx}+u_{yy}=0. \tag{7.1.1}$$

The Cartesian coordinates can be represented by the polar coordinates as follows;

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{7.1.2}$$

The partial derivatives of x,y w.r.t r,θ:

$$\begin{cases} \frac{\partial x}{\partial r}=\cos\theta, \quad \frac{\partial x}{\partial \theta}=-r\sin\theta \\ \frac{\partial y}{\partial r}=\sin\theta, \quad \frac{\partial x}{\partial \theta}=r\cos\theta \end{cases} \tag{7.1.3}$$

Let us compute $\frac{\partial^2 u}{\partial r^2}$ using Chain Rule;

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right)\\
&= \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)\\
&= \cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial x}+\sin\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial y}\\
&= \cos\theta\left(\frac{\partial x}{\partial r}\frac{\partial^2 u}{\partial x^2}+\frac{\partial y}{\partial r}\frac{\partial^2 u}{\partial y\partial x}\right)+\sin\theta\left(\frac{\partial x}{\partial r}\frac{\partial^2 u}{\partial x^2}+\frac{\partial y}{\partial r}\frac{\partial^2 u}{\partial x\partial y}\right)\\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2}+\cos\theta\sin\theta\frac{\partial^2 u}{\partial y\partial x}+\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}\\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Let us compute $\frac{\partial^2 u}{\partial \theta^2}$;

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) \\
&= \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial x}{\partial \theta} + r\cos\theta\frac{\partial y}{\partial \theta}\right) \\
&= -r\cos\theta\frac{\partial x}{\partial \theta} - r\sin\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial y} \\
&= -r\left(\cos\theta\frac{\partial x}{\partial \theta} + \sin\theta\frac{\partial u}{\partial y}\right) - r\sin\theta\left(\frac{\partial x}{\partial \theta}\frac{\partial^2 u}{\partial x^2} + \frac{\partial y}{\partial \theta}\frac{\partial^2 u}{\partial y \partial x}\right) + r\cos\theta\left(\frac{\partial x}{\partial \theta}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial y}{\partial \theta}\frac{\partial^2 u}{\partial y^2}\right) \\
&= -r\left(\cos\theta\frac{\partial x}{\partial \theta} + \sin\theta\frac{\partial u}{\partial y}\right) - r\sin\theta\left(-r\sin\theta\frac{\partial^2 u}{\partial x^2} + r\cos\theta\frac{\partial^2 u}{\partial y \partial x}\right) \\
&+ r\cos\theta\left(-r\sin\theta\frac{\partial^2 u}{\partial x \partial y} + r\cos\theta\frac{\partial^2 u}{\partial y^2}\right) \\
&= -r\left(\cos\theta\frac{\partial x}{\partial \theta} + \sin\theta\frac{\partial u}{\partial y}\right) + r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} \\
\Longrightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial r^2} &= -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} \\
&+ \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} \\
\Longrightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial r^2} &= -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
\Longrightarrow u_{xx} + u_{yy} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0
\end{aligned}
$$

7.2 In Spherical Coordinates

The spherical coordinates can be represented by the polar coordinates as follows;

$$
\begin{cases}
x = r\sin\theta\cos\varphi & \theta \in [0, \pi] \\
y = r\sin\theta\sin\varphi & \varphi \in [0, 2\pi] \\
z = r\cos\theta
\end{cases}
\tag{7.2.1}
$$

Laplace equation in $\mathbb{R}^3$ in terms of (x,y,z) coordinates system is:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=u_{xx}+u_{yy}+u_{zz}=0 \tag{7.2.2}$$

The partial derivatives of x,y,z w.r.t r,θ,φ is:

$$\begin{array}{ccc}
\frac{\partial x}{\partial r}=\sin\theta\cos\varphi, & \frac{\partial y}{\partial r}=\sin\theta\sin\varphi, & \frac{\partial z}{\partial r}=\cos\theta \\
\frac{\partial x}{\partial \theta}=r\cos\theta\cos\varphi, & \frac{\partial y}{\partial \theta}=r\cos\theta\sin\varphi, & \frac{\partial z}{\partial \theta}=-r\sin\theta \\
\frac{\partial x}{\partial \varphi}=-r\sin\theta\sin\varphi, & \frac{\partial y}{\partial \varphi}=r\sin\theta\cos\varphi, & \frac{\partial z}{\partial \varphi}=0
\end{array}$$

Let us compute $\frac{\partial^2 u}{\partial r^2}$ by using Chain Rule:

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2}&=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}\right)\\
&=\frac{\partial u}{\partial r}\left(\sin\theta\cos\varphi\frac{\partial u}{\partial x}+\sin\theta\sin\varphi\frac{\partial u}{\partial y}+\cos\theta\frac{\partial u}{\partial z}\right)\\
&=\sin\theta\cos\varphi\frac{\partial}{\partial r}\frac{\partial u}{\partial x}+\sin\theta\sin\varphi\frac{\partial}{\partial r}\frac{\partial u}{\partial y}+\cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial z}\\
&=\sin\theta\cos\varphi\left(\frac{\partial x}{\partial r}\frac{\partial^2 u}{\partial x^2}+\frac{\partial y}{\partial r}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial z}{\partial r}\frac{\partial^2 u}{\partial z\partial x}\right)+\sin\theta\sin\varphi\left(\frac{\partial x}{\partial r}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial y}{\partial r}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial r}\frac{\partial^2 u}{\partial z\partial y}\right)\\
&+\cos\theta\left(\frac{\partial x}{\partial r}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial y}{\partial r}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial z}{\partial r}\frac{\partial^2 u}{\partial z^2}\right)\\
&=\sin\theta\cos\varphi\left(\sin\theta\cos\varphi\frac{\partial^2 u}{\partial x^2}+\sin\theta\sin\varphi\frac{\partial^2 u}{\partial y\partial x}+\cos\theta\frac{\partial^2 u}{\partial z\partial x}\right)\\
&+\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial y}+\sin\theta\sin\varphi\frac{\partial^2 u}{\partial y^2}+\cos\theta\frac{\partial^2 u}{\partial z\partial y}\right)\\
&+\cos\theta\left(\sin\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial z}+\sin\theta\sin\varphi\frac{\partial^2 u}{\partial y\partial z}+\cos\theta\frac{\partial^2 u}{\partial z^2}\right)\\
&=\sin^2\theta\cos^2\varphi\frac{\partial^2 u}{\partial x^2}+\sin^2\theta\sin^2\varphi\frac{\partial^2 u}{\partial y^2}+\cos^2\theta\frac{\partial^2 u}{\partial z^2}+2\sin^2\theta\cos\varphi\sin\varphi\frac{\partial^2 u}{\partial x\partial y}\\
&+2\sin\theta\cos\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial z}+2\sin\theta\cos\theta\sin\varphi\frac{\partial^2 u}{\partial y\partial z}
\end{aligned}$$

Let us compute $\frac{\partial^2 u}{\partial \theta^2}$:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial\theta^2} &= \frac{\partial u}{\partial\theta}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial\theta}\right)\\
&= \frac{\partial u}{\partial\theta}\left(r\cos\theta\cos\varphi\frac{\partial u}{\partial x}+r\cos\theta\sin\varphi\frac{\partial u}{\partial y}-r\sin\theta\frac{\partial u}{\partial z}\right)\\
&= -r\sin\theta\cos\varphi\frac{\partial u}{\partial x}+r\cos\theta\cos\varphi\frac{\partial}{\partial x}\frac{\partial u}{\partial\theta}-r\sin\theta\sin\varphi\frac{\partial u}{\partial y}\\
&+r\cos\theta\sin\varphi\frac{\partial}{\partial y}\frac{\partial u}{\partial\theta}-r\cos\theta\frac{\partial u}{\partial z}-r\sin\theta\frac{\partial}{\partial\theta}\frac{\partial u}{\partial z}\\
&= -r\left(\sin\theta\cos\varphi\frac{\partial u}{\partial x}+\sin\theta\sin\varphi\frac{\partial u}{\partial y}+\cos\theta\frac{\partial u}{\partial z}\right)\\
&+r\cos\theta\cos\varphi\left(\frac{\partial x}{\partial\theta}\frac{\partial^2 u}{\partial x^2}+\frac{\partial y}{\partial\theta}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial z}{\partial\theta}\frac{\partial^2 u}{\partial z\partial x}\right)+r\cos\theta\sin\varphi\left(\frac{\partial x}{\partial\theta}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial y}{\partial\theta}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial\theta}\frac{\partial^2 u}{\partial z\partial y}\right)\\
&-r\sin\theta\left(\frac{\partial x}{\partial\theta}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial y}{\partial\theta}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial z}{\partial\theta}\frac{\partial^2 u}{\partial z^2}\right)\\
&= -r\left(\sin\theta\cos\varphi\frac{\partial u}{\partial x}+\sin\theta\sin\varphi\frac{\partial u}{\partial y}+\cos\theta\frac{\partial u}{\partial z}\right)\\
&+r\cos\theta\cos\varphi\left(r\cos\theta\cos\varphi\frac{\partial^2 u}{\partial x^2}+r\cos\theta\sin\varphi\frac{\partial^2 u}{\partial y\partial x}-r\sin\theta\frac{\partial^2 u}{\partial z\partial x}\right)\\
&+r\cos\theta\sin\varphi\left(r\cos\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial y}+r\cos\theta\sin\varphi\frac{\partial^2 u}{\partial y^2}-r\sin\theta\frac{\partial^2 u}{\partial z\partial y}\right)\\
&-r\sin\theta\left(r\cos\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial z}+r\cos\theta\sin\varphi\frac{\partial^2 u}{\partial y\partial z}-r\sin\theta\frac{\partial^2 u}{\partial z^2}\right)\\
&= -r\frac{\partial u}{\partial r}+r^2\left[\cos^2\theta\cos^2\varphi\frac{\partial^2 u}{\partial x^2}+\cos^2\theta\sin^2\varphi\frac{\partial^2 u}{\partial y^2}+\sin^2\theta\frac{\partial^2 u}{\partial z^2}\right]\\
&+2r^2\left[\cos^2\theta\cos\varphi\sin\varphi\frac{\partial^2 u}{\partial x\partial y}-\cos\theta\sin\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial z}-\cos\theta\sin\theta\sin\varphi\frac{\partial^2 u}{\partial y\partial z}\right]
\end{aligned}
$$

Let us compute $\frac{\partial^2 u}{\partial \varphi^2}$:

$$\begin{aligned}
\frac{\partial^2 u}{\partial \varphi^2} &= \frac{\partial u}{\partial \varphi}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \varphi}\right)\\
&= \frac{\partial u}{\partial \varphi}\left(-r\sin\theta\sin\varphi\frac{\partial u}{\partial x}+r\sin\theta\cos\varphi\frac{\partial u}{\partial y}\right)\\
&= -r\sin\theta\cos\varphi\frac{\partial u}{\partial x}-r\sin\theta\sin\varphi\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial x}-r\sin\theta\sin\varphi\frac{\partial u}{\partial y}+r\sin\theta\cos\varphi\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial y}\\
&= -r\sin\theta\left(\cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\frac{\partial u}{\partial y}\right)-r\sin\theta\sin\varphi\left(\frac{\partial x}{\partial \varphi}\frac{\partial^2 u}{\partial x^2}+\frac{\partial y}{\partial \varphi}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial z}{\partial \varphi}\frac{\partial^2 u}{\partial z\partial x}\right)\\
&+r\sin\theta\cos\varphi\left(\frac{\partial x}{\partial \varphi}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial y}{\partial \varphi}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial \varphi}\frac{\partial^2 u}{\partial z\partial y}\right)\\
&= -r\sin\theta\left(\cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\frac{\partial u}{\partial y}\right)-r\sin\theta\sin\varphi\left(-r\sin\theta\sin\varphi\frac{\partial^2 u}{\partial x^2}+r\sin\theta\cos\varphi\frac{\partial^2 u}{\partial x\partial y}\right)\\
&+r\sin\theta\cos\varphi\left(-r\sin\theta\sin\varphi\frac{\partial^2 u}{\partial x\partial y}+r\sin\theta\cos\varphi\frac{\partial^2 u}{\partial y^2}\right)\\
&= -r\sin\theta\left(\cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\frac{\partial u}{\partial y}\right)+r^2\sin^2\theta\left(\sin^2\varphi\frac{\partial^2 u}{\partial x^2}+\cos^2\varphi\frac{\partial^2 u}{\partial y^2}-2\sin\varphi\cos\varphi\frac{\partial^2 u}{\partial x\partial y}\right)
\end{aligned}$$

Consequently, we have:

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

$$\begin{aligned}
&= \left(\sin^2\theta\cos^2\varphi+\cos^2\theta\cos^2\varphi+\sin^2\varphi\right)\frac{\partial^2 u}{\partial x^2}+\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\sin^2\varphi+\cos^2\varphi\right)\frac{\partial^2 u}{\partial y^2}\\
&+\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 u}{\partial z^2}+\left(\frac{2\sin\theta\cos\varphi}{r}+\frac{\cos^2\theta\cos\varphi}{r\sin\theta}-\frac{\sin\theta\cos\varphi}{r}-\frac{\cos\varphi}{r\sin\theta}\right)\frac{\partial u}{\partial x}\\
&+\left(\frac{2\sin\theta\sin\varphi}{r}+\frac{\cos^2\theta\sin\varphi}{r\sin\theta}-\frac{\sin\theta\sin\varphi}{r}-\frac{\sin\varphi}{r\sin\theta}\right)\frac{\partial u}{\partial y}\\
&+\left(2\sin^2\theta\cos\varphi\sin\varphi+2\cos^2\theta\cos\varphi\sin\varphi-2\cos\varphi\sin\varphi\right)\frac{\partial^2 u}{\partial x\partial y}\\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x,y,x)
\end{aligned}$$

7.3 Laplace Operator in Polar Coordinates in Several Dimension

Let $n \geq 2$ be an integer. Consider $\mathbb{R}^n$. The Laplace operator $\mathbb{R}^n$ is

$$L_n = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{7.3.1}$$

For k with $1 \leq k \leq n$ define

$$r_k^2 = \sum_{i=1}^{k} x_i^2 \tag{7.3.2}$$

and for $2 \leq k \leq n$ write

$$x_k = r_k \cos\theta_k, \quad r_{k-1} = r_k \sin\theta_k \tag{7.3.3}$$

The polar coordinates of the point $(x_1, x_2, \ldots, x_n)$ will be $(r_n, \theta_2, \theta_3, \ldots, \theta_n)$.
In case $n = 2$;$(k = 1,2)$, $(r_2 = r)$, $(\theta_2 = \theta)$

$$x_1 = r_2 \sin\theta_2 = r_2 \sin\theta = y$$

$$x_2 = r_2 \cos\theta_2 = r_2 \cos\theta = x$$

$$r^2 = x_1^2 + x_2^2 = y^2 + x^2$$

T be a linear map such that

$$T : (x_1, x_2, \ldots, x_n) \longrightarrow (r_n, \theta_2, \ldots, \theta_n)$$

Calculate the Jacobian matrix with $k = n$:

$$\frac{\partial(r_{n-1}, x_n)}{\partial(r_n, \theta_n)} = \begin{bmatrix} \frac{\partial r_{n-1}}{\partial r_n} & \frac{\partial x_n}{\partial r_n} \\ \frac{\partial r_{n-1}}{\partial \theta_n} & \frac{\partial x_n}{\partial \theta_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial(r_n \sin\theta_n)}{\partial r_n} & \frac{\partial(r_n \cos\theta_n)}{\partial r_n} \\ \frac{\partial(r_n \sin\theta_n)}{\partial \theta_n} & \frac{\partial(r_n \cos\theta_n)}{\partial \theta_n} \end{bmatrix} = \begin{bmatrix} \sin\theta_n & \cos\theta_n \\ r_n \cos\theta_n & -r_n \sin\theta_n \end{bmatrix}$$

Calculate the Jacobian matrix of $\frac{\partial(r_n, \theta_n)}{\partial(r_{n-1}, x_n)}$ using Chain Rule, that is the inverse of Jacobian matrix of $\frac{\partial(r_{n-1}, x_n)}{\partial(r_n, \theta_n)}$:

$$\begin{bmatrix} \frac{\partial r_n}{\partial r_{n-1}} & \frac{\partial \theta_n}{\partial r_{n-1}} \\ \frac{\partial r_n}{\partial x_n} & \frac{\partial \theta_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial r_n}{\partial r_n}\frac{\partial r_n}{\partial r_{n-1}} + \frac{\partial r_n}{\partial \theta_n}\frac{\partial \theta_n}{\partial r_{n-1}} & \frac{\partial \theta_n}{\partial r_n}\frac{\partial r_n}{\partial r_{n-1}} + \frac{\partial \theta_n}{\partial \theta_n}\frac{\partial \theta_n}{\partial r_{n-1}} \\ \frac{\partial r_n}{\partial r_n}\frac{\partial r_n}{\partial x_n} + \frac{\partial r_n}{\partial \theta_n}\frac{\partial \theta_n}{\partial x_n} & \frac{\partial \theta_n}{\partial r_n}\frac{\partial r_n}{\partial x_n} + \frac{\partial \theta_n}{\partial \theta_n}\frac{\partial \theta_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \sin\theta_n & \frac{\cos\theta_n}{r_n} \\ \cos\theta_n & -\frac{\sin\theta_n}{r_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial r_{n-1}}{\partial r_n} & \frac{\partial x_n}{\partial r_n} \\ \frac{\partial r_{n-1}}{\partial \theta_n} & \frac{\partial x_n}{\partial \theta_n} \end{bmatrix}^{-1}$$

We want to evaluate the differential operator $D^2_{r_{n-1}} + D^2_{x_n}$.

$$\begin{aligned}
D^2_{r_{n-1}} &= \left(\sin\theta_n D_{r_n} + \frac{\cos\theta_n}{r_n} D_{\theta_n}\right)\left(\sin\theta_n D_{r_n} + \frac{\cos\theta_n}{r_n} D_{\theta_n}\right)\\
&= (\sin\theta_n D_{r_n})^2 + \sin\theta_n D_{r_n}\left(\frac{\cos\theta_n}{r_n} D_{\theta_n}\right) + \frac{\cos\theta_n}{r_n} D_{\theta_n}(\sin\theta_n D_{r_n}) + \left(\frac{\cos\theta_n}{r_n} D_{\theta_n}\right)^2\\
&= \sin^2\theta_n D^2_{r_n} - \frac{\sin\theta_n\cos\theta_n}{r_n^2} D_{\theta_n} + \frac{\sin\theta_n\cos\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\cos^2\theta_n}{r_n} D_{r_n}\\
&+ \frac{\cos\theta_n\sin\theta_n}{r_n} D_{\theta_n} D_{r_n} - \frac{\cos\theta_n\sin\theta_n}{r_n^2} D_{\theta_n} + \frac{\cos^2\theta_n}{r_n^2} D_{\theta_n}\\
&= \sin^2\theta_n D^2_{r_n} + \frac{2\sin\theta_n\cos\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\cos^2\theta_n}{r_n^2} D_{\theta_n} - \frac{2\sin\theta_n\cos\theta_n}{r_n^2} D_{\theta_n} + \frac{\cos^2\theta_n}{r_n} D_{r_n}.
\end{aligned}$$

$$\begin{aligned}
D^2_{x_n} &= \left(\cos\theta_n D_{r_n} - \frac{\sin\theta_n}{r_n} D_{\theta_n}\right)\left(\cos\theta_n D_{r_n} - \frac{\sin\theta_n}{r_n} D_{\theta_n}\right)\\
&= (\cos\theta_n D_{r_n})^2 - \cos\theta_n D_{r_n}\left(\frac{\sin\theta_n}{r_n} D_{\theta_n}\right) - \frac{\sin\theta_n}{r_n} D_{\theta_n}(\cos\theta_n D_{r_n}) + \left(\frac{\sin\theta_n}{r_n} D_{\theta_n}\right)^2\\
&= \cos^2\theta_n D^2_{r_n} + \frac{\cos\theta_n\sin\theta_n}{r_n^2} D_{\theta_n} - \frac{\cos\theta_n\sin\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\sin^2\theta_n}{r_n} D_{r_n}\\
&- \frac{\sin\theta_n\cos\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\sin\theta_n\cos\theta_n}{r_n^2} D_{\theta_n} + \frac{\sin^2\theta_n}{r_n^2} D_{\theta_n}.
\end{aligned}$$

$$\begin{aligned}
D^2_{r_{n-1}} + D^2_{x_n} &= \sin^2\theta_n D^2_{r_n} + \frac{2\sin\theta_n\cos\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\cos^2\theta_n}{r_n^2} D_{\theta_n} - \frac{2\sin\theta_n\cos\theta_n}{r_n^2} D_{\theta_n} + \frac{\cos^2\theta_n}{r_n} D_{r_n}\\
&+ \cos^2\theta_n D^2_{r_n} + \frac{\cos\theta_n\sin\theta_n}{r_n^2} D_{\theta_n} - \frac{\cos\theta_n\sin\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\sin^2\theta_n}{r_n} D_{r_n}\\
&- \frac{\sin\theta_n\cos\theta_n}{r_n} D_{r_n} D_{\theta_n} + \frac{\sin\theta_n\cos\theta_n}{r_n^2} D_{\theta_n} + \frac{\sin^2\theta_n}{r_n^2} D_{\theta_n}\\
&= D^2_{r_n} + \frac{1}{{r_n}^2} D^2_{\theta_n} + \frac{1}{r_n} D_{r_n}.
\end{aligned}$$

In case $n = 2$, we have:

$$\begin{aligned}
r_1 &= x_n = y\\
x_n &= x\\
\theta_n &= \theta\\
r_n &= (x^2 + y^2)^{1/2}.
\end{aligned}$$

So we obtain the Laplacian $D_{x_1}^2 + D_{x_n}^2$.

$$\begin{aligned} D_x{}^2 + D_y{}^2 &= D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\theta^2 \\ &= \frac{1}{r}D_r(rD_r) + \frac{1}{r^2}D_\theta. \end{aligned}$$

In case $n = 3$, we have :
$x_1 = y, \quad \theta_2 = \theta\ , \quad \rho = \sqrt{x^2+y^2} = r_2 = r\sin\varphi$
$x_2 = x, \quad \theta_3 = \varphi,$
$x_3 = z, \quad r_3 = r.$

$$\begin{aligned} D = D_x{}^2 + D_y{}^2 + D_z{}^2 &= D_\rho{}^2 + \frac{1}{\rho}D_\rho + \frac{1}{\rho^2}D_\theta{}^2 + D_z{}^2 \\ &= \frac{1}{r\sin\varphi}D_\rho + \frac{1}{r^2\sin^2\varphi}D_\theta + D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\varphi^2 \\ &= \frac{1}{r\sin\varphi}\left(\sin\varphi D_r + \frac{\cos\varphi}{r}D_\varphi\right) + \frac{1}{r^2\sin^2\varphi}D_\theta^2 + D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\varphi^2 \\ &= D_r^2 + \frac{2}{r}D_r + \frac{\cos\varphi}{r^2}\sin\varphi D_\varphi + \frac{1}{r^2}D_\varphi^2 + \frac{1}{r^2\sin^2\varphi}D_\theta^2. \end{aligned}$$

8 Solution of Partial Differential Equations by Fourier Transform

We have studied the Fourier Transform in the previous chapters. Now we will determine the solution of differential equations using the Fourier Transform. For this purpose, we will make use of an important theorem for differential equations.

Remark 8.0.1. *Let $f \in \mathbf{L}^2(\mathbb{R}^n)$. Various variants can be found in the literature for the Fourier transform of f, for example (Y. Choquet-Bruhat)*

$$\begin{cases} \mathscr{F} f(y) & = (2\pi)^{-n/2} \displaystyle\int_{\mathbb{R}^n} e^{-i<x,y>} f(x)dx \\ \overline{\mathscr{F}} f(y) & = (2\pi)^{-n/2} \displaystyle\int_{\mathbb{R}^n} e^{i<x,y>} f(x)dx \end{cases} \tag{8.0.1}$$

*$\mathscr{F} f(y)$ is **the Fourier transform of** f, and $\overline{\mathscr{F}} f(y)$ is **the inverse Fourier transform of** f.*

This form is used in exercises.

8.1 Example in $\mathbb{R}^3$

Definition 8.1.1. *The Delta function is often also referred to as the Dirac delta function, defined as follows:*

- $\delta(x) = 0$, $x \neq 0$,
- $\displaystyle\int_{\mathbb{R}^n} \delta(y-x)f(x)dx = f(y)$, $\forall f \in \mathscr{S}(\mathbb{R}^n)$.

Example 8.1.1. *Let*

$$\Delta E = \sum_{i=1}^{3} \frac{\partial^2}{(\partial x_i)^2} E = \delta. \tag{8.1.1}$$

Let us determine a tempered elementary solution of Δ in $\mathbb{R}^3$ using Fourier transform. Let $\rho = |y|^2 = (y_1)^2 + (y_2)^2 + (y_3)^2$. We have the following equivalences:

$$\begin{aligned}\Delta E = \delta &\iff \mathscr{F}\,\Delta E(y) = \mathscr{F}\,\delta(y)\\ &\iff -\rho^2 \mathscr{F}\,E = (2\pi)^{-3/2}.1\\ &\iff \mathscr{F}\,E = \frac{-(2\pi)^{-3/2}}{\rho^2}.\end{aligned}$$

By the inversion theorem, we know that $E = \overline{\mathscr{F}}(FE)$, *then*

$$\begin{aligned}E = \overline{\mathscr{F}}(\mathscr{F}\,E) = \overline{\mathscr{F}}(\frac{-(2\pi)^{-3/2}}{\rho^2})\\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i<x,y>}\frac{-(2\pi)^{-3/2}}{\rho^2}dy\\ &= -(2\pi)^{-3}\int_{\mathbb{R}^3}\frac{e^{i<x,y>}}{\rho^2}dy.\end{aligned}$$

We use polar coordinates $\rho > 0$, $\theta \in [0,\pi]$ *and* $\varphi \in [0,2\pi]$;

$$\begin{aligned}&|x| = r,\\ &y_1 = \rho\sin\theta\cos\varphi,\\ &y_2 = \rho\sin\theta\sin\varphi,\\ &y_3 = \rho\cos\theta.\end{aligned}$$

We have $< x, y >= |x||y|\cos\theta = r\rho\cos\theta$ *and* $dy = dy_1 dy_2 dy_3$. *Thus, we obtain:*

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{e^{i<x,y>}}{\rho^2} dy &= \int_{\mathbb{R}^3} \frac{e^{ir\rho\cos\theta}}{\rho^2} \frac{\partial(y_1, y_2, y_3)}{\partial(\rho, \theta, \varphi)} dy \\
&= \int_{\mathbb{R}^3} \frac{e^{ir\rho\cos\theta}}{\rho^2} \rho^2 \sin\theta d\rho d\theta d\varphi \\
&= \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_{\rho=0}^{\rho=+\infty} \int_{\theta=0}^{\theta=\pi} \frac{e^{ir\rho\cos\theta}}{\rho^2} \rho^2 \sin\theta d\theta d\rho \\
&= 2\pi \int_{\rho=0}^{\rho=+\infty} \int_{\theta=0}^{\theta=\pi} \frac{e^{ir\rho\cos\theta}}{\rho^2} \rho^2 \sin\theta d\theta d\rho \\
&= 2\pi \int_{\rho=0}^{\rho=+\infty} \Big(\frac{1}{-r\rho} e^{ir\rho\cos\theta} \Big|_{\theta=0}^{\pi} \Big) d\rho \\
&= 2\pi \int_{0}^{+\infty} \frac{1}{-r\rho} (e^{-ir\rho} - e^{ir\rho}) d\rho \\
&= 2\pi \int_{0}^{+\infty} \frac{2\sin r\rho}{r\rho} d\rho \\
&= \frac{4\pi}{r} \int_{0}^{+\infty} \frac{\sin R}{R} dR, \quad \textit{where } R = r\rho \\
&= \frac{4\pi}{r} \frac{\pi}{2} \\
&= \frac{2\pi^2}{r}.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
E = \overline{\mathscr{F}}(\mathscr{F} E) &= \overline{\mathscr{F}}\Big(\frac{-(2\pi)^{-3/2}}{\rho^2}\Big) \\
&= -(2\pi)^{-3} \int_{\mathbb{R}^3} \frac{e^{i<x,y>}}{\rho^2} dy \\
&= -\frac{1}{(2\pi)^3} \frac{2\pi^2}{r} \\
&= -\frac{1}{4\pi r}.
\end{aligned}$$

8.2 Elementary Solution of The Wave Equation

Example 8.2.1. *Let us consider the elementary solution* $E(t, x)$ *of the wave equation in* $\mathbb{R}^4$*:*

$$\Box E = \frac{\partial^2 E}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2 E}{(\partial x_i)^2} = \delta(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3. \tag{8.2.1}$$

We define the Fourier transform of E with respect to the variables x in $\mathbb{R}^3$*, denoted by* $\varepsilon(t,y) := \mathscr{F}\, E(t,x)$*. Let* $\delta(t,x) = \delta(t)\times\delta(x)$*.*
Let us calculate the Fourier transform of $\Box E$*:*

$$\begin{aligned}
\mathscr{F}(\Box E) &= \mathscr{F}\Big(\frac{\partial^2 E}{\partial t^2} - \sum_{i=1}^{3}\frac{\partial^2 E}{(\partial x_i)^2}\Big)\\
&= \mathscr{F}\Big(\frac{\partial^2 E}{\partial t^2}\Big) - \mathscr{F}\Big(\sum_{i=1}^{3}\frac{\partial^2 E}{(\partial x_i)^2}\Big)\\
&= \frac{\partial^2}{\partial t^2}\mathscr{F}(E) - \Big(\sum_{i=1}^{3}\mathscr{F}\Big(\frac{\partial^2 E}{(\partial x_i)^2}\Big)\Big)\\
&= \frac{\partial^2}{\partial t^2}\varepsilon + \sum_{i=1}^{3}(y_i)^2\varepsilon\\
&= \frac{\partial^2}{\partial t^2}\varepsilon + \rho^2\varepsilon, \quad \sum_{i=1}^{3}(y_i)^2 = \rho^2\\
&= \Big(\frac{\partial^2}{\partial t^2} + \rho^2\Big)\varepsilon\\
&= (2\pi)^{-3/2}\delta(t).
\end{aligned}$$

As we have $(\frac{\partial^2}{\partial t^2} + \rho^2)\varepsilon = (2\pi)^{-3/2}\delta(t)$*, we apply the previous theorem. We find the homogeneous solution of* ε *that satisfies* $f(0) = 0$ *and* $f'(0) = 1$*. These conditions give us:*

$$f(t) = \frac{\sin\rho t}{\rho}.$$

By this theorem, we have:

$$\begin{aligned}
\varepsilon = \tfrac{H(t)}{(2\pi)^{3/2}}\tfrac{\sin\rho t}{\rho} \quad &\Longrightarrow \overline{\mathscr{F}}\varepsilon = \overline{\mathscr{F}}\Big(\tfrac{H(t)}{(2\pi)^{3/2}}\tfrac{\sin\rho t}{\rho}\Big)\\
&\Longrightarrow E(x,t) = \tfrac{H(t)}{(2\pi)^{3/2}}\overline{\mathscr{F}}\Big(\tfrac{\sin\rho t}{\rho}\Big)
\end{aligned}$$

Now, let's calculate the inverse Fourier transform of $\frac{sin\rho t}{\rho}$ *in* $\mathbb{R}^3$*. For this, we will use polar coordinates* $\rho > 0$*,* $\theta \in [0,\pi]$ *and* $\varphi \in [0,2\pi]$*.*

$$\begin{aligned}
|x| &= r,\\
y_1 &= \rho\sin\theta\cos\varphi,\\
y_2 &= \rho\sin\theta\sin\varphi,\\
y_3 &= \rho\cos\theta
\end{aligned}$$

We have $<x,y>=|x||y|\cos\theta = r\rho\cos\theta$ *and* $dy = dy_1dy_2dy_3$. *Thus,*

$$\begin{aligned}
\overline{\mathscr{F}}(\frac{sin\rho t}{\rho})(x) &= (2\pi)^{-3/2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=+\infty}\int_{\theta=0}^{\theta=\pi}\frac{\sin\rho t}{\rho}e^{ir\rho\cos\theta}\rho^2\sin\theta d\varphi d\theta d\rho \\
&= (2\pi)^{-3/2}(2\pi)\int_{\rho=0}^{\rho=+\infty}\Big(\int_{\theta=0}^{\theta=\pi}e^{ir\rho\cos\theta}\sin\theta d\theta\Big)\rho\sin(\rho t)d\rho \\
&= (2\pi)^{-1/2}(\int_{\rho=0}^{\rho=+\infty}\Big(\frac{-1}{\rho r}e^{ir\rho\sin\theta}\Big|_{\theta=0}^{\theta=\pi}\Big)\rho\sin(\rho t)d\rho \\
&= (2\pi)^{-1/2}\int_{\rho=0}^{\rho=+\infty}\frac{2\sin\rho r}{\rho r}\rho\sin(\rho t)d\rho \\
&= \frac{(2\pi)^{-1/2}}{r}\int_0^{+\infty}\Big(\cos\rho(r-t)-\cos\rho(r+t)\Big)d\rho
\end{aligned}$$

To obtain the solution, we need to evaluate the integrals:

$$\begin{aligned}
\int_0^{+\infty}\cos\rho(r-t)d\rho &= \frac{1}{2}\int_0^{+\infty}e^{i\rho(r-t)}+e^{-i\rho(r-t)}d\rho \\
&= \frac{1}{2}\Big[\int_0^{+\infty}e^{i\rho(r-t)}d\rho+\int_0^{-\infty}e^{i\rho(r-t)}d(-\rho)\Big] \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}e^{i\rho(r-t)}d\rho \\
&= \pi\delta(r-t), \quad \textit{because} \quad \int_{-\infty}^{+\infty}e^{i\rho s}d\rho = 2\pi\delta(s).
\end{aligned}$$

Similarly, we find:

$$\begin{aligned}
\int_0^{+\infty}\cos\rho(r+t)d\rho &= \frac{1}{2}\int_0^{+\infty}e^{i\rho(r+t)}+e^{-i\rho(r+t)}d\rho \\
&= \frac{1}{2}\Big[\int_0^{+\infty}e^{i\rho(r+t)}d\rho+\int_0^{-\infty}e^{i\rho(r+t)}d(-\rho)\Big] \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}e^{i\rho(r+t)}d\rho \\
&= \pi\delta(r+t).
\end{aligned}$$

Thus,

$$\int_0^{+\infty}\cos\rho(r-t)-\cos\rho(r+t)d\rho = \pi\delta(r-t)-\pi\delta(r+t).$$

Therefore, we have:

$$\overline{\mathscr{F}}(\frac{sin\rho t}{\rho})(x) = \frac{(2\pi)^{-1/2}}{r}\Big(\pi\delta(r-t) - \pi\delta(r+t)\Big)$$
$$= \sqrt{\frac{\pi}{2}}\frac{1}{r}\Big(\delta(r-t) - \delta(r+t)\Big).$$

We know that $H(t)\delta(r+t) = 0$ for $r > 0$ and $t > 0$ by the definition of the Heaviside function. Therefore, the elementary solution is:

$$E(x,t) = \frac{H(t)}{(2\pi)^{3/2}}\overline{\mathscr{F}}(\frac{sin\rho t}{\rho})(x)$$
$$= \frac{H(t)}{(2\pi)^{3/2}}\sqrt{\frac{\pi}{2}}\frac{1}{r}\delta(r-t)$$
$$= \frac{H(t)}{4\pi r}\delta(r-t).$$

We also note that:

$$suppE(x,t) = \{(x,t); \sum_{i=1}^{3} x_i^2 - t^2 = 0, t \geq 0\}.$$

9 Iterated Laplace Operator

In this chapter, our aim is to obtain the elementary solution of the iterated Laplacian denoted by Δ^p

Recall that if E is the elementary solution of Δ^p, it must satisfy the condition $\Delta^p E = \delta$, where δ represents the Dirac delta function.

Consider a function f defined on the complement $\complement O$ of the origin in $\mathbb{R}^n$,$n \geq 2$, and suppose that f admits a partial derivative (in the usual sense) everywhere on $\complement O$.

$$g_i = \left\{ \frac{\partial f}{\partial x_i} \right\} \tag{9.0.1}$$

Given that f and g_i, when regarded as functions defined almost everywhere on $\mathbb{R}^n$, are locally integrable, it follows that g_i can be identified as the partial derivative of f with respect to x_i in the sense of distributions.

Indeed, it is evident that this result does not hold for $n = 1$. (example: The Heaviside function. We know $H' = \delta$ and δ is not a locally integrable function). We will now assume $n > 2$:

Γ denotes the function gamma. We recall that $\Gamma(x+1) = x\Gamma(x)$.

We put, in $\mathbb{R}^n$,

$$r = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{9.0.2}$$

We define the function $\{P_s\}$, where s is a real parameter, in $\complement O$ as follows:

$$\{P_s\} = \frac{r^s}{2^{s/2}\Gamma(\frac{s+n}{2})}. \tag{9.0.3}$$

Let a_s and b_s be constants (which depend on s) that will be determined. First, we show the following formulas, in the usual sense in $\complement O$:

$$\Delta\{P_{s+2}\} = a_{s+2}\{P_s\}. \tag{9.0.4}$$

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\{P_s\} = b_s\{P_s\}. \tag{9.0.5}$$

For better understanding, we examine $\{P_s\}$ and $\Delta\{P_s\}$ for $n=2$:

$$r=(x_1^2+x_2^2)^{1/2}, \quad \Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}. \tag{9.0.6}$$

The partial derivatives of r with respect to x_1 and x_2 are:

$$\frac{\partial r}{\partial x_1}=\frac{\partial(x_1^2+x_2^2)^{1/2}}{\partial x_1}=\frac{x_1}{r}, \quad \frac{\partial r}{\partial x_2}=\frac{\partial(x_1^2+x_2^2)^{1/2}}{\partial x_2}=\frac{x_2}{r}. \tag{9.0.7}$$

Let us calculate the partial derivatives of $\{P_s\}$ with respect to x_1 and x_2:

$$\begin{aligned}
\frac{\partial\{P_s\}}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\frac{r^s}{2^{s/2}\Gamma(\frac{s+n}{2})}\right)\\
&= \frac{1}{2^{s/2}\Gamma(\frac{s+n}{2})}\frac{\partial r^s}{\partial x_1}\\
&= \frac{1}{2^{s/2}\Gamma(\frac{s+n}{2})}\frac{\partial r^s}{\partial r}\frac{\partial r}{\partial x_1}\\
&= \frac{1}{2^{s/2}\Gamma(\frac{s+n}{2})}sr^{s-1}\frac{x_1}{r}\\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}x_1.
\end{aligned}$$

Similarly,

$$\frac{\partial\{P_s\}}{\partial x_2}=\frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}x_2.$$

Now, we shall compute the second partial derivative of P_s with respect to x_1 and x_2:

$$\begin{aligned}
\frac{\partial^2\{P_s\}}{\partial x_1^2} &= \frac{\partial}{\partial x_1}\left(\frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}x_1\right)\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_1\frac{\partial r^{s-2}}{\partial x_1}+r^{s-2}\right]\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_1\frac{\partial r^{s-2}}{\partial r}\frac{\partial r}{\partial x_1}+r^{s-2}\right]\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_1(s-2)r^{s-3}\frac{x_1}{r}+r^{s-2}\right]\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_1^2(s-2)r^{s-4}+r^{s-2}\right]\\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_1^2(s-2)r^{-2}+1\right].
\end{aligned}$$

$$\Longrightarrow \frac{\partial^2\{P_s\}}{\partial x_1^2} = \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_1^2(s-2)r^{-2}+1\right]. \tag{9.0.8}$$

In a similar way,

$$\frac{\partial^2\{P_s\}}{\partial x_2^2} = \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[(s-2)x_2^2r^{-2}+1\right]$$

Hence, by 9.0.8:

$$\begin{aligned}
\Delta\{P_s\} &= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[(s-2)x_1^2r^{-2}+1+(s-2)x_2^2r^{-2}+1\right] \\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[(s-2)(x_1^2+x_2^2)r^{-2}+2\right] \\
&= \frac{s^2r^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}.
\end{aligned}$$

Now, we can determine for n dimensions. Let us start with $\frac{\partial r}{\partial x_i}$:

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sqrt{\sum_{i=1}^{n} x_i^2}\right) = \frac{1}{2}\frac{2x_i}{\sqrt{\sum_{i=1}^{n} x_i^2}} = \frac{x_i}{r}.$$

Let us calculate the partial derivative of $\{P_s\}$ with respect to x_i:

$$\begin{aligned}
\frac{\partial\{P_s\}}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{r^s}{2^{s/2}\Gamma(\frac{s+n}{2})}\right) \\
&= \frac{1}{2^{s/2}\Gamma(\frac{s+n}{2})}\frac{\partial r^s}{\partial x_i} \\
&= \frac{1}{2^{s/2}\Gamma(\frac{s+n}{2})}\frac{\partial r^s}{\partial r}\frac{\partial r}{\partial x_i} \\
&= \frac{1}{2^{s/2}\Gamma(\frac{s+n}{2})}sr^{s-1}\frac{x_i}{r} \\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}x_i.
\end{aligned}$$

Now, we will determine the second partial derivative of $\{P_s\}$ with respect to x_i:

$$\begin{aligned}
\frac{\partial^2\{P_s\}}{\partial x_i^2} &= \frac{\partial}{\partial x_i}\left(\frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}x_i\right)\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_i\frac{\partial r^{s-2}}{\partial x_i}+r^{s-2}\right]\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_i\frac{\partial r^{s-2}}{\partial r}\frac{\partial r}{\partial x_i}+r^{s-2}\right]\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_i(s-2)r^{s-3}\frac{x_i}{r}+r^{s-2}\right]\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_i^2(s-2)r^{s-4}+r^{s-2}\right]\\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[x_i^2(s-2)r^{-2}+1\right].
\end{aligned}$$

Consequently, we have:

$$\begin{aligned}
\Delta\{P_s\} = \Delta &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\{P_s\}\\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\sum_{i=1}^{n} r^{s-2}\left[x_i^2(s-2)r^{-2}+1\right]\\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[(x_1^2(s-2)r^{-2}+1)+\cdots+(x_n^2(s-2)r^{-2}+1)\right]\\
&= \frac{sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[(s-2)r^{-2}(x_1^2+\cdots+x_n^2)+n\right]\\
&= \frac{sr^{s-2}(s-2+n)}{2^{s/2}\Gamma(\frac{s+n}{2})}.
\end{aligned}$$

Now, We can prove the formulas (9.0.4) and (9.0.5). By definition of Γ, we have:

$$\Gamma\left(\frac{s+n}{2}+1\right) = \frac{s+n}{2}\Gamma\left(\frac{s+n}{2}\right). \tag{9.0.9}$$

We can easily show them using this equality:

$$\begin{aligned}
\Delta\{P_{s+2}\} &= \frac{(s+2)r^s(s+n)}{2^{\frac{s+2}{2}}\Gamma(\frac{s+n+2}{2})} \\
&= \frac{(s+2)r^s(s+n)}{2^{\frac{s+2}{2}}(\frac{s+2}{2})\Gamma(\frac{s+n}{2})} \\
&= (s+2)\frac{r^s}{2^{s/2}\Gamma(\frac{s+n}{2})} \\
&= (s+2)\{P_s\}.
\end{aligned}$$

Hence, we have:

$$\Delta\{P_{s+2}\} = (s+2)\{P_s\}, \quad \text{with } a_{s+2} = s+2. \tag{9.0.10}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\{P_s\} &= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\left(\frac{r^s}{2^{s/2}\Gamma(\frac{s+n}{2})}\right) \\
&= \sum_{i=1}^{n} \frac{x_i}{2^{s/2}\Gamma(\frac{s+n}{2})} sr^{s-2}x_i \\
&= \sum_{i=1}^{n} \frac{x_i^2 sr^{s-2}}{2^{s/2}\Gamma(\frac{s+n}{2})} \\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\sum_{i=1}^{n} x_i^2 r^{s-2} \\
&= \frac{s}{2^{s/2}\Gamma(\frac{s+n}{2})}\left[2r^{s-2}(x_1^2+\cdots+x_n^2)\right] \\
&= \frac{sr}{2^{s/2}\Gamma(\frac{s+n}{2})} \\
&= s\{P_s\}.
\end{aligned}$$

We have then:

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\{P_s\} = b_s\{P_s\}, \quad \text{with } b_s = s. \tag{9.0.11}$$

For $s > -n$, $\{P_s\}$, considered as defined almost everywhere in $\mathbb{R}^n$, is locally integrable. Therefore, it defines a distribution on $\mathbb{R}^n$, which we will also denote as P_s. To show that equations (9.0.4) and (9.0.5) remain valid in the sense of distributions for $s > -n$ when replacing $\{P_s\}$ with P, we need to demonstrate that the operations involved in these equations are well-defined for distributions. Since $\{P_s\}$, $\frac{\partial\{P_{s+2}\}}{\partial x_i}$, $\frac{\partial^2\{P_{s+2}\}}{\partial x_i^2}$ are locally integrable, (9.0.4) is also valid in the sense of distribution:

- $$\Delta P_{s+2} = a_{s+2} P_s, \quad \text{with } a_{s+2} = s+2. \tag{9.0.12}$$

- $$\begin{aligned}
\left|\frac{\partial}{\partial x_i}(x_i\{P_s\})\right| &= \left|x_i\frac{\partial}{\partial x_i}\{P_s\} + \{P_s\}\right| \\
&\leq \left|x_i\frac{\partial}{\partial x_i}\{P_s\}\right| + |\{P_s\}| \\
&\leq \left|\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}\{P_s\}\right| + |\{P_s\}| \\
&\leq s\{P_s\}.
\end{aligned}$$

Since $\{P_s\}$ is locally integrable, $x_i\frac{\partial}{\partial x_i}\{P_s\}$ is also locally integrable.
Therefore, the validity in the sense of distributions of the equation (9.0.5):

$$\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}P_s = b_s P_s, \quad \text{with } b_s = s. \tag{9.0.13}$$

For any arbitrary real number s, our aim is to define the distribution P_s in $\mathbb{R}^n$ as the following formula (in the sense of distributions):

$$P_s = \frac{1}{(s+2)(s+4)\dots(s+2k)}\Delta^k P_{s+2k}, \quad k \text{ integer} \geq 0 \tag{9.0.14}$$

We will choose the integer k such that P_{s+2k} is defined according to the first question, i.e., $s+2 > -n$, but such that $s+2, s+4, \dots, s+2k$ are all non-zero. This is always possible, except in the exceptional case where $n = 2$ and s is a negative integer. We will temporarily set aside this exceptional case for now.
We will show that this definition of P_s is independent of the chosen integer k, that it gives back the previously defined P_s if $s > -n$, and that P_s coincides with the function $\{P_s\}$ at CO.
Suppose $s > -2$. For $k = 1$, we have:

$$\Delta P_{s+2} = (s+2)P_s. \tag{9.0.15}$$

The formula (9.0.14) coincides with the formula (9.0.12). Iterating of (9.0.12):

$$\Delta P_{s+2} = (s+2)P_s.$$

$$\begin{aligned}\Delta^2 P_{s+4} &= \Delta(\Delta P_{s+4})\\ &= \Delta((s+4)P_{s+2})\\ &= (s+4)\Delta P_{s+2}\\ &= (s+2)(s+4)P_s\\ &\vdots\end{aligned}$$

$$\begin{aligned}\Delta^{k-1} P_{s+2(k-1)} &= \Delta(\Delta^{k-2} P_{s+2(k-2)})\\ &=\Delta(s+2)(s+4)\dots(s+2(k-2))P_{s+2(k-1)})\\ &=(s+2)(s+4)\dots(s+2(k-2))(s+2(k-1))P_s.\end{aligned}$$

$$\begin{aligned}\Delta^k P_{s+2k} &= \Delta(\Delta^{k-1} P_{s+2(k-1)})\\ &= \Delta((s+2)(s+4)\dots(s+2(k-1))P_s)\\ &= (s+2)(s+4)\dots(s+2(k-1))(s+2k)P_s.\end{aligned}$$

$$\Longrightarrow P_s = \frac{1}{(s+2)(s+4)\dots(s+2k)}\Delta^k P_{s+2k}.$$

If $s > -n$, then there exists an integer k such that $s+2k > -n$ and P_{s+2k} is defined by the equation (9.0.14). If $k' > k$ integer such that $s+2k' > -n$, the formula (9.0.12) show that

$$\begin{aligned}\Delta^{k'} P_{s+2k'} &= (s+2)\dots(s+2k)(s+2(k+1))\dots(s+2k')P_s\\ &= (s+2(k+1))\dots(s+2k')\Delta^k P_{s+2k}.\end{aligned}$$

Consequently, P_s is independent of the chosen integer k.
Let us verifying (9.0.4) and (9.0.5) hold for distributions P for any s (except when $n=2$ and s is an even negative integer):

For $k=1$, the equation (9.0.14) becomes:

$$\begin{aligned} P_s &= \frac{1}{a_{s+2}} \Delta P_{s+2} \\ \Longrightarrow \Delta P_{s+2} &= a_{s+2} P_s. \end{aligned}$$

Then, according the equation (9.0.14), P_s verify (9.0.12). We show (9.0.13) by recurrence. Suppose that (9.0.14) hold for P_s.
Let T be any distribution. According to the laws of differentiation and product with a C^∞ function, we have:

- If $i \neq j$,

$$\begin{aligned} \Delta\left(x_i \frac{\partial T}{\partial x_i}\right) &= \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right)^2 \left(x_i \frac{\partial T}{\partial x_i}\right) \\ &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(x_i \frac{\partial}{\partial x_j} \frac{\partial T}{\partial x_i}\right) \\ &= \sum_{j=1}^{n} x_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j}\right)^2 T. \end{aligned}$$

- If $i = j$,

$$\begin{aligned} \Delta\left(x_i \frac{\partial T}{\partial x_i}\right) &= \left(\frac{\partial}{\partial x_i}\right)^2 \left(x_i \frac{\partial T}{\partial x_i}\right) \\ &= \frac{\partial T}{\partial x_i} \left[\frac{\partial T}{\partial x_i} + x_i \left(\frac{\partial}{\partial x_i}\right)^2 T\right] \\ &= 2\left(\frac{\partial}{\partial x_i}\right)^2 T + x_i \left(\frac{\partial}{\partial x_i}\right)^3 T. \end{aligned}$$

So,

$$\Delta\left(x_i \frac{\partial T}{\partial x_i}\right) = \sum_{j=1}^{n} x_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j}\right)^2 T + 2\left(\frac{\partial}{\partial x_i}\right)^2 T = x_i \frac{\partial}{\partial x_i} \Delta T + 2\left(\frac{\partial}{\partial x_i}\right)^2 T. \qquad (9.0.16)$$

$$\begin{aligned}\Delta\left(\sum_{i=1}^{n} x_i \frac{\partial T}{\partial x_i}\right) &= \sum_{i=1}^{n} \Delta\left(x_i \frac{\partial T}{\partial x_i}\right)\\ &= \sum_{i=1}^{n} 2\left(\frac{\partial}{\partial x_i}\right)^2 T + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\Delta T\\ &= 2\Delta T + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\Delta T\end{aligned}$$

$$\Longrightarrow \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\Delta T = \Delta\left(\sum_{i=1}^{n} x_i \frac{\partial T}{\partial x_i}\right) - 2\Delta T. \tag{9.0.17}$$

Using (9.0.12) and (9.0.13) for $s+2$, we obtain:

$$\begin{aligned}\sum_{i=1}^{n} x_i \frac{\partial P_s}{\partial x_i} &= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\left(\frac{1}{s+2}\Delta P_{s+2}\right)\\ &= \frac{1}{s+2}\left[\Delta\left(\sum_{i=1}^{n} x_i \frac{\partial P_{s+2}}{\partial x_i}\right) - 2\Delta P_{s+2}\right] \quad \text{by (9.0.17)}\\ &= \frac{1}{s+2}\left[\Delta\left((s+2)P_{s+2}\right) - 2\Delta P_{s+2}\right] \quad \text{by (9.0.13)}\\ &= \frac{1}{s+2} s\Delta P_{s+2}\\ &= sP_s \quad \text{by (9.0.12).}\end{aligned}$$

It is the relation (9.0.13) for P_s.
Consider a distribution T. The Laplacian of the product r^2T is the form:

$$\begin{aligned}\Delta(r^2T) &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}\right)^2\left(T\sum_{j=1}^{n} x_j^2\right)\\ &= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i}\sum_{j=1}^{n} x_j^2 + 2Tx_i\right)\\ &= \sum_{i=1}^{n}\left[\frac{\partial^2 T}{\partial x_i^2}\sum_{j=1}^{n} x_j^2 + \frac{\partial T}{\partial x_i}2x_i + 2x_i\frac{\partial T}{\partial x_i} + 2T\right]\\ &= r^2\Delta T + 4\sum_{i=1}^{n} x_i \frac{\partial T}{\partial x_i} + 2nT.\end{aligned}$$

$$\Longrightarrow r^2\Delta T = \Delta(r^2T) - 4\sum_{i=1}^{n} x_i\frac{\partial T}{\partial x_i} - 2nT. \tag{9.0.18}$$

We will show by recurrence that

$$r^2 P_s = (s+n)P_{s+2}. \tag{9.0.19}$$

Suppose that (9.0.19) holds for P_{s+2}, i.e.: $r^2 P_{s+2} = (s+n+2)P_{s+4}$. Then we have:

$$\begin{aligned}
r^2 P_s &= \frac{1}{s+2} r^2 \Delta P_{s+2} \\
&= \frac{1}{s+2}\left[\Delta(r^2 P_{s+2}) - 4\sum_{i=1}^{n} x_i \frac{\partial P_{s+2}}{\partial x_i} - 2nP_{s+2}\right] \\
&= \frac{1}{s+2}\left[\Delta(r^2 P_{s+2}) - (4s+8+2n)P_{s+2}\right] \\
&= \frac{1}{s+2}\left[(s+n+2)\Delta P_{s+4} - (4s+8+2n)P_{s+2}\right] \\
&= \frac{1}{s+2}\left[(s+n+2)(s+4)P_{s+2} - (4s+8+2n)P_{s+2}\right] \\
&= \frac{1}{s+2}\left[(s+2)(s+n)P_{s+2}\right] \\
&= (s+n)P_{s+2}.
\end{aligned}$$

By iterating, we obtain:

$$r^{2l} P_s = (s+n)(s+n+2)\dots(s+n+2(l-1))P_{s+2l}. \tag{9.0.20}$$

The distribution P_s is said to depend continuously on the parameter s if $< P_s, \varphi >$ is a continuous function of s for all $\varphi \in \mathscr{D}$.
If $s > -n$, we have:

$$\begin{aligned}
< P_s, \varphi > &= \int_{\mathbb{R}^n} P_s \varphi(x) dx \\
&= \int_{\Omega \in S^{n-1}} \left(\int_0^\infty P_s \varphi(r\Omega) r^{n-1} dr \right) d\Omega \\
&= \int_{supp\varphi} \frac{r^s \varphi}{2^{s/2}\Gamma(\frac{s+n}{2})} r^{n-1} dr d\Omega \\
&\leq \left| \int_{\Omega \in S^{n-1}} d\Omega \right| \left| \int_0^R c r^{s+n-1} dr \right| \\
&= area(S^{n-1}) c \frac{R^{s+n}}{s+n}
\end{aligned}$$

because,

$$\{|supp\varphi|\} \subset \{|x| \leq R\} \Longrightarrow |supp_{x\in\mathbb{R}^n}\varphi(x)| \leq c : \text{constant} \tag{9.0.21}$$

where $d\Omega$ is area of the sphere of radius 1 in R_n. The function to be integrated is a continuous function of s if $s > -n$ and of $x \in supp\varphi$ except at point $= 0$, bounded in modulus by an integrable function the integral is therefore a continuous function for $s > -n$, by the Lebesgue theorem.
For any s different from an even = negative integer if $n = 2$, we prove that P_s, depends continuously of s by using (9.0.14) and the fact that if $T \in \mathscr{D}'$ depends continuously on a parameter, it is the same for its partial derivatives, since

$$< \frac{\partial T}{\partial x_i}, \varphi >= - < T, \frac{\partial \varphi}{\partial x_i} >, \quad \forall\varphi, \quad \frac{\partial \varphi}{\partial x_i} \in \mathscr{D}. \tag{9.0.22}$$

Now, we will show that P_{-n-2p}, p integer ≥ 0, $n \neq 2$ is of the form $c_p\Delta^p\delta$, and we will deduce the limit (in the sens of distribution) of $(s+n)r^s$ when s tends to n, by values $> -n$, $n \neq 2$.

- **The case $s+n > 2$:**

If $s+n > 2$ is an integer odd, we have:

$$\begin{aligned}\Delta P_{s+2} = (s+2)P_s \Longrightarrow \Delta P_s = sP_{s-2}\\ \Longrightarrow \Delta\left(\frac{r^s}{2^{s/2}\Gamma(\frac{s+n}{2})}\right) = s\left(\frac{r^{s-2}}{2^{\frac{s-2}{2}}\Gamma(\frac{s+n-2}{2})}\right).\end{aligned}$$

$$\begin{aligned}\Longrightarrow \Delta(r^s) &= 2^{s/2}\Gamma(\frac{s+n}{2})\frac{sr^{s-2}}{2^{\frac{s-2}{2}}\Gamma(\frac{s+n-2}{2})}\\ &= 2sr^{s-2}\frac{\Gamma(\frac{s+n-2}{2}+1)}{\Gamma(\frac{s+n-2}{2})}\\ &= 2sr^{s-2}\frac{s+n-2}{2}\\ &= s(s+n-2)r^{s-2}.\end{aligned}$$

$$\Longrightarrow \Delta(r^s) = s(s+n-2)r^{s-2}. \tag{9.0.23}$$

- **The case $n > 2$:**

Consider $s = 2 - n$. We have then:

$$P_{2-n} = \frac{r^{2-n}}{2^{\frac{2-n}{2}}\Gamma(1)} = 2^{n/2-1}r^{2-n}.$$

We know that:

$$\Delta\frac{1}{r^{n-2}} = -(n-2)\frac{2(\sqrt{\pi})^n}{\Gamma(n/2)}\delta = -(n-2)S_n\delta \tag{9.0.24}$$

where S_n is the area of the sphere of radius 1 in R^n.
Using (9.0.12), we obtain:

$$\begin{aligned}\Delta P_s = sP_{s-2} &\Longrightarrow \Delta P_{2-n} = (2-n)P_{-n}\\ &\Longrightarrow \Delta(2^{n/2-1}r^{2-n}) = (2-n)P_{-n}\\ &\Longrightarrow -2^{n/2-1}(n-2)\frac{2(\sqrt{\pi})^n}{\Gamma(n/2)}\delta = (2-n)P_{-n}.\end{aligned}$$

$$\Longrightarrow P_{-n} = \frac{2^{n/2}(\sqrt{\pi})^n}{\Gamma(n/2)}\delta. \tag{9.0.25}$$

Using (9.0.14), we have also:

$$P_s = \frac{1}{(s+2),(s+4),\dots,(s+2k)}\Delta^k P_{s+2k}.$$

$$\begin{aligned}\Longrightarrow P_{-n-2p} &= \frac{1}{(-n-2p+2)(-n-2p+4)\dots(-n)}\Delta^p P_{-n}\\ &= \frac{(-1)^p}{(n+2p-2)\dots n}\Delta^p\left(\frac{2^{n/2}(\sqrt{\pi})^n}{\Gamma(n/2)}\delta\right)\\ &= \frac{(-1)^p}{(n+2p-2)\dots n}\frac{2^{n/2}(\sqrt{\pi})^n}{\Gamma(n/2)}\Delta^p\delta\end{aligned}$$

Consequently, P_{-n-2p}, $n \neq 2$ is of the form $c_p\Delta^p\delta$:

$$\Longrightarrow P_{-n-2p} = c_p\Delta^p\delta \tag{9.0.26}$$

where

$$c_p = \frac{(-1)^p}{(n+2p-2)\dots n}\frac{2^{n/2}(\sqrt{\pi})^n}{\Gamma(n/2)}.$$

Thus, we obtain the elementary solution of iterated Laplacian Δ^p:

$$\Delta^p\left(\frac{P_{-n+2p}}{c_{-p}}\right)=\delta. \tag{9.0.27}$$

In conclusion, we have:

$$\Delta^p E=\delta \tag{9.0.28}$$

where $E=\Delta^p\left(\frac{P_{-n+2p}}{c_{-p}}\right)=\delta$ is a elementary solution of Δ^p.

- **The case $s>-n$:**

We have:

$$\begin{aligned}(s+n)r^s &= (s+n)P_s 2^{s/2}\Gamma(\frac{s+n}{2})\\ &= 2^{s/2+1}\Gamma(\frac{s+n}{2}+1)P_s \quad \text{by definition of Gamma function.}\end{aligned}$$

So, by definition of P_{-n}, we obtain:

$$\lim_{s\to -n} 2^{s/2+1}\Gamma(\frac{s+n}{2}+1)P_s=2^{1-n/2}P_{-n}=S_n\delta. \tag{9.0.29}$$

- **The case $n=2$ ($s>-2$):**

By replacing, in (9.0.4), P_{s+2} by $P_{s+2}-P_0$, we show that similar for $n=2$, P_s has a limit in D' when s tend to 2, and we will calculate this limit when s tend to $-2p$, p positive integer; we will call P_{-2p} this limit so that now P_s is defined for all s and all $n>2$.

$$P_s=\frac{r^s}{2^{s/2}\Gamma(\frac{s}{2}+1)},\quad P_0=1. \tag{9.0.30}$$

By the equation (9.0.12),

$$\begin{aligned}P_s=\frac{\Delta P_{s+2}}{s+2} &\Longrightarrow P_s=\Delta\left(\frac{P_{s+2}-P_0}{s+2}\right)\\ &\Longrightarrow \lim_{s+2\to 0}\Delta\left(\frac{P_{s+2}-P_0}{s+2}\right)=\lim_{s\to -2}P_s=P_{-2}\end{aligned}$$

On the other hand, we have:

$$\lim_{s+2\to 0}\frac{P_{s+2}-P_0}{s+2}=\left\{\frac{\partial P_s}{\partial s}\right\}_{s=0} \tag{9.0.31}$$

Taking integration by part, we conclude this limit:

$$\begin{aligned}\frac{\partial}{\partial s}(r^s 2^{-s/2}) &= 2^{-s/2}(\frac{\partial}{\partial s}r^s) + r^s(\frac{\partial}{\partial s})2^{-s/2} \\ &= 2^{-s/2}\left(\frac{\partial e^u}{\partial u}\frac{\partial u}{\partial s}\right) + r^s\left(\frac{\partial(2^{-s/2})}{\partial(-s/2)}\frac{\partial(-s/2)}{\partial s}\right) \quad \text{where } s\log r = u \\ &= 2^{-s/2}(r^s\log r) + r^s(2^{-s/2}\log 2).\end{aligned}$$

$$\Longrightarrow \left\{\frac{\partial}{\partial s}(r^s 2^{-s/2})\right\}_{s=0} = \log r + K, \quad K\text{:constant.} \tag{9.0.32}$$

Consequently, we obtain:

$$P_{-2} = \Delta \log r = 2\pi\delta. \tag{9.0.33}$$

$$P_{-2p} = \frac{2\pi}{(-2p+2)\ldots(-2)}\Delta^{p-1}\delta. \tag{9.0.34}$$

10 Conclusion

This thesis has explored significant mathematical topics including Schwartz distributions, Fourier transform, and the iterated Laplacian. Additionally, important concepts such as the Hadamard finite part, Cauchy principal values, and the Laplace operator have been thoroughly examined. The impact of these concepts on differential equations and their application has been emphasized. Particularly, the importance of the iterated Laplacian has been highlighted, and its potential in solving differential equations has been discussed.

Furthermore, the Fourier transform has been studied on the real line, $\mathbb{R}$, and in $\mathbb{R}^n$. The definition and properties of the Fourier transform on $\mathbb{R}$ have been elucidated, with a focus on spaces such as the Schwartz space and the space of rapidly decreasing functions. Similarly, the Fourier transform on $\mathbb{R}^n$ has been addressed, discussing its properties in the $\mathbf{L}^1$, $\mathscr{S}$, and $\mathbf{L}^2$ spaces.

In the subsequent sections of the thesis, the influence of the Hadamard finite part, Cauchy principal values, Laplace operator, and the iterated Laplacian in different coordinate systems has been investigated. Moreover, the practical applications of the Fourier transform in solving differential equations, particularly providing elementary solutions for the wave equation and the iterated Laplacian, have been discussed.

In conclusion, this thesis has delved into important topics in mathematical analysis, including Schwartz distributions, Fourier transform, the iterated Laplacian, and the solution of differential equations.

References

Aubin,J.P.(1987). *Analyse Fonctionnelle Appliquée*, PUF, .

Choquet-Bruhant, Y.(1973). *Distributions Théorie et Problèmes*.

Hörmander, L. (1983). *The Analysis of Linear Partial Differential Operators*, Springer.

Rudin, W. (1987). *Real and Complex Analysis*.

Schwartz, L. (1966). *Théorie des Distibutions*, Hermann.

Schwartz, L. (1993). *Analyse IV,Application à la Théorie de la Mesure*, Hermann.

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