

MANIPULATION VIA INFORMATION IN LARGE ELECTIONS

A Master's Thesis

By
İLHAN SEZER

Department of Economics
Bilkent University
Ankara

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MANIPULATION VIA INFORMATION IN LARGE ELECTIONS

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İLHAN SEZER

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Semih Koray
Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Asst. Prof. Tarık Kara
Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Asst. Prof. Emre Berk
Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Erdal Erel
Director

ABSTRACT

MANIPULATION VIA INFORMATION IN LARGE ELECTIONS

Sezer, İlhan

M.A., Department of Economics

Supervisor: Prof. Dr. Semih Koray

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This thesis studies manipulations of equilibria by candidates in two-alternative elections along with their effects on voter turnout, winner of the election and social welfare where voters have common values, and both voting and manipulating are costly. We show that manipulation is not desirable for the society, and the candidates' incentives for manipulating can be mitigated by appropriately sequencing the order of manipulations. We present some results of a manipulation game which may rather unexpected under the assumption that the candidates have prior beliefs about each others' manipulations. Finally we determine the set of manipulations which can be prevented by informed voters for a given composition of society.

Keywords: Voting, Manipulation, Social Welfare, Voter Turnout, Trust Ratio

ÖZET

SEÇMEN SAYISI BÜYÜK SEÇİMLERDE BİLGİ ÜZERİNDEN MANİPÜLASYON

Sezer, İlhan

Yüksek Lisans, Ekonomi Bölümü

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Bu çalışmamızda oy kullanma ve manipülasyon yapmanın maliyetli olduğu, ortak değerli iki alternatifli seçimler çerçevesinde adayların denge manipülasyonlarını, bunların oy kullanma oranı, seçimi, galibi ve toplumsal refah üstündeki etkileriyle birlikte inceledik. Manipülasyonun toplum için istenilir olmadığını ve adayları manipülasyona özendiren etkenlerin uygun bir manipülasyon yapma sıralaması ile yumuşatılabileceğini gösterdik. Adayların birbirlerinin manipülasyonları hakkında önsel bir sanıya sahip oldukları varsayımı altında oynanan bir manipülasyon oyununun şaşırtıcı olabilecek bazı sonuçlarına ulaştık. Son olarak verilmiş bir toplum bileşimi için bu konuda bilgi sahibi seçmenler tarafından önlenebilecek manipülasyon kümesini belirledik.

Anahtar Kelimeler: Oy kullanma, Manipülasyon, Toplumsal Refah, Seçmen Katılımı, İtimat Oranı

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CHAPTER I

INTRODUCTION

Elections have become an important part of our life in democratic societies. In our modern world election is the most frequently used mechanism by which the offices in the executive, judiciary, regional or local government are filled. Elections are also held in many other settings, ranging from clubs and societies to business. The idea which results this situation is that the person who is supported by the majority should be the executive, and elections are held for determining this person. There are two sides in each election, the candidates and the voters. It is the interaction between these two sides, which determines the outcome of an election. On one side, the candidates try to influence voters' preferences along with their decisions to participate in the elections. They do this by giving information about their future plans or relevant properties of themselves or their parties, where this information need not be true. On the other side, the voters do affect the candidates: lots through the voters they cast or by simply not casting any vote at all. So the underlying reasons for a voter to support a candidate and cast a vote in his favor, as well as the possible efforts of the candidates to influence the results of an election should be researched, aiming to reach socially desirable results in the elections.

There have been many studies concerning these issues from various aspects. In this study we will focus on the effects of possible manipulations on the part of the candidates. However, such an analysis cannot be done without analyzing voters' behavior. In the present setup, we refer to Taylor and Yildirim (2005), who consider voting in two alternative elections with common value. They present a theory of strategic voting, and figure out the effects of relevant information on voters' behaviors. In their setting, the relevant information is restricted to the distribution of political preferences, and they regard voting to be costly. A voter casts a vote for his preferred candidate only if the effect of his vote, which is proportional to the probability of his vote being the decisive vote, is not less than the cost of voting for him. In our study, we analyze candidate's behavior, while regarding how voters behave in this setting we borrow the results of Taylor and Yildirim (2005).

In the light of Taylor and Yildirim (2005), we consider in our study two different cases. In the first case of informed voters, the voters are assumed to be informed about the distribution of political preferences. The second case is that of uninformed voters, where the voters are assumed to be symmetrically ignorant about the distribution of types. The voters are not aware of the manipulations in either of the settings. In the first case if the candidates manipulate simultaneously, then cooperation among the candidates is impossible. We try to predict what manipulations will take place along with the result of the election, when the candidates have a prior belief about each others' manipulation. On the other hand when the candidates manipulate sequentially, there are equilibria in

which the order influences the cost of the election both the candidates and the society. In the second setting, there turns out to be a positive probability that one of the candidates wins without manipulation whatever the other candidate does. Whenever this is the case, cooperation is obviously possible, but otherwise cooperation is again impossible. Using a setup where candidates have prior beliefs about each others' manipulations, we get some unexpected results: A candidate who has the support of a majority may lose the election. When the candidates manipulate sequentially, we again determine the effect of sequencing on the cost and the winner. We examine what properties a society, which can undo the effects of manipulations is endowed with, under the assumption that the people who are not manipulated know the exact number of manipulated voters. We observe that the society can prevent some manipulations, while manipulation is always possible as long as the society is large and there are manipulable voters manipulation is always possible.

The rest of the thesis is organized as follows. In the next chapter, we survey the literature on voting concerned with turnout, manipulation and social welfare. In chapter 3, the framework and findings of Taylor and Yildirim (2005) are summarized. Chapter 4 deals with manipulation via information on the part of candidates along with its welfare implications. Chapter 5 concludes our study.

CHAPTER II

LITERATURE SURVEY

Voting and manipulation are important in democracy and have been studied in many papers. A significant observation about democratic political process is that only a fraction of eligible voters turn out to vote, which gives rise to an important question: “Why do some voters not vote?”. To answer this question we have to accept that voting is costly for the voters, for otherwise they would vote for certain as this would increase the probability of victory of the candidate they support. When voting has a positive cost, the fraction of the electorate participating in an election will depend on the magnitude of cost, and this will of course have an impact upon welfare of the society. In the voting literature, the works of Ledyard (1981), Palfrey and Rosenthal (1983, 1985) are important for understanding the theory of participation, While Ledyard (1984), Borgers (2004) and Campbell (1999) can be consulted regarding the welfare aspect.

Ledyard (1981) models simultaneous voting. In his model, the citizens have rational expectations about the probability of being decisive; they know the size of the electorate and their own preferences. The probability distribution from which the other voters’

preferences are drawn is assumed to be common knowledge. The major result of this paper is that turnout is positive, when the cost of voting is very small for some of the citizens, when compared with their valuation for their candidates' victory. Ledyard's conclusions about turnout are rather limited. But Palfrey and Rosenthal (1983) obtain a richer variety of conclusions using a similar approach. They consider a costly voting game-theoretic model under complete information. They focus on the voter turnouts at some equilibria of the voting game which they characterize in their study. Their predictions about the size of voter turnout are rather weak for small numbers of voters, but for large electorates they conclude that the percentage of voter turnout approaches zero or the turnout percentage of the majority side is smaller than that of the minority side. In their 1985 paper, the citizens are assumed to be privately informed about their own cost of voting, and it is shown that this leads to a unique equilibrium at which only citizens with costs below an endogenous threshold vote.

Ledyard (1984) develops a Bayesian model of a two candidate competition to deal with the game theoretic considerations on the part of the voter. In his model the voters vote deterministically, they can vote for one of the candidates or abstain. The cost of voting is a random variable. His major result is that, in large elections, if voting costs are non-negative there is an equilibrium at which the social welfare is optimal. At this equilibrium the candidates choose identical positions which maximize welfare, and at which no one votes. But Ledyard (1984) does not study the possibility of multiple equilibria. Campbell (1999) studies a model in which the voters in the minority group

possess stronger political preferences (lower costs) than the ones in the majority. He finds that when the number of voters tends to infinity, the minority group wins the election with a probability which is at least $\frac{1}{2}$. This result may seem a bit unexpected but it is supported by other recent papers on this issue. Borgers (2004) investigates an informed voter setting in which it is common knowledge that each citizen is equally likely to support either candidate. He proves the existence of a unique symmetric equilibrium when preferences are independent, and shows that equilibrium turnout may be excessive, which in term means that these equilibria are suboptimal.

Goeree and Grosser (2004), Taylor and Yildirim (2005) are two recent studies, which are closely related to the present study of ours. Goeree and Grosser (2004) present a simple two candidate voting model to answer the following question: “Why do the polls predict the wrong outcome?” Their results support the intuition that after releasing the results of the pre-election polls, the members of the minority group participate in the election more frequently to offset the advantage of the majority group. They also find that polls raise the expected turnout but reduce the expected welfare by increasing the effect of the minority group, because the election leads to in a tie in terms of expected values. Their results support Campbell (1999) too. Taylor and Yildirim (2005) consider the impact of public information on expected turnout and electoral outcomes. They present a strategic voting model with two candidates, which predicts the probability of victory for the majority group to be less than that for the minority group (democratic inefficiency),

and voter turnout turns out to be higher (economic inefficiency) when citizens possess more information about the composition of the electorate. Thus social welfare is higher when the citizens possess less information. A fine point of their model is that it can easily be extended to deal with other issues.

There is also a recent empirical study of Levine and Palfrey (2005) on voter participation. They test three voter turnout predictions of the rational choice Palfrey-Rosenthal model of participation with asymmetric information: turnout goes down in larger electorates (size effect), turnout is higher in elections in which the votes the candidates receive are expected to be close (competition effect), voters supporting the less popular alternative have higher turnout rates (underdog effect).

Taylor and Yildirim (2005) concerns the effect of information on voters' behaviors. Here we continue this story by examining how the candidates can use the effect of information, by manipulating the knowledge of voters. Thus another important notion, manipulation, is included into our study. Gibbard (1973) proves that any non-dictatorial voting scheme with at least three possible outcomes is subject to individual manipulation. Following Gibbard (1973), there are some papers which compare the vulnerability of different election methods, exemplified by Lepelley and Mbih (1994), Dummett (1998) and Favardin, Lepelley and Serais (2002). Some papers study the complexity of manipulating existing protocols, as Conitzer and Sandholm (2002), Conitzer, Lang and Sandholm (2003); Finally, papers as Franzese (1999) and Drazen and Eslava (2005)

study the effects of manipulations in different models and some other studies the effects of manipulations in different models.

Lepelley and Mbih (1994) define vulnerability of a social choice function to coalitional manipulation of preferences as the proportion of voting situations in which the social choice function can be manipulated by a coalition of individuals. They provide exact relations yielding the vulnerability of four specific social choice functions (plurality rule, anti-plurality rule, plurality with runoff, anti-plurality with runoff) in three-alternative elections and show that plurality with runoff is less vulnerable than the others. Dummett (1998) draws attention to the objection that Borda count is subject to agenda manipulation and describes two possible ways for fixing it by introducing revised and adjusted Borda scores. He shows that these will often, but not always, undo the agenda manipulation effect, and questions whether it is desirable at all to undo it altogether. Favardin, Lepelley and Serais (2002) characterize the voting situations at which the Borda rule or the Copeland method can be manipulated in three-alternative elections and derive some analytical representations measuring the vulnerability of these rules to strategic misrepresentation of preferences. They show that the Borda rule is significantly more vulnerable than the Copeland method.

Conitzer and Sandholm (2002) deal with the level of difficulty of manipulation for practical multi-agent settings where the number of voters can be large but the number of candidates is small. They show that individual and coalitional manipulations-with

unweighted voters- are easy with complete information about others' votes. However, constructive coalitional manipulation with weighted voters turns out to be intractable for all of the voting protocols under study, except for the nonrandomized cup. Destructive manipulation is shown out to be relatively easy, while randomizing over instantiations of the protocols makes the manipulation hard. Finally, they show that if weighted coalitional manipulation with complete information about the others' votes is hard in some voting protocol, then individual and unweighted manipulation is also hard when there is uncertainty about the others' votes. Conitzer, Lang and Sandholm (2003) are concerned with the necessary number of candidates to make elections hard to manipulate. They show that the voting protocols they study become hard to manipulate with three candidates, four candidates, seven candidates or never. Conitzer and Sandholm (2003) show how to tweak existing protocols to make manipulation hard, where the tweak consists of adding one elimination preround to the election.

Franzese (1999) examines the manipulation of public debt, and suggests an equally rational-strategic logic for partisan manipulation. Drazen and Eslava (2005) present a model of the political budget cycle in which they consider manipulation via expenditure composition. They predict that election-year shifts of the budget improve the incumbent's chances of being re-elected, since voters are manipulated about the incumbent's true preferences for types of spending with a positive probability. They also find that the incumbent party is penalized for running large deficits before elections and is rewarded for increasing the amount of targeted spending.

Although most of the researchers think that manipulation is a serious problem, there are authors who do not worry about manipulations. Hees and Dowding (2005) call certain manipulations as 'sincere' and 'transparent' manipulations, and claim that they are virtues rather than vices. They show that a class of familiar voting procedures only allows sincere and transparent manipulation, so there is no need to fear manipulations.

CHAPTER III

SUMMARY OF TAYLOR AND YILDIRIM (2005)

3.1 The Model

Suppose that there are $n \geq 2$ voters who may vote in an election between two candidates, A and B. Each voter is one of the two types: A voter prefers A, and B voter prefers B. A t voter receives payoff 1 if t is implemented and 0 otherwise, for $t=A,B$ where cost of voting is $c \in (0,1]$ for all voters. Thus all voters have a common value for the victory of their candidates, and each possesses two actions, to abstain or to vote for his preferred candidate. Voters simultaneously choose whether to vote or not. Majority rule is used in the election and ties are broken by a fair coin toss. Each voter privately knows his type but believes that another voter is an A voter with probability $\lambda \in (0,1)$. Taylor and Yildirim compare the symmetric Bayesian Nash Equilibrium (BNE) outcomes of the voting games described across two informational settings. The value of λ is common knowledge among the voters in the informed-voter setting. All the voters have a

non-degenerate common prior over possible values of λ in the uninformed-voter setting.

All the results listed in this chapter are borrowed from Taylor and Yildirim (2005)

3.2 Informed Voters

Denote the probability that a t voter casts a vote with φ_t . A symmetric BNE is a pair (φ_a, φ_b) such that it is optimal for a t voter to vote with probability φ_t when all other voters use this strategy. Note that voters compare the expected payoff from voting with the payoff from abstaining, to find such an equilibrium; and for the equilibria in totally mixed strategies these payoffs must be equal. Let the ex ante probability that an agent votes for candidate A and B be denoted respectively by α_a and α_b , whence $\alpha_a = \lambda\varphi_a$ and $\alpha_b = (1-\lambda)\varphi_b$. The number of ways k_a voters can vote for A, k_b voters can vote for

B and $n-1-k_a-k_b$ can abstain is $\binom{n-1}{k_a, k_b, n-1-k_a-k_b} = \frac{(n-1)!}{k_a!k_b!(n-1-k_a-k_b)!}$.

Lemma 1: The net expected utility to a type t agent from voting can be written as

$\frac{1}{2}P_t(\alpha_t, \alpha_{t'}) - c$ where

$$P_t(\alpha_t, \alpha_{t'}) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[\binom{n-1}{k, k, n-1-2k} \alpha_t^k \alpha_{t'}^k (1-\alpha_t-\alpha_{t'})^{n-1-2k} \right] \\ + \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \left[\binom{n-1}{k, k+1, n-2-2k} \alpha_t^k \alpha_{t'}^{k+1} (1-\alpha_t-\alpha_{t'})^{n-2-2k} \right] = 2c$$

for $t=a, b$, $t \neq t'$, and $\lfloor \cdot \rfloor$ is the usual operator that rounds a number to the lower integer when necessary.

Proposition 1 (Neutrality): Suppose (φ_a, φ_b) is a symmetric BNE in totally mixed strategies; i.e., $0 < \varphi_t < 1$ for $t = a, b$.

(i) The ex ante probability that an agent votes for alternative A equals the ex ante probability that he votes for B, i.e. $\lambda \varphi_a = (1 - \lambda) \varphi_b$.

(ii) Both outcomes are equally likely in equilibrium, i.e.

$$\Pr\{A \text{ wins} \mid \lambda\} = \Pr\{B \text{ wins} \mid \lambda\} = \frac{1}{2}.$$

Let α denote the ex ante probability that an agent votes for one of the two candidates in a symmetric totally mixed-strategy BNE when one exists, so he casts a vote with probability 2α , which means $\alpha \in (0, 0.5]$. Thus to find a mixed-strategy equilibrium, we have to find $\alpha \in (0, 0.5]$ solving $\frac{1}{2}P(\alpha, n) - c = 0$ where $P(\alpha, n) = P_t(\alpha, \alpha)$.

Lemma 2: $P(\alpha, n)$ is strictly decreasing in $\alpha \in (0, 0.5]$.

An important implication of this result is that there is at most one solution to

$\frac{1}{2}P(\alpha, n) - c = 0$. Furthermore there is a such solution if and only if $\frac{1}{2}P(0, n) > c$ and

$\frac{1}{2}P(\frac{1}{2}, n) \leq c$. Let $\alpha(c, n)$ denote the solution when it exists.

Proposition 2 (Characterization): There exists a unique symmetric mixed-strategy

BNE if and only if $c(n) \leq c \leq \frac{1}{2}$ and $\alpha(c, n) < \lambda < 1 - \alpha(c, n)$ where

$$c(n) = \begin{cases} \left(\frac{n-1}{2} \right) \left(\frac{1}{2} \right)^n & \text{if } n \text{ is odd} \\ \left(\frac{n}{2} \right) \left(\frac{1}{2} \right)^n & \text{if } n \text{ is even.} \end{cases}$$

Lemma 3: (i) $c(n)$ is decreasing and converges to zero as $n \rightarrow \infty$.

(ii) $\alpha(c, n)$ is decreasing and converges to zero as $n \rightarrow \infty$.

Theorem 1 (Neutrality in Large Elections): For any voting cost $c \in (0, 0.5)$ and any distribution of preferences $\lambda \in (0, 1)$, there exists a critical population size $\bar{n}$ such that $n \geq \bar{n}$ implies the existence of a unique symmetric BNE (φ_a, φ_b) . Moreover, in this equilibrium $\lambda \varphi_a = (1 - \lambda) \varphi_b$ and $\Pr\{A \text{ wins} \mid \lambda\} = \Pr\{B \text{ wins} \mid \lambda\} = \frac{1}{2}$.

The disturbing results in Proposition 1 and Theorem 1 are a consequence of the strategic voting behavior that arises as a result of agents having information about voter preferences. When agents do not know the actual value of λ , the alternative favored by the majority is more likely to win. This is demonstrated in the following setting, which considers an uninformed voter where n is respectively finite and infinite.

3.3 Uninformed Voters

In this setting λ is not common knowledge. Suppose that before learning their types, the agents' beliefs about λ correspond to a non-degenerate common prior distribution. Assume that the prior is symmetric and defined over a finite set of values, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_r$ ($r \geq 2$). From the symmetry of the prior we have $\lambda_i = 1 - \lambda_{r+1-i}$ and $\Pr\{\lambda = \lambda_i\} = \Pr\{\lambda = 1 - \lambda_i\} = \theta_i \in (0,1)$ for $\forall i \in \{1, \dots, n\}$. Note that that $\theta_i = \theta_{r-i}$ for $\forall i \in \{1, \dots, n\}$ and $\sum_{i=1}^r \theta_i = 1$. After learning their types, agents' updated beliefs are $P(\lambda = \lambda_i | A) = 2\theta_i \lambda_i$ and $P(\lambda = \lambda_i | B) = 2\theta_i (1 - \lambda_i)$ for $\forall i \in \{1, \dots, n\}$. As in the informed voter setting a symmetric BNE is a pair of voting probabilities (φ_a, φ_b) , and (φ_a, φ_b) is a symmetric BNE in totally mixed strategies if and only if it satisfies $\frac{1}{2}E[P_a(\lambda\varphi_a, (1-\lambda)\varphi_b) | A] - c = 0$ and $\frac{1}{2}E[P_b((1-\lambda)\varphi_b, \lambda\varphi_a) | B] - c = 0$.

Proposition 3 (Non-Neutrality): Suppose (φ_a, φ_b) is symmetric BNE in totally mixed strategies.

(i) Both types of agents vote with the same probability $\varphi_a = \varphi_b = \varphi$.

(ii) For $n < \infty$, the candidate favored by the expected majority is more likely to win

the election, $\Pr\{A \text{ wins} | \lambda\} > \frac{1}{2} \Leftrightarrow \lambda > \frac{1}{2}$.

To characterize a symmetric BNE in totally mixed strategies in the uninformed voter setting, recall that the agents vote with the same probability φ . Given λ , the probability that an agent's vote is pivotal is $\bar{P}_a(\varphi | \lambda, n) = P_a(\lambda\varphi, (1-\lambda)\varphi)$ if he is an A voter and $\bar{P}_b(\varphi | \lambda, n) = P_a((1-\lambda)\varphi, \lambda\varphi)$ if he is a B voter. Under the symmetric prior, each type of agent expects to be pivotal with probability $Q(\varphi, n | \theta) = E[P_t(\varphi | \lambda, n) | t]$

$$= \sum_{i=1}^{\left\lfloor \frac{r+1}{2} \right\rfloor} 2\theta_i T(\varphi, n, \lambda_i) \text{ for } t=a,b \text{ where } T(\varphi, n, \lambda_i) = \lambda_i \bar{P}_a(\varphi | \lambda_i, n) + (1-\lambda_i) \bar{P}_a(\varphi | 1-\lambda_i, n)$$

for $\lambda_i \neq 0.5$ and $T(\varphi, n, 0.5) = \bar{P}_a(\varphi | 0.5, n) / 2$. Hence a symmetric BNE in totally mixed strategies corresponds to a solution $\varphi(n) \in (0,1)$ which of the equation

$$\frac{1}{2} Q(\varphi, n | \theta) - c = 0.$$

Proposition 4 (Characterization): There exists a unique symmetric BNE in totally mixed strategies if and only if $\underline{c}(n) < c < 0.5$, where

$$\underline{c}(n) = \begin{cases} c(n) 2^n E[\lambda^{\frac{n+1}{2}} (1-\lambda)^{\frac{n-1}{2}}], & \text{if } n \text{ is odd} \\ c(n) 2^n E[(\lambda(1-\lambda))^{\frac{n}{2}}], & \text{if } n \text{ is even.} \end{cases}$$

Corollary 1: For any n , $\underline{c}(n) < c(n)$, and hence $\lim_{n \rightarrow \infty} \underline{c}(n) = 0$.

This result indicates that if a symmetric BNE in totally mixed strategies obtains in the informed setting, then an analogous BNE obtains in the uninformed-voter setting too. Moreover, for n sufficiently large, existence and uniqueness of these equilibria are assured.

Proposition 5 (Expected Turnout): Suppose there is a symmetric BNE in totally mixed strategies in each of the two informational settings. Then the expected equilibrium voter turnout is higher in the informed-voter setting than in the uninformed-voter setting, i.e. $2n\alpha(c, n) > n\phi(n)$.

Definition 1 (Mean-Preserving Spread): Let $\theta' = (\theta'_1, \theta'_2, \dots, \theta'_r)$ and $\theta = (\theta_1, \theta_2, \dots, \theta_r)$ be two symmetric distributions over $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$. Distribution θ' is said to be a mean-preserving spread of θ if there is some $i_0 \in \{1, 2, \dots, \lfloor (r+1)/2 \rfloor\}$ such that

$$\begin{cases} \theta'_i \geq \theta_i, & \text{if } i = 1, \dots, i_0 \\ \theta'_i \leq \theta_i, & \text{if } i = i_0 + 1, \dots, \lfloor (r+1)/2 \rfloor. \end{cases}$$

Proposition 6 (Uncertainty): Let θ' and θ be two symmetric priors that respectively induce the equilibrium voting probabilities ϕ'^* and ϕ^* . If θ' is a mean-preserving spread of θ , then $\phi'^* \leq \phi^*$.

This result indicates that voters are less likely to vote when there is more uncertainty about the distribution of voter preferences.

Theorem 2 (Welfare): Suppose a symmetric BNE in totally mixed strategies obtains in either informational setting. Then, for any given λ , expected equilibrium welfare is higher in the uninformed-voter setting than in the informed-voter setting.

This indicates that for a fixed n , expected welfare is higher when voters are uninformed. In particular in the uninformed-voter setting, the candidate preferred by the majority is more likely to win the election and expected voting costs are strictly less.

3.4 Large Elections

In this section the uninformed-voter setting is studied in the important case as the number of agents tends to infinity.

Lemma 4: (i) $\varphi^*(n)$ is decreasing.

(ii) $\lim_{n \rightarrow \infty} \varphi^*(n) = 0$.

(iii) $\lim_{n \rightarrow \infty} n\varphi^*(n) < \infty$.

For a given n and λ , let X_a and X_b be the random variables representing the equilibrium number of votes for candidates A and B, respectively.

Lemma 5: Let $\lim_{n \rightarrow \infty} n\varphi^*(n) = m$. Then, the limiting marginal distributions of X_a and X_b are independent Poisson distributions with respective means λm and $(1 - \lambda)m$.

Lemma 6: Let X_a and X_b be two independent random variables that follow Poisson distributions with respective means λm and $(1 - \lambda)m$. Then:

(i) $\lim_{m \rightarrow \infty} \Pr\{X_a = X_b + k\} = 0$ for any finite integer k ;

(ii) $\lim_{m \rightarrow \infty} \Pr\{X_a > X_b\} = 1$ if and only if $\lambda > \frac{1}{2}$.

Notice that $T(\varphi^*(n), n, \lambda_i) = \Pr\{X_b = X_a\} + 2\lambda_i \Pr\{X_b = X_a + 1\}$. Applying the results of lemma 5 we get $\lim_{n \rightarrow \infty} T(\varphi^*(n), n, \lambda_i) = \lim_{m \rightarrow \infty} [\Pr\{X_b = X_a\} + 2\lambda_i \Pr\{X_b = X_a + 1\}]$

$$= e^{-m} \left[\sum_{k=0}^{\infty} (\lambda_i (1 - \lambda_i))^k \frac{m^{2k}}{(k!)^2} + 2 \sum_{k=0}^{\infty} (\lambda_i (1 - \lambda_i))^{k+1} \frac{m^{2k+1}}{k!(k+1)!} \right] = J(m, \lambda_i). \quad \text{Thus, expected}$$

equilibrium voter turnout in the limit, m^* , is determined by $\sum_{i=0}^{\lfloor (r+1)/2 \rfloor} \theta_i J(m, \lambda_i) - c = 0$.

Lemma 7: (i) For $c \in (0, 0.5)$ there exists a unique and finite expected equilibrium voter turnout in the limit, m^* .

(ii) m^* is strictly decreasing in c .

(iii) For any $M < \infty$ there exists $\bar{c}(M) > 0$ such that $c < \bar{c}(M)$ implies $m^* > M$.

Theorem 3 (Non-Neutrality in Large Elections): (i) If $n \rightarrow \infty$, then the candidate favored by the majority is more likely to win the election, $\Pr\{A \text{ wins} \mid \lambda\} > \frac{1}{2} \Leftrightarrow \lambda > \frac{1}{2}$.

(ii) If $n \rightarrow \infty$, then the probability that the alternative favored by the majority wins is arbitrarily close to one when the cost of voting is sufficiently small:

$$\lim_{c \rightarrow 0} \Pr\{A \text{ wins} \mid \lambda\} = 1 \Leftrightarrow \lambda > \frac{1}{2}.$$

Theorems 1, 2 and 3 tell us that whether elections are large or small, a setting with uninformed voters results in electoral outcomes possessing higher expected social benefits and lower expected social costs. The difference between the two informational regimes is most stark in the case of large elections in which the cost of voting is very small.

In this chapter we have reported some results from Taylor and Yildirim (2005) out the behavior of the voters when they have some information or belief about the distribution of political preferences. In the next chapter we will consider the behavior of the candidates, when they have the power to influence some of the voters' information about the distribution of political preferences, where their influence is restricted in such a fashion that a symmetric BNE in totally mixed-strategy obtains for all the voters.

CHAPTER IV

MANIPULATION VIA INFORMATION IN LARGE ELECTIONS

4.1 The Manipulation Game of Two Candidates with Informed Voters

The model is almost the same as the one in section 3.2, except that the voters will now learn λ from the companies which predict the distribution of political preferences. There are three different types of company: A, B and fair type. For $i \in \{A, B\}$, i type company supports candidate i , which allows candidate i to release wrong information via such a company and manipulate voters' information. None of the i voters trust the other candidate's company. Only a portion of the i voters will believe what the i type company says. The ratio of voters who believe in the i company is denoted by n_i . We refer to n_i as the trust ratio of candidate i . The remaining i voters believe in the fair type company's prediction. The only restriction on the manipulation is its cost, which is given by $c_i : [0, 1] \rightarrow \mathbb{R}^+$ for candidate $i \in \{A, B\}$. $c_i(\lambda_i)$ is to be interpreted as the cost of making the voters who trust i to believe that λ_i is the proportion of i voters in the society, while the true value λ for this proportion is given and fixed. We assume that this manipulation

cost function is continuous, increases with the distance between the manipulated value and λ , and is zero when there is no manipulation, or in other words the manipulated value is λ . When A manipulates $\lambda_a \in (0,1)$ and B releases $\lambda_b \in (0,1)$, the profit function of candidate $i \in \{A, B\}$ is given by:

$$\pi_i(c, n, \lambda_a, \lambda_b) = P(i \text{ wins} | (c, n, \lambda_a, \lambda_b)) - c_i(\lambda_i) \quad (1)$$

So the cost of manipulation is expressed in units of winning probability. The candidates manipulate so as to maximize their profit functions. In order to find the probabilities of winning for the candidates, we need to find the voting probabilities of all the voters after the manipulations. To be able to use Proposition 1 for this purpose, we should guarantee the existence of a symmetric BNE in totally mixed strategies in the voter's game voters for all the three predictions of political preferences. As a result of Proposition 2 the existence of such a BNE is guaranteed when $c(n) \leq c \leq 0.5$, and $\lambda, \lambda_a, \lambda_b \in (\alpha(c, n), 1 - \alpha(c, n))$. We assume that c is in the interval $[c(n), 0.5]$ and the cost functions satisfy $c_i(\alpha(c, n)) \geq 1$ and $c_i(1 - \alpha(c, n)) \geq 1$ for $i \in \{A, B\}$ to prevent the manipulations outside the interval $(\alpha(c, n), 1 - \alpha(c, n))$. With these assumptions, using Proposition 1, we observe that A voters who believe in the A type company vote with probability $\alpha(c, n) / \lambda_a$, B voters who believe in the B type company vote with probability $\alpha(c, n) / (1 - \lambda_b)$, the remaining A voters vote with probability $\alpha(c, n) / \lambda$ and the remaining B voters vote with probability $\alpha(c, n) / (1 - \lambda)$. So

$$P(A \text{ voter votes for candidate A}) = n_a \frac{\alpha(c, n)}{\lambda_a} + (1 - n_a) \frac{\alpha(c, n)}{\lambda}$$

$$P(B \text{ voter votes for candidate B}) = n_b \frac{\alpha(c, n)}{1 - \lambda_b} + (1 - n_b) \frac{\alpha(c, n)}{1 - \lambda}$$

We denote the total number of people who cast a vote in the election –voter turnout- by k and define

$$\begin{aligned} K(\lambda_a, \lambda_b) &= \frac{P(\text{any voter votes for candidate A})}{P(\text{any voter votes for candidate B})} \\ &= \frac{\lambda}{1 - \lambda} \frac{P(A \text{ voter votes for candidate A})}{P(B \text{ voter votes for candidate B})} = \frac{n_a(\lambda / \lambda_a) + 1 - n_a}{n_b((1 - \lambda)/(1 - \lambda_b)) + 1 - n_b} \end{aligned}$$

Lemma 8: The respective probabilities of A and B winning the election are given by:

$$P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \begin{cases} \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j}}{\sum_{j=0}^k \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j}}, & \text{if } k \text{ is odd} \\ \frac{\left(\sum_{j=0}^{\frac{k-2}{2}} \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j} \right) + \left(\frac{k}{2} \right) \frac{K(\lambda_a, \lambda_b)^{\frac{k}{2}}}{2}}{\sum_{j=0}^k \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j}}, & \text{if } k \text{ is even} \end{cases} \quad (2)$$

$$P(B \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \begin{cases} \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} \left(\frac{1}{K(\lambda_a, \lambda_b)} \right)^{k-j}}{\sum_{j=0}^k \binom{k}{j} \left(\frac{1}{K(\lambda_a, \lambda_b)} \right)^{k-j}}, & \text{if } k \text{ is odd} \\ \frac{\left(\sum_{j=0}^{\frac{k-2}{2}} \binom{k}{j} \left(\frac{1}{K(\lambda_a, \lambda_b)} \right)^{k-j} \right) + \frac{1}{2} \left(\frac{k}{2} \right) \left(\frac{1}{K(\lambda_a, \lambda_b)} \right)^{\frac{k}{2}}}{\sum_{j=0}^k \binom{k}{j} \left(\frac{1}{K(\lambda_a, \lambda_b)} \right)^{k-j}}, & \text{if } k \text{ is even} \end{cases} \quad (3)$$

where λ_a and λ_b are values declared by companies A and B, respectively.

This lemma relates the probabilities of winning for the candidates to the probability that a voter votes for candidate A and the probability that a voter votes for candidate B. Notice that although the probability that a voter votes for candidate $i \in \{A, B\}$ depends on n and c , the probability of winning for the candidates depends only on $\lambda, \lambda_a, \lambda_b$ and k , but not n and c .

Lemma 9: When the candidates A and B let λ_a, λ_b declared respectively,

$$K(\lambda_a, \lambda_b) = 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \frac{1}{2}$$

$$K(\lambda_a, \lambda_b) > 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 1$$

$$K(\lambda_a, \lambda_b) < 1 \Rightarrow \lim_{k \rightarrow \infty} P(B \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 1$$

Lemma 9 tells us that when the voter turnout is large, there is sharp cut off in the manipulation game. The probability of winning the election for any of the candidates can be 0, 0.5 or 1, but nothing else is possible. From now on we will assume that the voter turnout is large, i.e. k tends to ∞ . Then, however, we have to show that this is possible because voter turnout is endogenous.

Lemma 10: For any $\varepsilon > 0$, any $M < \infty$, there exists $c(M, \varepsilon) > 0$, $N(M, \varepsilon) \in \mathfrak{R}$ such that for $\forall c < c(M, \varepsilon)$ and $\forall n > N(M, \varepsilon)$, $P(\text{voter turnout} < M) < \varepsilon$.

To interpret this result suppose that the society is large, i.e. n can be taken as large as we wish. Then for sufficiently small voting costs, the probability that the voter turnout is higher than a certain value can be made arbitrarily close to 1, and as c approaches to

zero the voter turnout goes to infinity with a probability which approaches to 1. As we want to make calculations when $k \rightarrow \infty$, these calculations will be true only if c goes to zero. When $k \rightarrow \infty$, we conclude the following proposition.

Proposition 7: The manipulation game between two candidates does not have any Nash equilibrium when $k \rightarrow \infty$.

As there is no Nash equilibrium cooperation of the candidates in the manipulation game is impossible. If you oblige the candidates to meet and decide on a procedure of manipulation, at least one of them has the incentive to deviate from the designated procedure as it is not the best that it can afford.

Now we will consider what happens when the candidates have some beliefs about each others' manipulations. We will use the probability distributions about each others' manipulations as the belief structures. Furthermore, we assume that none of the candidates knows what the other candidate thinks about his manipulation for the consistency of belief structures. Depending on their prior belief, candidate A maximizes his profit by letting declared, where λ_a solves:

$$\begin{aligned} \max \{ \pi_a(\lambda_a) : \lambda_a \in (0,1) \} &= \max \{ \text{probability of A wins} - c_a(\lambda_a) : \lambda_a \in (0,1) \} \\ &= \max \{ P(K(\lambda_a, \lambda_b) > 1) + \frac{1}{2} P(K(\lambda_a, \lambda_b) = 1) - c_a(\lambda_a) : \lambda_a \in (0,1) \}, \end{aligned}$$

and similarly candidate B solves the following problem:

$$\max \{ \pi_b(\lambda_b) : \lambda_b \in (0,1) \} = \max \{ \text{probability of B wins} - c_b(\lambda_b) : \lambda_b \in (0,1) \}$$

$$= \max \{P(K(\lambda_a, \lambda_b) < 1) + \frac{1}{2}P(K(\lambda_a, \lambda_b) = 1) - c_b(\lambda_b) : \lambda_b \in (0,1)\}$$

Now we have to figure out the changes in the magnitude of the function K , as the winner of the election is determined by the magnitude of K at the manipulated values.

$$K(\lambda_a, \lambda_b) = \frac{n_a \frac{\lambda}{\lambda_a} + 1 - n_a}{n_b \frac{1-\lambda}{1-\lambda_b} + 1 - n_b} > 1 \Leftrightarrow \lambda_b < 1 - \frac{(1-\lambda)n_b}{n_b - n_a + \frac{\lambda}{\lambda_a} n_a}$$

$$K(\lambda_a, \lambda_b) = \frac{n_a \frac{\lambda}{\lambda_a} + 1 - n_a}{n_b \frac{1-\lambda}{1-\lambda_b} + 1 - n_b} < 1 \Leftrightarrow \lambda_a > \frac{\lambda n_a}{n_a - n_b + \frac{1-\lambda}{1-\lambda_b} n_b}$$

Using these observations, and assuming that A believes that the manipulations of B have a continuous distribution function f_b , and B believes that the manipulations of A have a continuous distribution function f_a , the problem of the candidates can be stated in the terms of cumulative distribution functions. Candidate A chooses $\lambda_a \in (0, \lambda]$ which maximizes:

$$\pi_a(\lambda_a) = F_b \left(1 - \frac{(1-\lambda)n_b}{n_b - n_a + \frac{\lambda}{\lambda_a} n_a} \right) - c_a(\lambda_a)$$

and candidate B chooses $\lambda_b \in [\lambda, 1)$ which maximizes:

$$\pi_b(\lambda_b) = 1 - F_a \left(\frac{\lambda n_a}{n_a - n_b + \frac{1-\lambda}{1-\lambda_b} n_b} \right) - c_b(\lambda_b)$$

Notice that there can be multiple solutions to these maximization problems. To single out a unique solution, we assume that the candidates prefer the manipulation with the least cost most among the expected profit maximizers. If $K(\lambda_a, \lambda_b) > 1$, the winner of the election is candidate A; if $K(\lambda_a, \lambda_b) = 1$, each candidate can win the election with a probability of 0.5, and otherwise candidate B wins the election. The following example will clarify how these implications work.

Example 1: Let B believe that A's manipulation is uniformly distributed on $(0, \lambda]$, and let A believe that B's manipulation is uniformly distributed on $[\lambda, 1]$. Moreover, let

$$\text{the cost function be affine, namely, } c_a(x) = c_b(x) = \begin{cases} \frac{\lambda - x}{\lambda}, & x \in [0, \lambda] \\ \frac{x - \lambda}{1 - \lambda}, & x \in (\lambda, 1] \end{cases}.$$

Using the density and cost functions for maximizing $\pi_a(\lambda_a), \pi_b(\lambda_b)$, it turns out that A manipulates λ_a which is an element of $\{0, \lambda\} \cup \Lambda_a^{\text{int}}$ and B manipulates λ_b which is an element of $\{\lambda, 1\} \cup \Lambda_b^{\text{int}}$, where Λ_a^{int} is the set of all interior extreme points $\lambda_a^m \in (0, \lambda)$, which satisfy:

$$\begin{aligned} \frac{\partial \pi_a(\lambda_a^m)}{\partial \lambda_a} &= -f_b \left(1 - \frac{(1 - \lambda)n_b}{n_b - n_a + \frac{\lambda}{\lambda_a^m} n_a} \right) \frac{(1 - \lambda)\lambda n_a n_b}{(\lambda_a^m(n_b - n_a) + \lambda n_a)^2} - c'_a(\lambda_a^m) \\ &= -\frac{\lambda n_a n_b}{(\lambda_a^m(n_b - n_a) + \lambda n_a)^2} + \frac{1}{\lambda} = 0 \end{aligned}$$

and similarly Λ_b^{int} is the set of all $\lambda_b^m \in (\lambda, 1)$ which satisfy:

$$\begin{aligned}\frac{\partial \pi_b(\lambda_b^m)}{\partial \lambda_a} &= f_a \left(\frac{\lambda n_a}{n_a - n_b + \frac{1-\lambda}{1-\lambda_a^m} n_b} \right) \frac{(1-\lambda) \lambda n_a n_b}{((1-\lambda_b^m)(n_a - n_b) + (1-\lambda) n_b)^2} - c'_b(\lambda_b^m) \\ &= \frac{(1-\lambda) n_a n_b}{((1-\lambda_b^m)(n_a - n_b) + (1-\lambda) n_b)^2} - \frac{1}{1-\lambda} = 0.\end{aligned}$$

Choosing expected profit maximizers from the given sets, we get the following results:

$$n_a = n_b \Rightarrow \pi_a(\lambda_a) = 0 \quad \text{for} \quad \forall \lambda_a \in [0, \lambda], \quad \pi_b(\lambda_b) = 0 \quad \text{for} \quad \forall \lambda_b \in [\lambda, 1] \Rightarrow \lambda_a = \lambda,$$

$\lambda_b = \lambda \Rightarrow$ A and B has equal probabilities for winning the election.

$$n_a > n_b \Rightarrow \lambda_a = \frac{\lambda \sqrt{n_a}}{\sqrt{n_a} + \sqrt{n_b}}, \quad \lambda_b = \lambda \Rightarrow K(\lambda_a, \lambda_b) > 1 \Rightarrow \text{A wins the election.}$$

$$n_a < n_b \Rightarrow \lambda_a = \lambda, \quad \lambda_b = 1 - \frac{(1-\lambda) \sqrt{n_b}}{\sqrt{n_a} + \sqrt{n_b}} \Rightarrow K(\lambda_a, \lambda_b) < 1 \Rightarrow \text{B wins the election.}$$

This result differs from those in previous researches. The probabilities of winning the election for both of the candidates are equal in Campbell (1999), Goeree and Grosser (2004), Taylor and Yildirim (2005). The power of a candidate's manipulation is normally expected to increase if his trust ratio increases. In this example as the cost function is same for both of the players and all possible manipulations have a cost less than 1, the analysis is not disturbed by cost functions. We conclude that the winner depends only on the trust ratios. On the other hand if some disturbing results occur for different cost functions, the underlying reason will be that the candidates do not know anything about each others' beliefs and so their guesses about each others profit maximizing manipulation are not perfect.

The analysis of the setup with belief can be improved. But we will not pursue such an analysis any further and turn to exploring other possible equilibria of the manipulation game. Until now we assumed that the candidates manipulate simultaneously. Now we will consider what happens when they manipulate sequentially. We will assume that there is a social organizer who has the power to oblige the candidates to manipulate in a given order. One's intuition tells that if the candidate who starts the game makes a small manipulation, the other candidate will win the game with manipulation which is a bit large, and thus the first manipulation turns out to be meaningless. If the first manipulator increases the size of his manipulation, the other may match it accordingly. But they both have a constraint on the feasible set of manipulations, since the cost of manipulation should not be more than 1. Depending on their feasible sets the result of the game changes. If the second mover can win against any feasible manipulation of the candidate who moves first then, being rational, the first player should not make any manipulation, and the second candidate would win with a small manipulation. Otherwise, the first candidate has a feasible manipulation against which the second candidate has no feasible manipulation that would yield him a positive profit. In that case by making that manipulation, the first candidate wins the election. Now let us clarify the borders of these possible results with a proposition. But before stating the proposition, we will denote the manipulations which have unit cost with $a^* \in (0, \lambda)$ and $b^* \in (\lambda, 1)$ respectively, i.e. $c_a(a^*) = 1$ and $c_b(b^*) = 1$.

Proposition 8: For sufficiently large k , the following results are satisfied for $\forall c < 0.5$.

(I) $(\lambda - a^*)n_a(1 - b^*) > (b^* - \lambda)n_b a^* \Rightarrow$ A always wins. If A manipulates first,

$\left(\frac{\lambda n_a(1 - b^*)}{n_a(1 - b^*) + (b^* - \lambda)n_b}, \lambda \right)$ is a Stackelberg equilibrium; and when B manipulates first

there is an ε -equilibrium. In this case, A wants to be second player to reduce the cost of manipulation, which order also increases the social welfare by decreasing the voter turnout.

(II) $(\lambda - a^*)n_a(1 - b^*) < (b^* - \lambda)n_b a^* \Rightarrow$ B always wins. If B manipulates first,

$\left(\lambda, \frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*} \right)$ is a Stackelberg equilibrium; and when A manipulates first there

is an ε -equilibrium. In this case, B wants to be second player to reduce the cost of manipulation, which order also increases the social welfare by decreasing the voter turnout.

(III) $(\lambda - a^*)n_a(1 - b^*) = (b^* - \lambda)n_b a^* \Rightarrow$ No matter who manipulates first there is an ε -equilibrium, but both of the candidates want to be the second player to win the election.

This proposition shows that the effect of manipulation can be made as small as we want but still positive by an appropriately sequencing the manipulations. The unit cost manipulations are the only crucial values in determining the winner, where the particular forms of the cost functions are not effective. The candidate, for whom the cost of

manipulation is less in the sense given above, wins the election; and he should manipulate as the second for the benefit of both himself and the society. On the other hand, there is a competition for second movement when none of the candidates is superior to each other. The above proposition shows that in fact neither the candidates nor the society like the manipulations, but the candidates have the incentive to manipulate in order to guarantee their victory.

The results so far are under the assumption that the voters believe that they are well informed about the composition of political preferences in the society. In the following section, we consider what happens if the voters are not well informed about the distributions of voter preferences.

4.2 The Manipulation Game of Two Candidates with Uninformed Voters

The model is the same as the one in section 3.2 except that the agents' learn θ from the companies which predict the distribution of political preferences, where the companies have three different types: A, B and fair type. For $i \in \{A, B\}$, i type company supports candidate i , which allows candidate i to release false information to his favor. Similar to the previous section no voter believes the announcement of the other candidate's company and n_i represents the trust ratio of candidate i . The remaining i voters believe in the fair type company's prediction, θ , which is the true value without

manipulation. Manipulation again is restricted since it is costly. The cost is given by $\bar{c}_i : \Delta^r \rightarrow \mathbb{R}^+$ for candidate $i \in \{A, B\}$. We assume that $\bar{c}_i(\theta_m) \geq 1$, $\bar{c}_i(\theta) = 0$, $\bar{c}_i(x)$ increases with spreading x and $\bar{c}_i(x) = \bar{c}_i(y)$ for $\forall x, y \in \Delta^r$ resulting in the same probability of voting for all types of agents. Here θ_m is the minimum symmetric spread, so for odd values of r $\theta_m = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the $(\frac{r+1}{2})$ th column, and for even values of r $\theta_m = (0, \dots, 0, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$ with $\frac{1}{2}$ in the $(\frac{r}{2})$ th and $(\frac{r+2}{2})$ th columns. When A chooses $\theta_a \in \Delta^r$ and B chooses $\theta_b \in \Delta^r$, where $\theta_{ai} = \theta_{a(r+1-i)}$, $\theta_{bi} = \theta_{b(r+1-i)}$ for $\forall i \in \{1, \dots, n\}$, the profit function of candidate $i \in \{A, B\}$ is:

$$\pi_i(\theta_a, \theta_b) = P(i \text{ wins} | (\theta_a, \theta_b)) - \bar{c}_i(\theta_i) \quad (4)$$

The candidates manipulate so as to maximize their profits. We will find the probability of winning for the candidates, by using the voting probabilities of the voters after the manipulations. From proposition 3, we know that $\bar{\varphi} \in (0, 1)$, the probability of voting for the voters who believes in $\bar{\theta}$, solves

$$P^*(\bar{\theta}, \bar{\varphi}, n) = Q(\bar{\varphi}, n | \bar{\theta}) = \sum_{k=0}^{\left\lceil \frac{n-1}{2} \right\rceil} \left[\binom{n-1}{k, k, n-1-2k} \sum_{i=1}^r 2\bar{\theta}_i (1-\lambda_i)^{k+1} \lambda_i^k \bar{\varphi}^{-2k} (1-\bar{\varphi})^{n-1-2k} \right] \\ + \sum_{k=0}^{\left\lceil \frac{n-2}{2} \right\rceil} \left[\binom{n-1}{k, k+1, n-2-2k} \sum_{i=1}^r 2\bar{\theta}_i (1-\lambda_i)^{k+1} \lambda_i^{k+1} \bar{\varphi}^{-2k+1} (1-\bar{\varphi})^{n-2-2k} \right] = 2c$$

when these voting probabilities constitute a symmetric BNE in totally mixed strategies.

To be able to use this result, we have to guarantee the existence of such equilibria in the

game voters' for all the three predictions of political preferences. As a result of Proposition 4 this is guaranteed when $\underline{c}(n) < c < 0.5$. From the Corollary 1 of Proposition 4, we know that $\lim_{n \rightarrow \infty} \underline{c}(n) \rightarrow 0$. Thus when n tends to infinity, the existence conditions in Proposition 4 are satisfied for $\forall c \in (0, 0.5)$, all possible sets of possible values of λ and all possible symmetric manipulations. Note that $\bar{\varphi}$ solves $P^*(\bar{\theta}, \bar{\varphi}, n) = 2c$ with $\bar{\varphi} = \varphi(c, n, \bar{\theta})$. Then the voters believing in θ_a vote with probability $\varphi_a = \varphi(c, n, \theta_a)$. Similarly, voters believing in θ_b vote with probability $\varphi_b = \varphi(c, n, \theta_b)$, and the remaining agents vote with probability $\varphi = \varphi(c, n, \theta)$. Thus a voter votes for candidate A with probability $\lambda(\varphi_a n_a + \varphi(1 - n_a))$, and for candidate B with probability $(1 - \lambda)(\varphi_b n_b + \varphi(1 - n_b))$. Using these probabilities, we get that candidate A wins the election with the following probability:

$$P(A \text{ wins} \mid (\theta_a, \theta_b)) = \begin{cases} \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} K(\varphi_a, \varphi_b)^{k-j}}{\sum_{j=0}^k \binom{k}{j} K(\varphi_a, \varphi_b)^{k-j}}, & \text{if } k \text{ is odd} \\ \frac{\left(\sum_{j=0}^{\frac{k-2}{2}} \binom{k}{j} K(\varphi_a, \varphi_b)^{k-j} \right) + \left(\frac{k}{2} \right) \frac{K(\varphi_a, \varphi_b)^{\frac{k}{2}}}{2}}{\sum_{j=0}^k \binom{k}{j} K(\varphi_a, \varphi_b)^{k-j}}, & \text{if } k \text{ is even} \end{cases}$$

$$\text{where } K(\varphi_a, \varphi_b) = \frac{P(\text{any voter votes for candidate A})}{P(\text{any voter votes for candidate B})} = \frac{\lambda(\varphi_a n_a + \varphi(1 - n_a))}{(1 - \lambda)(\varphi_b n_b + \varphi(1 - n_b))}.$$

Using the same argument as in Lemma 9 we get

$$K(\varphi_a, \varphi_b) > 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (\theta_a, \theta_b)) = 1$$

$$K(\varphi_a, \varphi_b) < 1 \Rightarrow \lim_{k \rightarrow \infty} P(B \text{ wins} \mid (\theta_a, \theta_b)) = 1$$

$$K(\varphi_a, \varphi_b) = 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (\theta_a, \theta_b)) = \lim_{k \rightarrow \infty} P(B \text{ wins} \mid (\theta_a, \theta_b)) = \frac{1}{2}$$

Let's examine the function $K(\varphi_a, \varphi_b)$ as it determines the effect of manipulation.

$$\frac{\partial K(\varphi_a, \varphi_b)}{\partial \varphi_a} = \frac{\lambda n_a}{(1 - \lambda)(\varphi_b n_b + \varphi(1 - n_b))} > 0$$

$$\frac{\partial K(\varphi_a, \varphi_b)}{\partial \varphi_b} = -\frac{\lambda(\varphi_a n_a + \varphi(1 - n_a))n_b}{(1 - \lambda)(\varphi_b n_b + \varphi(1 - n_b))^2} < 0$$

The above calculations show us that A wants to increase φ_a , and similarly B wants to increase φ_b to win the election. Proposition 6 tells us that both of the candidates will have the aim of minimizing the symmetric spread of θ when manipulation is profitable for them. As in the previous section, we will assume that the voter turnout is large. Again we have to show that this is possible, since the voter turnout is endogenous.

Lemma 11: When $n \rightarrow \infty$, for any $\varepsilon > 0$, any $M < \infty$, there exists $c(M, \varepsilon) > 0$ such that for $\forall c < c(M, \varepsilon)$, $P(\text{voter turnout} < M) < \varepsilon$.

We can interpret this lemma such that $k \rightarrow \infty$ when $c \rightarrow 0$. Noticing this fact, from now on we will assume that k tends to ∞ . For simplicity in the calculations we will denote the manipulation A whose cost is 1 by φ_a^* , the manipulation by B whose cost is 1

by φ_b^* and the cost functions with $c_i : [0,1] \rightarrow \mathfrak{R}^+$ where $c_i(x) = \bar{c}_i(y)$ for $\forall y$ such that $x = \varphi(c, y)$ for $i \in \{a, b\}$.

Proposition 9: If $\varphi_a^* \leq \varphi \frac{(1-\lambda) - (1-n_a)\lambda}{\lambda n_a}$ or $\varphi_b^* \leq \varphi \frac{\lambda - (1-n_b)(1-\lambda)}{(1-\lambda)n_b}$, then

$(\varphi_a, \varphi_b) = (\varphi, \varphi)$ is a Nash equilibrium. Otherwise there is no Nash equilibrium.

This proposition tells us that cooperation is impossible unless one of the candidates wins the election even without making any manipulation, whatever manipulation is undertaken by the other candidate with a cost less than 1. Similar to the informed setup, the candidates do not have any idea about each other, and we can improve our setup by assuming that each candidate has a prior belief about the probability distribution of the other candidate's manipulation. We assume that that they do not know what the other candidate believes about them to make their beliefs consistent. Otherwise, if the candidates know each others' beliefs, then they will exactly know what the other will play. But then they will also know that the other knows what he will play for sure, in which case at least for one of them deviating from his strategy would be profitable which will also contradict that he played his profit maximizing strategy. Assuming that voter turnout is large, A chooses λ_a which results in the voting probability φ_a that solves:

$$\max \{\pi_a(\varphi_a) : \varphi_a\} = \max \left\{ P(K(\varphi_a, \varphi_b) > 1) + \frac{P(K(\varphi_a, \varphi_b) = 1)}{2} - c_a(\varphi_a) : \varphi_a \right\},$$

and similarly B chooses λ_b which results in the voting probability φ_b that solves:

$$\max \{\pi_b(\varphi_b) : \varphi_b\} = \max \left\{ P(K(\varphi_a, \varphi_b) < 1) + \frac{P(K(\varphi_a, \varphi_b) = 1)}{2} - c_b(\varphi_b) : \varphi_b \right\}$$

We now turn to examining the magnitude of the function K as it determines the winning probabilities of the candidates.

$$K(\varphi_a, \varphi_b) = \frac{\lambda(\varphi_a n_a + \varphi(1 - n_a))}{(1 - \lambda)(\varphi_b n_b + \varphi(1 - n_b))} > 1 \Leftrightarrow \varphi_b < \frac{\lambda\varphi_a n_a + \varphi(\lambda(1 - n_a) - (1 - \lambda)(1 - n_b))}{(1 - \lambda)n_b}$$

$$K(\varphi_a, \varphi_b) < 1 \Leftrightarrow \varphi_a < \frac{(1 - \lambda)\varphi_b n_b + \varphi((1 - \lambda)(1 - n_b) - \lambda(1 - n_a))}{\lambda n_a}$$

Assuming candidate A's belief about candidate B's manipulations has a continuous distribution function f_b , and candidate B's belief has a continuous distribution function f_a , and using the above observation, the candidates' problem becomes the following: Candidate A chooses the manipulation which causes the voters believing him to vote with the probability $\varphi_a \in [\varphi, \varphi_a^*]$ that maximizes:

$$\pi_a(\varphi_a) = F_b\left(\frac{\lambda\varphi_a n_a + \varphi(\lambda(1 - n_a) - (1 - \lambda)(1 - n_b))}{(1 - \lambda)n_b}\right) - c_a(\varphi_a),$$

and candidate B chooses the appropriate manipulation which causes a corresponding voting probability $\varphi_b \in [\varphi, \varphi_b^*]$ that maximizes:

$$\pi_b(\varphi_b) = F_a\left(\frac{(1 - \lambda)\varphi_b n_b + \varphi((1 - \lambda)(1 - n_b) - \lambda(1 - n_a))}{\lambda n_a}\right) - c_b(\varphi_b).$$

Notice that there may be multiple solutions to these maximization problems. The candidates prefer the manipulation with the smallest cost in these cases. Let us denote the solution to this problem by $(\varphi_a^m, \varphi_b^m)$. Then, A wins the election if $K(\varphi_a^m, \varphi_b^m) > 1$; when

$K(\varphi_a^m, \varphi_b^m) = 1$ each candidate can win with equal probability and B wins the election when $K(\varphi_a^m, \varphi_b^m) < 1$. The following example will clarify how these implications work:

Example 2: Let B believe that A's manipulation is uniformly distributed on $[\varphi, \bar{\varphi}_a]$, and let A believe that B's manipulation is uniformly distributed on $[\varphi, \bar{\varphi}_b]$, where the maximum reasonable manipulations are $\bar{\varphi}_a = \inf(\{\varphi_a^*\} \cup \{\varphi_a \in [\varphi, \varphi_a^*] : K(\varphi_a, \varphi_b) > 1 \text{ for } \forall \varphi_b \in [\varphi, \varphi_b^*]\})$ and $\bar{\varphi}_b = \inf(\{\varphi_b^*\} \cup \{\varphi_b \in [\varphi, \varphi_b^*] : K(\varphi_a, \varphi_b) > 1 \text{ for } \forall \varphi_a \in [\varphi, \varphi_a^*]\})$. The cost functions are affine, such that $c_a(x) = \frac{x - \varphi}{\varphi_a^* - \varphi}$ where $x \in [\varphi, \varphi_a^*]$ and $c_b(x) = \frac{x - \varphi}{\varphi_b^* - \varphi}$ where

$x \in [\varphi, \varphi_b^*]$. Define $K^* = \frac{\lambda(\varphi_a^* n_a + \varphi(1 - n_a))}{(1 - \lambda)(\varphi_b^* n_b + \varphi(1 - n_b))}$, and $D = \frac{\varphi_a^* - \varphi}{\varphi_b^* - \varphi} - \frac{(1 - \lambda)n_b}{\lambda n_a}$ where

K^* is the value of function K when both of the candidates resort to manipulations with unit cost, or in other words the most costly manipulations possible. We know that the profit functions are maximized either by interior solutions where the derivative with respect to the voting probability is zero or by values on the boundary. Using this information with the given density and cost functions we get the following results:

$K^* = 1 \Rightarrow \varphi_a = \varphi, \varphi_b = \varphi \Rightarrow K(\varphi_a, \varphi_b) = \frac{\lambda}{1 - \lambda} \Rightarrow$ if $\lambda > \frac{1}{2}$ A wins the election, if

$\lambda = \frac{1}{2}$, A and B have equal probability of winning; and if $\lambda < \frac{1}{2}$, B wins the election;

$K^* < 1$ and $D \geq 0 \Rightarrow \varphi_a = \varphi, \varphi_b = \varphi \Rightarrow K(\varphi_a, \varphi_b) = \frac{\lambda}{1 - \lambda} \Rightarrow$ same results with $K^* = 1$

$$K^* < 1 \text{ and } D < 0 \Rightarrow \varphi_a = \varphi, \varphi_b = \bar{\varphi}_b = \varphi_a^* \frac{\lambda n_a}{(1-\lambda)n_b} + \varphi \frac{\lambda(1-n_a) - (1-\lambda)(1-n_b)}{(1-\lambda)n_b}$$

$$\Rightarrow K(\varphi_a, \varphi_b) < 1 \Rightarrow \text{B wins the election}$$

$$K^* > 1 \text{ and } D \leq 0 \Rightarrow \varphi_a = \varphi, \varphi_b = \varphi \Rightarrow K(\varphi_a, \varphi_b) = \frac{\lambda}{1-\lambda} \Rightarrow \text{same results with } K^* = 1$$

$$K^* > 1 \text{ and } D > 0 \Rightarrow \varphi_a = \bar{\varphi}_a = \varphi_b^* \frac{(1-\lambda)n_b}{\lambda n_a} + \varphi \frac{(1-\lambda)(1-n_b) - \lambda(1-n_a)}{\lambda n_a}, \varphi_b = \varphi$$

$$\Rightarrow K(\varphi_a, \varphi_b) > 1 \Rightarrow \text{A wins the election}$$

After observing the results of the example above, we change our model in the same manner as we have done in the informed voters setting, and oblige the candidates to manipulate sequentially one after other. The results are similar, but the main difference is that in the present setup the candidates take care of the voting probabilities obtained after the manipulations, as the function K depends on the voting probabilities, not the manipulated ratios of A voters in the society.

Proposition 10: Assume that $\varphi_a^* \geq \varphi \frac{(1-\lambda) - (1-n_a)\lambda}{\lambda n_a}$ and

$$\varphi_b^* \geq \varphi \frac{\lambda - (1-n_b)(1-\lambda)}{(1-\lambda)n_b}.$$

(I) $K^* > 1 \Rightarrow$ A always wins. If A manipulates first

$(\varphi_a, \varphi_b) = \left(\varphi_b^* \frac{n_b(1-\lambda)}{n_a\lambda} + \varphi \frac{(1-n_b)(1-\lambda) - (1-n_a)\lambda}{n_a\lambda}, \varphi \right)$ is a Stackelberg equilibrium,

and if B manipulates first there exists an ε -equilibrium. A wants to be the second player to reduce the cost of manipulation, which also increases the social welfare.

(II) $K^* < 1 \Rightarrow$ B always wins. If B manipulates first

$(\varphi_a, \varphi_b) = \left(\varphi, \frac{\varphi_a^* n_a \lambda}{n_b (1 - \lambda)} + \varphi \frac{(1 - n_a) \lambda - (1 - n_b)(1 - \lambda)}{n_b (1 - \lambda)} \right)$ is a Stackelberg equilibrium. If A

manipulates first there exists an ε -equilibrium. B wants to be the second player to reduce the cost of manipulation, which also increases the social welfare.

(III) $K^* = 1 \Rightarrow$ there exists an ε -equilibrium no matter who manipulates first. Both of the candidates want to be the second player to win the election.

The voting probabilities, caused by unit cost manipulations, determine the winner of the elections similar to the informed case. The shape of the cost function just affects the cost of manipulations for the candidates, and not the winner. Neither the candidates nor the society like manipulation, but the candidates have an incentive to deviate a bit for winning, and when the candidate who has the power to win the election manipulates first, he has to choose his maximal reasonable value to guarantee a victory. Thus, for decreasing the effect of manipulations a social organizer should oblige an appropriate order of manipulation. As a result of this proposition, the results of example 2 turn out to be disturbing too. When A is the stronger candidate, so that $K^* > 1$, B can win the game. Similarly when B is the stronger candidate, A can win the game. These cases occur because the candidates' guesses about each other are very bad in extreme cases where the boundary values are manipulated.

So far we have assumed that the society is ignorant of the candidates' manipulations. In the next section we examine whether the society can prevent the effects of the manipulations when they have more information, where the candidates are unaware of this knowledge of the society.

4.3 The Properties of a Society Which Can Prevent the Manipulation

4.3.1 - Informed Voters

Suppose that there are three different regions each containing one of the three types of companies. In the region where the t type company is located, the portion of the t voters who live in that region is given by n_t for $t \in \{a, b\}$, and in the region where fair type company is located the remaining voters live. For $t \in \{a, b\}$, we assume that the t type company can manipulate only all people in its region. Moreover, people living in same region as the fair type company know the manipulations as well as n_a and n_b , exactly as the fair type company reveals this information in its region. But the candidates are assumed to be unaware of this fact. We assume that the necessary conditions for the existence of a symmetric Bayesian Nash Equilibrium in totally mixed strategies are satisfied and $\{n_a, n_b\} \subset (0, 1)$. As a result of the unawareness of the candidates, the voters living in the same region with the fair type company may prevent the effects of

manipulations if the manipulations are small in the sense stated in the following proposition.

Proposition 11: The society can prevent the manipulation (λ_a, λ_b) if $n_a < \frac{\lambda_a}{\lambda}$ and

$$n_b < \frac{1 - \lambda_b}{1 - \lambda}.$$

Using this proposition in Example 1, we can calculate that the society can prevent manipulation in the following cases:

$$n_a > n_b \Rightarrow \lambda_a = \frac{\lambda \sqrt{n_a}}{\sqrt{n_a} + \sqrt{n_b}}, \quad \lambda_b = \lambda \quad \text{are chosen by the candidates, so when}$$

$$n_a < \frac{\sqrt{n_a}}{\sqrt{n_a} + \sqrt{n_b}} \quad \text{and} \quad n_b < 1 \quad \text{the society can prevent the manipulations. The second}$$

inequality is always satisfied and the first one is satisfied only when $n_a + \sqrt{n_a n_b} < 1$.

$$n_a < n_b \Rightarrow \lambda_a = \lambda, \quad \lambda_b = 1 - \frac{(1 - \lambda) \sqrt{n_b}}{\sqrt{n_a} + \sqrt{n_b}} \quad \text{are chosen by the candidates. When } n_a < 1 \text{ and}$$

$$n_b < \frac{\sqrt{n_b}}{\sqrt{n_a} + \sqrt{n_b}} \quad \text{the society can prevent manipulation. The first inequality is always}$$

satisfied, the second is satisfied only when $n_b + \sqrt{n_a n_b} < 1$.

As a result of this simple example we can say that when the trust ratios of the candidates are less than half, i.e. $n_a < 0.5$ and $n_b < 0.5$, manipulation can be prevented.

Corollary 2: When $n_a \leq \frac{\alpha(c,n)}{\lambda}$ and $n_b \leq \frac{\alpha(c,n)}{1-\lambda}$ the society can prevent all manipulations which have a cost less than 1.

Corollary 3: It is always possible to manipulate in large societies, as long as all the manipulations on the (0,1) interval have a cost less than one.

Corollary 3 shows us that large societies are more manipulable. But we shall not forget that the cost of manipulation increases with the size of the society, and the candidates think that the society is ignorant about their manipulations. Thus, it is still possible that large societies may prevent manipulation as well. Now we will consider the uninformed-voters setting and see what the society's power is in preventing manipulation when the society is ignorant about the distribution of the political preferences.

4.3.2 - Uninformed Voters

Let $\{n_a, n_b\} \subset (0,1)$, and assume that the voters who are not manipulated, know exactly what portions of A and B voters are manipulated. In other words, they know n_a and n_b . The candidates are unaware of the voters' knowledge, and this unawareness is used by the voters to prevent manipulation. The underlying scenario is the same as in the informed-voters setting. First of all let us define the:

$\varphi_{\min}$: The minimal value of the voting probability of a person that can be reached by candidates' manipulation

$\varphi_{\max}$: The maximum value of the voting probability of a person that can be reached by the candidates' manipulation

We know that $P^*(\theta_{\max}, \varphi_{\min}, n) = 2c$ and $P^*(\theta_{\min}, \varphi_{\max}, n) = 2c$, where $\theta_{\max}$ is the maximum symmetric spread of the prior, $\theta_{\max} = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2})$, and $\theta_{\min}$ is the minimum symmetric prior, $\theta_{\min} = \theta_m$. These two equations give us the values of $\varphi_{\min}$ and $\varphi_{\max}$. The following proposition clarifies the manipulations that can be prevented by the society by using mixed strategies, and the corollaries show necessary conditions for preventing all possible effects of manipulations about the winner of the election.

Proposition 12: The society can prevent the manipulation (θ_a, θ_b) when $n_a < \min\left\{\frac{\varphi}{\varphi_a}, \frac{1-\varphi}{1-\varphi_a}\right\}$ and $n_b < \min\left\{\frac{\varphi}{\varphi_b}, \frac{1-\varphi}{1-\varphi_b}\right\}$ where $\varphi_a = \varphi(c, n, \theta_a)$, $\varphi_b = \varphi(c, n, \theta_b)$ and $\varphi = \varphi(c, n, \theta)$.

Corollary 4: The society can prevent all possible manipulations when $n_a < \min\left\{\frac{\varphi}{\varphi_{\max}}, \frac{1-\varphi}{1-\varphi_{\min}}\right\}$ and $n_b < \min\left\{\frac{\varphi}{\varphi_{\max}}, \frac{1-\varphi}{1-\varphi_{\min}}\right\}$.

Corollary 5: It is always possible to manipulate large societies when $\varphi_{\max} - \varphi \neq 0$.

Corollary 5 shows us that in large societies, so long as there is a positive probability of changing the voting probabilities of some voters, the aware voters can prevent only some of the manipulations but not all of them. As a result of this, if the candidates become aware of the voters' knowledge, they can affect the results of the elections.

CHAPTER V

CONCLUSION

In this study we presented a simple model where voters have probabilistic preferences over two candidates, and examined the behavior of candidates regarding the impact of their manipulations of the distribution of political preferences on equilibrium voting behavior, winner of the election and social welfare. Our results, for two different settings-informed and uninformed voters- demonstrate that both the candidates and the society suffer from manipulations. In order to mitigate the negative effects of manipulation in equilibrium, a social organizer has to have sufficient information about the candidates' cost functions and set the rules for the manipulation game accordingly. Moreover, when the voters who are not manipulated know the manipulated values together with the number of manipulated people, the society can prevent the effects of certain manipulations on the winner of the election. Our results are derived under the assumption that voters' benefits are correctly measured by the chance of being pivotal, and the parameters satisfy the necessary conditions for the existence of symmetric Bayesian Nash Equilibrium in mixed strategies.

Further studies can examine the effects of the repetition of the manipulation game to explain why many countries ban making statements about the distribution of voters' preferences before the elections. The effects of combined manipulations which have been studied separately in the literature so far naturally deserve further consideration. Such studies may also lead to more profitable manipulations on the part of candidates. But the most valuable contributions of these studies would be to design institutions that prevent losses in social welfare due to manipulations, possibly by revealing true information about the composition of the society along with the possible manipulations this composition results in.

APPENDIX

Proof of Lemma 8: Let's say that A voters who believe λ_a vote with probability φ_{a1} , the remaining A voters vote with probability φ_{a2} , B voters who believe λ_b vote with probability φ_{b1} , the remaining B voters vote with probability φ_{b2} . Using Proposition 1 we have:

$$\lambda_a \varphi_{a1} = \alpha(c, n) \text{ , } \lambda \varphi_{a2} = \alpha(c, n) \text{ , } (1 - \lambda_b) \varphi_{b1} = \alpha(c, n) \text{ , } (1 - \lambda) \varphi_{b2} = \alpha(c, n)$$

$$\Rightarrow \varphi_{a1} = \frac{\alpha(c, n)}{\lambda_a} \text{ , } \varphi_{a2} = \frac{\alpha(c, n)}{\lambda} \text{ , } \varphi_{b1} = \frac{\alpha(c, n)}{1 - \lambda_b} \text{ , } \varphi_{b2} = \frac{\alpha(c, n)}{1 - \lambda}$$

$$\Rightarrow P(\text{any voter votes for A}) = P_a(c, n, \lambda_a) = \lambda n_a \varphi_{a1} + \lambda(1 - n_a) \varphi_{a2} = \left(\frac{\lambda n_a}{\lambda_a} + 1 - n_a \right) \alpha(c, n)$$

$$\Rightarrow P(\text{any voter votes for B}) = P_b(c, n, \lambda_b) = (1 - \lambda) n_b \varphi_{b1} + (1 - \lambda)(1 - n_b) \varphi_{b2}$$

$$= (1 - \lambda) n_b \frac{\alpha(c, n)}{1 - \lambda_b} + (1 - n_b) \alpha(c, n)$$

Using these equations we can find the probability of A winning the election. Let's denote the total number of agents who vote in the election by k .

$$P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} P_a(c, n, \lambda_a)^{k-j} P_b(c, n, \lambda_b)^j}{\sum_{j=0}^k \binom{k}{j} P_a(c, n, \lambda_a)^{k-j} P_b(c, n, \lambda_b)^j} \quad \text{if } k \text{ is odd, and otherwise}$$

$$P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \frac{\left(\sum_{j=0}^{\frac{k-2}{2}} \binom{k}{j} P_a(c, n, \lambda_a)^{k-j} P_b(c, n, \lambda_b)^j \right) + \binom{k}{\frac{k}{2}} \frac{P_a(c, n, \lambda_a)^{\frac{k}{2}} P_b(c, n, \lambda_b)^{\frac{k}{2}}}{2}}{\sum_{j=0}^k \binom{k}{j} P_a(c, n, \lambda_a)^{k-j} P_b(c, n, \lambda_b)^j}$$

Dividing both the nominator and denominator by $(P_b(c, n, \lambda_b))^k$, we get the desired result for $P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b))$. On the other hand, as $P(B \text{ wins} \mid (\lambda_a, \lambda_b)) = 1 - P(A \text{ wins} \mid (\lambda_a, \lambda_b))$, just by dividing the numerator and the denominator by $K(\lambda_a, \lambda_b)^k$, we get equation (3).

Proof of Lemma 9: We will prove this fact in two parts, k is odd and k is even.

First let's assume that k is odd. Then $K(\lambda_a, \lambda_b) = 1 \Rightarrow$

$$\lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \lim_{k \rightarrow \infty} \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j}}{\sum_{j=0}^k \binom{k}{j}} = \lim_{k \rightarrow \infty} \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j}}{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} + \binom{k}{k-j}} = \lim_{k \rightarrow \infty} \frac{\sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j}}{2 \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j}} = \frac{1}{2}$$

Now assume that $K(\lambda_a, \lambda_b) > 1$. Let's fix $r \in N$ and define $X_r = \frac{X_r^n}{X_r^d}$, where

$$X_r^n = \sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j} \text{ and } X_r^d = \sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \binom{k}{j} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j)$$

$$X_r = \frac{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \left(\frac{k!}{(k-j)! j!} \right) K(\lambda_a, \lambda_b)^{k-j} \left(\frac{k+1+2r}{2} \right)! \left(\frac{k-1}{2} \right)!}{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \left(\frac{k!}{(k-j)! j!} \right) (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j) \left(\frac{k+1+2r}{2} \right)! \left(\frac{k-1}{2} \right)!}$$

$$= \frac{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \prod_{i=k-j+1}^{\frac{k+1+2r}{2}} i \prod_{l=j+1}^{\frac{k-1}{2}} l K(\lambda_a, \lambda_b)^{k-j}}{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \prod_{i=k-j+1}^{\frac{k+1+2r}{2}} i \prod_{l=j+1}^{\frac{k-1}{2}} l (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j)}$$

$$= \frac{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \prod_{i=k-j+1}^{\frac{k+1+2r}{2}} \frac{i}{k/2} \prod_{l=j+1}^{\frac{k-1}{2}} \frac{l}{k/2} K(\lambda_a, \lambda_b)^{k-j}}{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \prod_{i=k-j+1}^{\frac{k+1+2r}{2}} \frac{i}{k/2} \prod_{l=j+1}^{\frac{k-1}{2}} \frac{l}{k/2} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j)} \quad \text{(I)}$$

$$\Rightarrow \lim_{k \rightarrow \infty} X_r = \lim_{k \rightarrow \infty} \frac{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} K(\lambda_a, \lambda_b)^{k-j}}{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j)}$$

$$\begin{aligned}
& \sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} K(\lambda_a, \lambda_b)^{k-j-\frac{k-2r-1}{2}} \\
&= \lim_{k \rightarrow \infty} \frac{\sum_{j=\frac{k-2r-1}{2}}^{\frac{k-1}{2}} \left(K(\lambda_a, \lambda_b)^{k-j-\frac{k-2r-1}{2}} + K(\lambda_a, \lambda_b)^{j-\frac{k-2r-1}{2}} \right)}{\sum_{j=0}^{2r+1} K(\lambda_a, \lambda_b)^j} \\
&= \lim_{k \rightarrow \infty} \frac{\sum_{j=r+1}^{2r+1} K(\lambda_a, \lambda_b)^j}{\sum_{j=0}^{2r+1} K(\lambda_a, \lambda_b)^j} = \frac{K(\lambda_a, \lambda_b)^{r+1} \left(\frac{K(\lambda_a, \lambda_b)^{r+1} - 1}{K(\lambda_a, \lambda_b) - 1} \right)}{\frac{K(\lambda_a, \lambda_b)^{2r+2} - 1}{K(\lambda_a, \lambda_b) - 1}} = \frac{K(\lambda_a, \lambda_b)^{2r+2} - K(\lambda_a, \lambda_b)^{r+1}}{K(\lambda_a, \lambda_b)^{2r+2} - 1} \quad \text{(II)}
\end{aligned}$$

Let's define $Y_j = \frac{Y_j^n}{Y_j^d}$ for $\forall j < \frac{k-1-2r}{2}$ where $Y_j^n = \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j}$ and

$Y_j^d = \binom{k}{j} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j)$. As we've

$$\frac{\partial Y_j}{\partial j} = \frac{-2K(\lambda_a, \lambda_b)^{k-2j} \ln K(\lambda_a, \lambda_b)}{(K(\lambda_a, \lambda_b)^{k-2j} + 1)^2} < 0 \Rightarrow$$

$$Y_j = \frac{K(\lambda_a, \lambda_b)^{k-2j}}{K(\lambda_a, \lambda_b)^{k-2j} + 1} > \frac{K(\lambda_a, \lambda_b)^{2r+1}}{K(\lambda_a, \lambda_b)^{2r+1} + 1} \quad \text{(III)}.$$

Notice that $P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \frac{X_r^n + \sum_{j=0}^{\frac{k-3-2r}{2}} Y_j^n}{X_r^d + \sum_{j=0}^{\frac{k-3-2r}{2}} Y_j^d}$. By using equation (III), we get

$$P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) \geq \min \left\{ X_r, \frac{K(\lambda_a, \lambda_b)^{2r+1}}{K(\lambda_a, \lambda_b)^{2r+1} + 1} \right\}$$

Letting $k \rightarrow \infty$ and using (II)

$$\lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) \geq \min \left\{ \frac{K(\lambda_a, \lambda_b)^{2r+2} - K(\lambda_a, \lambda_b)^{r+1}}{K(\lambda_a, \lambda_b)^{2r+2} - 1}, \frac{K(\lambda_a, \lambda_b)^{2r+1}}{K(\lambda_a, \lambda_b)^{2r+1} + 1} \right\} \quad \text{for}$$

$\forall r \in N$ and when $r \rightarrow \infty$

$$\lim_{r \rightarrow \infty} \frac{K(\lambda_a, \lambda_b)^{2r+2} - K(\lambda_a, \lambda_b)^{r+1}}{K(\lambda_a, \lambda_b)^{2r+2} - 1} = \lim_{r \rightarrow \infty} \frac{K(\lambda_a, \lambda_b)^{2r+1}}{K(\lambda_a, \lambda_b)^{2r+1} + 1} = 1$$

$$\Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) \geq 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 1$$

$K(\lambda_a, \lambda_b) < 1 \Rightarrow \frac{1}{K(\lambda_a, \lambda_b)} > 1$ and using the above calculations in equation **(3)** we get

$$\lim_{k \rightarrow \infty} P(B \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 0.$$

To complete the proof, let's assume that k is even. Then $K(\lambda_a, \lambda_b) = 1 \Rightarrow$

$$\lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \lim_{k \rightarrow \infty} \frac{\left(\sum_{j=0}^{\frac{k-2}{2}} \binom{k}{j} \right) + \frac{1}{2} \binom{k}{\frac{k}{2}}}{\sum_{j=0}^k \binom{k}{j}} = \lim_{k \rightarrow \infty} \frac{\sum_{j=0}^{\frac{k-2}{2}} \binom{k}{j} + \frac{1}{2} \binom{k}{\frac{k}{2}}}{\sum_{j=0}^{\frac{k-2}{2}} \left(\binom{k}{j} + \binom{k}{k-j} \right) + \binom{k}{\frac{k}{2}}} = \frac{1}{2}$$

Now we can consider the case, where $K(\lambda_a, \lambda_b) > 1$. Let's fix $r \in N$ and define $X_r = \frac{X_r^n}{X_r^d}$

$$\text{where } X_r^n = \left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \binom{k}{j} K(\lambda_a, \lambda_b)^{k-j} \right) + \binom{k}{\frac{k}{2}} \frac{K(\lambda_a, \lambda_b)^{\frac{k}{2}}}{2}, \text{ and}$$

$$X_r^d = \left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \binom{k}{j} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j) \right) + \left(\frac{k}{2} \right) K(\lambda_a, \lambda_b)^{\frac{k}{2}}.$$

$$X_r = \frac{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \frac{k!}{(k-j)! j!} K(\lambda_a, \lambda_b)^{k-j} \right) + \left(\frac{k}{2} \right) \frac{K(\lambda_a, \lambda_b)^{\frac{k}{2}}}{2}}{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \frac{k!}{(k-j)! j!} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j) \right) + \left(\frac{k}{2} \right) K(\lambda_a, \lambda_b)^{\frac{k}{2}} \frac{\left(\frac{k+2r}{2} \right)! \left(\frac{k}{2} \right)!}{\left(\frac{k+2r}{2} \right)! \left(\frac{k}{2} \right)!}}$$

$$= \frac{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \prod_{i=k-j+1}^{\frac{k+2r}{2}} i \prod_{l=j+1}^{\frac{k}{2}} l K(\lambda_a, \lambda_b)^{k-j} \right) + \frac{\frac{k+2r}{2} K(\lambda_a, \lambda_b)^{\frac{k}{2}}}{2}}{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \prod_{i=k-j+1}^{\frac{k+2r}{2}} i \prod_{l=j+1}^{\frac{k}{2}} l (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j) \right) + \prod_{i=\frac{k+2}{2}}^{\frac{k+2r}{2}} i K(\lambda_a, \lambda_b)^{\frac{k}{2}}}$$

$$= \frac{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \prod_{i=k-j+1}^{\frac{k+2r}{2}} \frac{i}{k/2} \prod_{l=j+1}^{\frac{k}{2}} \frac{l}{k/2} K(\lambda_a, \lambda_b)^{k-j} \right) + \prod_{i=\frac{k+2}{2}}^{\frac{k+2r}{2}} \frac{i}{k/2} \frac{K(\lambda_a, \lambda_b)^{\frac{k}{2}}}{2}}{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} \prod_{i=k-j+1}^{\frac{k+2r}{2}} \frac{i}{k/2} \prod_{l=j+1}^{\frac{k}{2}} \frac{l}{k/2} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j) \right) + \prod_{i=\frac{k+2}{2}}^{\frac{k+2r}{2}} \frac{i}{k/2} K(\lambda_a, \lambda_b)^{\frac{k}{2}}} \quad \text{(IV)}$$

$$\Rightarrow \lim_{k \rightarrow \infty} X_r = \lim_{k \rightarrow \infty} \frac{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} K(\lambda_a, \lambda_b)^{k-j} \right) + \frac{K(\lambda_a, \lambda_b)^{\frac{k}{2}}}{2}}{\left(\sum_{j=\frac{k-2r}{2}}^{\frac{k-2}{2}} (K(\lambda_a, \lambda_b)^{k-j} + K(\lambda_a, \lambda_b)^j) \right) + K(\lambda_a, \lambda_b)^{\frac{k}{2}}}$$

$$= \lim_{k \rightarrow \infty} \frac{\frac{K(\lambda_a, \lambda_b)^{\frac{k}{2}} \left(\frac{2K(\lambda_a, \lambda_b)^{r+1} - K(\lambda_a, \lambda_b) - 1}{K(\lambda_a, \lambda_b) - 1} \right)}{2}}{K(\lambda_a, \lambda_b)^{\frac{k-2r}{2}} \left(\frac{K(\lambda_a, \lambda_b)^{2r+1} - 1}{K(\lambda_a, \lambda_b) - 1} \right)} = \frac{2K(\lambda_a, \lambda_b)^{r+1} - K(\lambda_a, \lambda_b) - 1}{2K(\lambda_a, \lambda_b)^{r+1} - 2K(\lambda_a, \lambda_b)^{-r}} \quad (\text{V})$$

As Y_j decreases in j , for each j satisfying $j < \frac{k-2r}{2}$, we get the following inequality:

$$Y_j = \frac{K(\lambda_a, \lambda_b)^{k-2j}}{K(\lambda_a, \lambda_b)^{k-2j} + 1} > \frac{K(\lambda_a, \lambda_b)^{2r}}{K(\lambda_a, \lambda_b)^{2r} + 1} \quad (\text{VI})$$

$$\text{From (VI), } P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = \frac{X_r^n + \sum_{j=0}^{\frac{k-2-2r}{2}} Y_j^n}{X_r^d + \sum_{j=0}^{\frac{k-2-2r}{2}} Y_j^d} \geq \min \left\{ \frac{K(\lambda_a, \lambda_b)^{2r}}{K(\lambda_a, \lambda_b)^{2r} + 1}, X_r \right\}$$

Letting $k \rightarrow \infty$ and using (V)

$$\lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) \geq \min \left\{ \frac{2K(\lambda_a, \lambda_b)^{r+1} - K(\lambda_a, \lambda_b) - 1}{2K(\lambda_a, \lambda_b)^{r+1} - 2K(\lambda_a, \lambda_b)^{-r}}, \frac{K(\lambda_a, \lambda_b)^{2r}}{K(\lambda_a, \lambda_b)^{2r} + 1} \right\} \quad \text{for}$$

$\forall r \in N$, and when $r \rightarrow \infty$

$$\lim_{r \rightarrow \infty} \frac{2K(\lambda_a, \lambda_b)^{r+1} - K(\lambda_a, \lambda_b) - 1}{2K(\lambda_a, \lambda_b)^{r+1} - 2K(\lambda_a, \lambda_b)^{-r}} = \lim_{r \rightarrow \infty} \frac{K(\lambda_a, \lambda_b)^{2r}}{K(\lambda_a, \lambda_b)^{2r} + 1} = 1$$

$$\Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) \geq 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 1$$

$K(\lambda_a, \lambda_b) < 1 \Rightarrow \frac{1}{K(\lambda_a, \lambda_b)} > 1$ and with similar calculations as in equation (3) we get

$$\lim_{k \rightarrow \infty} P(B \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 1 \Rightarrow \lim_{k \rightarrow \infty} P(A \text{ wins} \mid (c, n, \lambda_a, \lambda_b)) = 0 \quad \text{which completes the}$$

proof.

Proof of Lemma 10:

$$P(a \text{ voter votes for candidate } A) = \alpha(c, n)(n_a(\lambda / \lambda_a) + 1 - n_a)$$

$$P(a \text{ voter votes for candidate } B) = \alpha(c, n)(n_b((1 - \lambda)/(1 - \lambda_b)) + 1 - n_b)$$

$$\Rightarrow P(a \text{ voter votes}) = p_v = \alpha(c, n) \left(2 + n_a \frac{\lambda - \lambda_a}{\lambda_a} + n_b \frac{\lambda_b - \lambda}{1 - \lambda_b} \right) \in [2\alpha(c, n), 1]$$

We know that $\alpha(c, n)$ solves the following equation:

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{k, k, n-1-2k} \alpha^{2k} (1-2\alpha)^{n-1-2k} + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{k, k+1, n-2-2k} \alpha^{2k+1} (1-2\alpha)^{n-2-2k} = 2c$$

Using the observation $\binom{n-1}{a, b, n-1-a-b} \geq \binom{n-1}{a+b, n-1-a-b}$ in the above equation,

$$\text{we get } (1 - \alpha(c, n))^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k, n-1-k} \alpha(c, n)^k (1 - 2\alpha(c, n))^{n-1-k} \leq 2c.$$

Claim: For $\forall K \in \mathfrak{R}^+$, $\exists c_K \in \mathfrak{R}^+$, $N_K \in \mathfrak{N}^+$ such that for $\forall n > N_K$, $\forall c < c_K$

$$\alpha(c, n)n > K.$$

Proof: Take $c_K = \frac{1}{4}e^{-K-1}$. Assume there exists no $N_K \in \mathfrak{N}^+$ satisfying the desired

conditions. Then there exists a sequence $\{n_i\}_{i=1}^{\infty}$ such that $z_i = \alpha(c, n_i)n_i \leq K$. Then

$\{z_i\}_{i=1}^{\infty}$ is a bounded sequence in the closed set $[0, K]$. So there exists a convergent

subsequence. Without loss of generality treat the sequence itself as a convergent

sequence. Then $\lim_{i \rightarrow \infty} z_i = A \leq K$. Make a new sequence $\{\beta_i\}_{i=1}^{\infty}$ such that $\beta_i = \alpha(c, n_i)$ if

$i \in \{n_i\}_{i=1}^{\infty}$, and $\beta_i = A/i$ otherwise. Then $\lim_{i \rightarrow \infty} i\beta_i = A \Rightarrow \lim_{i \rightarrow \infty} (1 - \beta_i)^i = \exp(-\lim_{i \rightarrow \infty} i\beta_i)$

$= e^{-A} \geq e^{-M}$. So $\exists N \in \mathbb{N}^+$ such that $(1 - \beta_i)^i > (e^{-M} + e^{-M+1})/2$ for $\forall i \geq N$. Then for $\forall \eta \in \{n_i\}_{i=1}^\infty$ and $\eta \geq N$, $\beta_\eta = \alpha(c_K, \eta)$ and so $(1 - \alpha(c_K, \eta))^\eta > (e^{-K} + e^{-K-1})/2$. This contradicts that $(1 - \alpha(c_K, \eta))^\eta \leq (1 - \alpha(c_K, \eta))^{\eta-1} \leq 2c_K = e^{-K-1}/2$. So our claim is true.

As $\varepsilon > 0$, there exists $\ell \in \mathbb{N}^+$ such that $\varepsilon > 1/\ell$. For $K = \ell M$, $\exists c_K \in \mathbb{R}^+, N_K \in \mathbb{N}^+$ such that for $\forall n > N_K$, $\forall c < c_K$ we have $\alpha(c, n)n > K$. Notice also that $n > K$. We have $np_v \geq 2\alpha(c, n)n > 2K$, and for $i < K$ we know $ip_v < K \Rightarrow (n-i)p_v > K$ and

$$(i+1)(1-p_v) < K. \quad \text{So} \quad \frac{\binom{n}{i} p_v^i (1-p_v)^{n-i}}{\binom{n}{i+1} p_v^{i+1} (1-p_v)^{n-i-1}} = \frac{(i+1)(1-p_v)}{(n-i)p_v} < 1 \quad \text{for } i < K. \quad \text{So}$$

$\binom{n}{i} p_v^i (1-p_v)^{n-i}$ is an increasing function of i as long as $i < K$

$$\Rightarrow 1 = \sum_{i=0}^n \binom{n}{i} p_v^i (1-p_v)^{n-i} > \sum_{i=0}^{K-1} \binom{n}{i} p_v^i (1-p_v)^{n-i} > \ell \sum_{i=0}^{M-1} \binom{n}{i} p_v^i (1-p_v)^{n-i}$$

$$\Rightarrow P(\text{voter turnout} < M) = \sum_{i=0}^{M-1} \binom{n}{i} p_v^i (1-p_v)^{n-i} < 1/\ell < \varepsilon \quad \text{for } \forall c < c_K \text{ and } \forall n > N_K.$$

So take $c(M, \varepsilon) = c_K$ and $N(M, \varepsilon) = N_K$.

Proof of Proposition 7: Let's assume that this game has a Nash equilibrium,

(λ_a, λ_b) .

$K(\lambda_a, \lambda_b) < 1 \Rightarrow$ B wins the game, so A manipulates $\lambda_a = \lambda$ to maximize $\pi_a(c, n, \lambda_a, \lambda_b)$ by definition of Nash equilibrium. Then $K(\lambda_a, \lambda_b) < 1 \Rightarrow \lambda_b > \lambda \Rightarrow \lambda_b > \frac{\lambda_b + \lambda}{2} > \lambda$. If B had chosen $\frac{\lambda_b + \lambda}{2}$, $K(\lambda_a, \frac{\lambda_b + \lambda}{2}) < 1 \Rightarrow P(B \text{ wins} | (c, n, \lambda_a, \frac{\lambda_b + \lambda}{2})) = 1$, and $c_b(\lambda_b) > c_b(\frac{\lambda_b + \lambda}{2})$ because $c_b(x)$ increases with increasing $x \in (\lambda, 1)$. Thus B increases his profit by playing $\frac{\lambda_b + \lambda}{2}$, so $\frac{\lambda_b + \lambda}{2}$ strictly dominates λ_b which contradicts that (λ_a, λ_b) is a Nash equilibrium.

$K(\lambda_a, \lambda_b) > 1 \Rightarrow$ A wins the game, so B chooses $\lambda_b = \lambda$ to maximize his profit. Then we have $K(\lambda_a, \lambda_b) > 1 \Rightarrow \lambda_a < \lambda \Rightarrow \lambda_a < \frac{\lambda_a + \lambda}{2} < \lambda \Rightarrow K(\frac{\lambda_a + \lambda}{2}, \lambda_b) > 1 \Rightarrow P(A \text{ wins} | (c, n, \frac{\lambda_a + \lambda}{2}, \lambda_b)) = 1$. On the other hand as $c_a(\lambda_a) > c_a(\frac{\lambda_a + \lambda}{2})$, playing $\frac{\lambda_a + \lambda}{2}$ strictly dominates λ_a which contradicts that (λ_a, λ_b) is a Nash equilibrium.

$$K(\lambda_a, \lambda_b) = 1 \Rightarrow P(A \text{ wins} | (c, n, \lambda_a, \lambda_b)) = P(B \text{ wins} | (c, n, \lambda_a, \lambda_b)) = \frac{1}{2}.$$

Candidates guarantee that $\pi_a(c, n, \lambda_a, \lambda_b) \geq 0$ and $\pi_b(c, n, \lambda_a, \lambda_b) \geq 0$ as they have the chance of not manipulating, and (λ_a, λ_b) is a Nash equilibrium. Thus, $c_a(\lambda_a) \leq \frac{1}{2}$ and

$c_b(\lambda_b) \leq \frac{1}{2}$. As $c_a(x)$ is a continuous function there exists x such that $0 < x < \lambda_a$ and

$\frac{0.6 + c_a(\lambda_a)}{2} > c_a(x) > c_a(\lambda_a)$. As K is a decreasing function in its first variable we have

$$K(x, \lambda_b) > 1 \Rightarrow P(A \text{ wins} | (c, n, x, \lambda_b)) = 1 \Rightarrow \pi_a(c, n, x, \lambda_b) > 1 - \frac{0.6 + c_a(\lambda_a)}{2}$$

$= 0.7 - \frac{c_a(\lambda_a)}{2} > \frac{1}{2} - c_a(\lambda_a)$. Thus (x, λ_b) strictly dominates (λ_a, λ_b) which contradicts

that (λ_a, λ_b) is Nash equilibrium. Thus there isn't any Nash equilibrium of this game.

Proof of Proposition 8: Let A start the game and choose λ_a , and as a best response

B choose λ_b . First of all let's analyze how K changes with respect to its variables.

$$\frac{\partial K(\lambda_a, \lambda_b)}{\partial \lambda_b} = \frac{-(1-\lambda)n_b \frac{1}{(1-\lambda_b)^2} \left(\lambda n_a \frac{1}{\lambda_a} + (1-n_a) \right)}{\left((1-\lambda)n_b \frac{1}{1-\lambda_b} + (1-n_b) \right)^2} \leq 0$$

$$\frac{\partial K(\lambda_a, \lambda_b)}{\partial \lambda_a} = \frac{-\lambda n_a \frac{1}{\lambda_a^2}}{(1-\lambda)n_b \frac{1}{1-\lambda_b} + (1-n_b)} \leq 0$$

A wants to make $K(\lambda_a, \lambda_b) > 1$ and B wants to make $K(\lambda_a, \lambda_b) < 1$, so A decreases

$\lambda_a \geq a^*$ till $K(\lambda_a, \lambda_b) > 1$ and if this isn't possible he doesn't manipulate. Similarly B

increases $\lambda_b \leq b^*$ till $K(\lambda_a, \lambda_b) < 1$ and if this isn't possible he doesn't manipulate. So

the critical value of the function K is:

$$K^* = K(a^*, b^*) = \frac{\lambda n_a \frac{\alpha(c, n)}{a^*} + (1 - n_a) \alpha(c, n)}{(1 - \lambda) n_b \frac{\alpha(c, n)}{1 - b^*} + (1 - n_b) \alpha(c, n)}$$

(i) $K^* = 1 \Leftrightarrow (\lambda - a^*) n_a (1 - b^*) = (b^* - \lambda) n_b a^* \Rightarrow$ whatever $\lambda_i \notin \{\lambda, i^*\}$ the first player $i \in \{a, b\}$ chooses the second player j has the ability of making $K(\lambda_i, \lambda_j) > 1$ and $K(\lambda_i, \lambda_j) < 1$ by choosing appropriate $\lambda_j \neq j^*$ so that j wins the election. So choosing $\lambda_i \notin \{\lambda, i^*\}$ causes only a positive cost to i , so i doesn't choose $\lambda_i \notin \{\lambda, i^*\}$. If $\lambda_i = \lambda$ then for $\forall \varepsilon > 0$ there exists $x \in (j^*, \lambda)$ such that $P(j \text{ wins} | (c, n, \lambda, x)) = 1$ and $c_j(x) < \varepsilon$ so that $\pi_j(c, n, \lambda, x) > 1 - \varepsilon$ since $c_j(x)$ is a continuous function of x . On the other hand if i chooses $\lambda_i = i^*$, then the best response of j is λ , which makes $\pi_i(c, n, i^*, \lambda) = \pi_j(c, n, i^*, \lambda) = 0$. But if i chooses $\lambda_i = \lambda$, then an ε -best best response of j will be close to λ , which makes $\pi_i(c, n, \lambda, \lambda_j) = 0, \pi_j(c, n, \lambda, \lambda_j) \cong 1$. Thus λ dominates i^* , and so i doesn't choose i^* . As a result i doesn't manipulate. On the other hand the maximum possible value of the profit is 1 and j can achieve a profit arbitrarily close to this value. So there is an ε -equilibrium.

(ii) $K^* > 1 \Leftrightarrow (\lambda - a^*) n_a (1 - b^*) > (b^* - \lambda) n_b a^* \Rightarrow$ If A is the second player whatever $\lambda_b \in [\lambda, b^*]$ B chooses, $\exists a_* \in (a^*, \lambda)$ such that $K(a_*, \lambda_b) = 1$. So A guarantees

$\pi_a(c, n, \lambda_a, \lambda_b) \geq \pi_a(c, n, a_*, \lambda_b) = \frac{1}{2} - c_a(a_*)$ and for $\forall \varepsilon \in (0, 0.5)$ there exists $\bar{a} \in (a^*, a_*)$ such that $c_a(\bar{a}) < c_a(a_*) + \varepsilon$, and $\bar{a} < a_* \Rightarrow K(\bar{a}, \lambda_b) > 1 \Rightarrow \pi_a(c, n, \bar{a}, \lambda_b) = 1 - c_a(\bar{a}) > 1 - c_a(a_*) - \varepsilon \Rightarrow \pi_a(c, n, \bar{a}, \lambda_b) > \pi_a(c, n, a_*, \lambda_b)$. On the other hand $\pi_a(c, n, \bar{a}, \lambda_b) = 1 - c_a(\bar{a}) > 0$, so winning the election is better for A than a tie in election or loosing the election. Knowing this fact chooses $\lambda_b = \lambda$ to minimize his manipulation cost and thus to maximize his profit. Then for $\forall \varepsilon > 0$ there exists $a_\varepsilon \in (a^*, \lambda)$ such that $c_a(a_\varepsilon) < \varepsilon$, and $a_\varepsilon < \lambda_* \Rightarrow K(a_\varepsilon, \lambda) > 1 \Rightarrow \pi_a(c, n, a_\varepsilon, \lambda) = 1 - c_a(a_\varepsilon) > 1 - \varepsilon$, where the maximal value of π_a is 1. Thus we have an ε -equilibrium.

Let A be the first player. Now $\exists a_* \in (a^*, \lambda)$ such that $K(a_*, b^*) = 1$. So when A chooses $\lambda_a \in (a_*, \lambda]$, $\exists b_* \in [\lambda, b^*)$ such that $K(\lambda_a, b_*) = 1$. So B guarantees a profit of $\pi_b(c, n, \lambda_a, b_*) = \frac{1}{2} - c_b(b_*)$ and $\forall \varepsilon \in (0, 0.5)$ there exists $\bar{b} \in (b_*, b^*)$ such that $c_b(\bar{b}) < c_b(b_*) + \varepsilon$, and as K decreases in its second variable $\bar{b} > b_* \Rightarrow K(\lambda_a, \bar{b}) < 1 \Rightarrow \pi_b(c, n, \lambda_a, \bar{b}) = 1 - c_b(\bar{b}) > 1 - c_b(b_*) - \varepsilon > \pi_b(c, n, \lambda_a, b_*)$. B guarantees a positive profit as $\pi_b(c, n, \lambda_a, \bar{b}) = 1 - c_b(\bar{b}) > 0$. So winning the election is better for B than a tie in election or loosing the election. As a result, if A chooses $\lambda_a \in (a_*, \lambda)$, then $\pi_a \leq 0$. If A manipulates $\lambda_a = a_*$, then by choosing $\lambda_b = b^*$, B can

make $K(a_*, \lambda_b) = 1$ and so $\pi_b(c, n, a_*, \lambda_b) = \frac{1}{2} - c_b(b^*) = -\frac{1}{2}$, or by choosing

$\lambda_b \in [\lambda, b^*)$, $\pi_b(c, n, a_*, \lambda_b) = -c_b(\lambda_b)$, and as is easily seen the best response of B is to choose $\lambda_b = \lambda$. Thus $\pi_a(c, n, a_*, \lambda) = 1 - c_a(a_*)$. If A chooses $\lambda_a < a_*$, then for $\forall \lambda_b \in [\lambda, b^*]$ $K(\lambda_a, \lambda_b) > 1$, so B doesn't manipulate. Thus $\pi_a(c, n, \lambda_a, \lambda) = 1 - c_a(\lambda_a)$.

From all the possible strategies it is seen that choosing $\lambda_a = a_*$ maximizes the profit of

A. Now let's determine the exact value of a_* . $K(a_*, b^*) = 1$

$$\Rightarrow a_* = \frac{\lambda n_a(1 - b^*)}{n_a(1 - b^*) + (b^* - \lambda)n_b} \Rightarrow \left(\frac{\lambda n_a(1 - b^*)}{n_a(1 - b^*) + (b^* - \lambda)n_b}, \lambda \right) \text{ is a Stackelberg}$$

equilibrium.

(iii) $K^* < 1 \Leftrightarrow (\lambda - a^*)n_a(1 - b^*) < (b^* - \lambda)n_b a^* \Rightarrow$ If B is the second player

whatever $\lambda_a \in [a^*, \lambda^*]$ A chooses, there exists $b_* \in (\lambda, b^*)$ such that $K(\lambda_a, b_*) = 1$. So B

guarantees $\pi_b(c, n, \lambda_a, b_*) = \frac{1}{2} - c_b(b_*)$. For $\forall \varepsilon \in (0, 0.5)$ there exists $\bar{b} \in (b_*, b^*)$ such

that $c_b(\bar{b}) < c_b(b_*) + \varepsilon$, and as K decreases with respect to its second variable is negative,

$\bar{b} < b_* \Rightarrow K(\lambda_a, \bar{b}) < 1 \Rightarrow \pi_b(c, n, \lambda_a, \bar{b}) = 1 - c_b(\bar{b}) > 1 - c_b(b_*) - \varepsilon > \pi_b(c, n, \lambda_a, b_*)$. On

the other hand, $\pi_b(c, n, \lambda_a, \bar{b}) = 1 - c_b(\bar{b}) > 0$, so winning the election is better for B than

a tie in election or loosing the election. Thus, A chooses $\lambda_a = \lambda$ to minimize $c_a(\lambda_a)$.

Then for $\forall \varepsilon > 0$ there exists $b_\varepsilon \in (\lambda, b^*)$ such that $c_b(b_\varepsilon) < \varepsilon$, and

$b_\varepsilon > \lambda \Rightarrow K(\lambda, b_\varepsilon) < 1 \Rightarrow \pi_b(c, n, \lambda, b_\varepsilon) = 1 - c_b(b_\varepsilon) > 1 - \varepsilon$. Thus we have an ε -equilibrium.

Let B be the first player. Now $\exists b_* \in (\lambda, a^*)$ such that $K(a^*, b_*) = 1$. If B chooses $\lambda_b \in [\lambda, b_*)$, then $\exists a_* \in (a^*, \lambda]$ such that $K(a_*, \lambda_b) = 1$. So A guarantees $\pi_a(c, n, a_*, \lambda_b) = \frac{1}{2} - c_a(a_*)$. There exists $\bar{a} \in (a^*, a_*)$ such that $c_a(\bar{a}) < c_a(a_*) + \varepsilon$ for $\forall \varepsilon \in (0, 0.5)$, and $\bar{a} < a_* \Rightarrow K(\bar{a}, \lambda_b) > 1 \Rightarrow \pi_a(c, n, \bar{a}, \lambda_b) = 1 - c_a(\bar{a}) > 1 - c_a(a_*) - \varepsilon > \pi_a(c, n, a_*, \lambda_b)$. Taking into consideration that $\pi_a(c, n, \bar{a}, \lambda_b) = 1 - c_a(\bar{a}) > 0$, it can be seen that winning the election is better for A than a tie or defeat in the election. If B chooses $\lambda_b \in (\lambda, b_*)$ he earns a negative profit. If he chooses $\lambda_b = b_*$, then the best response of A is to choose $\lambda_a = \lambda$. Because by choosing $\lambda_a = a^*$ A can make $K(\lambda_a, b_*) = 1$ and so $\pi_a(c, n, \lambda_a, b_*) = \frac{1}{2} - c_a(a^*) = -\frac{1}{2}$, or by choosing $\lambda_a \in (a^*, \lambda]$, $\pi_a(c, n, \lambda_a, b_*) = -c_a(\lambda_a)$. Thus, $\pi_b(c, n, \lambda, b_*) = 1 - c_b(b_*)$. If B chooses $\lambda_b > b_*$, then as for $\forall \lambda_a \in [a^*, \lambda]$ $K(\lambda_a, \lambda_b) > 1$, A doesn't manipulate, and B gets $\pi_b(c, n, \lambda, \lambda_b) = 1 - c_b(\lambda_b)$. From all the possible strategies it is seen that playing $\lambda_b = b_*$ is the best manipulation for B. Let's find the exact value of b_* .

$K(a^*, b_*) = 1 \Rightarrow b_* = \frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*} \Rightarrow \left(\lambda, \frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*} \right)$ is a Stackelberg equilibrium.

On the other hand when **(I)** is the case and A is the first player

$$\pi_a\left(c, n, \frac{\lambda n_a(1-b^*)}{n_a(1-b^*) + (b^* - \lambda)n_b}, \lambda\right) = 1 - c_a\left(\frac{\lambda n_a(1-b^*)}{n_a(1-b^*) + (b^* - \lambda)n_b}\right) < 1 - \varepsilon \quad \text{for}$$

$$\varepsilon < c_a\left(\frac{\lambda n_a(1-b^*)}{n_a(1-b^*) + (b^* - \lambda)n_b}\right). \text{ Thus being the second player A can reduce the cost of}$$

manipulation and increase his profit.

The cost to the society, when (λ_a, λ_b) is played, is:

$$C(\lambda_a, \lambda_b) = ncP(a \text{ voter votes in the election}) = nc\alpha(c, n)\left(\lambda \frac{n_a}{\lambda_a} + (1-\lambda)\frac{n_b}{(1-\lambda_b)} + 2 - n_a - n_b\right)$$

$$\text{Using this } C\left(\frac{\lambda n_a(1-b^*)}{n_a(1-b^*) + (b^* - \lambda)n_b}, \lambda\right) = nc\alpha(c, n)\left(2 + n_b \frac{b^* - \lambda}{1-b^*}\right). \text{ When A is the second}$$

$$\text{player, let } (\lambda_a, \lambda) \text{ be played, where } \lambda_a \in \left(\frac{\lambda n_a(1-b^*)}{n_a(1-b^*) + (b^* - \lambda)n_b}, \lambda\right). \text{ Then}$$

$$C(\lambda_a, \lambda) = nc\alpha(c, n)\left(2 + n_a \frac{\lambda - \lambda_a}{\lambda_a}\right) < C\left(\frac{\lambda n_a(1-b^*)}{n_a(1-b^*) + (b^* - \lambda)n_b}, \lambda\right) \Leftrightarrow nc\alpha(c, n) \neq 0. \text{ Let's}$$

show that $\lim_{n \rightarrow \infty} nc\alpha(c, n) \neq 0$. We know that $(1-\alpha)^{n-1} \leq 2c$ and by using

$$1 - \alpha(c, n)n \leq 1 - \alpha(c, n)(n-1) \leq (1 - \alpha(c, n))^{n-1} \leq 2c \text{ we have that } \alpha(c, n)n \geq 1 - 2c. \text{ So}$$

$$\lim_{n \rightarrow \infty} n\alpha(c, n) \geq 1 - 2c > 0 \text{ since } c < 1/2. \text{ Thus } \lim_{n \rightarrow \infty} nc\alpha(c, n) > 0, \text{ and when A is the second}$$

player the cost of the selection decreases.

Similarly when **(II)** is the case and B starts the manipulation game

$$\pi_b\left(c, n, \lambda, \frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*}\right) = 1 - c_b \left(\frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*}\right) < 1 - \varepsilon \quad \text{when}$$

$\varepsilon < c_b \left(\frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*}\right)$, thus being the second player B can increase his profit. The

cost to the society when B is the first player is

$$C\left(\lambda, \frac{n_a(\lambda - a^*) + n_b \lambda a^*}{n_a(\lambda - a^*) + n_b a^*}\right) = nc\alpha(c, n) \left(2 + n_a \frac{\lambda - a^*}{a^*}\right). \text{ When B is the second player, if}$$

(λ, λ_b) is played where $\lambda_b \in (\lambda, \frac{\lambda n_b a^* + n_a(\lambda - a^*)}{n_b a^* + n_a(\lambda - a^*)})$, then the cost of this to the society is

$$C(\lambda, \lambda_b) = nc\alpha(c, n) \left(2 + n_b \frac{\lambda_b - \lambda}{1 - \lambda_b}\right) < C\left(\lambda, \frac{\lambda n_b a^* + n_a(\lambda - a^*)}{n_b a^* + n_a(\lambda - a^*)}\right) \Leftrightarrow nc\alpha(c, n) \neq 0. \text{ Thus}$$

when A is the first player social welfare increases.

Proof of Lemma 11: As $\varepsilon > 0$, there exists $\ell \in \mathbb{N}^+$ such that $\varepsilon > 1/\ell$. For

$$K = \left\lfloor \frac{\ell M}{\lambda} + 1 \right\rfloor, \text{ by Lemma 7 there exists } c_K > 0 \text{ such that for } \forall c < c_K \text{ we have } m^* > K.$$

Lemma 5 indicates that the number of votes for A- X_a - and B- X_b - have Poisson

distributions with respective means λm^* and $(1 - \lambda)m^*$. So,

$$P(\text{voter turnout} < M) = P(X_a + X_b < M) \leq P(X_a < M) = \sum_{i=0}^{M-1} \exp(-\lambda m^*) \frac{(\lambda m^*)^i}{i!}$$

Notice that $\lambda m^* > \lambda K > \ell M$. So $\frac{\exp(\lambda m^*) \frac{(\lambda m^*)^{k+1}}{(k+1)!}}{\exp(\lambda m^*) \frac{(\lambda m^*)^k}{k!}} = \frac{\lambda m^*}{k+1} > 1$ when $k \leq \ell M - 1$. Thus

$$1 = \sum_{i=0}^{\infty} \exp(\lambda m^*) \frac{(\lambda m^*)^k}{k!} \geq \sum_{i=0}^{\ell M - 1} \exp(\lambda m^*) \frac{(\lambda m^*)^k}{k!} \geq \ell \sum_{i=0}^{M-1} \exp(\lambda m^*) \frac{(\lambda m^*)^k}{k!}$$

$\geq \ell P(\text{voter turnout} < M) \Rightarrow P(\text{voter turnout} < M) \leq 1/\ell < \varepsilon$ for $\forall c < c_K$. So taking

$c(M, \varepsilon) = c_K$ will complete the proof of our lemma.

Proof of Proposition 9: If $\varphi_b^* \leq \varphi \frac{\lambda - (1 - n_b)(1 - \lambda)}{(1 - \lambda)n_b}$, then as K increases with φ_a

and decreases with φ_b we have $\min_{\varphi_a \in [\varphi, \varphi_a^*], \varphi_b \in [\varphi, \varphi_b^*]} K(\varphi_a, \varphi_b) = \frac{\lambda \varphi}{(1 - \lambda)(\varphi_b^* n_b + \varphi(1 - n_b))}$

$$\geq \frac{\lambda \varphi}{(1 - \lambda)(\varphi \frac{\lambda - (1 - n_b)(1 - \lambda)}{(1 - \lambda)n_b} n_b + \varphi(1 - n_b))} = 1$$

If B chooses $\varphi_b = \varphi_b^*$ the $K(\varphi_a, \varphi_b) \geq 1 \Rightarrow \pi_b \leq \frac{1}{2} - 1 < 0$. Now

$\varphi_b < \varphi_b^* \Rightarrow K(\varphi_a, \varphi_b) > 1 \Rightarrow \pi_b = -c_b(\varphi_b)$. So B does not manipulate whatever A

chooses. Thus A always wins the election, so A does not manipulate either and this leads

to a Nash and also a Stackelberg equilibrium.

If $\varphi_a^* \leq \varphi \frac{(1-\lambda)-(1-n_a)\lambda}{\lambda n_a}$, then by using the same observations as above, we get

$$\max_{\varphi_a \in [\varphi, \varphi_a^*], \varphi_b \in [\varphi, \varphi_b^*]} K(\varphi_a, \varphi_b) = \frac{\lambda(\varphi_a^* n_a + \varphi(1-n_a))}{(1-\lambda)\varphi} \leq \frac{\lambda n_a \varphi \frac{(1-\lambda)-(1-n_a)\lambda}{\lambda n_a} + \lambda \varphi(1-n_a)}{(1-\lambda)\varphi} = 1$$

If A chooses $\varphi_a = \varphi_a^*$ the $K(\varphi_a, \varphi_b) \leq 1$, so profit is at most $(1/2) - 1 < 0$. Now $\varphi_a < \varphi_a^*$
 $\Rightarrow K(\varphi_a, \varphi_b) < 1 \Rightarrow \pi_a = -c_a(\varphi_a)$. So A makes no manipulation and B always wins the election. So B does not manipulate either, and this leads a Nash and also a Stackelberg equilibrium.

Let's show that there is not any other Nash equilibrium. Assume not, and let (φ_a, φ_b) be a Nash equilibrium.

$K(\varphi_a, \varphi_b) > 1 \Rightarrow$ A wins, and as φ_b is the best response of B, $\varphi_b = \varphi$ and

$\pi_b(\varphi_a, \varphi_b) = 0$. If $\varphi_a = \varphi$, then, as $\varphi_b^* > \varphi \frac{\lambda - (1-n_b)(1-\lambda)}{(1-\lambda)n_b}$, choosing some value

closer to φ_b^* , B makes a positive profit. So $\varphi_a > \varphi$. But then, as K is a continuous function, by decreasing φ_a a bit ($> \varphi$) A can increase his profit, which contradicts that (φ_a, φ_b) is a Nash equilibrium.

$K(\varphi_a, \varphi_b) < 1 \Rightarrow B$ wins and $\varphi_a = \varphi$. Now $\varphi_a^* > \varphi \frac{(1-\lambda) - (1-n_a)\lambda}{\lambda n_a} \Rightarrow \varphi_b > \varphi$. By

decreasing φ_b a bit, B can increase his profit, which contradicts that (φ_a, φ_b) is a Nash equilibrium.

$K(\varphi_a, \varphi_b) = 1 \Rightarrow c_a(\varphi_a) \leq \frac{1}{2}, c_b(\varphi_b) \leq \frac{1}{2}$. If A chooses $\varphi^* > \varphi_a$ such that

$c_a(\varphi^*) - c_a(\varphi_a) < \frac{1}{2}$, then $\pi_a(\varphi_a, \varphi_b) = \frac{1}{2} - c_a(\varphi_a) < 1 - c_a(\varphi^*) = \pi_a(\varphi^*, \varphi_b)$,

contradicting that (φ_a, φ_b) is a Nash equilibrium.

Proof of Proposition 10: Note that in the given domain of φ_a^*, φ_b^* we

have $\min_{\varphi_a \in [\varphi, \varphi_a^*], \varphi_b \in [\varphi, \varphi_b^*]} K(\varphi_a, \varphi_b) < 1$ and $\max_{\varphi_a \in [\varphi, \varphi_a^*], \varphi_b \in [\varphi, \varphi_b^*]} K(\varphi_a, \varphi_b) > 1$.

(i) $K^* = 1 \Rightarrow$ whatever $\varphi_i \notin \{\varphi, \varphi_i^*\}$ the first player $i \in \{a, b\}$ chooses the second player j has the ability of making $P(j \text{ wins} | (\varphi_i, \varphi_j)) = 1$ by choosing an appropriate $\varphi_j \neq \varphi_j^*$. So j wins the election. Choosing $\varphi_i \notin \{\varphi, \varphi_i^*\}$ causes only a positive cost to i .

If $\varphi_i = \varphi$, then for $\forall \varepsilon > 0$ there exists $x \in (\varphi, \varphi_j^*)$ such that

$P(j \text{ wins} | (\varphi_i, \varphi_j) = (\varphi, x)) = 1$ and $c_j(x) < c_j(\varphi \frac{\lambda_i - (1-n_j)\lambda_j}{\lambda_j n_j}) + \varepsilon$, so that

$\pi_j(\varphi, x) > 1 - c_j(\varphi \frac{\lambda_i - (1 - n_j)\lambda_j}{\lambda_j n_j}) - \varepsilon$ since $c_j(x)$ is a continuous function of x . On the

other hand, if i chooses $\varphi_i = \varphi_i^*$, then best response of j is φ , which makes

$\pi_i(\varphi_i^*, \varphi) = \pi_j(\varphi_i^*, \varphi) = 0$. So φ dominates φ_i^* and as a result i chooses $\varphi_i = \varphi$. As the

maximum value for π_j is $1 - c_j(\varphi \frac{\lambda_i - (1 - n_j)\lambda_j}{\lambda_j n_j})$ and j can make

$\pi_j > 1 - c_j(\varphi \frac{\lambda_i - (1 - n_j)\lambda_j}{\lambda_j n_j}) - \varepsilon$ for any ε , there is an ε -equilibrium. For

$\forall \varepsilon \in \left(0, 1 - c_j(\varphi \frac{\lambda_i - (1 - n_j)\lambda_j}{\lambda_j n_j})\right)$, $\pi_j > 0$, so both of the candidates want to be the

second player.

(ii) Consider the case where $K^* > 1$. If A is the second player whatever $\varphi_b \in [\varphi, \varphi_b^*]$

B chooses, $\exists \varphi_{a^*} \in [\varphi, \varphi_a^*)$ such that $K(\varphi_{a^*}, \varphi_b) = 1$ or $\varphi_{a^*} = \varphi$ and $K(\varphi_{a^*}, \varphi_b) > 1$. For

$K(\varphi, \varphi_b) > 1$, knowing that he will lose the election B chooses $\varphi_b = \varphi$. For first case,

$K(\varphi_{a^*}, \varphi_b) = 1$, where A guarantees $\pi_a(\varphi_a, \varphi_b) \geq \pi_a(\varphi_{a^*}, \varphi_b) = \frac{1}{2} - c_a(\varphi_{a^*})$ and for

$\forall \varepsilon \in (0, 0.5)$ $\exists \varphi_{aa} \in (\varphi_{a^*}, \varphi_a^*)$ such that $c_a(\varphi_{aa}) < c_a(\varphi_{a^*}) + \varepsilon$, and

$\varphi_{a^*} < \varphi_{aa} \Rightarrow K(\varphi_{aa}, \varphi_b) > 1 \Rightarrow \pi_a(\varphi_{aa}, \varphi_b) > 1 - c_a(\varphi_{a^*}) - \varepsilon$

$\Rightarrow \pi_a(\varphi_{aa}, \varphi_b) > \pi_a(\varphi_{a^*}, \varphi_b)$. On the other hand $\pi_a(\varphi_{aa}, \varphi_b) = 1 - c_a(\varphi_{aa}) > 0$. So

winning the election is better for A than a tie in election or loosing the election. Knowing

this fact B chooses $\varphi_b = \varphi$. Then for $\forall \varepsilon > 0 \exists \varphi_\varepsilon \in (\varphi \frac{(1-\lambda)-(1-n_a)\lambda}{\lambda n_a}, \varphi_a^*)$ such that

$$c_a(\varphi_\varepsilon) < c_a(\varphi \frac{(1-\lambda)-(1-n_a)\lambda}{\lambda n_a}) + \varepsilon, \quad \text{and} \quad \varphi_\varepsilon > \varphi \frac{(1-\lambda)-(1-n_a)\lambda}{\lambda n_a} \Rightarrow K(\varphi_\varepsilon, \varphi) > 1$$

$$\Rightarrow \pi_a(\varphi_\varepsilon, \varphi) = 1 - c_a(\varphi_\varepsilon) > 1 - c_a(\varphi \frac{(1-\lambda)-(1-n_a)\lambda}{\lambda n_a}) - \varepsilon. \quad \text{Thus we have an } \varepsilon -$$

equilibrium.

Let A be the first player. Now $\exists \varphi_* \in (\varphi, \varphi_a^*)$ such that $K(\varphi_*, \varphi_b^*) = 1$. So when A chooses $\varphi_a \in [\varphi, \varphi_*)$, $\exists \varphi_{b^*} \in [\varphi, \varphi_b^*)$ such that $K(\varphi_a, \varphi_{b^*}) = 1$. So B guarantees

$$\pi_b(\varphi_a, \varphi_{b^*}) = \frac{1}{2} - c_b(\varphi_{b^*}) \quad \text{and} \quad \forall \varepsilon \in (0, 0.5) \quad \exists \varphi_{bb} \in (\varphi_{b^*}, \varphi_b^*) \quad \text{such that}$$

$$c_b(\varphi_{bb}) < c_b(\varphi_{b^*}) + \varepsilon, \quad \text{and}$$

$$\varphi_{bb} > \varphi_{b^*} \Rightarrow K(\varphi_a, \varphi_{bb}) < 1 \Rightarrow \pi_b(\varphi_a, \varphi_{bb}) = 1 - c_b(\varphi_{bb}) > 1 - c_b(\varphi_{b^*}) - \varepsilon > \pi_b(\varphi_a, \varphi_{b^*}).$$

On the other hand $\pi_b(\varphi_a, \varphi_{bb}) = 1 - c_b(\varphi_{bb}) > 0$. So winning the election is better for B than a tie in election or loosing the election. As a result if A chooses $\varphi_a \in [\varphi, \varphi_*)$ then

$\pi_a \leq 0$. If A chooses $\varphi_a = \varphi_*$, then by choosing $\varphi_b = \varphi_b^*$ B can make $K(\varphi_*, \varphi_b) = 1$ and

$$\pi_b(\varphi_*, \varphi_b) = \frac{1}{2} - c_b(\varphi_b^*) = -\frac{1}{2} \quad \text{or by choosing } \varphi_b \in [\varphi, \varphi_b^*), \pi_b(\varphi_*, \varphi_b) = -c_b(\varphi_b). \quad \text{Thus}$$

the best response of B is to choose $\varphi_b = \varphi$ which maximizes π_b . Thus

$\pi_a(\varphi_*, \varphi) = 1 - c_a(\varphi_*)$. If A chooses $\varphi_a > \varphi_*$ then as for $\forall \varphi_b \in [\varphi, \varphi_b^*]$, $K(\varphi_a, \varphi_b) > 1$,

so B chooses $\varphi_b = \varphi$. Thus $\pi_a(\varphi_a, \varphi) = 1 - c_a(\varphi_a)$. From all the possible strategies of A,

$$\varphi_a = \varphi_*$$

maximizes π_a .

Now

$$K(\varphi_*, \varphi_b^*) = 1 \Rightarrow \varphi_* = \varphi_b^* \frac{n_b(1-\lambda)}{n_a\lambda} + \varphi \frac{(1-n_b)(1-\lambda) - (1-n_a)\lambda}{n_a\lambda} \Rightarrow$$

$$\left(\varphi_b^* \frac{n_b(1-\lambda)}{n_a\lambda} + \varphi \frac{(1-n_b)(1-\lambda) - (1-n_a)\lambda}{n_a\lambda}, \varphi \right) \text{ is a Stackelberg equilibrium.}$$

$$\text{When } \varepsilon < c_a \left(\varphi_b^* \frac{n_b(1-\lambda)}{n_a\lambda} + \varphi \frac{(1-n_b)(1-\lambda) - (1-n_a)\lambda}{n_a\lambda} \right) - c_a \left(\varphi \frac{(1-\lambda) - (1-n_a)\lambda}{\lambda n_a} \right)$$

we get $\pi_a(\varphi_\varepsilon, \varphi) > \pi_a(\varphi_*, \varphi) \Rightarrow c_a(\varphi_\varepsilon) < c_a(\varphi_*) \Rightarrow \varphi_\varepsilon < \varphi_*$. So A wants to be the second player to reduce the cost of manipulation, and this also decreases the probability of voting for the people who believe in A, thus increasing the social welfare as well.

(iii) Now assume that $K^* < 1$. If B is the second player, whatever $\varphi_a \in [\varphi, \varphi_a^*]$ A chooses, $\exists \varphi_{b^*} \in [\varphi, \varphi_b^*)$ such that $K(\varphi_a, \varphi_{b^*}) = 1$ or $\varphi_{b^*} = \varphi$ and $K(\varphi_a, \varphi_{b^*}) < 1$. For $K(\varphi_a, \varphi) < 1$, knowing that he will lose the election, A chooses $\varphi_a = \varphi$. For the first

case, $K(\varphi_a, \varphi_{b^*}) = 1$, B guarantees $\pi_b(\varphi_a, \varphi_b) \geq \pi_b(\varphi_a, \varphi_{b^*}) = \frac{1}{2} - c_b(\varphi_{b^*})$ and for

$$\forall \varepsilon \in (0, 0.5) \quad \exists \varphi_{bb} \in (\varphi_b, \varphi_{b^*}^*) \text{ such that } c_b(\varphi_{bb}) < c_b(\varphi_{b^*}) + \varepsilon, \quad \text{and}$$

$$\varphi_{b^*} < \varphi_{bb} \Rightarrow K(\varphi_a, \varphi_{bb}) < 1 \Rightarrow \pi_b(\varphi_a, \varphi_{bb}) > 1 - c_b(\varphi_{b^*}) - \varepsilon \Rightarrow \pi_b(\varphi_a, \varphi_{bb}) > \pi_b(\varphi_a, \varphi_{b^*}).$$

On the other hand, $\pi_b(\varphi_a, \varphi_{bb}) = 1 - c_b(\varphi_{bb}) > 0$. So winning the election is better for B than a tie in election or loosing the election. Knowing this fact A chooses $\varphi_a = \varphi$. Then

for $\forall \varepsilon > 0$ $\exists \varphi_\varepsilon \in (\varphi \frac{\lambda - (1 - n_b)(1 - \lambda)}{(1 - \lambda)n_b}, \varphi_b^*)$ such that

$$c_b(\varphi_\varepsilon) < c_b(\varphi \frac{\lambda - (1 - n_b)(1 - \lambda)}{(1 - \lambda)n_b}) + \varepsilon, \quad \text{and} \quad K(\varphi, \varphi_\varepsilon) < 1$$

$$\Rightarrow \pi_b(\varphi, \varphi_\varepsilon) = 1 - c_b(\varphi_\varepsilon) > 1 - c_b(\varphi \frac{\lambda - (1 - n_b)(1 - \lambda)}{(1 - \lambda)n_b}) - \varepsilon. \quad \text{Thus we have an } \varepsilon -$$

equilibrium.

Let B be the first player. Now $\exists \varphi_* \in (\varphi, \varphi_b^*)$ such that $K(\varphi_a^*, \varphi_*) = 1$. So when B chooses $\varphi_b \in [\varphi, \varphi_*)$, $\exists \varphi_{a*} \in [\varphi, \varphi_a^*)$ such that $K(\varphi_{a*}, \varphi_b) = 1$. A guarantees

$$\pi_a(\varphi_{a*}, \varphi_b) = \frac{1}{2} - c_a(\varphi_{a*}) \quad \text{and} \quad \forall \varepsilon \in (0, 0.5) \quad \exists \varphi_{aa} \in (\varphi_{a*}, \varphi_a^*) \quad \text{such that}$$

$$c_a(\varphi_{aa}) < c_a(\varphi_{a*}) + \varepsilon, \quad \text{and} \quad \varphi_{aa} > \varphi_{a*} \Rightarrow K(\varphi_{aa}, \varphi_b) > 1 \Rightarrow \pi_a(\varphi_{aa}, \varphi_b) = 1 - c_a(\varphi_{aa})$$

$$> 1 - c_a(\varphi_{a*}) - \varepsilon > \pi_a(\varphi_{a*}, \varphi_b). \quad \text{On the other hand, } \pi_a(\varphi_{aa}, \varphi_b) = 1 - c_a(\varphi_{aa}) > 0. \quad \text{So}$$

winning the election is better for A than a tie in election or loosing the election. As a result, if B chooses $\varphi_b \in [\varphi, \varphi_*)$, then $\pi_b \leq 0$. If B chooses $\varphi_b = \varphi_*$, then by choosing

$$\varphi_a = \varphi_a^*, \text{ A can make } K(\varphi_a, \varphi_*) = 1 \text{ and } \pi_a(\varphi_a, \varphi_*) = \frac{1}{2} - c_a(\varphi_a^*) = -\frac{1}{2} \text{ or by choosing}$$

$$\varphi_a \in [\varphi, \varphi_a^*), \pi_a(\varphi_a, \varphi_*) = -c_a(\varphi_a). \quad \text{Thus } \varphi_a = \varphi \text{ and } \pi_b(\varphi, \varphi_*) = 1 - c_b(\varphi_*). \quad \text{If B}$$

chooses $\varphi_b > \varphi_*$, then for $\forall \varphi_a \in [\varphi, \varphi_a^*)$ $K(\varphi_a, \varphi_b) > 1$. So $\varphi_a = \varphi$ is the best for A.

Thus $\pi_b(\varphi, \varphi_b) = 1 - c_b(\varphi_b)$. From all the possible strategies of B, $\varphi_b = \varphi_*$ maximizes

$$\pi_b. \quad \text{Now} \quad K(\varphi_a^*, \varphi_*) = 1 \Rightarrow \varphi_* = \frac{\varphi_a^* n_a \lambda}{n_b (1 - \lambda)} + \varphi \frac{(1 - n_a) \lambda - (1 - n_b)(1 - \lambda)}{n_b (1 - \lambda)} \Rightarrow$$

$$\left(\varphi, \frac{\varphi_a^* n_a \lambda}{n_b (1 - \lambda)} + \varphi \frac{(1 - n_a) \lambda - (1 - n_b)(1 - \lambda)}{n_b (1 - \lambda)} \right) \text{ is a Stackelberg equilibrium.}$$

$$\text{When} \quad \varepsilon < c_b \left(\frac{\varphi_a^* n_a \lambda}{n_b (1 - \lambda)} + \varphi \frac{(1 - n_a) \lambda - (1 - n_b)(1 - \lambda)}{n_b (1 - \lambda)} \right) - c_b \left(\varphi \frac{\lambda - (1 - n_b)(1 - \lambda)}{(1 - \lambda) n_b} \right),$$

we get $\pi_b(\varphi, \varphi_\varepsilon) > \pi_a(\varphi, \varphi_*) \Rightarrow c_b(\varphi_\varepsilon) < c_b(\varphi_*) \Rightarrow \varphi_\varepsilon < \varphi_*$. So B wants to be the second player to reduce the cost of manipulation. This also decreases the probability of voting for the people who believe in B, they increasing the social welfare as well.

Proof of Proposition 11: The voters who believe λ determine their strategy - φ_{a2}

and φ_{b2} - such that:

$$\alpha(c, n) = \lambda(\varphi_{a1} n_a + \varphi_{a2} (1 - n_a)) = (1 - \lambda)(\varphi_{b1} n_b + \varphi_{b2} (1 - n_b)) \quad \text{where} \quad \varphi_{a1} = \frac{\alpha(c, n)}{\lambda_a} \quad \text{and}$$

$$\varphi_{b1} = \frac{\alpha(c, n)}{1 - \lambda_b} \Rightarrow \varphi_{a2} = \frac{\alpha(c, n) \left(1 - \frac{\lambda}{\lambda_a} n_a \right)}{\lambda(1 - n_a)}, \quad \varphi_{b2} = \frac{\alpha(c, n) \left(1 - \frac{1 - \lambda}{1 - \lambda_a} n_b \right)}{(1 - \lambda)(1 - n_b)}$$

If $\{\varphi_{a2}, \varphi_{b2}\} \subset (0, 1)$ then the society can prevent manipulation using totally mixed

strategies and $P(A \text{ wins}) = P(B \text{ wins}) = \frac{1}{2}$ occurs as the result of the election. Now

$$0 < \varphi_{a2} = \frac{\alpha(c, n) \left(1 - \frac{\lambda}{\lambda_a} n_a\right)}{\lambda(1 - n_a)} < 1 \Leftrightarrow 0 < \alpha(c, n) \left(1 - \frac{\lambda}{\lambda_a} n_a\right) < \lambda(1 - n_a) \Leftrightarrow \lambda_a > \lambda n_a, \text{ and if}$$

$$\alpha(c, n) - \lambda(1 - n_a) > 0, \text{ then } \lambda_a < \frac{\alpha(c, n) \lambda n_a}{\alpha(c, n) - \lambda(1 - n_a)} \text{ or } \alpha(c, n) - \lambda(1 - n_a) \leq 0. \text{ Now}$$

$$\lambda_a < \frac{\alpha(c, n) \lambda n_a}{\alpha(c, n) - \lambda(1 - n_a)} \Leftrightarrow n_a < \frac{\lambda_a (\lambda - \alpha(c, n))}{\lambda(\lambda_a - \alpha(c, n))} \text{ and using the fact that } \lambda_a \in (\alpha(c, n), \lambda]$$

$$\text{we have, } \lambda \alpha(c, n) \geq \lambda_a \alpha(c, n) \Rightarrow \lambda_a \lambda - \lambda_a \alpha(c, n) \geq \lambda_a \lambda - \lambda \alpha(c, n) > 0$$

$$\Rightarrow \frac{\lambda_a (\lambda - \alpha(c, n))}{\lambda(\lambda_a - \alpha(c, n))} \geq 1 > n_a. \text{ So } 0 < \varphi_{a2} < 1 \text{ doesn't depend on the sign of}$$

$$\alpha(c, n) - \lambda(1 - n_a) \text{ and thus } 0 < \varphi_{a2} < 1 \Leftrightarrow n_a < \frac{\lambda_a}{\lambda}. \text{ Now}$$

$$0 < \varphi_{b2} = \frac{\alpha(c, n) \left(1 - \frac{1 - \lambda}{1 - \lambda_b} n_b\right)}{(1 - \lambda)(1 - n_b)} < 1 \Leftrightarrow 0 < \alpha(c, n) \left(1 - \frac{1 - \lambda}{1 - \lambda_b} n_b\right) < (1 - \lambda)(1 - n_b)$$

$$\Rightarrow \lambda_b < 1 - (1 - \lambda)n_b, \text{ and if } \alpha(c, n) - (1 - \lambda)(1 - n_b) > 0, \text{ then}$$

$$\lambda_b > \frac{\alpha(c, n) - (1 - \lambda)(1 - n_b) - \alpha(c, n)(1 - \lambda)n_b}{\alpha(c, n) - (1 - \lambda)(1 - n_b)} \text{ or } \alpha(c, n) - (1 - \lambda)(1 - n_b) \leq 0. \text{ Now}$$

$$\lambda_b > \frac{\alpha(c, n) - (1 - \lambda)(1 - n_b) - \alpha(c, n)(1 - \lambda)n_b}{\alpha(c, n) - (1 - \lambda)(1 - n_b)} \Leftrightarrow n_b < \frac{(1 - \lambda_b)(1 - \lambda - \alpha(c, n))}{(1 - \lambda)(1 - \lambda_b - \alpha(c, n))} \text{ and as}$$

$$\lambda_b \in [\lambda, 1 - \alpha(c, n)) \text{ we have } \lambda_b \alpha(c, n) \geq \lambda \alpha(c, n)$$

$$\Rightarrow \lambda_b \alpha(c, n) + 1 - \lambda - \alpha(c, n) - \lambda_b + \lambda_b \lambda \geq \lambda \alpha(c, n) + 1 - \lambda - \alpha(c, n) - \lambda_b + \lambda_b \lambda$$

$$\Rightarrow (1 - \lambda_b)(1 - \lambda - \alpha(c, n)) \geq (1 - \lambda)(1 - \lambda_b - \alpha(c, n)) \Rightarrow \frac{(1 - \lambda_b)(1 - \lambda - \alpha(c, n))}{(1 - \lambda)(1 - \lambda_b - \alpha(c, n))} \geq 1 > n_b.$$

So $0 < \varphi_{b2} < 1$ when $n_b < \frac{1 - \lambda_b}{1 - \lambda}$.

Proof of Corollary 2: We get the desired result by using Proposition 11 together with the fact that $\lambda_a \in (\alpha(c, n), \lambda]$ and $\lambda_b \in [\lambda, 1 - \alpha(c, n))$.

Proof of Corollary 3: If $\lim_{n \rightarrow \infty} \alpha(c, n) = 0$ then the society can prevent all manipulations only when $n_a = n_b = 0$. But as $\{n_a, n_b\} \subset (0, 1)$, the society cannot prevent the manipulations $\{\lambda_a, \lambda_b\}$ satisfying $n_a \geq \frac{\lambda_a}{\lambda}$ and $n_b \geq \frac{1 - \lambda_b}{1 - \lambda}$.

Proof of Proposition 12: The voters who believe $P(\lambda = \lambda_i | A) = 2\theta_i \lambda_i$ vote with probability φ_{aa} which satisfies:

$$\varphi = n_a \varphi_a + (1 - n_a) \varphi_{aa} \Leftrightarrow \varphi_{aa} = \frac{\varphi - n_a \varphi_a}{1 - n_a},$$

and the voters who believe $P(\lambda = \lambda_i | B) = 2\theta_i (1 - \lambda_i)$ vote with probability φ_{bb} which satisfies:

$$\varphi = n_b \varphi_b + (1 - n_b) \varphi_{bb} \Leftrightarrow \varphi_{bb} = \frac{\varphi - n_b \varphi_b}{1 - n_b}$$

If $\{\varphi_{aa}, \varphi_{bb}\} \subset (0,1)$, then it is possible for the society to prevent manipulation and get the same result in the election with the case without manipulation.

$$0 < \frac{\varphi - n_a \varphi_a}{1 - n_a} < 1 \Leftrightarrow 0 < \varphi - n_a \varphi_a < 1 - n_a \Leftrightarrow n_a < \min \left\{ \frac{\varphi}{\varphi_a}, \frac{1 - \varphi}{1 - \varphi_a} \right\}$$

$$0 < \frac{\varphi - n_b \varphi_b}{1 - n_b} < 1 \Leftrightarrow 0 < \varphi - n_b \varphi_b < 1 - n_b \Leftrightarrow n_b < \min \left\{ \frac{\varphi}{\varphi_b}, \frac{1 - \varphi}{1 - \varphi_b} \right\}$$

Proof of Corollary 4: We get the desired result by using Proposition 12 together with the fact that $\{\varphi_a, \varphi_b\} \subset [\varphi_{\min}, \varphi_{\max}]$. When we take $\{\varphi_a, \varphi_b\} \subset (\varphi, \varphi_{\max}]$, as the candidates are rational, then the society can prevent all possible manipulations, when

$$n_a < \frac{\varphi}{\varphi_{\max}} \text{ and } n_b < \frac{\varphi}{\varphi_{\max}}, \text{ which follows from Proposition 12.}$$

Proof of Corollary 5: If $\lim_{n \rightarrow \infty} \varphi = 0$, then the society can prevent manipulation only when $n_a = n_b = 0$. But as $\{n_a, n_b\} \subset (0,1)$, the society cannot prevent the manipulations

$\{\theta_a, \theta_b\}$ satisfying $n_a \geq \frac{\varphi}{\varphi_a} = 0$ and $n_b \geq \frac{\varphi}{\varphi_b} = 0$. Now as the candidates can manipulate

such that $\varphi_a \in (\varphi, \varphi_{\max}]$ or $\varphi_b \in (\varphi, \varphi_{\max}]$, they have the power to manipulate and win the election.

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