SENSITIVITY ANALYSIS OF DISTANCE CONSTRAINTS AND OF MULTIFACILITY MINIMAX LOCATION ON TREE NETWORKS

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by

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ABSTRACT

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AND OF MULTIFACILITY MINIMAX LOCATION ON TREE NETWORKS

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M.S. in Industrial Engineering

Supervisor: Assoc. Prof. Barbaros Tansel

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In this thesis, the main concern is to investigate the use of consistency conditions of distance constraints in sensitivity analysis of certain network location problems. The interest is in minimax type of objective functions. A single parametric approach is adopted in the sensitivity analysis for the m-facility minimax location problem on tree networks. Apart from the traditional sensitivity analysis approach, a conceptual framework for imprecision in distance constraints is developed.

ÖZET

AĞAÇ TİPİ SERİMLERDEKİ UZAKLIK KISITLARI VE
ÇOKTESİSLİ ENKÜÇÜK - ENBÜYÜK YERSEÇİMİ PROBLEMLERİ ÜZERİNDE
DUYARLILIK ÇÖZÜMLEMESİ

Esra Doğan
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Serimler üzerindeki yerseçimi problemleri, amaç fonksiyonunun tanım kümesinin düzgülenmiş doğrusal bir uzayda (normed linear space) değil de özel bir metrik uzayında olduğu enküçükleme problemleri örnekleridir. Bir uzaklık kısıtı, genellikle belirtilmiş iki tesisin yerleşim noktaları arasındaki uzaklığın veya yolculuk süresinin üst sınırını belirler. Bu çalışmanın temelini, uzaklık kısıtları üzerindeki tutarlılık koşullarının duyarlılık çözümlemesindeki önemi ve kullanımı oluşturmaktadır. Göz önüne alınan amaç fonksiyonları enbüyüğün enküçüklenmesi şeklindedir. Ağaç tipi serimler üzerindeki çoktesisli enküçük-enbüyük yerseçimi probleminin duyarlılık çözümlemelerinde tek değişken yaklaşımına bağlı kalınmıştır. Elde edilen sonuçların, özellikle çokdeğişkenli duyarlılık çözümlemeleri için metod geliştirme sürecinde önemli rol oynayacağı düşünülmektedir. Ayrıca bu çalışmada, alışılmış duyarlılık çözümlemesi yaklaşımının dışında, uzaklık kısıtlarının belirsizliği üzerine kavramsal bir çatı geliştirilmiştir.

To my family,

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TABLE OF CONTENTS

1.INTRODUCTION

Most facility location decisions are strategic decisions for which the input data represents the future state of affairs several years ahead. Problems of facility location on networks are examples of minimization problems where the domain of the function being minimized is not a normed linear space but is a special metric space.

In this study, our main concern is to develop sensitivity analysis techniques for certain network location problems. In these problems, we try to find the locations of new facilities with respect to existing facilities, so as to minimize some objective function. Our particular interest is in minimax type of objectives. Upper bounds on distances between pairs of facilities may also be present, and they constitute the distance constraints. We will be performing sensitivity analysis on distance constraints and on "weights" that are assigned to interfacility interactions.

Models formulated for facility location decisions generally involve the parameters of transportation costs, interfacility interactions, and server charecteristics . Values of such parameters are usually hard to estimate due to a lack of organized and complete raw data. Sometimes, data may not be available at all. Despite such difficulties, it may usually be possible to obtain some estimates (though they are "rough") or a range of possible values for input data. In these circumstances, the questions that arise are: Given a range of possible values for input data, can we still get some meaningful solutions from our models? If not, how should we refine our estimates? We try to give a general framework to deal with questions of this sort in Chapter 3.

Some other questions that we adress are the following: Given a solution based on a possible realization of data, how does the optimal solution and the optimal objective value change as deviations occur from the given realization of data? How can we find the new solution to the problem without starting from scratch? These questions are related with traditional sensitivity analysis.

Sensitivity analysis within the context of network location problems with distance constraints has not received much attention in the literature. Even though consistency conditions of distance constraints have been fairly well studied in the literature, their relation to sensitivity analysis has not been studied. Our aim in the thesis is to fill this gap for at least certain location problems.

We now give an overview of the thesis. Following the introduction, we give a review of the related literature in Chapter 2. In this chapter the emphasis is given to network location problems with distance constraints and to the solution procedures so far developed for the tree network case. Chapter 3 is concerned with the imprecision in distance constraints and gives a conceptual framework to deal with tree network location problems with imprecise distance constraints. Here, imprecision in distance constraints means that we do not know exactly what right hand side values of the constraints are but, instead, we know lower and upper bounds on their possible realizations. Another approach that can be adopted in the case of imprecise data defining the distance constraints is to perform sensitivity analysis. We deal with the sensitivity analysis of basically two types of network location problems on a tree, namely minimax location problem with interfacility interactions (also referred as problems with mutual communication) and a two parameter version of this problem. These are discussed in Chapter 5. Chapter 4 defines the range analysis in this context and is actually an introductory chapter to Chapter 5. The primary focus of our study is on the trajectory analysis of the optimal objective value, and the response of the optimal location vector to changes in parameters. In Chapter 6 we provide concluding remarks and give suggestions for future research.

2.REWIEV OF THE RELATED LITERATURE

In this chapter, we review the existing theory for multifacility tree network location models. This theory will subsequently be used for sensitivity analysis in these models.

The problem of finding locations for new facilities in an imbedded network with respect to existing facilities, with upper bounds on distances between pairs of facilities is defined by Dearing, Francis, and Lowe [5].

The idea behind upper bounds imposed on interfacility distances is to prevent facilities from being "too far apart". A bound on the distance between an existing and a new facility may represent tolerable travel distance, or time, to render some service. In the location of ambulance and fire stations, the relavance of such bounds is apparent. Likewise, a bound on the distance between two new facilities may be appropriate if these service facilities interact or can provide back-up service in the case of a breakdown of the other. Interfacility transportation may also impose constraints on the distance between two new facilities.

It is also possible that distance constraints are imposed only on distances between existing facilities and their nearest new facilities. These models are called covering models (see [19, 20, 21]). In such models, new facilities are "indistinguishable" in the sense that they all provide the same kind of service. Unlike covering problems and their relatives, network location problems with interfacility distance constraints allow different upper bouns for distinct pairs of facilities [9, 17, 19, 20, 21]. In network location problems with distance constraints, it is assumed that new facilities are "distinguishable" in terms of kinds of services they provide and that each new facility may serve part or all of the network [5, 7, 21]. However, in covering, p-center, and p-median problems, all new facilities provide the same kind of service, and each existing facility is assumed to be served by a nearest service (new) facility. Furthermore, the objective function of a network location problem with interfacility distance constraints may contain terms for distances between new facilities, while the classical p-center, p-median, and covering problems deal only with the distances between existing facilities and the closest new facility [10, 11, 20, 21].

2.1. Notation and Some Definitions

As in [5, 20], let N represent an imbedded network with the vertex set $V = \{v_i, i \in I\}$ where $I = \{1,\dots,n\}$ is the set of vertex indices, and the edge set E. We assume any two edges in E intersect at most one point, a vertex, and that each edge of N has a positive arc length and is rectifiable in the sense that there is a one-to-one mapping between each edge and the interval [0,1]. That is, if x is any point on an edge of length e_{ij} joining v_i and v_j, then there exist a unique real number $w_{ij}(x) \in [0,1]$ such that $w_{ij}(x)e_{ij}$ is the length of the subedge (v_i,x) and $[1-w_{ij}(x)]e_{ij}$ is the length of the subedge (x,v_j).

As is customary, the distance between any two points x and y in N, d(x,y), is defined to be the length of a shortest path in N joining these two points. The distance function d(.,.) satisfies all the properties that a metric should have. Hence, N and d(.,.) together constitute a metric space [5].

If N is a network with no cycles,that is to say there is a unique path (which is ,by definition, the shortest path) between any two points x and y in N, then we call N a tree and denote it by T instead of N.

Consider a location vector $X = (x_1,...,x_m)$ with $x_i \in N$ (or T) , i=1,...,m. X belongs to N^m (or T^m), the m-fold cartesian product of N (or T) with itself. For X and Y in N^m, if we define the distance function $d_m(X,Y)$ to be $\sum_{i=1}^{m} d(x_i.y_i)$, it is known that (N^m,d_m) forms a metric space [5]. For m=1, we write (N,d) instead of (N^1,d_1).

For every Y and Z in N^m, Dearing et al [5] define the line segment joining Y and Z, L(Y,Z) as follows:

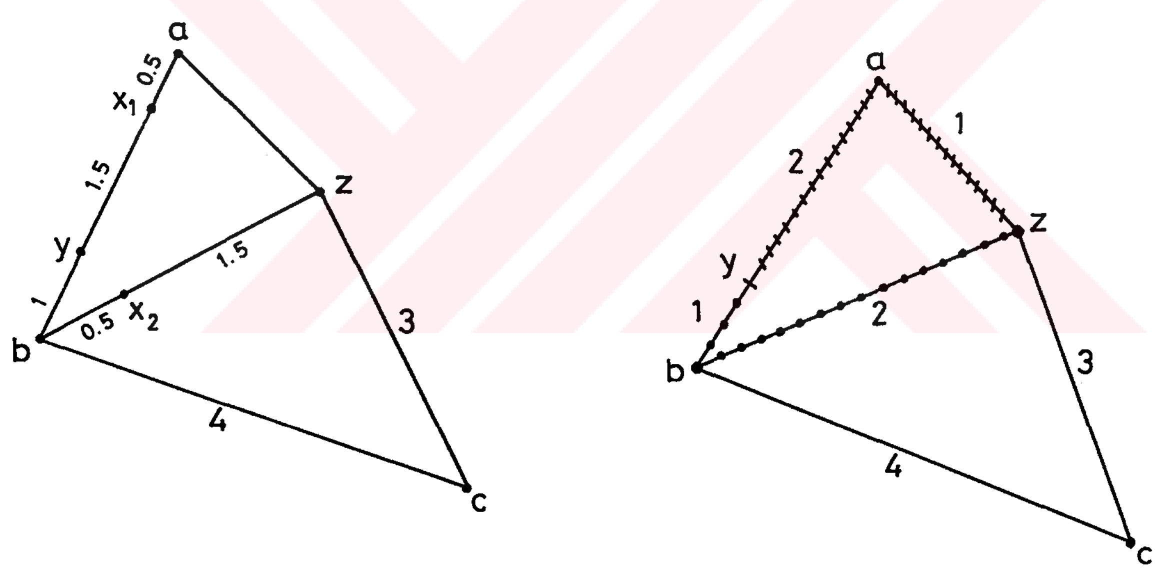
$$L(Y,Z) \equiv \bigcup \{ L_\lambda(Y,Z): 0 \le \lambda \le 1 \}$$

where, for $\lambda \in [0,1]$

$$L_\lambda(Y,Z) \equiv \{ X \in N^m : d(y_i,x_i) + d(x_i,z_i) = d(y_i,z_i);$$
$$d(x_i,z_i) = \lambda d(y_i,z_i),\ 1 \le i \le m \}.$$

When m is equal to 1, $L(y,z) \equiv \{ x \in N: d(y,x) + d(x,z) = d(y,z) \}$ is the union of all shortest paths each connecting y and z.

EXAMPLE 2.1. Consider Fig. 2.1 . For λ=0.5, $L_{0.5}(y,z) = \{x_1,x_2\}$, (Fig. 2.1 (a)); while the line segment joining y and z is; $L(y,z) = S_1(y,z) \cup S_2(y,z)$, (Fig. 2.1 (b)).



λ =0.5 ; $d(x_i,z)$=1.5 $S_1(y,z)$

$d(y,z)$=3 $S_2(y,z)$

(a) (b)

Fig. 2.1

In [5], Dearing et al also define convex sets and convex functions on the metric space (N^m,d_m) as follows: A subset S of N^m

is said to be convex if, for every $Y,Z \in S$, $L(Y,Z) \subseteq S$. When $S \subseteq N$, to say that S is convex is equivalent to requiring S to be connected [12]. If S is a convex subset of N^m and f is a real valued function with domain S, then f is said to be convex on S, if given any Y, Z in S, $f(X) \leq \lambda f(Y) + (1-\lambda)f(Z)$ for every $X \in L_\lambda(Y,Z)$ and every λ with $0 \leq \lambda \leq 1$. The followings have been proved by Dearing et al in [5]:

* (N,d) is compact,since each imbedded edge of N is compact
* $d(x,a)$ is continuous in x for any fixed $a \in N$
* for any two points $x,y \in N$, and with $0 \leq r \leq 1$,there exists a point z in any $P(x,y)$ for which $d(x,z)=r$ where $P(x,y)$ denotes any shortest path joining x and y
* for tree networks, connectivity implies convexity; that is to say ,any connected subset S of T is convex, but this is not true for networks having cycles. The reverse implication (convexity implies connectivity) is true for both tree and cyclic networks.

2.2. Multifacility Network Location Problems with Distance Constraints

Suppose m new facilities are to be located at points $x_1,...,x_m$ in an imbedded network N with respect to existing facilities at known vertex locations $v_1,...,v_n$.

Transport costs incurred are nondecreasing linear functions of the distances between facilities with v_{jk} and w_{ij} being the nonnegative constants of proportionality for $d(x_j,x_k)$ and $d(x_i,v_j)$, respectively. We refer to the constants v_{jk} and w_{ij} as weights.

Let $I_B \subseteq \{ (j,k): 1 \le j < k \le m \}$ and $I_C \subseteq \{ (i,j): 1 \le i \le m, 1 \le j \le n \}$ specify pairs of facility indices for which we have an upper bound on the distance. Given that $(j,k) \in I_B$ and $(i,j) \in I_C$, new facilities j and k can be a distance of at most b_{jk} apart while new facility i and existing facility j can be a distance of at most c_{ij} apart.

Consider the following two network location problems:

(P1) Multifacility Minisum Network Location Problem with Distance Constraints

$$\text{MIN } g(X) \equiv \sum_{1 \le j < k \le m} v_{jk} d(x_j,x_k) + \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} d(x_i,v_j)$$

s.t.

$$d(x_j,x_k) \le b_{jk};\ (j,k) \in I_B \qquad \text{(DC1)}$$

$$d(x_i,v_j) \le c_{ij};\ (i,j) \in I_C \qquad \text{(DC2)}$$

$$x_i \in N \qquad ;\ i=1,\ldots,m$$

(P2) Multifacility Minimax Network Location Problem with Distance Constraints

$$\text{MIN } f(X) \equiv \max \{ \max \{ v_{jk} d(x_j,x_k) + \beta_{jk}: 1 \le j < k \le m \},$$
$$\max \{ w_{ij} d(x_i,v_j) + \gamma_{ij}: 1 \le i \le m, 1 \le j \le n \} \}$$

s.t.

b_{jk}. Any two nodes N_i and E_j for which $(i,j) \in I_C$ are joined by an undirected arc of length c_{ij}. The following example demonstrates the construction of NBC.

EXAMPLE 2.2. Assume we have 3 existing facilities on a network, locations of which are the vertices of the network, v_1, v_2, v_3. Also assume that we are going to locate 2 new facilities on the network, whose locations are denoted by x_1 and x_2, subject to the following constraints: $d(x_1,x_2)\le 2$; $d(x_1,v_1)\le 3$; $d(x_1,v_3)\le 4$; $d(x_2,v_2)\le 1$; $d(x_2,v_3)\le 2$. The corresponding NBC is given in Fig. 2.3.

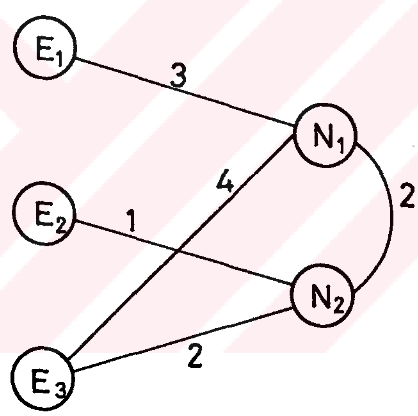


Fig. 2.3

Network BC is one of the main tools we use in our research, especially in the range analysis of distance constraints (Chapter 4), and in sensitivity analysis of minimax location problems (Chapter 5).

Letting f_p and f_q be nodes of NBC such that $(p,q) \in I_B \cup I_C$, the length of $arc(f_p,f_q)$ is denoted by $l(f_p,f_q)$ (if $f_p=N_i$ and

$f_q=N_j(E_j)$, then $l(f_p,f_q)=b_{ij}(c_{ij})$). A direct path in NBC is a path whose starting and ending nodes are E-nodes and whose intermediate nodes are N-nodes [7,18]. $P(E_s,E_t)$ denotes a path between E_s and E_t in NBC and $LP(E_s,E_t)$ is the length of $P(E_s,E_t)$. $L(E_s,E_t)$ denotes the length of a shortest path in NBC between E_s and E_t. We will denote the length of a shortest path between N_i and E_j in NBC by $L(N_i,E_j)$. $P(E_s,E_t)$ is called a tight path if $LP(E_s,E_t)=d(v_s,v_t)$, in which case $L(E_s,E_t)=LP(E_s,E_t)$ [17].

PROPERTY 2.1. At least one shortest path in NBC between E_p and E_q is a direct path between E_p and E_q.

The above property is proven in [18]. It should be noted that there may be other shortest paths in NBC that are not direct paths. The following theorem is proven in [9].

THEOREM 2.1. If the distance constraints (DC) are consistent, then

$$d(v_s,v_t) \leq L(E_s,E_t); \quad 1\leq s<t\leq n$$

The inequalities in Theorem 2.1. are called the Separation Conditions which are the necessary conditions for the consistency of (DC). They become sufficient also when the network (on which the new facilities are to be located) is a tree(see the following theorem).

THEOREM 2.2. For tree networks, the distance constraints are consistent if and only if the Separation Conditions hold.

The proof of Theorem 2.2. is given in [9]. The necessity proof uses the triangle inequality, and the sufficiency proof uses the so called "Sequential Location Procedure (SLP)".

The Sequential Location Procedure developed in [9] locates new facilities, one at a time, in a tree network T. It determines whether a feasible solution to distance constraints exists, constructs a feasible solution if one exists, and is the fundamental procedure used to prove that satisfaction of Separation Conditions is sufficient for the consistency of (DC) on a tree. SLP is an $O(m(n+m))$ algorithm.

Consider the unconstrained case of (P2) given in section 2.2., which is referred as the m-facility minimax problem with mutual communication in [20,21]: Given the nonempty sets I_B and I_C specifying pairs of facilities for which the distances are of interest, the m-facility minimax problem with mutual communication is to find a location vector $X^* \in N^m$ such that $f(X^*) = \min \{ f(X): X \in N^m \}$, where $f(X) = \max \{ \max \{v_{jk}d(x_j,x_k): (j,k) \in I_B\}, \max \{w_{ij}d(x_i,v_j): (i,j) \in I_B\} \}$. On a general network the problem is shown to be NP-hard by Kolen [12]. In the the case of a tree network the problem is solved by Francis et al [9] by using the following equivalent formulation.

(PMM) MIN z

s.t.

$$d(x_j,x_k) \le z/v_{jk} \;;\; (j,k) \in I_B \qquad (1)$$

$$d(x_i,v_j) \le z/w_{ij} \;;\; (i,j) \in I_C \qquad (2)$$

Francis et al [9] give a solution procedure which computes z^* first and constructs an optimal location vector X^* by applying SLP of [9] to the constraints (1) and (2) above, with $z = z^*$. The procedure to find z^* is as follows: Construct the corresponding network BC for the constraints (1) and (2) with $z = 1$, which we denote by NBC(1) (NBC(z) is the corresponding network BC for a given value of z). Let $L(E_s,E_t)$ denote the length of a shortest path between E_s and E_t in NBC(1). The minimum z for which (1) and (2) are consistent is that z^* for which the Separation Conditions hold. The Separation Conditions for a given z is $d(v_s,v_t) \le zL(E_s,E_t)$, $1 \le s < t \le n$. It follows that $z^* = \max \{d(v_s,v_t)/L(E_s,E_t):\ 1 \le s<t \le n\}$. The distances $d(v_s,v_t)$ can be computed in $O(n^2)$ operations for a tree network followed by $O(n^3)$ operations to compute $\{\ L(E_s,E_t):\ 1 \le s<t \le n\ \}$ [6].

As stated in [21], the procedure given for computing z^* demonstrates a rudimentary duality, since the minimum objective function value is equal to the maximum of a collection of terms. Also it is reasonable to assume NBC(1) is connected, since otherwise the problem of computing z^* decomposes into a collection of smaller, independent problems (see Connectivity Assumptions 1 and 2 made by Tansel at al in [18]). An important property established by Tansel et al in [17], which explores the relationship between z^* and tight paths (in NBC), is stated below.

PROPERTY 2.2. Let (X,z) be a feasible solution to (PMM)

(a). (X,z) is an optimum feasible solution to (PMM) if and only if at least one path in NBC(z) is tight, that is, for some $P(E_j,E_k)$, $d(v_j,v_k) = zLP(E_j,E_k)$.

(b). For any tigth path, the facilities whose nodes lie on the path are uniquely located, and their locations have the same ordering and spacing in T as their nodes have in the corresponding path in NBC.

It should be noted that,in this research, the emphasis will be on the sensitivity analysis of (PMM), and of distance constraints. In this context, we will consider the following case. Given an optimal location vector X^* to (PMM) with the optimal objective value z^*, if we perturb any one of the weights (w_{pq}, $(p,q) \in I_C$ or v_{pq}, $(p,q) \in I_B$) by $\varepsilon>0$ ($w_{pq} \rightarrow w_{pq}+\varepsilon$ or $v_{pq} \rightarrow v_{pq}+\varepsilon$), what will be the new optimal objective value? Trajectory analysis of optimal objective value will be given in Chapter 5. Since the procedure to solve (PMM) first computes the optimal objective value and then finds an optimal location vector using SLP, whenever we know the optimal objective value for a given ε, we can easily construct an optimal location vector for that ε by employing SLP.

For location problems some parametric analyses have been given recently in the literature. These are related with the trajectories of optimal facility locations. As an example, Brandeau and Chiu in [1,3] examine the optimal location of a single facility on a tree network with the objective to minimize the sum of weighted distances from each node measured by an L_p-norm-based [16] cost function. The possible trajectory paths of the optimal location are characterized in [1] when the L_p-norm parameter p varies from one to infinity, where the cost function is $\min \sum [w_i d(x,v_i)^p]^{1/p}$, $x \in N$. The weights w_i can be interpreted as the relative customer demands at nodes v_i, $i=1,\ldots,n$. Brandeau and Chiu also analyse the trajectory results for the optimal facility location as a function of customer call rate in Stochastic Queue Median Location Problem in a planar region with a rectilinear travel metric [2]. They summarize the results for parametric analysis of optimal facility location and report on a number of solved and unsolved problems in [4]. Erkut and Tansel consider the optimal location of a single facility with respect to a set of demand points on a tree, with both linear and nonlinear demand functions in [8]. They make use of the analytical properties of tree networks and sensitivity analysis, and devise efficient algorithms to construct the optimal trajectory of the parametric location problem. Although the location problems dealt with by these authors are considerably different from ours, their parametric approach to location problems (on networks and/or on a planar region) gave us the idea that the trajectory of optimal objective value as a function of the perturbation amount ε can be analyzed in the m-facility minimax location problem with mutual communication.

Recently, Erkut, Francis, and Tamir [7] have developed two algorithms to solve (P2) (see section 2.2.) on tree networks. Both of the solution procedures they develop use the Separation Conditions. Note that (P2) can equivalently be written in the following compact form:

(P2)′ Distance Constrained Multi-Center Problem

MIN z

s.t.

$$d(x_j, x_k) \le b_{jk} \quad ; \quad (j.k) \in I_B$$

$$d(x_i, v_j) \le c_{ij} \quad ; \quad (i,j) \in I_C$$

$$d(x_j, x_k) \le z/v_{jk} \quad ; \quad v_{jk} > 0, \ 1 \le j < k \le n$$

$$d(x_i, v_j) \le z/w_{ij} \quad ; \quad w_{ij} > 0, \ 1 \le i \le m, \ 1 \le j \le n$$

The first algorithm of [7] to solve (P2)′ is based on binary search, and uses representation of the problem data as rational numbers. The complexity of this algorithm depends polynomially on the problem data. The second algorithm is strongly polynomial (polynomial in m and n), and employs the general parametric approach suggested by Megiddo [13, 14, 15].

3. IMPRECISION IN DISTANCE CONSTRAINTS

3.1. Single Facility Case

Let T be a tree network and $V=\{v_1,..,v_n\}$ be the set of existing facility locations in T (the set of vertices in T). Let x denote the location of a new facility to be located on T and consider the following distance constraints of type (DC2).

$$(dc) : \quad d(x,v_i) \le c_i \; : \; i=1,..,n$$

where all c_i are nonnegative.

The composite neiborhood, defined as in [9], is the set of all points $x \in T$ that solve (dc), and is given by

$$N(a,r) = \bigcap_{i=1}^{n} N(v_i,c_i).$$

Here, $N(v_i,c_i)$ denotes all points $x \in T$ within a distance at most c_i of point v_i, i.e. $N(v_i,c_i) = \{x \in T : d(x,v_i) \le c_i\}$. We refer to $N(v_i,c_i)$ as a neighborhood with center v_i, and radius c_i, and to $N(a,r)$, the intersection of all $N(v_i,c_i)$, $i=1,..,n$, as the composite neighborhood.

Let us assume now that the distance constraints (dc) are imprecise in the sense that we do not know exactly what each c_i is, but instead we have lower and upper bounds on their possible realizations. That is to say, we have a range of possible values for each c_i; $c_i \in [lc_i,uc_i]$, $i=1,...,n$.

Let $d(x) \in \mathbb{R}^n$ be the vector whose i-th coordinate is $d(x,v_i)$ and let $c \in \mathbb{R}^n$ be the vector whose i-th coordinate is c_i. The vectors lc and uc are similarly defined. The constraints (dc) can now be written as

$$d(x) \le c \quad ; \quad c \in [lc,uc].$$

Let $C \equiv \{c \in \mathbb{R}^n: lc \le c \le uc\}$ denote the set of all possible realizations of c. Then C is an n-cell (a hyperrectangle) in $\mathbb{R}^n$.

We define a point $x \in T$ to be weakly feasible if x satisfies $d(x) \le c$ for at least one possible realization of $c \in C$. We say a point $x \in T$ is permanently feasible if x satisfies $d(x) \le c$ for all c in C. We denote the set of all weakly feasible points to (dc) by W and the set of all permanently solutions to (dc) by P. The following two theorems characterize P and W as the composite neighborhoods defined by lc and uc, respectively.

THEOREM 3.1. $P = N(\underline{a},\underline{r})$, where $N(\underline{a},\underline{r}) \equiv \bigcap_{i=1}^{n} N(v_i,lc_i)$.

Proof: To show $P \subseteq N(\underline{a},\underline{r})$, let $x' \in P$ (if $P=\emptyset$, true). Then, $d(x') \le c$ for all c in C. In particular, $d(x') \le lc$. Hence, $x' \in \bigcap_{i=1}^{n} N(v_i,lc_i)$. To show $N(\underline{a},\underline{r}) \subseteq P$, let $x' \in N(\underline{a},\underline{r})$ (if $N(\underline{a},\underline{r})=\emptyset$, true). Then $x' \in N(v_i,lc_i)$ $\forall$ i, and this implies $d(x') \le lc$. But for all c in C, we have $d(x') \le lc \le c$. Thus, x' is feasible for all $c \in C$, implying that $x' \in P$.□

THEOREM 3.2. $W = N(\bar{a},\bar{r})$, where $N(\bar{a},\bar{r}) \equiv \bigcap_{i=1}^{n} N(v_i,uc_i)$

Proof: To show $W \subseteq N(\bar{a},\bar{r})$, let $x' \in W$ (if $W=\emptyset$, true). Then, $d(x') \le c$

for at least one c in C. But, we have $d(x')\le c\le uc$ $\forall$ c $\in$ C. Hence $x' \in \bigcap_{i=1}^{n} N(v_i, uc_i)$. To show $N(\bar{a},\bar{r})\subseteq W$, let $x' \in N(\bar{a},\bar{r})$ (if $N(\underline{a},\underline{r})=\emptyset$, true). Then, $d(x')\le uc$, implying that x′ is feasible to (dc) for at least one c in C. Hence, $x' \in W$.□

THEOREM 3.3. $P \subseteq W$

Proof: If $P=\emptyset$, true. Otherwise, let $x' \in P$. Then, x′ solves $d(x)\le c$ for all c in C. In particular, $d(x')\le uc$ implying that $x' \in W$. □

It is quite possible that for a collection of vectors c in C, the distance constraints (dc) is inconsistent. Then the following question arises: How can we find the set of all vectors c in C, for which $d(x)\le c$ is consistent?

There are two extreme cases:

1. If $d(x)\le lc$ is consistent, then for all c in C, $d(x)\le c$ is consistent, and $P = N(\underline{a},\underline{r})$ is nonempty. This is actually the ideal case in which we can find the set of all permanently feasible points in T.

2. If $d(x)\le uc$ is inconsistent, then for all c in C, $d(x)\le c$ is inconsistent, i.e $W=\emptyset$. This is the worst case in which we can not find any feasible solution to the problem whatever c is (i.e for all c with $lc\le c\le uc$, $d(x)\le c$ is inconsistent).

Now, let us consider the case with $P=\emptyset$, but $W\ne\emptyset$. Since C is the set of all possible instances of c, we may refer to C as the range of imprecision associated with c (the larger C is, the more imprecision c has). It may be possible to refine our estimates on the range of c and confine the possible instances of c to a smaller set than C.

DEFIFITION 3.1. Given two sets C and C′, we call C′ a refinement of C if $C' \subseteq C$, and a proper refinement of C if $C' \subset C$.

DEFINITION 3.2. Given C with an empty set of permanently feasible points in T with respect to (dc), a proper refinement C′ of C will be called a good refinement if the the set of permanently feasible points associated with C′, $\{x \in T: d(x) \leq c \ \forall \ c \in C'\}$, is nonempty.

DEFINITION 3.3. A good refinement C′ will be called a critical refinement of C if C′ is a maximal subset of C having the property of being a good refinement.

The following theorem was proven by Francis et al in [9].

THEOREM 3.4 Given $N(a_1,r_1),\ldots,N(a_n,r_n)$. if we use SIP to obtain $\bigcap_{i=1}^{n} N(a_i,c_i) = N(a,r) \neq \emptyset$, then for any ε ,with $0 \leq \varepsilon \leq r$, we have

$$\bigcap_{i=1}^{n} N(a_i,c_i-\varepsilon) = N(a,r-\varepsilon) \neq \emptyset.$$

That is , if the radius of each neigborhood is reduced by ε, the net effect is to reduce the radius of the composite neighborhood by ε.

Given the set of permanently feasible points in T with respect to set C is empty, we will show that we can find a critical refinement set for C.

PROPERTY 3.1. For any vector $y = (y_1,...,y_n)$ selected in such away that $0 \le y_i \le \min\{\bar{r}, uc_i - lc_i\}$ $\forall$ i, the set $G(c) \equiv \{c \in \mathbb{R}^n: uc-y \le c \le uc\}$ is a good refinement.

Proof: By definition of $N(\bar{a},\bar{r})$, for such a choice of y, $uc-y \ge lc$ implying that uc-y is in the set C. Hence G(c) is a refinement of C. We should show that the set of permanently feasible points associated with G(c) is nonempty. By definition of y, $y_i \le \bar{r}$ $\forall$i. Hence, $\bigcap_{i=1}^{n} N(v_i, uc_i - y_i) \supseteq \bigcap_{i=1}^{n} N(v_i, uc_i - \bar{r})$. By Theorem 3.4., we have $\bigcap_{i=1}^{n} N(v_i, uc_i - \bar{r}) = N(\bar{a}, \bar{r} - \bar{r}) = \bar{a}$ implying that $\bigcap_{i=1}^{n} N(v_i, uc_i - y_i) \ne \emptyset$. Hence $d(x) \le uc-y$ is consistent. By Theorem 3.1. the permanent set associated with G(c) is nonempty implying that G(c) is a good refinement.□

THEOREM 3.5. If $y=(y_1,..,y_n)$ is such that $y_i = \min\{\bar{r}, uc_i - lc_i\}$, then the set $C' = \{c \in \mathbb{R}^n: uc-y \le c \le uc\}$ is a critical refinement of C.

Proof: By Property 3.1. C′ is a good refinement of C, and $d(x) \le uc-y$ is consistent. Hence, for all c in C, with $c \ge uc-y$, $d(x) \le c$ is consistent. To show maximality of C′ we should show that for all c in C with $c \le uc-y$, $c \ne uc-y$, $d(x) \le c$ is inconsistent. This is equivalent to showing that for all $y' \ge y$, $y' \ne y$, $d(x) \le uc-y'$ is inconsistent. For a given y′, with $y' \ge y$ and $y' \ne y$, for at least one entry, say k-th, $y'_k > y_k$. If, $y_k = uc_k - lc_k$, then $y'_k > y_k = uc_k - lc_k$, implying that $uc_k - lc_k < y_k'$. Hence uc-y′is not in the set C. If $y_k = \bar{r}$, then $y_k' > \bar{r} = \min\{\bar{r}, uc_k - lc_k\}$. Also if $\bar{r} = uc_k - lc_k$, then uc-y′ is not in the set C. Otherwise, i.e. $\bar{r} < uc_k - lc_k$, we have two cases to consider:

1. $\bar{r} < uc_k - lc_k < y_k'$, in which case $uc-y'$ is not in the set C.

2. $\bar{r} < y_k' \leq uc_k - lc_k$, in which case by theorem 3.4. and by the construction of the vector y, $\bigcap_{i=1}^{n} N(v_i, uc_i - y_i') = \emptyset$ implying that $d(x) \leq uc-y'$ is inconsistent.□

3.2. Multifacility Case

Now we will consider the multifacility case in which we locate m new facilities in T with respect to existing facilities at the vertices of T. Denoting the vector of new facility locations by $X=(x_1,...,x_m)$, consider the constraints (DC):

$$d(x_j,x_k) \leq b_{jk} \ ; \ (j,k) \in I_B \text{..........(DC1)}$$

$$d(x_i,v_j) \leq c_{ij}; \ (i,j) \in I_C \text{..........(DC2)}$$

Here, I_B and I_C are the sets of pairs of facility indices for which the upper bounds on distances are of interest. For notational convenience, let us assume an ordering of the members of I_B and I_C. Let $D_B(X)$ be the vector of all $d(x_j,x_k)$, and b be the corresponding vector of b_{jk} values defined by the assumed ordering on I_B. Similarly, $D_C(X)$ will denote the vector of all $d(x_i,v_j)$ and c will be the corresponding vector of c_{ij} values defined by the assumed ordering on I_C. Then the constraints (DC) can be written in the following form:

$$D(X) \leq e$$

where $D(X)=(D_B(X),D_C(X))$, and $e=(b,c)$.

Assume we do not know what e is exactly, but we have lower and upper bounds, $\underline{e}$ and $\bar{e}$ respectively on its possible realizations. Let $E = \{e: \underline{e} \leq e \leq \bar{e}\}$, the set of all possible realizations of e. Since, $e \in \mathbb{R}^k$, where k is the total number of constraints in (DC), E is a k-cell (or k-dimensional hyperrectangle) in $\mathbb{R}^k$.

DEFINITION 3.4. Given $e \in E$, the set of all location vectors $X \in N^m$ satisfying $D(X) \leq e$ is called the feasible set of location vectors for the distance constraints DC(e), and is denoted by S(e). Hence, $S(e) = \{X: X \text{ solves } D(X) \leq e,\ X \in T^m\}$.

PROPERTY 3.2. Given $e_1, e_2 \in E$, with $S(e_1) \neq \emptyset$, $S(e_2) \neq \emptyset$, if $e_1 \leq e_2$ then $S(e_1) \subseteq S(e_2)$.

Proof: Let $X \in S(e_1)$. Then, $D(X) \leq e_1 \leq e_2$, implying that $X \in S(e_2)$.□

Note that the set E defines the range of imprecision for distance constraints DC(e). A location vector $X \in T^m$ is said to be a weakly feasible location vector if X is feasible to DC(e) for at least one $e \in E$, and a permanently feasible location vector if X is feasible to DC(e) for all $e \in E$. We will refer to weakly feasible location vectors simply as weak solutions, and permanently feasible location vectors as permanent solutions. We now characterize weak and permanent solution sets.

THEOREM 3.6 $X \in T^m$ is a weak solution $\Leftrightarrow$ X solves $DC(\bar{e})$.

Proof:

($\Rightarrow$) If X is a weak solution, then $X \in S(e)$ for at least one e in E. Since $e \leq \bar{e}$, by Property 3.3. $S(e) \subseteq S(\bar{e})$. Hence, $X \in S(\bar{e})$ implying that X solves $DC(\bar{e})$.

($\Leftarrow$) If X solves $DC(\bar{e})$, then X is a weak solution by definition (it solves $DC(e)$ for at least one e in E, which is $\bar{e}$).□

THEOREM 3.7. $X \in T^m$ is a permanent solution $\Leftrightarrow$ X solves $DC(\underline{e})$.

Proof:

($\Rightarrow$) If X is a permanent solution, then X solves $DC(e)$ for all e in E, in particular X solves $DC(\underline{e})$.

($\Leftarrow$) If X solves $DC(\underline{e})$, then $X \in S(\underline{e})$. Since $\underline{e} \le e$ $\forall$ $e \in E$, using Property 3.3. $S(\underline{e}) \subseteq S(e)$ $\forall$ $e \in E$. So, $X \in S(e)$ $\forall$ $e \in E$ implying X solves $DC(e)$ $\forall$ $e \in E$. Hence, X is a permanent solution. □

Again assuming that the permanent solution set P associated with E is empty and that the weak solution set W is nonempty, we want to characterize the critical refinements of E.

Conjecture 3.1. If E' is a critical refinement of E, then E' is a hyperrectangle (a k-cell) contained in E.

If P is empty, by Theorem 3.6. there exists no X in T^m satisfying $D(X) \le \underline{e}$. Assuming Conjecture 3.1. is true, to find a critical refinement of E we need to look for a vector $y' \in \mathbb{R}^k$ such that $D(X) \le \underline{e}+y'$ is consistent while $D(X) \le \underline{e}+y$ is inconsistent for all y such that $y \le y'$, $y \ne y'$, and $\underline{e}+y \in E$. Then any y' satisfying the above conditions will be called a critical increment for $\underline{e}$. Let Y be the set of all critical increments for $\underline{e}$. For any $y' \in Y$, $E' \equiv \{e' : \underline{e}+y' \le e' \le \bar{e}\}$ will be a critical refinement of E.

If Conjecture 3.1. is true, we can show that there are infinitely many such y' satisfying the conditions given above, hence infinitely many critical refinements of E.

Conjecture 3.2. There are infinitely many critical refinements of E for the distance constraints with 2 or more new facilities.

In contrast to the single facility case, it is not easy to construct a critical refinement set in the multifacility case. Our future research interest is to develop these theoretical concepts and to find an efficient algorithm that constructs a critical refinement of E. We have started to analyze these with a procedure to find a critical refinement of E, which uses the network BC corresponding to the distance constraints defined by the lower bound of the set E. But in its current form, the procedure is nonpolynomial and hence is not included here. So, modifications are required as the theoretical concepts are further developed.

4.RANGE ANALYSIS FOR DISTANCE CONSTRAINTS

In this chapter, we will consider the constraints DC(e), which are defined (in Chapter 3) as

$$D(X) \le e$$

where $X \in T^m$. The following two cases are of interest.

(*) Assume we have a set of consistent distance constraints. How much can we decrease the right hand side of one of the constraints so that the system of constraints will remain consistent? Since increasing the RHS of one of the consistent distance constraints will not cause any inconsistency (see Lemma 3.1.) we concentrate on the perturbation of RHS by decreasing it ε amount, $\varepsilon>0$.

(*) Assume we have a feasible location vector $X \in T^m$ satisfying a given set of distance constraints $D(X) \le e$. Denoting the vector of reduction amounts in e by Δ what is the possible range for $\Delta \in R^k$ (k is the number of constraints in DC(e)), for which X is still feasible to $D(X) \le e-\Delta$?

Tansel,Francis ,and Lowe [17] established the following properties which will be our main reference points in this chapter. Let NBC have arc lengths defined by the RHS vector e.

PROPERTY 4.1 Let $P(E_p,E_q)$ be a tight path in NBC, then the nodes representing facilities in the path occur with the same ordering and spacing in the path as do the locations representing facilities in the unique path $L(v_p,v_q)$ joining v_p and v_q in T. Further, every facility represented by a node in $P(E_p,E_q)$ is uniquely located.

PROPERTY 4.2. Let DC(e) be consistent. Let (f_i,f_j) be any arc in NBC, of length e_{ij}, whose length is reduced by some positive amount ε. Let $DC_\varepsilon(NBC_\varepsilon)$ be the distance constraints (network BC) obtained from DC(e) (NBC) by replacing e_{ij} by $e_{ij}-\varepsilon$.

(a). Every path containing (f_i,f_j) is slack if and only if ε can be chosen (with $\varepsilon>0$) so that DC_ε is consistent.

(b). Whenever every path containing (f_i,f_j) is slack, ε can be chosen, with $\varepsilon>0$, so that DC_ε is consistent and at least one of the following is true:

(i). at least one path in NBC_ε containing (f_i,f_j) is tight;

(ii). the length of (f_i,f_j) in NBC_ε can be reduced to zero.

4.1. Perturbation in RHS

Assume we have a set of consistent distance constraints on a tree:

(DC)
$$d(x_j,x_k) \le b_{jk};\ (j,k) \in I_B$$
$$d(x_i,v_j) \le c_{ij};\ (i,j) \in I_C$$

We adress the following question: Let $\varepsilon>0$ be the amount of reduction in a given RHS. What is the range of values for ε for which the system of constraints is consistent?

Let NBC be the corresponding network BC having the arc lengths of b_{jk} and c_{ij} for all pairs of (N_j,N_k), $(j,k) \in I_B$ and (N_i,E_j), $(i,j) \in I_C$, respectively. Since the constraints (DC) are consistent, the separation conditions hold, i.e.

$$L\ (E_p,E_q) \geq d(v_p,v_q)\ ;\forall(p,q),\ 1\leq p<q\leq n$$

Let f_p and f_q be the nodes in NBC corresponding to the pair of facilities for which the upper bound on distance is to be decreased; let e_{pq} be the length of the arc (f_p,f_q) in NBC. Note that e_{pq} is the entry of e corresponding to the constraint whose RHS will be perturbed. $DC_\varepsilon(NBC_\varepsilon)$ will denote distance constraints (network BC) obtained from DC(NBC) by replacing e_{pq} by $e_{pq}-\varepsilon$.

The following definition gives the condition for a distance constraint to be binding (for all feasible solutions).

DEFINITION 4.1. Given a constraint $d(x_j,x_k) \leq b_{jk}$ (or $d(x_i,v_j) \leq c_{ij}$) in (DC), if the corresponding arc (N_j,N_k) (or (N_i,E_j)) in NBC is on at least one tight path, then we call this constraint a binding constraint.

We remark that a binding constraint holds as an equality in all feasible solutions to (DC) (as apposed to a given feasible solution). This follows from property 4.1..

DEFINITION 4.2. Among the paths between any two E-nodes in NBC that contain a specified arc, say (f_p,f_q), one with the smallest length will be called a restricted shortest path containing the arc (f_p,f_q).

A restricted shortest path is allowed to be nonsimple. If a shortest path between two E-nodes contains the arc under consideration, then a restricted shortest path containing (f_p,f_q) turns out to be a shortest path between these two E-nodes. We will denote a restricted shortest path between E_i and E_j in NBC containing the specified arc (f_p,f_q) by $P_R(E_i,E_j)$ and denote its length by $LP_R(E_i,E_j)$.

LEMMA 4.1. If at least one of the paths containing the arc (f_p,f_q) in NBC is tight, then decreasing RHS value of the corresponding constraint in (DC) by $\varepsilon>0$ will violate the Separation Conditions. Hence, DC_ε will be inconsistent.

Proof: Assume (f_p,f_q) is on the tight path $P(E_i,E_j)$ between E_i and E_j. Since $P(E_i,E_j)$ is tight for (DC), $LP(E_i,E_j)=d(v_i,v_j)$. If we subtract $\varepsilon>0$ from e_{pq}, the length of the path $P(E_i,E_j)$ will be decreased by ε in NBC_ε. So, $LP_\varepsilon(E_i,E_j) = LP(E_i,E_j)-\varepsilon = d(v_i,v_j)-\varepsilon < d(v_i,v_j)$. Hence, separation conditions are violated for the constraint set DC_ε □

LEMMA 4.2. If every path $P(E_i,E_j)$ containing the arc (f_p,f_q) is slack, then there exists $\varepsilon>0$ such that DC_ε is consistent, with $0<\varepsilon\leq\min\{\varepsilon',e_{pq}\}$ where $\varepsilon'\equiv \min\{ LP(E_i,E_j)-d(v_i,v_j):\ P(E_i,E_j)$ contains $(f_p,f_q)\ \}$.

Proof: Since every path containing (f_p,f_q) is slack,

$LP(E_i,E_j)-d(v_i,v_j) > 0$ for all $P(E_i,E_j)$ containing (f_p,f_q). If ε' is defined and the range for ε are as in the lemma, then we want to show that $LP_\varepsilon(E_i,E_j) \geq d(v_i,v_j)$, $1 \leq i < j \leq n$ (1) (i.e. DC_ε is consistent). Since $LP_\varepsilon(E_i,E_j)=LP(E_i,E_j)-\varepsilon$, for (1) to hold we should show that $\varepsilon \leq LP(E_i,E_j)-d(v_i,v_j)$, $1 \leq i < j \leq n$.(2)
Since $0 < \varepsilon \leq \min\{\varepsilon',e_{pq}\}$ and by definition of ε', (2) holds for the paths containing (f_p,f_q). So, $LP_\varepsilon(E_i,E_j) \geq d(v_i,v_j)$ $\forall$ $P(E_i,E_j)$ containing the arc (f_p,f_q)..(3). Any path that does not contain the arc (f_p,f_q) in NBC has its length unchanged in NBC_ε. Since (DC) is consistent, initially, such paths have lengths no smaller than the distance between their terminal nodes in the tree. Because their lengths are unaffected by ε, we have $LP(E_i,E_j) \geq d(v_i,v_j)$ $\forall$ $P(E_i,E_j)$ not containing the arc (f_p,f_q)..(4)
Hence, by (3) and (4), DC_ε is consistent if $0 \leq \varepsilon \leq \min\{\varepsilon',e_{pq}\}$ □

In view of lemmas 4.1. and 4.2. we conclude that

(i). If the constraint under consideration is binding (i.e. the corresponding node in NBC is on at least one tight path) then we can not decrease the RHS of the constraint since otherwise separation conditions will be violated and DC_ε will be inconsistent.

(ii). If the constraint under consideration is non-binding (i.e every path containing the corresponding arc in NBC is slack) then we can find $\varepsilon>0$ such that DC_ε is consistent. In this case, the range of ε for which the system remains consistent is the interval $[0, \min\{\varepsilon',e_{pq}\}]$, where ε' is the minimum slack among all paths containing the pertubed arc.

Non-binding constraints (for which decrease in RHS is possible) are related with new facilities that are not uniquely located. The decrease in RHS of a non-binding constraint can be made until the corresponding arc in NBC is on a tight path, i.e until one or more of the non-uniquely located facilities become uniquely located. In the following we present a procedure to find ε', for the case when the perturbed constraint is non-binding.

Case 1 : $c_{pq} \rightarrow c_{pq} - \varepsilon$

The constraint under consideration is $d(v_p,v_q) \leq c_{pq}$, and the corresponding arc in NBC is (N_p,E_q). To find ε', we need to compute the slacks associated with all paths containing (N_p,E_q). From property 2.1., we need only to consider direct paths that contain (N_p,E_q). All such direct paths will have one end point at some E_k, $k \neq q$. So, there are (n-1) possible pairs of E-nodes to be considered. Given an existing facility E_k $(k \neq q)$, there may be many direct paths between E_q and E_k that contain the arc (N_p,E_q). The definition of ε' (as in Lemma 4.2.) implies

$\varepsilon' = \min \{LP(E_q,E_k) - d(v_q,v_k) : P(E_q,E_k)$ contains the arc (N_p,E_q); $\forall k \neq q\}$. To find ε' we first compute ε_k, for $k=1,\ldots,n$; $k \neq q$, where ε_k is $\min\{LP(E_q,E_k) - d(v_q,v_k) : P(E_q,E_k)$ contains the arc $(N_p,E_q)\}$. Then, $\varepsilon' = \min \{\varepsilon_k : k=1,\ldots,n;\ k \neq q\}$.

In general, there is a nonpolynomial number of paths that contain the perturbed arc (N_p,E_q) and a brute force enumeration on these paths would supply the value of ε_k in exponential time. However, we can do much better by observing that $d(v_q,v_k)$ is a constant for a given pair of existing facilities (E_q,E_k) so that

$$\varepsilon_k = LP_R(E_q,E_k) - d(v_q,v_k)$$

where $LP_R(E_q,E_k)$ is the length of a restricted shortestpath between E_q and E_k in NBC containing the arc (N_p,E_q). Note that $LP_R(E_q,E_k)=c_{pq}+ L(N_p,E_k)$, giving $\varepsilon_k = c_{pq}+ L(N_p,E_k) - d(v_q,v_k)$.

Let the matrix A be an m by n matrix having the entries a_{ij}, where a_{ij} is the length of a shortest path between N_i and E_j (i.e. $a_{ij}= L(N_i,E_j)$) in NBC. Then, $\varepsilon_k= c_{pq}+ a_{pk}- d(v_q,v_k)$, and

$$\varepsilon'= \min \{\varepsilon_k: k=1,..,n;\ k\neq q\}$$

The distances $d(v_p,v_q)$ can be computed in $O(n^2)$ operations for a tree network. For a given new facility N_p, we can find all a_{pk}, $k=1,...,n$,by Dijkstra's algorithm in $O((m+n)^2)$ operations [6]. So, the matrix A can be constructed in $O(m(m+n)^2)$ operations. Once A and the distance matrix of T (representing the distances between vertices of the tree, T) are computed, they can be used for computing any ε_k, $k\neq q$. Hence, the worst case time bound for computing ε' for all possible perturbations is $O(\ m(m+n)^2)$.

So, if $0 \leq \varepsilon \leq \min\{\varepsilon',c_{pq}\}$, then DC_ε will be consistent, given ε' is found as described above. Note that $\varepsilon'>0$ and hence $\min\{\varepsilon',c_{pq}\} >0$, if all paths containing (N_p,E_q) are slack.

EXAMPLE 4.1. Given the tree network in Fig.4.1 with three vertices (existing facilities), we are going to locate three new facilities subject to the following constraints:

$$d(x_1,v_1) \le 1 \ldots\ldots (1)$$

$$d(x_1,v_3) \le 5 \ldots\ldots (2)$$

$$d(x_2,v_1) \le 4 \ldots\ldots (3)$$

$$d(x_2,v_3) \le 2 \ldots\ldots (4)$$

$$d(x_3,v_1) \le 5 \ldots\ldots (5)$$

$$d(x_3,v_2) \le 6 \ldots\ldots (6)$$

$$d(x_1,x_2) \le 3 \ldots\ldots (7)$$

$$d(x_1,x_3) \le 5 \ldots\ldots (8)$$

$$d(x_2,x_3) \le 1 \ldots\ldots (9)$$

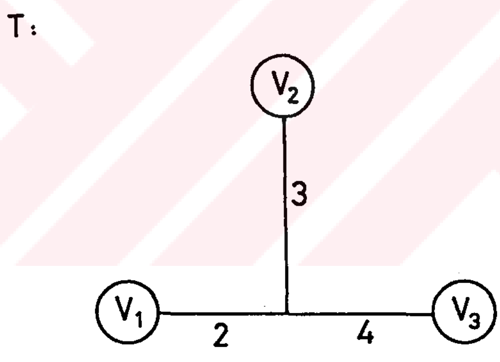


Fig. 4.1

Given the network with vertices v_1, v_2, v_3 (Fig. 4.1), $d(v_1,v_2)=5$, $d(v_1,v_3)=6$, $d(v_2,v_3)=7$. The corresponding NBC is given in Fig.4.2. The constraints (1), (2), (3), (4), and (7) are binding.

NBC:

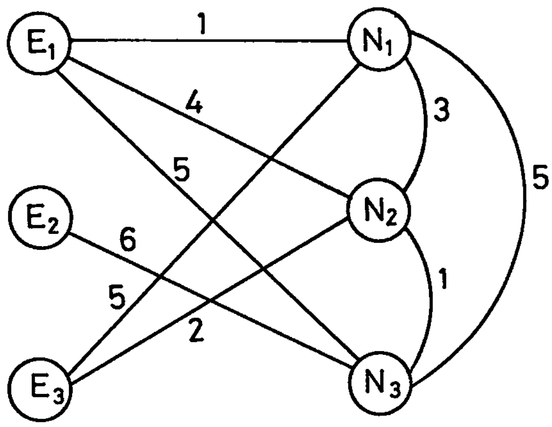


Fig. 4.2

Since $L(E_1,E_2) = 11 > d(v_1,v_2)$; $L(E_1,E_3) = 6 = d(v_1,v_3)$; $L(E_2,E_3) = 9 > d(v_2,v_3)$, the distance constraints are consistent. Given NBC in Fig. 4.2, the corresponding A matrix is given below:

$$A = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 1 & 10 & 5 \\ 4 & 7 & 2 \\ 5 & 6 & 3 \end{bmatrix} \end{array}$$

Consider the constraint (5), $d(x_3,v_1) \le 5$. What is the range of values for ε, for which DC_ε, obtained by replacing c_{31} ($=5$) by $c_{31}-\varepsilon$, is consistent? The arc under consideration in NBC is (N_3,E_1) and all paths containing this arc are slack. Note that $q=1$, then for $k=1$,and 2:

$$\varepsilon_2 = c_{31} + a_{32} - d(v_1,v_2) = 5+6-5 = 6$$

$$\varepsilon_3 = c_{31} + a_{33} - d(v_1,v_3) = 5+3-6 = 2$$

Then, $\varepsilon' = 2$, and $0 \le \varepsilon \le \min\{2,5\}$. So, for $0 \le \varepsilon \le 2$ DC_ε is consistent.

Consider the case with ε=2. Then, in the new set of constraints, RHS of constraint (5) is 3, instead of 5. $NBC_\varepsilon = NBC_2$ is given in Fig. 4.3.

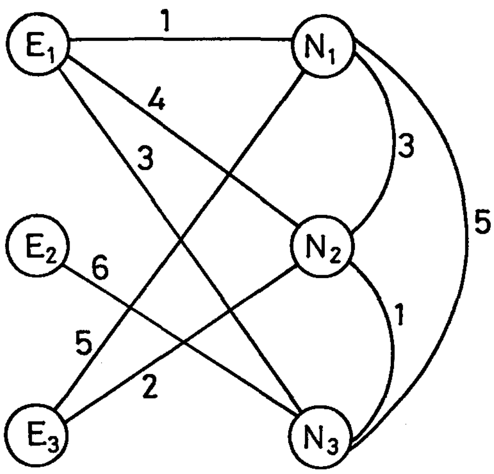


Fig. 4.3

Since, $L(E_1,E_2) = 9 > d(v_1,v_2)$; $L(E_1,E_3) = 6 = d(v_1,v_3)$; $L(E_2,E_3) = 9 > d(v_2,v_3)$, DC_2 is consistent. Note that, in NBC the new facilities N_1 and N_2 were uniquely located, while N_3 was not. In NBC_2 the new facility 3 is also uniquely located. Moreover, the binding constraints of DC_2 are (1), (2), (3), (4), (5), (6), (7), and (9). The only non-binding constraint is (8): $d(x_1,x_3) \leq 5$.

Case 2: $b_{pq} \rightarrow b_{pq} - \varepsilon$

The constraint to be considered is $d(x_p,x_q) \leq b_{pq}$ and the corresponding arc in NBC is (N_p,N_q). By a similar argument as in Case 1, we need to compute the slacks associated with all direct paths in NBC containing (N_p,N_q), connecting E_i and E_j, $1 \leq i < j \leq n$. So, we can compute $\varepsilon_{ij} = \min\{LP(E_i,E_j) - d(v_i,v_j) : P(E_i,E_j)$ contains $(N_p,N_q)\}$ for all $1 \leq i < j \leq n$. Then $\varepsilon' = \min\{\varepsilon_{ij} : 1 \leq i < j \leq n\}$. If we let

$LP_R(E_i,E_j)$ denote the length of a restricted shortest path between E_i and E_j in NBC, containing the arc (N_p,N_q), then

$$LP_R(E_i,E_j) = b_{pq} + \min\{ (L(N_p,E_i)+L(N_q,E_j)),\ (L(N_p,E_j)+L(N_q,E_i)) \}$$
$$= b_{pq} + \min\{ (a_{pi}+a_{qj}),\ (a_{pj}+a_{qi}) \} \text{ (see Fig 4.4)}$$

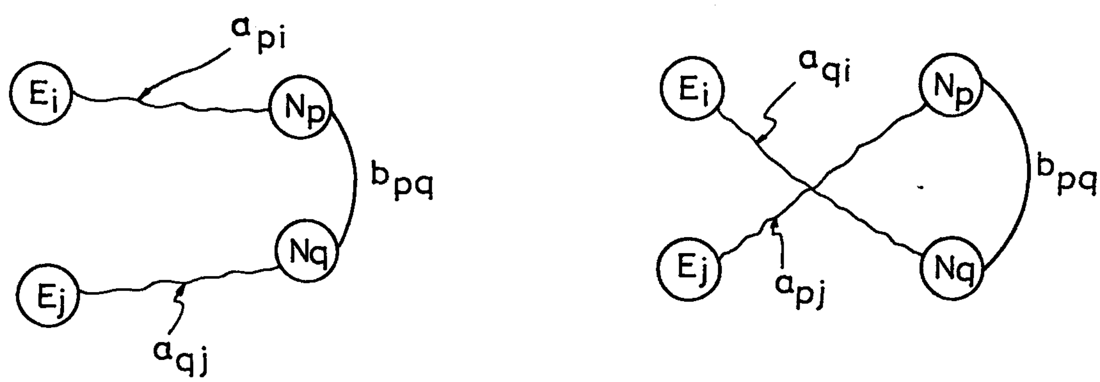


Fig. 4.4

Then, $\varepsilon_{ij} = b_{pq} + \min\{ (a_{pi}+a_{qj}),\ (a_{pj}+a_{qi}) \} - d(v_i,v_j)$, and

$$\varepsilon' = \min\{ \varepsilon_{ij} : 1 \le i < j \le n \}.$$

So, for $0 \le \varepsilon \le \min\{\varepsilon', b_{pq}\}$, DC_ε is consistent. Again, ε' and $\min\{\varepsilon', b_{pq}\}$ is positive if no path containing (N_p,N_q) is tight (i.e. the corresponding constraint is non-binding).

EXAMPLE 4.2. In example 4.1. consider the constraint $d(x_2,x_3) \le 1$ (constraint (9)). What is the range of values for ε for which DC_ε, obtained by replacing b_{23} (=1) by $b_{23} - \varepsilon$, is consistent?
The arc under consideration is (N_2,N_3) and all paths containing this arc in NBC are slack. Note that p=2, q=3.

$$\varepsilon_{12} = b_{23} + \min\{(a_{21}+a_{32}),\ (a_{22}+a_{31})\} - d(v_1,v_2)$$
$$= 1 + \min\{(4+6),\ (7+5)\} - 5$$
$$= 6 \quad \text{(see Fig. 4.5)}$$

Similarly,

ε_{13}= 1+ min{(4+3), (2+5)} - 6 = 2

ε_{23}= 1+ min{(7+3), (2+6)} - 7 = 2

Hence, ε'= min{6, 2, 2} = 2; $0\le\varepsilon\le$min{2, 1}. So, for $0\le\varepsilon\le1$, DC_ε is consistent. Note that in the case when ε=1, the length of the arc can be reduced to zero; $d(x_2,x_3)=0$, i.e. x_2 and x_3 should be located in the same place, $x_2=x_3$. We should treat those two facilities as if a single facility in NBC_ε. If we consider the constraints (1),...,(8) we see that (5) and (6) become redundant in NBC_ε when ε=1. NBC_1 is given in Fig. 4.6.

Since $L(E_1,E_2) = 10 > d(v_1,v_2)$; $L(E_1,E_3) = 6 = d(v_1,v_3)$; $L(E_2,E_3) = 8 > d(v_2,v_3)$, DC_1 is consistent as claimed.

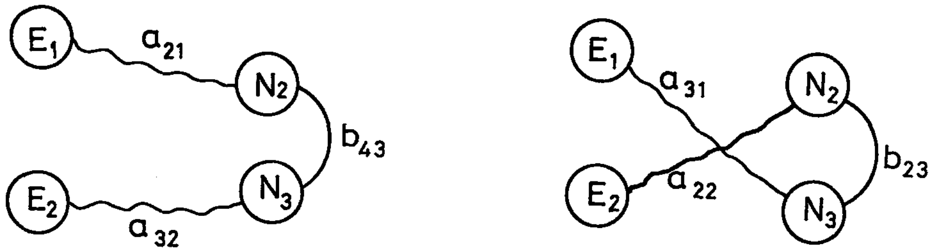


Fig. 4.5

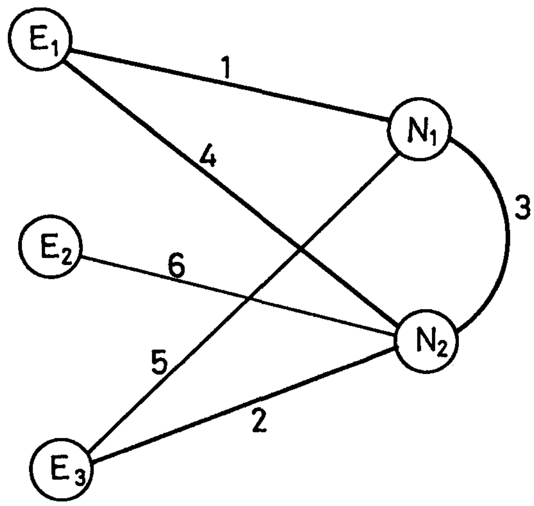


Fig. 4.6

4.2. Perturbation in RHS of Distance Constraints With Respect To a Feasible Location Vector

In this section, we are interested in the second question adressed at the beginning of the chapter. Given a location vector $X \in T^m$ satisfying a given set of distance constraints DC(e), for what range of values of Δ will X remain feasible to $D(X) \le e-\Delta$?

LEMMA 4.3. Assume we have a feasible location vector $Y \in T^m$, satisfying a given set of constraints, $D(X) \le e$. Y will remain feasible as long as the reduction amounts in e is a vector Δ for which $0 \le \Delta \le e-D(Y)$.

Proof: Let $0 \le \Delta \le e-D(Y)$. Then $D(Y) \le e-\Delta$ implying that Y will remain feasible when e is reduced by Δ. Note also that if we choose

Δ=e-D(Y), then D(Y)=e-Δ so that any further reduction in any component of e will cause Y to become infeasible.□

EXAMPLE 4.3. In example 4.1., consider the location vector, $X=(x_1,x_2,x_3)$ for which x_1, x_2 and x_3 are as in Fig. 4.7.

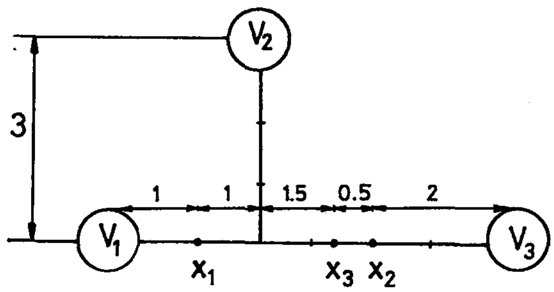


Fig. 4.7

Then, D(X) = (1, 5, 4, 2, 3.5, 4.5, 3, 2.5, 0.5) ≤ (1, 5, 4, 2, 5, 6, 3, 5, 1). So, for $\underline{0} \le \Delta \le$ (0, 0, 0, 0, 1.5, 1.5, 0, 2.5, 0.5), X is a feasible location vector to D(X)≤e-Δ.

5. SENSITIVITY ANALYSIS FOR MINIMAX LOCATION ON A TREE

5.1. M-Facility Minimax Location Problem With Mutual Communication on a Tree

Consider (PMM) on a tree network:

(PMM) MIN z

s.t.

$$d(x_j, x_k) \le z/v_{jk} \; ; \; (j,k) \in I_B$$

$$d(x_i, v_j) \le z/w_{ij} \; ; \; (i,j) \in I_C$$

with $X = (x_1, \ldots, x_m) \in T^m$.

Using the procedure described in section 2.3., the minimum objective function value for (PMM) turns out to be $z^* = \max\{d(v_p, v_q)/L(E_p, E_q) : 1 \le p < q \le n\}$. Property 2.2. (which was established by Tansel, Francis, and Lowe in [17]) clarifies the relationship between the optimal objective value z^* and tight paths in the corresponding network BC.

Here the weights v_{jk} and w_{ij} usually represent the relative density of interactions between pairs of facilities. They are the estimated parameters to the problem. Estimates on such exogeneous parameters usually have a range of allowable values, due to the inaxactness of raw data. For example, assume we are to locate a hospital and/or a fire station with respect to the districts of a metropolitan area. The proportionality constants between these districts and the hospital can be determined using the relative populations and health statistics of the districts. However, we may not have reliable forecasts on the expected population growth of different parts of the city in the future. Also, the required statistics may not be available. Relative frequencies of fires in these districts, on the other hand, may be a factor in determining the weight associated with each district and the fire station. Lack of such statistics will make it difficult to predict the future demand behaviour of each district. Then, it is apparent that estimates of these uncontrollable parameters to the problem are not always reliable.

With the above motivation, given a solution to (PMM) with respect to estimated values of the weights, it is quite possible that the actual realizations of some or all of the weights will be different from what we expect, and the optimal objective value and the corresponding optimal location vector might no longer be optimal. We wish to carry out a sensitivity analysis for weights and the perturbation will be made one at a time. The results of such a study may also be helpful in multiparametric analysis in which case all of the weights are subject to change simultaneously.

Noting that for a given realization of weights the existence of an optimal solution to (PMM) is guaranteed, we are going to perturb one of the weights. Our aim is to investigate the effect of such a change on the optimal objective value z^*, and on the optimal location vector X^*. We have two cases to consider: (1). $w_{pq} \to w_{pq} + \varepsilon$; (2). $v_{pq} \to v_{pq} + \varepsilon$.

Referring to section 2.3., the network BC associated with the constraints of (PMM), with z=1, will be denoted by NBC(1). The arc lengths of NBC(1) are reciprocal weights. $NBC_{\varepsilon}(1)$ will denote the network BC corresponding to the constraints of (PMM) obtained from NBC(1) by replacing w_{pq}(or v_{pq}) by $w_{pq} + \varepsilon$ (or $v_{pq} + \varepsilon$).

Let P be a direct path connecting some two E-nodes, say E_p and E_q, in NBC(1). Recall that a direct path in network BC is a simple path connecting two E-nodes which contains only these two E-nodes. There are finitely many direct paths joining two specified E-nodes, E_p and E_q. Let LP(ε) be the length of P as a function of ε and define $z_P(\varepsilon)$ as follows:

$$z_P(\varepsilon) = d(v_p, v_q)/LP(\varepsilon).$$

Here, $d(v_p, v_q)$ is a constant that depends only on p and q but not on the path connecting E_p and E_q. For notational ease, let us write d for $d(v_p, v_q)$ and $z(\varepsilon)$ for $z_P(\varepsilon)$ with the understanding that the pair E_p, E_q is fixed as terminal nodes of all such paths P under consideration. Hence,

$$z(\varepsilon) = d/LP(\varepsilon).$$

The arc in NBC(1) corresponding to the perturbed weight will be called the perturbed arc. In section 5.1.1. the trajectories of LP(ε) and $z(\varepsilon)$ will be given. Section 5.1.2 investigates the possible trajectories of z^* as a function of ε, $z^*(\varepsilon)$. The results will be illustrated on an example problem given in appendix.

5.1.1. Trajectories of LP(ε) and z(ε)

Case 1:

P does not contain the perturbed arc. Then, $LP(\varepsilon)=LP(0)$ and $z(\varepsilon)=z(0)=d/LP(0)$, where LP(0) is the length of P in NBC(1). Since P does not contain the perturbed arc the length of this path will not change whatever ε is (see Fig. 5.1).

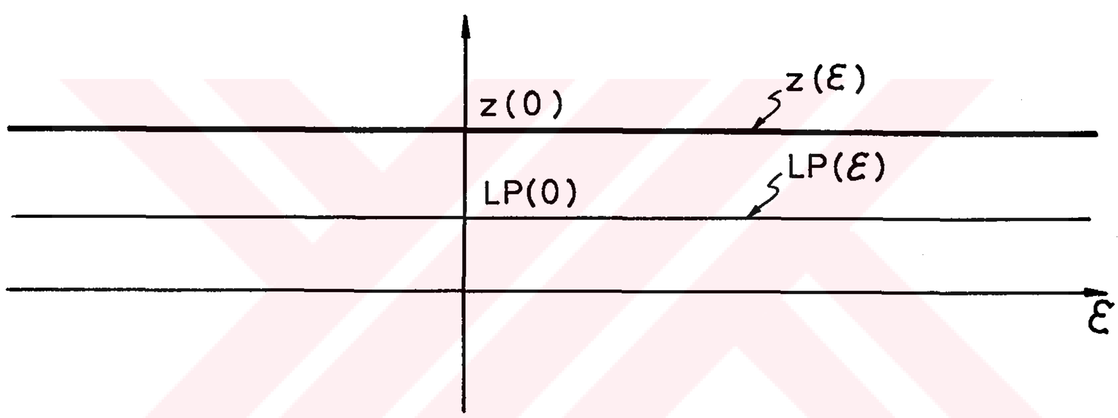


Fig. 5.1

Case 2:

In the second case, P contains the perturbed arc. Let u denote the perturbed weight,then

$$LP(\varepsilon) = (1/(u+\varepsilon))+\alpha$$

where α is a constant (depending on the path P) and, by definition, it denotes the sum of the arc lengths (reciprocal weights) in P other than the perturbed arc. Note that the length of the perturbed arc is 1/u before the perturbation and $1/(u+\varepsilon)$ after the perturbation.

$LP(\varepsilon)$ is a hyperbolic function with a vertical asymptote (pole) at $\varepsilon=-u$ and a horizontal asymptote at $LP(\varepsilon)=\alpha$ (see Fig 5.2).

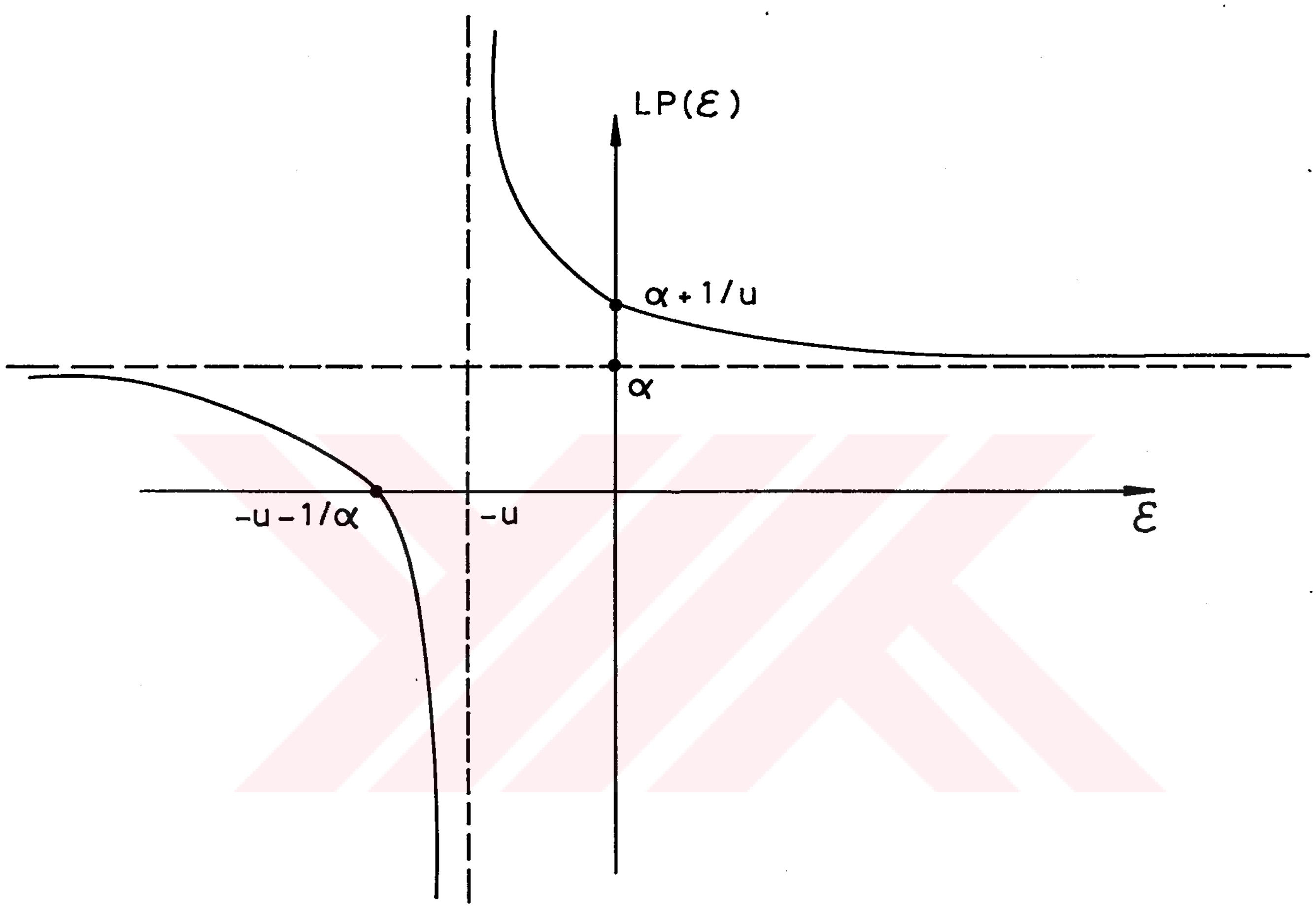


Fig. 5.2

Since, within the context of minimax problems, we deal with positive weights, the left hand portion of the curve is not of interest for our purposes.

The right hand portion of the curve is monotone decreasing in ε. As ε approaches $-u$ from the right, the path length approaches infinity and the path will definitely be a slack path for ε

sufficiently close to $-u$ (see the trajectory of $z(\varepsilon)$ given in Fig. 5.3).

As defined previously $z(\varepsilon)=d/LP(\varepsilon)$, i.e. $z(\varepsilon)=d/[(1/u+\varepsilon)+\alpha]$. Hence,

$$z(\varepsilon) = d(u+\varepsilon)/[1+\alpha(u+\varepsilon)]$$

$z(\varepsilon)$ is also a hyperbola with the vertical asymptote at $\varepsilon=-u-(1/\alpha)$ and the horizontal asymptote at $z(\varepsilon)=d/\alpha$ (see Fig. 5.3).

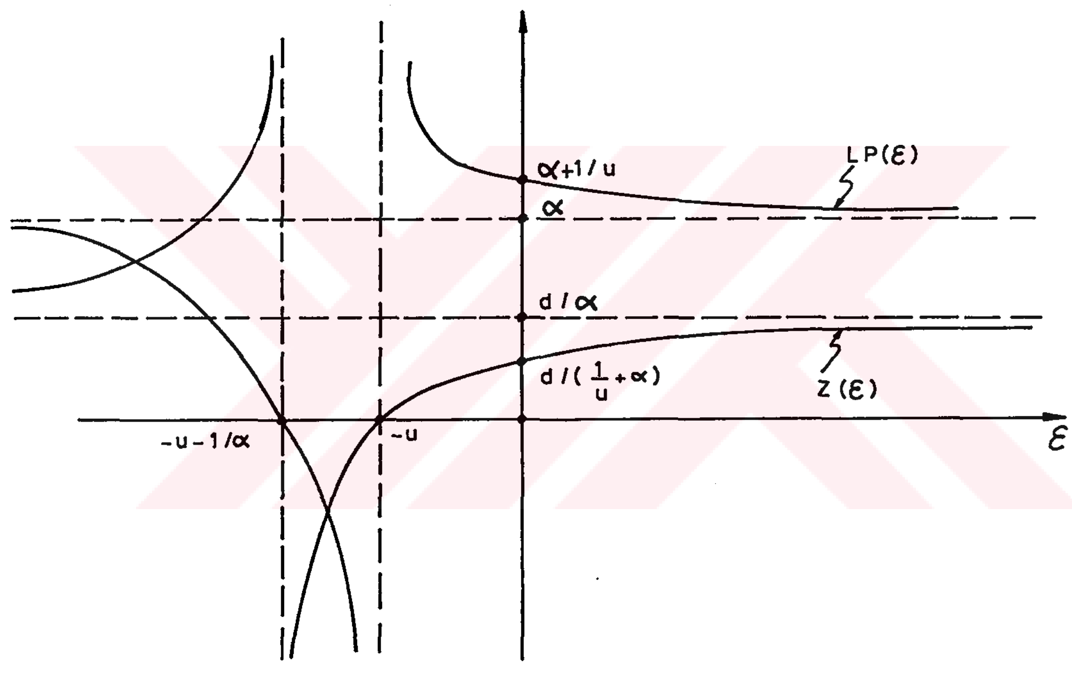


Fig. 5.3

Let S be the set of all direct paths containing the perturbed arc. Let these paths be numbered 1 to r as $P_1,\ldots,P_r$ in such a way that $\alpha_1\le\alpha_2\le\ldots\le\alpha_r$ where α_i is the α value associated with the path P_i containing the perturbed arc (that is, α_i is the sum of the arc lengths of P_i other than the perturbed arc).

PROPERTY 5.1. Given the paths $P_1,\ldots,P_r$ containing the perturbed arc numbered in such a way that $\alpha_1\le\alpha_2\le\ldots\le\alpha_r$, the curves $LP_1(\varepsilon),\ldots,LP_r(\varepsilon)$ will never intersect unless their α values are equal.

Proof: The proof of the above property follows from the fact that ε affects all of those r paths in exactly the same way. That is, for $1\le i<j\le r$,

$$LP_i(\varepsilon)-LP_j(\varepsilon) = \{(1/(u+\varepsilon))+\alpha_j\}-\{(1/(u+\varepsilon))+\alpha_i\} = \alpha_j-\alpha_i$$

Hence, if $\alpha_i<\alpha_j$, the curves corresponding to $LP_j(\varepsilon)$ and $LP_i(\varepsilon)$ are always $\alpha_j-\alpha_i>0$ apart. If $\alpha_j=\alpha_i$, the two curves concide.□

As a consequence, whichever path is shortest at $\varepsilon=0$, it will remain shortest at $\varepsilon\neq0$, and the same applies to second shortest, third shortest etc. (see Fig.5.4).

Consider any two paths P_1, P_2 containing the perturbed arc, with $z_1(\infty)=d_1/\alpha_1 \le z_2(\infty)=d_2/\alpha_2$. Here, d_i is the d value corresponding to P_i. The question adressed is the follwing: Given $\varepsilon>0$ what is the condition for the curves, $z_1(\varepsilon)$ and $z_2(\varepsilon)$ to intersect at an ε', with $\varepsilon'>0$?

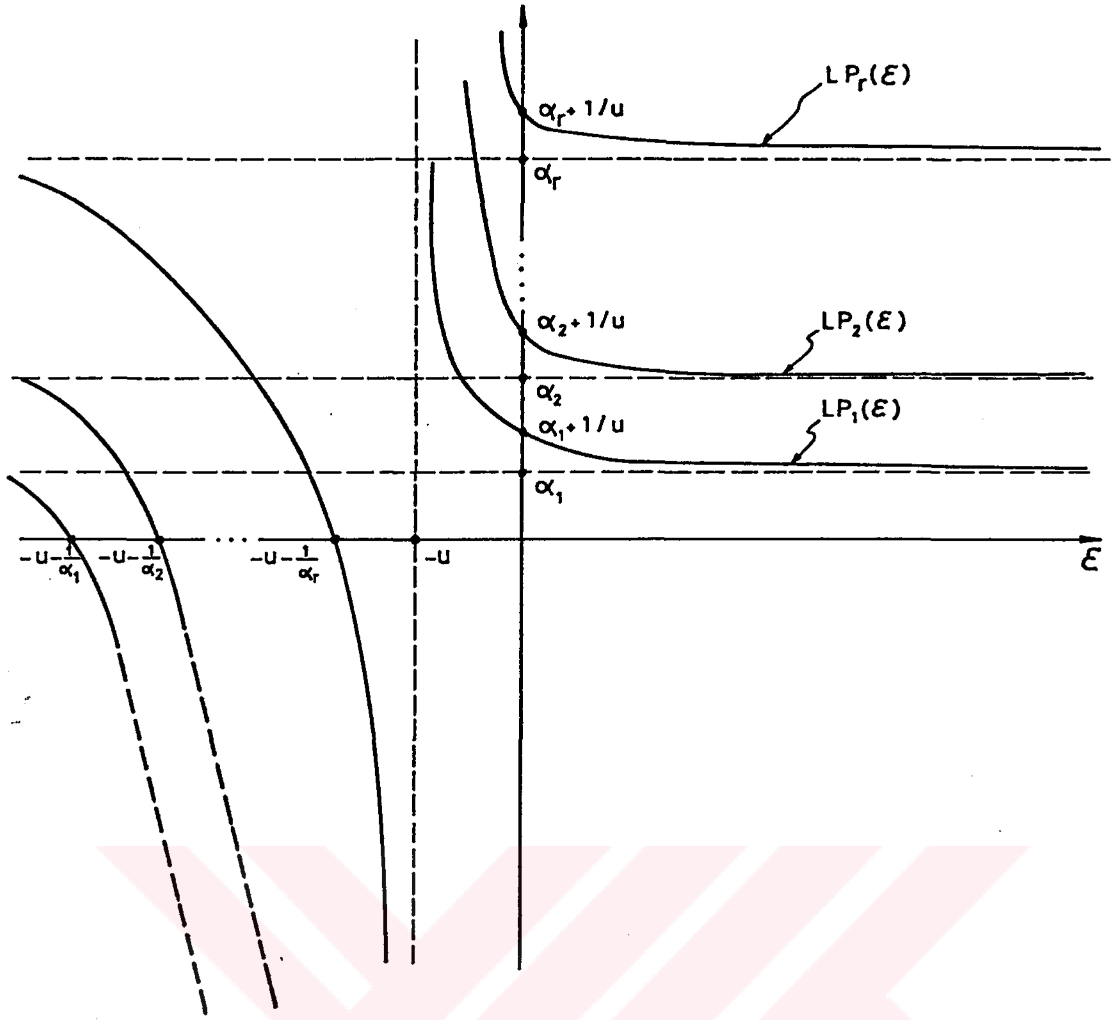
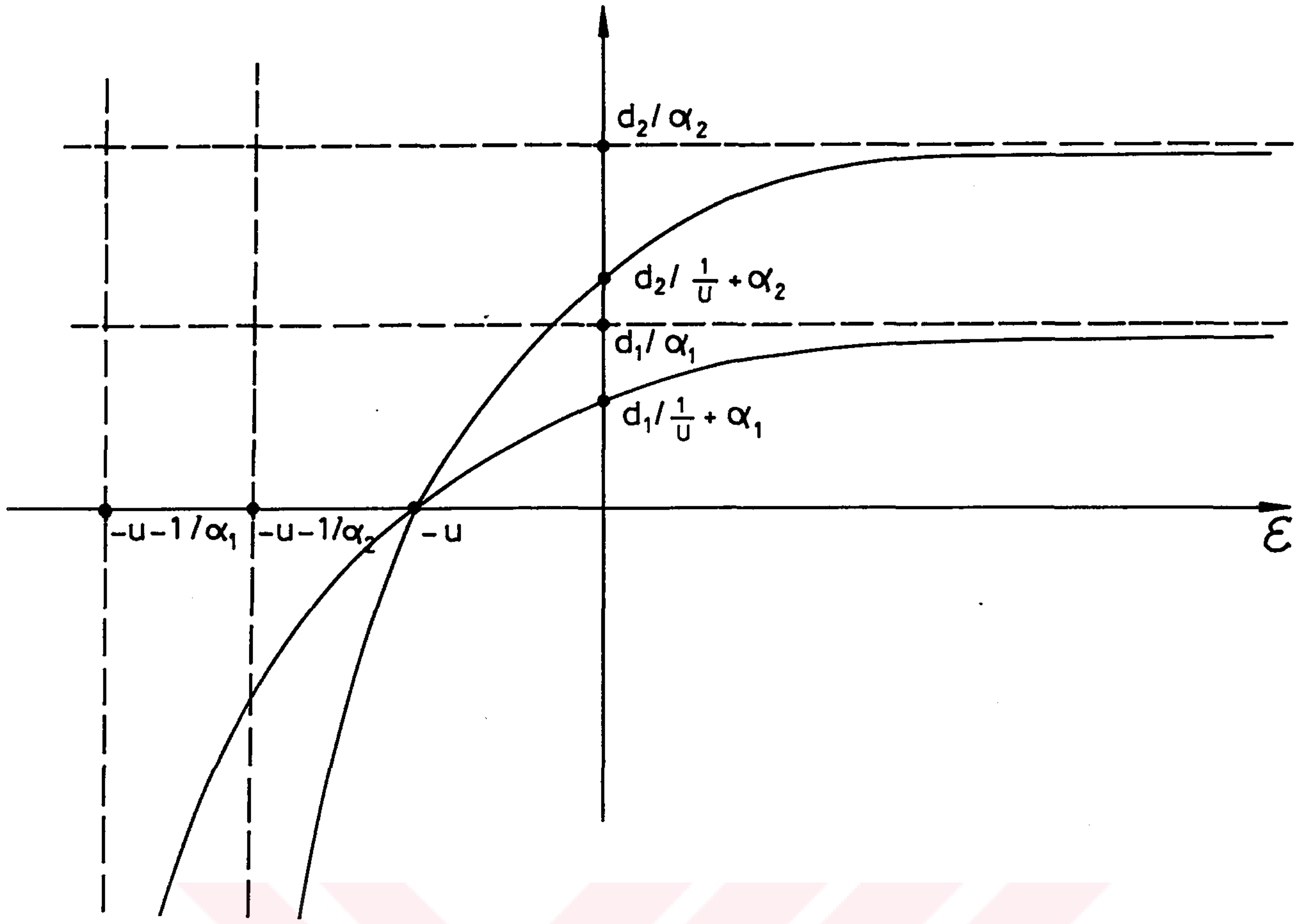


Fig. 5.4

To find ε', we solve $z_1(\varepsilon') = z_2(\varepsilon')$ and get $\varepsilon' = [(d_1-d)_2/(d_2\alpha_1-d_1\alpha_2)]-u$. For ε' to be greater than 0, $(d_1-d)_2/(d_2\alpha_1-d_1\alpha_2)$ should be greater than u.

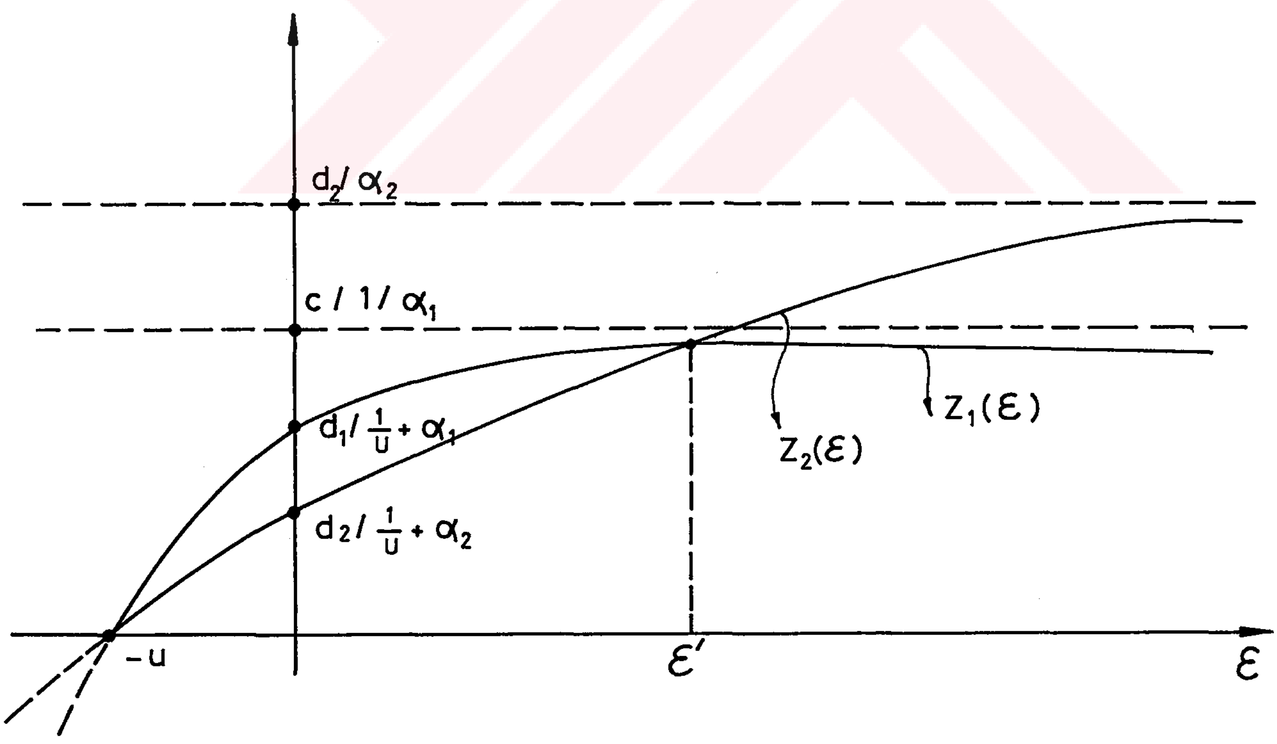
Given $z_1(\infty) < z_2(\infty)$, if, in addition, $z_1(0)=d_1/((1/u)+\alpha_1) < z_2(0)=d_2/((1/u)+\alpha_2)$ then these curves will never intersect for $\varepsilon'>0$ (see Fig. 5.5). Subsequently, we prove Property 5.2. which gives the conditions for many such curves not to intersect.

Given, $z_1(\infty)<z_2(\infty)$, if $z_1(0)>z_2(0)$ then these curves will intersect at ε', with $\varepsilon'>0$ (see Fig. 5.6).



d_2/α_2
$d_2/\frac{1}{U}+\alpha_2$
d_1/α_1
$d_1/\frac{1}{U}+\alpha_1$
$-u-1/\alpha_1$
$-u-1/\alpha_2$
$-u$
ε

Fig. 5.5



d_2/α_2
$c/1/\alpha_1$
$d_1/\frac{1}{U}+\alpha_1$
$d_2/\frac{1}{U}+\alpha_2$
$Z_1(\varepsilon)$
$Z_2(\varepsilon)$
$-u$
ε'
ε

Fig. 5.6

5.1.2 Trajectory of $z^*(\varepsilon)$

Given the problem (PMM), NBC(1) has the arc lenghts of reciprocal weights and the optimal objective value to the problem (with no perturbed weight) is $z^*(0)$. It is known that

$$z^*(0) = \max\{d(v_i,v_j)/L(E_i,E_j): 1\le i<j\le n\}$$

where $L(E_i,E_j)$ is the length of a shortestpath between E_i and E_j in NBC(1).

Now assume a weight u (which is either w_{ij}, $(i,j)\in I_C$ or v_{jk}, $(j,k)\in I_B$) is replaced by $u+\varepsilon$, $\varepsilon>-u$. Let us denote the optimal objective value to perturbed version of (PMM) by $z^*(\varepsilon)$. We wish to construct the trajectory of $z^*(\varepsilon)$.

First we propose a procedure to find $z^*(\varepsilon)$ in 5.1.2(a). Then we investigate possible trajectories of $z^*(\varepsilon)$.

5.1.2(a) Construction of $z^*(\varepsilon)$

Consider all the direct paths containing the perturbed arc, $P_1,\ldots,P_r$. From section 5.1.1. we know that, for a given ε

$$z_i(\varepsilon)=d_i/(1/(u+\varepsilon)+\alpha_i) \quad ;i=1,\ldots,r$$

where d_i is the distance between the existing facilities on the tree network corresponding to the E nodes connected by P_i, and α_i is as defined before.

If $\max\{z_i(\varepsilon): 1\le i\le r\}\ge z^*(0)$, then $z^*(\varepsilon)=\max\{z_i(\varepsilon): 1\le i\le r\}$ since for all other direct paths P that do not contain the perturbed arc $z^*(\varepsilon) \ge z(0)$. On the other hand, if $\max\{z_i(\varepsilon): 1 \le i \le r\} < z^*(0)$, then $z^*(0)$ is determined by a path that does not contain the perturbed arc. Therefore, $z^*(\varepsilon) = \max\{z^*(0), \max_{1 \le i \le r} z_i(\varepsilon)\}$.

Let $Q(\varepsilon) = \max_{1 \le i \le r} z_i(\varepsilon)$. We may compute $Q(\varepsilon)$ by enumaration on all paths $P_1, \ldots, P_r$ (i.e. all $z_i(\varepsilon)$ values for $1 \le i \le r$), but this has a nonpolynomial computational time, since r is an exponential function of n and m. In what follows a procedure is developed to find $Q(\varepsilon)$ in polynomial time.

Given a specific pair of existing facilities (E_p, E_q), let $\{P_{pq1}, \ldots, P_{pqk}\} \subseteq \{P_1, \ldots, P_r\}$ be the set of all direct paths which join E_p and E_q and which contain the perturbed arc. Assume the numbering is done in such a way that

$$\alpha_{pq1} \le \ldots \le \alpha_{pqk} \qquad (1)$$

We also know that

$$z_{pqj} = d_{pq} / (1/(u+\varepsilon) + \alpha_{pqj}) \quad ; j = 1, \ldots, k \qquad (2)$$

where $d_{pq} = d(v_p, v_q)$ is the same constant for all of the paths $P_{pq1}, \ldots, P_{pqk}$. Then by (1) and (2)

$$z_{pq1}(\varepsilon) \ge z_{pq2}(\varepsilon) \ldots \ge z_{pqk}(\varepsilon)$$

Define,

$$z_{pq}(\varepsilon) = \max\{z_{pqj}(\varepsilon): 1 \le j \le k\}.$$

Note that $z_{pq}(\varepsilon)$ is defined by a resticted shortest path between E_p and E_q in NBC(1) containing the perturbed arc. This follows from the following fact: $z_{pq}(\varepsilon) = z_{pq1}(\varepsilon)$, i.e $z_{pq}(\varepsilon)$ is defined by P_{pq1}, for which $\alpha_{pq1} = \min\{\alpha_{pqj}: 1 \le j \le k\}$. Then,

$$Q(\varepsilon) = \max\{z_{pq}(\varepsilon): 1 \le p < q \le n\}, \text{ and}$$

$$z^*(\varepsilon) = \max\{z^*(0), Q(\varepsilon)\}$$

We have m path lengths to be considered each corresponding to a restricted shortest path connecting a pair E_p, E_q, $1 \le p < q \le n$, where $m = n(n-1)/2$ if the perturbed arc is (N_i, N_j), $(i,j) \in I_B$, and $m = n-1$ if the perturbed arc is (N_i, E_j), $(i,j) \in I_C$.

CASE 1: $w_{pq} \to w_{pq} + \varepsilon$

The arc under consideration in NBC(1) is (N_p, E_q). So we should consider the restricted shortest paths joining E_k to E_q $(k \neq q)$. So, for $k = 1, \ldots, n; k \neq q$,

$$z_{kq}(\varepsilon) = d(v_k, v_q)/(1/(w_{pq}+\varepsilon)+\alpha_{pk})$$

where $\alpha_{pk} = a_{pk}$ = the length of a shortest path between N_p and E_k (see Fig. 5.7). Note that a_{pk} is the corresponding entry of matrix A defined in Chapter 4. So, it follows that

$$z_{kq}(\varepsilon) = d(v_k, v_q)/(1/(w_{pq}+\varepsilon)+a_{pk}) \quad ; k \neq q.$$

Hence,

$$Q(\varepsilon) = \max \{z_{kq}(\varepsilon) : 1 \le k \le n, \ k \neq q\}, \text{ and}$$
$$z^*(\varepsilon) = \max\{z^*(0), \ Q(\varepsilon)\}$$

Given that the matrix A and the distance matrix of the tree are available, the computation of $Q(\varepsilon)$ is done in $O(n)$ since it requires computing the maximum of $(n-1)$ numbers.

CASE 2: $v_{pq} \to v_{pq} + \varepsilon$

The arc under consideration in NBC(1) is (N_p, N_q). We should consider all restricted shortest paths $P_R(E_i, E_j)$ containing

(N_p, N_q), $1 \le i < j \le n$. Then,

$$z_{ij}(\varepsilon) = d(v_i, v_j)/(1/(v_{pq}+\varepsilon)+\alpha_{ij}),$$

where $\alpha_{ij} = \min\{(a_{qj}+a_{pi}),\ (a_{qi}+a_{pj})\}$ (see Fig. 5.8). Hence,

$$z_{ij}(\varepsilon) = d(v_i, v_j)/\{\ 1/(v_{pq}+\varepsilon)+\min\{(a_{qj}+a_{pi}),\ (a_{qi}+a_{pj})\}\ \},$$

$$Q(\varepsilon) = \max\{z_{ij}(\varepsilon):\ 1 \le i < j \le n\},\ \text{and}$$

$$z^*(\varepsilon) = \max\{z^*(0),\ Q(\varepsilon)\}.$$

In this case, the computation of $Q(\varepsilon)$ requires computing the maximum of $n(n-1)/2$ numbers.

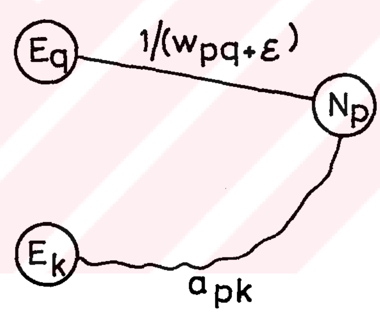


Fig. 5.7

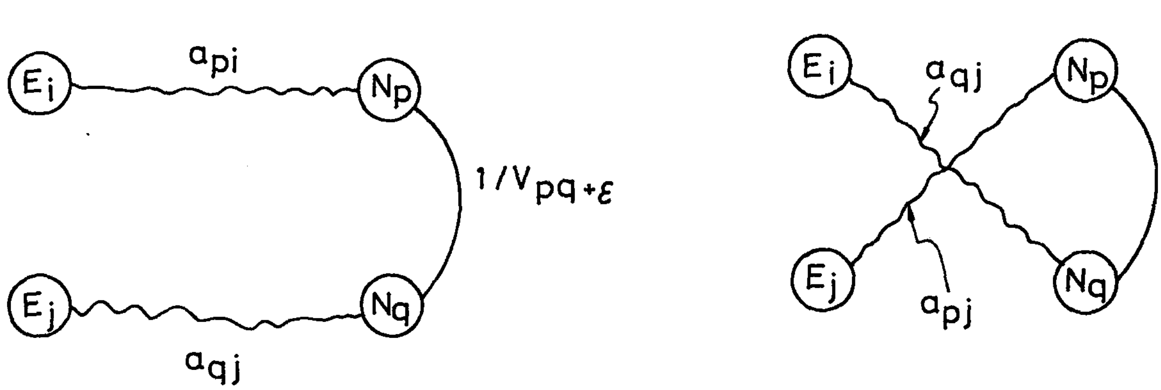


Fig. 5.8

5.1.2(b) Possible Trajectories of $z^*(\varepsilon)$

Consider all restricted shortest paths $P_R(E_i,E_j)$ in NBC(1) containing the perturbed arc. If, for each pair of E-nodes, we pick one such path, then we have m such paths, where m=(n-1) or n(n-1)/2, depending on the perturbed weight. Let us denote these paths by $P_{R1},\ldots,P_{Rm}$. Then considering 5.1.2(a),

$$z_i(\varepsilon)=d_i/(1/(u+\varepsilon)+\alpha_i) \quad ;i=1,\ldots,m$$

where u is the perturbed weight, α_i is the sum of the arc lengths in P_{Ri} except the length of the perturbed arc, and d_i is the distance between the existing facilities on the tree corresponding to E-nodes joined by P_{Ri}. From our foregoing results, we know that

$$z^*(\varepsilon)=\max\{z^*(0), \max_i z_i(\varepsilon)\}.$$

We have five cases related with the trajectory of $z^*(\varepsilon)$, they are explained in what follows. Related properties will be given prior to the discussion of each case.

CASE 1: Recalling that $z_i(\infty)=d_i/\alpha_i$, if $z^*(0)\geq d_i/\alpha_i$ for all $i=1,..,n$, then $z^*(0)$ will remain as the optimal objective value, $\forall$ $\varepsilon\geq 0$ (Fig. 5.9).

The following property will be helpful in investigating the other possible trajectories of $z^*(\varepsilon)$.

PROPERTY 5.2. Assume $P_{R1},\ldots,P_{Rm}$ are numbered in such a way that $d_1/\alpha_1 \leq d_2/\alpha_2 \leq\ldots\leq d_m/\alpha_m$. If also, $z_1(0)<z_2(0)<\ldots<z_m(0)$, where

$z_i(0)=d_i/(1/u+\alpha_i)$, then the curves $z_i(\varepsilon)$, $\varepsilon\geq 0$, will never intersect.

Proof: Select k and l such that $1\leq k<l\leq m$. Then,

$$z_k(0) < z_l(0) \quad \text{(a)}$$

$$z_k(\infty) \leq z_l(\infty) \quad \text{(b)}$$

by (a) $$d_k\alpha_l - d_l\alpha_k < (d_l-d_k)/u \quad (1)$$

by (b) $$d_k\alpha_l - d_l\alpha_k \leq 0 \quad (2)$$

Assume for $\varepsilon' > 0$, we have $z_k(\varepsilon') > z_l(\varepsilon')$ (i.e. there is an intersection point at some ε, $0\leq\varepsilon<\varepsilon'$ where $z_k(\varepsilon)=z_l(\varepsilon)$). Then,

$$d_k\alpha_l - d_l\alpha_k > (d_l-d_k)/(u+\varepsilon') \quad (3)$$

Using (2) and (3), we obtain

$$(d_l-d_k)/(u+\varepsilon') < 0.$$

But $u+\varepsilon' > 0$ $(u>0,\ \varepsilon'>0)$, implying that $d_l-d_k < 0$

Using (1) and (3), we obtain

$$(d_l-d_k)/u > d_k\alpha_l - d_l\alpha_k > (d_l-d_k)/(u+\varepsilon')$$

so that $$(d_l-d_k)/u > (d_l-d_k)/(u+\varepsilon').$$

Since $d_l-d_k < 0$, $(u+\varepsilon')/u < 1$, implying that $\varepsilon' < 0$, which is a contradiction.□

Assume the restricted shortest paths containing the perturbed arc, $P_{R1},\ldots,P_{Rm}$, numbered in such a way that $d_1/\alpha_1 \leq d_2/\alpha_2 \leq\ldots\leq d_m/\alpha_m$, if also $z_1(0) < z_2(0) < \ldots < z_m(0)$, then $z_1(\varepsilon) < z_2(\varepsilon) <\ldots< z_m(\varepsilon)$ for all $\varepsilon > 0$. So, $Q(\varepsilon) = z_m(\varepsilon) = \max\{z_i(\varepsilon):\ 1\leq i\leq m\}$. Given such a situation, Property 5.2. is used in the following two cases.

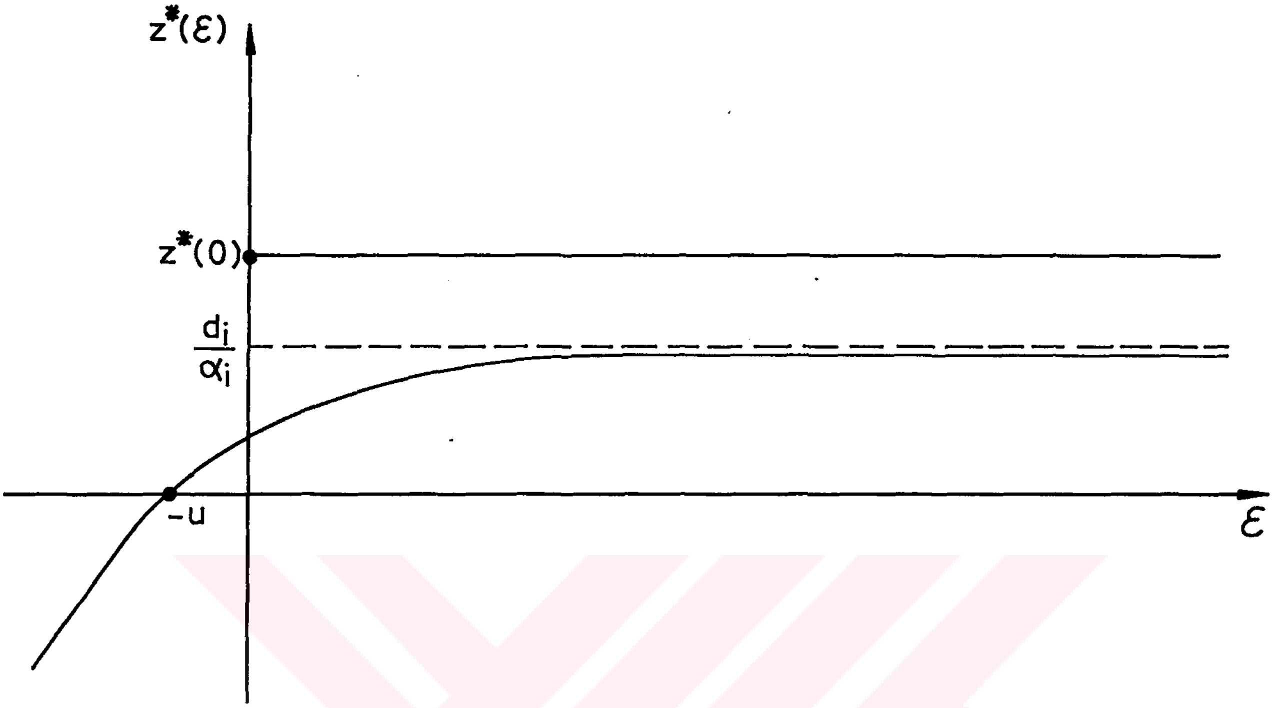


Fig.5.9

CASE 2: If $z^*(0) = z_m(0)$, i.e. P_{Rm} is a tight path in NBC(1), then $z^*(\varepsilon) = z_m(\varepsilon)$ for all $\varepsilon \geq 0$ (Fig. 5.10).

CASE 3: If $z_m(0) < z^*(0) < d_m/\alpha_m$, then there exists $\varepsilon' > 0$ for which $z_m(\varepsilon') = z^*(0)$ (Fig. 5.11). In this case,
$\varepsilon' = \{z^*(0) / (d_m - z^*(0)\alpha_m)\} - u$, and $z^*(\varepsilon) = z^*(0)$ for $0 \leq \varepsilon \leq \varepsilon'$,
$z^*(\varepsilon) = z_m(\varepsilon)$ for $\varepsilon > \varepsilon'$.

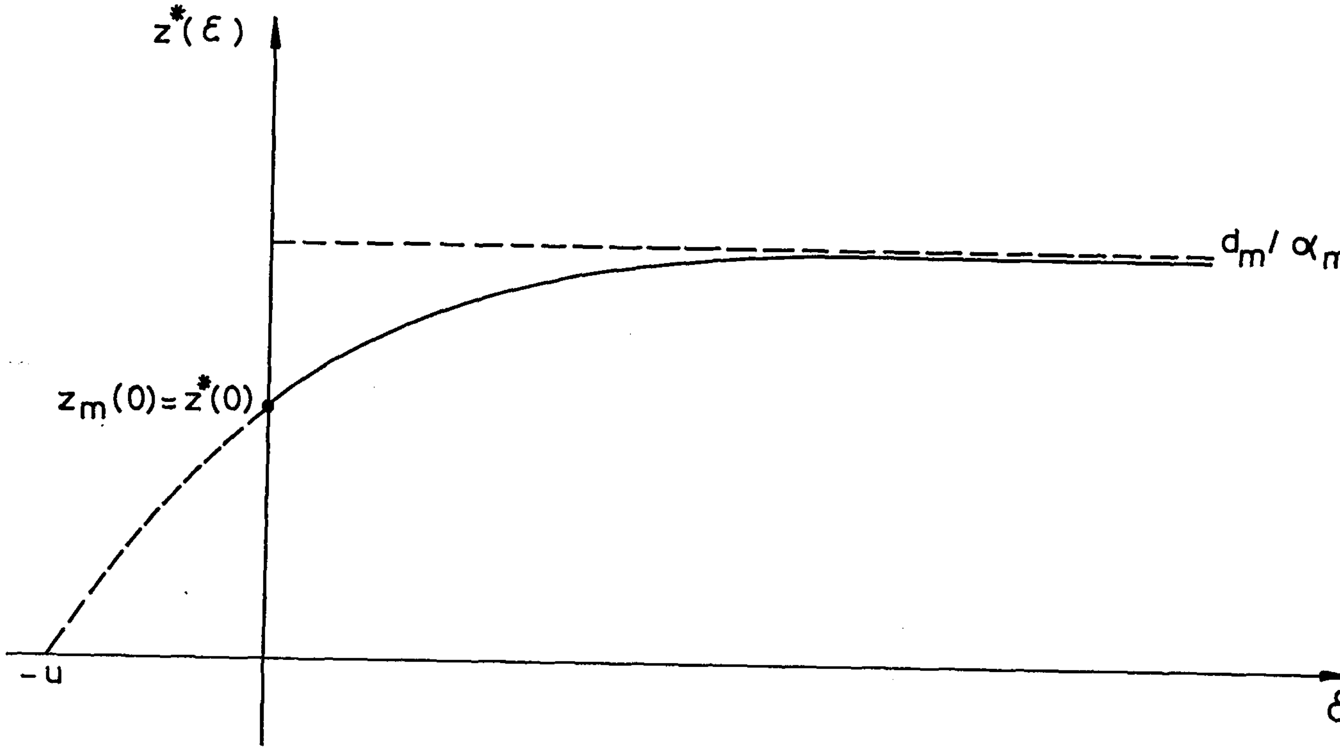


Fig. 5.10

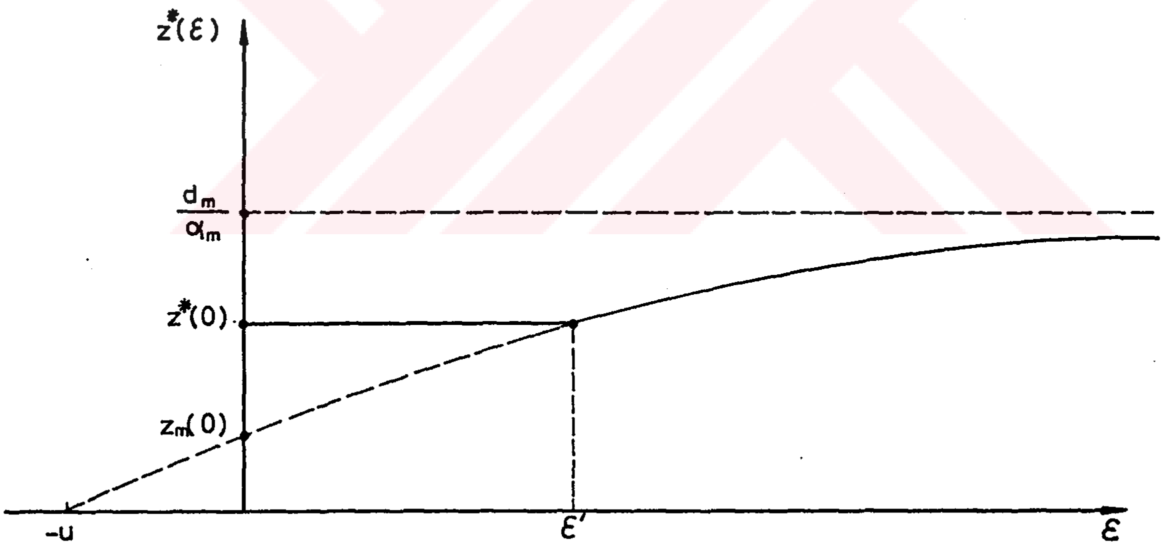


Fig. 5.11

In the following, we consider the cases that arise when $Q(\varepsilon)$ is defined by various restricted shortest paths for different values of ε. Referring to the discussion at the end of section 5.1.2(a), it is possible for $Q(\varepsilon)$, $\varepsilon>0$, to be defined by k different shortest paths (k>1) in k different intervals. Without loss of generality, we assume that the paths $P_{R1},..,P_{Rk}$, $P_{Rk+1},..,P_{Rm}$ are numbered in such away that

$$Q(\varepsilon) = \max \{ z_i(\varepsilon) : 1 \le i \le m \}$$

$$= \begin{cases} z_1(\varepsilon) & : 0 \le \varepsilon \le \varepsilon_1 \\ z_2(\varepsilon) & : \varepsilon_1 < \varepsilon \le \varepsilon_2 \\ & \\ z_{k-1}(\varepsilon) & : \varepsilon_{k-2} < \varepsilon \le \varepsilon_{k-1} \\ z_k(\varepsilon) & : \varepsilon > \varepsilon_{k-1} \end{cases}$$

in which case $z_1(0) > z_2(0) > \ldots z_k(0)$ while $z_1(\infty) \le \ldots \le z_k(\infty)$, for the above assumption to hold. An example trajectory of $Q(\varepsilon)$ is given for k=3 (i.e. 3 different paths define $Q(\varepsilon)$ for $\varepsilon>0$) in Fig. 5.12).

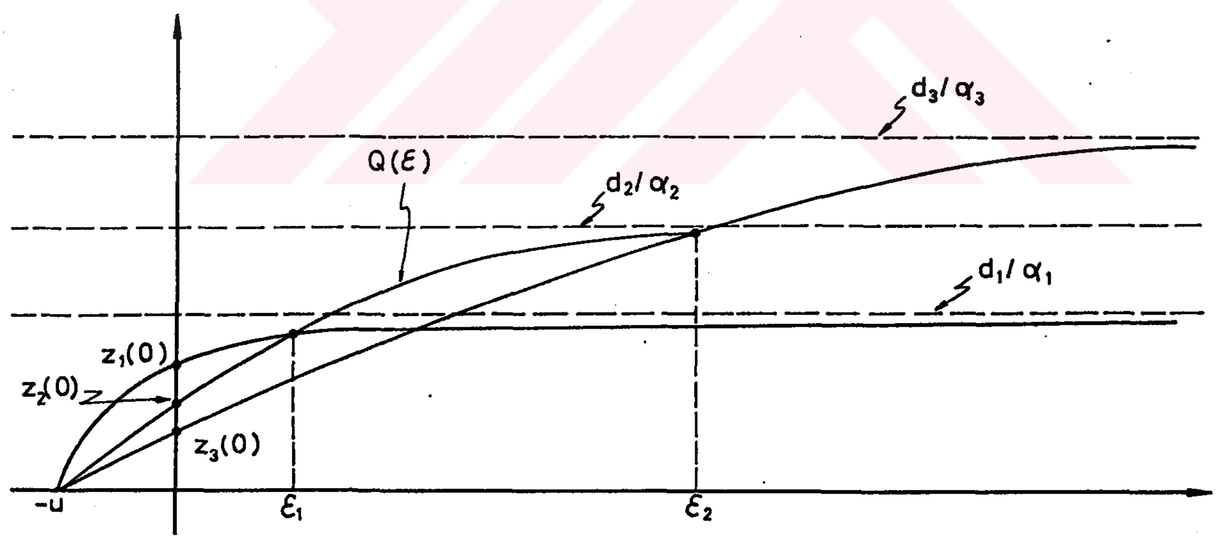


Fig. 5.12

CASE 4: $z^*(0) = z_1(0)$, i.e. P_{R1} is a tight path in NBC(1). Then,

$$z^*(\varepsilon) = Q(\varepsilon) = \begin{cases} z_1(\varepsilon) & : 0 \le \varepsilon \le \varepsilon_1 \\ & \\ z_k(\varepsilon) & : \varepsilon > \varepsilon_{k-1} \end{cases}$$

The illustration of this case is given in Fig. 5.13.

CASE 5: For some $\varepsilon'>0$, $z_p(\varepsilon') = z^*(0)$ where $1\le p\le k$. Then, for $\varepsilon_{p-1}< \varepsilon < \varepsilon_p$, $z_{p-1}(\varepsilon') < z^*(0) < z_p(\varepsilon)$, and

$$z^*(\varepsilon) = \begin{cases} z^*(0) & : 0<\varepsilon\le\varepsilon' \\ z_{p+1}(\varepsilon) & : \varepsilon'<\varepsilon\le\varepsilon_{p+1} \\ & \\ z_k(\varepsilon) & : \varepsilon>\varepsilon_{k-1} \end{cases}$$

where $\varepsilon' = z^*(0)/(d_p-z^*(0)\alpha_p)-u$ (see Fig. 5.14).

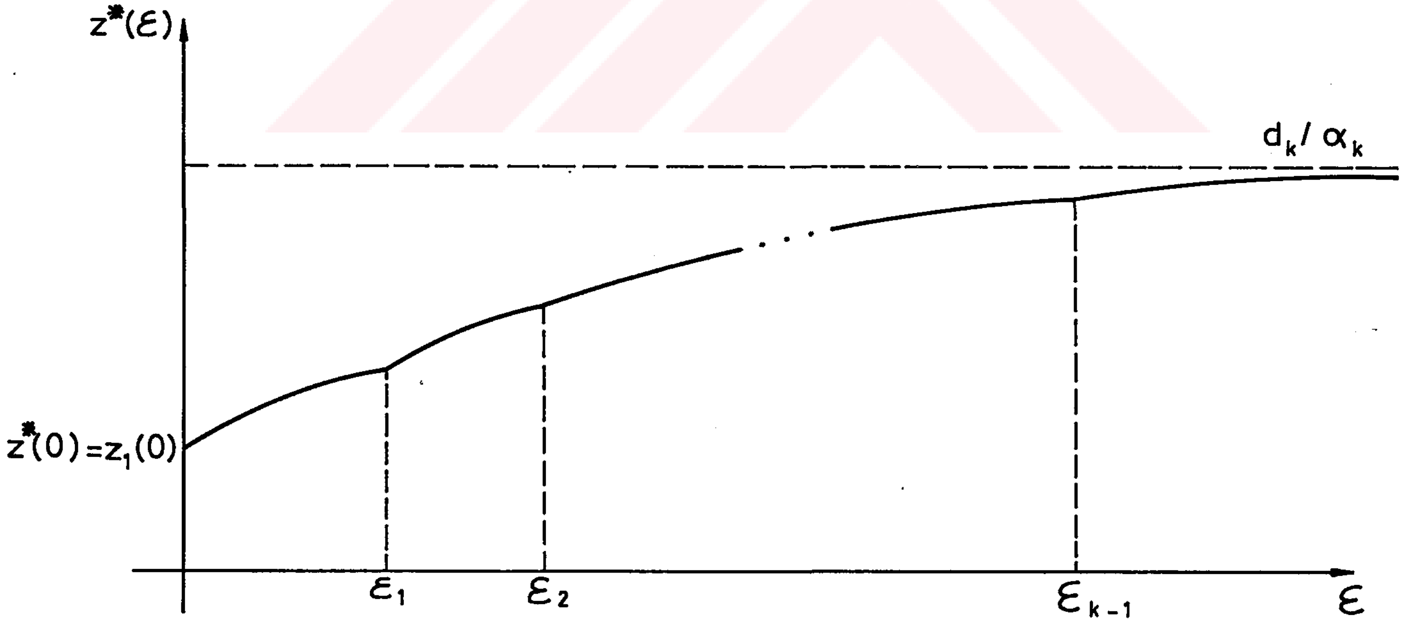


Fig. 5.13

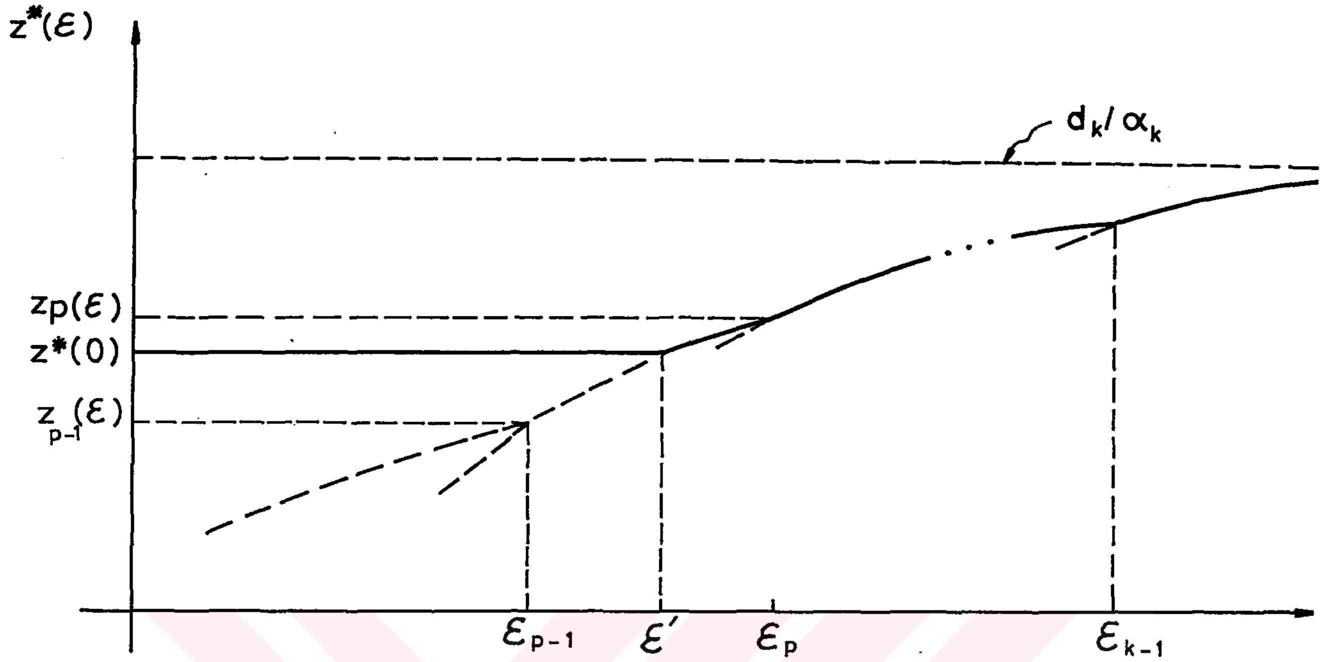


Fig. 5.14

An example problem is given in Appendix to illustrate how a perturbation affects the optimal objective value in the multifacility minimax location problem on tree networks.

If we analyze the changes in $X^*(\varepsilon)$ we observe the followings. In cases 3 and 5, $z^*(\varepsilon) = z^*(0)$ (a constant) for $0 \le \varepsilon \le \varepsilon'$, and the only change in NBC($z^*(0)$) is due to the decrease in the length of the arc associated with the perturbed weight (whithin this interval of values for ε). So, we can make use of the results of Chapter 4 and conclude that if the perturbed weight corresponds to a non-uniquely located facility in NBC(1), for $\varepsilon = \varepsilon'$ this facility will become uniquely located. But,we are not able to predict the status of an originally non-uniquely located facility at $\varepsilon = \varepsilon'$, if the perturbed arc is not adjacent to this facility in NBC(1).

For $\varepsilon>\varepsilon'$, the behaviour of $z^*(\varepsilon)$ is the same as in cases 2 and 4, i.e. as the length of the perturbed arc changes , $z^*(\varepsilon)$ also changes. Analysis of the behavior of the optimal location vector as a function of ε is a potential future research area in sensitivity analysis of multifacility minimax location problems. It should be noted that, in all of these cases for a given ε, $X^*(\varepsilon)$ can be found, by employing NBC($z^*(\varepsilon)$) and using SLP developed in [9].

It should also be noted that case 1 occurs whenever the perturbed arc will never affect the original tight path defining $z^*(0)$. That is to say, no one of the paths containing the perturbed arc will become tight for all $\varepsilon>0$.

5.2. 2-Parameter Minimax Location Problems

In the m-facility minimax location problem on a tree network, assume that there are two types of weights :w, a positive constant, is the weight associated with any new facility-existing facility pair, and v, another positive constant, is the weight associated with any pair of new facilities.

With rescaling if necessary we can take w=1 and investigate the problem as v varies. For v>0, the problem is as follows.

$$PMM2(v) \qquad z^*(v) = \text{MIN} \quad z$$
$$\text{s.t}$$
$$d(x_i, v_j) \le z \quad ; \ (i,j) \in I_B$$
$$d(x_j, x_k) \le z/v \ ; \ (j,k) \in I_C$$

If each new facility represents a service facility, then the pairs of facility indices in I_B may be thought of as providing back-up service (or support) for one another. The interaction between a new facility and an existing facility would correspond to "real" service. The new facility-existing facility pairs between which we have a real service type interaction is assumed to be given by I_C. Since w=1, if $v \in (0,1)$, we will be concerned with the case in which real service is more important than back up (support) service. Otherwise ($v>1$), we can say that support service is more important. The latter case may be applicable in the case of ambulance and fire stations, police patrol units and defense units in a military context. Each unit having a given capacity serves (defends) existing facilities while providing back up support for some of the other service units. If the number of such service units is relatively few with respect to the number of existing facilities, the back up service may be as important as real services or may be more important, especially in such kind of service facilities.

Considering PMM2(v), the aim, here, is to analyze the trajectory of $z^*(v)$. Note that NBC(1) has arc lenghs 1 for the arcs (N_i, E_j), $(i,j) \in I_C$ and $1/v$ for the arcs (N_j, N_k), $(j,k) \in I_B$.

We observe that if $I_C=\{(i,j): 1\le i\le m, 1\le j\le n\}$, i.e.each service facility gives service to every existing facility, then the length of a shortest path between E_p and E_q, $p\ne q$ in NBC(1) is 2 (Fig.5.15). That is,

$$L(E_p,E_q) = 2 \quad ;\forall(p,q): 1\le p<q\le n$$

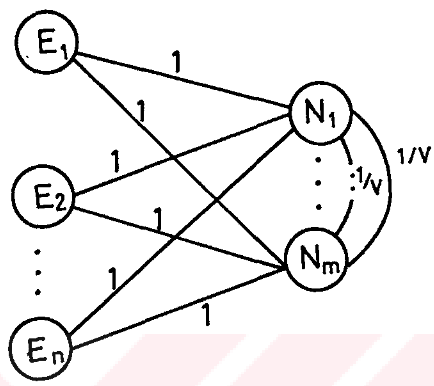


Fig. 5.15

In this case, $z^*(v) = \max\{d(v_p,v_q)/L(E_p,E_q): 1\le p<q\le n\} = \max\{d(v_p,v_q)/2: 1\le p<q\le n\}$, which is independent of the value of v ,implying that back up service is not of interest. This is because, for a given existing facility, each service facility is accessible.

In the general case, we have $I_C\subseteq \{(j,k): 1\le i\le m, 1\le j\le n\}$, and $I_B\subseteq \{(j,k): 1\le i\le m, 1\le j\le n\}$. Considering E_p, E_q in NBC(1), $p\ne q$, if there exists an N_j node such that $(j,p) \in I_C$ and $(j,q) \in I_C$, then the shortest path length between E_p and E_q is again 2 (see Fig. 5.16). Otherwise, the shortest path $P(E_p,E_q)$ visits more than one, say k, N-nodes in NBC(1). Then, $L(E_p,E_q) = 2+((k-1)/v)$ (see Fig. 5.17).

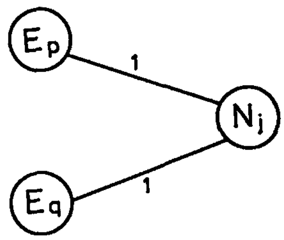


Fig. 5.16

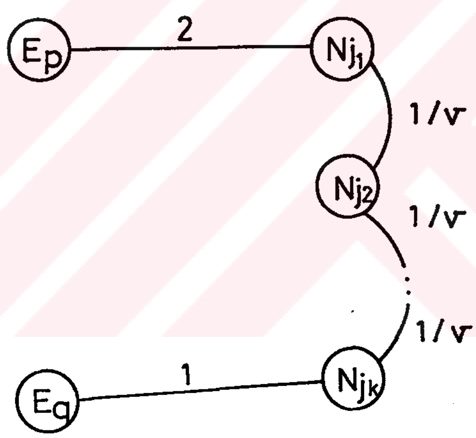


Fig. 5.17.

Let k_{pq} be the number of N-nodes visited by a shortest path joining E_p and E_q $(1 \le k_{pq} \le m)$. We have, $L(E_p, E_q) = 2 + ((k_{pq}-1)/v)$, and

$$z^*(v) = \max\{f_{pq}(v)\}, \text{ where}$$

$$f_{pq}(v) = d_{pq}/[2+((k_{pq}-1)/v)] \ , \text{ and } \ d_{pq} = d(v_p, v_q)$$

Trajectory of $f_{pq}(v)$:

If $k_{pq}=1$, then $f_{pq}(v)=d_{pq}/2$, a constant. If $k_{pq}>1$, then $f_{pq}(v)$ is a hyperbola with a horizontal asymptote at $f_{pq}(\infty) = d_{pq}/2$, and a vertical asymptote at $v=(1-k_{pq})/2 <0$ (see Fig. 5.18).

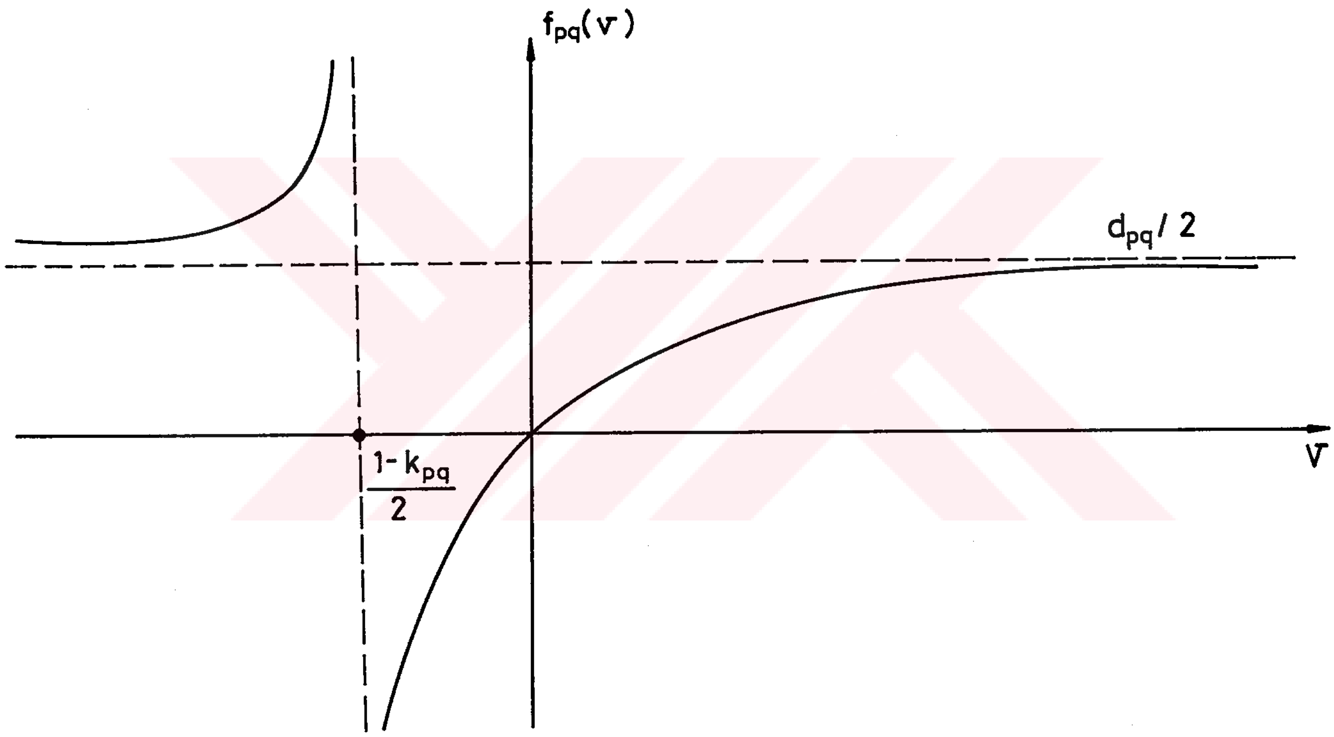


Fig. 5.18

Since we are dealing with v>0, only the related portion of this curve is of interest (Fig. 5.19).

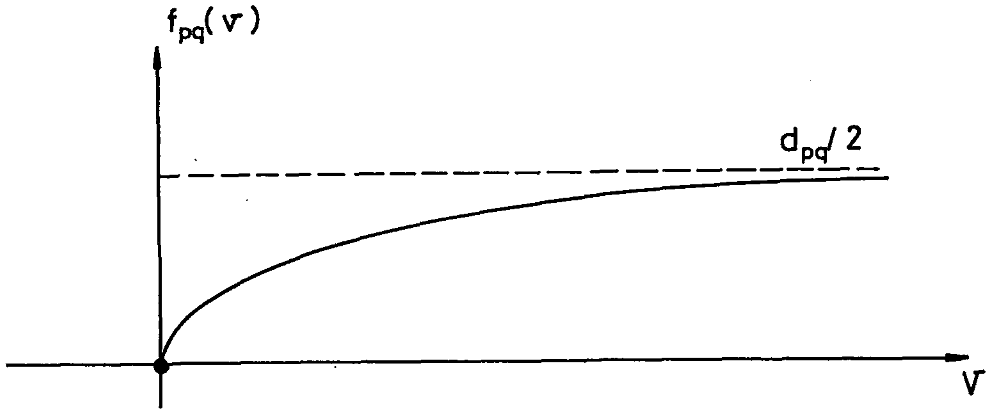


Fig. 5.19

The function is monotone increasing in v for v>0. There are n(n-1)/2 functions $f_{pq}(v)$ to be considered to find $z^*(v)$. Since, $z^*(v)$ is the pointwise maximum of these curves ,the general trajectory of $z^*(v)$ is as given in Fig. 5.20, where β=max $\{d_{ij}/2 : 1 \le i < j \le n\}$.

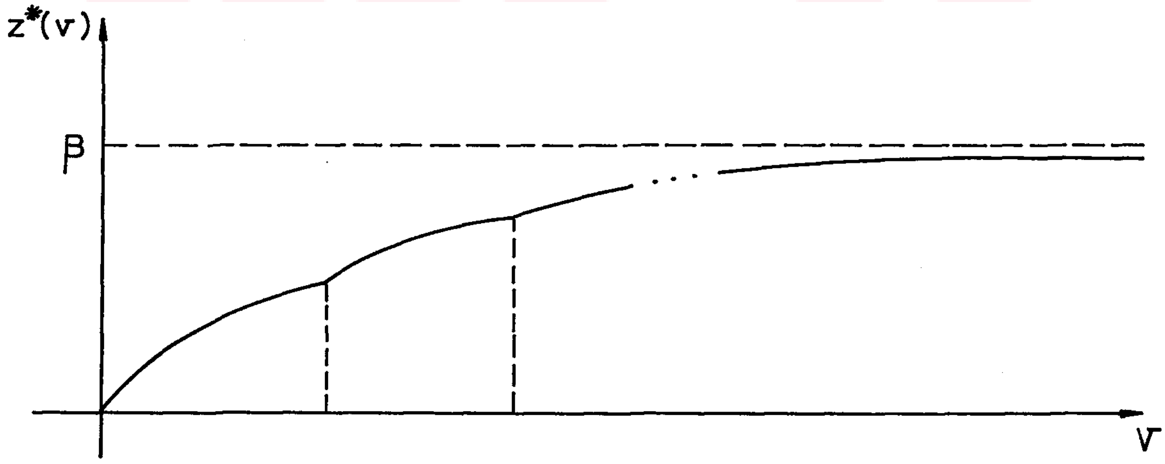


Fig. 5.20

Note that when v approaches infinity, all the constraints related with the service facilities will be of the form

$$d(x_j, x_k) \leq 0^+; \quad (j,k) \in I_B$$

in which case the differentiation between new facilities will be lost as well as that between real and back up services, and the problem reduces to a single facility minimax network location problem on a tree, with all weights associated with the existing facilities being equal. Then, β is the optimal objective value to this problem, meaning that we should locate the new facility in the middle of the path that connects two existing facility locations v_i and v_j that are farthest apart.

6. SUGGESTIONS FOR FUTURE RESEARCH

Given that the parameters defining the distance constraints have a given set of possible realizations, say the set $E \subset \mathbb{R}^k$ (where k is the number of constraints), the concepts of weakly and permanently feasible solutions have been developed for both single and multifacility tree network location contexts.

In many real world applications, the set of all permanently feasible solutions to distance constraints (a subset of T (or T^m)) might be empty. That is to say, we usually do not have a location vector $X \in T^m$ which is feasible to distance constraints for all possible realizations of RHS values. Also, it may be possible to confine the set of all possible realizations of the parameters to a smaller set reducing the range of imprecision on the distance constraints. We call all such sets refinements of E. What we refer to as a critical refinement set is a maximal subset of E for which the associated set of all permanently feasible location vectors is nonempty. In the case of single facility location with imprecise distance constrains on a tree, it has been shown that we can find a critical refinement of the set E.

The conjectures related with the m-facility case are the following: It is proposed that , in the m-facility tree network location problems with imprecise distance constraints, there are infinitely many critical refinements of the given set of realizations of the parameters (the set E), and these refinement sets are k-cells contained in E if E itself is a k-cell. Altough these arguments are not proved, they constitute promising future research areas in this context. Developing an efficient algorithm to find a critical refinement of E is the next aim. We have started to analyze this case with a procedure to find a critical refinement of E, which uses the network BC corresponding to the distance constraints defined by the lower bound of the set E, adopting the conjecture that a critical refinement is a k-cell in E. However, in its current form the procedure is non polynomial. What we can say here is that instead of dealing with NBC, we may make use of the ideas introduced by the Sequential Location Procedure of [9] to find a lower bound for the critical refinement of the set E. Hence, the algorihm is left to be modified as the required theoretical issues are further developed, and is not included in this thesis.

In the case of sensitivity analysis of distance constraints in the equivalent formulation of the m-facility minimax problem with mutual communication on a tree, the perturbations have been made one at a time. We have found that the optimal objective value, $z^*(\varepsilon)$, as a function of the perturbation amount ε, approaches a constant as ε approaches infinity. The change in the optimal location vector $X^*(\varepsilon)$ for a given ε can be analyzed by employing $NBC(z^*(\varepsilon))$ with the arc length corresponding to the perturbed weight modified, and then using SLP of [9].

In the case of two-parameter minimax problem, with the weight corresponding to the real service being 1 (w=1), we have analyzed the optimal objective function value as a function of v>0, the weight for back-up service. We observed that the two trajectories, namely $z^*(\varepsilon)$ and $z^*(v)$ are similar, and $z^*(v)$ also approaches a positive real value as v approaches infinity. If the weight of back up service is too large, the differentiation between real and back-up services will be lost, and the problem can be considered as that of locating a single real service center with respect to existing demand points in T. This follows from the fact that, the problem is reduced (in the limit) to single facility minimax location problem with all weigths associated with existing facilities being equal.

In all of these cases, we have considered the trajectory of some function with respect to a single parameter. The following problem which incorporates a more general parametric approach will be one of the subsequent research issues in this context.

In the m-facility minimax problem, assume each weight is a continiuous function of a parameter t for $t \in [0,u]$. Here, t may denote the time, and each proportionality constant may be assumed to change through time with respect to a given function. This function may be a linear function or may be nonlinear. Such time dependent weights may reflect possible shifts in service and interfacility interactions in response to changes in traffic conditions and population shifts within a metropolitan area during the course of the day. Note that if we locate the facilities by cosidering only the values of weigths at t=0, we will certainly be suboptimal as weights change later. If we denote the optimal objective value of this minimax network location problem by $z^*(t)$, its trajectory can be analyzed for the purposes of making location decisions to minimize the costs incurred due to inefficient use of those service facilities in the future.

The analysis techniques given in the thesis for the trajectory of $z^*(\varepsilon)$ in response to perturbations in a single weight might prove useful in the analysis of the case that allow simultaneaus changes in many weights as a function of a single parameter.

The sensitivity analysis of the distance constrained version of this problem (Distance Constrained Multi-center Problem) is another potential research area. This research calls for the integration of our results developed independently for distance constraints and for weight perturbations. Computational aspects of these perturbations will most likely require the use of procedures similar to those developed in [7].

One final research issue is the analysis of imprecision associated with arc lengths of the network on which travel takes place. If we interpret arc lengths as travel times along arcs, it is apparent that such travel times may undergo frequent changes as dependent on daily traffic conditions. The location of mobile service units such as police patrol cars may require the incorporation of such variations in arc travel times.

APPENDIX

EXAMPLE: Consider the tree network with four existing facility locations, v_1, v_2, v_3, v_4, in Fig.1.

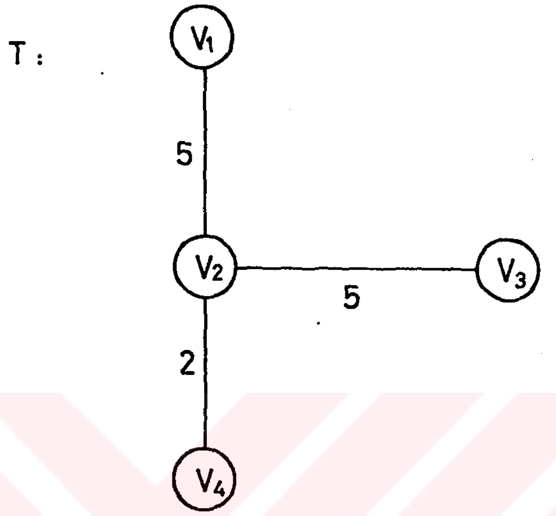


Fig.1

Assume 3 new facilities are to be located on T with $I_B=\{(1,2),\ (1,3),\ (2,3)\}$, and $I_C=\{(1,1),\ (1,3),\ (2,1),\ (2,2),\ (2,4),\ (3,2),\ (3,4)\}$. The corresponding weights are: $w_{11}=0.1$, $w_{13}=0.4$, $w_{21}=0.8$, $w_{22}=0.5$, $w_{24}=0.2$, $w_{32}=0.1$, $w_{34}=2$, $v_{12}=0.2$, $v_{13}=1$, $v_{23}=0.2$. NBC(1) is given in Fig.2.

Therefore, $L(E_1,E_2)=3.25$, $L(E_1,E_3)=8.75$, $L(E_1,E_4)=6.25$, $L(E_2,E_3)=9.5$, $L(E_2,E_4)=7$, $L(E_3,E_4)=4$;and $z^*(0)=1.75$, the matrix A is given in the following.

$$A = \begin{bmatrix} 6.25 & 7 & 2.5 & 1.5 \\ 1.25 & 2 & 7.5 & 5 \\ 6.25 & 7 & 3.5 & 0.5 \end{bmatrix}$$

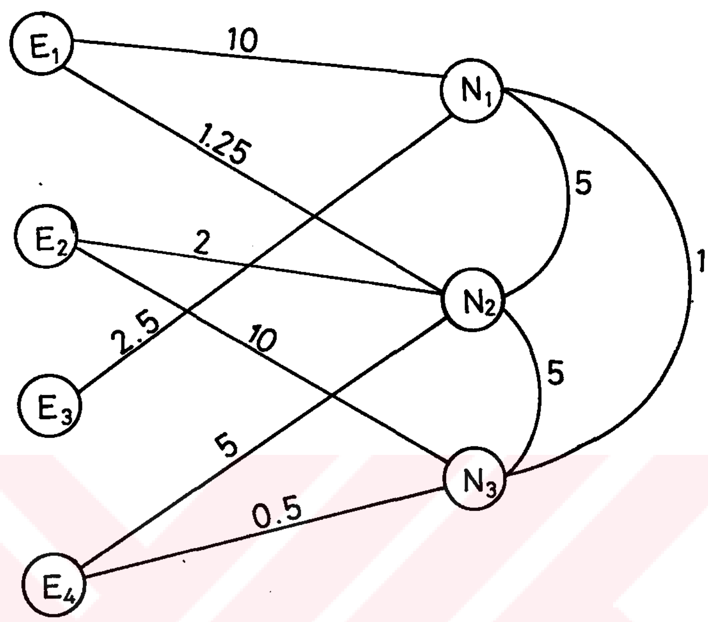


Fig.2

Assume we want to perturb $w_{11}=0.1$; $w_{11} \to w_{11} + \varepsilon \Rightarrow p=1, q=1$. The arc under consideration in NBC(1) is (N_1, E_1). There are 3 restricted shortest paths containing this arc, which will be denoted by P_k, $k=1,2,3$.

k	P_k	d_k	α_k	d_k/α_k	$z_k(0)$
1	$E_1 \to N_1 \to N_2 \to E_2$	5	7	0.71	0.294
2	$E_1 \to N_1 \to E_3$	10	2.5	4	0.8
3	$E_1 \to N_1 \to N_3 \to E_4$	7	1.5	$4.\overline{66}$	0.609

Since $d_1/\alpha_1 = 0.71 < 1.75 = z^*(0)$, the path P_1 never determines $z^*(\varepsilon)$, as ε approaches infinity. And since $d_1/\alpha_1 < d_2/\alpha_2$ and $z_1(0) < z_2(0)$, $z_2(\varepsilon)$ dominates $z_1(\varepsilon)$ (they will never intersect and $z_2(\varepsilon) > z_1(\varepsilon) \quad \forall\ \varepsilon>0$). Similarly, $z_3(\varepsilon)$ dominates $z_1(\varepsilon)$. Then, $Q(\varepsilon)$ will be defined either by $z_2(\varepsilon)$ or $z_3(\varepsilon)$. We have $d_2/\alpha_2 < d_3/\alpha_3$ and $z_2(0) > z_3(0)$, hence at $\varepsilon=1.1 > 0$, $z_2(1.1) = z_3(1.1) = 3$. Therefore, $Q(\varepsilon)=z_2(\varepsilon)$ for $0 \le \varepsilon \le 1.1$, and $Q(\varepsilon)=z_3(\varepsilon)$ for $\varepsilon>1.1$ (see Fig.3).

For $\varepsilon<0.211$, $z^*(0)=1.75$ will remain as optimal objective value $z^*(\varepsilon)$, for $\varepsilon>0.211$ $z^*(\varepsilon)$ will be $Q(\varepsilon)$. The trajectory of $z^*(\varepsilon)$ is given in Fig.4.

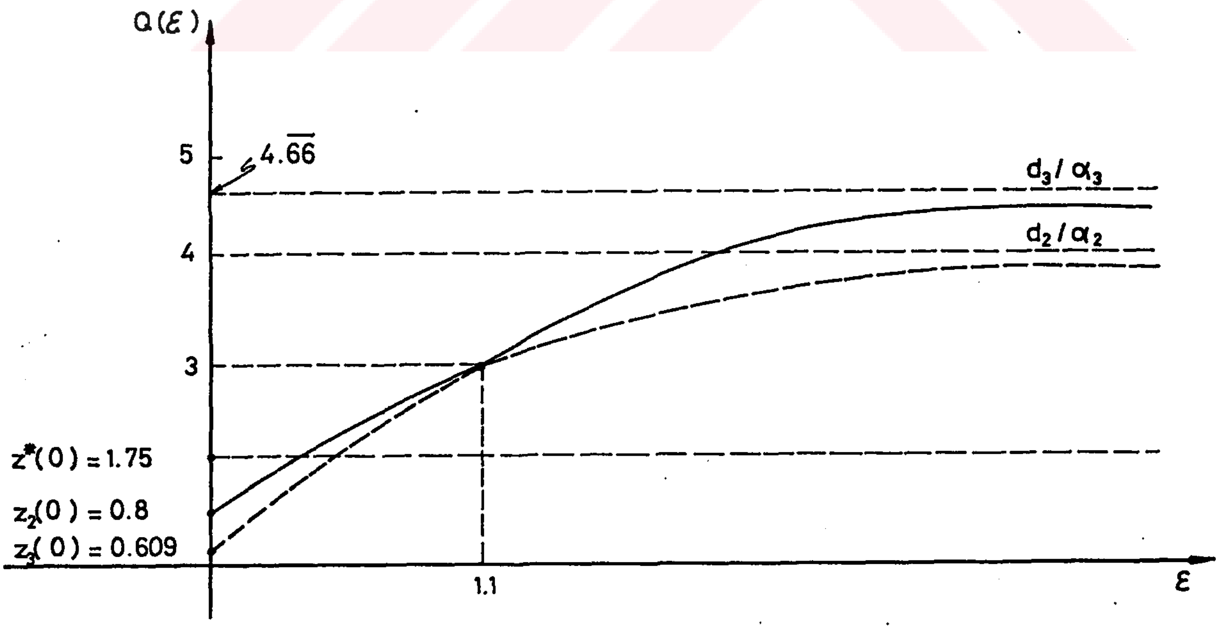


Fig.3

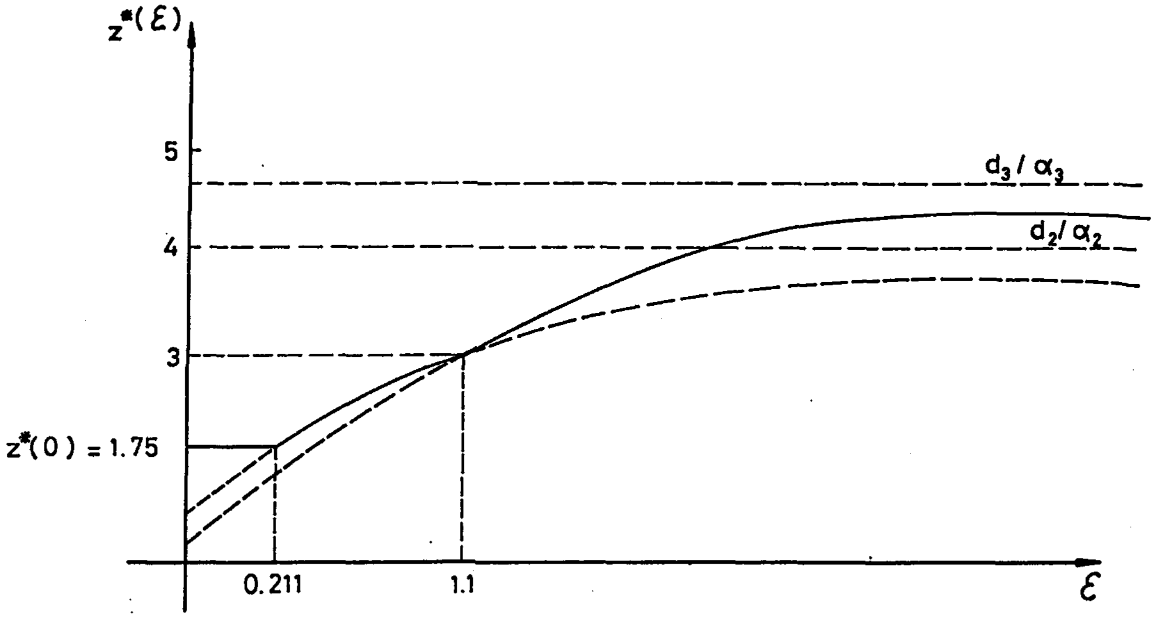


Fig.4

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