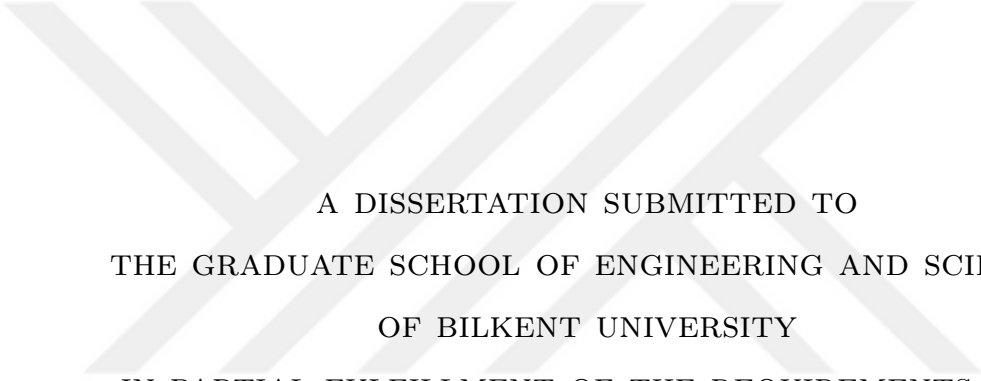


# ADAPTIVE CONTROL OF CYBERPHYSICAL HUMAN SYSTEMS



A DISSERTATION SUBMITTED TO  
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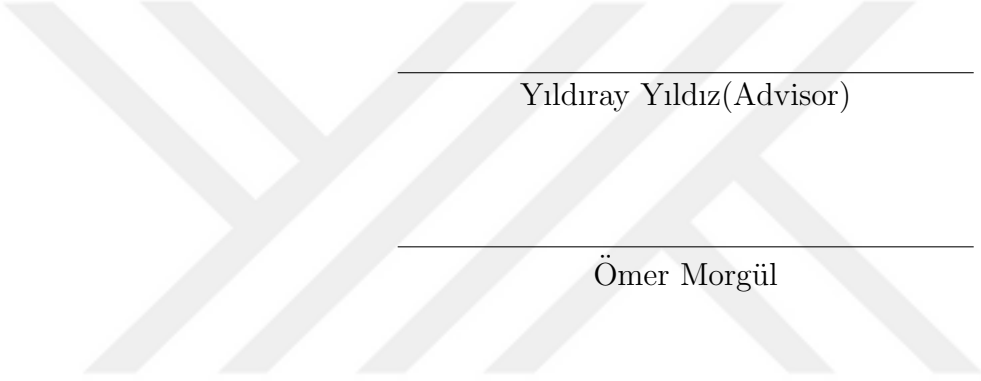
By  
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August 2021

Adaptive control of cyberphysical human systems

By Seyed Shahabaldin Tohidi

August 2021

We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.



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# ABSTRACT

## ADAPTIVE CONTROL OF CYBERPHYSICAL HUMAN SYSTEMS

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This dissertation focuses on the control of cyberphysical human systems in the presence of actuators' redundancy and constraints. A novel adaptive control technique is proposed to allocate control signals among redundant actuators in the presence of uncertainty and actuator saturation. The proposed method does not require any uncertainty identification or persistency of excitation assumption. The stability of the proposed method is guaranteed using Lyapunov stability analysis. In addition, a modified projection operator that can be implemented to the adaptive control allocation is proposed. This operator enables the allocator to handle both magnitude and rate limits of actuators. A novel sliding mode controller with time-varying sliding surface is designed to complement the adaptive allocator and guarantee stability and reference tracking in the presence of uncertainty and actuator saturation. This controller is robust to both adaptive control allocation error and external disturbance. Furthermore, an adaptive human model is proposed to mimic the human control response in the presence of uncertainty. The proposed structure is based on the model reference adaptive control, and the adaptive laws are obtained using the Lyapunov-Krasovskii stability criteria. To validate this model, an experimental setup is employed to collect data and a statistical analysis is conducted to measure the predictive power of the pilot model. Finally, the stability limits of a human-in-the-loop closed loop control system, where the plant to be controlled has redundant actuators with uncertain dynamics, are demonstrated. Various human models with and without time delays are investigated. Simulation results are provided to demonstrate the effectiveness of the proposed methods in each chapter.

*Keywords:* Adaptive control, cyberphysical human systems, control allocation, constrained systems, human-in-the-loop stability analysis.

## ÖZET

# SİBER FİZİKSEL İNSAN SİSTEMLERİNİN UYARLAMALI KONTROLÜ

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Bu tez, yedeklemeli ve kısıtlanmış eyleyicilere sahip siber-fiziksel insan sistemlerinin kontrolü üzerine odaklanmaktadır. Belirsizlik ve eyleyici doygunluğu varlığında kontrol sinyallerinin yedeklemeli eyleyiciler arasında paylaşılması amacıyla yenilikçi bir uyarlamalı kontrol tekniği önerilmektedir. Önerilen metod belirsizlik saptaması ya da sürekli uyarım gerektirmez. Verilen metodun kararlılığı Lyapunov kararlılık analizi kullanarak garanti altına alınmıştır. Buna ek olarak, uyarlamalı kontrol paylaşımına uygulanabilecek bir modifiyeli projeksiyon operatörü önerilmektedir. Bu operatör paylaştırıcının eyleyicilerin hem büyüklük hem de hız limitlerini idare etmesine olanak sağlar. Belirsizlik ve eyleyici doygunluğu varlığında uyarlamalı paylaştırıcıyı tamamlamak ve kararlılık ve referans takibini garanti altına almak için, zamana bağımlı kayan yüzeyleri olan yenilikçi bir kayan mod kontrolcü tasarlanmıştır. Bu kontrolcü hem uyarlamalı kontrol paylaşımı hatalarına hem de dış müdahalelere karşı gürbüzdür. Ayrıca, belirsizlik varlığında insan kontrol cevabını taklit etmek amacıyla uyarlamalı bir insan modeli önerilmiştir. Önerilen yapı, model referans uyarlamalı kontrole dayandırılmış olup uyarlama yasaları Lyapunov-Krasovskii kararlılık kriterleri kullanılarak elde edilmiştir. Bu modeli doğrulama amacıyla, veri toplamak için bir deney düzeneği kurulmuş ve pilot modelinin öngörü gücünü ölçmek için istatistiksel analiz yapılmıştır. Son olarak, belirsiz dinamiklere ve yedekleyici eyleyicilere sahip, ve insanın çevrim içi olduğu kapalı döngü kontrol sisteminin kararlılık limitleri gösterilmiştir. Bu sistemin kararlılığı gösterilirken zaman gecikmelerine sahip olan ve olmayan çeşitli insan modelleri göz önüne alınmıştır. Simülasyon sonuçları önerilen metotların etkinliğini göstermek için sunulmuştur.

*Anahtar sözcükler:* Uyarlamalı kontrol, siber-fiziksel insan sistemleri, kontrol paylaşımı, kısıtlı sistemler, döngüde-insan kararlılık analizi.

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# Chapter 1

## Introduction

This thesis introduces methodologies for the control of cyberphysical human systems in the presence of uncertainty, constraints and redundancy.

Control allocation, which is the process of distributing control signals among redundant actuators, improves fault tolerability and modularity of the overall control system while permitting to exploit actuator redundancy for improved maneuverability. Furthermore, control allocation is an appealing approach for the design of active fault-tolerant control systems [1–5]. However, the existing control allocation methods require fault estimation and persistency of excitation assumption. One of the contributions of this thesis is an adaptive control allocation method that can compensate actuator faults without fault estimation and persistency of excitation assumption requirements.

Actuator constraints such as magnitude and rate limits play a prominent role in advanced control systems. These limits induce nonlinear behavior which may lead to performance degradation, occurrence of limit cycles, multiple equilibria, and even instability [6, 7]. Projection algorithm is an appealing approach in robust adaptive control design. Restricting adaptive parameters while ensuring the stability of the closed loop system, simultaneously, is a prominent benefit of employing this algorithm in adaptive systems. It is noted that existing projection

algorithms ([8,9]) bound adaptive parameters' magnitudes and thus do not have a straightforward utility to handle actuator rate limits. The second contribution of this thesis is a projection algorithm that can be used in adaptive control allocation implementations, where actuators are both magnitude and rate limited.

Although control allocation methods enable modularity for the overall control system design, as they separate the generation of the control signal and its distribution, control allocation errors can be significant in transients and degrade the performance. Therefore, a controller must be designed for the system to be robust to the control allocation error, as well as external disturbances. The robustness characteristics of the sliding mode control motivated its use in combination with many control allocation implementations [10–12]. The third contribution of this thesis is a novel sliding mode controller with a time-varying sliding surface for constrained multi-input-multi-output systems. The proposed sliding surface contains the initial conditions of the states, and consequently guarantees the robustness during the entire response by omitting the reaching phase.

Humans' unique abilities such as adaptive behavior in dynamic environments, and social interaction and moral judgment capabilities, make them essential elements of many control loops. As stated in [13], “the purposes of proposing human models are to summarize behavioral data, to provide a basis for rationalization and understanding of pilot control actions, and, most important of all, to be used in conjunction with vehicle dynamics in forming predictions or in explaining the behavior of pilot-aircraft systems”. Adaptive models, such as the ones proposed in [14, 15] and [16], aim to replicate the adaptation capability of humans in uncertain and dynamics environments. In [14] and [15], adaptation rules are proposed based on expert knowledge. In this thesis, as the fourth contribution, we build upon the earlier successful pilot models and propose an adaptive human pilot model that modifies its behavior based on plant uncertainties. This model distinguishes itself from earlier adaptive models by having mathematically derived laws to achieve a cross-over-model-like behavior, instead of employing expert knowledge. This allows a rigorous stability proof, using the Lyapunov-Krasovskii stability criteria, of the overall closed loop system. To validate the model, a setup, including a joystick and a monitor, is used. The participant data

collected through this experimental setup is subjected to visual and statistical analyses to evaluate the accuracy of the proposed model.

Investigation of human in the loop dynamics help develop safe control mechanisms, and provide a better realization and understanding of human control actions and limitations [17–20]. The stability of a human-in-the-loop closed loop system in the presence of adaptive control allocation and sliding mode controller is investigated in this thesis. Various common human models are considered such as pure gain, lag filter, general transfer function without time delay, and the one with time delay. As the fifth contribution, criteria leading to the human-in-the-loop closed loop system stability has been provided.

This thesis is organized as follows. Section 2 introduces the adaptive control allocation for constrained systems. Section 3 presents the novel sliding mode controller with time-varying sliding surface applicable to constrained systems. A new projection operator is proposed in Section 4 to handle actuator magnitude and rate saturation. Stability of the closed loop system considering the presence of human is analyzed in Section 5. Finally, in Section 6, an innovative human model is proposed to mimic the human pilot behaviors in the presence of uncertainty.

# Chapter 2

## Adaptive control allocation for constrained systems

### 2.1 Introduction

Control allocation is the process of distributing control signals among redundant actuators which is employed in many engineering domains such as aerial vehicles [1, 21–29], marine vehicles [30–34], automobiles [35, 36], robots [37], and power systems [38, 39]. In particular, in aircraft and spacecraft applications, the flight control law determines forces and moments while control actuator settings are determined through control allocation. Control allocation improves fault tolerability and modularity of the overall control system while permitting to exploit actuator redundancy for improved maneuverability.

Control allocation methods can be categorized into the following three categories: Pseudo-inverse-based methods, optimization based methods and dynamic control allocation. Given a mapping between a control input  $v$  and the actuator input vector  $u$  defined as  $Bu = v$ , in pseudo inverse based control allocation [10, 40–42], the control input is distributed to the individual actuators by the pseudo inverse mapping, i.e.,  $u = B^+v$ , which is the minimum 2-norm solution

among infinitely many control allocation solutions. Such a method can be extended to account for actuator saturation [40–46]. In optimization based control allocation [47–52], control allocation is performed by minimizing the cost function  $\|Bu - v\| + J_0$ , where  $J_0$  represents a secondary objective such as minimizing actuator deflections. For a recent optimization based computationally efficient control allocation example see [53]. In dynamic control allocation [54–59], the control signals are distributed among actuators using a set of rules dictated by differential equations. A survey of control allocation methods can be found in [60].

Control allocation is an appealing approach also for the design of active fault-tolerant control systems [1–5]. For instance, it is used in [35] to improve the steering performance in faulty automotive vehicles. In another study [30], control allocation distributes propulsive forces of an autonomous underwater vehicle among redundant thrusters so that faults are accommodated. In [28], experimental results of employing control allocation for quadrotor helicopter are reported. In several applications, fault detection and isolation methods are employed in parallel with control allocation [1]. In [10], a sliding mode controller is coupled with a pseudo inverse based control allocation to obtain a fault tolerant controller wherein faults are assumed to be estimated. Similarly in [4], it is assumed that there exists a fault detection and isolation scheme which is able to estimate and identify stuck-in-place, hard-over, loss of effectiveness and floating actuator faults. In [61], an unknown input observer is applied to identify actuator and effector faults. A family of unknown input observers is proposed in [62] to detect and isolate faults in systems with redundant actuators. In [42] and [49], faults are estimated adaptively using recursive least squares, and an online dither generation method is proposed to guarantee the persistence of excitation.

In this thesis we consider the following problem: Allocate a control input vector  $v$  among redundant constrained actuators in the presence of actuator effectiveness uncertainty. To solve this problem, two subproblems need to be addressed:

**(i) Actuator uncertainty:** Determine  $u$  such that  $B\Lambda u + \bar{d} = v$ , where  $\Lambda$  represents uncertainty and  $\bar{d}$  represents an unknown bounded disturbance. In this thesis, we address this problem by proposing an adaptive control allocation

method which does not need uncertainty estimation unlike prior approaches.

**(ii) Actuator constraints:** Determine  $u$  such that  $B\Lambda u + \bar{d} = v$ , where the elements of the actuator signal vector,  $u$ , are constrained. One approach to prevent actuators from getting saturated is to determine a set for the control input vector elements,  $v_i$ ,  $i = 1, \dots, r$ , and confine them to this set [7, 63, 64]. However, when an adaptive approach is utilized to solve (i), restricting control signals in an attainable set may not prevent actuator saturation. Hence, in addition to restricting control signals to an attainable set, an element-wise non-symmetric projection algorithm is employed in this thesis to constrain adaptive parameters.

Other adaptive approaches to control allocation have been described in [55] and [56]. As compared to these references, the approach in this thesis eliminates the need for uncertainty estimation and persistence of excitation assumption. Furthermore, in the proposed approach, a closed loop reference model [65] is employed to achieve fast convergence without inducing undesired oscillations. Apart from these contributions, we also show that it is possible to employ the projection algorithm [66] in a stable manner even if the ideal adaptive parameters are not inside the projection boundaries. To the best of our knowledge, such a result is not available in the prior literature. Preliminary results were reported in our conference papers [57], [58] and [29].

This chapter is organized as follows. Section 2.2 introduces notations and preliminary results. Section 2.3 presents the uncertain over-actuated plant dynamics and the proposed model reference adaptive control allocation approach with a closed loop reference model. A discussion of actuator saturation and its effects on the control limits together with the projection algorithm are given in Section 2.4. The ADMIRE model is used in Section 2.5 to demonstrate the effectiveness of the proposed approach in the simulation environment. Finally, a summary is given in Section 2.6.

## 2.2 Notations and preliminaries

In this section, we collect several definitions and basic results which are exploited in the following sections. Throughout this chapter,  $\|\cdot\|$  refers to the Euclidean norm for vectors and induced 2-norm for matrices, and  $\|\cdot\|_F$  refers to the Frobenius norm.  $\lambda_{\min}(\cdot)$ ,  $\lambda_{\max}(\cdot)$  and  $\lambda_i(\cdot)$  refer to the minimum, maximum and  $i^{\text{th}}$  eigenvalue of a matrix, respectively. Also,  $\text{row}_i(\cdot)$  and  $\text{column}_i(\cdot)$  refer to the  $i^{\text{th}}$  row and column of a matrix, respectively, and  $\text{vec}(\cdot) : \mathbb{R}^{r \times m} \rightarrow \mathbb{R}^{rm}$  puts the elements of a matrix in a column vector.  $I_r$  is an identity matrix of dimension  $r \times r$  and  $\text{tr}(\cdot)$  refers to the trace operation.

The element-wise projection  $\text{Proj}(\theta_{v_{i,j}}, Y_{i,j}, f) : \mathbb{R} \times \mathbb{R} \times \mathcal{F} \rightarrow \mathbb{R}$  is defined as

$$\begin{aligned} & \text{Proj}(\theta_{v_{i,j}}, Y_{i,j}, f) \\ & \equiv \begin{cases} Y_{i,j} - Y_{i,j} f(\theta_{v_{i,j}}) & \text{if } f(\theta_{v_{i,j}}) > 0 \text{ \& } Y_{i,j} \left( \frac{df}{d\theta_{v_{i,j}}} \right) > 0 \\ Y_{i,j} & \text{otherwise,} \end{cases} \end{aligned} \quad (2.1)$$

where  $f(\cdot) \in \mathcal{F}(\mathbb{R} \rightarrow \mathbb{R})$  is a convex and continuously differentiable ( $C^1$ ) function defined as

$$f(\theta_{v_{i,j}}) = \frac{(\theta_{v_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})(\theta_{v_{i,j}} - \theta_{\max_{i,j}} + \zeta_{i,j})}{(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}}, \quad (2.2)$$

and where  $\zeta_{i,j}$  is the projection tolerance of the  $(i, j)^{\text{th}}$  element of  $\theta_v$  that should be chosen as  $0 < \zeta_{i,j} < 0.5(\theta_{\max_{i,j}} - \theta_{\min_{i,j}})$ .  $\theta_{\max_{i,j}}$  and  $\theta_{\min_{i,j}}$  are the upper and lower bound of the  $(i, j)^{\text{th}}$  element of  $\theta_v$ . These bounds also form the projection set,  $\Omega_{\text{proj}} = \{\theta_v : \theta_{v_{i,j}} \in [\theta_{\min_{i,j}}, \theta_{\max_{i,j}}]\}$  (see Figure 2.1). The projection algorithm (2.1) is element-wise and the proposed convex function in (2.2) considers the cases where  $\theta_{\min_{i,j}} \neq -\theta_{\max_{i,j}}$ . In the convex function (2.2),  $f(\theta_{v_{i,j}}) = 0$  when  $\theta_{v_{i,j}} = \theta_{\max_{i,j}} - \zeta_{i,j}$  or  $\theta_{v_{i,j}} = \theta_{\min_{i,j}} + \zeta_{i,j}$ , and  $f(\theta_{v_{i,j}}) = 1$  when  $\theta_{v_{i,j}} = \theta_{\max_{i,j}}$  or  $\theta_{v_{i,j}} = \theta_{\min_{i,j}}$ . A useful subset of  $\Omega_{\text{proj}}$  is defined as  $\hat{\Omega}_{\text{proj}} = \{\theta_v : \theta_{v_{i,j}} \in [\theta_{\min_{i,j}} + \zeta_{i,j}, \theta_{\max_{i,j}} - \zeta_{i,j}]\}$ . For  $\theta_v \in \Omega_{\text{proj}}$  and  $\theta_v^* \in \hat{\Omega}_{\text{proj}}$ , the following inequality holds [66]:

$$\text{tr}((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}(\theta_v, Y, f))) \leq 0, \quad (2.3)$$

where  $\theta_v \in \Omega_{\text{proj}} \subset \mathbb{R}^{r \times m}$ ,  $Y \in \mathbb{R}^{r \times m}$ .

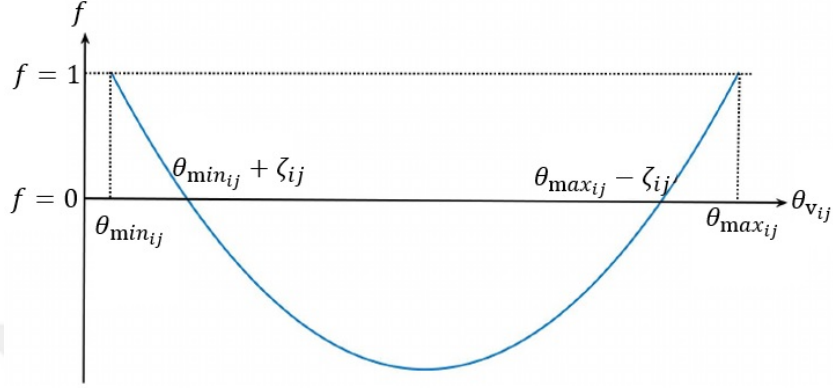


Figure 2.1: Convex function  $f(\theta_{v,i,j})$ .

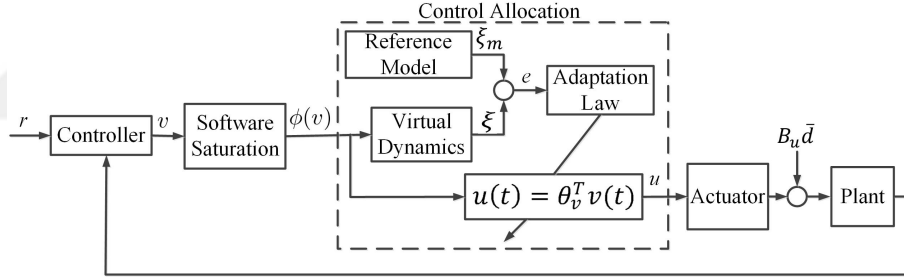


Figure 2.2: Block diagram of the closed loop system with the proposed control allocation method.

## 2.3 Control Allocation Problem

The closed loop system studied in this chapter is presented in Figure 2.2. The plant dynamics are given by

$$\dot{x}(t) = Ax(t) + B_u(\Lambda u(t) + d_u(t)), \quad (2.4)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$  is the actuator input vector, and where  $A \in \mathbb{R}^{n \times n}$  and  $B_u \in \mathbb{R}^{n \times m}$  are known matrices and  $d_u \in \mathbb{R}^m$  is a bounded disturbance input. The matrix  $\Lambda \in \mathbb{R}^{m \times m}$  is assumed to be diagonal, with positive elements representing actuator effectiveness uncertainty. It is assumed that the pair  $(A, B_u \Lambda)$  is controllable.

Due to actuator redundancy, the input matrix is rank deficient, that is  $\text{Rank}(B_u) = r < m$ . Consequently,  $B_u$  can be written as  $B_u = B_v B$ , where  $B_v \in \mathbb{R}^{n \times r}$  is a full column rank matrix, and  $B \in \mathbb{R}^{r \times m}$  is a full row rank matrix. This decomposition of  $B_u$  helps exploit the actuator redundancy using control allocation. Employing this decomposition, (2.4) can be rewritten as

$$\dot{x}(t) = Ax(t) + B_v(B\Lambda u(t) + \bar{d}(t)), \quad (2.5)$$

where  $\bar{d}(t) = Bd_u(t)$  is assumed to have an upper bound. The actuator range limits are  $u_i \in [-u_{\max_i}, u_{\max_i}]$ ,  $u_{\max_i} > 0, i = 1, \dots, m$ . The actuator saturation will be addressed both in the design of our adaptive control allocation and by passing the output of the controller,  $v$ , through a software saturation function,  $\phi(\cdot) : \mathbb{R}^r \rightarrow \mathbb{R}^r$  in Figure 2.2, the requirement for latter will be discussed in Section 2.4. The adaptive control allocation task is to achieve

$$B\Lambda u(t) + \bar{d}(t) = \phi(v(t)), \quad (2.6)$$

which would lead to

$$\dot{x}(t) = Ax(t) + B_v \phi(v(t)). \quad (2.7)$$

**Remark 1.** *The focus of this chapter is not the adaptive control of the plant (2.4), for which various solutions exist [67, 68] but the control allocation problem of achieving (2.6) when  $\Lambda$  is unknown and there are actuator limits. Our objective is to characterize the mismatch between  $\phi(v(t))$  and  $B\Lambda u(t) + \bar{d}(t)$  due to transients in control allocation dynamics so that it can be accounted for in the design of the controller which generates the signal  $v$ . In typical applications, such as flight control, the controller may not be adaptive. Hence, it is desirable to handle actuator uncertainty within the control allocation layer.*

**Remark 2.** *The treatment of nonlinear time-varying uncertainties represents a direction for future work.*

In order to achieve (2.6), the following virtual dynamics,

$$\dot{\xi}(t) = A_m \xi(t) + B\Lambda u(t) + \bar{d}(t) - \phi(v(t)), \quad (2.8)$$

where  $A_m \in \mathbb{R}^{r \times r}$  is a stable matrix, and the reference model

$$\dot{\xi}_m(t) = A_m \xi_m(t), \quad (2.9)$$

is constructed. Defining the actuator input as a mapping from  $v$  to  $u$ ,

$$u(t) = \theta_v^T(t)\phi(v(t)), \quad (2.10)$$

where  $\theta_v \in \mathbb{R}^{r \times m}$  represents the adaptive parameter matrix to be determined, and substituting (2.10) into (2.8), we obtain

$$\dot{\xi}(t) = A_m \xi(t) + (B\Lambda\theta_v^T(t) - I_r)\phi(v(t)) + \bar{d}(t). \quad (2.11)$$

Let  $\theta_v^* \in \mathbb{R}^{r \times m}$  be the ideal  $\theta_v$  matrix, such that

$$B\Lambda\theta_v^{*T} = I_r, \quad (2.12)$$

and note that under our assumptions such  $\theta_v^*$  always exists. Defining  $e = \xi - \xi_m$  and subtracting (2.9) from (2.11), it follows that

$$\dot{e}(t) = A_m e(t) + B\Lambda\tilde{\theta}_v^T(t)\phi(v(t)) + \bar{d}(t), \quad (2.13)$$

where  $\tilde{\theta}_v(t) = \theta_v(t) - \theta_v^*$ .

**Theorem 1.** *Consider (2.8) and (2.9), and assume that there exists a positive scalar  $D$  such that  $\|\bar{d}(t)\| \leq D$  for all  $t \geq 0$ . Suppose that the adaptive parameter matrix is updated using the following adaptive law,*

$$\dot{\theta}_v(t) = \Gamma_\theta \text{Proj}(\theta_v(t), -\phi(v(t))e^T(t)PB, f), \quad (2.14)$$

where the symmetric positive definite matrix  $P$  satisfies  $A_m^T P + P A_m = -Q$ ,  $Q$  is a symmetric positive definite matrix, the projection operator “Proj” is defined in (2.1), with convex function  $f \in C^1$  in (2.2), and  $\Gamma_\theta = \gamma_\theta I_r$ ,  $\gamma_\theta > 0$ . Then given any initial condition  $e(0) \in \mathbb{R}^r$ ,  $\theta_v(0) \in \Omega_{\text{proj}}$ , and  $\theta_v^* \in \hat{\Omega}_{\text{proj}}$ ,  $e(t)$  and  $\tilde{\theta}_v(t)$  remain uniformly bounded for all  $t \geq 0$  and their trajectories converge exponentially to the set

$$E_1 = \{(e, \tilde{\theta}_v) : \|e\|^2 \leq (\frac{s\tilde{\theta}_{\max}^2}{\gamma_\theta \lambda_{\min}(Q)} + \frac{2\chi^4 D^2 \|Q\|^2}{\sigma^2 \lambda_{\min}(Q)^2}) \frac{4s\chi^2 \|Q\|}{\sigma \lambda_{\min}(Q)}, \|\tilde{\theta}_v\| \leq \tilde{\theta}_{\max}\}, \quad (2.15)$$

where  $s = -\min_i(\lambda_i(A_m + A_m^T)/2)$ ,  $\sigma = -\max_i(\text{Real}(\lambda_i(A_m)))$ ,  $\chi = \frac{3}{2}(1 + 4\frac{a}{\sigma})^{(r-1)}$ ,  $a = \|A_m\|$  and  $\|\tilde{\theta}_v(t)\|_F \leq \tilde{\theta}_{\max} \equiv \sqrt{\sum_{i,j}(\theta_{\max i,j} - \theta_{\min i,j} - \zeta_{i,j})^2}$ . In addition, if  $\bar{d}(t) = 0$  for  $t \geq t'$  for some  $t' \geq 0$  and  $\phi(v(t))$  is uniformly continuous as a function of  $t \in [t', +\infty)$ , then  $B\Lambda u(t) \rightarrow \phi(v(t))$  as  $t \rightarrow \infty$ , i.e., (2.6) is achieved asymptotically.

*Proof.* See Appendix A.1. □

Theorem 1 implies that  $\theta_v$  and  $e$  are bounded. Considering that  $\phi(v)$  is bounded, (2.10) implies that  $u$  is bounded. Since  $A_m$  is Hurwitz, the variable  $\xi$ , whose dynamics is given in (2.8), is also bounded. Therefore, all the signals in the adaptive control allocation system are bounded.

**Remark 3.** *To realize (2.8), the signal  $B\Lambda u + \bar{d}$ , is required. In motion control applications, this signal corresponds to the net forces and moments (see [69], for example, for an aerospace application), that can be obtained via an inertial measurement unit (IMU). Examples of measuring/estimating this signal, without introducing delay or noise amplification, and employing it in real applications can be found in [70, 71].*

**Remark 4.** *A sufficient but not necessary condition for the uniform continuity of  $\phi(v(t))$  is that its time rate of change is bounded, which can be ensured, e.g., by extending the software saturation function to include the rate limiting. (see Figure 2.2).*

**Remark 5.** *The proposed control allocation method uses the signal  $\phi(v(t))$ , which is the output of the software saturation. The saturation limits are determined based on the attainable set for  $v$ , the details of which are given in Section 2.4. The control allocator then determines the actuator signals,  $u_i, i = 1, 2, \dots, m$ , that do not saturate the actuators. Note that putting a bound on  $v$  using soft saturation does not necessarily imply that the actuators will not be saturated by  $u_i$ . To prevent actuator saturation, the boundaries of the projection operator in (2.14) must be selected carefully, as further detailed in Section 2.4.*

**Remark 6.** *In the case the assumption  $d(t) = 0$ , for  $t \geq t'$  for some  $t' \geq 0$  is not made, a bound similar to  $E_1$  can be derived for the control allocation error,  $B\Lambda u + \bar{d} - \phi(v)$ . However, note that even though only a boundedness result can be given for the control allocation error, the tracking error for the overall closed loop system can still be forced to converge to zero by an appropriately designed controller (see Remark 11).*

**Remark 7.**  $\theta_v^* \in \mathbb{R}^{r \times m}$  is the ideal parameter matrix that should satisfy (2.12). Since  $\Lambda$  is unknown,  $\theta_v^*$  is also unknown. However, although the diagonal matrix  $\Lambda$  is unknown, its elements are in the interval  $(0, 1]$ . Thus, using (2.12), the set  $\Omega^*$  that contains all possible values of  $\theta_v^*$  can be defined.

Conventionally, it is assumed that the projection boundaries,  $\theta_{\min_{i,j}}$  and  $\theta_{\max_{i,j}}$ , are chosen such that  $\theta_v^* \in \Omega^* \subset \hat{\Omega}_{\text{proj}} \subset \Omega_{\text{proj}}$ . However, such a choice may not be consistent with the actuator limits as further discussed in Section 2.4.

**Remark 8.** Let  $\theta_v^* \in \Omega^*$ ,  $\Omega^* \not\subset \hat{\Omega}_{\text{proj}}$  and consider the projection algorithm (2.1) with convex function (2.2). Then,  $\tilde{\theta}_{\max}$ , defined in Theorem 1, should be redefined as

$$\tilde{\theta}_{\text{MAX}} \equiv \sqrt{\sum_{i,j} (\max(|\theta_{\max_{i,j}}^* - \theta_{\min_{i,j}} - \zeta_{i,j}|, |\theta_{\min_{i,j}}^* - \theta_{\max_{i,j}} + \zeta_{i,j}|))^2}. \quad (2.16)$$

We note that to differentiate two different cases (the ideal parameter  $\theta_v^*$  being inside or outside the projection bounds), the maximum value of adaptive parameter deviation from its ideal value is designated by  $\tilde{\theta}_{\max}$  for the former and  $\tilde{\theta}_{\text{MAX}}$  for the latter case. Below, we provide a lemma and a theorem, regarding the stability of the control allocation for the latter case, i.e. when  $\theta_v^* \notin \hat{\Omega}_{\text{proj}}$ .

**Lemma 1.** For  $\theta_v^* \notin \hat{\Omega}_{\text{proj}}$ ,  $\theta_{v_{i,j}} \in \mathbb{R}^{r \times m}$ ,  $Y \in \mathbb{R}^{r \times m}$  with  $r \leq m$  and the projection algorithm (2.1)-(2.2), the following inequality holds:

$$\text{tr}((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}(\theta_v, Y))) \leq \sqrt{r} \tilde{\theta}_{\text{MAX}} \|Y\|. \quad (2.17)$$

*Proof.* See Appendix A.2. □

**Theorem 2.** Consider (2.8), (2.9), (2.10), and (2.14). Assume that there exist positive scalars  $M$  and  $D$  such that  $\|\phi(v(t))\| \leq M$  and  $\|\bar{d}(t)\| \leq D$  for all  $t \geq 0$ . Then, for any initial condition  $e(0) \in \mathbb{R}^r$ ,  $\theta_v(0) \in \Omega_{\text{proj}}$ , and  $\theta_v^* \notin \hat{\Omega}_{\text{proj}}$ ,  $e(t)$  and  $\tilde{\theta}_v(t)$  are uniformly bounded for all  $t \geq 0$  and their trajectories converge

exponentially to the closed and bounded set

$$\begin{aligned} \hat{E}_1 = \{ (e, \tilde{\theta}_v) : \|\tilde{\theta}_v\| \leq \tilde{\theta}_{MAX}, \|e\|^2 \leq & (\frac{s\tilde{\theta}_{MAX}^2}{\gamma_\theta \lambda_{min}(Q)} \\ & + \frac{4\chi^4 \|Q\|^2 (D^2 + r\tilde{\theta}_{MAX}^2 \|B\|^2 M^2)}{\sigma^2 \lambda_{min}(Q)^2}) \frac{4s\chi^2 \|Q\|}{\sigma \lambda_{min}(Q)} \}, \end{aligned} \quad (2.18)$$

where  $s = -\min_i(\lambda_i(A_m + A_m^T)/2)$ ,  $\sigma = -\max_i(\text{Real}(\lambda_i(A_m)))$ ,  $\chi = \frac{3}{2}(1 + 4\frac{a}{\sigma})^{(r-1)}$ ,  $a = \|A_m\|$  and  $\tilde{\theta}_{MAX}$  is defined in (2.16).

*Proof.* See Appendix A.3. □

To obtain fast convergence without introducing excessive oscillations, (2.9) is modified to obtain the following closed loop reference model [65].

$$\dot{\xi}_m(t) = A_m \xi_m(t) - L(\xi(t) - \xi_m(t)), \quad (2.19)$$

where  $A_m \in \mathbb{R}^{r \times r}$  is Hurwitz,  $L = -\ell I_r$ ,  $\ell > 0$ . Defining  $\bar{A}_m = A_m + L$ , and subtracting (2.19) from (2.11), we get

$$\dot{e}(t) = \bar{A}_m e(t) + B \Lambda \tilde{\theta}_v^T(t) \phi(v(t)) + \bar{d}(t). \quad (2.20)$$

We assume that the matrix  $\bar{A}_m$  is Hurwitz through an appropriate selection of  $L$ .

**Theorem 3.** Consider (2.8), the reference model (2.19), the actuator signal (2.10), and the adaptive law

$$\dot{\theta}_v(t) = \Gamma_\theta \text{Proj}(\theta_v(t), -\phi(v(t))e(t)^T \bar{P}B, f), \quad (2.21)$$

where the symmetric positive definite matrix  $\bar{P}$  satisfies  $\bar{A}_m^T \bar{P} + \bar{P} \bar{A}_m = -\bar{Q}$ ,  $\bar{Q}$  is a symmetric positive definite matrix,  $\Gamma_\theta^{-1} = (1/\gamma_\theta)I_r$ ,  $\gamma_\theta > 0$  and the projection is defined by (2.1) and (2.2). Assume that there exist positive scalars  $M$  and  $D$  such that  $\|\phi(v(t))\| \leq M$  and  $\|\bar{d}(t)\| \leq D$  for all  $t \geq 0$ . Then, for any initial condition  $e(0) \in \mathbb{R}^r$ , and  $\theta_v(0) \in \Omega_{proj}$ ,  $e(t)$  and  $\tilde{\theta}(t)$  are uniformly bounded for all  $t \geq 0$  and their trajectories converge exponentially to a closed and bounded set

$$\begin{aligned} E_2 = \{ (e, \tilde{\theta}_v) : \|e\|^2 \leq & (\frac{(s + \ell)\tilde{\theta}_{max}^2}{\gamma_\theta \lambda_{min}(Q)} + \frac{2\bar{\chi}^4 D^2 \|Q\|^2}{\bar{\sigma}^2 \lambda_{min}(Q)^2}) \\ & \times \frac{4(s + \ell)\bar{\chi}^2 \|Q\|}{\bar{\sigma} \lambda_{min}(Q)}, \|\tilde{\theta}_v\| \leq \tilde{\theta}_{max} \}, \end{aligned}$$

if  $\theta_{v_{i,j}}^* \in \hat{\Omega}_{proj}$ , or to closed and bounded set

$$\begin{aligned} \hat{E}_2 = \{ (e, \tilde{\theta}_v) : \|\tilde{\theta}_v\| \leq \tilde{\theta}_{MAX}, \|e\|^2 \leq & \left( \frac{(s + \ell)\tilde{\theta}_{MAX}^2}{\gamma_{\theta}\lambda_{min}(Q)} \right. \\ & \left. + \frac{4\bar{\chi}^4\|Q\|^2(D^2 + r\tilde{\theta}_{MAX}^2\|B\|^2M^2)}{\bar{\sigma}^2\lambda_{min}(Q)^2} \right) \frac{4(s + \ell)\bar{\chi}^2\|Q\|}{\bar{\sigma}\lambda_{min}(Q)} \}, \end{aligned}$$

if  $\theta_{v_{i,j}}^* \notin \hat{\Omega}_{proj}$ , where  $\bar{\sigma} = -\max_i(\text{Real}(\lambda_i(\bar{A}_m)))$ ,  $s = -\min_i(\lambda_i(A_m + A_m^T)/2)$ ,  $\bar{a} = \|\bar{A}_m\|$  and  $\bar{\chi} = \frac{3}{2}(1 + 4\frac{\bar{a}}{\bar{\sigma}})^{(r-1)}$ .

*Proof.* The proof procedure is similar to the one for Theorem 1 and 2, and is omitted for brevity.  $\square$

## 2.4 Determination of the projection set

In the previous section, the control allocator was designed based on the projection operator. In this section, the determination of the projection set,  $\Omega_{proj}$ , which defines the bounds on adaptive control parameters, is detailed. The projection set is determined to satisfy two requirements: 1) The actuator command signals,  $u_i, i = 1, \dots, m$ , should not saturate the actuators and 2) a specific condition (to be introduced shortly) which is necessary for the controller and control allocation integration, should be satisfied. The design procedure to achieve these goals is composed of three main steps. First, an attainable set for control signal vector  $v$  is found for  $\Lambda = I_m$ , which is used to determine the software saturation function  $\phi(\cdot)$ . Secondly, a compact set for the adaptive parameter  $\theta_v$  is calculated, to satisfy  $-u_{max} \leq \theta_v^T \phi(v) \leq u_{max}$ . Thirdly, the projection set that satisfies the specific requirement necessary for the controller and control allocation integration is determined.

**Step 1:** Realizable values of control signals are found.

The actuator constraints are known:  $u(t) \in \Omega_u$ , where  $\Omega_u = \{[u_1, \dots, u_m]^T : -u_{max_i} \leq u_i \leq u_{max_i}, i = 1, \dots, m\}$ ,  $u_{max_i} > 0, i = 1, \dots, m$ . Using  $\Omega_u$ , the

set  $\Omega_v$ , defining all realizable values of the control input  $v$ , can be obtained as  $\Omega_v = \{v : v = Bu, u \in \Omega_u, B^\dagger v \in \Omega_u\}$ , where  $(\cdot)^\dagger$  refers to the pseudo inverse of a non-square matrix. Furthermore, there exist  $M_i$ ,  $i = 1, \dots, r$ , such that  $\hat{\Omega}_v \equiv \{v : v_i \in [-M_i, M_i], i = 1, \dots, r\} \subset \Omega_v$ . The set  $\hat{\Omega}_v$  can be used to define the constraints which are enforced using a software saturation function, whose output is  $\phi(v(t))$ . See Figure 2.2 and Remark 5.

**Step 2:** The boundaries for the adaptive parameters to ensure that the actuator signal vector  $u$  does not saturate the actuators are calculated.

A subset of the attainable set for the control signals,  $\hat{\Omega}_v$ , was obtained in Step 1. The actuator limits are known. With this information, the set  $\Omega_\theta = \{\theta_v : -u_{max} \leq \theta_v^T \phi(v) \leq u_{max}, \phi(v) \in \hat{\Omega}_v\}$  can be obtained. Let  $\theta_I^* = B^\dagger$  which corresponds to an adaptive parameter value in the absence of uncertainty, i.e.,  $\Lambda = I$ . Since  $\phi(v) \in \hat{\Omega}_v \subset \Omega_v$ , we have  $B^\dagger \phi(v) = \theta_I^{*T} \phi(v) \in \Omega_u$ , for all  $\phi(v) \in \hat{\Omega}_v$ . Therefore,  $\theta_I^* \in \Omega_\theta$ , which is required to subsequently solve the optimization problem (2.25) (see Remark 9).

**Step 3:** A subset of  $\Omega_\theta$ , which satisfies a necessary condition for a stabilizing controller, is obtained. This subset of  $\Omega_\theta$  also determines the ultimate projection boundaries, and is denoted by  $\Omega_{proj}$ .

The plant dynamics (2.5), can be rewritten, by using (2.10), (2.12) and defining  $\tilde{\theta}_v = \theta_v - \theta_v^*$ , as

$$\begin{aligned} \dot{x} &= Ax + B_v(B\Lambda u + \bar{d}) = Ax + B_v(B\Lambda \theta_v^T \phi(v) + \bar{d}) \\ &= Ax + B_v(I + B\Lambda \tilde{\theta}_v^T) \phi(v) + B_v \bar{d}, \end{aligned} \quad (2.22)$$

where  $\phi(v) \in \hat{\Omega}_v$  and  $\bar{d}$  is a bounded disturbance. Note that the projection algorithm guarantees that the actuators do not saturate, and the closed loop system does not see any residual disturbance due to saturation of the actuators. Defining  $\Delta B(t) \equiv B\Lambda \tilde{\theta}_v^T(t)$ , and substituting in (2.22), it follows that

$$\dot{x}(t) = Ax(t) + B_v(\phi(v(t)) + d(t)), \quad (2.23)$$

where  $d(t) = \Delta B(t)\phi(v(t)) + \bar{d}(t) \in \mathbb{R}^r$ .

To be able to design a stabilizing controller, which will produce the control input  $v$ , one must make sure that each element of the disturbance vector  $d = [d_1, \dots, d_r]^T$  in (2.23), is smaller in absolute value than the upper bound of the corresponding element of the saturated control input,  $\phi(v)$ , that is  $|d_i| < M_i$ ,  $i = 1, \dots, r$ . Since  $d_i = \text{row}_i(\Delta B)v + \bar{d}_i$ ,  $i = 1, \dots, r$ , where  $\bar{d}_i$  is the  $i$ th element of  $\bar{d}$ , satisfying the following condition ensures that  $|d_i| < M_i$ ,  $i = 1, \dots, r$ :

$$M_i - \|\text{row}_i(\Delta B)\| M_{\max} > |\bar{d}_i|, \quad i = 1, \dots, r, \quad (2.24)$$

where  $M_{\max} = \max_i M_i$ . A necessary condition for satisfying the inequality (2.24) is  $\|\text{row}_i(\Delta B)\| = \|\text{row}_i(B\Lambda\tilde{\theta}_v^T)\| < M_i/M_{\max}$  for all  $i = 1, \dots, r$ . (Sufficient conditions required to satisfy (2.24) is discussed later in Remark 10). Thus, the elements of the matrix  $\theta_v$  should be properly bounded in order to satisfy the necessary condition  $\|\text{row}_i(B\Lambda\tilde{\theta}_v^T)\| < M_i/M_{\max}$  for all  $i = 1, \dots, r$ , and for all  $\Lambda \in \hat{\Omega}_\Lambda$ , where  $\hat{\Omega}_\Lambda \subset \Omega_\Lambda$ , and  $\Omega_\Lambda$  is the set of all  $m \times m$  diagonal matrices with elements in  $(0, 1]$ . Thus  $\hat{\Omega}_\Lambda$  has diagonal elements  $\lambda_i \in (\gamma, 1]$ , where  $\gamma$  is precisely defined later in Theorem 4.

Since  $B\Lambda\tilde{\theta}_v^T = B\Lambda(\theta_v^T - \theta_v^{*T}) = B\Lambda\theta_v^T - I_r$ , the solution,  $R$ , of the optimization problem,

$$R^2 = \min_{\theta_v} (\text{vec}(\theta_v - \theta_I^*)^T \text{vec}(\theta_v - \theta_I^*)) \quad (2.25)$$

$$s.t. \quad \|\text{row}_i(B\Lambda\theta_v^T - I_r)\|^2 = \frac{M_i^2}{M_{\max}^2} - \epsilon, \quad i = 1, \dots, r,$$

$$\Lambda \in \hat{\Omega}_\Lambda, \quad \theta_v \in \Omega_\theta,$$

which is solved offline, is the minimum distance from the  $\text{vec}(\theta_I^*)$  to the boundary of the set

$$\begin{aligned} \Psi &= \{\theta_v : \|\text{row}_i(B\Lambda\theta_v^T - I_r)\|^2 \leq \frac{M_i^2}{M_{\max}^2} - \epsilon, \\ &\quad \Lambda \in \hat{\Omega}_\Lambda, \theta_v \in \Omega_\theta, i = 1, \dots, r\}, \end{aligned} \quad (2.26)$$

where  $\epsilon$  is a small positive constant used to have a closed set, since typical numerical optimizers optimize only over a closed set. It is noted that the corresponding  $\theta_{\max_{i,j}}$  and  $\theta_{\min_{i,j}}$  are not unique, and different boundaries can be found by defining different cost functions in (2.25). With the calculated  $\theta_{\max_{i,j}}$  and  $\theta_{\min_{i,j}}$ , the

projection set is obtained as,

$$\Omega_{\text{proj}} = \{\theta_v : \theta_{i,j} \in [\theta_{\min_{i,j}}, \theta_{\max_{i,j}}]\}. \quad (2.27)$$

**Remark 9.** For all elements of  $\hat{\Omega}_\Lambda$ , the optimization problem (2.25) finds the largest neighborhood of  $\theta_I^*$  in  $\Omega_\theta$  that satisfies  $\|\text{row}_i(\Delta B)\| < M_i/M_{\max}$ ,  $i = 1, \dots, r$ . This neighborhood is an  $n$ -sphere, with the center at  $\text{vec}(\theta_I^*)$  and with the radius  $R$ .

To show that the optimization problem (2.25) is feasible, it should be proven that the set  $E$  always includes  $\text{vec}(\theta_I^*)$ .

**Theorem 4.** The set  $\Upsilon = \{\text{vec}(\theta_v) : \|\text{row}_i(B\Lambda\theta_v^T - I_r)\|^2 \leq \frac{M_i^2}{M_{\max}^2} - \epsilon, \Lambda \in \hat{\Omega}_\Lambda \subset \Omega_\Lambda, i = 1, \dots, r\} \cap \text{vec}(\theta_I^*) \neq \emptyset$  when  $\lambda_{\min}(\Lambda) \geq \gamma \equiv \max_i(1 - \sqrt{\gamma_{M_i}/\gamma_{B_i}})$ , where  $\gamma_{B_i} \equiv \|\text{row}_i(B)\| \|B^T(BB^T)^{-1}\|$  and  $\gamma_{M_i} \equiv \frac{M_i^2}{M_{\max}^2} - \epsilon$ .

*Proof.* See Appendix A.4. □

Based on the definition of  $\gamma_{B_i}$ ,  $\gamma_{M_i}$  and  $\gamma$  given in Theorem 4,  $\hat{\Omega}_\Lambda$  is defined as

$$\hat{\Omega}_\Lambda = \{\Lambda : \Lambda \in \mathbb{D}^{m \times m}, \text{diag}_i(\Lambda) \in (\gamma, 1], i = 1, \dots, m\}, \quad (2.28)$$

where  $\mathbb{D}$  denotes the set of real diagonal matrices, and  $\text{diag}_i(\cdot) : \mathbb{R}^{m \times m} \rightarrow \mathbb{R}$  provides the  $i^{\text{th}}$  diagonal element of square matrices.

**Remark 10.** Using  $\hat{\Omega}_\Lambda$ , defined in (2.28), and  $\Omega_{\text{proj}}$ , defined in (2.27), the upper bound on  $\|\text{row}_i(\Delta B)\| = \|\text{row}_i(B\Lambda\tilde{\theta}_v^T)\|$  can be found, which we denote as  $\rho_i$ ,  $i = 1, \dots, r$ . Therefore, recalling that  $M_i$  is defined as the upper bound on the absolute value of  $i^{\text{th}}$  control signal  $v_i$ , and considering (2.24),  $|\bar{d}_i|$  should be smaller than  $M_i - \rho_i M_{\max}$ , i.e.  $|\bar{d}_i| < M_i - \rho_i M_{\max}$ . Note that  $\rho_i < \frac{M_i}{M_{\max}}$  is guaranteed by the solution of (2.25). Therefore, the condition  $|d_i| < M_i$  is satisfied.

**Remark 11.** A step by step procedure for the control allocation design is provided in Appendix A.5. Note that in the presence of the control allocator, the system dynamics takes the form of (2.23). The controller, which will produce the

control signal  $v$ , needs to be designed in the presence of the saturation nonlinearity represented by  $\phi(\cdot)$  and the bounded disturbance  $d$ . Tracking controllers for (23) can be designed using, for instance, the methods presented in [63]. See also [57, 58] which illustrate the process of integrating an initial version of our control allocation scheme and a controller. We omit details for lack of space.

## 2.5 Simulation results

The Aerodata Model in Research Environment (ADMIRE), which represents the dynamics of an over-actuated aircraft model, is used to demonstrate the effectiveness of the adaptive control allocation in the presence of uncertainty and actuator constraints. The linearized ADMIRE model [48] is given below:

$$\begin{aligned}
\dot{x} &= Ax + B_u u = Ax + B_v v, \\
v &= Bu, \quad B_u = B_v B, \quad B_v = [0_{3 \times 2} \ I_{3 \times 3}]^T, \\
x &= [\alpha \ \beta \ p \ q \ r]^T, \\
y &= [p \ q \ r]^T, \\
u &= [u_c \ u_{re} \ u_{le} \ u_r]^T,
\end{aligned} \tag{2.29}$$

where  $\alpha$ ,  $\beta$ ,  $p$ ,  $q$  and  $r$  are the angle of attack, sideslip angle, roll rate, pitch rate and yaw rate, respectively. The vector  $u$  includes the commanded control surfaces' deflection. The control surfaces  $u_c$ ,  $u_{re}$ ,  $u_{le}$  and  $u_r$  are the canard wings, right and left elevons and the rudder, respectively. The magnitude and rate limits of the commanded control surfaces are given as  $u_c \in [-55, 25] \times \frac{\pi}{180}(\text{rad})$ ,  $u_{re}, u_{le}, u_r \in [-30, 30] \times \frac{\pi}{180}(\text{rad})$  and  $\dot{u}_c, \dot{u}_{re}, \dot{u}_{le}, \dot{u}_r \in [-40, 40] \times \frac{\pi}{180}(\text{rad/sec})$ . The state and control matrices which are provided by [50], are given as

$$A = \begin{bmatrix} -0.5432 & 0.0137 & 0 & 0.9778 & 0 \\ 0 & -0.1179 & 0.2215 & 0 & -0.9661 \\ 0 & -10.5123 & -0.9967 & 0 & 0.6176 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2127 \end{bmatrix},$$

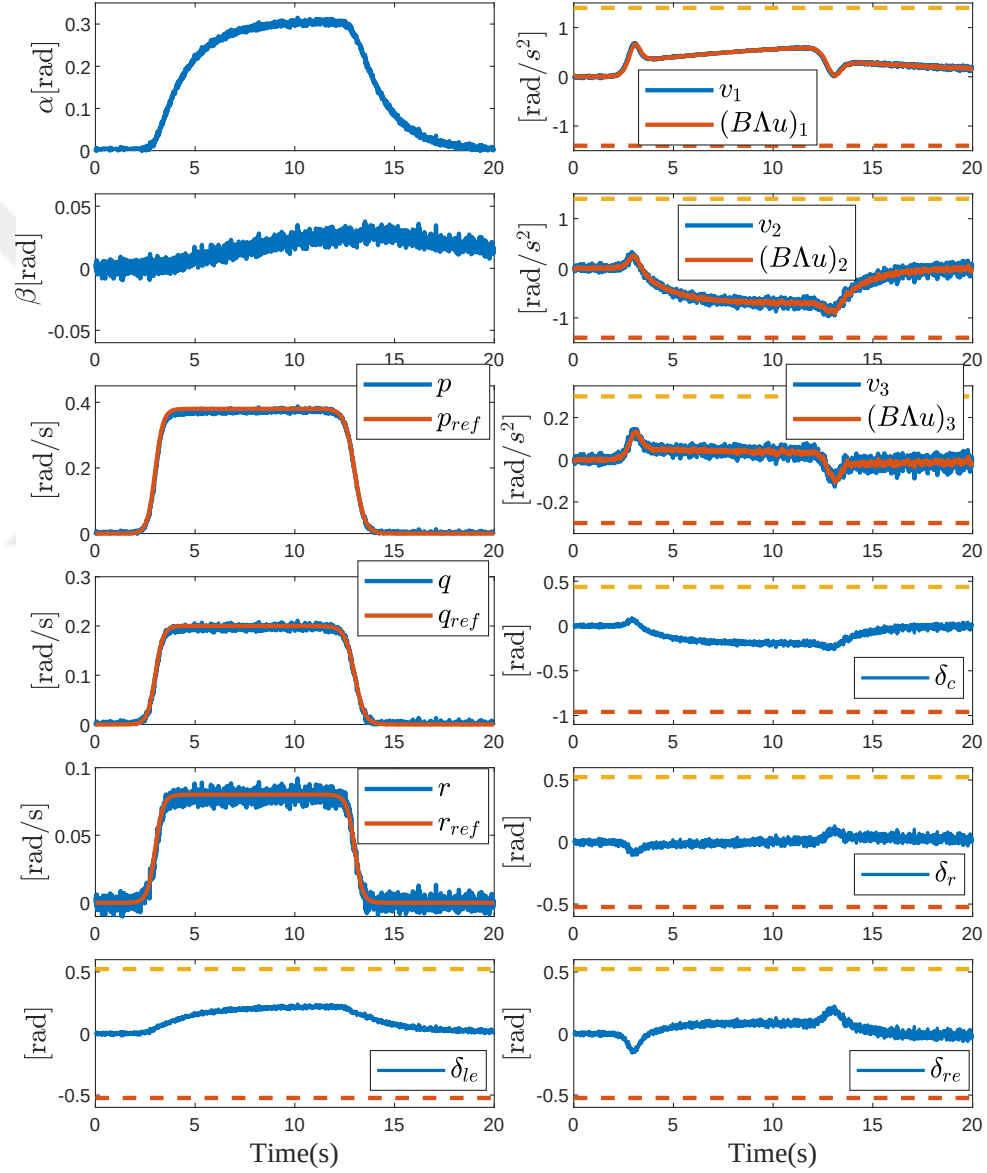


Figure 2.3: System states, control signals and actuators' deflections using conventional control allocation when actuator loss of effectiveness is  $\Lambda_1 = I$ .

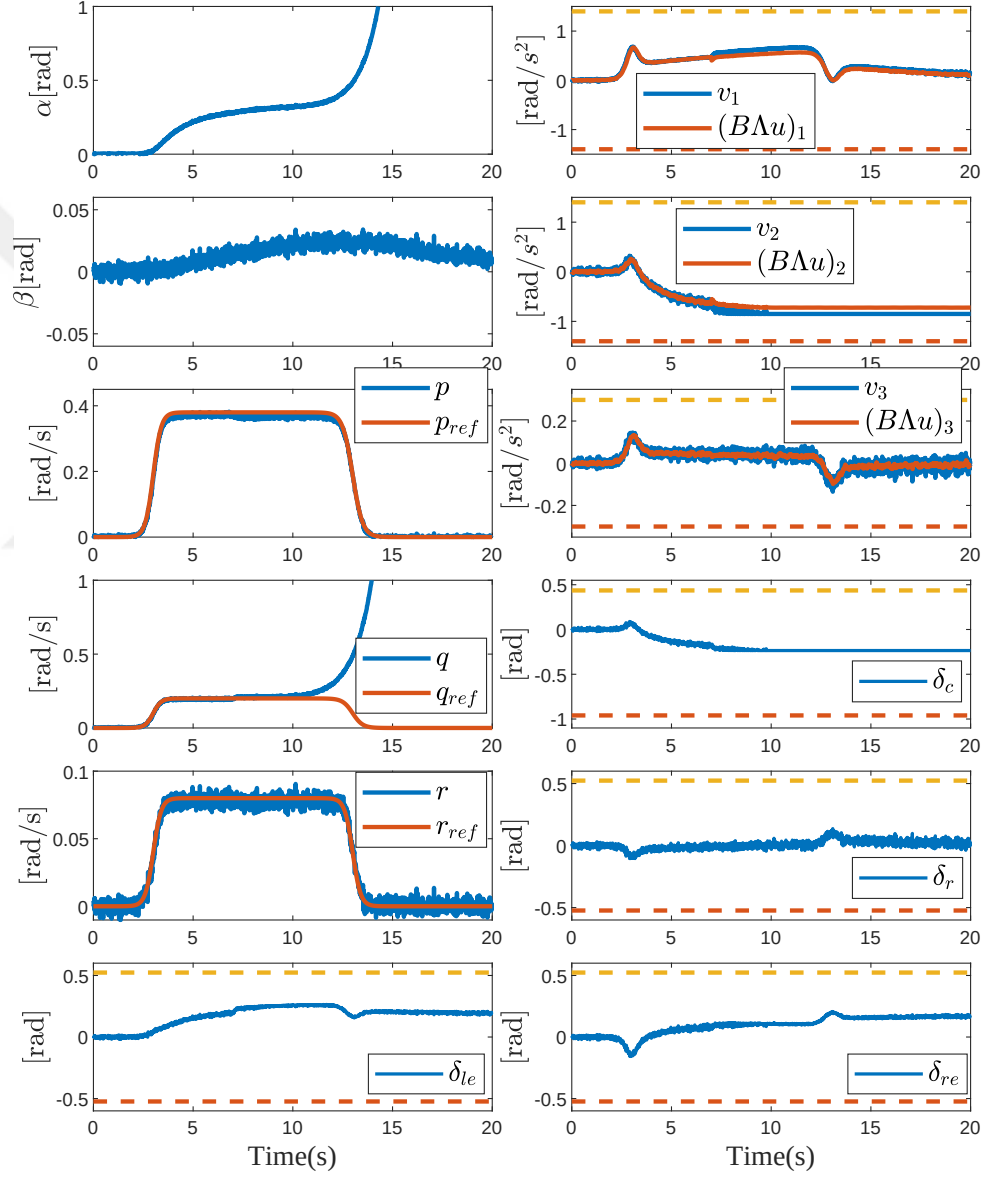


Figure 2.4: System states, control signals and actuators' deflections using conventional control allocation when actuator loss of effectiveness is 15% after  $t = 7$ s.

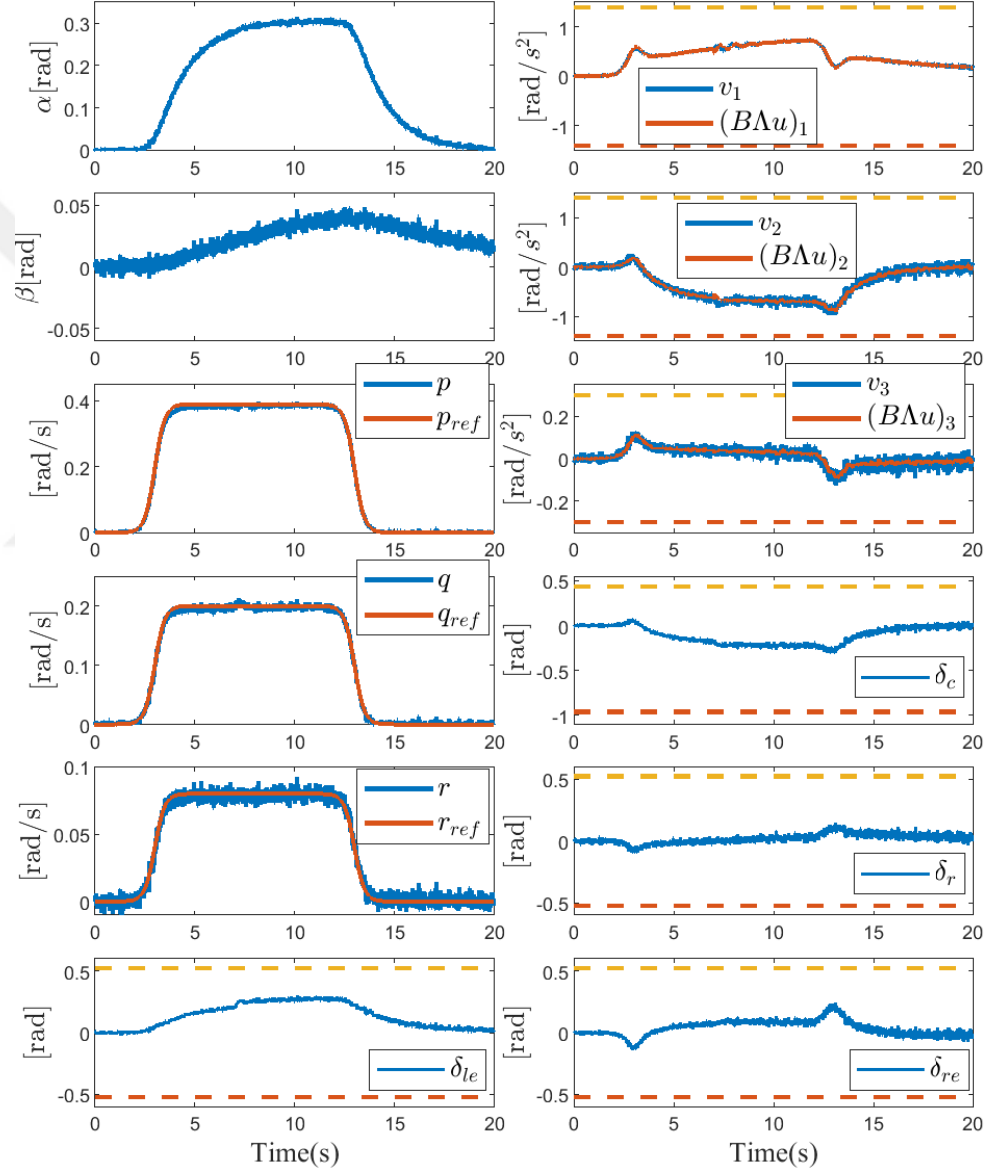


Figure 2.5: System states, control signals and actuators' deflections using adaptive control allocation, when actuator loss of effectiveness is 15% after  $t = 7$ s.

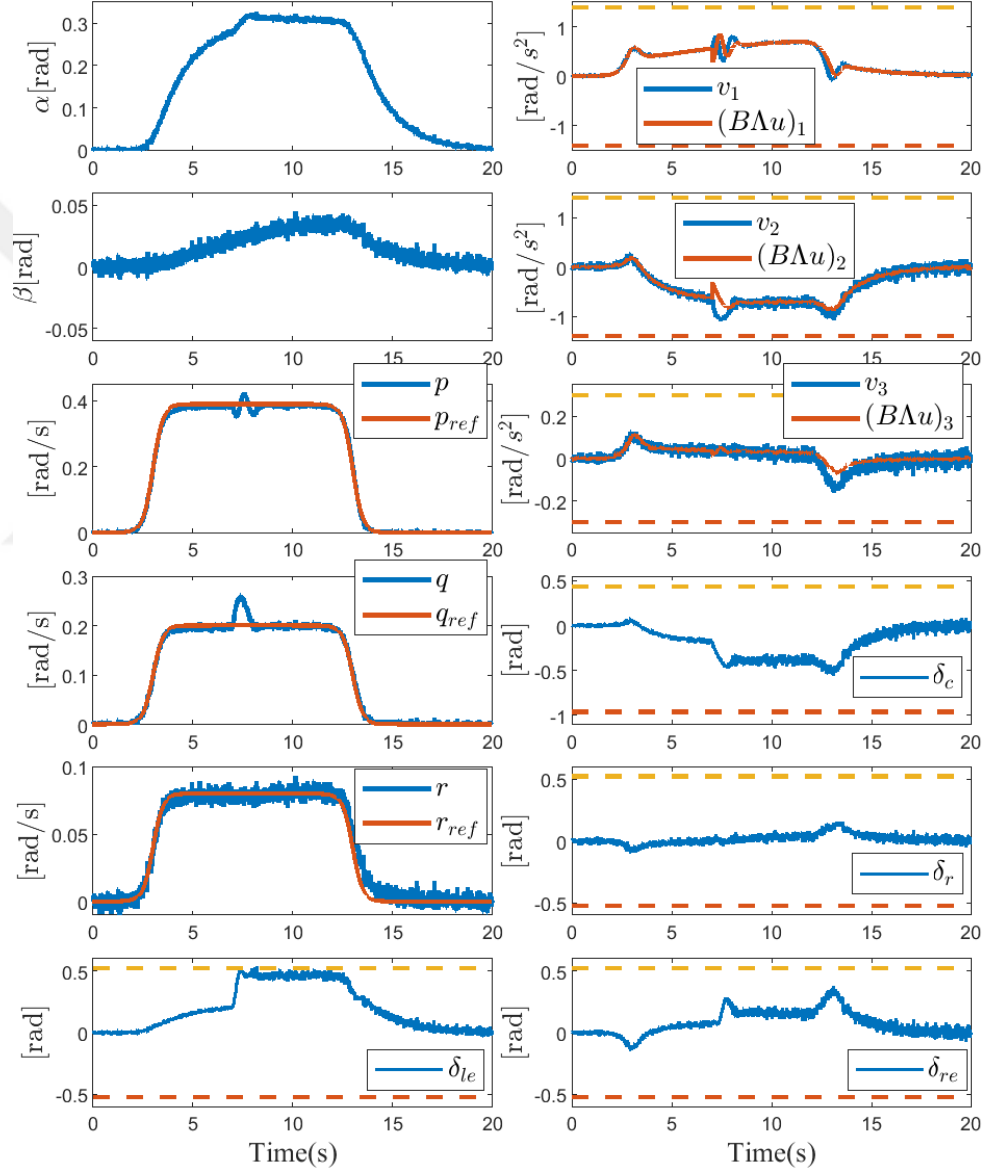


Figure 2.6: System states, control signals and actuators' deflections using adaptive control allocation when actuator loss of effectiveness is 50% after  $t = 7$ s.

$$B = \begin{bmatrix} 0 & -4.2423 & 4.2423 & 1.4871 \\ 1.6532 & -1.2735 & -1.2735 & 0.0024 \\ 0 & -0.2805 & 0.2805 & -0.8823 \end{bmatrix}. \quad (2.30)$$

To represent actuator loss of effectiveness and disturbance, a diagonal matrix  $\Lambda$  and a vector  $d_u$ , respectively, are augmented to the model (2.29) as

$$\begin{aligned} \dot{x} &= Ax + B_u \Lambda u + B_u d_u = Ax + B_v v + B_v \bar{d} \\ v &= B \Lambda u, \quad \bar{d} = B d_u, \quad B_u = B_v B, \quad B_v = [0_{3 \times 2} \quad I_{3 \times 3}]^T. \end{aligned} \quad (2.31)$$

In the simulations, the disturbance  $\bar{d}$  is a sinusoidal function with amplitude 0.1 and frequency 1 rad/s while zero-mean white Gaussian noise with standard deviation  $\sigma_{p,q,r} = 0.0017$  rad/s for the angular rates and  $\sigma_{\alpha,\beta} = 0.0044$  rad for angles represent the measurement noise [70]. For all of the simulations a sliding mode controller is used to create the control signal  $v$ . The simulation results for a conventional pseudo inverse based control allocation are reported in Figure 2.3 for the case without actuator loss of effectiveness, that is,  $\Lambda_1 = I$ . With the actuator loss of effectiveness modeled as changing the diagonal elements of  $\Lambda$  from 1 to 0.85 at  $t = 7$ s, the simulation results for the conventional control allocation are given in Figure 2.4. It is seen that 15% loss of effectiveness in all actuators at  $t = 7$  s causes instability. The proposed adaptive control allocation introduced in Section 2.3 is used in the simulations. Using Steps 1-3 in Section 2.4, the values of  $M_1 = 1.4$ ,  $M_2 = 1.4$  and  $M_3 = 0.3$  are obtained and  $\Omega_{\text{proj}}$ , which determines the maximum and minimum of each element of the adaptive parameter matrix is computed corresponding to  $\theta_{v_{1,1}} \in [-0.0129, 0.0129]$ ,  $\theta_{v_{1,2}} \in [0.0307, 0.5225]$ ,  $\theta_{v_{1,3}} \in [-0.1357, 0.1371]$ ,  $\theta_{v_{1,4}} \in [-0.212, 0]$ ,  $\theta_{v_{2,1}} \in [-0.3149, -0.1113]$ ,  $\theta_{v_{2,2}} \in [-0.217, -0.1416]$ ,  $\theta_{v_{2,3}} \in [-0.0241, 0.2363]$ ,  $\theta_{v_{2,4}} \in [-0.4162, -0.01]$ ,  $\theta_{v_{3,1}} \in [0.1587, 0.1977]$ ,  $\theta_{v_{3,2}} \in [0.0673, 0.0675]$ ,  $\theta_{v_{3,3}} \in [-0.001, 0.001]$ ,  $\theta_{v_{3,4}} \in [-1.2755, -0.7641]$ . We use a closed loop reference model with  $l = 4$  and  $A_m$  selected as  $A_m = \text{diag}([-0.2, -0.1, -0.1])$ . Figure 2.5 shows the simulation results for the system with adaptive control allocation and 15% actuator loss of effectiveness at  $t = 7$ s. It is seen that the first two states,  $\alpha$  and  $\beta$ , are bounded, and the other states ( $p$ ,  $q$  and  $r$ ) follow the reference inputs ( $p_{ref}$ ,  $q_{ref}$  and  $r_{ref}$ ) after the introduction of 15% actuator loss of effectiveness at  $t = 7$  sec. Also, it is seen

that the elements of  $(B\Lambda u)_i$  for  $i = 1, 2, 3$ , converge to the control signal elements  $v_i$  for  $i = 1, 2, 3$ . Finally, as seen from Figures 2.3, 2.4, and 2.5, the existence of noise does not affect the adaptive control allocation performance compared to the non-adaptive fixed control allocation. Another scenario is considered next, where the diagonal elements of  $\Lambda$  change from 1 to 0.5 at  $t = 7$ s. It is seen in Figure 2.6 that the system remains stable even after the introduction of the 50% loss of effectiveness.

## 2.6 Summary

An adaptive control allocation for uncertain over-actuated systems with actuator saturation is proposed in this chapter. The method needs neither uncertainty identification nor persistence of excitation assumption. Simulation results with the ADMIRE model show the effectiveness of the proposed method.

## Chapter 3

# Time-varying sliding mode controller for over-actuated systems in the presence of constrained and uncertain actuators

### 3.1 Introduction

Redundancy in actuators is beneficial for improving functionality and fault tolerance of engineered systems. Such systems require an algorithm to distribute control signals among redundant actuators, which is referred to as control allocation. Systems where control allocation is employed include aircraft/spacecraft [1,21–29], marine vessels [30–34,72], automobiles [35,73–75], robots [37] and power systems. [38,39]

Control allocation methods can be categorized into the following categories: Pseudo-inverse-based methods, optimization based methods, and dynamic control

allocation. Pseudo-inverse-based control allocation methods [10, 40–42] rely on manipulating the null space of the control input matrix, and have the lowest computational complexity. Optimization-based control allocation methods [47–52] solve an optimization problem at each time instant and can be computationally intense. In dynamic control allocation methods [54–59, 76], the control signals are distributed among actuators using a set of rules dictated by differential equations. A survey of control allocation methods can be found in [60]. In the presence of actuator uncertainty, the pseudo-inverse and optimization-based control allocation methods require fault detection and identification as well as persistency of excitation assumption for the input signals. Adaptive control allocation methods [55–57, 76] on the other hand, can handle actuator uncertainties without the need for fault identification or persistency of excitation assumption.

Actuator limits can induce nonlinear behavior and lead to performance degradation, limit cycles, multiple equilibria, and even instability [7, 25, 29, 51]. Several control allocation methods have been proposed in the literature that can handle actuator saturation. These include direct control allocation [40], daisy chaining [44], pseudo-inverse-based control allocation [77] and iterative approaches that use the null space of the control input matrix [42]. Optimization-based control allocation is another commonly used method of accounting for actuator magnitude and rate constraints [21, 25, 32, 47, 48, 50, 78, 79]. Furthermore, a control allocation approach by Naderi et al [64], employs model predictive control to handle actuator magnitude constraints. In order to allocate control signals in the presence of uncertainty as well as actuator constraints, an adaptive control allocator for constrained systems has been developed in [58, 76]. An adaptive control allocator which exploits a modified projection algorithm to handle magnitude and rate constraints in over-actuated systems is proposed by Tohidi et al [80].

Although control allocation methods enable modularity for the overall control system design, as they separate the generation of the control signal and its distribution, control allocation errors can be significant in transients and degrade the performance. In vehicle and flight applications, the goal of the control allocation is to match the commanded ( $v$ ) and the actual ( $Bu$ ) control moments/forces,

where  $u$  designates a vector of actuator positions. However, the equality  $Bu = v$  may not be satisfied in the case of dynamic methods in transients or in the case of optimization-based methods if time to compute the solution online is insufficient. This transient control allocation error may not be negligible especially in the presence of actuator limits. Therefore, the controller must be designed to be robust to the control allocation error, as well as external disturbances. The robustness characteristics of the sliding mode control motivated its use in combination with many control allocation implementations [10–12].

Inspired by [63], this thesis proposes a sliding mode controller with a time-varying sliding surface which guarantees stability and tracking, and which is robust not only to bounded disturbances but also to the adaptive control allocation error. Different from [63], the proposed sliding surface and sliding mode controller provide reference tracking without any requirement for error augmentation. This reduces the computational burden. To the best of authors’ knowledge, controllers that are robust to both the adaptive control allocator error and to external disturbances have not been proposed in the prior literature. Preliminary results of this study are submitted earlier as a conference paper [81], where the detailed mathematical proofs and implementation details are missing.

This chapter is organized as follows. Section 3.2 describes the problem of controlling an over-actuated uncertain system in the presence of an adaptive control allocator. Section 3.3 presents the sliding mode controller design. The ADMIRE model is used in Section 3.4 to demonstrate the effectiveness of the proposed approach in the simulation environment. Finally, a conclusion is given in Section 3.5.

## 3.2 Problem statement and preliminaries

We consider the adaptive control allocation setting in Figure 3.1 and the following Plant dynamics [76],

$$\dot{x} = Ax + B_u(\Lambda u + d_u), \quad (3.1)$$

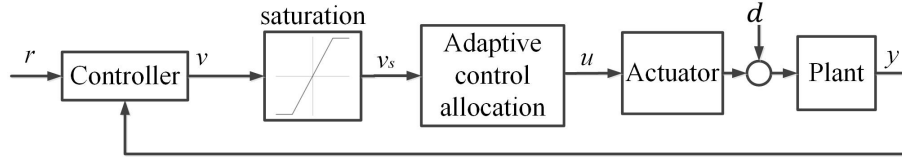


Figure 3.1: Block diagram of the closed loop system.

where  $x \in \mathbb{R}^n$  is the state vector,  $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$  is the actuator input vector whose elements are constrained as  $u_i \in [-u_{\max_i}, u_{\max_i}]$ ,  $A \in \mathbb{R}^{n \times n}$  is a known state matrix,  $B_u \in \mathbb{R}^{n \times m}$  is a known input matrix and  $d_u \in \mathbb{R}^m$  is an unknown bounded disturbance input. The matrix  $\Lambda \in \mathbb{R}^{m \times m}$  is assumed to be diagonal with positive elements representing actuator effectiveness uncertainty. It is assumed that the pair  $(A, B_u \Lambda)$  is controllable. Due to actuator redundancy, the input matrix is rank deficient, that is  $\text{Rank}(B_u) = \ell < m$ . Consequently,  $B_u$  can be written as  $B_u = B_v B$ , where  $B_v \in \mathbb{R}^{n \times \ell}$  is a full column rank matrix, i.e.  $\text{Rank}(B_v) = \ell$ , and  $B \in \mathbb{R}^{\ell \times m}$ . The decomposition of  $B_u$  helps exploit the actuator redundancy using control allocation. Employing this decomposition, (3.1) can be rewritten as

$$\dot{x} = Ax + B_v(B\Lambda u + \bar{d}), \quad (3.2)$$

where  $\bar{d}(t) = B d_u(t)$  with an upper bound  $\|\bar{d}(t)\| \leq D$ , for all  $t \geq 0$ . Throughout this chapter,  $\|\cdot\|$  refers to the Euclidean norm for vectors and induced 2-norm for matrices. The control allocation task is to achieve

$$B\Lambda u + \bar{d} = v_s, \quad (3.3)$$

where  $v_s \in \mathbb{R}^\ell$  is the output of the saturation block, which is receiving the control signal  $v$  as the input. (See Figure 3.1).

Considering the following dynamics,

$$\dot{y} = A_m y + B\Lambda u + \bar{d} - v_s, \quad (3.4)$$

where  $A_m \in \mathbb{R}^{\ell \times \ell}$  is a stable matrix, a reference model is constructed as

$$\dot{y}_m = A_m y_m. \quad (3.5)$$

Defining the actuator input as a mapping from  $v_s$  to  $u$ ,

$$u = \theta_v^T v_s, \quad (3.6)$$

where  $\theta_v \in \mathbb{R}^{\ell \times m}$  represents the adaptive parameter matrix to be determined, and substituting (3.6) into (3.4), we obtain

$$\dot{y} = A_m y + (B\Lambda\theta_v^T - I_\ell)v_s + \bar{d}, \quad (3.7)$$

where  $I_\ell$  is an identity matrix of dimension  $\ell \times \ell$ . It is assumed that there exists an ideal matrix  $\theta_v^*$  such that

$$B\Lambda\theta_v^{*T} = I_\ell. \quad (3.8)$$

Defining  $e = y - y_m$  and subtracting (3.5) from (3.7), it follows that

$$\dot{e} = A_m e + B\Lambda\tilde{\theta}_v^T v_s + \bar{d}, \quad (3.9)$$

where  $\tilde{\theta}_v = \theta_v - \theta_v^*$ .

**Theorem 5.** *Consider (3.4) and (3.5). Suppose that the adaptive parameter matrix is updated using the adaptive law,*

$$\dot{\theta}_v(t) = \Gamma_\theta \text{Proj}(\theta_v(t), -v_s(t)e^T(t)PB, f), \quad (3.10)$$

where the symmetric positive definite matrix  $P$  satisfies  $A_m^T P + P A_m = -Q$ ,  $Q$  is a symmetric positive definite matrix, “Proj” is the projection operator [9, 80] with a convex function  $f \in C^1$ , and  $\Gamma_\theta = \gamma_\theta I_\ell$ , where  $\gamma_\theta$  is a positive scalar. Then, given any initial condition  $e(0) \in \mathbb{R}^\ell$ ,  $e(t)$  and  $\tilde{\theta}_v(t)$  remain uniformly bounded for all  $t \geq 0$  and their trajectories converge exponentially to the set

$$E_1 = \{(e, \tilde{\theta}_v) : \|e\|^2 \leq \left( \frac{s\tilde{\theta}_{max}^2}{\gamma_\theta \lambda_{min}(Q)} + \frac{2\chi^4 D^2 \|Q\|^2}{\sigma^2 \lambda_{min}(Q)^2} \right) \frac{4s\chi^2 \|Q\|}{\sigma \lambda_{min}(Q)}, \|\tilde{\theta}_v\| \leq \tilde{\theta}_{max}\}, \quad (3.11)$$

where  $s = -\min_i(\lambda_i(A_m + A_m^T)/2)$ ,  $\sigma = -\max_i(\text{Real}(\lambda_i(A_m)))$ ,  $\chi = \frac{3}{2}(1 + 4\frac{a}{\sigma})^{(\ell-1)}$ ,  $a = \|A_m\|$  and  $\|\tilde{\theta}_v(t)\|_F \leq \tilde{\theta}_{max} \equiv \sqrt{\sum_{i,j} (\theta_{max_{i,j}} - \theta_{min_{i,j}} - \zeta_{i,j})^2}$ . In addition, if  $\bar{d}(t) = 0$  for  $t \geq t'$  for some  $t' \geq 0$ , and  $v_s(t)$  is uniformly continuous as a function of  $t \in [t', +\infty)$ , then  $\lim_{t \rightarrow \infty} B\Lambda u(t) = v_s(t)$ , i.e., (3.3) is achieved asymptotically.

*Proof.* See Chapter 2. □

By substituting (3.6) and (3.8) into (3.2), it follows that

$$\begin{aligned}
 \dot{x} &= Ax + B_v(B\Lambda u + \bar{d}) \\
 &= Ax + B_v(B\Lambda\theta_v^T v_s + \bar{d}) \\
 &= Ax + B_v(I + B\Lambda\tilde{\theta}_v^T)v_s + B_v\bar{d}.
 \end{aligned} \tag{3.12}$$

Defining  $\Delta B(t) \equiv B\Lambda\tilde{\theta}_v^T(t)$ , and substituting in (3.12), it follows that

$$\dot{x}(t) = Ax(t) + B_v(v_s(t) + d(t)), \tag{3.13}$$

where  $d(t) = \Delta B(t)v_s(t) + \bar{d}(t) \in \mathbb{R}^\ell$  is the sum of the disturbance and the control allocation error. Therefore, the controller to be designed should be robust to both the disturbances *and* the control allocation errors. It is noted that since  $\Delta B(t)$ ,  $v_s(t)$  and  $\bar{d}(t)$  are bounded,  $d(t)$  is also bounded.

Up until here, we introduced the plant dynamics with constrained uncertain actuators, the adaptive control allocation, and the resulting system dynamics (3.13) after the inclusion of the control allocator. What remains to be done is the design of a controller that generates the signal  $v$  (see Figure 3.1). The controller needs to be robust to the control allocation error and the disturbances, in the presence of a software saturation. In the proceeding sections, the controller that satisfies these requirements are presented, where a sliding mode control approach is used.

### 3.3 Controller design

In this section, a design procedure for the controller that generates the virtual control signal  $v$  is proposed (see Figure 3.1).

The following two assumptions are made:

**Assumption 1.** *The dynamics in (3.13) can be written as*

$$\begin{bmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + B_v(v_s + d), \quad (3.14)$$

where  $A_{1,1} \in \mathbb{R}^{(n-\ell) \times (n-\ell)}$  is a Hurwitz matrix,  $A_{1,2} \in \mathbb{R}^{(n-\ell) \times \ell}$ ,  $A_{2,1} \in \mathbb{R}^{\ell \times (n-\ell)}$  and  $A_{2,2} \in \mathbb{R}^{\ell \times \ell}$  are parts of the state matrix, and  $x^{(1)} \in \mathbb{R}^{(n-\ell)}$  and  $x^{(2)} \in \mathbb{R}^\ell$  constitute the state vector. We are interested in the vector  $x^{(2)}$  as the system output  $y$ . Therefore,

$$y = C \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}, \quad (3.15)$$

where  $C = [0_{\ell \times (n-\ell)} \ I_\ell]$ .

**Assumption 2.** *The matrix  $B_v \in \mathbb{R}^{n \times \ell}$  is in the form  $[0_{\ell \times (n-\ell)} \ I_\ell]^T$ .*

Both of the above assumptions are justified for typical aircraft models. [1,82] In the simulation section, these assumptions are validated for the AeroData Model in Research Environment (ADMIRE) [21,48].

**Remark 12.** *For systems where Assumption 2 does not hold, given that  $B_v$  has full column rank, it is possible to find a transformation matrix,  $T_B$ , such that  $\hat{B}_v = T_B B_v = [0_{\ell \times (n-\ell)} \ I_\ell]^T$ . [63, 83] However, employing this transformation may lead to a state space realization which violates Assumption 1.*

**Remark 13.** *Since  $A_{1,1}$  is assumed to be Hurwitz, showing that the states  $x^{(2)}$  are bounded will be sufficient to demonstrate the boundedness of  $x^{(1)}$ .*

### 3.3.1 Time Varying Sliding Surface

The sliding surface, inspired by [63], is given as

$$s(x^{(2)}(t), x^{(2)}(t_0), t) = x^{(2)}(t) - x^{(2)}(t_0)e^{-\bar{\lambda}(t-t_0)} - \frac{2}{\pi}r(t)\tan^{-1}(\bar{\lambda}(t-t_0)) = 0, \quad (3.16)$$

where  $\bar{\lambda} > 0$  is a scalar parameter,  $x^{(2)} \in \mathbb{R}^\ell$  is defined in (3.14), and  $r(t) \in \mathbb{R}^\ell$  is the twice continuously differentiable ( $C^2$ ) reference to be tracked.

The response of a system controlled by a sliding mode controller includes two phases. [84] The first phase is called the reaching phase. During this phase, the controller drives the system towards the sliding surface. In the second phase, which is called the sliding phase, the trajectory evolves on the sliding manifold. For the sliding surface (3.16), no reaching phase exists since the sliding surface is a function of the initial condition and the trajectories are on the sliding surface at  $t = t_0$ , i.e.  $s(x^{(2)}(t), x^{(2)}(t_0), t_0) = 0$ . Below, we analyze the behavior of the system trajectories on the sliding surface and show that the trajectories remain on the sliding surface for all  $t \geq t_0$ .

### 3.3.2 Motion on the sliding surface

Using (3.16), the trajectories of  $x^{(2)}$  on the sliding surface satisfy

$$x^{(2)}(t) = x^{(2)}(t_0)e^{-\bar{\lambda}(t-t_0)} + \frac{2}{\pi}r(t)\tan^{-1}(\bar{\lambda}(t-t_0)). \quad (3.17)$$

Substituting (3.17) into (3.14), it follows that

$$\dot{x}^{(1)} = A_{1,1}x^{(1)} + A_{1,2} \left( x^{(2)}(t_0)e^{-\bar{\lambda}(t-t_0)} + \frac{2}{\pi}r(t)\tan^{-1}(\bar{\lambda}(t-t_0)) \right). \quad (3.18)$$

By defining  $G_1 \equiv A_{1,2}x^{(2)}(t_0)$ , and  $G_2(t) \equiv \frac{2}{\pi}A_{1,2}r(t)\tan^{-1}(\bar{\lambda}(t-t_0))$ , (3.18) can be rewritten as

$$\dot{x}^{(1)} = A_{1,1}x^{(1)} + G_1e^{-\bar{\lambda}(t-t_0)} + G_2(t) = A_{1,1}x^{(1)} + g(t), \quad (3.19)$$

where  $g(t) \equiv G_1e^{-\bar{\lambda}(t-t_0)} + G_2(t)$ .

**Lemma 2.** *When  $x^{(2)}(t)$  is on the sliding surface (3.16),  $\|x^{(1)}(t)\| \leq k\bar{x}^{(1)}(t_0) + K_2\bar{x}^{(2)}(t_0) + K_2\bar{r}$ , where  $K_2 = \frac{k}{\xi}\|A_{1,2}\|$ ,  $k$  and  $\xi$  are positive constants, and  $\bar{x}^{(1)}(t_0)$ ,  $\bar{x}^{(2)}(t_0)$  and  $\bar{r}$  are the upper bounds of  $\|x^{(1)}(t_0)\|$ ,  $\|x^{(2)}(t_0)\|$  and  $\|r(t)\|$ , respectively. Furthermore,  $\lim_{t \rightarrow \infty} y(t) = r(t)$ .*

*Proof.* Per Assumption 1,  $A_{1,1}$  is Hurwitz, hence the homogeneous system  $\dot{x}_h^{(1)}(t) = A_{1,1}x_h^{(1)}(t)$  is globally exponentially stable at the origin. The solution of this system is given as  $x_h^{(1)}(t) = \Phi(t, t_0)x_h^{(1)}(t_0)$ , where  $\Phi(t, t_0)$  is the state transition matrix and there exist constants  $k > 0$  and  $\xi > 0$  such that

$$\|\Phi(t, t_0)\| \leq ke^{-\xi(t-t_0)}, \quad \forall t \geq t_0, \quad (3.20)$$

where  $\xi = \frac{1}{2\|X\|}$ ,  $k = \sqrt{\|X^{-1}\|\|X\|}$  and the positive definite matrix  $X \in \mathbb{R}^{(n-\ell) \times (n-\ell)}$  satisfies the Lyapunov equation  $A_{1,1}^T X + X A_{1,1} = -I_{n-\ell}$  [85].

Since the state transition matrices of the dynamics  $\dot{x}_h^{(1)}(t) = A_{1,1}x_h^{(1)}(t)$  and  $\dot{x}^{(1)}(t) = A_{1,1}x^{(1)}(t) + g(t)$  are the same, we use the state transition matrix  $\Phi(t, t_0)$  used in (3.20) to provide the solution of (3.19) as

$$x^{(1)}(t) = \Phi(t, t_0)x^{(1)}(t_0) + \int_{t_0}^t \Phi(t, \eta)g(\eta)d\eta. \quad (3.21)$$

Taking the norm of both sides of (3.21) and using the triangle inequality, we obtain that

$$\|x^{(1)}(t)\| \leq \|\Phi(t, t_0)x^{(1)}(t_0)\| + \int_{t_0}^t \|\Phi(t, \eta)\| \|g(\eta)\| d\eta. \quad (3.22)$$

Using the definition of  $g(t)$ , given after (3.19), it follows that  $\|g(t)\| = \|G_1 e^{-\bar{\lambda}t} + G_2(t)\| \leq \|G_1\| + \sup_{t \geq t_0} \|G_2(t)\|$ . Note that, since  $G_2(t)$  is a function of the reference input  $r(t)$ ,  $\sup_{t \geq t_0} \|G_2(t)\|$  exists (see the definition of  $G_2$  given after (3.18)). Therefore,  $\|g(t)\| \leq \|A_{1,2}\|\bar{x}^{(2)}(t_0) + \|A_{1,2}\|\bar{r}$ . Defining  $K_1 = \|A_{1,2}\|\bar{x}^{(2)}(t_0) + \|A_{1,2}\|\bar{r}$ , and using (3.20), (3.22) can be rewritten as,

$$\begin{aligned} \|x^{(1)}(t)\| &\leq k\bar{x}^{(1)}(t_0)e^{-\xi(t-t_0)} + kK_1 \int_{t_0}^t e^{-\xi(t-\eta)} d\eta \\ &\leq k\bar{x}^{(1)}(t_0)e^{-\xi(t-t_0)} + kK_1 \frac{1}{\xi} (1 - e^{-\xi(t-t_0)}) \\ &\leq k\bar{x}^{(1)}(t_0) + kK_1 \frac{1}{\xi} \\ &= k\bar{x}^{(1)}(t_0) + k(\|A_{1,2}\|\bar{x}^{(2)}(t_0) + \|A_{1,2}\|\bar{r}) \frac{1}{\xi} \\ &\leq k\bar{x}^{(1)}(t_0) + K_2\bar{x}^{(2)}(t_0) + K_2\bar{r}, \end{aligned} \quad (3.23)$$

where  $K_2 = \frac{k}{\xi}\|A_{1,2}\|$ , and  $\bar{x}^{(1)}(t_0)$  and  $\bar{x}^{(2)}(t_0)$  represent the upper bounds on  $\|x^{(1)}(t_0)\|$  and  $\|x^{(2)}(t_0)\|$ , respectively. Since the reference signal  $r(t)$ ,  $x^{(1)}(t_0)$

and  $x^{(2)}(t_0)$  are bounded, (3.23) shows that  $x^{(1)}(t)$  is bounded. Since  $x(t_0)$  and  $r(t)$  are bounded, it can be shown, using (3.17), that  $x^{(2)}(t)$  is bounded and converges to  $r(t)$ . Since  $y = x^{(2)}$ , this completes the proof.  $\square$

### 3.3.3 Control Law

Figure 3.1 shows that once the control signal  $v$  is generated by the controller, it is passed through a software saturation block, whose output is represented by  $v_s$ . In this subsection, it is assumed that  $v_s = v$  to demonstrate that with the proposed control law, trajectories stay on (3.16) and hence properties of Lemma 2 hold. This requires that  $v$  always stays within saturation bounds. A condition for this assumption to hold will be presented in the next subsection.

**Definition 1.**  $sign_v(a)$ , where  $a$  is a column vector, is a diagonal matrix whose elements are the signs of the elements of the vector  $a$ . For example,  $sign_v([a_1 \ a_2]^T) = diag(sign(a_1), sign(a_2))$ , where  $a_1$  and  $a_2$  are scalars.

**Definition 2.**  $|a|_v \equiv sign_v(a)a$  and  $|a^T|_v \equiv a^T sign_v(a)$ , where  $a$  is a column vector and  $sign_v(.)$  is defined in Definition 1. For example,  $|[a_1 \ a_2]|_v = [a_1 \ a_2]sign_v([a_1 \ a_2]^T) = [|a_1| \ |a_2|]$ , where  $a_1$  and  $a_2$  are scalars.

**Theorem 6.** Consider the dynamics in (3.14) with the control law,

$$\begin{aligned} v(t) = & -A_{2,1}x^{(1)}(t) - A_{2,2}x^{(2)}(t) - \bar{\lambda}x^{(2)}(0)e^{-\bar{\lambda}t} + \frac{2}{\pi}\dot{r}(t)\tan^{-1}(\bar{\lambda}t) + \frac{2}{\pi}r(t)\frac{\bar{\lambda}}{1 + \bar{\lambda}^2t^2} \\ & - sign_v(s(x^{(2)}(t), x^{(2)}(0), t)) \rho, \end{aligned} \quad (3.24)$$

where  $\rho \in R^r$  contains the absolute upper bounds of the elements of the disturbance vector  $d$ , and  $s(x^{(2)}(t), x^{(2)}(0), t)$  is the sliding surface (3.16). Assume  $v_s(t) = v(t)$  for all  $t$ . Then, the trajectories of  $x^{(2)}$  stay on the sliding surface (3.16).

*Proof.* Consider a Lyapunov function candidate  $V_2(s) = \frac{1}{2}s^Ts$ , where the arguments of  $s(x^{(2)}(t), x^{(2)}(t_0), t)$  are dropped for clarity. By taking the time-derivative

of  $V_2$  along the system trajectories, and using (3.16) with  $t_0 = 0$ , we obtain

$$\dot{V}_2 = s^T \dot{s} = s^T \left( \dot{x}^{(2)}(t) + \bar{\lambda} x^{(2)}(0) e^{-\bar{\lambda} t} - \frac{2}{\pi} \dot{r}(t) \tan^{-1}(\bar{\lambda} t) - \frac{2}{\pi} r(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \right). \quad (3.25)$$

Using (3.14) and Assumption 2, we have  $\dot{x}^{(2)}(t) = A_{2,1}x^{(1)}(t) + A_{2,2}x^{(2)}(t) + v + d$ . Therefore, (3.25) can be rewritten as

$$\begin{aligned} \dot{V}_2 = s^T & \left( A_{2,1}x^{(1)}(t) + A_{2,2}x^{(2)}(t) + v + d + \bar{\lambda} x^{(2)}(0) e^{-\bar{\lambda} t} \right. \\ & \left. - \frac{2}{\pi} \dot{r}(t) \tan^{-1}(\bar{\lambda} t) - \frac{2}{\pi} r(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \right). \end{aligned} \quad (3.26)$$

By substituting the control law (3.24) into (3.26), and using Definitions 1 and 2, it follows that

$$\dot{V}_2 = s^T [d - \text{sign}_v(s)\rho] = s^T d - |s^T|_v \rho \leq |s^T|_v (|d|_v - \rho). \quad (3.27)$$

Since the elements of  $|d|_v - \rho$  are non-positive,  $\dot{V}_2 \leq 0$ . Therefore,  $x^{(2)}$  trajectories which are on the sliding surface at  $t = t_0$ , will remain there for all  $t > 0$ .  $\square$

### 3.3.4 Bounding the control signals

In this section, we provide a method inspired by the work of Corradini et al. [63] to make sure that  $|v_i| \leq M_i$ , where  $v_i$  refers to the  $i^{\text{th}}$  element of the control signal  $v$ , and  $M_i$  is a positive scalar,  $i = 1, 2, \dots, \ell$ , which is a predefined soft saturation limit. This ensures that  $v_s = v$ , which is an assumption used in the previous section. We have observed in our simulation studies that the controller is robust for a range of cases where this assumption is violated. However, to have formal stability guarantees, we present sufficient conditions in this section that can be used to ensure  $v_s = v$ . Later in the simulations section, we show that these conditions are indeed sufficient, not necessary, by demonstrating that the equality  $v_s = v$  is satisfied even when they are violated.

**Lemma 3.** *The control signals  $v_i$ ,  $i = 1, 2, \dots, \ell$ , are bounded by  $M_i$ ,  $i = 1, 2, \dots, \ell$ ,*

i.e.  $|v_i(t)| \leq M_i$  for all  $t$  if the inequality

$$\begin{aligned}
& \left| - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} a_{2i,\omega} \phi_{\omega,j}(t) x_j^{(1)}(0) - \int_0^t \left( \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} a_{2i,\omega} \phi_{\omega,j}(t-\eta) \right. \right. \\
& \left. \left[ \sum_{k=n-\ell+1}^n a_{1j,k} \left( x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} + r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) \right) \right] \right) d\eta \\
& - \sum_{j=n-\ell+1}^n a_{2i,j} \left( x_{j-n+\ell}^{(2)}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} r_{j-n+\ell}(t) \tan^{-1}(\bar{\lambda}t) \right) - \bar{\lambda} x_{n-\ell+i}(0) e^{-\bar{\lambda}t} \\
& \left. + \frac{2}{\pi} \dot{r}_i(t) \tan^{-1}(\bar{\lambda}t) + \frac{2}{\pi} r_i(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \right| \leq M_i - \rho_i, \quad i = 1, \dots, \ell
\end{aligned} \tag{3.28}$$

is satisfied for all  $t \geq 0$ , where  $\rho$  is defined in Theorem 6, and  $\phi_{i,j}(t)$  is the  $(i, j)^{th}$  element of the state transition matrix  $\Phi(t, t_0)$  in (3.20), with  $t_0 = 0$ .

*Proof.* See Appendix B.1. □

**Remark 14.** For (3.28) to be mathematically meaningful, its right hand side,  $M_i - \rho_i$ , must be positive. It is shown in Tohidi et al. [76] that this is indeed the case.

The adjustable parameters in (3.28) are the initial values of the states  $x(0) = [x^{(1)T}(0), x^{(2)T}(0)]^T$  and the scalar design parameter  $\bar{\lambda}$ . Assuming that the reference,  $r(t)$ , input vector and its rate of change are bounded, all of the terms in (3.28) are either bounded or converge to zero. Since  $A_{1,1}$  is stable, the elements of state transition matrix,  $\phi_{i,j}(t)$ , converge to zero exponentially fast (see (3.20)). Specifically for the investigated flight control problem, the state transition matrix of  $x^{(1)}$  dynamics (see (3.14)), where  $x^{(1)} = [\alpha, \beta]^T$  and  $\alpha$  and  $\beta$  are the angle of attack and the sideslip angle, respectively, consists of sums of decaying exponentials due to distinct eigenvalues. In this case, calculating the feasible values of the adjustable parameters becomes easier since the inequality (3.28) takes a simpler form, which is explained in the following corollary.

**Corollary 1.** *If the elements of the state transition matrix are formed as sums of exponential functions (which is the case in several flight control problems), that is,  $\phi_{i,j}(t) = \sum_{\kappa=1}^{\bar{n}} c_{i,j,\kappa} e^{-h_{i,j,\kappa} t}$  with  $h_{i,j,\kappa} > 0$  and  $\bar{n}$  a positive integer, then the*

control signals  $v_i$ ,  $i = 1, \dots, \ell$  are bounded by  $M_i$ ,  $i = 1, \dots, \ell$ , i.e.  $|v_i| \leq M_i$  if the inequality

$$\begin{aligned}
W_i \equiv & \left| - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} a_{2i,\omega} c_{\omega,j_\kappa} e^{-h_{\omega,j_\kappa} t} x_j^{(1)}(0) \right. \\
& - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n a_{2i,\omega} c_{\omega,j_\kappa} a_{1j,k} x_{k-n+\ell}^{(2)}(0) q_{\omega,j_\kappa}(t) \\
& - \sum_{j=n-\ell+1}^n a_{2i,j} \left( x_i^{(2)}(0) e^{-\bar{\lambda} t} + \frac{2}{\pi} r_i(t) \tan^{-1}(\bar{\lambda} t) \right) \\
& - \bar{\lambda} x_{n-\ell+i}(0) e^{-\bar{\lambda} t} + \frac{2}{\pi} \dot{r}_i(t) \tan^{-1}(\bar{\lambda} t) + \frac{2}{\pi} r_i(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \\
& \left. + \int_0^t \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n -a_{2i,\omega} c_{\omega,j_\kappa} e^{-h_{\omega,j_\kappa}(t-\eta)} a_{1j,k} r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda} \eta) d\eta \right| \\
& \leq M_i - \rho_i,
\end{aligned} \tag{3.29}$$

$i = 1, \dots, \ell$ , where

$$q_{\omega,j_\kappa}(t) \equiv \begin{cases} t e^{-h_{\omega,j_\kappa} t}, & \bar{\lambda} = h_{\omega,j_\kappa}, \\ \frac{e^{-\bar{\lambda} t} - e^{-h_{\omega,j_\kappa} t}}{h_{\omega,j_\kappa} - \bar{\lambda}}, & \bar{\lambda} \neq h_{\omega,j_\kappa}, \end{cases} \tag{3.30}$$

is satisfied for all  $t \geq 0$ .

*Proof.* See Appendix B.2. □

**Remark 15.** Given the upper and lower bounds of the reference signals and their derivatives, such that  $r_i(t) \in [r_i^-, r_i^+]$  and  $\dot{r}_i(t) \in [\dot{r}_i^-, \dot{r}_i^+]$ ,  $r_i^- = -r_i^+$ ,  $\dot{r}_i^- = -\dot{r}_i^+$ , one way to check the inequality (3.29) is by using the triangle inequality and finding an upper bound for  $W_i$  as  $\sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} |a_{2i,\omega} c_{\omega,j_\kappa} x_j^{(1)}(0)| + \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n |a_{2i,\omega} c_{\omega,j_\kappa} a_{1j,k} x_{k-n+\ell}^{(2)}(0) \bar{q}_{\omega,j_\kappa}| + \sum_{j=n-\ell+1}^n |a_{2i,j} x_i^{(2)}(0)| + \sum_{j=n-\ell+1}^n |a_{2i,j} r_i^+| + |\bar{\lambda} x_{n-\ell+i}(0)| + |\dot{r}_i^+| + |\frac{2}{\pi} r_i^+ \bar{\lambda}| + \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n |a_{2i,\omega} c_{\omega,j_\kappa} a_{1j,k} \frac{r_{k-n+\ell}^+}{h_{\omega,j_\kappa}}| \leq M_i - \rho_i$ , where  $\bar{q}_{\omega,j_\kappa} = \max\{e^{-1} h_{\omega,j_\kappa}^{-1}, \frac{2}{|h_{\omega,j_\kappa} - \bar{\lambda}|}\}$ . This inequality can be simplified as  $W_{i,1} + \bar{\lambda} W_{i,2} \leq 0$ , where  $W_{i,1} = \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} |a_{2i,\omega} c_{\omega,j_\kappa} x_j^{(1)}(0)| + \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n |a_{2i,\omega} c_{\omega,j_\kappa} a_{1j,k} x_{k-n+\ell}^{(2)}(0) q_{\omega,j_\kappa}(t)| + \sum_{j=n-\ell+1}^n |a_{2i,j} (x_i^{(2)}(0))| + \sum_{j=n-\ell+1}^n |a_{2i,j} r_i^+| + |\dot{r}_i^+| + \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n |a_{2i,\omega} c_{\omega,j_\kappa} a_{1j,k} \frac{r_{k-n+\ell}^+}{h_{\omega,j_\kappa}}| - M_i + \rho_i$  and  $W_{i,2} =$

$|x_{n-\ell+i}(0)| + |\frac{2}{\pi}r_i^+|$ . Note that  $W_{i,1}$  and  $W_{i,2}$  are functions of  $x^{(1)}(0)$ ,  $x^{(2)}(0)$  and  $r_i^+$ , and remain constant along the closed-loop trajectory. Since  $W_{i,2}$  is positive, a value of  $\bar{\lambda} > 0$  satisfying  $W_{i,1} + \bar{\lambda}W_{i,2} \leq 0$  can always be found if  $W_{i,1} < 0$ , which can be realized by putting suitable bounds on the elements of  $x^{(1)}(0)$ ,  $x^{(2)}(0)$  and  $r$ . This procedure is conservative due to the nature of the triangular inequality. On the other hand, if the reference signal trajectories are known, one can check the inequality (3.29) numerically. We demonstrate the application of the latter approach in the simulation results sections.

For a constant reference input vector  $r$ , the inequality (3.29) can be simplified further. This is described in the following corollary.

**Corollary 2.** *If, in addition to the conditions given in Corollary 1, the reference signals are constant, that is,  $r_i(t) = R_i$ ,  $i = 1, \dots, \ell$ , then the control signals  $v_i$ ,  $i = 1, \dots, \ell$  are bounded by  $M_i$ ,  $i = 1, \dots, \ell$ , i.e.  $|v_i| \leq M_i$ , if the inequality*

$$\begin{aligned} & \hat{W}_i \\ & \equiv \left| - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} a_{2_{i,\omega}} c_{\omega,j_\kappa} e^{-h_{\omega,j_\kappa} t} x_j^{(1)}(0) - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n a_{2_{i,\omega}} c_{\omega,j_\kappa} a_{1_{j,k}} \right. \\ & \times x_{k-n+\ell}^{(2)}(0) q_{\omega,j_\kappa}(t) - \sum_{j=n-\ell+1}^n a_{2_{i,j}} (x_i^{(2)}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} R_i \tan^{-1}(\bar{\lambda}t)) - \bar{\lambda} x_{n-\ell+i}(0) e^{-\bar{\lambda}t} \\ & \left. + \frac{2}{\pi} R_i \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \right| + \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \left| \sum_{k=n-\ell+1}^n -a_{2_{i,\omega}} c_{\omega,j_\kappa} a_{1_{j,k}} R_{k-n+\ell} \right| \frac{1 - e^{-h_{\omega,j_\kappa} t}}{h_{\omega,j_\kappa}} \\ & \leq M_i - \rho_i, \quad i = 1, \dots, \ell, \end{aligned} \tag{3.31}$$

where  $q_{\omega,j_\kappa}(t)$  is defined in (3.30), is satisfied for all  $t \geq 0$ .

*Proof.* The proof is similar to that of Corollary 1. □

**Remark 16.** *The left hand side of the inequality (3.31) can be written as  $|k_1 e^{-\bar{\lambda}t} + k_2 e^{-h_{\omega,j_\kappa} t} + k_3 \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} + k_4 \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) + k_5|$ , where  $k_i$ ,  $i = 1, 2, \dots, 5$  are the appropriate constant coefficients. Since all of the terms are monotonic functions, it's enough to check (3.31) at  $t = 0$ ,  $t = \infty$  and at the extremum points in between, if any. Extremum points can be found by taking the derivative and equating it to zero and finding the roots numerically.*

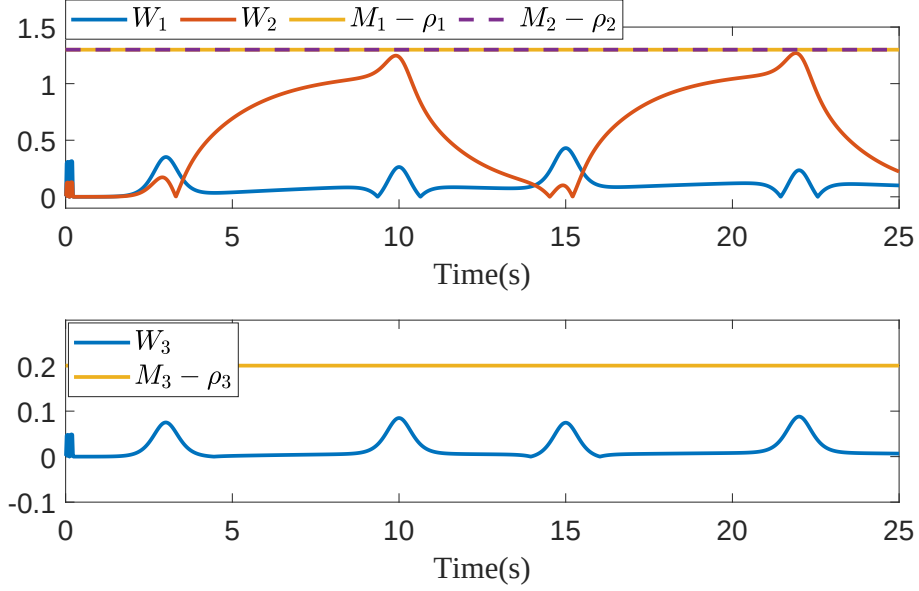


Figure 3.2: Validation of the inequality (3.29).

### 3.4 Simulation results

The Aerodata Model in Research Environment (ADMIRE), which represents the dynamics of an over-actuated aircraft model, is used to demonstrate the effectiveness of the proposed controller. The linearized ADMIRE model is used as the plant and the state and control matrices are provided in Chapter 2. To represent actuator loss of effectiveness and disturbance, a diagonal matrix  $\Lambda$  and a vector  $d_u$ , respectively, are introduced to the model as

$$\begin{aligned} \dot{x} &= Ax + B_u \Lambda u + B_u d_u = Ax + B_v v_s + B_v \bar{d} \\ v_s &= B \Lambda u + \bar{d}, \quad \bar{d} = B d_u, \quad B_u = B_v B, \quad B_v = [0_{3 \times 2} \quad I_{3 \times 3}]^T. \end{aligned} \quad (3.32)$$

A sinusoidal function with amplitude of 0.1 and frequency of 1 rad/s is considered as the disturbance  $\bar{d}$  in the simulation. Saturation limits are calculated as  $M_1 = 1.4$ ,  $M_2 = 1.4$  and  $M_3 = 0.3$ . Furthermore, the maximum and minimum range of the elements of  $\theta_v$  for the projection algorithm are taken similar to the values given in Chapter 2. (see [76] for how these values can be calculated). The proposed sliding mode controller is implemented using  $\bar{\lambda} = 3$  (see (3.24)). To avoid chattering, the boundary layer approach [84] is used. The actuator loss of

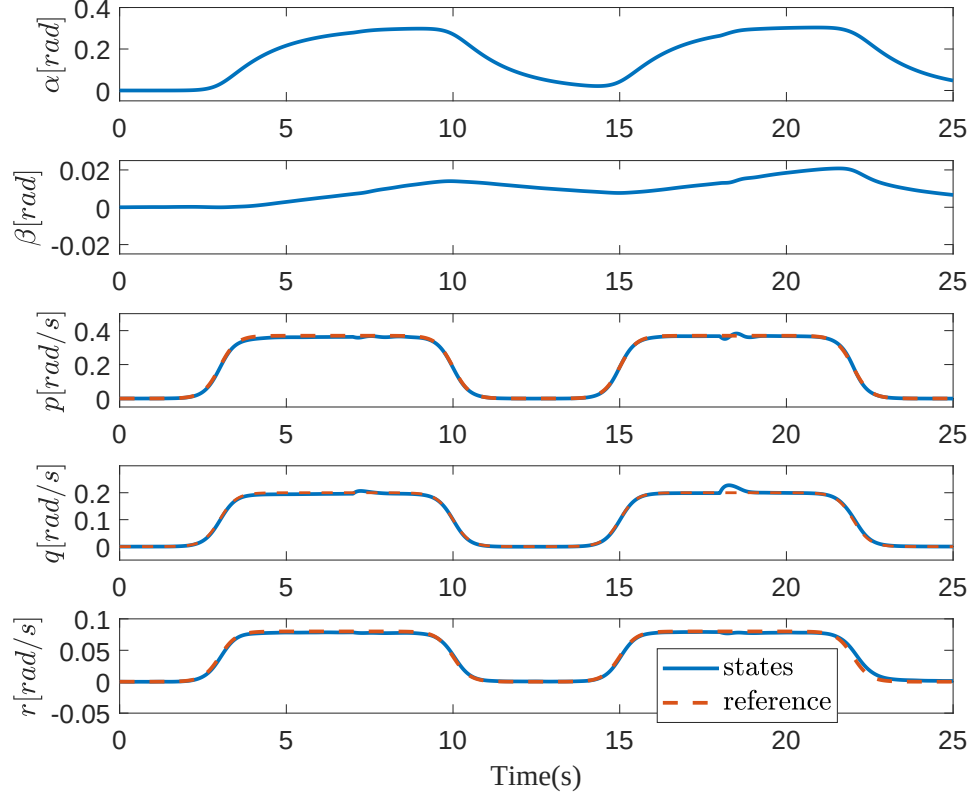


Figure 3.3: Time evolution of the aircraft states and the references, in the presence of actuator uncertainty. At  $t = 7$ s and  $t = 18$ s, 15% and 50% loss of effectivenesses are introduced, respectively.

effectiveness is modeled as

$$\Lambda(t) = \begin{cases} \text{diag}(1, 1, 1, 1) & \text{for } t < 7, \\ \text{diag}(0.85, 0.85, 0.85, 0.85) & \text{for } 7 \leq t \leq 18, \\ \text{diag}(0.5, 0.5, 0.5, 0.5) & \text{for } 18 < t. \end{cases}$$

Since  $A_{1,1}$  in the ADMIRE model has distinct real eigenvalues, we can use corollary 1, which requires (3.29) to be satisfied. The reference signals are chosen as

$$r_i(t) = \gamma_i \left( \frac{1}{1 + e^{-4(t-3)}} - \frac{1}{1 + e^{-4(t-10)}} + \frac{1}{1 + e^{-4(t-15)}} - \frac{1}{1 + e^{-4(t-22)}} \right), \quad i = 1, 2, 3, \quad (3.33)$$

where  $\gamma_1 = 0.37$ ,  $\gamma_2 = 0.2$  and  $\gamma_3 = 0.08$ . It is seen in Figure 3.2 that the inequality (3.29) is satisfied for  $i = 1, 2, 3$ .

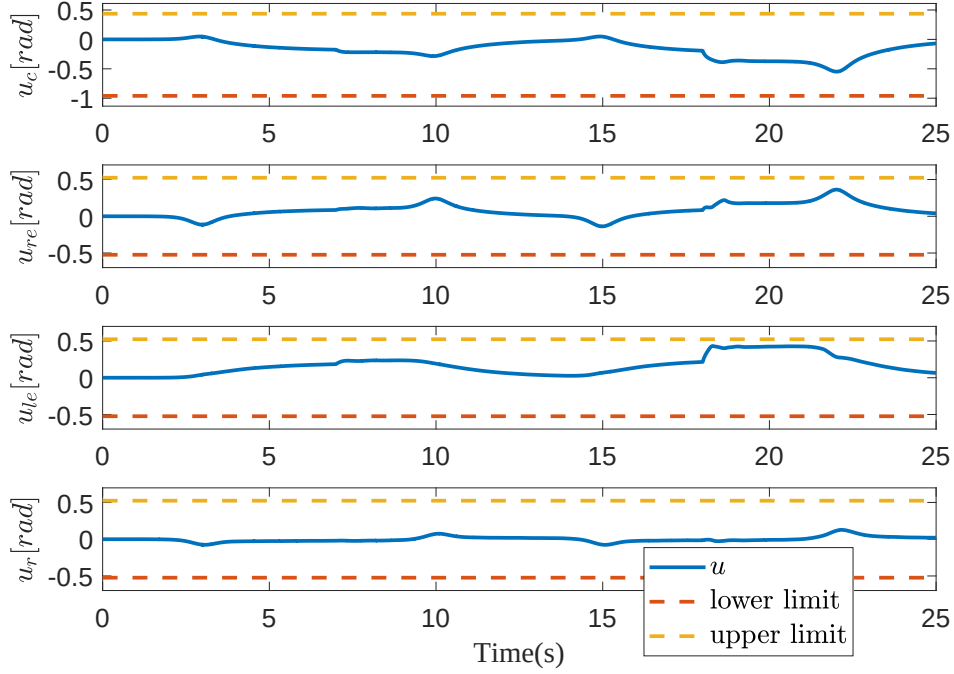


Figure 3.4: Time evolution of the control surfaces.

Figure 3.3 illustrates the time evolution of the states when 15% actuator loss of effectiveness occurs at  $t = 7s$  and 50% at  $t = 18s$ . The states remain bounded and  $p$ ,  $q$  and  $r$  follow the reference signals. Figure 3.4 demonstrates the control surface deflections together with the saturation limits. The actuators stay within their limits. The performance of the control allocation is depicted in Figure 3.5. It shows that the control allocation determines the actuator input vector  $u$  in such a way that  $BAu + \bar{d}$  follows the control signal  $v_s$ . Figure 3.6 shows the sliding surface trajectories, which tend to grow at the times of uncertainty injections but converge to a neighborhood of zero afterward, without any chattering effect.

In order to show the robustness of the proposed method, we examine the case where the inequality (3.29) is violated. The violation is realized by setting the parameters  $\gamma_i$ s in (3.33) to  $\gamma_1 = 0.38$ ,  $\gamma_2 = 0.32$  and  $\gamma_3 = 0.09$ . It is seen in Figure 3.7 that the inequality (3.29) is violated by  $W_2$ . Figures 3.8–3.11 illustrate that even though (3.29) is not satisfied, the system remains stable, while tracking its references.

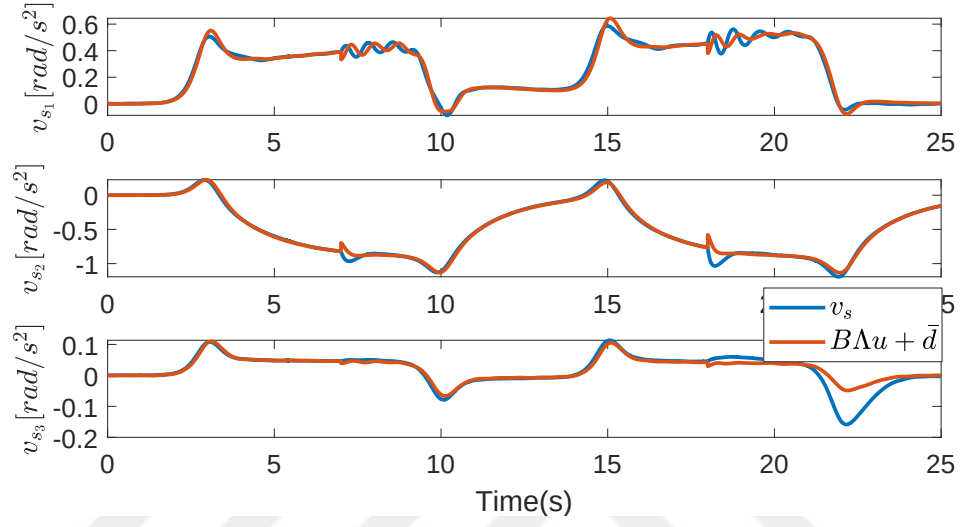


Figure 3.5: Control allocation performance.

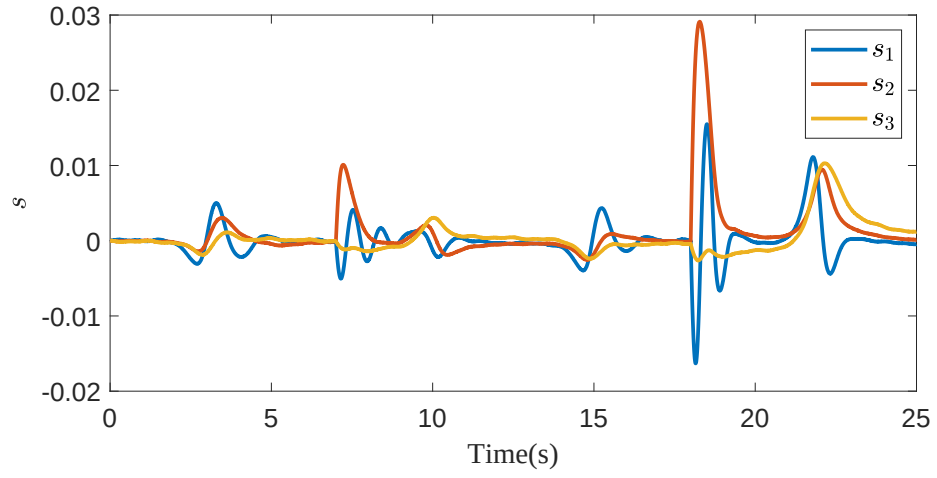


Figure 3.6: The evolution of the sliding surfaces.

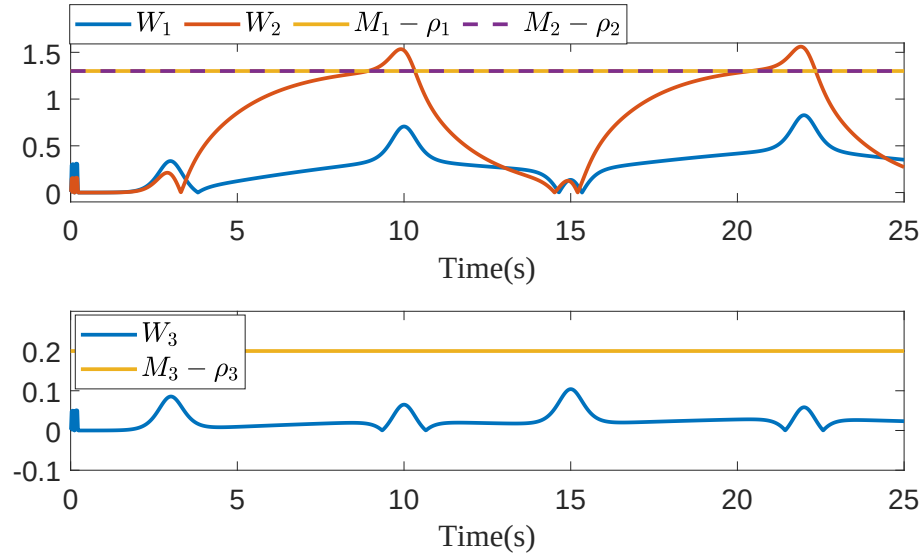


Figure 3.7: Validation of the inequality (3.29).

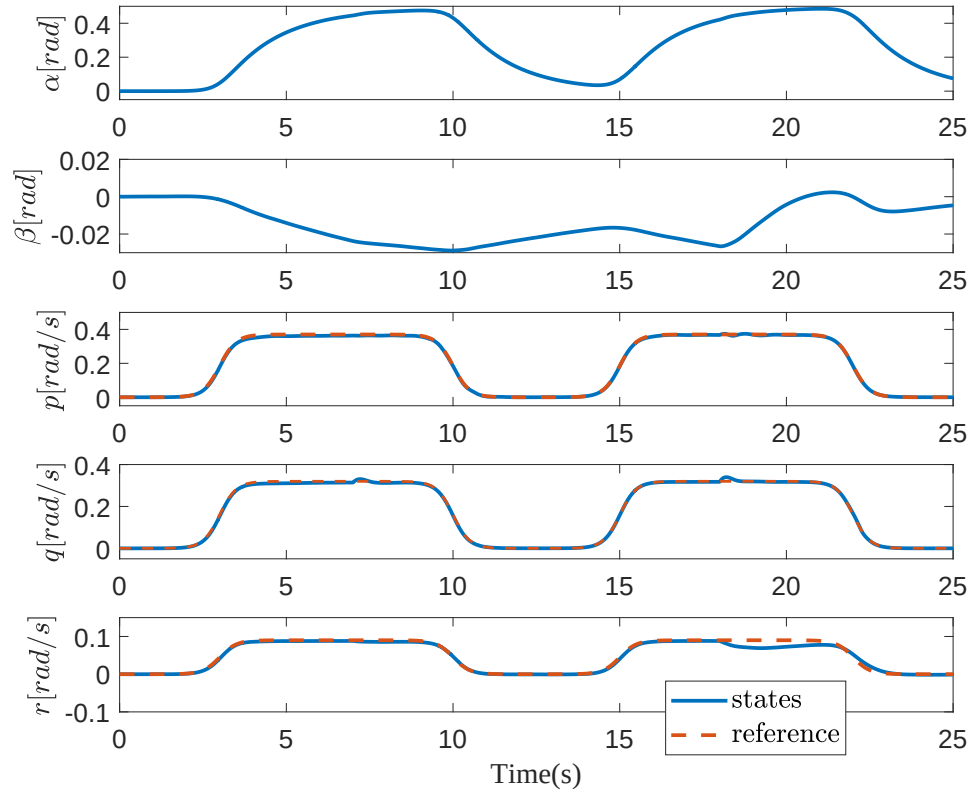


Figure 3.8: Time evolution of the states when (3.29) is violated.

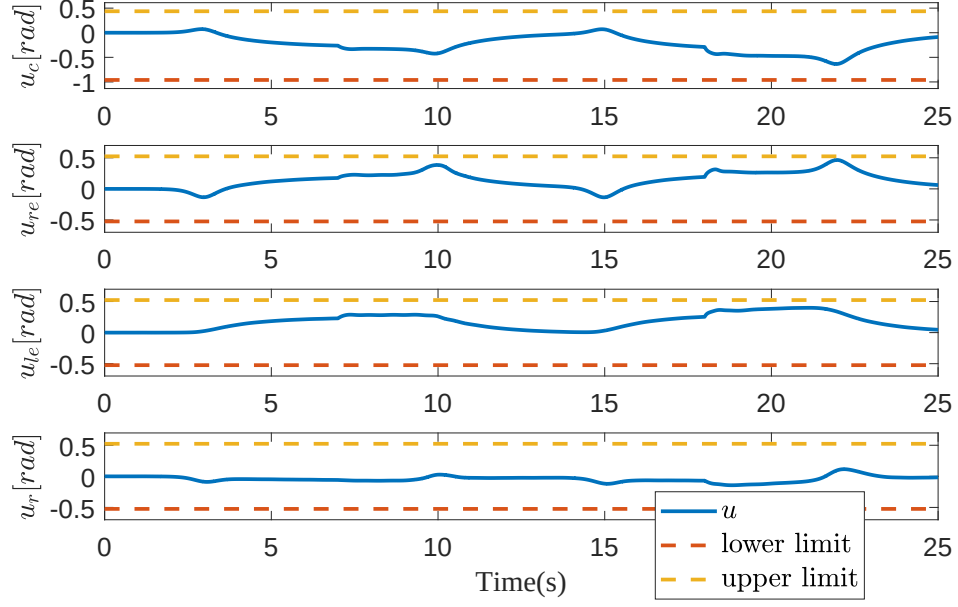


Figure 3.9: Time evolution of the control surfaces when (3.29) is violated.

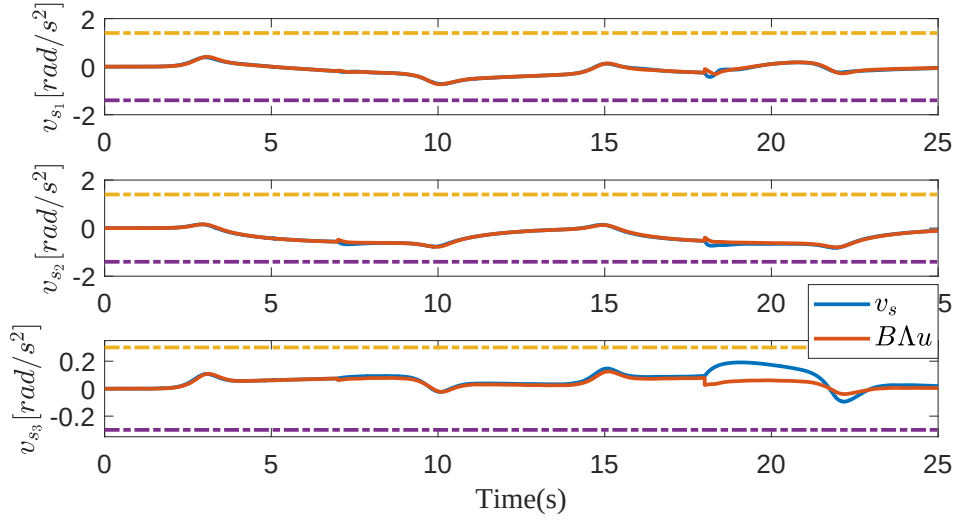


Figure 3.10: Control allocation performance when (3.29) is violated. Control allocation signal  $v_s$  does not saturate, and therefore  $v_s = v$  (see Figure 3.1).

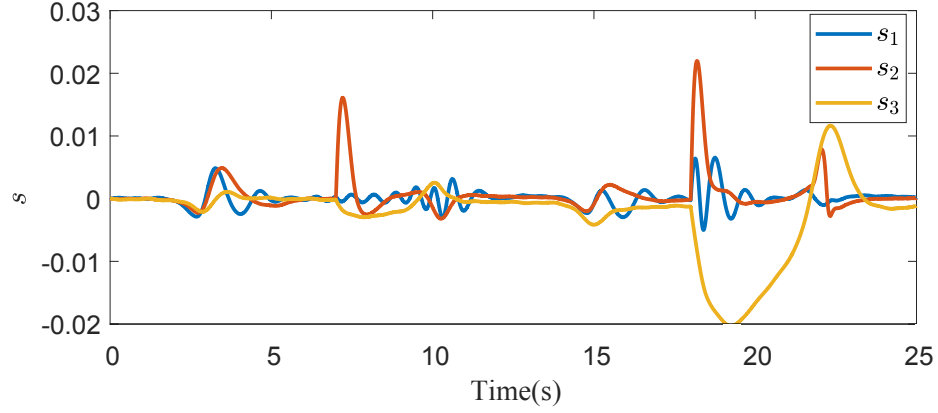


Figure 3.11: The evolution of the sliding surfaces when (3.29) is violated.

### 3.5 Summary

In this chapter, a sliding mode controller with a time-varying sliding surface is proposed to complement an adaptive control allocator for uncertain over-actuated systems with actuator saturation. Stability of the overall closed-loop system is shown using Lyapunov arguments. Simulation results with the ADMIRE model show the effectiveness of the proposed method.

## Chapter 4

# Handling actuator magnitude and rate saturation in uncertain over-actuated systems: A modified projection algorithm approach

### 4.1 Introduction

Actuator constraints such as magnitude and rate limits play a prominent role in advanced control systems. These limits induce nonlinear behavior which may lead to performance degradation, occurrence of limit cycles, multiple equilibria, and even instability [6, 7]. Actuator rate limits, specifically, introduce phase lags, which act as time delays, that can lead to persistent undesired oscillations called Pilot Induced Oscillations (PIO) [21, 25, 26, 29, 51, 52, 86]. These oscillations generally occur due to an abnormal coupling between the pilot and the aircraft, instigated by various factors such as high pilot gains, actuator rate saturation and control mode switches [87].

For systems with uncertainties, various adaptive controllers that account for actuator magnitude limits exist in the literature [88–91]. There are also adaptive approaches related to the problem of handling actuators that are constrained in both magnitude and rate. In the paper by [92], the approach presented by [90] and [91] is extended for systems with rate and magnitude limits. In the method proposed by [93], the reference inputs as well as the control signals are modified adaptively in order to guarantee the stability in the presence of magnitude and rate limits. In a recent work by [94], plant dynamics is augmented with the actuator dynamics, and an adaptive controller is introduced to compensate the effect of actuator magnitude and rate limits.

With the reduction of actuator costs due to advances in microprocessors, and with the help of actuator miniaturization, the utilization of redundant actuators have been growing in recent years. Actuator redundancy can improve the performance, maneuverability and the ability to tolerate system faults. The process of distributing control signals among redundant actuators is performed by control allocation. A study on control allocation that considers actuator magnitude constraints is conducted by [40] by using direct allocation method. Daisy chain control allocation method, which handles actuator magnitude limit, is employed by [95]. Actuator magnitude saturation of an unmanned underwater vehicle is considered using pseudo inverse based control allocation ([77]). An iterative approach based on the null space of the control matrix is proposed by [42], which handles actuator magnitude limits. Optimization based control allocation is one of the most common methods of accounting for actuator magnitude and rate constraints [21, 25, 32, 47, 48, 50, 78]. A sequential algorithm to solve optimization based control allocation is proposed by [79]. A survey on control allocation methods can be found in the study conducted by [60]. A recent control allocation study is presented by [64], where model predictive control is employed to handle actuator magnitude constraints.

When a system has uncertain dynamics, together with redundant actuators, it is natural to consider an adaptive control allocator to achieve the task of distributing the total control effort among actuators. There exists few approaches

presented in the literature that addresses the topic of adaptive control allocation. The method proposed by [55] reduces the difference between virtual and actual control signals, and guarantees that the control signals ultimately converge to an optimal set. An adaptive control allocation for a hexacopter system is proposed by [56]. A model reference adaptive control allocation structure is proposed by [57]. This method is also extended to handle actuator magnitude limits [58, 76, 96].

Projection algorithm is an appealing approach in robust adaptive control design. Restricting adaptive parameters while ensuring the stability of the closed loop system, simultaneously, is a prominent benefit of employing this algorithm in adaptive systems. It is noted that existing projection algorithms ([8, 9]) bound adaptive parameters' magnitudes and thus do not have a straightforward utility to handle actuator rate limits. In this thesis, we propose a projection algorithm that can be used in adaptive control allocation implementations, where actuators are both magnitude and rate limited. Therefore, the contribution of this chapter is a projection algorithm that can handle magnitude and rate limited redundant actuators for systems with uncertain dynamics, where a control allocator is utilized in the controller structure. We show that, the existence and uniqueness of the solution of the differential equation describing the proposed projection algorithm can be guaranteed. Furthermore, we provide a performance guarantee, in terms of error bounds, for the exploited adaptive control allocation, which is possible thanks to the proposed projection algorithm.

To summarize, we propose an answer to this question: “How can we modify the conventional projection algorithm, so that we can employ it in adaptive control allocation implementations where actuators are both magnitude and rate saturated?” To the best of our knowledge, this question is not answered earlier. It needs to be emphasized that a control allocator is not a controller and cannot be replaced as a controller. The duty of the control allocation is distributing the controller signal, or the total control input, among redundant actuators. The method proposed in this chapter is for the systems where an adaptive control allocator is used in the loop. We are not proposing a new controller or a new control allocation method.

This chapter is organized as follow. Notations used throughout this chapter and the conventional, element-wise projection algorithm and its properties are given in Section 4.2. Section 4.3 presents the uncertain over-actuated system along with the adaptive control allocation utilizing the conventional projection algorithm. The proposed modified projection algorithm and its characteristics are presented in Section 4.4. The ADMIRE model is used in Section 4.5 to illustrate the effectiveness of the proposed methodology in the simulation environment. Finally, a summary is given in Section 4.6.

## 4.2 Notations and preliminaries

Throughout this work,  $\text{tr}(\cdot)$  refers to the trace operation. The over-dot notation will be used for time derivatives only, i.e.  $(\dot{\cdot}) = d(\cdot)/dt$ .

Consider  $Y \in \mathbb{R}^{r \times m}$  and  $\theta_v \in \mathbb{R}^{r \times m}$ . The element-wise projection operator  $\text{Proj}(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$\text{Proj}(\theta_{v_{i,j}}, Y_{i,j}) \equiv \begin{cases} Y_{i,j} - Y_{i,j} f_{i,j} & \text{if } f_{i,j} > 0 \quad \& \quad Y_{i,j} \left( \frac{df_{i,j}}{d\theta_{v_{i,j}}} \right) > 0 \\ Y_{i,j} & \text{otherwise,} \end{cases} \quad (4.1)$$

where  $\theta_{v_{i,j}}$  and  $Y_{i,j}$  refer to the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $\theta_v$  and  $Y$ , respectively, and where  $f_{i,j}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is a convex and continuously differentiable function defined as

$$f_{i,j} = f(\theta_{i,j}) = \frac{(\theta_{v_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})(\theta_{v_{i,j}} - \theta_{\max_{i,j}} + \zeta_{i,j})}{(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}}, \quad (4.2)$$

where  $\zeta_{i,j} \in \mathbb{R}^+$  is the projection tolerance of  $\theta_{v_{i,j}}$  such that  $\zeta_{i,j} < 0.5(\theta_{\max_{i,j}} - \theta_{\min_{i,j}})$ ,  $\theta_{\max_{i,j}} - \zeta_{i,j} > 0$  and  $\theta_{\min_{i,j}} + \zeta_{i,j} < 0$ .  $\theta_{\max_{i,j}} > 0$  and  $\theta_{\min_{i,j}} < 0$  are the upper and lower bounds of the  $(i, j)^{\text{th}}$  element of  $\theta_v$ . Therefore, the projection operator  $\text{Proj}(\theta_v, Y)$  operates on the elements of  $\theta_v$  and  $Y$  using (4.1) and (4.2).

The following lemmas are useful in proving the main theorems where projection algorithm is used ([8, 9, 67]).

**Lemma 4.** *If an adaptive algorithm with adaptive law  $\dot{\theta}_{v_{i,j}} = \text{Proj}(\theta_{v_{i,j}}, Y_{i,j})$  and initial conditions  $\theta_{v_{i,j}}(0) \in \Omega_{i,j} = \{\theta_{v_{i,j}} \in \mathbb{R} | f(\theta_{v_{i,j}}) \leq 1\}$ , where  $f(\theta_{v_{i,j}}) : \mathbb{R} \rightarrow \mathbb{R}$  is defined as in (4.2), then  $\theta_{v_{i,j}} \in \Omega_{i,j}$  for  $\forall t \geq 0$ .*

*Proof.* The proof of Lemma 4 can be found in [9]. □

**Lemma 5.** *Let  $\theta_{v_{i,j}}^* \in [\theta_{\min_{i,j}} + \zeta_{i,j}, \theta_{\max_{i,j}} - \zeta_{i,j}]$ , and consider the projection algorithm in (4.1) with convex function (4.2), the following inequality holds:*

$$\text{tr}((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}(\theta_v, Y))) \leq 0. \quad (4.3)$$

*Proof.* The proof of Lemma 5 can be found in [9]. □

### 4.3 Problem statement

In this section, firstly, the over-actuated plant with constrained uncertain actuators is introduced. Then, the adaptive control allocation utilizing the conventional projection algorithm (4.1), which can bound only the magnitude of actuators input signals, is presented. Finally, the problem statement motivating the proposed projection algorithm is given.

Consider the following uncertain over-actuated plant dynamics

$$\begin{aligned} \dot{x} &= Ax + B_u \Lambda u \\ &= Ax + B_v B \Lambda u \\ &= Ax + B_v v_s, \end{aligned} \quad (4.4)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$  is the magnitude constrained actuator command vector, where  $u_j \in [u_{\min_j}, u_{\max_j}]$  with  $u_{\max_j} > 0$  and  $u_{\min_j} < 0$ . The matrix  $A \in \mathbb{R}^{n \times n}$  is the known state matrix and  $B_u = B_v B \in \mathbb{R}^{n \times m}$  is the known rank deficient control input matrix which is decomposed into the known matrices  $B_v \in \mathbb{R}^{n \times r}$  and  $B \in \mathbb{R}^{r \times m}$  such that  $\text{rank}(B) = \text{rank}(B_v) = r$ . The actuator loss of effectiveness is modeled as a

diagonal matrix  $\Lambda \in \mathbb{R}^{m \times m}$  with uncertain positive elements. The goal of the static control allocation methods in the absence of uncertainty, where  $\Lambda = I_m$ , is to distribute the total control effort  $v_s \in \mathbb{R}^r$ , produced by a controller, to the redundant actuators such that  $Bu = v_s$ . In the presence of uncertainty, the static control allocation methods are not applicable since the goal of the control allocation becomes

$$B\Lambda u = v_s. \quad (4.5)$$

One way to achieve (4.5) is by employing the following control allocation system proposed by [57]

$$\dot{\xi} = A_m \xi + B\Lambda u - v_s, \quad (4.6a)$$

$$\dot{\xi}_m = A_m \xi_m, \quad (4.6b)$$

$$\dot{\theta}_v = g(\theta_v, Y(v_s, e)), \quad (4.6c)$$

$$u = \theta_v^T v_s, \quad (4.6d)$$

where  $\xi \in \mathbb{R}^r$  is the output of the virtual dynamics,  $\theta_v \in \mathbb{R}^{r \times m}$  is the adaptive parameter to be updated,  $\xi_m \in \mathbb{R}^r$  is the output of the reference model,  $e = \xi - \xi_m$ , (4.6b) is the reference model with a Hurwitz matrix  $A_m \in \mathbb{R}^{r \times r}$ , (4.6c) is the adaptive law where  $g(.,.) : \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times m} \rightarrow \mathbb{R}^{r \times m}$  is a projection algorithm, and  $u$  is the control allocation signal, or the actuator command signal. It can be shown that ([57]), in the absence of actuator limits,  $e$  converges to zero and thus the control allocation goal (4.5) is achieved. In the presence of actuator magnitude limits,  $e$  converges to a predetermined compact set ([76,96]).

In the presence of actuator magnitude limits, if the control signal  $v_s$  is bounded, then (4.6d) shows that in order to produce actuator command signals  $u_j, j = 1, \dots, m$ , that respect the actuator saturation bounds, such that  $u_j \in [u_{\min_j}, u_{\max_j}]$ , the elements of the adaptive parameter matrix  $\theta_v$  should be appropriately bounded. It is shown in [76,96] that this could be achieved, together with the stability of the overall system dynamics, by using the conventional projection operator (4.1) as the function  $g$  in (4.6c).

Problem statement: If the actuators in (4.4) are not only magnitude saturated but also rate saturated, i.e.  $\dot{u}_j \in [\dot{u}_{\min_j}, \dot{u}_{\max_j}]$ ,  $j = 1, \dots, m$ , how should the

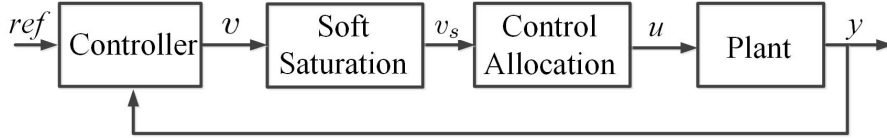


Figure 4.1: Closed loop control system.

projection algorithm (4.1), which is used as the function  $g$  in (4.6c), be modified to handle this additional condition?

To address the above problem, we need to reconstruct the conventional projection algorithm (4.1) such that not only the magnitude but also the rate of change of the elements of the matrix  $\theta_v$  become bounded. This problem needs to be solved in such a way that the new projection algorithm must have useful properties similar to the ones given in Lemma 4 and Lemma 5, to ensure the stability of the closed loop control system. In the next section, this new projection algorithm is introduced.

## 4.4 Modified projection algorithm

The structure of the overall closed loop control system considered in this chapter, consisting of the controller, the control allocator and the plant, is presented in Figure 4.1. The soft saturation introduced after the controller ensures that the input of the control allocator,  $v_s$ , and its derivative,  $\dot{v}_s$ , are bounded. From (4.6c) and (4.6d), it can be seen that one way to obtain a bounded actuator command signal  $u$  is to restrict both the magnitude and the rate of change of the adaptive parameter matrix  $\theta_v$ . This restriction must be achieved while ensuring the boundedness of all the signals in the closed loop control system. It is noted that a rate and magnitude bounded total control input  $v_s$  does not guarantee a rate and magnitude bounded actuator input signal vector  $u$ , due to the nature of the adaptation in the control allocator.

The approach proposed in this chapter for bounding the adaptive parameter matrix  $\theta_v$  in terms of both magnitude and rate is based on projecting  $Y_{i,j}$  and  $\theta_{v_{i,j}}$ , simultaneously. In this method, apart from the function  $f_{i,j}$  introduced in (4.2), another convex and continuously differentiable function given as

$$h_{i,j} = h(Y_{i,j}) = \frac{(Y_{i,j} - Y_{\min_{i,j}} - \epsilon_{i,j})(Y_{i,j} - Y_{\max_{i,j}} + \epsilon_{i,j})}{(Y_{\max_{i,j}} - Y_{\min_{i,j}} - \epsilon_{i,j})\epsilon_{i,j}} \quad (4.7)$$

is introduced, where  $Y_{\max_{i,j}} > 0$  and  $Y_{\min_{i,j}} < 0$  are the allowable maximum and minimum bounds of  $Y_{i,j}$ , respectively, and  $\epsilon_{i,j} \in \mathbb{R}^+$  is the projection tolerance such that  $Y_{\max_{i,j}} - \epsilon_{i,j} > 0$  and  $Y_{\min_{i,j}} + \epsilon_{i,j} < 0$ .

Using (4.2) and (4.7), an element-wise, modified projection algorithm is proposed as

$$\begin{aligned} & \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) \\ & \equiv \begin{cases} Y_{i,j}(1 - \hat{f}_{i,j})(1 - \hat{h}_{i,j}) & \text{if } f_{i,j} \geq 0 \quad \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0 \quad \& \quad h_{i,j} \geq 0 \\ Y_{i,j}(1 - \hat{f}_{i,j}) & \text{if } f_{i,j} > 0 \quad \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} > 0 \\ Y_{i,j}(1 - \hat{h}_{i,j}) & \text{if } h_{i,j} > 0 \\ Y_{i,j} & \text{otherwise,} \end{cases} \end{aligned} \quad (4.8)$$

where  $\hat{f}_{i,j} = \min\{1, f_{i,j}\}$  and  $\hat{h}_{i,j} = \min\{1, h_{i,j}\}$ .

Using this projection algorithm, the adaptive law is given as  $\dot{\theta}_{v_{i,j}} = \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$ . In the proposed projection algorithm defined in (4.8), when  $\theta_{v_{i,j}}$  reaches its boundary value ( $\theta_{\max_{i,j}}$  or  $\theta_{\min_{i,j}}$ ),  $f_{i,j}$  reaches 1, and from the first and second conditions of (4.8),  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  reaches zero. When  $Y_{i,j}$  reaches its boundary value ( $Y_{\max_{i,j}}$  or  $Y_{\min_{i,j}}$ ),  $h_{i,j}$  reaches 1, and from the first and third conditions of (4.8),  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  reaches zero. In addition, since  $f_{i,j}$  and  $h_{i,j}$  cannot exceed one, the magnitude and rate of  $\theta_{v_{i,j}}$  are both bounded. A formal proof is given below, in Lemma 6. It is noted that it is not necessary to take the time derivative of any signal to implement the proposed projection algorithm.

**Lemma 6.** *Given the adaptive law  $\dot{\theta}_{v_{i,j}} = \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$ , where the projection operator is given in (4.8), together with convex and continuously differentiable*

functions (4.2) and (4.7), if the initial conditions are defined as  $\theta_{v_{i,j}}(0) \in \Omega_{i,j} = \{\theta_{v_{i,j}} \in \mathbb{R} | f(\theta_{v_{i,j}}) \leq 1\}$  and  $Y_{i,j}(0) \in \bar{\Omega}_{i,j} = \{Y_{i,j} \in \mathbb{R} | h(Y_{i,j}) \leq 1\}$ , then  $\theta_{v_{i,j}}(t) \in \Omega_{i,j}$  and  $Y_{i,j}(t) \in \bar{\Omega}_{i,j}$  for all  $t \geq 0$ .

*Proof.* See Appendix C.1. □

Below, in Lemma 7, a property of the proposed projection algorithm, which is analogous to Lemma 5, is given, which will be useful later in the stability investigation.

**Lemma 7.** Let  $\theta_{v_{i,j}}^* \in [\theta_{\min_{i,j}} + \zeta_{i,j} \quad \theta_{\max_{i,j}} - \zeta_{i,j}]$ ,  $Y_{i,j}(0) \in \bar{\Omega}_{i,j} = \{Y_{i,j} \in \mathbb{R} | h(Y_{i,j}) \leq 1\}$ , and consider the projection algorithm (4.8) with convex functions (4.2) and (4.7). The inequality

$$\text{tr}((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}_m(\theta_v, Y))) \leq \|\tilde{\theta}_{\max}\|_F \|Y_{MAX}\|_F \quad (4.9)$$

holds, where  $\tilde{\theta}_{\max}$  and  $Y_{MAX}$  are the matrices whose elements constitute the upper bounds of the absolute values of the elements of  $\tilde{\theta}$  and  $Y$ , respectively.

*Proof.* See Appendix C.2. □

Discontinuity in the projection algorithm is not desirable and may cause numerical problems. In the following lemma, we prove that the proposed projection algorithm is continuous.

**Lemma 8.** For continuous  $\theta_{v_{i,j}}$  and  $Y_{i,j}$ , the function  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) : S_\theta \times S_Y \rightarrow \mathbb{R}$ , where  $S_\theta, S_Y \subset \mathbb{R}$ , is continuous.

*Proof.* See Appendix C.3. □

The final step before presenting the main theorem of this study is showing that the solution of the differential equation providing the parameter adaptation law  $\dot{\theta}_{v_{i,j}} = \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$ , actually exists and is unique. Considering that  $\theta_{v_{i,j}}$  and  $Y_{i,j}$  are piecewise continuous functions of time, it is enough to prove that  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  is locally Lipschitz to show existence and uniqueness.

**Lemma 9.** *The function  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) : S_\theta \times S_Y \rightarrow \mathbb{R}$ , where  $S_\theta, S_Y \subset \mathbb{R}$ , is locally Lipschitz.*

*Proof.* See Appendix C.4. □

After defining the modified projection algorithm, proving its properties that will be useful in the stability analysis of the closed loop system, and proving the existence and uniqueness of the solution of the differential equation describing the algorithm, we provide the main theorem below, stating that when the proposed projection algorithm is employed, all the signals in the adaptive control allocation system, in the presence of actuator magnitude and rate saturation, remains bounded and the control allocation error converges to a predetermined closed set.

**Theorem 7.** *Consider the actuator command signal  $u$  produced by the adaptive control allocation (4.6) with  $g(\theta_v, Y(v_s, e)) = \Gamma \text{Proj}_m(\theta_v, Y(v_s, e))$ , where  $\Gamma$  is a diagonal positive definite matrix and the projection operator is defined in (4.8) with convex functions (4.2) and (4.7). If  $Y = -v_s e^T P B$ , where  $P$  is the positive definite symmetric matrix solution of the Lyapunov equation  $A_m^T P + P A_m = -Q$  with a symmetric positive definite matrix  $Q$ , then  $\tilde{\theta}_v$  and  $e$  remain bounded and converge to the compact set*

$$E_2 = \{(e, \tilde{\theta}_v) : \|e\|^2 \leq \frac{2\|\tilde{\theta}_v\|_F^2 \|Y_{MAX}\|_F}{\lambda_{\min}(Q)}, \|\tilde{\theta}\| \leq \tilde{\theta}_{\max}\}. \quad (4.10)$$

Moreover, the design parameters  $\theta_{\min_{i,j}}$ ,  $\theta_{\max_{i,j}}$ ,  $Y_{\min_{i,j}}$  and  $Y_{\max_{i,j}}$  in (4.2) and (4.7) can be chosen such that for  $v_s \in \Omega_v = \{v \mid -M_i \leq v_i \leq M_i, -L_i \leq \dot{v}_i \leq L_i, i = 1, \dots, r\}$ , where  $M_i$  and  $L_i$  are positive scalars for  $i = 1, \dots, r$ ,  $u$  remains in  $\Omega_u = \{u \mid u_{\min_j} \leq u_j \leq u_{\max_j}, \bar{u}_{\min_j} \leq \dot{u}_j \leq \bar{u}_{\max_j}, j = 1, \dots, m\}$ , where  $u_{\min_j}$ ,  $u_{\max_j}$ ,  $\bar{u}_{\min_j}$ ,  $\bar{u}_{\max_j}$  are actuator magnitude and rate constraints.

*Proof.* Substituting (4.6d) into (4.6a), we obtain that

$$\dot{\xi} = A_m \xi + (B \Lambda \theta_v^T - I) v_s. \quad (4.11)$$

It is assumed that there exists an ideal adaptive parameter,  $\theta_v^*$ , such that

$$B \Lambda \theta_v^{*T} = I. \quad (4.12)$$

Since  $B\Lambda$  is a full row rank matrix, this assumption is always valid. Defining  $\theta_v^T = \theta_v^{*T} + \tilde{\theta}_v^T$ , where  $\tilde{\theta}_v^T$  is the deviation of  $\theta_v^T$  from its ideal value, (4.11) can be rewritten as

$$\dot{\xi} = A_m \xi + B\Lambda \tilde{\theta}_v^T v_s. \quad (4.13)$$

Using (4.6b) and (4.13), the error dynamics is obtained as

$$\dot{e} = A_m e + B\Lambda \tilde{\theta}_v^T v_s. \quad (4.14)$$

Consider a Lyapunov function candidate

$$V = e^T P e + \text{tr}(\tilde{\theta}_v^T \Gamma^{-1} \dot{\tilde{\theta}}_v \Lambda). \quad (4.15)$$

The derivative of  $V$  along the trajectories of (4.6) can be calculated as

$$\begin{aligned} \dot{V} &= e^T (A_m^T P + P A_m) e + 2e^T P B \Lambda \tilde{\theta}_v^T v_s + 2\text{tr}(\tilde{\theta}_v^T \Gamma^{-1} \dot{\tilde{\theta}}_v \Lambda) \\ &= -e^T Q e + 2e^T P B \Lambda \tilde{\theta}_v^T v_s + 2\text{tr}(\tilde{\theta}_v^T \Gamma^{-1} \dot{\tilde{\theta}}_v \Lambda). \end{aligned} \quad (4.16)$$

Using the property of the trace operation  $a^T b = \text{tr}(b a^T)$  where  $a$  and  $b$  are vectors, (4.16) can be rewritten as

$$\dot{V} = -e^T Q e + 2\text{tr}(\tilde{\theta}_v^T (v_s e^T P B + \Gamma^{-1} \dot{\tilde{\theta}}_v) \Lambda). \quad (4.17)$$

Substituting modified adaptive control law (4.6c) into (4.17), the derivative of the Lyapunov function candidate is obtained as

$$\dot{V} = -e^T Q e + 2\text{tr}(\tilde{\theta}_v^T (v_s e^T P B + \text{Proj}_m(\theta_v, -v_s e^T P B)) \Lambda). \quad (4.18)$$

By using Lemma 7, we get

$$\dot{V} \leq -\lambda_{\min}(Q) \|e\|^2 + 2\|\tilde{\theta}_v\|_F^2 \|Y_{MAX}\|_F, \quad (4.19)$$

where  $\lambda_{\min}(\cdot)$  denotes the minimum eigenvalue.  $\dot{V} \leq 0$  for  $\|e\|^2 \geq (2\|\tilde{\theta}_v\|_F^2 \|Y_{MAX}\|_F) / (\lambda_{\min}(Q))$ . Therefore, for any initial conditions  $e(0)$  and  $\tilde{\theta}_v(0)$ , if  $\|\tilde{\theta}_v(0)\| \leq \tilde{\theta}_{max}$ , where  $\tilde{\theta}_{max}$  is the predetermined upper bound for  $\tilde{\theta}_v$ ,  $e(t)$  and  $\tilde{\theta}_v(t)$  are bounded for all  $t \geq 0$  and their trajectories converge to the following compact set ([67]),

$$E_2 = \{(e, \tilde{\theta}_v) : \|e\|^2 \leq \frac{2\|\tilde{\theta}_v\|_F^2 \|Y_{MAX}\|_F}{\lambda_{\min}(Q)}, \|\tilde{\theta}_v\| \leq \tilde{\theta}_{max}\}. \quad (4.20)$$

Using Lemma 6, if the initial conditions are defined as  $\theta_{v_{i,j}}(0) \in \Omega_{i,j} = \{\theta_{v_{i,j}} \in \mathbb{R} | f(\theta_{v_{i,j}}) \leq 1\}$  and  $Y_{i,j}(0) \in \bar{\Omega}_{i,j} = \{Y_{i,j} \in \mathbb{R} | h(Y_{i,j}) \leq 1\}$ , then  $\theta_{v_{i,j}}(t) \in \Omega_{i,j}$  and  $Y_{i,j}(t) \in \bar{\Omega}_{i,j}$  for all  $t \geq 0$ . For a bounded  $v_s \in \Omega_v$ , suitable values of  $\theta_{max_{i,j}}$ ,  $\theta_{min_{i,j}}$ ,  $Y_{max_{i,j}}$  and  $Y_{min_{i,j}}$  can be found to be used in  $f(\theta_{i,j})$  and  $h(Y_{i,j})$  that ensure  $u_j \in [u_{min_j}, u_{max_j}]$  and  $\dot{u}_j \in [\bar{u}_{min_j}, \bar{u}_{max_j}]$ ,  $j = 1, \dots, m$  for all  $t \geq 0$ .  $\square$

**Remark 17.** *It should be noted that control allocation's task is to distribute the total control effort produced by a controller among redundant actuators. The investigated control allocation method and the proposed projection algorithm in this chapter can be used with various different types of controllers.*

**Remark 18.** *Although the employment of the proposed projection algorithm is exemplified on an adaptive control allocation implementation, the proposed method can be extended to be used for other adaptive systems where the actuators are both magnitude and rate saturated.*

## 4.5 Simulation results

The closed loop control structure depicted in Figure 4.1, with ADMIRE model as the plant to be controlled, is used for the simulations. The reference signal is  $ref = [p_{ref}, q_{ref}, r_{ref}]^T$ , where  $p_{ref}$ ,  $q_{ref}$  and  $r_{ref}$  are the desired roll, pitch and yaw rates, respectively. The effectiveness of the actuators are reduced by 30% at  $t = 6s$ . The total control input (see Figure 4.1)  $v \in \mathbb{R}^r$  can be designed using a proper control method. For the simulations conducted in this chapter, we use the controller provided by [76, 96].

Three different cases are simulated. Figure 4.2 shows the evolution of the system states, total control input signals,  $v_i$ ,  $i = 1, 2, 3$ , and the adaptive parameters,  $\theta_v$ , in the presence of actuator magnitude saturation and conventional projection algorithm (4.1). It is seen that all the signals are bounded and  $p$ ,  $q$  and  $r$  track their references. Also, the total control input  $v$  is realized reasonably well.

In the second case, actuators are both magnitude and rate limited and again

the conventional projection algorithm is used. It is shown in Figure 4.3 that the overall closed loop system shows oscillatory behavior under these conditions.

Finally, in the third case, the proposed projection algorithm is applied in the presence of both magnitude and rate saturation. Figure 4.4 demonstrates the resulting stable and oscillation-free system response.

The effect of the conventional and the proposed projection algorithms on the actuator input signals are presented separately, in Figures 4.5-4.7, to emphasize the ability of the latter to limit the signal rates. Figure 4.5 shows that the conventional projection algorithm is able to limit the actuator signals within predefined values, when the actuators are only magnitude limited. When actuators are both magnitude and rate limited, the conventional projection algorithm fails to limit the rate of change of actuator signals. This is shown in Figure 4.6, where  $u_{le}$  (yellow line) and  $u_r$  (purple line) increase faster than the rate limit (dashed green line). Finally, Figure 4.7 shows that the proposed projection algorithm is capable of limiting both the magnitude and the rate of actuator signals. This can be deduced from the observation that the rate of change of the fastest growing actuator signal,  $u_{le}$  (yellow line), grows still slower than the rate limit (dashed green line).

## 4.6 Summary

A modified projection algorithm that is capable of bounding both the magnitude and rate of change of adaptive parameters is proposed in this chapter. This method can be combined with an adaptive control allocator for the control of uncertain over-actuated systems with constrained actuators. The existence and uniqueness of the solutions of the differential equation describing the proposed projection algorithm are shown. Furthermore, properties of the modified projection algorithm that are instrumental for the stability analysis are proven. The performance of the exploited control allocator, in terms of the error bounds, is

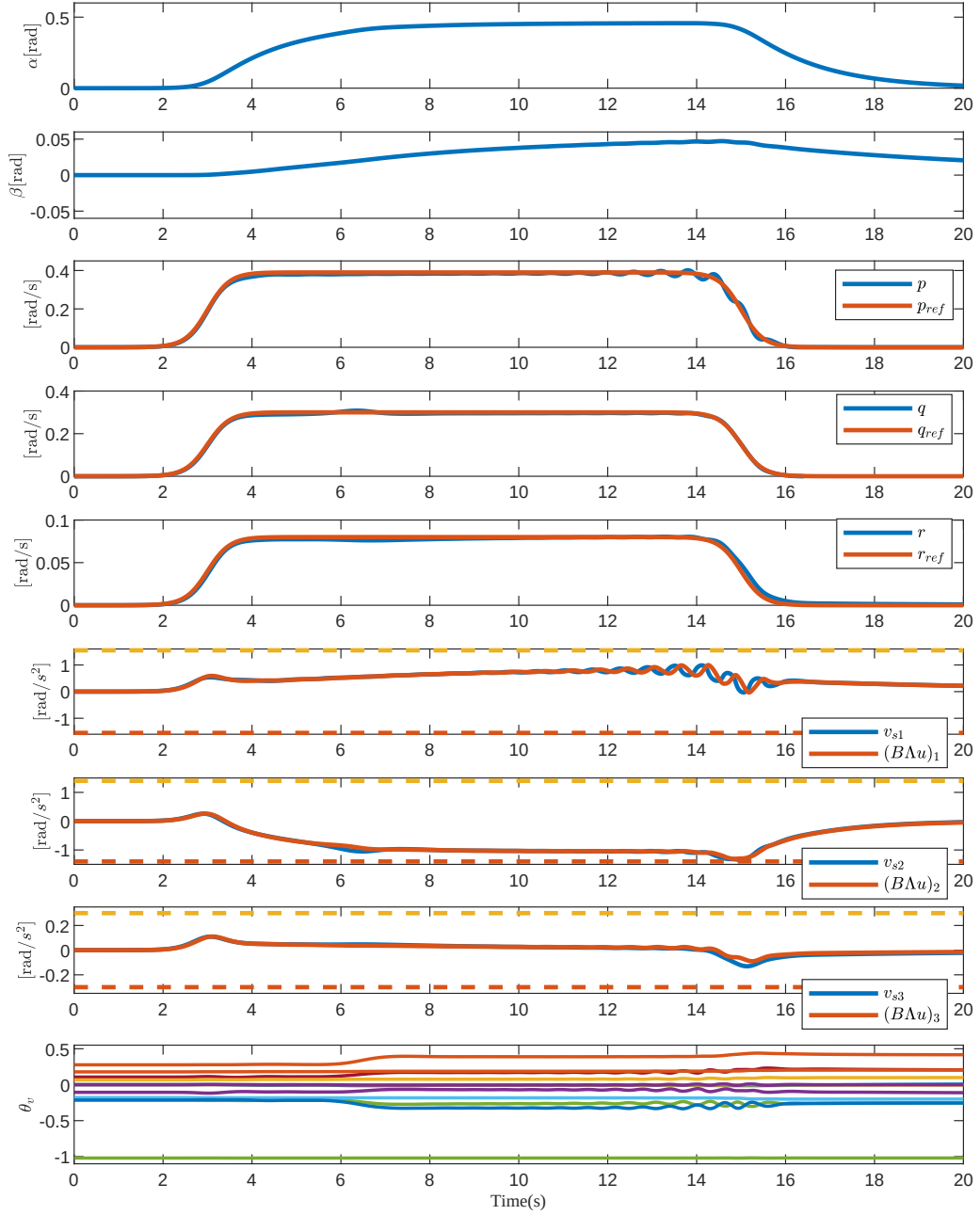


Figure 4.2: Case I: Evolution of the states, total control inputs and adaptive parameters in the presence of **magnitude saturation**, using the **conventional projection method**.

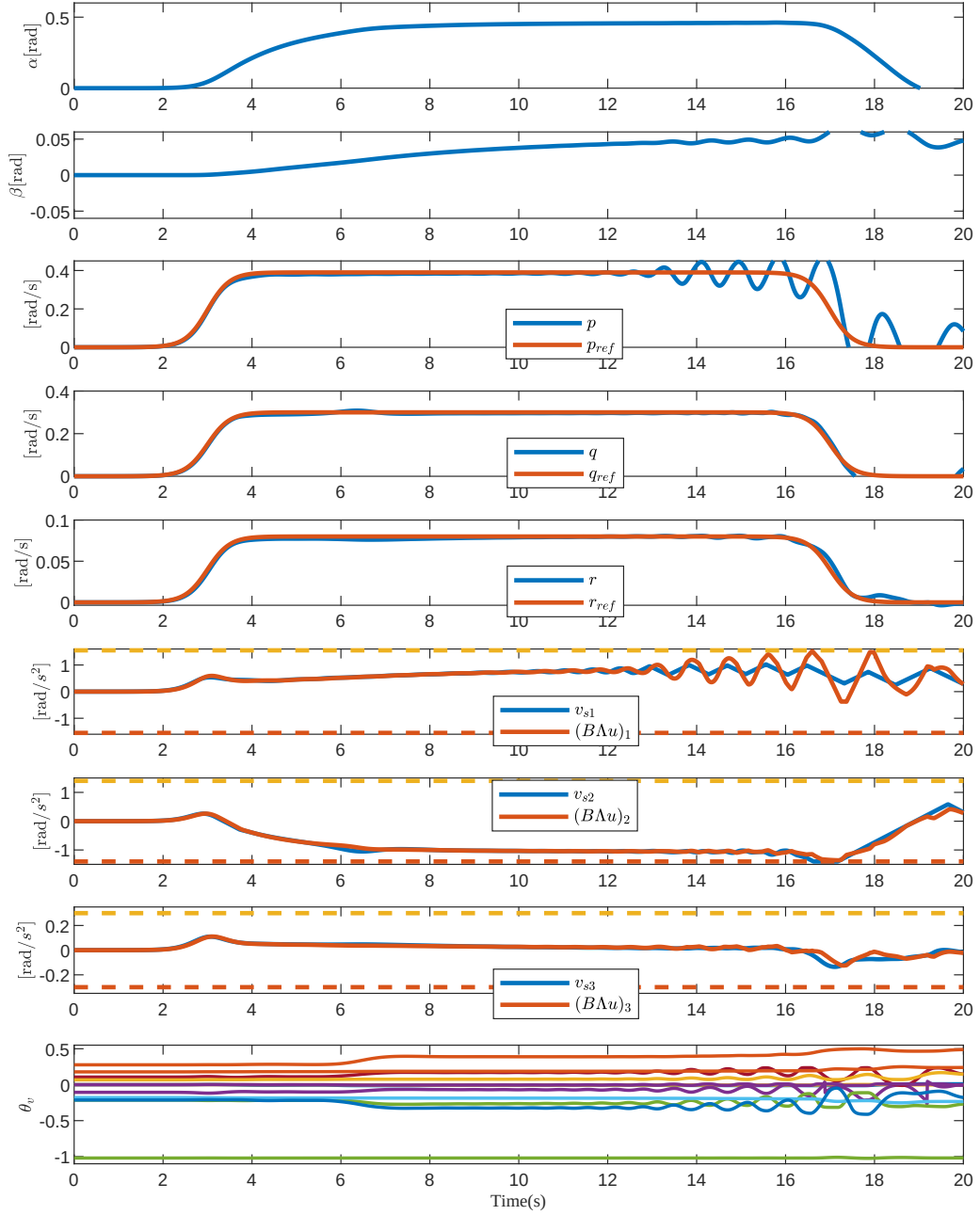


Figure 4.3: Case II: Evolution of the states, total control inputs and adaptive parameters in the presence of **both magnitude and rate saturation**, using the **conventional projection method**.

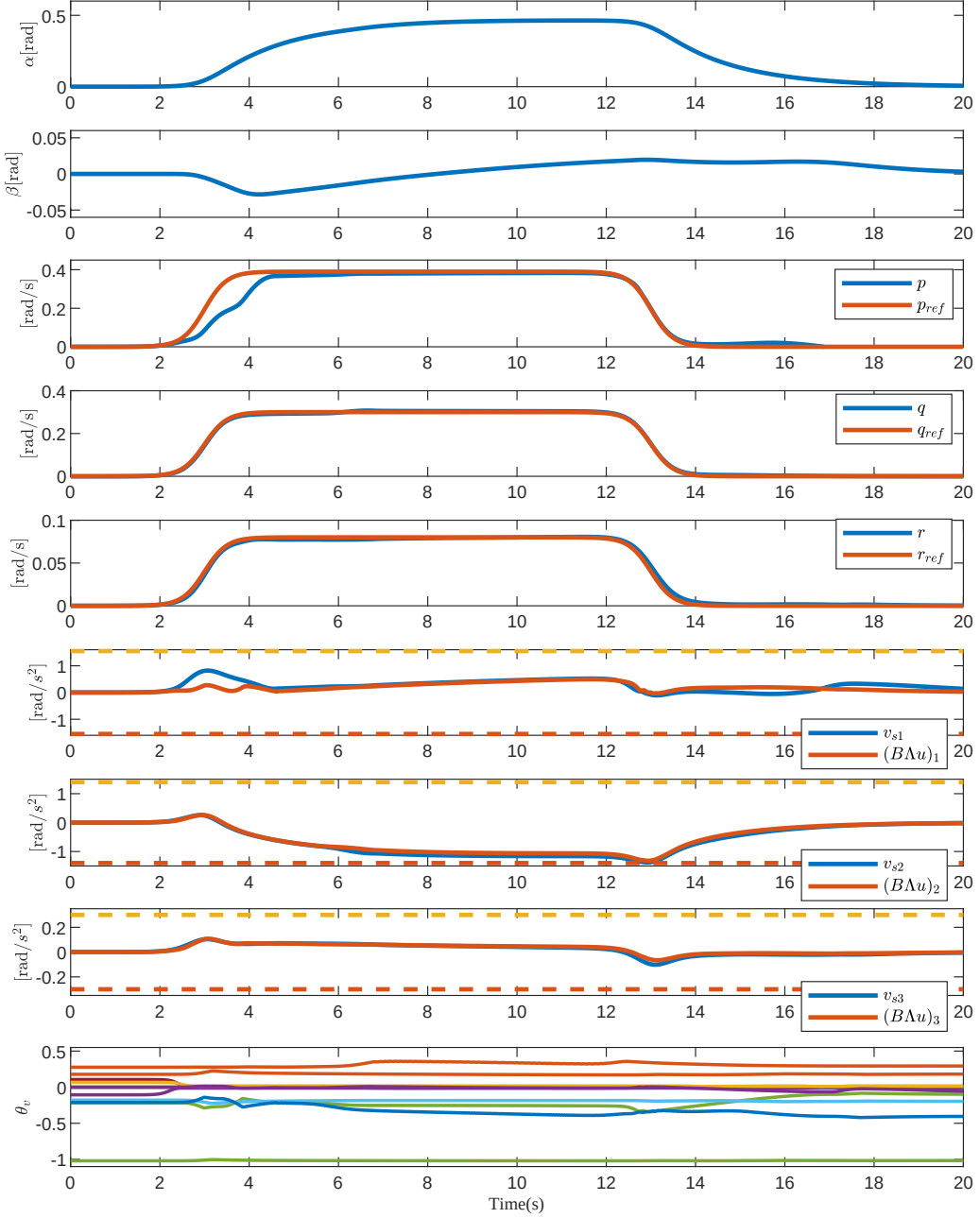


Figure 4.4: Case III: Evolution of the states, total control inputs and adaptive parameters in the presence of **both magnitude and rate saturation**, using the **proposed projection algorithm**.

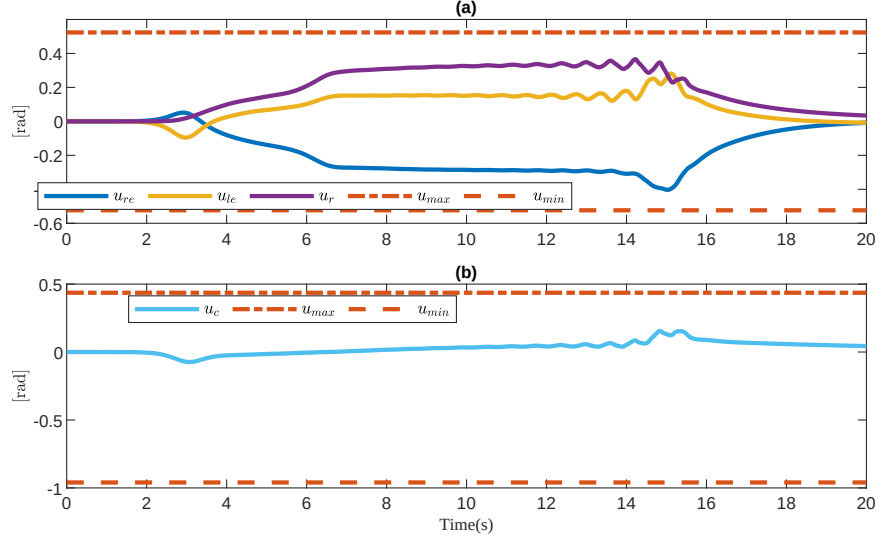


Figure 4.5: Case I: Evolution of the actuator inputs in the presence of **magnitude saturation**, using the **conventional projection method**.

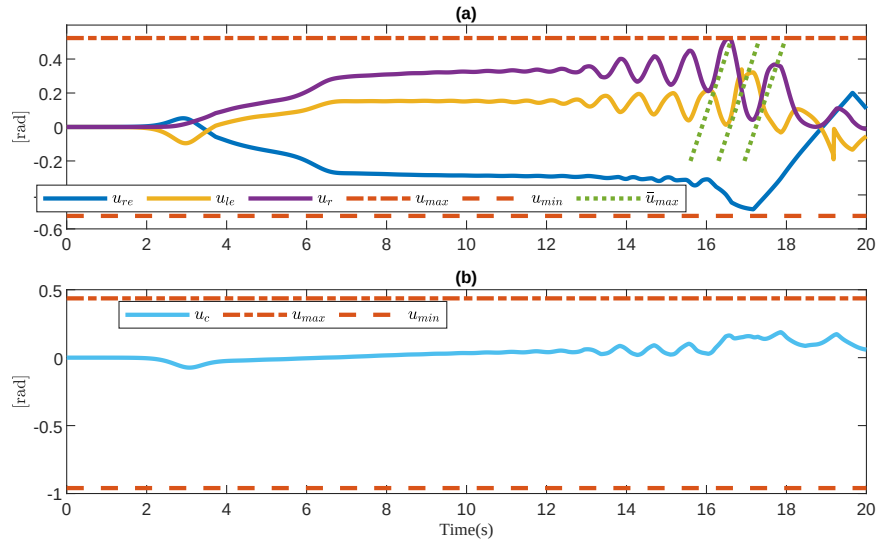


Figure 4.6: Case II: Evolution of the actuator inputs in the presence of **both magnitude and rate saturation**, using the **conventional projection method**.

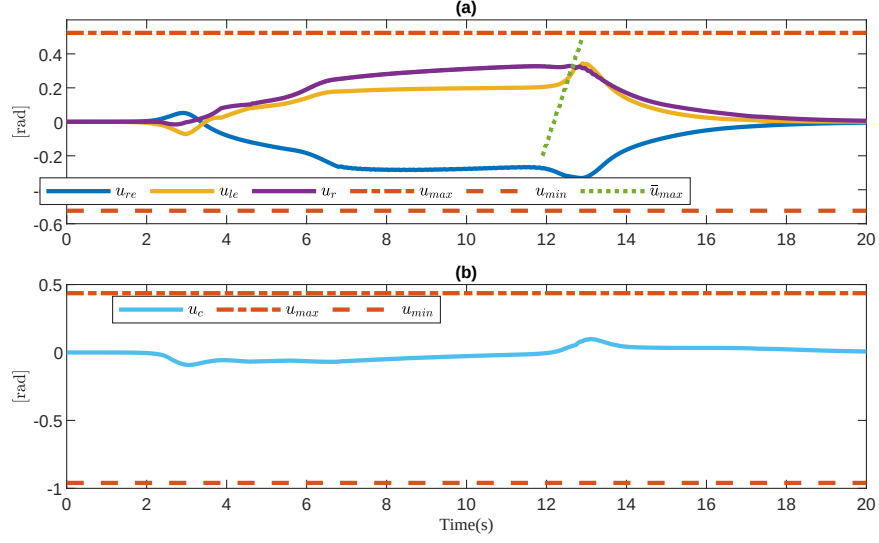


Figure 4.7: Case III: Evolution of the actuator inputs in the presence of **both magnitude and rate saturation**, using the **proposed projection algorithm**.

also guaranteed with the help of the presented projection method. The simulation results with the ADMIRE aircraft model are provided to demonstrate the efficacy of the proposed algorithm.

# Chapter 5

## Adaptive control allocation: human-in-the-loop stability analysis

### 5.1 Introduction

Humans' unique skills, such as adaptive behavior in dynamic environments, social interaction and moral judgment capabilities, make them indivisible elements of many control systems. Investigation of human in the loop dynamics help develop safe control mechanisms, and provide a better realization and understanding of human control actions and limitations [17–20].

Modeling human behavior in its entire complexity is an impossible task given our existing technological capabilities. This is also unnecessary for the purpose of most engineering systems. That's why many studies focus on developing models that mimic the human behavior for specific tasks. For example, in situations that require very accurate control commands, human operators are shown to behave like a pure gain [97, 98]. The magnitude of the gain shows human attention level,

aggressiveness, or task urgency. Human models with a gain and a pole (lag filter), which captures the humans' limitation of not being able to provide adequate control inputs at high frequencies, are also used for stability analysis [99]. The concept of describing function for human behavior is used by [100]. Quasi-linear model, proposed by [101], consists of a describing function and a remnant signal accounts for nonlinear behavior. An overview of this pilot model is provided by [102]. In some applications where the linear behavior may be dominant, the nonlinear part of this model can be ignored, and the resulting lead-lag-type compensator is used in closed loop stability analysis [103]. Crossover model, proposed by [104], is a prominent human model in aeronautics. An investigation into the crossover model is provided by [105]. [106–109] propose optimal human models, assuming that a well trained human operator behaves in an optimal manner. Adaptive human pilot behavior is formulated by [14,15], [16] and [110]. A survey on various pilot models can be found in the works of [111] and [112].

The capabilities of the controllers while interfaced with a human operator can be confined. Human reaction delay is a prominent parameter that can cause instability. The limitations of, for example, the model reference adaptive control in the presence of a human operator with reaction time delay is provided by [17] and [19]. Communication time delay, in addition to the operator delay, exacerbates the controller limitations. [113] analyze the stability of a human-in-the-loop telerobotics system. The effect of the adaptive controller adaptation time to recover nominal aircraft dynamics on pilots is investigated by [114]. A specific example of operator behavior possibly being detrimental to stability is observed for tasks that require increased attention and focus, where humans tend to provide high gain control inputs that can cause undesired oscillations. For example, undesired and sustained oscillations, called pilot induced oscillations (PIO), can emerge due to an abnormal coupling between the aircraft and the pilot [51,97,98].

Control allocation (CA) can be used to distribute control signals among redundant actuators [21,25,40,42,47,52,60]. While conventional control allocations require an identification method to identify fault in uncertain systems, adaptive control allocation methods proposed by [55] and [57,58,96] do not require an

identification method or assumption on the persistence of excitation of signals. In this chapter, the stability of a human-in-the-loop closed loop system in the presence of an adaptive control allocator is analyzed. The plant is assumed to have uncertain redundant actuators and controlled by a sliding mode controller that feeds the adaptive CA with a desired control input vector. The CA then distributes this control input among redundant actuators whose actuator effectiveness are uncertain. Different human models, the properties and specific use cases of which are discussed above, are considered: a pure gain which represents the human operator behavior in high gain control tasks, a lag filter which models the human operator's inability to respond to high frequency inputs, a general transfer function without time delay, and the one with time delay. To the best of authors' knowledge, human-in-the-loop stability analysis for an over-actuated system with uncertain actuator dynamics, in the presence of an adaptive control allocator, has not been conducted earlier in the literature.

This chapter is organized as follows. Section 5.2 presents the over-actuated system dynamics where the degree of actuator loss of effectiveness is uncertain. Adaptive control allocation as well as the sliding mode control design are also presented in this section. Closed loop dynamics including the uncertain plant, control allocation and the controller are given in Section 5.3. Human-in-the-loop stability analysis for the closed loop systems incorporating two different human operator models are provided in Section 5.4. Simulation results are presented in Section 5.4.5, and a summary is given in Section 5.5.

## 5.2 Background

To be able to develop the fundamental stability limits of the human-in-the-loop (HIL) closed loop system, we first need to introduce the components of this system, consisting of the plant, the controller, the adaptive control allocator and the human operator. We've recently proposed an adaptive control allocator and an accompanying controller solutions in [76] and [81] for systems with redundant actuators. In this chapter, we focus on the question "How do the human operator

dynamics effect the overall system stability properties?”.

### 5.2.1 Over-actuated uncertain plant

Consider the following uncertain over-actuated plant dynamics

$$\dot{x} = Ax + B_u \Lambda u = Ax + B_v B \Lambda u, \quad (5.1)$$

where  $x \in \mathbb{R}^n$  is the system states vector,  $u \in \mathbb{R}^m$  is the control input vector,  $A \in \mathbb{R}^{n \times n}$  is the known state matrix and  $B_u = B_v B \in \mathbb{R}^{n \times m}$  is the known rank deficient control input matrix which is decomposed into the known matrices  $B_v \in \mathbb{R}^{n \times \ell}$  and  $B \in \mathbb{R}^{\ell \times m}$ . The actuator loss of effectiveness is modeled as a diagonal matrix  $\Lambda \in \mathbb{R}^{m \times m}$  with unknown positive elements. Throughout this chapter, the over-dot notation will be used for time derivatives only, i.e.  $(\dot{\cdot}) = d(\cdot)/dt$ .

The goal of the control allocation in an over-actuated system is to distribute the total control effort  $v \in \mathbb{R}^\ell$ , produced by an outer loop controller, to the redundant actuators such that

$$B \Lambda u = v \quad (5.2)$$

is achieved. If this task is achieved perfectly, the system “seen” by the outer loop controller will have the form

$$\dot{x} = Ax + B_v v. \quad (5.3)$$

**Remark 19.** *The assumption that the matrix  $A$  is known can be removed. However, the focus of this chapter is the human-in-the-loop stability analysis for a control system where control allocation is employed to distribute the total control effort among redundant actuators. In this setting, matrix  $A$  being known or not does not effect the development of the control allocation. As seen from (5.2), only the uncertainty in the input matrix effects the control allocation development. Therefore, to keep the development concise, we assume that  $A$  is known.*

The details about the control allocation and the controller used in this chapter can be found in Chapters 2 and 3, respectively.

### 5.3 Nonlinear time-varying dynamics

The system to be controlled by the human operator contains the uncertain over-actuated plant, adaptive control allocation and the sliding mode controller introduced in Section 5.2. Although the plant is time invariant, existence of the adaptive control allocator and the sliding mode controller leads to a time-varying closed loop system.

Substituting (3.24) into (3.14), the nonlinear time-varying system dynamics can be obtained as

$$\begin{cases} \dot{x}^{(1)}(t) = A_{1,1}x^{(1)}(t) + A_{1,2}x^{(2)}(t) \\ \dot{x}^{(2)}(t) = -\bar{\lambda}x^{(2)}(0)e^{-\bar{\lambda}t} + \frac{2}{\pi}\dot{z}(t)\tan^{-1}(\bar{\lambda}t) \\ \quad + \frac{2}{\pi}z(t)\frac{\bar{\lambda}}{1+\bar{\lambda}^2t^2} - \text{sign}_v(s(x^{(2)}(t), x^{(2)}(0), t))\rho + d(t) \\ y(t) = x^{(2)}(t). \end{cases} \quad (5.4)$$

The ADMIRE [48], which is an over-actuated aircraft model, is utilized as the plant to be controlled. The ADMIRE model can be written in the form of (3.14), with  $x^{(1)} = [\alpha \ \beta]^T$  and  $x^{(2)} = [p \ q \ r]^T$ , where  $\alpha$ ,  $\beta$ ,  $p$ ,  $q$  and  $r$  are the angle of attack, sideslip angle, roll rate, pitch rate and yaw rate, respectively. Also, the reference signal,  $z(t)$ , is taken as  $z = [p_d, \ q_d, \ r_d]^T$ , where  $p_d$ ,  $q_d$  and  $r_d$  are the desired roll, pitch and yaw rates, respectively. Therefore, (5.4) can be written as

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \end{bmatrix} = A_{1,1} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} + A_{1,2} \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \quad (5.5)$$

$$\begin{aligned} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \\ \dot{r}(t) \end{bmatrix} &= -\bar{\lambda}e^{-\bar{\lambda}t} \begin{bmatrix} p(0) \\ q(0) \\ r(0) \end{bmatrix} + \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) \begin{bmatrix} \dot{p}_d(t) \\ \dot{q}_d(t) \\ \dot{r}_d(t) \end{bmatrix} \\ &\quad + \frac{2}{\pi} \frac{\bar{\lambda}}{1 + \bar{\lambda}^2t^2} \begin{bmatrix} p_d(t) \\ q_d(t) \\ r_d(t) \end{bmatrix} - \text{sign}_v(s)\rho + d(t) \end{aligned} \quad (5.6)$$

$$y(t) = [p(t) \ q(t) \ r(t)]^T, \quad (5.7)$$

where the arguments of  $s(x(2)(t); x(2)(t); t)$  are dropped for clarity.

## 5.4 Human-in-the-loop stability analysis

In Chapter 3, it is shown that the control signal (3.24) keeps the trajectories of  $p(t)$ ,  $q(t)$  and  $r(t)$  on the sliding surface (3.16) for all  $t \geq t_0$ . Furthermore, on the sliding surface (3.16),  $p(t)$ ,  $q(t)$  and  $r(t)$  remain bounded and track their references  $p_d(t)$ ,  $q_d(t)$  and  $r_d(t)$ , assuming that the references are bounded. Therefore, using (5.5) it can be concluded that  $\alpha(t)$  and  $\beta(t)$  remain bounded, given that  $p(t)$ ,  $q(t)$  and  $r(t)$  are bounded. With this three useful observations, namely, the trajectories of  $p(t)$ ,  $q(t)$  and  $r(t)$  starting and remaining on the sliding surface, they remaining bounded for bounded references  $p_d(t)$ ,  $q_d(t)$  and  $r_d(t)$ , and  $\alpha(t)$  and  $\beta(t)$  being bounded for bounded  $p(t)$ ,  $q(t)$  and  $r(t)$ , in this section, we integrate the pilot in the control system and analyze the stability of the overall human-in-the-loop closed loop dynamics. The resulting control structure is shown in Figure 5.1.

**Remark 20.** *It is noted that in the following analysis, although the references  $p_d(t)$  and  $r_d(t)$  are assumed to be bounded, it will not be assumed that  $q_d(t)$  is also bounded. Therefore, the boundedness of  $q(t)$ ,  $\alpha(t)$  and  $\beta(t)$  need to be shown.*

### 5.4.1 Pure gain pilot model in the loop

In the control system structure given in Figure 5.1, the pilot controls the pitch angle,  $\theta(t)$ , and track the pitch angle reference ( $\theta_d(t)$ ) by giving a pitch rate stick input, which is represented by  $q_d(t)$ . It is assumed that the other attitude rate references,  $p_d(t)$  and  $r_d(t)$ , are produced based on the requirements of the operation, and they are bounded. The pilot compares the pitch angle,  $\theta(t)$ , with the reference input,  $\theta_d(t)$ , and provides the reference pitch rate,  $q_d(t)$ , for the inner loop controller, that is  $q_d(t) = (\theta_d(t) - \theta(t))K$ , where  $K$  is the pilot gain.

The dynamics of  $p(t)$ ,  $q(t)$  and  $r(t)$  can then be rewritten using (5.6) as

$$\begin{aligned} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \\ \dot{r}(t) \end{bmatrix} &= -\bar{\lambda}e^{-\bar{\lambda}t} \begin{bmatrix} p(0) \\ q(0) \\ r(0) \end{bmatrix} + \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) \begin{bmatrix} \dot{p}_d(t) \\ (\dot{\theta}_d(t) - \dot{\theta}(t))K \\ \dot{r}_d(t) \end{bmatrix} \\ &+ \frac{2}{\pi} \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \begin{bmatrix} p_d(t) \\ (\theta_d(t) - \theta(t))K \\ r_d(t) \end{bmatrix} - \text{sign}_v(s)\rho + d. \end{aligned} \quad (5.8)$$

It is seen in (5.8) that the specific sliding mode controller structure (3.24) employed to control the system, decouples  $p(t)$ ,  $q(t)$  and  $r(t)$  dynamics and the effect of the pilot can only be observed on  $q(t)$  dynamics. Therefore, it can be shown [96] that  $p(t)$  and  $q(t)$  are bounded and follow their references. So, to show the boundedness of all signals in the overall closed loop dynamics, the remaining task is to show that  $\theta(t)$  and  $q(t)$  are also bounded, which will also ensure the boundedness of  $\alpha(t)$  and  $\beta(t)$ . It is reminded that unlike the structure studied by [96], here we cannot assume that  $q_d(t)$  is bounded since it is produced by the human operator. For simplicity, we make the small angle assumption and assume that  $\dot{\theta}(t) = q(t)$ .

Using the fact that  $p(t)$ ,  $q(t)$  and  $r(t)$  remain on the sliding surface for all  $t \geq t_0$ , regardless of the boundedness of their references, and using (3.16) and  $q_d(t) = (\theta_d(t) - \theta(t))K$ , it can be concluded that the trajectories of  $q(t)$  on the sliding surface satisfy

$$\begin{aligned} q(t) &= q(0)e^{-\bar{\lambda}t} + \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)q_d(t) \\ &= q(0)e^{-\bar{\lambda}t} + \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)(\theta_d(t) - \theta(t))K. \end{aligned} \quad (5.9)$$

By using  $\dot{\theta}(t) = q(t)$ , (5.9) can be rewritten as

$$\dot{\theta} = \frac{-2K}{\pi}\tan^{-1}(\bar{\lambda}t)\theta(t) + q(0)e^{-\bar{\lambda}t} + \frac{2K}{\pi}\tan^{-1}(\bar{\lambda}t)\theta_d(t). \quad (5.10)$$

Defining  $A_\theta(t) = \frac{-2K}{\pi}\tan^{-1}(\bar{\lambda}t)$  and  $B_\theta(t) = q(0)e^{-\bar{\lambda}t} + \frac{2K}{\pi}\tan^{-1}(\bar{\lambda}t)\theta_d(t)$ , (5.10) can be rewritten in the following compact form

$$\dot{\theta}(t) = A_\theta(t)\theta(t) + B_\theta(t). \quad (5.11)$$

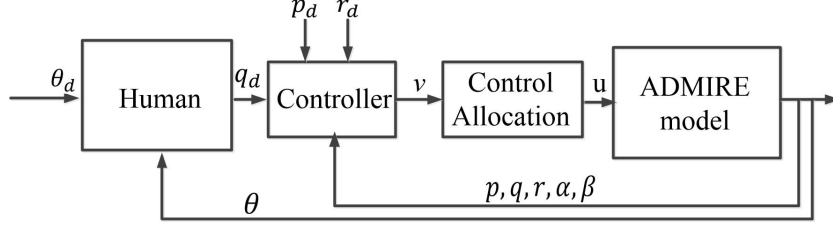


Figure 5.1: Closed loop system including human.

**Theorem 8.** *The time-varying system  $\dot{\theta}(t) = A_{\theta}(t)\theta(t)$ , where  $A_{\theta}(t) = \frac{-2K}{\pi}\tan^{-1}(\bar{\lambda}t)$ , is asymptotically stable if  $K > 0$ .*

*Proof.* The solution to the differential equation  $\dot{\theta}(t) = A_{\theta}(t)\theta(t)$  can be obtained as

$$\begin{aligned}
 \theta(t) &= \theta(0)e^{\int_0^t \frac{-2K}{\pi}\tan^{-1}(\bar{\lambda}\tau)d\tau} \\
 &= \theta(0)e^{\frac{-2K}{\pi}\left(t\times\tan^{-1}(\bar{\lambda}t) - \frac{1}{2\bar{\lambda}}\ln(1+\bar{\lambda}^2t^2)\right)} \\
 &= \theta(0)\frac{(1+\bar{\lambda}^2t^2)^{\frac{K}{\pi\bar{\lambda}}}}{e^{\frac{2K}{\pi}t\times\tan^{-1}(\bar{\lambda}t)}} = \theta(0)\Phi(t,0), \tag{5.12}
 \end{aligned}$$

where  $\Phi(t,0)$  is the state transition function, mapping  $\theta(0)$  to  $\theta(t)$ . Let  $\Phi(t,0) = \phi(t)e^{\frac{-K}{\pi}t\times\tan^{-1}(\bar{\lambda}t)}$ , where  $\phi(t) = \frac{(1+\bar{\lambda}^2t^2)^{\frac{K}{\pi\bar{\lambda}}}}{e^{\frac{K}{\pi}t\times\tan^{-1}(\bar{\lambda}t)}}$  is a positive function. As  $t \rightarrow \infty$  the growth rate of  $a^t$  is larger than  $t^b$  for positive constants  $a$  and  $b$ . Therefore, as  $t \rightarrow \infty$  the growth rate of  $e^{\frac{K}{\pi}t\times\tan^{-1}(\bar{\lambda}t)}$  is larger than  $(1+\bar{\lambda}^2t^2)^{\frac{K}{\pi\bar{\lambda}}}$ , which leads to  $\lim_{t \rightarrow \infty} \phi(t) = 0$ . In addition, using the fact that  $\phi(t)$  has a nonzero denominator and a polynomial numerator, it can be shown that  $\phi(t)$  does not have a finite escape time. Therefore,  $\phi(t)$  is bounded for all  $t \geq 0$ . Let  $\sup_{t \geq 0} \phi(t) = \delta$ , we have

$$|\theta(t)| \leq |\theta(0)|\delta e^{\frac{-K}{\pi}t\times\tan^{-1}(\bar{\lambda}t)}, \quad t \geq 0. \tag{5.13}$$

Therefore,  $|\theta(t)| \leq \theta(0)\delta$  and  $\lim_{t \rightarrow \infty} \theta(t) = 0$ . Hence,  $\theta(t)$  is asymptotically stable, in the sense of Lyapunov, for  $K > 0$ .  $\square$

**Theorem 9.** *The trajectory of  $\theta(t)$  satisfying the differential equation  $\dot{\theta}(t) = A_{\theta}(t)\theta(t) + B_{\theta}(t)$ , where  $A_{\theta}(t) = \frac{-2K}{\pi}\tan^{-1}(\bar{\lambda}t)$  and  $B_{\theta}(t) = q(0)e^{-\bar{\lambda}t} +$*

$\frac{2K}{\pi} \tan^{-1}(\bar{\lambda}t) \theta_d(t)$ , is bounded if  $K > 0$ . An upper bound for  $|\theta(t)|$  can be obtained as

$$|\theta(t)| \leq \theta_{max} \equiv \begin{cases} \delta(|\theta(0)| + \frac{|q(0)|}{\lambda} + (1 + t_2^*)K\theta_{d_{max}}) & \text{if } t_2^* > 1 \\ \delta(|\theta(0)| + \frac{|q(0)|}{\lambda} + 2K\theta_{d_{max}}) & \text{otherwise,} \end{cases} \quad (5.14)$$

where  $t_2^*$  is the finite solution of the equation  $t_2^{*2} = e^{\frac{K}{\pi} t_2^* \times \tan^{-1}(\bar{\lambda} t_2^*)}$ , if any.

*Proof.* The solution of the differential equation  $\dot{\theta}(t) = A_\theta(t)\theta(t) + B_q(t)$  can be calculated as

$$\theta(t) = \theta(t_0)\Phi(t, t_0) + \int_{t_0}^t \Phi(t - \tau)B_\theta(\tau)d\tau. \quad (5.15)$$

By using (5.12) and (5.13), an upper bound for (5.15) with  $t_0 = 0$  can be obtained as

$$\begin{aligned} |\theta(t)| &\leq |\theta(0)|\delta e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} + \int_0^t \delta e^{\frac{-K}{\pi}(t-\tau)\tan^{-1}(\bar{\lambda}(t-\tau))}|B_\theta(\tau)|d\tau \\ &= |\theta(0)|\delta e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} + \delta \int_0^t e^{\frac{-K}{\pi}(t-\tau)\tan^{-1}(\bar{\lambda}(t-\tau))}|q(0)|e^{-\bar{\lambda}\tau}d\tau \\ &\quad + \frac{2K}{\pi}\delta \int_0^t e^{\frac{-K}{\pi}(t-\tau)\tan^{-1}(\bar{\lambda}(t-\tau))}\tan^{-1}(\bar{\lambda}\tau)|\theta_d(\tau)|d\tau \\ &\leq |\theta(0)|\delta e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} + \delta \int_0^t e^{\frac{-K}{\pi}(t-\tau)\tan^{-1}(\bar{\lambda}(t-\tau))}|q(0)|e^{-\bar{\lambda}\tau}d\tau \\ &\quad + K\theta_{d_{max}}\delta \int_0^t e^{\frac{-K}{\pi}(t-\tau)\tan^{-1}(\bar{\lambda}(t-\tau))}d\tau, \end{aligned} \quad (5.16)$$

where  $\theta_{d_{max}}$  is an upper bound for  $|\theta_d|$ . It should be noted that  $\sup_{0 \leq \tau \leq t} e^{\frac{-K}{\pi}(t-\tau)\tan^{-1}(\bar{\lambda}(t-\tau))} = 1$ . By using this and the change of variable,  $\xi = t - \tau$ , (5.16) can be rewritten as

$$|\theta(t)| \leq |\theta(0)|\delta e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} + \delta \int_0^t |q(0)|e^{-\bar{\lambda}\tau}d\tau + K\theta_{d_{max}}\delta \int_0^t e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)}d\xi. \quad (5.17)$$

By using the growth rate discussion, which mentions that as  $t \rightarrow \infty$  the growth rate of  $a^t$  is larger than  $t^b$  for positive constants  $a$  and  $b$ , we know that  $e^{\frac{K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)}$  is greater than  $t^2$  as  $t \rightarrow \infty$ . To continue the analysis, we consider

the following three cases: **(1)** If positive values for  $K$  and  $\bar{\lambda}$  are chosen such that  $e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} < t^{-2}$ , for  $\forall t \geq 0$ , an upper bound on (5.17) can be calculated as

$$\begin{aligned}
|\theta(t)| &\leq \delta(|\theta(0)| + \frac{|q(0)|}{\bar{\lambda}} + K\theta_{d_{max}}(\int_0^1 e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi + \int_1^t e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi)) \\
&\leq |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + K\theta_{d_{max}}\delta(1 + \int_1^\infty e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi) \\
&\leq |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + K\theta_{d_{max}}\delta(1 + \int_1^\infty \xi^{-2} d\xi) \\
&= |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + 2K\theta_{d_{max}}\delta.
\end{aligned} \tag{5.18}$$

**(2)** If positive values for  $K$  and  $\bar{\lambda}$  are chosen such that  $e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} < t^{-2}$ , for  $\forall t \geq t_1^*$ ,  $t_1^* \leq 1$ , then an upper bound on (5.17) can be obtained as

$$\begin{aligned}
|\theta(t)| &\leq \delta(|\theta(0)| + \frac{|q(0)|}{\bar{\lambda}} + K\theta_{d_{max}}(\int_0^{t_1^*} e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi \\
&\quad + \int_{t_1^*}^1 e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi + \int_1^t e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi)) \\
&\leq \delta(|\theta(0)| + \frac{|q(0)|}{\bar{\lambda}} + K\theta_{d_{max}}(\int_0^1 e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi + \int_1^\infty e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi)) \\
&\leq |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + K\theta_{d_{max}}\delta(1 + \int_1^\infty \xi^{-2} d\xi) \\
&= |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + 2K\theta_{d_{max}}\delta.
\end{aligned} \tag{5.19}$$

**(3)** If positive values for  $K$  and  $\bar{\lambda}$  are chosen such that  $e^{\frac{-K}{\pi}t \times \tan^{-1}(\bar{\lambda}t)} < t^{-2}$ , for  $\forall t \geq t_2^*$ ,  $1 < t_2^*$ , then an upper bound for (5.17) can be obtained as

$$\begin{aligned}
|\theta(t)| &\leq \delta(|\theta(0)| + \frac{|q(0)|}{\bar{\lambda}} + K\theta_{d_{max}}(\int_0^{t_2^*} e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi + \int_{t_2^*}^t e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi)) \\
&\leq |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + K\theta_{d_{max}}\delta(t_2^* + \int_{t_2^*}^\infty e^{\frac{-K}{\pi}\xi \times \tan^{-1}(\bar{\lambda}\xi)} d\xi) \\
&\leq |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + K\theta_{d_{max}}\delta(t_2^* + \int_1^\infty \xi^{-2} d\xi) \\
&= |\theta(0)|\delta + \frac{|q(0)|}{\bar{\lambda}}\delta + K\theta_{d_{max}}\delta(t_2^* + 1).
\end{aligned} \tag{5.20}$$

It should be noted that  $t_2^*$  is a function of  $K$  and  $\bar{\lambda}$ . Therefore,  $\theta(t)$  is bounded, and its bound is a function of  $\theta(0)$ ,  $q(0)$ ,  $\theta_{d_{max}}$ ,  $\bar{\lambda}$  and  $K$ . Assuming that  $\theta_d(t)$  is bounded, the boundedness of  $\theta(t)$  proves that  $e_\theta(t) = \theta_d(t) - \theta(t)$  is bounded as well.  $\square$

**Remark 21.** Assuming that  $\theta_d(t)$ , which is the reference given to the operator to follow, is bounded, by using  $q_d(t) = (q_d(t) - q(t))K$  and the boundedness of  $q(t)$ , which was proved in Theorem 9 for  $K \geq 0$ , it can be concluded that  $q_d(t)$  is also bounded. Assuming that the reference signals,  $p_d(t)$  and  $r_d(t)$ , are bounded, it can then be shown that  $p(t)$ ,  $q(t)$  and  $r(t)$  are bounded and track their references [96]. Using (5.5), it can be concluded that the boundedness of  $p(t)$ ,  $q(t)$  and  $r(t)$  leads to the boundedness of  $\alpha(t)$  and  $\beta(t)$ . Therefore, the boundedness of all system signals is ensured.

#### 5.4.2 Lag filter pilot model in the loop

In this section, we consider human dynamics as a pole and a gain as  $K/(s + a)$ . Similar to the previous case, the human operator tries to track the pitch angle reference,  $\theta_d(t)$ , by producing a pitch rate input,  $q_d(t)$ , for the controller (see Figure 5.1).

It is assumed that bounded reference signals  $p_d(t)$ ,  $r_d(t)$  and  $\theta_d(t)$  are provided for the controller and the operator, respectively. Considering that the input to the operator dynamics is  $\theta_d(t) - \theta(t)$  and the output is  $q_d(t)$ , the differential equation describing the human operator dynamics can be given as

$$\dot{q}_d(t) = -aq_d(t) + (\theta_d(t) - \theta(t))K. \quad (5.21)$$

As shown earlier, the sliding mode controller (3.24) guarantees that the trajectories of  $p(t)$ ,  $q(t)$  and  $r(t)$  remain on the sliding surface for all  $t \geq t_0$ . Using (3.16) with  $t_0 = 0$ , the trajectory of  $q(t)$  on the sliding surface satisfies

$$q(t) = q(0)e^{-\bar{\lambda}t} + \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) q_d(t). \quad (5.22)$$

By using  $\dot{\theta}(t) = q(t)$ , (5.21), and (5.22), the dynamics of the pitch angle,  $q(t)$ , and pitch rate reference,  $q_d(t)$ , can be written as

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{q}_d(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) \\ -K & -a \end{bmatrix}}_{\hat{A}(t)} \underbrace{\begin{bmatrix} \theta(t) \\ q_d(t) \end{bmatrix}}_{\hat{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ K \end{bmatrix}}_{\hat{B}} \theta_d(t) + \underbrace{\begin{bmatrix} q(0)e^{-\bar{\lambda}t} \\ 0 \end{bmatrix}}_{\omega(t)}. \quad (5.23)$$

By defining  $\hat{x}(t) = [\theta(t) \ q_d(t)]^T$ , (5.23) can be written in the following compact form

$$\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + \hat{B}\theta_d(t) + \omega(t). \quad (5.24)$$

**Theorem 10.** *The time-varying system  $\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t)$ , where  $\hat{A}(t)$  and  $\hat{x}(t)$  are given in (5.23), is uniformly exponentially stable if  $a > 0$  and  $K > 0$ .*

*Proof.* The time-varying matrix  $\hat{A}(t)$  can be written as

$$\hat{A}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -K & -a \end{bmatrix}}_{\hat{A}_1} + \underbrace{\begin{bmatrix} 0 & \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) - 1 \\ 0 & 0 \end{bmatrix}}_{\hat{A}_2(t)}, \quad (5.25)$$

where  $\hat{A}_1$  is a constant Hurwitz matrix when  $K > 0$  and  $a > 0$ . Therefore, the origin of the differential equation  $\dot{\hat{x}}(t) = \hat{A}_1\hat{x}(t)$  is exponentially stable. Therefore, there exist a scalar function  $V(x)$  satisfying [6]

$$c_1\|\hat{x}\|^2 \leq V \leq c_2\|\hat{x}\|^2 \quad (5.26)$$

$$\frac{dV}{d\hat{x}}\hat{A}_1\hat{x}(t) \leq -c_3\|\hat{x}\|^2 \quad (5.27)$$

$$\left\|\frac{dV}{d\hat{x}}\right\| \leq c_4\|\hat{x}\|, \quad (5.28)$$

where  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$  are positive constants. Considering the system  $\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) = \hat{A}_1\hat{x}(t) + \hat{A}_2(t)\hat{x}(t)$  and using (5.26)-(5.28), an upper bound on the time derivative of  $V$  can be obtained as

$$\begin{aligned} \dot{V} &= \frac{dV}{d\hat{x}}\hat{A}_1\hat{x}(t) + \frac{dV}{d\hat{x}}\hat{A}_2(t)\hat{x}(t) \\ &\leq -c_3\|\hat{x}\|^2 + c_4\left|\frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) - 1\right|\|\hat{x}\|^2 \\ &\leq -\frac{c_3}{c_2}V + \frac{c_4}{c_1}\left|\frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) - 1\right|V \\ &= -\left(\frac{c_3}{c_2} - \frac{c_4}{c_1}\left|\frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) - 1\right|\right)V. \end{aligned} \quad (5.29)$$

Using the comparison lemma [6], we get

$$V \leq e^{-(\frac{c_3}{c_2}t - \frac{c_4}{c_1} \int_0^t |\frac{2}{\pi}\tan^{-1}(\bar{\lambda}\tau) - 1| d\tau)} V(\hat{x}(0)). \quad (5.30)$$

Using (5.26), (5.30) can be written as

$$\|\hat{x}(t)\|^2 \leq \frac{c_2}{c_1} e^{-(\frac{c_3}{c_2}t - \frac{c_4}{c_1} \int_0^t |\frac{2}{\pi} \tan^{-1}(\bar{\lambda}\tau) - 1| d\tau)} \|\hat{x}(0)\|^2, \quad (5.31)$$

which leads to

$$\|\hat{x}(t)\| \leq \sqrt{\frac{c_2}{c_1}} e^{-(\frac{c_3}{2c_2}t - \frac{c_4}{2c_1} \int_0^t |\frac{2}{\pi} \tan^{-1}(\bar{\lambda}\tau) - 1| d\tau)} \|\hat{x}(0)\|. \quad (5.32)$$

Let  $\gamma(t) = |\frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) - 1|$ , it should be noted that  $\gamma(t) \geq 0$  and  $\lim_{t \rightarrow \infty} \gamma(t) = 0$ . Also, the time derivative of  $\gamma(t)$  is bounded for  $\forall t \geq 0$ , and  $\frac{d}{dt}\gamma(t) = \frac{-2\bar{\lambda}}{\pi(1+\bar{\lambda}^2t^2)} < 0$ . Therefore, there exist positive constants  $T$  and  $\epsilon$  such that  $\gamma(t) \leq \epsilon < \frac{c_3c_1}{c_2c_4}$ ,  $\forall t \geq T$ . Also, from the boundedness of  $\gamma(t)$  we have  $\int_0^T \gamma(t) = \eta$ , where  $\eta$  is a positive constant. The following two cases should be considered: **(1)** For  $t \leq T$ ,  $\int_0^t \gamma(\tau) d\tau \leq \eta$ . Thus, using (5.32), an upper bound for  $\|\hat{x}\|$  can be obtained as

$$\|\hat{x}(t)\| \leq \sqrt{\frac{c_2}{c_1}} \kappa e^{-(\frac{c_3}{2c_2}t)} \|\hat{x}(0)\|, \quad (5.33)$$

where  $\kappa = e^{\frac{c_4}{2c_1}\eta}$ . **(2)** For  $t > T$ ,  $\int_0^t \gamma(\tau) d\tau = \int_0^T \gamma(\tau) d\tau + \int_T^t \gamma(\tau) d\tau \leq \epsilon t + \eta$ . Substituting this result in (5.32), an upper bound for  $\|\hat{x}\|$  can be obtained as

$$\|\hat{x}(t)\| \leq \sqrt{\frac{c_2}{c_1}} \kappa e^{-(\frac{c_3}{2c_2}t - \frac{c_4}{2c_1}\epsilon t)} \|\hat{x}(0)\| \quad (5.34)$$

Since  $\epsilon < \frac{c_3c_1}{c_2c_4}$ , the origin of  $\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t)$  is exponentially stable.  $\square$

**Theorem 11.** *The solution of the linear time-varying system (5.24), where  $\hat{A}(t)$ ,  $\hat{B}$ ,  $\omega(t)$  and  $\hat{x}(t)$  are given in (5.23), is bounded if  $a > 0$ , and  $K > 0$ .*

*Proof.* The solution of the linear time-varying system (5.24) is

$$\hat{x}(t) = \Phi(t, t_0)\hat{x}(t_0) + \int_{t_0}^t \Phi(t, \tau)(\hat{B}\theta_d(\tau) + \omega(\tau)), \quad (5.35)$$

where  $\Phi(t, t_0)$  is the state transition matrix. By using the definitions of  $\hat{B}$  and  $\omega(t)$  given in (5.23), it can be obtained that  $\sup_{0 \leq \tau \leq t} \|\hat{B}\theta_d(\tau) + \omega(\tau)\| = K\theta_{d_{max}} + |q(0)|$ . Also, it is proved in Theorem 10 that for  $a$ ,  $K > 0$ , the origin of  $\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t)$  is exponentially stable, that is, there exist finite positive constants  $k_1$

and  $k_2$  such that  $\|\Phi(t, t_0)\| \leq k_1 e^{-k_2(t-t_0)}$ , for  $\forall t \geq t_0$ . Therefore, considering  $t_0 = 0$ , an upper bound on  $\|\hat{x}(t)\|$  can be obtained as

$$\begin{aligned} \|\hat{x}(t)\| &\leq k_1 e^{-k_2 t} |\hat{x}(0)| + k_1 (K\theta_{d_{max}} + |q(0)|) \int_0^t e^{-k_2(t-\tau)} d\tau \\ &= k_1 e^{-k_2 t} |\hat{x}(0)| + \frac{k_1}{k_2} (K\theta_{d_{max}} + |q(0)|) (1 - e^{-k_2 t}) \\ &\leq k_1 |\hat{x}(0)| + \frac{k_1}{k_2} (K\theta_{d_{max}} + |q(0)|). \end{aligned} \quad (5.36)$$

Therefore,  $\theta(t)$  and  $q_d(t)$  are bounded. Assuming that  $\theta_d(t)$  is bounded, the boundedness of  $\theta(t)$  proves that  $e_\theta(t) = \theta_d(t) - \theta(t)$  is bounded as well.  $\square$

**Remark 22.** *The upper bound on the tracking error,  $e_\theta(t)$ , albeit conservative, can be calculated using (5.14) and (5.36) for different human operator reaction dynamics.*

### 5.4.3 Stability analysis of general human model without delay consideration

In the previous sections, two famous human control theoretic models were considered and the stability of human-in-the-loop was investigated. In this section, a general human model without considering delay is employed for stability analysis.

Consider the transfer function

$$\frac{q_d(s)}{\theta_d(s) - \theta(s)} = \frac{b_m s^{\hat{m}} + b_{m-1} s^{\hat{m}-1} + \dots + b_0}{s^{\hat{n}} + a_{n-1} s^{\hat{n}-1} + \dots + a_0}, \quad (5.37)$$

as human model, where  $\hat{m}$  and  $\hat{n}$  are real constants such that  $\hat{n} > 0$ ,  $\hat{m} \geq 0$  and  $\hat{m} \leq \hat{n}$ , and  $a_i$  and  $b_j$  for  $i = 0, \dots, \hat{n} - 1$  and  $j = 0, \dots, \hat{m}$ , are real constants. The transfer function (5.37) can be represented in the minimal state space form as

$$\dot{x}_h(t) = A_h x_h(t) + B_h (\theta_d(t) - \theta(t)), \quad (5.38)$$

$$q_d(t) = C_h x_h(t) + D_h (\theta_d(t) - \theta(t)), \quad (5.39)$$

where  $x_h \in \mathbb{R}^{\hat{n}}$  is the vector including the states of the human model,  $A_h \in \mathbb{R}^{\hat{n} \times \hat{n}}$  is the system matrix,  $B_h \in \mathbb{R}^{\hat{n}}$  is the control vector,  $C_h \in \mathbb{R}^{1 \times \hat{n}}$  is the output

vector,  $D_h \in \mathbb{R}$  is the feedforward gain,  $q_d \in \mathbb{R}$  is the reference pitch angle and the output of the human model,  $\theta_d \in \mathbb{R}$  is the desired pitch angle and  $\theta \in \mathbb{R}$  is the current pitch angle.

Using (5.22), which is the trajectory of  $q(t)$  on the sliding surface (3.16), (5.39) and considering  $\dot{\theta}(t) = q(t)$ , the pitch angle dynamics can be written as

$$\dot{\theta}(t) = q(0)e^{-\bar{\lambda}t} + \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)C_h x_h(t) + \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)D_h(\theta_d(t) - \theta(t)). \quad (5.40)$$

Augmenting the dynamics of general human model (5.38) and pitch angle (5.40), we have

$$\begin{aligned} \begin{bmatrix} \dot{\theta}(t) \\ \dot{x}_h(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} -\frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)D_h & \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)C_h \\ -B_h & A_h \end{bmatrix}}_{\bar{A}(t)} \underbrace{\begin{bmatrix} \theta(t) \\ x_h(t) \end{bmatrix}}_{\bar{x}(t)} \\ &+ \underbrace{\begin{bmatrix} \frac{2}{\pi}\tan^{-1}(\bar{\lambda}t)D_h \\ B_h \end{bmatrix}}_{\bar{B}} \theta_d(t) + \underbrace{\begin{bmatrix} q(0)e^{-\bar{\lambda}t} \\ 0 \end{bmatrix}}_{\omega(t)}. \end{aligned} \quad (5.41)$$

By defining  $\bar{x}(t) = [\theta(t) \ x_h^T(t)]^T$ , (5.41) can be written in the following compact form

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{B}\theta_d(t) + \omega(t). \quad (5.42)$$

**Theorem 12.** *The time-varying system  $\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t)$ , where  $\bar{A}(t)$  and  $\bar{x}(t)$  are given in (5.41), is uniformly exponentially stable if*

$$\lambda_i \left( \begin{bmatrix} -D_h & C_h \\ -B_h & A_h \end{bmatrix} \right) \in \mathbb{C}^-, \quad i = 1, \dots, \hat{n} + 1, \quad (5.43)$$

where  $\lambda_i(\cdot)$  provides  $i$ th eigenvalue of the matrix.

*Proof.* The time-varying matrix  $\bar{A}(t)$  can be written as

$$\hat{A}(t) = \underbrace{\begin{bmatrix} -D_h & C_h \\ -B_h & A_h \end{bmatrix}}_{\bar{A}_1} + \underbrace{\begin{bmatrix} (-\frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) + 1)D_h & (\frac{2}{\pi}\tan^{-1}(\bar{\lambda}t) - 1)C_h \\ 0 & 0 \end{bmatrix}}_{\bar{A}_2(t)}. \quad (5.44)$$

If all of the eigenvalues of  $\bar{A}_1$  are in  $\mathbb{C}^-$ , then the origin of the differential equation  $\dot{\bar{x}}(t) = \bar{A}_1 \bar{x}(t)$  is exponentially stable. Therefore, there exist a scalar function  $\bar{V}(x)$  satisfying [6]

$$\bar{c}_1 \|\bar{x}\|^2 \leq \bar{V} \leq \bar{c}_2 \|\bar{x}\|^2 \quad (5.45)$$

$$\frac{d\bar{V}}{d\bar{x}} \bar{A}_1 \bar{x}(t) \leq -\bar{c}_3 \|\bar{x}\|^2 \quad (5.46)$$

$$\left\| \frac{d\bar{V}}{d\bar{x}} \right\| \leq \bar{c}_4 \|\bar{x}\|, \quad (5.47)$$

where  $\bar{c}_1$ ,  $\bar{c}_2$ ,  $\bar{c}_3$  and  $\bar{c}_4$  are positive constants. Considering the system  $\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) = \bar{A}_1 \bar{x}(t) + \bar{A}_2(t)\bar{x}(t)$  and using (5.45)-(5.47), an upper bound on the time derivative of  $\bar{V}$  can be obtained as

$$\begin{aligned} \dot{\bar{V}} &= \frac{d\bar{V}}{d\bar{x}} \bar{A}_1 \bar{x}(t) + \frac{d\bar{V}}{d\bar{x}} \bar{A}_2(t) \bar{x}(t) \\ &\leq -\bar{c}_3 \|\bar{x}\|^2 + \bar{c}_4 \bar{c}_5 \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) - 1 \right| \|\bar{x}\|^2 \\ &\leq -\frac{\bar{c}_3}{\bar{c}_2} \bar{V} + \frac{\bar{c}_4 \bar{c}_5}{\bar{c}_1} \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) - 1 \right| \bar{V} \\ &= -\left( \frac{\bar{c}_3}{\bar{c}_2} - \frac{\bar{c}_4 \bar{c}_5}{\bar{c}_1} \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) - 1 \right| \right) \bar{V}, \end{aligned} \quad (5.48)$$

where  $\bar{c}_5 = (\|D_h\|^2 + \|C_h\|^2)^{1/2}$ . Using the comparison lemma [6], we get

$$\bar{V} \leq e^{-\left(\frac{\bar{c}_3}{\bar{c}_2}t - \frac{\bar{c}_4 \bar{c}_5}{\bar{c}_1} \int_0^t \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\tau) - 1 \right| d\tau\right)} \bar{V}(\hat{x}(0)). \quad (5.49)$$

Using (5.45), (5.49) can be written as

$$\|\bar{x}(t)\|^2 \leq \frac{\bar{c}_2}{\bar{c}_1} e^{-\left(\frac{\bar{c}_3}{\bar{c}_2}t - \frac{\bar{c}_4 \bar{c}_5}{\bar{c}_1} \int_0^t \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\tau) - 1 \right| d\tau\right)} \|\bar{x}(0)\|^2, \quad (5.50)$$

which leads to

$$\|\bar{x}(t)\| \leq \sqrt{\frac{\bar{c}_2}{\bar{c}_1}} e^{-\left(\frac{\bar{c}_3}{2\bar{c}_2}t - \frac{\bar{c}_4 \bar{c}_5}{2\bar{c}_1} \int_0^t \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\tau) - 1 \right| d\tau\right)} \|\bar{x}(0)\|. \quad (5.51)$$

Let  $\gamma(t) = \left| \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) - 1 \right|$ , it should be noted that  $\gamma(t) \geq 0$  and  $\lim_{t \rightarrow \infty} \gamma(t) = 0$ . Also, the time derivative of  $\gamma(t)$  is bounded for  $\forall t \geq 0$ , and  $\frac{d}{dt} \gamma(t) = \frac{-2\bar{\lambda}}{\pi(1+\bar{\lambda}^2 t^2)} < 0$ . Therefore, there exist positive constants  $\bar{T}$  and  $\bar{\epsilon}$  such that  $\gamma(t) \leq \bar{\epsilon} < \frac{\bar{c}_3 \bar{c}_1}{\bar{c}_2 \bar{c}_4 \bar{c}_5}$ ,  $\forall t \geq \bar{T}$ . Also, from the boundedness of  $\gamma(t)$  we have  $\int_0^{\bar{T}} \gamma(t) dt = \bar{\eta}$ , where  $\bar{\eta}$  is a positive constant. The following two cases should be considered: **(1)** For  $t \leq \bar{T}$ ,

$\int_0^t \gamma(\tau) d\tau \leq \bar{\eta}$ . Thus, using (5.51), an upper bound for  $\|\bar{x}\|$  can be obtained as

$$\|\bar{x}(t)\| \leq \sqrt{\frac{\bar{c}_2}{\bar{c}_1}} \bar{\kappa} e^{-(\frac{\bar{c}_3}{2\bar{c}_2} t)} \|\bar{x}(0)\|, \quad (5.52)$$

where  $\bar{\kappa} = e^{\frac{\bar{c}_4 \bar{c}_5}{2\bar{c}_1} \bar{\eta}}$ . **(2)** For  $t > \bar{T}$ ,  $\int_0^t \gamma(\tau) d\tau = \int_0^{\bar{T}} \gamma(\tau) d\tau + \int_{\bar{T}}^t \gamma(\tau) d\tau \leq \bar{\epsilon} t + \bar{\eta}$ . Substituting this result in (5.51), an upper bound for  $\|\bar{x}\|$  can be obtained as

$$\|\bar{x}(t)\| \leq \sqrt{\frac{\bar{c}_2}{\bar{c}_1}} \bar{\kappa} e^{-(\frac{\bar{c}_3}{2\bar{c}_2} t - \frac{\bar{c}_4 \bar{c}_5}{2\bar{c}_1} \bar{\epsilon} t)} \|\bar{x}(0)\| \quad (5.53)$$

Since  $\bar{\epsilon} < \frac{\bar{c}_3 \bar{c}_1}{\bar{c}_2 \bar{c}_4 \bar{c}_5}$ , the origin of  $\dot{\bar{x}}(t) = \bar{A}(t) \bar{x}(t)$  is exponentially stable.  $\square$

**Theorem 13.** *The solution of the linear time-varying system (5.42), where  $\bar{A}(t)$ ,  $\bar{B}$ ,  $\omega(t)$  and  $\bar{x}(t)$  are given in (5.41), is bounded if*

$$\lambda_i \left( \begin{bmatrix} -D_h & C_h \\ -B_h & A_h \end{bmatrix} \right) \in \mathbb{C}^-, \quad i = 1, \dots, \hat{n} + 1. \quad (5.54)$$

*Proof.* The solution of the linear time-varying system (5.42) is

$$\bar{x}(t) = \bar{\Phi}(t, t_0) \bar{x}(t_0) + \int_{t_0}^t \bar{\Phi}(t, \tau) (\bar{B} \theta_d(\tau) + \omega(\tau)) d\tau, \quad (5.55)$$

where  $\bar{\Phi}(t, t_0)$  is the state transition matrix. By using the definitions of  $\bar{B}$  and  $\omega(t)$  given in (5.41), it can be obtained that  $\sup_{0 \leq \tau \leq t} \|\bar{B} \theta_d(\tau) + \omega(\tau)\| = \|\bar{B}\| \theta_{d_{max}} + |q(0)|$ . Also, it is proved in Theorem 12 that if  $\lambda_i(\bar{A}_1) \in \mathbb{C}^-$ ,  $i = 1, \dots, \hat{n} + 1$ , the origin of  $\dot{\bar{x}}(t) = \bar{A}(t) \bar{x}(t)$  is exponentially stable, that is, there exist finite positive constants  $\bar{k}_1$  and  $\bar{k}_2$  such that  $\|\bar{\Phi}(t, t_0)\| \leq \bar{k}_1 e^{-\bar{k}_2(t-t_0)}$ , for  $\forall t \geq t_0$ . Therefore, considering  $t_0 = 0$ , an upper bound on  $\|\bar{x}(t)\|$  can be obtained as

$$\begin{aligned} \|\bar{x}(t)\| &\leq \bar{k}_1 e^{-\bar{k}_2 t} \|\bar{x}(0)\| + \bar{k}_1 (\|\bar{B}\| \theta_{d_{max}} + |q(0)|) \int_0^t e^{-\bar{k}_2(t-\tau)} d\tau \\ &= \bar{k}_1 e^{-\bar{k}_2 t} \|\bar{x}(0)\| + \frac{\bar{k}_1}{\bar{k}_2} (\|\bar{B}\| \theta_{d_{max}} + |q(0)|) (1 - e^{-\bar{k}_2 t}) \\ &\leq \bar{k}_1 \|\bar{x}(0)\| + \frac{\bar{k}_1}{\bar{k}_2} (\|\bar{B}\| \theta_{d_{max}} + |q(0)|). \end{aligned} \quad (5.56)$$

Therefore,  $\theta(t)$  and  $x_h(t)$  are bounded. Assuming that  $\theta_d(t)$  is bounded, the boundedness of  $\theta(t)$  proves that  $e_\theta(t) = \theta_d(t) - \theta(t)$  is bounded as well.  $\square$

**Remark 23.** Different state space representations of (5.37) do not change the analysis provided in Theorems 12 and 13. To show this, consider the state space representation

$$\dot{x}_h(t) = T_x^{-1}A_hT_x x_h(t) + T_x^{-1}B_h(\theta_d(t) - \theta(t)), \quad (5.57)$$

$$q_d(t) = C_hT_x x_h(t) + D_h(\theta_d(t) - \theta(t)), \quad (5.58)$$

where  $T_x$  is the linear transformation matrix. Let  $M = \begin{bmatrix} -D_h & C_h \\ -B_h & A_h \end{bmatrix}$  and  $M_T = \begin{bmatrix} -D_h & C_hT_x \\ -T_x^{-1}B_h & T_x^{-1}A_hT_x \end{bmatrix}$ , then  $M_T$  can be written as

$$M_T = \begin{bmatrix} I & 0 \\ 0 & T_x^{-1} \end{bmatrix} M \begin{bmatrix} I & 0 \\ 0 & T_x \end{bmatrix}. \quad (5.59)$$

By defining  $\bar{T}_x = \begin{bmatrix} I & 0 \\ 0 & T_x \end{bmatrix}$ , we have

$$\begin{aligned} \det(sI - M_T) &= \det(sI - \bar{T}_x^{-1}M\bar{T}_x) \\ &= \det(\bar{T}_x^{-1}(sI - M)\bar{T}_x) \\ &= \det(\bar{T}_x^{-1})\det(sI - M)\det(\bar{T}_x) \\ &= \det(sI - M). \end{aligned} \quad (5.60)$$

#### 5.4.4 Stability analysis of general human model with delay consideration

In the previous section, a general human model without considering delay was employed and stability of the system including human was considered. In this section, the effects of time delay in the stability of the closed loop system is investigated.

Consider the transfer function

$$\frac{q_d(s)}{\theta_d(s) - \theta(s)} = \frac{b_ms^{\hat{m}} + b_{m-1}s^{\hat{m}-1} + \dots + b_0}{s^{\hat{n}} + a_{n-1}s^{\hat{n}-1} + \dots + a_0} e^{-\tau_h s}, \quad (5.61)$$

as human model, where  $\tau_h$  is a positive constant representing internal time delay,  $\hat{m}$  and  $\hat{n}$  are real constants such that  $\hat{n} > 0$ ,  $\hat{m} \geq 0$  and  $\hat{m} \leq \hat{n}$ , and  $a_i$  and  $b_j$  for  $i = 0, \dots, \hat{n} - 1$  and  $j = 0, \dots, \hat{m}$ , are real constants. The transfer function (5.37) can be represented in the minimal state space form as

$$\dot{x}_h(t) = A_h x_h(t) + B_h(\theta_d(t - \tau_h) - \theta(t - \tau_h)), \quad (5.62)$$

$$q_d(t) = C_h x_h(t) + D_h(\theta_d(t - \tau_h) - \theta(t - \tau_h)), \quad (5.63)$$

where  $x_h \in \mathbb{R}^{\hat{n}}$  is the vector including the states of the human model,  $A_h \in \mathbb{R}^{\hat{n} \times \hat{n}}$  is the system matrix,  $B_h \in \mathbb{R}^{\hat{n}}$  is the control vector,  $C_h \in \mathbb{R}^{1 \times \hat{n}}$  is the output vector,  $D_h \in \mathbb{R}$  is the feedforward gain,  $q_d \in \mathbb{R}$  is the reference pitch angle and the output of the human model,  $\theta_d \in \mathbb{R}$  is the desired pitch angle and  $\theta \in \mathbb{R}$  is the current pitch angle.

Using (5.22), which is the trajectory of  $q(t)$  on the sliding surface (3.16), (5.63) and considering  $\dot{\theta}(t) = q(t)$ , the pitch angle dynamics can be written as

$$\dot{\theta}(t) = q(0)e^{-\bar{\lambda}t} + \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) C_h x_h(t) + \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) D_h(\theta_d(t - \tau_h) - \theta(t - \tau_h)). \quad (5.64)$$

Augmenting the dynamics of general human model (5.62) and pitch angle (5.64), we have

$$\begin{aligned} \begin{bmatrix} \dot{\theta}(t) \\ \dot{x}_h(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) C_h \\ 0_{\hat{n} \times 1} & A_h \end{bmatrix}}_{\bar{A}(t)} \underbrace{\begin{bmatrix} \theta(t) \\ x_h(t) \end{bmatrix}}_{\bar{x}(t)} \\ &+ \underbrace{\begin{bmatrix} -\frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) D_h & 0_{1 \times \hat{n}} \\ -B_h & 0_{\hat{n} \times \hat{n}} \end{bmatrix}}_{\bar{A}_\tau(t)} \underbrace{\begin{bmatrix} \theta(t - \tau_h) \\ x_h(t - \tau_h) \end{bmatrix}}_{\bar{x}(t - \tau_h)} \\ &+ \underbrace{\begin{bmatrix} \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) D_h & e^{-\bar{\lambda}t} \\ B_h & 0_{\hat{n} \times 1} \end{bmatrix}}_{\bar{E}(t)} \underbrace{\begin{bmatrix} \theta_d(t - \tau_h) \\ q(0) \end{bmatrix}}_{\bar{\omega}(t)}. \end{aligned} \quad (5.65)$$

By defining  $\bar{x}(t) \equiv [\theta(t) \ x_h^T(t)]^T \in \mathbb{R}^{(\hat{n}+1)}$  and  $\bar{\omega} \equiv [\theta_d(t - \tau_h) \ q(0)]^T \in \mathbb{R}^2$ , (5.65) can be written in the following compact form

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{A}_\tau(t)\bar{x}(t - \tau_h) + \bar{E}(t)\bar{\omega}(t). \quad (5.66)$$

with initial condition

$$\bar{x}(t) = \bar{\phi}(t), \quad t \in [-\tau_h, 0], \quad (5.67)$$

where  $\bar{\phi} : [-\tau_h, 0] \rightarrow \mathbb{R}^{\hat{n}+1}$  and  $\dot{\bar{\phi}}$  is a square integrable function, that is,  $\dot{\bar{\phi}} \in \mathcal{L}_2$ . Equation (5.66) can be written as

$$\dot{\bar{x}}(t) = (\bar{A}_0 + \Delta \bar{A}_0(t))\bar{x}(t) + (\bar{A}_{\tau 0} + \Delta \bar{A}_{\tau 0}(t))\bar{x}(t - \tau_h) + \bar{E}(t)\bar{\omega}(t), \quad (5.68)$$

where

$$\begin{aligned} \bar{A}_0 &= \begin{bmatrix} -\alpha_G \beta_{H1} & -\alpha_G \gamma_{H1} 1_{1 \times \hat{n}} \\ -\alpha_G \beta_{H1} 1_{\hat{n} \times 1} & A_h - \alpha_G \gamma_{H1} 1_{\hat{n} \times \hat{n}} \end{bmatrix}, \\ \Delta \bar{A}_0(t) &= \begin{bmatrix} \alpha_G \beta_{H1} & \frac{2}{\pi} \tan^{-1}(\bar{\lambda} t) C_h + \alpha_G \gamma_{H1} 1_{1 \times \hat{n}} \\ \alpha_G \beta_{H1} 1_{\hat{n} \times 1} & \alpha_G \gamma_{H1} 1_{\hat{n} \times \hat{n}} \end{bmatrix}, \\ \bar{A}_{\tau 0} &= \begin{bmatrix} -\alpha_G \beta_{H2} & -\alpha_G \gamma_{H2} 1_{1 \times \hat{n}} \\ -\alpha_G \beta_{H2} 1_{\hat{n} \times 1} - B_h & -\alpha_G \gamma_{H2} 1_{\hat{n} \times \hat{n}} \end{bmatrix}, \\ \Delta \bar{A}_{\tau 0}(t) &= \begin{bmatrix} -\frac{2}{\pi} \tan^{-1}(\bar{\lambda} t) D_h + \alpha_G \beta_{H2} & \alpha_G \gamma_{H2} 1_{1 \times \hat{n}} \\ \alpha_G \beta_{H2} 1_{\hat{n} \times 1} & \alpha_G \gamma_{H2} 1_{\hat{n} \times \hat{n}} \end{bmatrix}. \end{aligned} \quad (5.69)$$

Also, the matrix  $\Delta \equiv [\Delta \bar{A}_0(t) \quad \Delta \bar{A}_{\tau 0}] \in \mathbb{R}^{(\hat{n}+1) \times (2\hat{n}+2)}$  can be decomposed into the matrices  $F \in \mathbb{R}^{(\hat{n}+1) \times 2}$ ,  $G(t) \in \mathbb{R}^{2 \times 2}$  and  $\hat{H} = [H_1 \quad H_2] \in \mathbb{R}^{2 \times (2\hat{n}+2)}$  as

$$\Delta = \underbrace{\begin{bmatrix} 1 & 1 \\ 0_{\hat{n} \times 1} & 1_{\hat{n} \times 1} \end{bmatrix}}_F \underbrace{\begin{bmatrix} \frac{2}{\pi} \tan^{-1}(\bar{\lambda} t) & 0 \\ 0 & \alpha_G \end{bmatrix}}_{G(t)} \underbrace{\begin{bmatrix} H_1 & H_2 \end{bmatrix}}_{\hat{H}} \quad (5.70)$$

where  $\alpha_G \in \mathbb{R}$  with  $|\alpha_G| < 1$ ,  $H_1 \in \mathbb{R}^{2 \times (\hat{n}+1)}$  and  $H_2 \in \mathbb{R}^{2 \times (\hat{n}+1)}$ , and are defined as

$$H_1 = \begin{bmatrix} 0 & C_h \\ \beta_{H1} & \gamma_{H1} 1_{1 \times \hat{n}} \end{bmatrix}, \quad H_2 = \begin{bmatrix} -D_h & 0_{1 \times \hat{n}} \\ \beta_{H2} & \gamma_{H2} 1_{1 \times \hat{n}} \end{bmatrix}, \quad (5.71)$$

where  $\beta_{H1} \in \mathbb{R}$ ,  $\beta_{H2} \in \mathbb{R}$ ,  $\gamma_{H1} \in \mathbb{R}$  and  $\gamma_{H2} \in \mathbb{R}$ . Notice that the design parameters  $\alpha_G$ ,  $\beta_{H1}$ ,  $\beta_{H2}$ ,  $\gamma_{H1}$  and  $\gamma_{H2}$ , can be chosen such that the following assumption holds.

**Assumption 3.** *The eigenvalues of the nominal system without delay, i.e.  $\Delta \bar{A}_0(t) = \Delta \bar{A}_{\tau 0}(t) = 0$  and  $\tau_h = 0$ , are in  $\mathbb{C}^-$ , that is,  $\lambda_i(\bar{A}_0 + \bar{A}_{\tau 0}) \in \mathbb{C}^-$ .*

**Assumption 4.** *The human model with  $\tau = 0$  satisfies (5.54).*

The following lemmas are useful for the stability analysis of the system.

**Lemma 10.** *The matrix  $G(t)$ , defined in (5.70), satisfies  $G(t)^T G(t) < I_2$  for all  $t > 0$ , if  $|\alpha_G| < 1$ .*

*Proof.* The proof can be found in Appendix D.1.  $\square$

**Lemma 11.** *For real constant matrices  $X$  and  $Y$ , and time-dependent matrix  $G(t)$ , such that  $G^T(t)G(t) \leq I$ , the following inequality holds*

$$XG(t)Y + Y^T G^T(t)X^T \leq \epsilon_G X X^T + \frac{1}{\epsilon_G} Y^T Y, \quad (5.72)$$

*for any positive constant  $\epsilon_G$ .*

*Proof.* The proof, which is given by [115], can also be found in Appendix D.2.  $\square$

**Lemma 12.** *[Schur complement] A symmetric matrix  $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$ , where  $S_{11}$  and  $S_{22}$  are square matrices, is negative definite if and only if*

$$S_{22} < 0, \quad S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0. \quad (5.73)$$

*Proof.* The proof of this lemma is given by [115].  $\square$

After introducing the general human model with delay and the useful lemmas, it is time to prove the main theorems about human-in-the-loop stability analysis. In the sequel, we are going to provide the conditions which ensure boundedness of the system.

**Theorem 14.** *[Lyapunov-Krasovskii stability theorem] Suppose  $u, v, w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are continuous non-decreasing functions,  $u(t)$ ,  $v(t)$  and  $w(t)$  are positive for*

$t > 0$ , and  $u(0) = v(0) = w(0) = 0$ . If there exists a continuous differentiable functional  $V : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}$ , where  $\mathcal{C}$  is the set of continuous functions, such that

$$u(\|\bar{x}(t)\|) \leq V \leq v\left(\sup_{\theta \in [-h, 0]} \|\bar{x}(t + \theta)\|\right), \quad (5.74)$$

$$\dot{V} \leq -w(\|\bar{x}(t)\|), \quad (5.75)$$

then the solution of  $\bar{x}(t) = f(t, x(t + \theta))$  with initial condition  $\bar{x}(t_0 + \theta)$ ,  $\theta \in [-h, 0]$ , is uniformly asymptotically stable.

**Theorem 15.** Consider that Assumptions 3 and 4 are satisfied. The solution of the differential equation  $\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{A}_\tau(t)\bar{x}(t - \tau_h)$ , with initial condition (5.67) and delay,  $\tau_h$ , such that  $0 < \tau_h \leq h$ , is uniformly asymptotically stable if there exist symmetric matrices  $0 < P \in \mathbb{R}^{(1+\hat{n}) \times (1+\hat{n})}$ ,  $0 \leq Q \in \mathbb{R}^{(1+\hat{n}) \times (1+\hat{n})}$ ,  $0 < Z \in \mathbb{R}^{(1+\hat{n}) \times (1+\hat{n})}$  and  $0 \leq X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \in \mathbb{R}^{(2+2\hat{n}) \times (2+2\hat{n})}$ , and free weighting matrices  $N_1 \in \mathbb{R}^{(1+\hat{n}) \times (1+\hat{n})}$  and  $N_2 \in \mathbb{R}^{(1+\hat{n}) \times (1+\hat{n})}$ , such that the following LMIs hold

$$\begin{bmatrix} X_{11} & X_{12} & N_1 \\ * & X_{22} & N_2 \\ * & * & Z \end{bmatrix} \geq 0, \quad (5.76)$$

$$\begin{bmatrix} \Sigma_{11} + H_1^T H_1 & \Sigma_{12} + H_1^T H_2 & h\bar{A}_0^T Z & PF \\ * & \Sigma_{22} + H_2^T H_2 & h\bar{A}_{\tau_0}^T Z & 0 \\ * & * & -hZ & hZF \\ * & * & * & -I_2 \end{bmatrix} < 0, \quad (5.77)$$

where  $F$ ,  $H_1$  and  $H_2$  are defined in (5.70), and

$$\begin{aligned} \Sigma_{11} &= \bar{A}_0^T P + P\bar{A}_0 + Q + N_1 + N_1^T + hX_{11}, \\ \Sigma_{12} &= P\bar{A}_{\tau_0} + N_2^T - N_1 + hX_{12}, \\ \Sigma_{22} &= -\epsilon_\tau Q - N_2 - N_2^T + hX_{22}, \end{aligned} \quad (5.78)$$

where  $0 < \epsilon_\tau < 1$ .

*Proof.* Consider the following Lyapunov-Krasovskii functional candidate

$$V = \bar{x}^T(t)P\bar{x}(t) + \int_{t-\tau_h}^t \bar{x}^T(s)Q\bar{x}(s)ds + \int_{-h}^0 \int_{t+\chi}^t \dot{\bar{x}}^T(s)Z\dot{\bar{x}}(s)dsd\chi, \quad (5.79)$$

where  $Z > 0$  is a symmetric matrix. It is obtained that  $\lambda_{\min}(P)\|\bar{x}(t)\|^2 \leq V$ . Also, using the equality

$$\int_{-h}^0 \int_{t+\chi}^t \dot{\bar{x}}^T(s)R\dot{\bar{x}}(s)dsd\chi = \int_{t-h}^t (s-t+h)\dot{\bar{x}}^T(s)R\dot{\bar{x}}(s)ds, \quad (5.80)$$

where  $R \in \mathbb{R}^{(\hat{n}+1) \times (\hat{n}+1)}$ , and (5.79), an upper bound for  $V$  can be derived as

$$\begin{aligned} V &\leq \lambda_{\max}(P)\|\bar{x}(t)\|^2 + h\lambda_{\max}(Q) \sup_{\theta \in [-h, 0]} \|\bar{x}(t+\theta)\|^2 + h\lambda_{\max}(R) \int_{t-h}^t \|\dot{\bar{x}}(\xi)\|^2 d\xi \\ &\leq \lambda_{\max}(P)\|\bar{x}(t)\|^2 + h\lambda_{\max}(Q) \sup_{\theta \in [-h, 0]} \|\bar{x}(t+\theta)\|^2 \\ &\quad + h\lambda_{\max}(R) \int_{t-h}^t \|\bar{A}(\xi)\bar{x}(\xi) + \bar{A}_\tau(\xi)\bar{x}(\xi - \tau_h)\|^2 d\xi \\ &\leq \lambda_{\max}(P)\|\bar{x}(t)\|^2 + h\lambda_{\max}(Q) \sup_{\theta \in [-h, 0]} \|\bar{x}(t+\theta)\|^2 \\ &\quad + 4h^2\lambda_{\max}(R)\lambda_1 \sup_{\theta \in [-h, 0]} \|\bar{x}(t+\theta)\|^2 \\ &\leq \lambda_2 \sup_{\theta \in [-h, 0]} \|\bar{x}(t+\theta)\|^2, \end{aligned} \quad (5.81)$$

where  $\lambda_1 = \max\{\sup_{\theta \in [-h, \infty)} \|\bar{A}(\theta)\|^2, \sup_{\theta \in [-h, \infty)} \|\bar{A}_\tau(\theta)\|^2\}$  and  $\lambda_2 = \max\{\lambda_{\max}(P), h\lambda_{\max}(Q), 4h^2\lambda_{\max}(R)\lambda_1\}$ . Therefore, (5.74) is satisfied. Notice that the equality (5.80) can be verified by showing that the right and left hand sides of (5.80) are equal at  $t = -h$  and time-derivative of both sides are the same.

Now, we want to satisfy (5.75). The time derivative of  $V$  along the trajectories of (5.68) is obtained as

$$\begin{aligned} \dot{V} &= \dot{\bar{x}}^T(t)P\bar{x}(t) + \bar{x}^T(t)P\dot{\bar{x}}(t) + \bar{x}^T(t)Q\bar{x}(t) - \bar{x}^T(t - \tau_h)Q\bar{x}(t - \tau_h(t)) \\ &\quad + h\dot{\bar{x}}^T(t)Z\dot{\bar{x}}(t) - \int_{t-h}^t \dot{\bar{x}}^T(s)Z\dot{\bar{x}}(s)ds. \end{aligned} \quad (5.82)$$

Using the formula  $\bar{x}(t) - \bar{x}(t - \tau_h(t)) = \int_{t-\tau_h}^t \dot{\bar{x}}(s)ds$  and the inequality  $h\eta_1^T(t)X\eta_1(t) - \int_{t-\tau_h}^t \eta_1^T(s)X\eta_1(s)ds \geq 0$  for any symmetric positive semi-definite

matrix  $X$ , an upper bound for (5.82) can be obtained as

$$\begin{aligned} \dot{V} &\leq \dot{\bar{x}}^T(t)P\bar{x}(t) + \bar{x}^T(t)P\dot{\bar{x}}(t) + \bar{x}^T(t)Q\bar{x}(t) - \epsilon_\tau \bar{x}^T(t - \tau_h(t))Q\bar{x}(t - \tau_h(t)) \\ &\quad + h\dot{\bar{x}}^T(t)Z\dot{\bar{x}}(t) - \int_{t-\tau_h}^t \dot{\bar{x}}^T(s)Z\dot{\bar{x}}(s)ds + 2(\bar{x}^T(t)N_1 + \bar{x}^T(t - \tau_h(t))N_2) \\ &\quad \times \left( \bar{x}(t) - \bar{x}(t - \tau_h) - \int_{t-\tau_h}^t \dot{\bar{x}}(s)ds \right) + h\eta_1^T(t)X\eta_1(t) - \int_{t-\tau_h}^t \eta_1^T(t)X\eta_1(t)ds, \end{aligned} \quad (5.83)$$

where  $\eta_1(t) = [\bar{x}^T(t) \ \bar{x}^T(t - \tau_h)]^T \in \mathbb{R}^{2\hat{n}+2}$  and  $0 < \epsilon_\tau < 1$ . By substituting  $\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{A}_\tau(t)\bar{x}(t - \tau_h)$  into (5.83), the inequality (5.83) can be rewritten in the following compact form

$$\dot{V} \leq \eta_1^T(t)(\hat{\Sigma} + \Sigma_Z)\eta_1(t) - \int_{t-\tau_h}^t \eta_2^T(t, s)\Psi\eta_2(t, s)ds \quad (5.84)$$

where  $\eta_2(t, s) = [\bar{x}^T(t) \ \bar{x}^T(t - \tau_h) \ \dot{\bar{x}}^T(s)]^T \in \mathbb{R}^{3\hat{n}+3}$ ,

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & N_1 \\ * & X_{22} & N_2 \\ * & * & Z \end{bmatrix} \in \mathbb{R}^{(3\hat{n}+3) \times (3\hat{n}+3)}, \quad (5.85)$$

$$\Sigma_Z = \begin{bmatrix} h\bar{A}^T(t)Z\bar{A}(t) & h\bar{A}^T(t)Z\bar{A}_\tau(t) \\ * & h\bar{A}_\tau^T(t)Z\bar{A}_\tau(t) \end{bmatrix}, \quad (5.86)$$

and  $\hat{\Sigma} = \begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ * & \hat{\Sigma}_{22} \end{bmatrix} \in \mathbb{R}^{(2\hat{n}+2) \times (2\hat{n}+2)}$ , where,

$$\begin{aligned} \hat{\Sigma}_{11} &= \bar{A}^T(t)P + P\bar{A}(t) + Q + N_1 + N_1^T + hX_{11}, \\ \hat{\Sigma}_{12} &= P\bar{A}_\tau(t) + N_2^T - N_1 + hX_{12}, \\ \hat{\Sigma}_{22} &= \Sigma_{22}. \end{aligned} \quad (5.87)$$

The right hand side of (5.84) can be written as

$$\begin{aligned} \dot{V} &\leq \eta_1^T(t) \left( \hat{\Sigma} - \begin{bmatrix} h\bar{A}^T(t)Z \\ h\bar{A}_\tau^T(t)Z \end{bmatrix} (-h^{-1}Z^{-1}) \begin{bmatrix} hZ\bar{A}(t) & hZ\bar{A}_\tau(t) \end{bmatrix} \right) \eta_1(t) \\ &\quad - \int_{t-\tau_h}^t \eta_2^T(t, s)\Psi\eta_2(t, s)ds, \end{aligned} \quad (5.88)$$

For the right hand side of (5.84) to be negative, it is required that  $\Psi \geq 0$  and  $\hat{\Sigma} + \Sigma_Z < 0$ . Using Lemma 12 and (5.88),  $\hat{\Sigma} + \Sigma_Z < 0$  is equivalent to

$$\begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} & h\bar{A}^T(t)Z \\ * & \hat{\Sigma}_{22} & h\bar{A}_{\tau}^T(t)Z \\ * & * & -hZ \end{bmatrix} < 0. \quad (5.89)$$

Using  $\bar{A}(t) = \bar{A}_0 + \Delta\bar{A}_0(t)$ ,  $\bar{A}_{\tau}(t) = \bar{A}_{\tau 0} + \Delta\bar{A}_{\tau 0}(t)$  and (5.70), the inequality (5.89) can be rewritten as

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & h\bar{A}_0^T Z \\ * & \Sigma_{22} & h\bar{A}_{\tau 0}^T Z \\ * & * & -hZ \end{bmatrix} + \begin{bmatrix} PFG(t)H_1 + H_1^T G^T(t)F^T P & PFG(t)H_2 & hH_1^T G^T(t)F^T Z \\ * & 0 & hH_2^T G^T(t)F^T Z \\ * & * & 0 \end{bmatrix} < 0. \quad (5.90)$$

The inequality (5.90) can be written as

$$\begin{aligned} & \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & h\bar{A}_0^T Z \\ * & \Sigma_{22} & h\bar{A}_{\tau 0}^T Z \\ * & * & -hZ \end{bmatrix} + \begin{bmatrix} PF \\ 0 \\ hZF \end{bmatrix} G(t) \begin{bmatrix} H_1 & H_2 & 0 \end{bmatrix} \\ & + \begin{bmatrix} H_1^T \\ H_2^T \\ 0 \end{bmatrix} G^T(t) \begin{bmatrix} F^T P & 0 & hF^T Z \end{bmatrix} < 0. \end{aligned} \quad (5.91)$$

Using (5.91) and applying Lemmas 10 and 11, it implies that, in order to satisfy (5.89), the inequality

$$\begin{aligned} & \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & h\bar{A}_0^T Z \\ * & \Sigma_{22} & h\bar{A}_{\tau 0}^T Z \\ * & * & -hZ \end{bmatrix} - \begin{bmatrix} PF \\ 0 \\ hZF \end{bmatrix} (-\epsilon_G I_2) \begin{bmatrix} F^T P & 0 & hF^T Z \end{bmatrix} \\ & + \epsilon_G^{-1} \begin{bmatrix} H_1^T \\ H_2^T \\ 0 \end{bmatrix} \begin{bmatrix} H_1 & H_2 & 0 \end{bmatrix} < 0 \end{aligned} \quad (5.92)$$

should be satisfied. Multiply both sides of (5.92) with  $\epsilon_G$ , we get

$$\begin{aligned} & \epsilon_G \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & h\bar{A}_0^T Z \\ * & \Sigma_{22} & h\bar{A}_{\tau_0}^T Z \\ * & * & -hZ \end{bmatrix} - \begin{bmatrix} \epsilon_G P F \\ 0 \\ \epsilon_G h Z F \end{bmatrix} (-I_2) \begin{bmatrix} \epsilon_G F^T P & 0 & \epsilon_G h F^T Z \end{bmatrix} \\ & + \begin{bmatrix} H_1^T \\ H_2^T \\ 0 \end{bmatrix} \begin{bmatrix} H_1 & H_2 & 0 \end{bmatrix} < 0. \end{aligned} \quad (5.93)$$

Replacing  $\epsilon_G P$ ,  $\epsilon_G Z$ ,  $\epsilon_G Q$ ,  $\epsilon_G N_1$ ,  $\epsilon_G N_2$  and  $\epsilon_G X$  with  $P$ ,  $Z$ ,  $Q$ ,  $N_1$ ,  $N_2$  and  $X$ , respectively, and using Lemma 12, (5.93) can be rewritten as (5.77). This implies that (5.75) is satisfied and therefore, based on Theorem 14, the system is uniformly asymptotically stable.  $\square$

After finding the LMI conditions to guarantee the stability of the system  $\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{A}_\tau(t)\bar{x}(t - \tau_h)$ , it is time to analyze the system (5.66). The following definitions are required in the next theorem.

**Definition 3.** [6] A continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$  is of class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ . It belongs to class  $\mathcal{K}_\infty$  if  $\alpha = \infty$  and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . A continuous function  $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$  is said to be of class  $\mathcal{KL}$  if, for each  $s$ , the mapping  $\beta(r, s)$  is of class  $\mathcal{K}$  and, for each fixed  $r$ , the mapping  $\beta(r, s)$  is decreasing and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ .

**Theorem 16.** Consider that Assumptions 3 and 4 are satisfied. The solution of the differential equation (5.66), with initial condition (5.67) and delay,  $\tau_h$ , such that  $0 < \tau_h \leq h$ , is uniformly bounded if the the LMIs (5.76) and (5.77) are satisfied, and  $\|\bar{E}(t)\bar{\omega}(t)\|$  is uniformly bounded by

$$\|\bar{E}(t)\bar{\omega}(t)\| < \frac{\bar{\theta}\lambda_3\lambda_{\min}(P)}{\lambda_2\lambda_4}\bar{r}, \quad (5.94)$$

where  $0 < \bar{\theta} < 1$

*Proof.* It was proved in Theorem 15 that satisfying the LMIs (5.76) and (5.77), the system  $\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{A}_\tau(t)\bar{x}(t - \tau_h)$  is uniformly asymptotically stable.

Thus, there exist a class  $\mathcal{KL}$  function  $\beta$  such that

$$\|\bar{x}(t, t_0, \bar{\phi})\| \leq \beta(\|\bar{\phi}\|_{\mathcal{W}}, t - t_0), \quad (5.95)$$

where (see [116])

$$\|\bar{\phi}\|_{\mathcal{W}} = \max_{\tau \in [-h, 0]} \|\bar{\phi}(\tau)\| + \left( \int_{-h}^0 \|\dot{\bar{\phi}}(s)\|^2 ds \right)^{\frac{1}{2}}. \quad (5.96)$$

Using converse theorems [6], it can be shown that there exists a Lyapunov function which satisfies

$$\bar{\alpha}_1(\|\bar{x}\|) \leq V \leq \bar{\alpha}_2(\|\bar{x}\|), \quad (5.97)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \bar{x}} (\bar{A}(t)\bar{x}(t) + \bar{A}_r(t)\bar{x}(t - \tau_h)) \leq -\bar{\alpha}_3(\|\bar{x}\|), \quad (5.98)$$

$$\left\| \frac{\partial V}{\partial \bar{x}} \right\| \leq \bar{\alpha}_4(\|\bar{x}\|). \quad (5.99)$$

This Lyapunov function is found in Theorem 15. Considering (5.79), (5.81) and (5.84), it can be obtained that  $\bar{\alpha}_1(\|\bar{x}\|) = \lambda_{\min}(P)\|\bar{x}\|^2$ ,  $\bar{\alpha}_2(\|\bar{x}(t)\|) \leq \bar{\alpha}_2(\sup_{\theta \in [-h, 0]} \|\bar{x}(\theta)\|) = \lambda_2 \sup_{\theta \in [-h, 0]} \|\bar{x}(\theta)\|^2$ ,  $\bar{\alpha}_3(\|\bar{x}(t)\|) \leq \bar{\alpha}_3(\sup_{\theta \in [-h, 0]} \|\bar{x}(\theta)\|) = \lambda_3 \sup_{\theta \in [-h, 0]} \|\bar{x}(\theta)\|^2$  and  $\bar{\alpha}_4(\|\bar{x}(t)\|) \leq \bar{\alpha}_4(\sup_{\theta \in [-h, 0]} \|\bar{x}(\theta)\|) = \lambda_4 \sup_{\theta \in [-h, 0]} \|\bar{x}(\theta)\|$ , where  $\lambda_3 = 2\max(\|\hat{\Sigma} + \Sigma_Z\|)$  and  $\lambda_4 = 2\|P\| + 2h\|Q\|$ . Using [6], if  $\bar{E}(t)\bar{\omega}(t)$  satisfies

$$\|\bar{E}(t)\bar{\omega}(t)\| \leq \bar{\delta} < \frac{\bar{\theta}\bar{\alpha}_3(\bar{\alpha}_2^{-1}(\bar{\alpha}_1(\bar{r})))}{\bar{\alpha}_4(\bar{r})} = \frac{\bar{\theta}\lambda_3\lambda_{\min}(P)}{\lambda_2\lambda_4}\bar{r}, \quad (5.100)$$

for all  $x \in \mathcal{D} \equiv \{\bar{x} \in \mathbb{R}^{\hat{n}+1} \mid \|\bar{x}\| < \bar{r}\}$ , with  $0 < \bar{\theta} < 1$ , then the trajectories remain bounded. Since for any  $\bar{\delta}$ , we can choose  $\bar{r}$  large enough to satisfy (5.100). Therefore, it can be concluded that for all uniformly bounded exogenous signal  $\bar{E}(t)\bar{\omega}(t)$ , the solution of (5.66) will be uniformly bounded.  $\square$

### 5.4.5 Simulation results

The closed loop control structure depicted in Figure 5.1 is used for the simulations, where the plant to be controlled is the ADMIRE model. The reference signals,

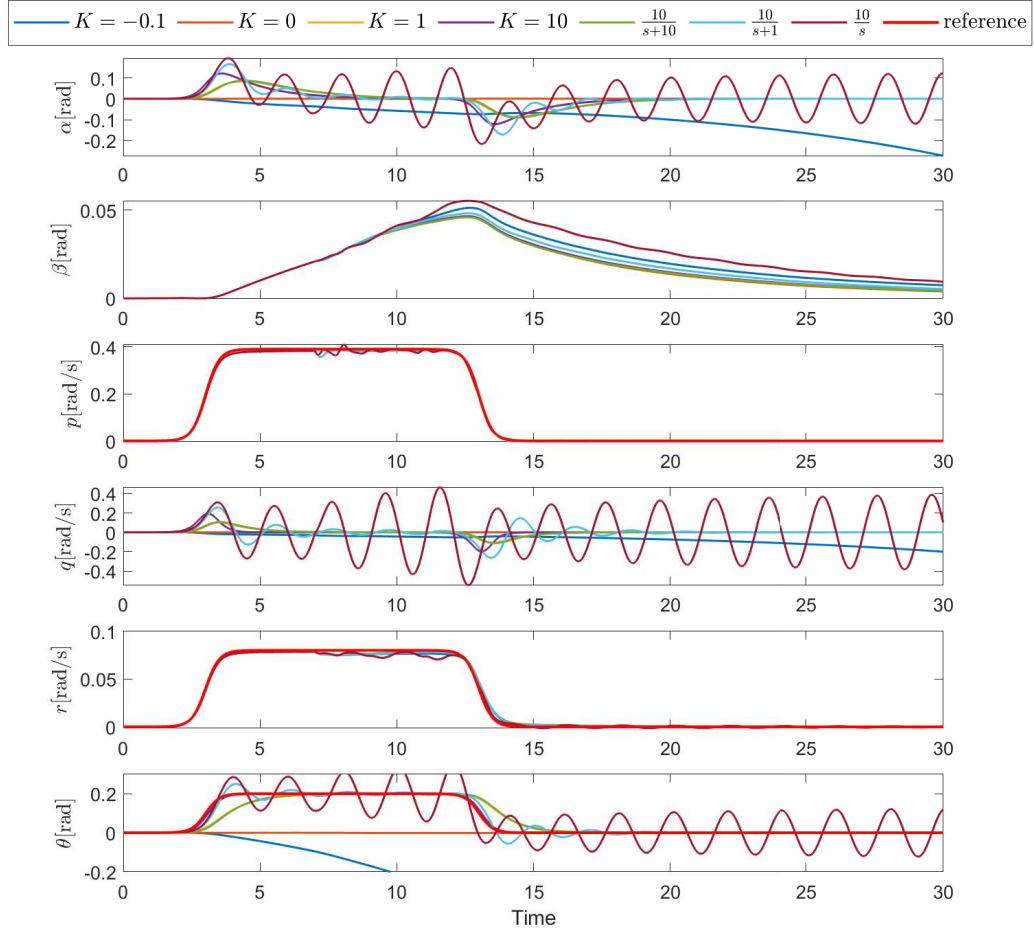


Figure 5.2: The evolution of states considering different human operator models.

$p_d(t)$ ,  $q_d(t)$  and  $r_d(t)$  are the desired roll, pitch and yaw rates, respectively. The signals  $p_d(t)$  and  $r_d(t)$  are provided to the controller externally, and  $q_d(t)$  is the human operator command. The effectiveness of the actuators are reduced by 30% at  $t = 7$ s.

Figure 5.2 illustrates the evolution of the states trajectories. Considering the pure gain human model, consistent with Theorem 9, it is seen that the states remain bounded for  $K = 0, 1, 10$ , and become unbounded when  $K = -0.1$ . Considering the lag filter human model, consistent with Theorem 11, it is seen

Table 5.1: Checking the requirements of Theorem 13

|     | Human transfer function              | $\lambda_i \left( \begin{bmatrix} -D_h & C_h \\ -B_h & A_h \end{bmatrix} \right)$ |
|-----|--------------------------------------|-----------------------------------------------------------------------------------|
| TF1 | $\frac{10}{s+10}$                    | $-1.127, -8.873$                                                                  |
| TF2 | $\frac{s+1}{(s+2)(s+3)}$             | $-0.16, -2.4 \pm 0.6j$                                                            |
| TF3 | $\frac{s+1}{(s-2)(s+3)}$             | $1.655, 0.21, -2.86$                                                              |
| TF4 | $\frac{s-1}{(s+2)(s+3)}$             | $0.13, -2.5 \pm 1.04j$                                                            |
| TF5 | $\frac{(s+1)(s+2)}{(s+3)(s+4)}$      | $-0.144, -2.678, -5.177$                                                          |
| TF6 | $\frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)}$ | $-0.03, -2.8, -4.6 \pm 0.96j$                                                     |

Table 5.2: Delay range for stability

|      | Human transfer function                            | Calculated delay range  |
|------|----------------------------------------------------|-------------------------|
| DTF1 | $\frac{10}{s+10} e^{-\tau_h s}$                    | $\tau_h \in [0, 0.861]$ |
| DTF2 | $\frac{s+1}{(s+2)(s+3)} e^{-\tau_h s}$             | $\tau_h \in [0, 0.861]$ |
| DTF3 | $\frac{s+1}{(s-2)(s+3)} e^{-\tau_h s}$             | — — —                   |
| DTF4 | $\frac{s-1}{(s+2)(s+3)} e^{-\tau_h s}$             | — — —                   |
| DTF5 | $\frac{(s+1)(s+2)}{(s+3)(s+4)} e^{-\tau_h s}$      | $\tau_h \in [0, 0.684]$ |
| DTF6 | $\frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)} e^{-\tau_h s}$ | $\tau_h \in [0, 0.863]$ |

that for  $a = 1$  and  $10$ , and  $K = 10$ , the states evolve in a bounded manner but they become unbounded when  $a = 0$ .

The poles of the matrix  $\begin{bmatrix} -D_h & C_h \\ -B_h & A_h \end{bmatrix}$  for the various human transfer function models, denoted as  $TF1, \dots, TF6$ , are provided in Table 1. Using Theorem 13 and the results given in Table 1, it can be predicted that the simulation results for human models  $TF1, TF2, TF5$  and  $TF6$  are stable. The simulation results for the general human models are illustrated in Figure 5.3. Figure 5.3 shows that only the results corresponding to the human models  $TF3$  and  $TF4$  are unstable.

A delay is added to the transfer functions in Table 1 and are denoted as  $DTF1, \dots, DTF6$  in Table 2. The calculated range of delay, which leads to stability, is also given in Table 2. Notice that the delay range for  $DTF3$  and  $DTF4$  are not given since they do not satisfy Assumption 4. The simulation results for the general human models with delay are illustrated in Figure 5.4.

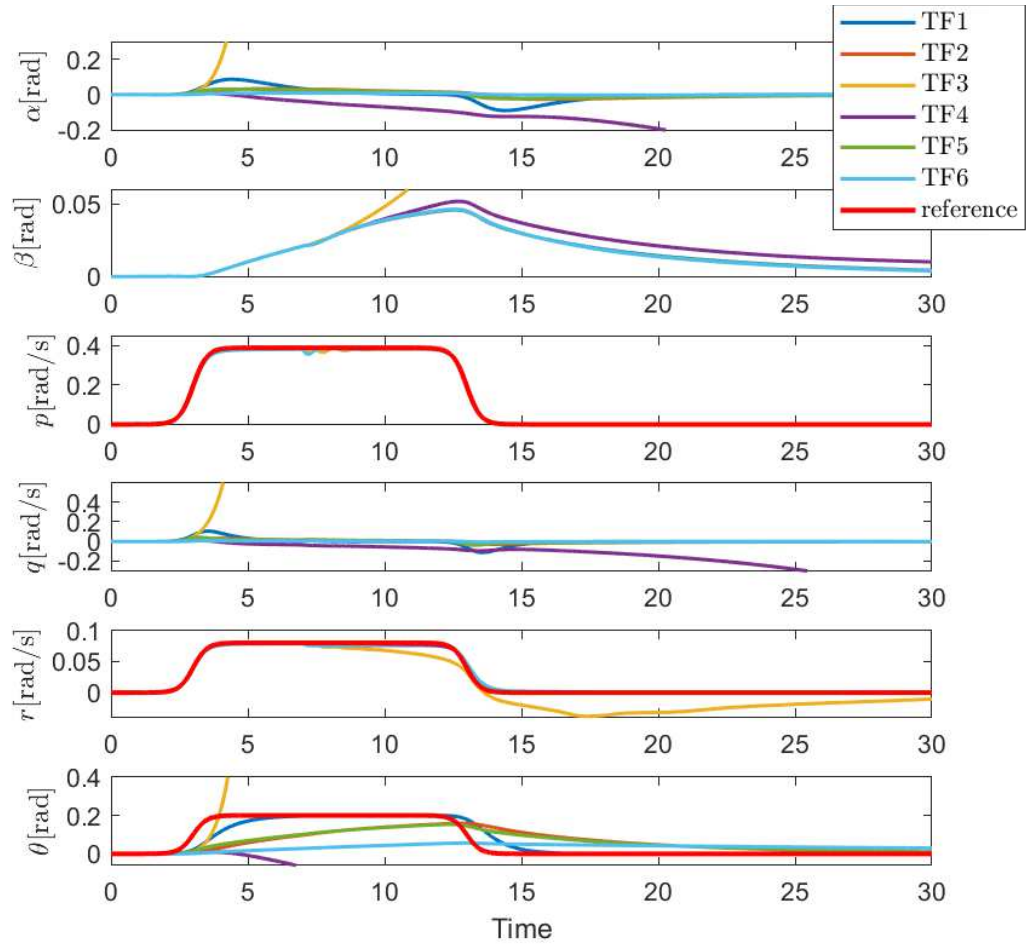


Figure 5.3: The evolution of states considering different human operator models.

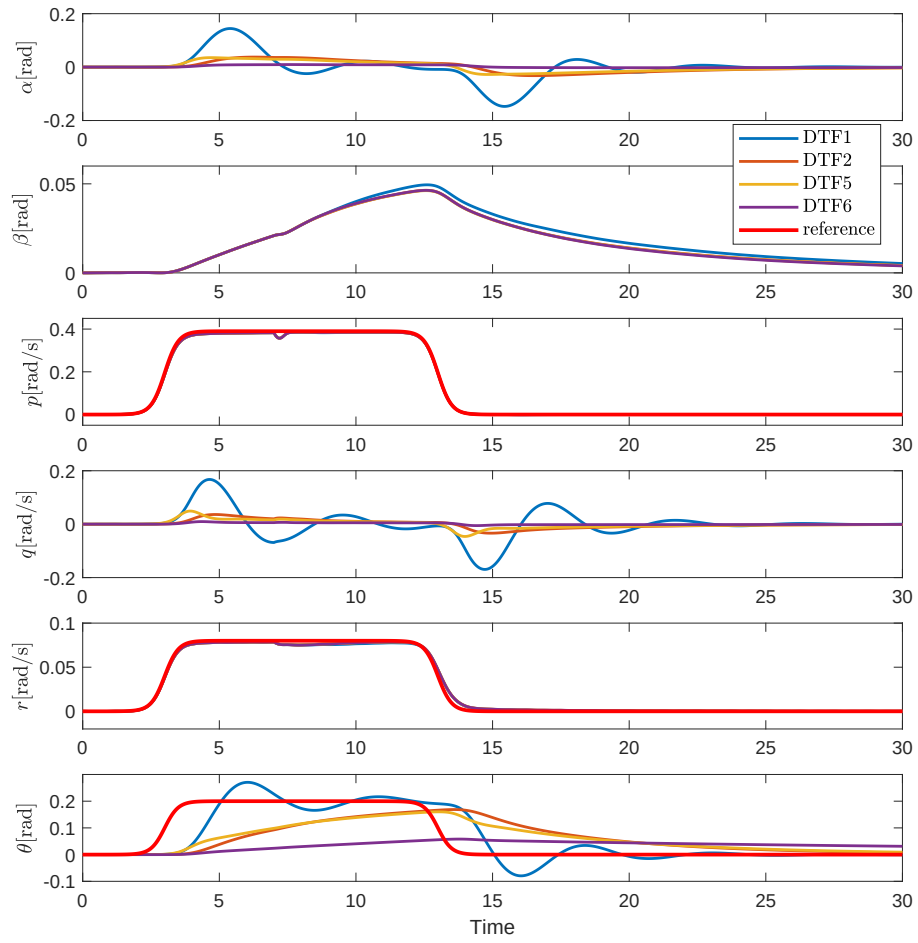


Figure 5.4: The evolution of states considering different human operator models.

## 5.5 Summary

The stability limits of a human-in-the-loop closed loop system for different human operator reaction models are analyzed. Operator models are selected to represent two different aspects of reaction dynamics. The first one, pure gain, represents operator behavior when the task at hand requires tight control, such as aerial refueling and aircraft landing. The second one, lag filter, models the inability of a human operator to respond to high frequency input. A sliding mode controller is used to guarantee reference tracking and boundedness of the states, when the controller receives bounded references. The adaptive control allocation is employed to distribute the total control signal vector among the redundant uncertain actuators. The simulations performed using two human operator models agree with the stability analysis.

## Chapter 6

# Control theoretical adaptive human pilot model: theory and experimental validation

### 6.1 Introduction

Humans' unique abilities such as adaptive behavior in dynamic environments, and social interaction and moral judgment capabilities, make them essential elements of many control loops. On the other hand, compared to humans, automation provides higher computational performance and multi-tasking capabilities without any fatigue, stress, or boredom [117,118]. Although they have their own individual strengths, humans and automation also demonstrate several weaknesses. Humans may have anxiety, fear and may become unconscious during an operation. Furthermore, in the tasks that require increased attention and focus, humans tend to provide high gain control inputs that can cause undesired oscillations. One example of this phenomenon, for example, is the occurrence of pilot induced oscillations (PIO), where undesired and sustained oscillations are observed due to an abnormal coupling between the aircraft and the pilot [21,26,29,51]. Similarly, automation may fail due to an uncertainty, fault or cyber-attack [119]. Thus,

it is more preferable to design systems where humans and automation work in harmony, complementing each other, resulting in a structure that benefits from the advantages of both.

To achieve a reliable human-automation harmony, a mathematically rigorous human operator model is paramount. A human operator model helps develop safe control systems, and provide a better prediction of human actions and limitations [17, 120–122]. Quasi-linear model [101] is one of the first human operator models, which consists of a describing function and a remnant signal accounting for nonlinear behavior. An overview of this model is provided in [102]. In some applications, where the linear behavior may be dominant, the nonlinear part of this model can be ignored, and the resulting lead-lag-type compensator is used in closed loop stability analysis [103]. The crossover model, proposed in [104], is another important human operator model in the aerospace domain. It is motivated from the empirical observations that human pilots adapt their responses in such a way that the overall system dynamics resembles that of a well designed feedback system [105]. A generalized crossover model which mimics human behavior when controlling a fractional order plant is proposed in [123]. In [124], crossover model is employed to provide information about the human intent for the controller. In [125], the dynamics of the operator is represented as a spring-damper-mass system.

Control theoretical operator models drawing from the optimal and adaptive control theories are also proposed by several authors. Optimal human models are based on the idea that a well trained human operator behaves in an optimal manner [106–109, 126]. On the other hand, adaptive models, such as the ones proposed in [14, 15] and [16], aim to replicate the adaptation capability of humans in uncertain and dynamics environments. In [14] and [15], adaptation rules are proposed based on expert knowledge. The adaptive model proposed in [14] is applied to change the parameters of the pilot model based on force feedback from a smart inceptor [127]. A survey on various pilot models can be found in [111] and [112].

Several approaches are also developed for human model parameter identification. In [128], a two-step method using wavelets and a windowed maximum likelihood estimation is exploited for the estimation of time-varying pilot model parameters. In [129], a linear parameter varying model identification framework is incorporated to estimate time-varying human state space representation matrices. Subsystem identification is used in [130] to model human control strategies. In [131], a human operator model for preview tracking tasks is derived from measurement data.

In this thesis, we build upon the earlier successful pilot models and propose an adaptive human pilot model that modifies its behavior based on plant uncertainties. This model distinguishes itself from earlier adaptive models by having mathematically derived laws to achieve a cross-over-model-like behavior, instead of employing expert knowledge. This allows a rigorous stability proof, using the Lyapunov-Krasovskii stability criteria, of the overall closed loop system. To validate the model, a setup, including a joystick and a monitor, is used. The participant data collected through this experimental setup is subjected to visual and statistical analyses to evaluate the accuracy of the proposed model. Initial research results of this study were presented in [16], where the details of the mathematical proof and human experimental validation studies were missing.

This chapter is organized as follows. In Section 6.2, the problem statement is given. Obtaining reference model parameters, which determine the properties of the cross-over model, is discussed in Section 6.3. Section 6.4 presents the human control strategy together with a stability analysis. Experimental set-up, results, and a statistical analysis are provided in Section 6.5. Finally, a summary is given in Section 6.6.

## 6.2 Problem statement

According to McRuer's crossover model [13], human pilots in the control loop behave in a way that results in an open loop transfer function

$$Y_{OL}(s) = Y_h(s)Y_p(s) = \frac{\omega_c e^{-\tau s}}{s}, \quad (6.1)$$

near the crossover frequency,  $\omega_c$ , where  $Y_h$  is the transfer function of the human pilot and  $Y_p$  is the transfer function of the plant.  $\tau$  is the effective time delay, including transport delays and high frequency neuromuscular lags.

Consider the following plant dynamics

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t), \quad (6.2)$$

where  $x_p \in \mathbb{R}^{n_p}$  is the plant state vector,  $u_p \in \mathbb{R}$  is the input vector,  $A_p \in \mathbb{R}^{n_p \times n_p}$  is an unknown state matrix and  $B_p \in \mathbb{R}^{n_p}$  is an unknown input matrix.

The human *neuromuscular model* [132, 133] is represented in state space form as

$$\begin{aligned} \dot{x}_h(t) &= A_h x_h(t) + B_h u(t - \tau) \\ y_h(t) &= C_h x_h(t) + D_h u(t - \tau), \end{aligned} \quad (6.3)$$

where  $x_h \in \mathbb{R}^{n_h}$  is the neuromuscular state vector,  $A_h \in \mathbb{R}^{n_h \times n_h}$  is the state matrix,  $B_h \in \mathbb{R}^{n_h}$  is the input matrix,  $C_h \in \mathbb{R}^{1 \times n_h}$  is the output matrix and  $D_h \in \mathbb{R}$  is the control output matrix.  $u \in \mathbb{R}$  is the neuromuscular input vector, which represents the control decisions taken by the human and fed to the neuromuscular system,  $y_h \in \mathbb{R}$  is the output vector, and  $\tau \in \mathbb{R}^+$  is a known, constant delay. The neuromuscular model parameters are assumed to be known and the output of the model,  $y_h$ , is used as the plant input  $u_p$  in (6.2), that is  $y_h = u_p$  (see Figure 6.1).

By aggregating the human pilot and plant states, we obtain the combined open loop human neuromuscular and plant dynamics as

$$\underbrace{\begin{bmatrix} \dot{x}_h(t) \\ \dot{x}_p(t) \end{bmatrix}}_{\dot{x}_{hp}(t)} = \underbrace{\begin{bmatrix} A_h & 0_{n_h \times n_p} \\ B_p C_h & A_p \end{bmatrix}}_{A_{hp}} \underbrace{\begin{bmatrix} x_h(t) \\ x_p(t) \end{bmatrix}}_{x_{hp}(t)} + \underbrace{\begin{bmatrix} B_h \\ B_p D_h \end{bmatrix}}_{B_{hp}} u(t - \tau), \quad (6.4)$$

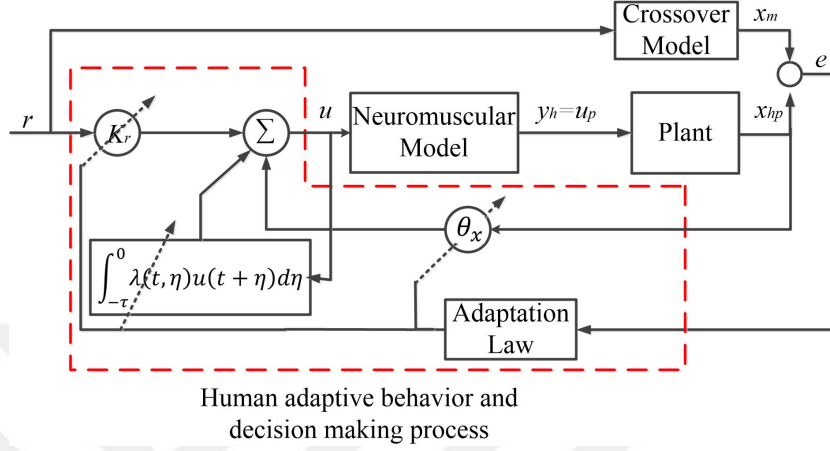


Figure 6.1: The block diagram of the human adaptive behavior and decision making in a closed loop system.

which can be written in the following compact form

$$\dot{x}_{hp}(t) = A_{hp}x_{hp}(t) + B_{hp}u(t - \tau), \quad (6.5)$$

where  $x_{hp} = [x_h^T \ x_p^T]^T \in \mathbb{R}^{(n_p+n_h)}$ ,  $A_{hp} \in \mathbb{R}^{(n_p+n_h) \times (n_p+n_h)}$ ,  $B_{hp} \in \mathbb{R}^{(n_p+n_h)}$ .

**Assumption 5.** The pair  $(A_{hp}, B_{hp})$  is controllable.

The goal is to obtain the input  $u(t)$  in (6.3), which is the *output of the human decision-making process*, such that the closed loop system consisting of the adaptive human pilot model and the plant follow the output of a unity feedback reference model with an open loop crossover model transfer function. It is noted that once  $u(t)$  is created, it is used as the input to the neuromuscular model, which creates the pilot motion. To summarize, assuming that the neuromuscular model, the parameters to determine the crossover model reference model and the structure of the plant with parametric uncertainties are available, we aim to find the evolution of the output of the human decision-making process, which is represented by the variable  $u(t)$ .

The closed loop transfer function of the reference model is therefore calculated as

$$G_{cl}(s) = \frac{\frac{\omega_c}{s} e^{-\tau s}}{1 + \frac{\omega_c}{s} e^{-\tau s}} = \frac{\omega_c e^{-\tau s}}{s + \omega_c e^{-\tau s}}. \quad (6.6)$$

An approximation of (6.6) can be given as

$$\hat{G}_{cl}(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} e^{-\tau s}, \quad (6.7)$$

where  $n = n_h + n_p$  and  $m \leq n$  are positive real constants, and  $a_i$  and  $b_j$  for  $i = 0, \dots, n-1$  and  $j = 0, \dots, m-1$ , are real constants to be estimated. The reference model then can be obtained as the state space representation of (6.7) as

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t - \tau), \quad (6.8)$$

where  $x_m \in \mathbb{R}^{(n_h+n_p)}$  is the reference model state vector,  $A_m \in \mathbb{R}^{(n_h+n_p) \times (n_h+n_p)}$  is the state matrix,  $B_m \in \mathbb{R}^{(n_h+n_p) \times m_h}$  is the input matrix, and  $r \in \mathbb{R}^{m_h}$  is the reference input.

### 6.3 Reference model parameters

The crossover transfer function (6.1) contains the crossover frequency,  $\omega_c$ , which is not known a priori. Experimental data, showing the reference input ( $r(t)$ ) frequency bandwidth,  $\omega_i$ , versus crossover frequency  $\omega_c$ , is provided in [105] and [13], for plant transfer functions  $K$ ,  $K/s$  and  $K/s^2$ . We fit polynomials to these experimental results to obtain the crossover frequency of the open loop transfer function given a reference input frequency bandwidth. These polynomials are given in Table I. It is noted that when the reference input has multiple frequency components, the highest frequency is used to calculate the crossover frequency.

**Remark 24.** *In this work, we use the polynomial relationships provided in Table I for zero, first and second order plant dynamics with nonzero poles and zeros. Further experimental work can be conducted to obtain a more precise relationship between the crossover and reference input frequencies, but this is currently out of the scope of this work.*

Table 6.1: Crossover frequency for different plant models

| Plant transfer function | Crossover frequency of the open loop transfer function (rad/s)                       |
|-------------------------|--------------------------------------------------------------------------------------|
| $K$                     | $\omega_c = 0.067\omega_i^2 + 0.099\omega_i + 4.8$                                   |
| $K/s$                   | $\omega_c = 0.14\omega_i + 4.3$                                                      |
| $K/s^2$                 | $\omega_c = -0.0031\omega_i^4 - 0.072\omega_i^3 + 0.29\omega_i^2 - 0.13\omega_i + 3$ |

## 6.4 Human pilot control decision command

The adaptive human pilot control decision command,  $u(t)$ , is determined as

$$u(t) = K_r K_x x_{hp}(t + \tau) + K_r r(t) \quad (6.9)$$

where  $K_x \in \mathbb{R}^{1 \times (n_h + n_p)}$ , and  $K_r \in \mathbb{R}^{m_h \times m_h}$ . Using (6.9) and (6.5), the closed loop dynamics can be obtained as

$$\dot{x}_{hp}(t) = (A_{hp} + B_{hp} K_r K_x) x_{hp}(t) + B_{hp} K_r r(t - \tau). \quad (6.10)$$

Equation (6.9) describes a non-causal decision command which requires future values of the states. This problem can be eliminated by solving the differential equation (6.5) as a  $\tau$ -seconds ahead predictor as

$$x_{hp}(t + \tau) = e^{A_{hp}\tau} x_{hp}(t) + \int_{-\tau}^0 e^{-A_{hp}\eta} B_{hp} u(t + \eta) d\eta. \quad (6.11)$$

**Assumption 6.** *There exist ideal parameters  $K_r^*$  and  $K_x^*$  satisfying the following matching conditions*

$$\begin{aligned} A_{hp} + B_{hp} K_r^* K_x^* &= A_m \\ B_{hp} K_r^* &= B_m. \end{aligned} \quad (6.12)$$

By substituting (6.11) into (6.9), the human pilot control decision input can be written as

$$u(t) = K_r K_x e^{A_{hp}\tau} x_{hp}(t) + K_r K_x \int_{-\tau}^0 e^{-A_{hp}\eta} B_{hp} u(t + \eta) d\eta + K_r r(t). \quad (6.13)$$

By defining  $\theta_x(t)$  and  $\lambda(t, \eta)$  as

$$\begin{aligned} \theta_x(t) &= K_r(t) K_x(t) e^{A_{hp}\tau}, \\ \lambda(t, \eta) &= K_r(t) K_x(t) e^{-A_{hp}\eta} B_{hp}, \end{aligned} \quad (6.14)$$

(6.13) can be rewritten as (see Figure 6.1)

$$u(t) = \theta_x(t)x_{hp}(t) + \int_{-\tau}^0 \lambda(t, \eta)u(t + \eta)d\eta + K_r(t)r(t). \quad (6.15)$$

The ideal values of  $\theta_x$  and  $\lambda$  can be obtained as

$$\begin{aligned} \theta_x^* &= K_r^* K_x^* e^{A_{hp}\tau} \\ \lambda^*(\eta) &= K_r^* K_x^* e^{-A_{hp}\eta} B_{hp}. \end{aligned} \quad (6.16)$$

Since  $A_{hp}$  and  $B_{hp}$  are unknown,  $\theta_x$  and  $\lambda$  need to be estimated. The closed loop dynamics can be obtained using (6.5) and (6.15) as

$$\begin{aligned} \dot{x}_{hp}(t) &= A_{hp}x_{hp}(t) + B_{hp}\theta_x(t - \tau)x_{hp}(t - \tau) \\ &+ \int_{-\tau}^0 B_{hp}\lambda(t - \tau, \eta)u(t + \eta - \tau)d\eta + B_{hp}K_r r(t - \tau), \end{aligned} \quad (6.17)$$

Defining the deviations of the adaptive parameters from their ideal values as  $\tilde{\theta}_x = \theta_x - \theta_x^*$  and  $\tilde{\lambda} = \lambda - \lambda^*$ , and adding and subtracting  $A_m x_{hp}(t)$  to (6.17), and using (6.12), we obtain that

$$\begin{aligned} \dot{x}_{hp}(t) &= A_m x_{hp}(t) - B_{hp}K_r^* K_x^* x_{hp}(t) \\ &+ B_{hp}K_r(t - \tau)K_x(t - \tau) \left( e^{A_{hp}\tau} x_{hp}(t - \tau) \right. \\ &+ \left. \int_{-\tau}^0 e^{-A_{hp}\eta} B_{hp}u(t + \eta - \tau)d\eta \right) \\ &+ B_{hp}K_r(t - \tau)r(t - \tau). \end{aligned} \quad (6.18)$$

Using (6.11), (6.18) is rewritten as

$$\begin{aligned} \dot{x}_{hp}(t) &= A_m x_{hp}(t) - B_{hp}K_r^* K_x^* x_{hp}(t) \\ &+ B_{hp}K_r(t - \tau)K_x(t - \tau)x_{hp}(t) \\ &+ B_{hp}K_r(t - \tau)r(t - \tau). \end{aligned} \quad (6.19)$$

Defining the tracking error as  $e(t) = x_{hp} - x_m$ , and subtracting (6.8) from (6.19), and using (6.12), and following a similar procedure given in [67], it is obtained

that

$$\begin{aligned}
\dot{e}(t) &= \dot{x}_{hp} - \dot{x}_m \\
&= A_m e(t) - B_{hp} K_r^* K_x^* x_{hp}(t) \\
&\quad + B_{hp} K_r(t - \tau) K_x(t - \tau) x_{hp}(t) \\
&\quad + B_{hp} (K_r(t - \tau) - K_r^*) r(t - \tau) \\
&= A_m e(t) + (-B_{hp} K_r^* K_x^* \\
&\quad + B_{hp} (K_r^* - K_r^* + K_r(t - \tau)) K_x(t - \tau)) x_{hp}(t) \\
&\quad + B_{hp} (K_r(t - \tau) - K_r^*) r(t - \tau) \\
&= A_m e(t) + B_m (K_x(t - \tau) - K_x^*) x_{hp}(t) \\
&\quad + B_m (K_r^{*-1} K_r(t - \tau) - 1) K_x(t - \tau) x_{hp}(t) \\
&\quad + B_m (K_r^{*-1} K_r(t - \tau) - 1) r(t - \tau) \\
&= A_m e(t) + B_m (\tilde{K}_x(t - \tau) x_{hp}(t) \\
&\quad + B_m (K_r^{*-1} - K_r^{-1}(t - \tau)) K_r(t - \tau) K_x(t - \tau) x_{hp}(t) \\
&\quad + B_m (K_r^{*-1} - K_r^{-1}(t - \tau)) K_r(t - \tau) r(t - \tau).
\end{aligned} \tag{6.20}$$

Using (6.11) and defining  $\Phi = K_r^{*-1} - K_r^{-1}$ , we can rewrite (6.20) as

$$\begin{aligned}
\dot{e}(t) &= A_m e(t) + B_m K_r^{*-1} (K_r^* K_x(t - \tau) - K_r^* K_x^*) \\
&\quad \times \left( e^{A_{hp}\tau} x_{hp}(t - \tau) + \int_{-\tau}^0 e^{-A_{hp}\eta} B_{hp} u(t + \eta - \tau) d\eta \right) \\
&\quad + B_m \Phi(t - \tau) \left( K_r(t - \tau) K_x(t - \tau) \left( e^{A_{hp}\tau} x_{hp}(t - \tau) \right. \right. \\
&\quad \left. \left. + \int_{-\tau}^0 e^{-A_{hp}\eta} B_{hp} u(t + \eta - \tau) d\eta \right) + K_r(t - \tau) r(t - \tau) \right).
\end{aligned} \tag{6.21}$$

Using (6.16) and (6.21), we obtain that

$$\begin{aligned}
\dot{e}(t) &= A_m e(t) + B_m K_x(t - \tau) \left( e^{A_{hp}\tau} x_{hp}(t - \tau) + \int_{-\tau}^0 e^{-A_{hp}\eta} B_{hp} u(t + \eta - \tau) d\eta \right) \\
&\quad - B_m K_r^{*-1} \left( \theta_x^* x_{hp}(t - \tau) + \int_{-\tau}^0 \lambda^*(\eta) u(t + \eta - \tau) d\eta \right) + B_m \Phi(t - \tau) u(t - \tau).
\end{aligned} \tag{6.22}$$

Using (6.14), (6.22) can be rewritten as

$$\begin{aligned} \dot{e}(t) = & A_me(t) + B_m \left( (K_r^{-1}(t-\tau)\theta_x(t-\tau) - K_r^{*-1}\theta_x^*) \right. \\ & \times x_{hp}(t-\tau) + \int_{-\tau}^0 (K_r^{-1}(t-\tau)\lambda(t-\tau, \eta) \\ & \left. - K_r^{*-1}\lambda^*(\eta))u(t+\eta-\tau)d\eta \right) + B_m\Phi(t-\tau)u(t-\tau). \end{aligned} \quad (6.23)$$

Defining  $\theta_1 = K_r^{-1}\theta_x$  and  $\lambda_1 = K_r^{-1}\lambda$ , and using their deviations from their ideal values,  $\tilde{\theta}_1 = \theta_1 - \theta_1^*$  and  $\tilde{\lambda}_1 = \lambda_1 - \lambda_1^*$ , where  $\theta_1^* = K_r^{*-1}\theta_x^*$  and  $\lambda_1^* = K_r^{*-1}\lambda^*$ , (6.23) can be rewritten as

$$\begin{aligned} \dot{e}(t) = & A_me(t) + B_m\tilde{\theta}_1(t-\tau)x_{hp}(t-\tau) \\ & + B_m \int_{-\tau}^0 \tilde{\lambda}_1(t-\tau, \eta)u(t+\eta-\tau)d\eta + B_m\Phi(t-\tau)u(t-\tau). \end{aligned} \quad (6.24)$$

The following lemma will be necessary to prove the main theorem of this article.

**Lemma 13.** *Suppose that the continuous function  $u(t)$  is given as*

$$u(t) = f(t) + \int_{-\tau}^0 \lambda(t, \eta)u(t+\eta)d\eta, \quad (6.25)$$

where  $u, f : [t_0 - \tau, \infty] \rightarrow R$ , and  $\lambda : [t_0, \infty) \times [-\tau, 0] \rightarrow R$ . Then

$$|u(t)| \leq 2(\bar{f} + c_0c_1)e^{c_0^2(t-t')}, \quad \forall t'_j \geq t'_i, \quad (6.26)$$

if constants  $t'_i, \bar{f}, c_0, c_1 \in R^+$  exist such that  $|f(t)| \leq \bar{f}$ ,

$$\int_{-\tau}^0 \lambda^2(t, \eta)d\eta \leq c_0^2 \quad \text{for } t \in [t'_i, t'_j], \quad (6.27)$$

and

$$\int_{-\tau}^0 u^2(t+\eta)d\eta \leq c_1^2 \quad \forall t \leq t'_i. \quad (6.28)$$

*Proof.* The proof of Lemma 13 can be found in [134]. □

**Theorem 17.** *Given the initial conditions  $\tilde{\theta}_1(\xi)$ ,  $\tilde{\lambda}_1(\xi, \eta)$ ,  $\Phi(\xi)$  and  $x_{hp}(\xi)$  for  $\xi \in [-\tau, 0]$ , and  $u(\zeta)$  for  $\zeta \in [-2\tau, 0]$ , there exists a  $\tau^*$  such that for all  $\tau \in [0, \tau^*]$ , the controller (6.15) with the adaptive laws*

$$\dot{\theta}_1^T(t) = -x_{hp}(t-\tau)e(t)^T P B_m, \quad (6.29)$$

$$\dot{\Phi}^T(t) = -u(t - \tau)e(t)^T P B_m, \quad (6.30)$$

$$\dot{\lambda}_1^T(t, \eta) = -u(t + \eta - \tau)e(t)^T P B_m, \quad (6.31)$$

where  $P$  is the symmetric positive definite matrix satisfying the Lyapunov equation  $A_m^T P + P A_m = -Q$  for a symmetric positive definite matrix  $Q$ , which can be employed to obtain controller parameters using  $\dot{K}_r = \text{Proj}(K_r \dot{\Phi} K_r)$ ,  $\theta_x(t) = K_r(t)\theta_1(t)$  and  $\lambda(t) = K_r(t)\lambda_1(t)$ , make the pilot neuromuscular and plant aggregate system (6.5) follow the crossover reference model (6.8) asymptotically, i.e.,  $\lim_{t \rightarrow \infty} x_{hp}(t) = x_m(t)$ , while keeping all the signals bounded.

*Proof.* The proof of this theorem is provided in Appendix E.1. □

### 6.4.1 Implementation guide

The implementation of the proposed model includes two parts, 1) reference model, 2) adaptive decision maker (See Figure 6.1). A step by step implementation guide is provided as:

#### Reference model determination

**Step 1:** Determine the maximum input frequency (frequency bandwidth), and the type of the plant transfer function.

**Step 2:** Using Table 1, calculate the crossover frequency.

**Step 3:** Substitute the calculated crossover frequency ( $\omega_c$ ) and a prediction for human delay ( $\tau$ ) in (6.6).

**Step 4:** Find an approximation for (6.6) in the form of (6.7).

**Step 5:** Find a state space representation of (6.7) as (6.8).

#### Adaptive decision maker

**Step 6:** Use (6.15) as adaptive decision command.

**Step 7:** Use (6.29)-(6.31) as adaptive laws along with  $\dot{K}_r = \text{Proj}(K_r \dot{\Phi} K_r)$ ,  $\theta_x(t) = K_r(t)\theta_1(t)$  and  $\lambda(t) = K_r(t)\lambda_1(t)$ .

**Remark 25.** *The control law (6.15) contains an integration which can be approximated using the method provided in [135]. Furthermore, one of the differential equations, as adaptive laws, requires projection algorithm. For more information about projection algorithm and its implementation, refer to [9, 80].*

## 6.5 Experimental results

### 6.5.1 Experimental environment

In order to test the proposed adaptive human model against data, an experimental setup consisting of a Logitech Extreme 3D Pro joystick and a Toshiba Portege-Z30-B laptop with Intel Core i7 CPU is used (see Figure 6.2).

The tracking task is performed by an operator monitoring the compensatory display, which provides information about the error between the target to be followed, and follower, which is the output of the plant (see Figure 6.3). The operator provides the input  $u_p$  (see Figure 6.1) through the joystick, which is fed to the plant using MATLAB SIMULINK (R2018b). In return, the response of the plant is calculated and shown on the laptop screen in real-time.

The reference signal  $r(t)$  is generated as a sum of the sinusoids with frequencies of 0.1, 0.3, 0.5, 0.7, 1, 1.3 and 1.5 rad/s with the same amplitude of 0.2 and without phase shift.

Three classes of plant models, having zero, first and second order transfer functions are used in the experiments. In this section, we first give a detailed analysis of the first order plant case and then provide a summary of the results of the other cases in tables. The nominal first order plant used in the experiments is  $Y_p(s) = \frac{4}{s+1}$ , which is similar to the one used in [14]. The uncertainty is introduced to the plant model by modifying the gain and the pole location by



Figure 6.2: Experimental setup.

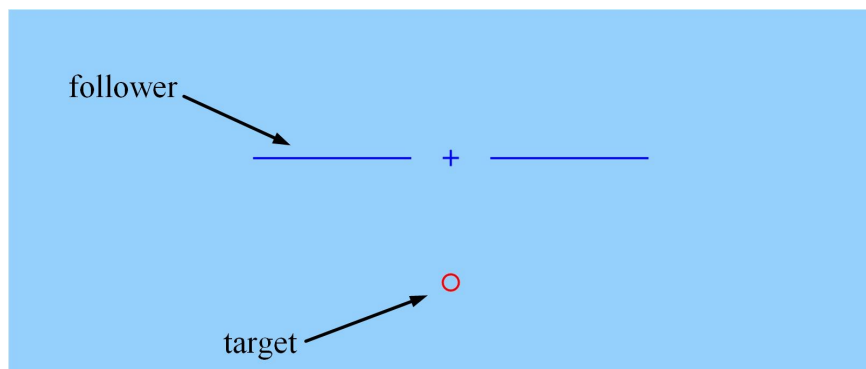


Figure 6.3: *Pursuit* display.

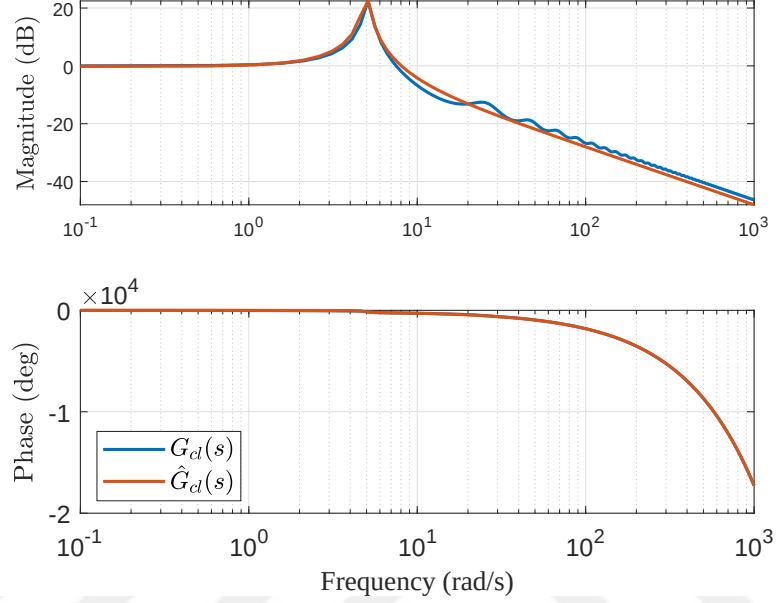


Figure 6.4: Bode plot of the reference model and its approximation.

50% to obtain  $Y_p(s) = \frac{6}{s+0.5}$ .

To form the reference model (6.8), two parameters, namely the crossover frequency and the time-delay, need to be determined. The highest frequency component of the reference signal is  $\omega_i = 1.5$  rad/s. Employing Table I for the first order plant  $Y_p$ , the crossover frequency is calculated as  $\omega_c = 4.5$  rad/s. The delay is determined by using the mean value of the operators' delay, which is  $\tau = 0.3$  s. Therefore, the closed loop transfer function of the reference model is calculated as

$$G_{cl}(s) = \frac{\frac{4.5}{s}e^{-0.3s}}{1 + \frac{4.5}{s}e^{-0.3s}} = \frac{4.5e^{-0.3s}}{s + 4.5e^{-0.3s}}. \quad (6.32)$$

Similar to (6.7), an approximate transfer function is obtained as

$$\hat{G}_{cl}(s) = \frac{3.881s + 24.24}{s^2 + 0.6834s + 24.72}e^{-0.3s}. \quad (6.33)$$

Figure 6.4 shows a comparison between (6.32) and (6.33), and demonstrates that the approximation works well for almost all frequencies.

The neuromuscular dynamics is taken as  $Y_h(s) = \frac{s+3}{s+2}e^{-0.3s}$ , where the time delay  $\tau = 0.3$  is the effective time delay, including human decision making delay and neuromuscular lags.

**Remark 26.** *In this chapter, we assume that the neuromuscular dynamics are given. The procedure for finding the neuromuscular model can be found in [132, 133].*

**Remark 27.** *In [15], where decoupled multi-axis tracking tasks are considered, a factor is introduced to represent the fact that the pilots are less aggressive in multi-axis tasks, compared to single-axis ones. In this work, although we cover the single-axis tracking task, an extension to two-axis tracking would not need any new model development or an increase in complexity, except using the same equations stated in the step-by-step implementation guide in the previous section, for the additional axis, with different gain and frequency parameters. A detailed analysis of single and double axis tracking tasks can be found in [136].*

### 6.5.2 The behavior of the adaptive model

The error between the plant output and the reference model is illustrated in Figure 6.5. The effect of uncertainty injection can be seen at  $t = 70$  s. Figures 6.6, 6.7, and 6.8 illustrates the adaptive human model parameters. To understand the amount of agreement between these results and the human experimental trials, visual and statistical analyses are provided in the following sections.

**Remark 28.** *The model is developed based on adaptive control principles and therefore even when the plant model is not changing, the adaptive parameters may continue to vary based on the changes in the reference inputs. This parameter variance, however, is not a problem since the model's tracking capability, as well as the boundedness of the adaptive parameters, is guaranteed by the Lyapunov analysis. We also know that unless the inputs are persistently exciting, the adaptive parameters may not converge to their ideal values. Therefore, different trajectories may result in different model parameters. This is not an issue since these parameters are tuned automatically by the adaptive process.*

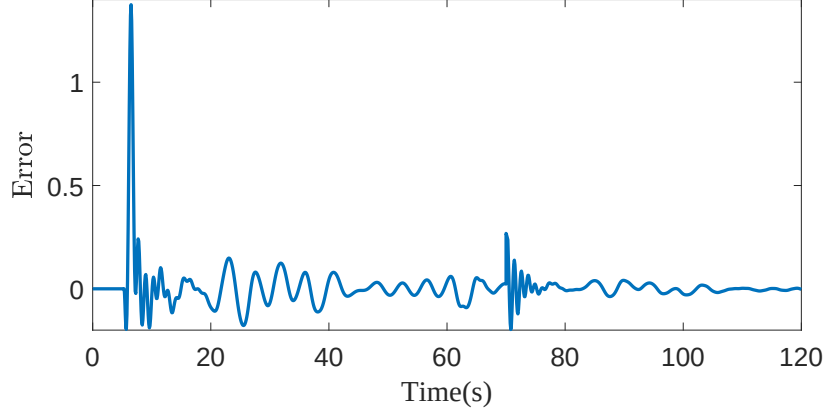


Figure 6.5: Time evolution of the error between the output of the plant controlled by the adaptive model, and the reference model output.

**Remark 29.** *Adaptation rates are important in terms of transient behavior, and therefore their selection is important to obtain a successful model. There are some guidelines that can be used in the implementation, at least to start at a good initial condition for the tuning process [135, 137]. Considering a generic adaptive law  $\dot{\theta} = -\Gamma e_1 \Omega$ , where  $\Gamma$  is the adaptation rate matrix,  $e_1$  is the tracking error, and  $\Omega$  is the vector of corresponding system signals. Assuming that  $e_1$  and elements of  $\Omega$  are the same order of magnitude as the reference signal  $r$ , the adaptation rate for a parameter  $\theta$  is chosen as  $\Gamma = \theta^* / (3\tau_m \bar{r}^2)$ , where  $\theta^*$  is an estimate of the desired adaptive parameter,  $\tau_m$  is the smallest time constant of the reference model, and  $\bar{r}$  is the maximum possible amplitude of the reference signal.*

### 6.5.3 Participants and experimental procedure

Eleven participants (6 women and 5 men) from the graduate and undergraduate student pools of Bilkent University participated the experiment. All of the participants read and signed the “informed consent to participate” document. This study is approved by Bilkent University Ethics Committee for research with human participants. Before the experiments, to familiarize the participants with

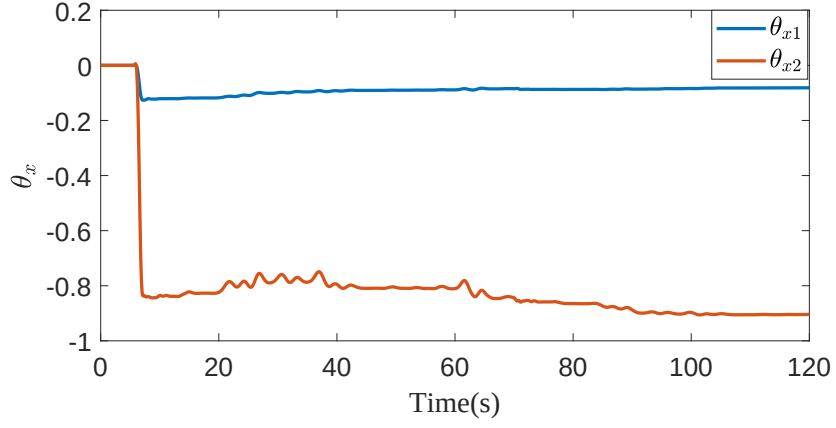


Figure 6.6: Evolution of human adaptive parameters  $\theta_{x1}$  and  $\theta_{x2}$ .

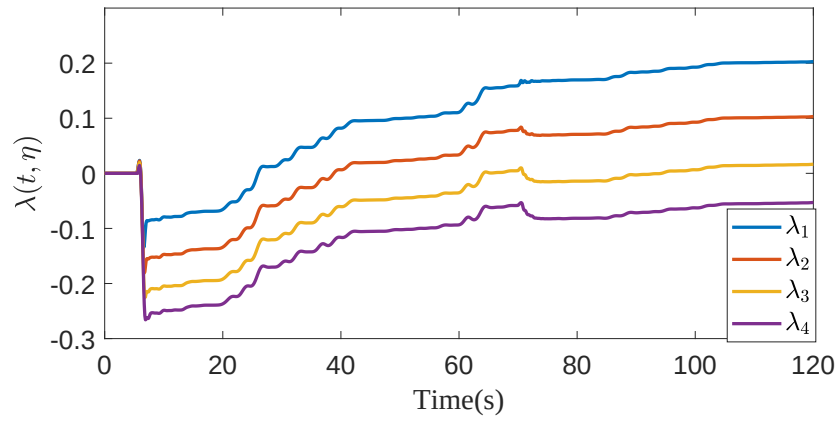


Figure 6.7: Evolution of human adaptive parameters  $\lambda_i$ ,  $i = 1, 2, 3$  and 4.

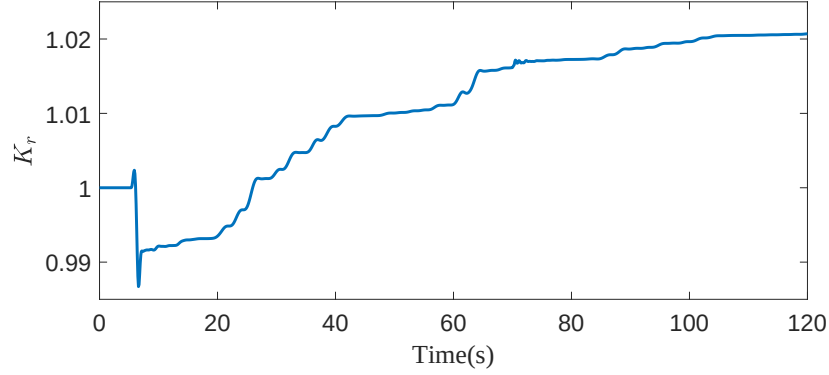


Figure 6.8: Evolution of human adaptive parameter  $K_r$ .

the experimental setup, and its environment, consisting of the display and the joystick, each participant was asked to follow a given reference via joystick inputs for the duration of 200 seconds. To prevent learning during these warm-up runs, the reference input, uncertainty injection times and the uncertainty types were chosen differently from the ones used in the real experimental runs. Specifically, the reference signal for the warm-up runs consisted of the sum of the sinusoids with frequencies of 0.1, 0.5, 1 and 1.5 rad/s with the same amplitude of 0.2 and without phase shift. The plant dynamics at the beginning of the warm-up run was  $\frac{2}{s^2+3s+2}$ . At  $t = 45$  s, the dynamics changed to  $\frac{5}{s+2}$  in a step like manner (suddenly). It changed to  $\frac{3}{s+1}$  at around  $t = 90$  s using a sigmoid function (gradually), and again changed to a zero order dynamics at 150 s (suddenly).

#### 6.5.4 Time and frequency domain analyses of the adaptive model

Let  $f_{p1}(t), f_{p2}(t), \dots, f_{pk}(t)$  be the plant outputs when participants  $p1, p2, \dots, pk$  are in the loop, respectively. For each  $f_{pi}(t)$ ,  $t = T_1, T_2, \dots, T_N$ , where  $T_j, j = 1, 2, \dots, N$ , represents a sampling instant. At each sampling instant  $T_j$ , the minimum, the maximum and the mean values of the plant outputs when participants

are in the loop can be obtained as

$$f_{p_{min}}(T_j) = \min_{i=1,2,\dots,k} f_{pi}(T_j), \quad j = 1, \dots, N, \quad (6.34)$$

$$f_{p_{max}}(T_j) = \max_{i=1,2,\dots,k} f_{pi}(T_j), \quad j = 1, \dots, N, \quad (6.35)$$

$$f_{p_{mean}}(T_j) = \frac{\sum_{i=1}^k f_{pi}(T_j)}{k}, \quad j = 1, \dots, N, \quad (6.36)$$

where  $k = 11$  is the number of participants. Figure 6.9 shows the evolutions of  $f_{p_{min}}$  and  $f_{p_{max}}$ , together with  $f_{ad}(t) \in \mathbb{R}^N$ , which is the plant output when adaptive human model is in the loop, where  $t = T_1, T_2, \dots, T_N$ . It is seen that the plant output when adaptive human model is in the loop almost always stays between the maximum and the minimum values of the plant output when participants are in the loop. Furthermore, Figure 6.10 demonstrates that  $f_{p_{mean}}$  and  $f_{ad}$  evolve reasonably close to each other.

Adaptive model and the mean participant data are also compared in the frequency domain both before and after the uncertainty introduction, in Figures 6.11 and 6.12, respectively. As seen from the figures, both the adaptive model and participant data show similar properties with the crossover model. However, the model starts to deviate from the human data for lower frequencies. The transfer functions used to plot these bode plots are obtained using the time-response data provided in Figure 6.10. For the systems run by the adaptive model and the participants, we obtained the open loop transfer functions  $\frac{3.5}{s}e^{-0.3s}$  and  $\frac{4}{s+0.8}e^{-0.3s}$ , respectively, before the uncertainty introduction, and  $\frac{4.5}{s}e^{-0.3s}$  and  $\frac{3.5}{s+0.1}e^{-0.3s}$ , after the uncertainty introduction. Figure 6.13 shows the close match between the closed loop time-responses of the estimated transfer functions and the data used to obtain them.

### 6.5.5 A comparison between the adaptive and fixed human models

In this subsection, to demonstrate the advantages of having an adaptive human model instead of a fixed one, we compare the behavior of these two models. As

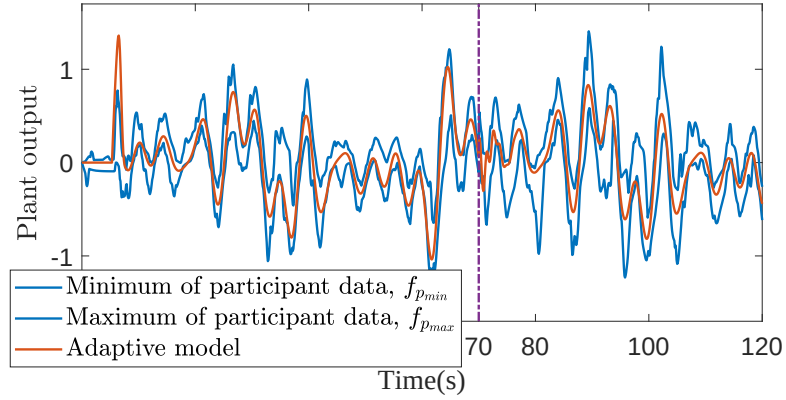


Figure 6.9: Plant output,  $x_{hp}$ , when adaptive human model is in the loop vs. minimum and maximum values of plant output when participants are in the loop.

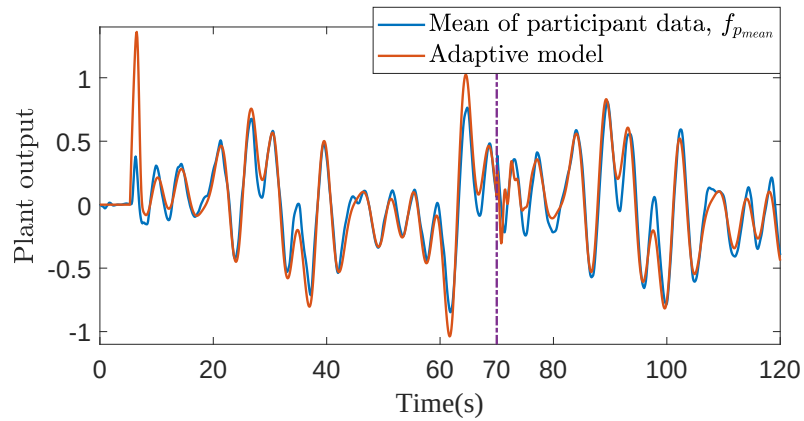


Figure 6.10: Plant output,  $x_{hp}$ , when adaptive human model is in the loop vs. mean value of plant output when participants are in the loop.

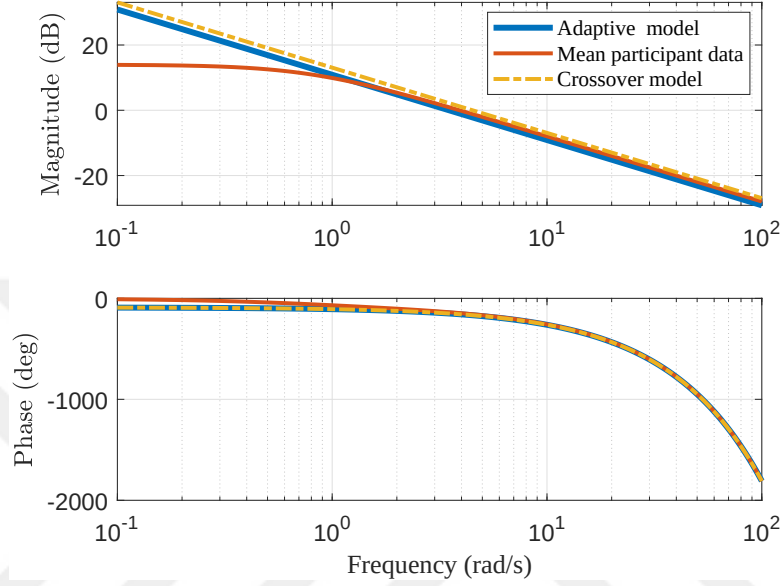


Figure 6.11: Frequency responses of the crossover model, the adaptive model and the mean human experimental data, before the uncertainty introduction.

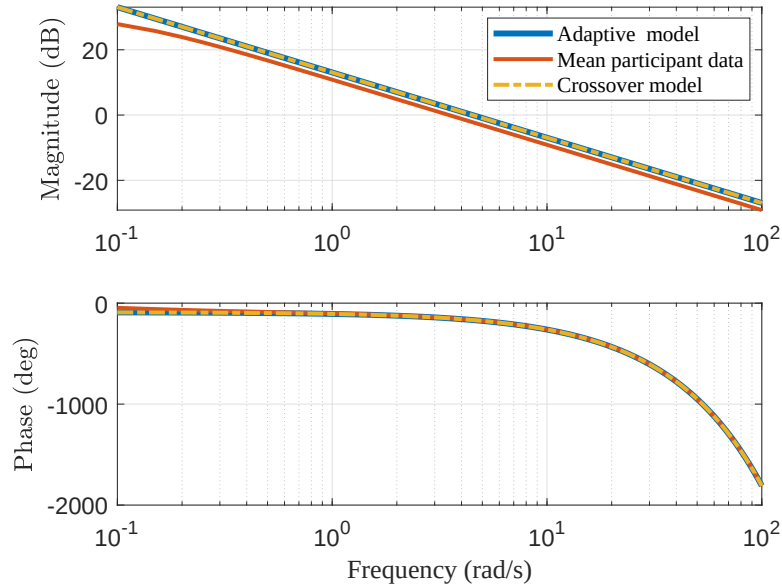
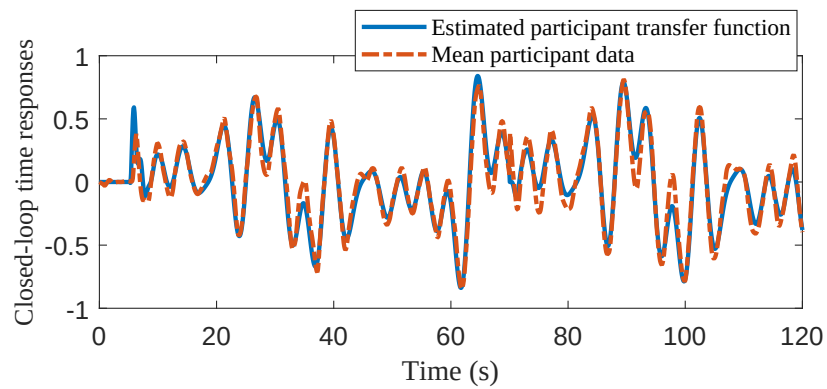
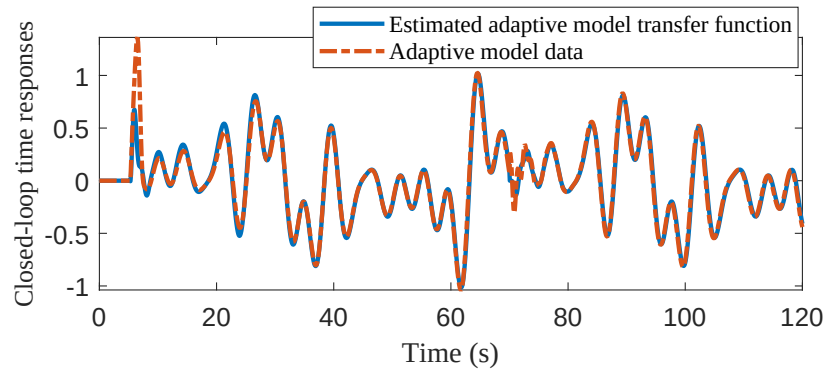


Figure 6.12: Frequency responses of the crossover model, the adaptive model and the mean human experimental data, after the uncertainty introduction.



a) Participant transfer function and data



b) Adaptive model transfer function and data

Figure 6.13: Comparison between the closed loop time response of estimated transfer functions and the data used to create them.

the fixed human model, we use the transfer function  $\frac{1.125(s^2+3s+2)}{s^2+3s}e^{-0.3s}$ , which makes the closed loop transfer function equal to the crossover model given in (6.32).

Figure 6.14-a shows that before the occurrence of uncertainty at  $t = 70$  s, both the fixed and the adaptive human models present a reasonable agreement with the experimental results. However, after the uncertainty introduction, the fixed model becomes unstable while the adaptive model continues to show acceptable performance. The result demonstrates the danger of using a fixed model in the presence of time delays. Using the method proposed in [138], one can find that the system after the uncertainty introduction becomes unstable for delay values larger than 0.22 s. To provide a comparison in the stable region, we plot the response of the fixed model for a delay value of 0.22 s in Figure 6.14-b. The plot shows that even though the response is stable, it fails to provide a reasonable prediction of the human pilot response after the uncertainty introduction.

### 6.5.6 Statistical analysis of the adaptive model using confidence intervals

The difference between the plant output when the  $i^{\text{th}}$  participant is in the loop and when the adaptive human model is in the loop is defined as

$$d_i \equiv f_{ad} - f_{pi}, \quad i = 1, \dots, k, \quad (6.37)$$

where  $d_i = [d_i(T_1), \dots, d_i(T_N)]^T \in \mathbb{R}^N, i = 1, \dots, k$ , is called the  $i^{\text{th}}$  difference. The mean and the standard deviation of the  $i^{\text{th}}$  difference is obtained as

$$\bar{d}_i = \frac{\sum_{j=1}^N d_i(T_j)}{N}, \quad i = 1, \dots, k, \quad (6.38)$$

$$s_i = \sqrt{\frac{\sum_{j=1}^N (d_i(T_j) - \bar{d}_i)^2}{N - 1}}, \quad i = 1, \dots, k. \quad (6.39)$$

The normal-scores plot for  $\bar{d}_i$  is given in Figure 6.15. The figure does not show any significant deviation from the normal distribution. This shows us that the

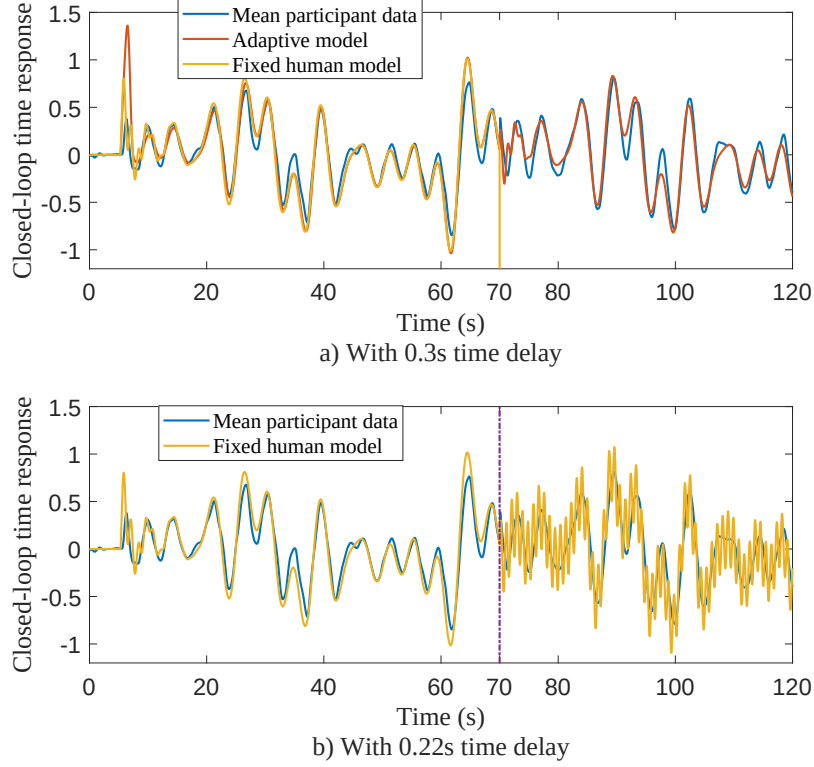


Figure 6.14: Plant output,  $x_{hp}$ , when adaptive human model is in the loop vs. fixed human model with 0.3 s and 0.22 s time delay is in the loop.

data do not suggest that the *population* of mean-errors,  $\bar{d}_i$ , deviates significantly from normal distribution. The *sample* mean and the *sample* standard deviation of  $\bar{d}_i$ 's can be obtained as

$$\bar{d} = \frac{\sum_{i=1}^k \bar{d}_i}{k}, \quad (6.40)$$

$$s = \sqrt{\frac{\sum_{i=1}^k (\bar{d}_i - \bar{d})^2}{k - 1}}. \quad (6.41)$$

Let  $\mu_0$  be the mean value of the *population* of mean-errors, given as

$$\mu_0 \equiv \frac{\sum_{i=1}^K \bar{d}_i}{K}, \quad (6.42)$$

where  $K$  is the *population* size. Since normal-scores plot, given in Figure 6.15, didn't provide any counter evidence, assuming that the distribution of the set of

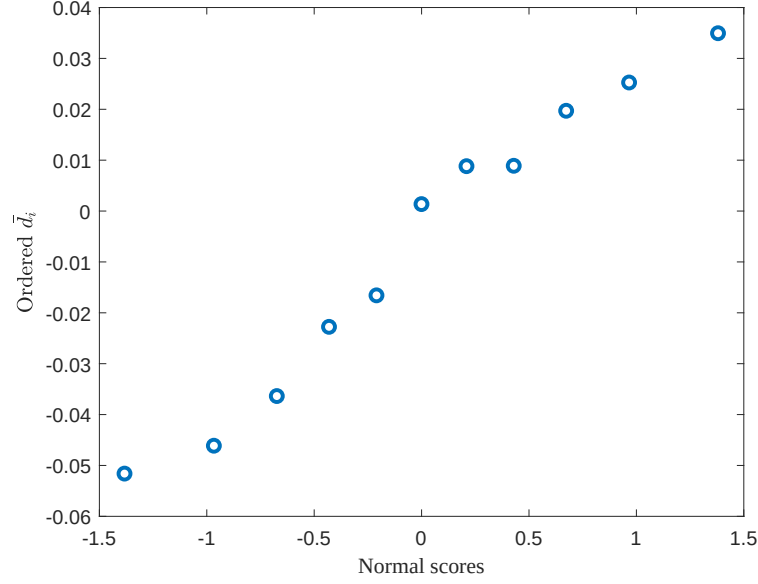


Figure 6.15: Normal-scores plot

data  $\{\bar{d}_1, \dots, \bar{d}_K\}$  is normal with mean  $\mu_0$ ,  $\mu_0$  satisfies the following probability [139]

$$P \left[ \bar{d} - t_{\alpha/2} \frac{s}{\sqrt{k}} < \mu_0 < \bar{d} + t_{\alpha/2} \frac{s}{\sqrt{k}} \right] = 1 - \alpha, \quad (6.43)$$

where  $\bar{d}$  and  $s$  are obtained from (6.40-6.41),  $k$  is the number of participants,  $\alpha$  is the significance level, and  $t_{\alpha/2}$  is the upper  $\alpha/2$  point of the  $t$  distribution with degree of freedom  $k - 1$ , which can be obtained from the  $t$ -distribution table. Since the number of participants,  $k = 11$ , is less than 30, it is appropriate to use the  $t$ -distribution. Using  $\alpha = 0.05$ , obtaining  $t_{\alpha/2}$  from the  $t$ -distribution table as 2.228, and calculating  $\bar{d}$  as  $-0.0068$  and  $s$  as  $0.0379$ , it can be concluded using (6.43) that we are 95% confident that  $\mu_0$  is in the interval  $(-0.0323, 0.0187)$ . This shows that the mean  $\mu_0$  of the *population's* mean deviation from the adaptive human model is reasonably close to zero.

Similarly, the variance,  $\sigma_0^2$ , of the *population's* mean deviation from the adaptive human model satisfies the following probability [139]

$$P \left[ \frac{(k-1)s^2}{\chi_{\alpha/2}^2} < \sigma_0^2 < \frac{(k-1)s^2}{\chi_{1-\alpha/2}^2} \right] = 1 - \alpha, \quad (6.44)$$

where  $\chi_{\alpha/2}^2$  is the upper  $\alpha/2$  point of the  $\chi^2$  distribution with degree of freedom  $k-1$  and can be obtained from the  $\chi^2$  distribution table. Calculating  $s$  from (6.41), using  $\alpha = 0.05$ , and obtaining  $\chi_{\alpha/2}^2$  and  $\chi_{1-\alpha/2}^2$  from the  $\chi^2$  table with 10 degrees of freedom, it can be concluded using (6.44) that we are 95% confident that  $\sigma_0$  is in the interval (0.0265, 0.0663). This shows that the standard deviation  $\sigma_0$  of the *population's* mean deviation from the adaptive human model is reasonably small.

### 6.5.7 Statistical analysis of the adaptive model using hypothesis testing

In this analysis, we test whether the hypothesis “the mean value of the *population* mean-errors, or the mean deviations from the adaptive model,” is zero. In other words, our null hypothesis,  $H_0$ , is given as

$$H_0 : \mu_0 = 0, \quad (6.45)$$

where  $\mu_0$  is defined in (6.42). The alternative hypothesis,  $H_1$ , is given as  $H_1 : \mu \neq 0$ . Similar to the confidence interval analysis, assuming that  $\mu_0$  is the mean of a normally distributed set of data  $\{\bar{d}_1, \dots, \bar{d}_K\}$  where  $K$  is the *population* size, the hypothesis  $H_0$  is rejected if,

$$\left| \frac{(\bar{d} - \mu_0)\sqrt{k}}{s} \right| \geq t_{\alpha/2}, \quad (6.46)$$

where  $\bar{d}$  and  $s$  are obtained from (6.40-6.41),  $k$  is the number of participants and  $t_{\alpha/2}$  is the upper  $\alpha/2$  point of the  $t$  distribution with degree of freedom  $k-1$  [139]. Using the significance level  $\alpha = 0.05$  and degree of freedom  $k-1 = 10$ , obtaining  $t_{0.025} = 2.228$  from the  $t$ -distribution table, calculating  $\bar{d} = -0.0068$  and  $s = 0.038$

Table 6.2: Sudden uncertainty

|                  | 0 order           | 1st order         | 2nd order                  |
|------------------|-------------------|-------------------|----------------------------|
| TF before 70 s   | 4                 | $\frac{4}{s+1}$   | $\frac{4}{(s+1)(s+5)}$     |
| TF after 70 s    | 6                 | $\frac{6}{s+0.5}$ | $\frac{6}{(s+0.5)(s+2.5)}$ |
| $\bar{d}$        | 0.0085            | -0.0068           | 0.0011                     |
| $s$              | 0.0339            | 0.0379            | 0.0252                     |
| Mean conf. int.  | (-0.014, 0.03)    | (-0.032, 0.02)    | (-0.016, 0.018)            |
| St.d. conf. int. | (0.024, 0.06)     | (0.026, 0.066)    | (0.018, 0.044)             |
| Hypothesis test  | $H_0$ is retained | $H_0$ is retained | $H_0$ is retained          |

using (6.40) and (6.41), respectively, and substituting  $\mu_0 = 0$  and  $k = 11$ , the left hand side of (6.46) can be calculated as 0.5935, which is less than  $t_{\alpha/2}$ . Therefore, we cannot reject  $H_0$ . We retain  $H_0$  and conclude that  $H_1$  fails to be proved.

Since we are retaining the null hypothesis, we want to minimize the probability  $\beta$  of incorrectly retaining the null hypothesis. This means that we want our test's power,  $1 - \beta$ , to be large, such as 0.95. What is the minimum required deviation of the *population* mean from 0, represented as  $\mu_1$ , that would make our test to incorrectly retain the null hypothesis with 0.05 probability, i.e.  $\beta = 0.05$ ? To calculate this, we first write the rejection region,  $R$ , using (6.46) as

$$R : \left| \frac{(\bar{d} - \mu_0)\sqrt{k}}{s} \right| \geq t_{\alpha/2} \implies R : |\bar{d}| \geq 0.0255. \quad (6.47)$$

Defining  $T = \frac{(\bar{d} - \mu_1)\sqrt{k}}{s}$ , for  $\beta = 0.05$ , we need

$$P \left[ \frac{(-0.0255 - \mu_1)\sqrt{k}}{s} < T < \frac{(0.0255 - \mu_1)\sqrt{k}}{s} \right] = \frac{\beta}{2} = 0.025. \quad (6.48)$$

Using the  $t$ -table, it can be found that the minimum  $|\mu_1|$  that satisfies (6.48) is 0.051. This means that our test can detect an 0.051 deviation from the mean value of the mean-error between the adaptive human model and the participant data when the probability of the test to incorrectly conclude that the model and the data are compatible ( $\mu_0 = 0$ ) is only 5%.

Analyses of the experimental results where a first order plant dynamics is used with a sudden uncertainty injection is provided above. All of

Table 6.3: Gradual uncertainty

|                  | 0 order           | 1st order         | 2nd order                  |
|------------------|-------------------|-------------------|----------------------------|
| TF before 70 s   | 4                 | $\frac{4}{s+1}$   | $\frac{4}{(s+1)(s+5)}$     |
| TF after 70 s    | 6                 | $\frac{6}{s+0.5}$ | $\frac{6}{(s+0.5)(s+2.5)}$ |
| $d$              | 0.0154            | 0.0026            | 0.004                      |
| $s$              | 0.038             | 0.034             | 0.03                       |
| Mean conf. int.  | (−0.01, 0.04)     | (−0.02, 0.025)    | (−0.016, 0.023)            |
| St.d. conf. int. | (0.026, 0.067)    | (0.0235, 0.06)    | (0.02, 0.05)               |
| Hypothesis test  | $H_0$ is retained | $H_0$ is retained | $H_0$ is retained          |

the results, including the ones for the other cases, where plants with different orders and sudden/gradual uncertainty injections, are summarized in Tables II and III. The data collected from the participants can be reached at <http://www.syslab.bilkent.edu.tr/research>.

## 6.6 Summary

In this chapter, an adaptive human pilot model based on model reference adaptive control principles is proposed. This model mimics the pilot decision making process by making sure that the overall closed loop system follows the crossover model in the presence of plant uncertainties. The stability of the system is shown using the Lyapunov-Krasovskii stability criteria. Furthermore, experiments with human operators are conducted to validate the model. Detailed visual and statistical analyses of the experimental results show that the adaptive model creates similar system responses as the human operators.

## Chapter 7

### Conclusion

In this thesis, an adaptive control allocation methodology is proposed to distribute control signals among redundant and uncertain actuators. This method does not require any uncertainty estimation or persistency of excitation of input signals. In addition, a novel projection operator is proposed to handle actuator magnitude and rate limits by bounding the magnitude and rate of change of the adaptive parameters. To guarantee the robustness of the closed loop system with respect to disturbance and control allocation error, a novel sliding mode controller with time-varying sliding surface is introduced. The proposed controller guarantees stability and reference tracking while ensuring robustness from the initial time. Furthermore, to consider the effect of human on the stability of the overall system, the stability of human-in-the-loop is analyzed for various human models. Finally, a control theoretical adaptive human pilot model is proposed. This model can mimic the adaptive behavior of humans in the presence of uncertainty. This model is validated using the statistical analysis of human data. The efficacy of each proposed methods are demonstrated using simulation results.

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# Appendix A

## Proofs of Chapter 2

### A.1 Proof of Theorem 1

Consider a Lyapunov function candidate,

$$V = e^T P e + \text{tr}(\tilde{\theta}_v^T \Gamma_\theta^{-1} \tilde{\theta}_v \Lambda). \quad (\text{A.1})$$

The time derivative of (A.1) along the trajectories of (2.13)-(2.14) can be calculated as

$$\dot{V} = -e^T Q e + 2e^T P B \Lambda \tilde{\theta}_v^T v + 2\text{tr}(\tilde{\theta}_v^T \Gamma_\theta^{-1} \dot{\tilde{\theta}}_v \Lambda) + 2e^T P \bar{d}. \quad (\text{A.2})$$

Using the property of the trace operation and the adaptive law (2.14), (A.2) can be written as

$$\begin{aligned} \dot{V} = & -e^T Q e + 2e^T P \bar{d} \\ & + 2\text{tr}(\tilde{\theta}_v^T (ve^T P B + \text{Proj}(\theta_v, -ve^T P B, f)) \Lambda). \end{aligned} \quad (\text{A.3})$$

If  $\bar{d}(t) = 0$  for  $t \geq 0$  (without loss of generality), it can be shown, by using (2.3), that  $\dot{V} \leq 0$  and therefore  $e(t)$  and  $\tilde{\theta}_v(t)$  are bounded. Since  $e \in \mathcal{L}_2$  and  $\dot{e} \in \mathcal{L}_\infty$ ,  $e(t)$  converges to zero as  $t \rightarrow \infty$  [68]. Since  $\theta_v$ ,  $e$  and  $\phi(v)$  are bounded,  $\dot{\theta}_v$  is also bounded by (2.14). In addition, since  $e$ ,  $\tilde{\theta}_v$  and  $v$  are bounded,  $\dot{e}$  is also bounded by (2.13). Therefore,  $e$  and  $\tilde{\theta}_v$  are uniformly continuous functions. If  $v$  is also

uniformly continuous, then  $\dot{e}$  is uniformly continuous by (2.13) since the multiplication of bounded and uniformly continuous functions are uniformly continuous. Then, according to Barbalat's lemma,  $\dot{e}(t)$  converges to zero as  $t \rightarrow \infty$ . Since both  $e$  and  $\dot{e}$  converges to zero, by (2.13)  $B\Lambda\tilde{\theta}_v v(t)$  converges to zero, and hence using (2.10) and (2.12), it can be shown that (2.6) is achieved asymptotically. When  $\bar{d} \neq 0$ , it can be shown that all the trajectories converge to a compact set  $E_1$ . To find  $E_1$ , we first introduce two properties of  $P$  [65] derived from the Lyapunov equation  $A_m^T P + P A_m = -Q$  as

$$\|P\| \leq \chi^2 \|Q\| / \sigma, \quad (\text{A.4})$$

$$\lambda_{\min}(P) \geq \lambda_{\min}(Q) / 2s. \quad (\text{A.5})$$

Using (A.1), it follows that

$$\begin{aligned} V &\leq \|e\|^2 \|P\| + \text{tr}(\tilde{\theta}_v^T \Gamma_\theta^{-1} \tilde{\theta}_v \Lambda) = \|e\|^2 \|P\| \\ &\quad + (1/\gamma_\theta) \text{tr}(\tilde{\theta}_v^T \tilde{\theta}_v \Lambda) \leq \|e\|^2 \|P\| + (1/\gamma_\theta) \tilde{\theta}_{\max}^2, \end{aligned} \quad (\text{A.6})$$

Since  $\theta_v^* \in \hat{\Omega}_{\text{proj}}$ , using (A.3) and (2.3), we have  $\dot{V} \leq -\lambda_{\min}(Q) \|e\|^2 + 2\|e\| \|P\bar{d}\|$ . By using Young's inequality, it follows that  $\dot{V} \leq -\frac{1}{2}\lambda_{\min}(Q) \|e\|^2 + 2\|P\|^2 D^2 / \lambda_{\min}(Q)$ . Thus, using this inequality and (A.6), we have

$$\dot{V}(t) \leq -\frac{\lambda_{\min}(Q)V}{2\|P\|} + \frac{\lambda_{\min}(Q)\tilde{\theta}_{\max}^2}{2\gamma_\theta\|P\|} + \frac{2\|P\|^2 D^2}{\lambda_{\min}(Q)} \leq -\omega_1 V + \omega_2, \quad (\text{A.7})$$

where  $\omega_1 = \frac{\sigma\lambda_{\min}(Q)}{2\chi^2\|Q\|}$  and  $\omega_2 = \frac{s}{\gamma_\theta} \tilde{\theta}_{\max}^2 + \frac{2\chi^4 D^2 \|Q\|^2}{\sigma^2 \lambda_{\min}(Q)}$ . By using the Gronwall inequality, (A.7) can be rewritten as

$$V(t) \leq \left(V(0) - \frac{\omega_2}{\omega_1}\right) e^{-\omega_1 t} + \frac{\omega_2}{\omega_1}. \quad (\text{A.8})$$

Using  $e(t)^T P e(t) \leq V(t) \leq \left(V(0) - \frac{\omega_2}{\omega_1}\right) e^{-\omega_1 t} + \frac{\omega_2}{\omega_1}$  and taking the limits of the leftmost and rightmost sides as  $t$  goes to infinity, we have

$$\limsup_{t \rightarrow \infty} e(t)^T P e(t) \leq \left(\frac{s\tilde{\theta}_{\max}^2}{\gamma_\theta} + \frac{2\chi^4 D^2 \|Q\|^2}{\sigma^2 \lambda_{\min}(Q)}\right) \frac{2\chi^2 \|Q\|}{\sigma \lambda_{\min}(Q)}. \quad (\text{A.9})$$

By using the inequality  $\lambda_{\min}(P) \|e\|^2 \leq e^T P e \leq \lambda_{\max}(P) \|e\|^2$  and (A.5), we have  $\frac{\lambda_{\min}(Q)}{2s} \|e\|^2 \leq \lambda_{\min}(P) \|e\|^2 \leq e^T P e$ . Thus, from (A.9) we have,

$$\limsup_{t \rightarrow \infty} \|e(t)\|^2 \leq \left(\frac{s\tilde{\theta}_{\max}^2}{\gamma_\theta \lambda_{\min}(Q)} + \frac{2\chi^4 D^2 \|Q\|^2}{\sigma^2 \lambda_{\min}(Q)}\right) \frac{4s\chi^2 \|Q\|}{\sigma \lambda_{\min}(Q)}.$$

Therefore, for the initial conditions  $e(0)$  and  $\theta_v(0) \in \Omega_{\text{proj}}$ ,  $e(t)$  and  $\tilde{\theta}_v(t)$  are uniformly bounded for all  $t \geq 0$  and system trajectories converge to the following compact set  $E_1$  given by (2.15).

## A.2 Proof of Lemma 1

For both cases in projection algorithm (2.1), we have

$$\begin{aligned}
& \text{tr}((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}(\theta_v, Y, f))) \\
&= \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*) (-Y_{i,j} + \text{Proj}(\theta_{v_{i,j}}, Y_{i,j}, f)) \\
&\leq \sum_{j=1}^m \sum_{i=1}^r |(\theta_{v_{i,j}} - \theta_{v_{i,j}}^*) Y_{i,j} f(\theta_{v_{i,j}})| \leq \sum_{j=1}^m \sum_{i=1}^r |\tilde{\theta}_{i,j} Y_{i,j}| \\
&= \text{tr}(|\tilde{\theta}_v^T| |Y|) \leq \|\tilde{\theta}_v\|_F \|Y\|_F \leq \sqrt{r} \tilde{\theta}_{MAX} \|Y\|,
\end{aligned}$$

where we used the property,  $\|Y\|_F \leq \sqrt{\min(r, m)} \|Y\|$ , and  $|\tilde{\theta}_v^T|$  and  $|Y|$  which are the matrices of absolute values of the elements of  $\tilde{\theta}_v^T$  and  $Y$ , respectively.

## A.3 Proof of Theorem 2

By using (A.1), (A.3), (2.17) with  $Y = -ve^T P B$ , and Young's inequality, we have

$$\dot{V} \leq -\frac{1}{2} \|e\|^2 + 4 \|P\|^2 D^2 + 4r \tilde{\theta}_{MAX}^2 \|P\|^2 \|B\|^2 M^2. \quad (\text{A.10})$$

Also, instead of (A.6), we have  $\frac{V}{\|P\|} - \frac{\tilde{\theta}_{MAX}^2}{\gamma_\theta \|P\|} \leq \|e\|^2$ . Using this inequality, (A.4), (A.5) and (A.10), we obtain

$$\begin{aligned}
\dot{V} &\leq -\frac{\lambda_{\min}(Q)V}{2\|P\|} + \frac{\lambda_{\min}(Q)\tilde{\theta}_{MAX}^2}{2\gamma_\theta \|P\|} + \frac{4\|P\|^2 D^2}{\lambda_{\min}(Q)} \\
&\quad + \frac{4r\tilde{\theta}_{MAX}^2 \|P\|^2 \|B\|^2 M^2}{\lambda_{\min}(Q)} \leq -\hat{\omega}_1 V + \hat{\omega}_2,
\end{aligned} \quad (\text{A.11})$$

where  $\hat{\omega}_1 = \frac{\lambda_{\min}(Q)\sigma}{2\chi^2 \|Q\|}$  and  $\hat{\omega}_2 = \frac{s\tilde{\theta}_{MAX}^2}{\gamma_\theta} + 4\frac{\chi^4 D^2 \|Q\|^2}{\sigma^2 \lambda_{\min}(Q)} + 4\frac{r\tilde{\theta}_{MAX}^2 \chi^4 \|B\|^2 M^2 \|Q\|^2}{\sigma^2 \lambda_{\min}(Q)}$ . Following the same procedure as for  $E_1$ ,  $\hat{E}_1$  is obtained and is given by (2.18).

## A.4 Proof of Theorem 4

To prove the non-emptiness of  $\Upsilon$ , we should show that  $\|\text{row}_i(B\Lambda\theta_I^{*T} - I_r)\|^2 \leq \frac{M_i^2}{M_{\max}^2} - \epsilon$ . Using (2.26) and the definition of  $\theta_I^{*T} = B^T(BB^T)^{-1}$ , we have,

$$\begin{aligned}
\|\text{row}_i(B\Lambda\theta_I^{*T} - I_r)\|^2 &= \|\text{row}_i(B\Lambda B^T(BB^T)^{-1} - I_r)\|^2 \\
&= \|\text{row}_i(B\Lambda B^T(BB^T)^{-1} - BB^T(BB^T)^{-1})\|^2 \\
&\leq \|\text{row}_i(B(\Lambda - I_m))\|^2 \|B^T(BB^T)^{-1}\|^2 \\
&\leq (\lambda_{\max}(\Lambda - I_m))^2 \|\text{row}_i(B)\|^2 \|B^T(BB^T)^{-1}\|^2 \\
&= (\lambda_{\min}(\Lambda) - 1)^2 \|\text{row}_i(B)\|^2 \|B^T(BB^T)^{-1}\|^2.
\end{aligned} \tag{A.12}$$

Therefore, in order to show that  $\|\text{row}_i(B\Lambda\theta_I^{*T} - I_r)\|^2 \leq \frac{M_i^2}{M_{\max}^2} - \epsilon$ , for all  $i = 1, \dots, r$ , we should satisfy

$$\begin{aligned}
(\lambda_{\min}(\Lambda) - 1)^2 \|\text{row}_i(B)\|^2 \|B^T(BB^T)^{-1}\|^2 &\leq \frac{M_i^2}{M_{\max}^2} - \epsilon \\
\Rightarrow -\sqrt{\gamma_{M_i}/\gamma_{B_i}} &\leq \lambda_{\min}(\Lambda) - 1 \leq \sqrt{\gamma_{M_i}/\gamma_{B_i}}, \quad i = 1, \dots, r,
\end{aligned}$$

where  $\gamma_{B_i} \equiv \|\text{row}_i(B)\| \|B^T(BB^T)^{-1}\|$  and  $\gamma_{M_i} = \frac{M_i^2}{M_{\max}^2} - \epsilon$  for all  $i = 1, \dots, r$ . Since the maximum value for the diagonal elements of  $\Lambda$  is one, the only condition that should be satisfied is that  $1 - \sqrt{\gamma_{M_i}/\gamma_{B_i}} \leq \lambda_{\min}(\Lambda)$  for  $i = 1, \dots, r$  or  $\gamma \equiv \max_i(1 - \sqrt{\gamma_{M_i}/\gamma_{B_i}}) \leq \lambda_{\min}(\Lambda)$ .

## A.5 Design procedure

The following procedure can be followed to compute the parameters of our control allocation scheme:

**Step 1-** Use Step 1 in Section 2.4, to determine  $M_i$ ,  $i = 1, \dots, r$ , thus,  $\hat{\Omega}_v$  is obtained.

**Step 2-** Calculate  $\Omega_\theta$  using Step 2 in Section 2.4.

**Step 3-** Using Theorem 4, calculate  $\gamma \equiv \max_i(1 - \sqrt{\gamma_{M_i}/\gamma_{B_i}})$ , where  $M_{\max} = \max_i M_i$ ,  $\gamma_{M_i} \equiv \frac{M_i^2}{M_{\max}^2} - \epsilon$  for a small positive  $\epsilon$ , and  $\gamma_{B_i} \equiv \|\text{row}_i(B)\| \|B^T(BB^T)^{-1}\|$ .

**Step 4-** Using  $\gamma$ , obtain  $\hat{\Omega}_\Lambda$  in (2.28).

**Step 5-** Solve the optimization problem (2.25), which leads to obtaining the  $\Omega_{\text{proj}}$ , which is defined in Section 2.4, step 3. This allows the implementation of (2.10).



# Appendix B

## Proofs of Chapter 3

### B.1 Proof of Lemma 3

Rewriting (3.24) using the matrices in (3.14), we have

$$\begin{aligned}
 \begin{bmatrix} v_1(t) \\ \vdots \\ v_\ell(t) \end{bmatrix} &= - \begin{bmatrix} a_{2_{1,1}} & \cdots & a_{2_{1,(n-\ell)}} \\ \vdots & \ddots & \vdots \\ a_{2_{\ell,1}} & \cdots & a_{2_{\ell,(n-\ell)}} \end{bmatrix} \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_{n-\ell}^{(1)}(t) \end{bmatrix} - \begin{bmatrix} a_{2_{1,(n-\ell+1)}} & \cdots & a_{2_{1,n}} \\ \vdots & \ddots & \vdots \\ a_{2_{\ell,(n-\ell+1)}} & \cdots & a_{2_{\ell,n}} \end{bmatrix} \begin{bmatrix} x_1^{(2)}(t) \\ \vdots \\ x_\ell^{(2)}(t) \end{bmatrix} \\
 &\quad - \bar{\lambda} e^{-\bar{\lambda} t} \begin{bmatrix} x_1^{(2)}(0) \\ \vdots \\ x_\ell^{(2)}(0) \end{bmatrix} + \frac{2}{\pi} \tan^{-1}(\bar{\lambda} t) \begin{bmatrix} \dot{r}_1(t) \\ \vdots \\ \dot{r}_\ell(t) \end{bmatrix} \\
 &\quad + \frac{2}{\pi} \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} \begin{bmatrix} r_1(t) \\ \vdots \\ r_\ell(t) \end{bmatrix} - \begin{bmatrix} \bar{s}_1(t) & 0 & 0 & \cdots & 0 \\ 0 & \bar{s}_2(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & \bar{s}_{\ell-1}(t) & 0 \\ 0 & \cdots & 0 & 0 & \bar{s}_\ell(t) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_\ell \end{bmatrix}, \quad (\text{B.1})
 \end{aligned}$$

where  $a_{2_{i,j}}$ ,  $i = 1, \dots, \ell$ ,  $j = 1, \dots, n - \ell$  are the elements of  $A_{2,1}$ , and  $a_{2_{i,j}}$ ,  $i = 1, \dots, \ell$ ,  $j = n - \ell + 1, \dots, n$  are the elements of  $A_{2,2}$ . Also,  $x_i^{(2)}$ ,  $r_i$ ,  $\dot{r}_i$  and  $\rho_i$ ,  $i = 1, \dots, \ell$  are the elements of the vectors  $x^{(2)}$ ,  $r$ ,  $\dot{r}$  and  $\rho$ , respectively, and  $\bar{s}_i$  are the  $i^{\text{th}}$

diagonal element of  $\text{sign}_v(s(x^{(2)}(t), x^{(2)}(0), t))$ . Thus,

$$\begin{aligned} v_i(t) = & -\sum_{j=1}^{n-\ell} a_{2i,j} x_j^{(1)}(t) - \sum_{j=n-\ell+1}^n a_{2i,j} x_{j-n+\ell}^{(2)}(t) - \bar{\lambda} x_i^{(2)}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} \dot{r}_i(t) \tan^{-1}(\bar{\lambda}t) \\ & + \frac{2}{\pi} r_i(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} - \bar{s}_i(t) \rho_i. \end{aligned} \quad (\text{B.2})$$

To ensure that  $|v_i| \leq M_i$  for  $i = 1, \dots, \ell$ , the inequality

$$\begin{aligned} & \left| -\sum_{j=1}^{n-\ell} a_{2i,j} x_j^{(1)}(t) - \sum_{j=n-\ell+1}^n a_{2i,j} x_{j-n+\ell}^{(2)}(t) - \bar{\lambda} x_i^{(2)}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} \dot{r}_i(t) \tan^{-1}(\bar{\lambda}t) \right. \\ & \left. + \frac{2}{\pi} r_i(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} - \bar{s}_i(t) \rho_i \right| \leq M_i \end{aligned} \quad (\text{B.3})$$

should be satisfied for all  $i = 1, \dots, \ell$ .

By considering the elements of  $\Phi(t, t_0)$  as  $\phi_{i,j}(t)$ ,  $i = 1, \dots, n-\ell$ ,  $j = 1, \dots, n-\ell$ , with  $t_0 = 0$ , and substituting these in (3.21), we have

$$\begin{aligned} \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_{n-\ell}^{(1)}(t) \end{bmatrix} = & \begin{bmatrix} \phi_{1,1}(t) & \cdots & \phi_{1,(n-\ell)}(t) \\ \vdots & \cdots & \vdots \\ \phi_{(n-\ell),1}(t) & \cdots & \phi_{(n-\ell),(n-\ell)}(t) \end{bmatrix} \begin{bmatrix} x_1^{(1)}(0) \\ \vdots \\ x_{n-\ell}^{(1)}(0) \end{bmatrix} \\ & + \int_0^t \begin{bmatrix} \phi_{1,1}(t-\eta) & \cdots & \phi_{1,(n-\ell)}(t-\eta) \\ \vdots & \cdots & \vdots \\ \phi_{(n-\ell),1}(t-\eta) & \cdots & \phi_{(n-\ell),(n-\ell)}(t-\eta) \end{bmatrix} \begin{bmatrix} g_1(\eta) \\ \vdots \\ g_{n-\ell}(\eta) \end{bmatrix} d\eta. \end{aligned} \quad (\text{B.4})$$

Remembering that  $g(t) \equiv G_1 e^{-\bar{\lambda}(t-t_0)} + G_2(t)$ , with  $G_1 \equiv A_{1,2} x^{(2)}(t_0)$ , and  $G_2(t) \equiv \frac{2}{\pi} A_{1,2} r(t) \tan^{-1}(\bar{\lambda}(t-t_0))$ , the elements of the vector  $g(t)$  can be written as

$$\begin{aligned} \begin{bmatrix} g_1(t) \\ \vdots \\ g_{n-\ell}(t) \end{bmatrix} = & \begin{bmatrix} a_{11,(n-\ell+1)} & \cdots & a_{11,n} \\ \vdots & \ddots & \vdots \\ a_{1(n-\ell),(n-\ell+1)} & \cdots & a_{1(n-\ell),n} \end{bmatrix} \begin{bmatrix} x_1^{(2)}(0) \\ \vdots \\ x_{n-\ell}^{(2)}(0) \end{bmatrix} e^{-\bar{\lambda}t} \\ & + \frac{2}{\pi} \tan^{-1}(\bar{\lambda}t) \begin{bmatrix} a_{11,(n-\ell+1)} & \cdots & a_{11,n} \\ \vdots & \ddots & \vdots \\ a_{1(n-\ell),(n-\ell+1)} & \cdots & a_{1(n-\ell),n} \end{bmatrix} \begin{bmatrix} r_1(t) \\ \vdots \\ r_{n-\ell}(t) \end{bmatrix}. \end{aligned} \quad (\text{B.5})$$

Substituting (B.5) into (B.4), the  $i^{\text{th}}$  element of the vector  $x^{(1)}(t)$  can be written as

$$x_i^{(1)}(t) = \sum_{j=1}^{n-\ell} \phi_{i,j}(t) x_j^{(1)}(0) + \int_0^t \left( \sum_{j=1}^{n-\ell} \phi_{i,j}(t-\eta) \left[ \sum_{k=n-\ell+1}^n a_{1,j,k} \left( x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} + r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) \right) \right] \right) d\eta, \quad i = 1, \dots, n-\ell, \quad (\text{B.6})$$

where  $a_{1,j,k}$  refer to the elements of  $A_1$ . Substituting (B.6) and the elements of  $x^{(2)}$  (see (3.17)) in (B.3), we have

$$\begin{aligned} |v_i(t)| = & \left| - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} a_{2,i,\omega} \phi_{\omega,j}(t) x_j^{(1)}(0) - \int_0^t \left( \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} a_{2,i,\omega} \phi_{\omega,j}(t-\eta) \right. \right. \\ & \times \left[ \sum_{k=n-\ell+1}^n a_{1,j,k} \left( x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} + r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) \right) \right] d\eta \\ & - \sum_{j=n-\ell+1}^n a_{2,i,j} \left( x_{j-n+\ell}^{(2)}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} r_{j-n+\ell}(t) \tan^{-1}(\bar{\lambda}t) \right) \\ & \left. - \bar{\lambda} x_{n-\ell+i}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} \dot{r}_i(t) \tan^{-1}(\bar{\lambda}t) + \frac{2}{\pi} r_i(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} - \bar{s}_i(x(t)) \rho_i \right| \leq M_i, \end{aligned} \quad (\text{B.7})$$

for  $i = 1, \dots, \ell$ . Using the triangle inequality, (3.28) is obtained.

## B.2 Proof of Corollary 1

The proof follows the same steps of the proof of Lemma 3 until (B.6).

Substituting  $\phi_{i,j}(t) = \sum_{\kappa=1}^{\bar{n}} c_{i,j,\kappa} e^{-h_{i,j,\kappa} t}$  into (B.6), we obtain that

$$\begin{aligned}
x_i^{(1)}(t) &= \sum_{j=1}^{n-\ell} \left( \sum_{\kappa=1}^{\bar{n}} c_{i,j,\kappa} e^{-h_{i,j,\kappa} t} \right) x_j^{(1)}(0) + \int_0^t \left( \sum_{j=1}^{n-\ell} \left( \sum_{\kappa=1}^{\bar{n}} c_{i,j,\kappa} e^{-h_{i,j,\kappa} (t-\eta)} \right) \right. \\
&\quad \times \left. \left[ \sum_{k=n-\ell+1}^n a_{1,j,k} \left( x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} + r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) \right) \right] \right) d\eta \\
&= \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} c_{i,j,\kappa} e^{-h_{i,j,\kappa} t} x_j^{(1)}(0) + \int_0^t \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n c_{i,j,\kappa} e^{-h_{i,j,\kappa} (t-\eta)} a_{1,j,k} \\
&\quad \times \left( x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} + r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) \right) d\eta \\
&= \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} c_{i,j,\kappa} e^{-h_{i,j,\kappa} t} x_j^{(1)}(0) \\
&\quad + \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n \left( \int_0^t c_{i,j,\kappa} e^{-h_{i,j,\kappa} (t-\eta)} a_{1,j,k} x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} d\eta \right. \\
&\quad \left. + \int_0^t c_{i,j,\kappa} e^{-h_{i,j,\kappa} (t-\eta)} a_{1,j,k} r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) d\eta \right), \quad i = 1, \dots, n-\ell.
\end{aligned} \tag{B.8}$$

Substituting (3.17) with  $t_0 = 0$  and (B.8) in (B.3), we get

$$\begin{aligned}
|v_i(t)| &= \left| - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} a_{2_{i,\omega}} c_{\omega,j,\kappa} e^{-h_{\omega,j,\kappa} t} x_j^{(1)}(0) \right. \\
&\quad - \sum_{\omega=1}^{n-\ell} \sum_{j=1}^{n-\ell} \sum_{\kappa=1}^{\bar{n}} \sum_{k=n-\ell+1}^n \left( \int_0^t a_{2_{i,\omega}} c_{\omega,j,\kappa} e^{-h_{\omega,j,\kappa} (t-\eta)} a_{1,j,k} x_{k-n+\ell}^{(2)}(0) e^{-\bar{\lambda}\eta} d\eta \right. \\
&\quad \left. + \int_0^t a_{2_{i,\omega}} c_{\omega,j,\kappa} e^{-h_{\omega,j,\kappa} (t-\eta)} a_{1,j,k} r_{k-n+\ell}(\eta) \frac{2}{\pi} \tan^{-1}(\bar{\lambda}\eta) d\eta \right) \\
&\quad - \sum_{j=n-\ell+1}^n a_{2_{i,j}} \left( x_i^{(2)}(0) e^{-\bar{\lambda}t} + \frac{2}{\pi} r_i(t) \tan^{-1}(\bar{\lambda}t) \right) - \bar{\lambda} x_{n-\ell+i}(0) e^{-\bar{\lambda}t} \\
&\quad \left. + \frac{2}{\pi} \dot{r}_i(t) \tan^{-1}(\bar{\lambda}t) + \frac{2}{\pi} r_i(t) \frac{\bar{\lambda}}{1 + \bar{\lambda}^2 t^2} - \bar{s}_i(x(t)) \rho_i \right| \leq M_i, \quad i = 1, \dots, \ell.
\end{aligned} \tag{B.9}$$

Defining  $q_{\omega,j_\kappa}(t)$  as

$$q_{\omega,j_\kappa}(t) = \int_0^t e^{-h_{\omega,j_\kappa}(t-\eta)} e^{-\bar{\lambda}\eta} d\eta = \begin{cases} te^{-h_{\omega,j_\kappa}t}, & \bar{\lambda} = h_{\omega,j_\kappa}, \\ \frac{e^{-\bar{\lambda}t} - e^{-h_{\omega,j_\kappa}t}}{h_{\omega,j_\kappa} - \bar{\lambda}}, & \bar{\lambda} \neq h_{\omega,j_\kappa}, \end{cases} \quad (\text{B.10})$$

remembering that  $\bar{s}_i$  is either 1 or -1, and using the triangle inequality, we can rewrite (B.9) as (3.29).



# Appendix C

## Proofs of Chapter 4

### C.1 Proof of Lemma 6

Taking the time derivative of the convex function  $f(\theta_{v_{i,j}})$  along the dynamics of  $\theta_{v_{i,j}}$ , we have

$$\begin{aligned} \frac{df_{i,j}}{dt} &= \frac{df_{i,j}}{d\theta_{v_{i,j}}} \frac{d\theta_{v_{i,j}}}{dt} = \frac{df_{i,j}}{d\theta_{v_{i,j}}} \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) \\ &= \begin{cases} \frac{df_{i,j}}{d\theta_{v_{i,j}}} Y_{i,j} (1 - \hat{f}_{i,j}) (1 - \hat{h}_{i,j}) & \text{if } f_{i,j} \geq 0 \quad \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0 \quad \& \quad h_{i,j} \geq 0 \\ \frac{df_{i,j}}{d\theta_{v_{i,j}}} Y_{i,j} (1 - \hat{f}_{i,j}) & \text{if } f_{i,j} > 0 \quad \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} > 0 \\ \frac{df_{i,j}}{d\theta_{v_{i,j}}} Y_{i,j} (1 - \hat{h}_{i,j}) & \text{if } h_{i,j} > 0 \\ \frac{df_{i,j}}{d\theta_{v_{i,j}}} Y_{i,j} & \text{otherwise} \end{cases} \end{aligned}$$

$$\Rightarrow \begin{cases} \frac{df_{i,j}}{dt} = 0 & \text{if } f_{i,j} = 1 & \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0 & \& \quad h_{i,j} = 1 \\ \frac{df_{i,j}}{dt} = 0 & \text{if } 0 \leq f_{i,j} < 1 & \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0 & \& \quad h_{i,j} = 1 \\ \frac{df_{i,j}}{dt} = 0 & \text{if } f_{i,j} = 1 & \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0 & \& \quad 0 \leq h_{i,j} < 1 \\ \frac{df_{i,j}}{dt} > 0 & \text{if } 0 \leq f_{i,j} < 1 & \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0 & \& \quad 0 \leq h_{i,j} < 1 \\ \frac{df_{i,j}}{dt} = 0 & \text{if } f_{i,j} = 1 & \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} > 0 \\ \frac{df_{i,j}}{dt} > 0 & \text{if } 0 < f_{i,j} < 1 & \& \quad Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} > 0 \\ \frac{df_{i,j}}{dt} = 0 & \text{if } h_{i,j} = 1 \end{cases} \quad (\text{C.1})$$

Also, when  $\hat{f}_{i,j} = 1$ ,  $\frac{df_{i,j}}{dt} = \frac{df_{i,j}}{d\theta_{v_{i,j}}} \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) \leq 0$ . Therefore, if  $\theta_{v_{i,j}}(0) \in \Omega_{i,j}$ ,  $\theta_{v_{i,j}}(t) \in \Omega_{i,j}$  for all  $t \geq 0$ . The same procedure can be followed for  $\frac{dh_{i,j}}{dt} = \frac{dh_{i,j}}{dY_{i,j}} \frac{dY_{i,j}}{d\theta_{v_{i,j}}} \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  to prove that if  $Y_{i,j}(0) \in \bar{\Omega}_{i,j}$ , then  $Y_{i,j}(t) \in \bar{\Omega}_{i,j}$  for all  $t \geq 0$ .

## C.2 Proof of Lemma 7

If  $f_{i,j} \geq 0$ ,  $Y_{i,j}(df_{i,j}/d\theta_{v_{i,j}}) \geq 0$  and  $h_{i,j} \geq 0$  (first condition), then

$$\begin{aligned} & \text{tr}((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}_m(\theta_v, Y))) \\ &= \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*) (-Y_{i,j} + \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})) \\ &= \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*) (-Y_{i,j} + Y_{i,j}(1 - \hat{f}_{i,j})(1 - \hat{h}_{i,j})) \\ &= \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*) (-Y_{i,j}\hat{f}_{i,j} - Y_{i,j}\hat{h}_{i,j} + Y_{i,j}\hat{f}_{i,j}\hat{h}_{i,j}). \end{aligned} \quad (\text{C.2})$$

$0 \leq \hat{h}_{i,j} \leq 1$  and  $0 \leq \hat{f}_{i,j} \leq 1$ , therefore  $|Y_{i,j}\hat{f}_{i,j}| \geq |Y_{i,j}\hat{f}_{i,j}\hat{h}_{i,j}|$  and  $|Y_{i,j}\hat{h}_{i,j}| \geq |Y_{i,j}\hat{f}_{i,j}\hat{h}_{i,j}|$ . Hence,

$$\begin{aligned} & \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*) (-Y_{i,j}\hat{f}_{i,j} - Y_{i,j}\hat{h}_{i,j} + Y_{i,j}\hat{f}_{i,j}\hat{h}_{i,j}) \\ & \leq \sum_{j=1}^m \sum_{i=1}^r \underbrace{(\theta_{v_{i,j}}^* - \theta_{v_{i,j}}) Y_{i,j}\hat{f}_{i,j}\hat{h}_{i,j}}_{<0} < 0. \end{aligned} \quad (\text{C.3})$$

If  $h_{i,j} > 0$  (third condition), then

$$\begin{aligned}
& tr((\theta_v^T - \theta_v^{*T})(-Y + \text{Proj}_m(\theta_v, Y))) \\
&= \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*)(-Y_{i,j} + \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})) \\
&= \sum_{j=1}^m \sum_{i=1}^r (\theta_{v_{i,j}} - \theta_{v_{i,j}}^*)(-Y_{i,j} + Y_{i,j}(1 - \hat{h}_{i,j})) \\
&\leq \sum_{j=1}^m \sum_{i=1}^r |\theta_{v_{i,j}}^* - \theta_{v_{i,j}}| Y_{MAX_{i,j}} \\
&= tr(|\tilde{\theta}_v^T| Y_{MAX}) \leq \|\tilde{\theta}_{max}\|_F \|Y_{MAX}\|_F.
\end{aligned} \tag{C.4}$$

Same procedure used in the proof of Lemma 5 can be employed to complete the proof for the second and fourth conditions.

### C.3 Proof of Lemma 8

We first decompose the set of feasible  $(Y_{i,j}, \theta_{v_{i,j}})$ , denoted as  $S = S_\theta \times S_Y \subset \mathbb{R}^2$ , into the following subsets:

$$\begin{aligned}
S_1 &= \bigcup_{\eta=1,2} S_{1,\eta} = \{(Y_{i,j}, \theta_{v_{i,j}}) | f_{i,j} > 0, Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} > 0\}, \\
S_2 &= \bigcup_{\eta=1,2} S_{2,\eta} = \{(Y_{i,j}, \theta_{v_{i,j}}) | h_{i,j} > 0\}, \\
S_3 &= \bigcup_{\eta=1,2,3,4} S_{3,\eta} = \{(Y_{i,j}, \theta_{v_{i,j}}) | f_{i,j}, h_{i,j} \geq 0, Y_{i,j} \frac{df_{i,j}}{d\theta_{v_{i,j}}} \geq 0\}, \\
S_0 &= S \setminus \left( \bigcup_{\eta=1,2,3} S_\eta \right),
\end{aligned} \tag{C.5}$$

which are illustrated in Figure C.1. Since  $Y_{i,j}$ ,  $\theta_{v_{i,j}}$ ,  $f_{i,j}$  and  $h_{i,j}$  are continuous functions, the proposed projection operator (4.8) is continuous in each subspace of  $S$ . Here, we will prove that the proposed projection is continuous also on the boundaries of these subsets.

Consider the boundary between  $S_0$  and  $S_{2,1}$  (see Figure C.1). Let the point  $(\theta_0, Y_{max_{i,j}} - \epsilon_{i,j}) \in S_0$  be an arbitrary point on the boundary. Notice that since  $S_0$

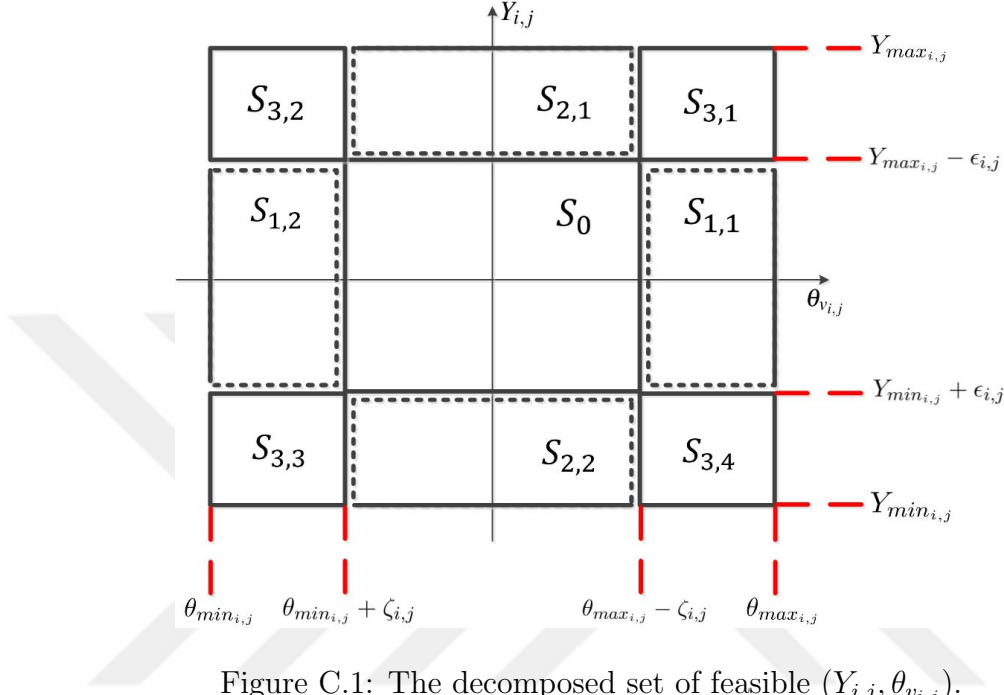


Figure C.1: The decomposed set of feasible  $(Y_{i,j}, \theta_{v,i,j})$ .

is a closed set, the points on the boundary of  $S_0$  and  $S_{2,1}$  belong to  $S_0$ . Therefore, in order to show that the proposed projection algorithm is continuous on the boundary of  $S_0$  and  $S_{2,1}$ , we should show that

$$\lim_{(\theta_{v,i,j}, Y_{i,j}) \rightarrow (\theta_0, Y_{max,i,j} - \epsilon_{i,j})} \text{Proj}_m(\theta_{v,i,j}, Y_{i,j}) = \text{Proj}_m(\theta_0, Y_{max,i,j} - \epsilon_{i,j}) = Y_{max,i,j} - \epsilon_{i,j} \quad (\text{C.6})$$

in both sets,  $S_0$  and  $S_{2,1}$ .

First, consider taking the limit in the set  $S_2$ . For any given  $\gamma > 0$ , there exists  $\delta_1 = \min\{\sqrt{2}\epsilon_{i,j}, \frac{\sqrt{2}\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{max,i,j}}\}$  such that for  $Y_{i,j} \in (Y_{max,i,j} - \epsilon_{i,j}, Y_{max,i,j} - \epsilon_{i,j} + \frac{\delta_1}{\sqrt{2}})$  and  $\theta_{v,i,j} \in (\theta_0 - \frac{\delta_1}{2\sqrt{2}}, \theta_0 + \frac{\delta_1}{2\sqrt{2}})$ ,  $0 < \sqrt{(\theta_{i,j} - \theta_0)^2 + (Y_{i,j} - Y_{max,i,j} + \epsilon_{i,j})^2} \leq \delta_1$ . Then using  $|Y_{i,j} - Y_{max,i,j} + \epsilon_{i,j}| < \frac{\delta_1}{\sqrt{2}}$  we have

$$\begin{aligned} |\text{Proj}_m(\theta_{v,i,j}, Y_{i,j}) - Y_{max,i,j} + \epsilon_{i,j}| &= |Y_{i,j}(1 - \hat{h}_{i,j}) - Y_{max,i,j} + \epsilon_{i,j}| \\ &\leq |Y_{i,j} - Y_{max,i,j} + \epsilon_{i,j}| + |Y_{i,j}\hat{h}_{i,j}| \\ &< \frac{\delta_1}{\sqrt{2}} + \left| \frac{Y_{i,j}(Y_{i,j} - Y_{min,i,j} - \epsilon_{i,j})(Y_{i,j} - Y_{max,i,j} + \epsilon_{i,j})}{(Y_{max,i,j} - Y_{min,i,j} - \epsilon_{i,j})\epsilon_{i,j}} \right|. \end{aligned} \quad (\text{C.7})$$

Considering  $Y_{i,j} \in (Y_{\max i,j} - \epsilon_{i,j}, Y_{\max i,j} - \epsilon_{i,j} + \frac{\delta_1}{\sqrt{2}})$ , an upper bound on (C.7) can be calculated as

$$\begin{aligned} & |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - Y_{\max i,j} + \epsilon_{i,j}| \\ & < \frac{\delta_1}{\sqrt{2}} + \left| \frac{(Y_{\max i,j} - \epsilon_{i,j} + \frac{\delta_1}{\sqrt{2}})(Y_{\max i,j} - Y_{\min i,j} - 2\epsilon_{i,j} + \frac{\delta_1}{\sqrt{2}})(\frac{\delta_1}{\sqrt{2}})}{(Y_{\max i,j} - Y_{\min i,j} - \epsilon_{i,j})\epsilon_{i,j}} \right|. \end{aligned} \quad (\text{C.8})$$

If  $\sqrt{2}\epsilon_{i,j} \leq \frac{\sqrt{2}\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}}$ , then  $\gamma \geq \epsilon_{i,j} + Y_{\max i,j}$ , and  $\delta_1 = \sqrt{2}\epsilon_{i,j}$ . Substituting  $\sqrt{2}\epsilon_{i,j}$  for  $\delta_1$  in (C.8) leads to

$$|\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - Y_{\max i,j} + \epsilon_{i,j}| < \epsilon_{i,j} + Y_{\max i,j} \leq \gamma. \quad (\text{C.9})$$

On the other hand, if  $\sqrt{2}\epsilon_{i,j} > \frac{\sqrt{2}\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}}$ , then  $\gamma < \epsilon_{i,j} + Y_{\max i,j}$ , and  $\delta_1 = \frac{\sqrt{2}\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}}$ . Substituting  $\frac{\sqrt{2}\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}}$  in (C.8) leads to

$$\begin{aligned} & |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - Y_{\max i,j} + \epsilon_{i,j}| < \frac{\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}} \\ & + \left| \frac{(Y_{\max i,j} - \epsilon_{i,j} + \frac{\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}})(Y_{\max i,j} - Y_{\min i,j} - 2\epsilon_{i,j} + \frac{\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}})(\frac{\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}})}{(Y_{\max i,j} - Y_{\min i,j} - \epsilon_{i,j})\epsilon_{i,j}} \right|. \end{aligned} \quad (\text{C.10})$$

Since  $Y_{\max i,j} - \epsilon_{i,j} > 0$  and  $Y_{\min i,j} + \epsilon_{i,j} < 0$ , we have  $Y_{\max i,j} - Y_{\min i,j} - 2\epsilon_{i,j} > 0$ . Using these inequalities, and the fact that  $\gamma < \epsilon_{i,j} + Y_{\max i,j}$ , (C.10) can be rewritten as

$$|\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - Y_{\max i,j} + \epsilon_{i,j}| < \frac{(\epsilon_{i,j} + Y_{\max i,j})\gamma}{\epsilon_{i,j} + Y_{\max i,j}} = \gamma. \quad (\text{C.11})$$

Therefore,  $\lim_{(\theta_{v_{i,j}}, Y_{i,j}) \rightarrow (\theta_0, Y_{\max i,j} - \epsilon_{i,j})} \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) = Y_{\max i,j} - \epsilon_{i,j}$  in set  $S_{2,1}$ .

Let us now consider the same limit operation in  $S_0$ . Again, for any  $\gamma > 0$ , there exist a  $\delta_1 = \min\{\sqrt{2}\epsilon_{i,j}, \frac{\sqrt{2}\epsilon_{i,j}\gamma}{\epsilon_{i,j} + Y_{\max i,j}}\}$  such that for  $Y_{i,j} \in (Y_{\max i,j} - \epsilon_{i,j} - \frac{\delta_1}{\sqrt{2}}, Y_{\max i,j} - \epsilon_{i,j})$  and  $\theta_{v_{i,j}} \in (\theta_0 - \frac{\delta_1}{2\sqrt{2}}, \theta_0 + \frac{\delta_1}{2\sqrt{2}})$ ,  $0 < \sqrt{(\theta_{i,j} - \theta_0)^2 + (Y_{i,j} - Y_{\max i,j} + \epsilon_{i,j})^2} \leq \delta_1$ . Then using  $|Y_{i,j} - Y_{\max i,j} + \epsilon_{i,j}| < \frac{\delta_1}{\sqrt{2}}$  we have

$$\begin{aligned} & |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - Y_{\max i,j} + \epsilon_{i,j}| = |Y_{i,j} - Y_{\max i,j} + \epsilon_{i,j}| \\ & < \frac{\delta_1}{\sqrt{2}} \leq \frac{\epsilon_{i,j}}{\epsilon_{i,j} + Y_{\max i,j}}\gamma < \gamma. \end{aligned} \quad (\text{C.12})$$

This shows that  $\lim_{(\theta_{v_{i,j}}, Y_{i,j}) \rightarrow (\theta_0, Y_{\max_{i,j}} - \epsilon_{i,j})} \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) = Y_{\max_{i,j}} - \epsilon_{i,j}$  in  $S_0$ . Therefore,  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  is continuous on the boundary of  $S_0$  and  $S_{2,1}$ .

Consider now the boundary between  $S_{1,1}$  and  $S_{3,1}$  (see Figure C.1). Let the point  $(\theta_1, Y_{\max_{i,j}} - \epsilon_{i,j})$  be an arbitrary point on the boundary of  $S_{1,1}$  and  $S_{3,1}$ . Notice that since  $S_{3,1}$  is a closed set, the points on the boundary of  $S_1$  and  $S_3$  belong to  $S_3$ . We should show that the limit of  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  when  $(\theta_{v_{i,j}}, Y_{i,j})$  approaches  $(\theta_1, Y_{\max_{i,j}} - \epsilon_{i,j})$  in  $S_{3,1}$  leads to the same value as  $(\theta_{v_{i,j}}, Y_{i,j})$  approaches to  $(\theta_1, Y_{\max_{i,j}} - \epsilon_{i,j})$  in  $S_{1,1}$ , and this value is equal to  $\text{Proj}_m(\theta_1, Y_{\max_{i,j}} - \epsilon_{i,j}) = (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{h}(Y_{\max_{i,j}} - \epsilon_{i,j}))(1 - \hat{f}(\theta_1)) = (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1))$ .

First, consider the limit in  $S_{3,1}$ . For any  $\gamma > 0$ , there exists  $\delta_2 = \min\{\sqrt{2}\epsilon_{i,j}, \sqrt{2}\gamma X^{-1}\}$ , where  $X = 1 + \hat{f}(\theta_1) + (\frac{2\theta_1 - \theta_{\max_{i,j}} - \theta_{\min_{i,j}} + \epsilon_{i,j}/2}{2(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}})Y_{\max_{i,j}}$ , such that for  $Y_{i,j} \in (Y_{\max_{i,j}} - \epsilon_{i,j}, Y_{\max_{i,j}} - \epsilon_{i,j} + \frac{\delta_2}{\sqrt{2}})$  and  $\theta_{v_{i,j}} \in (\theta_1 - \frac{\delta_2}{2\sqrt{2}}, \theta_1 + \frac{\delta_2}{2\sqrt{2}})$ ,  $0 < \sqrt{(\theta_{i,j} - \theta_0)^2 + (Y_{i,j} - Y_{\max_{i,j}} + \epsilon_{i,j})^2} \leq \delta_2$ . Then, we have

$$\begin{aligned}
& |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1))| \\
&= |Y_{i,j}(1 - \hat{f}_{i,j})(1 - \hat{h}_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1))| \\
&\leq |Y_{i,j}(1 - \hat{f}(\theta_1 - \frac{\delta_2}{2\sqrt{2}})) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1 + \frac{\delta_2}{2\sqrt{2}}))| \\
&\leq |Y_{i,j} - Y_{\max_{i,j}} + \epsilon_{i,j}| + |Y_{i,j}\hat{f}(\theta_1 - \frac{\delta_2}{2\sqrt{2}}) - (Y_{\max_{i,j}} - \epsilon_{i,j})\hat{f}(\theta_1 + \frac{\delta_2}{2\sqrt{2}})| \\
&< \frac{\delta_2}{\sqrt{2}} + |(Y_{\max_{i,j}} - \epsilon_{i,j} + \frac{\delta_2}{\sqrt{2}})\hat{f}(\theta_1 - \frac{\delta_2}{2\sqrt{2}}) - (Y_{\max_{i,j}} - \epsilon_{i,j})\hat{f}(\theta_1 + \frac{\delta_2}{2\sqrt{2}})|.
\end{aligned} \tag{C.13}$$

It can be shown that  $\hat{f}(\theta_1 - \frac{\delta_2}{2\sqrt{2}}) = \hat{f}(\theta_1) - \frac{\delta_2}{\sqrt{2}}(\frac{2\theta_1 - \theta_{\max_{i,j}} - \theta_{\min_{i,j}} + \frac{\delta_2}{2\sqrt{2}}}{2(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}})$  and  $\hat{f}(\theta_1 + \frac{\delta_2}{2\sqrt{2}}) = \hat{f}(\theta_1) + \frac{\delta_2}{\sqrt{2}}(\frac{2\theta_1 - \theta_{\max_{i,j}} - \theta_{\min_{i,j}} + \frac{\delta_2}{2\sqrt{2}}}{2(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}})$ . Therefore, an upper bound on (C.13) can be obtained as

$$\begin{aligned}
& |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{h}(Y_{\max_{i,j}} - \epsilon_{i,j}))(1 - \hat{f}(\theta_1))| \\
&< \frac{\delta_2}{\sqrt{2}} + \frac{\delta_2}{\sqrt{2}}\hat{f}(\theta_1) + \frac{\delta_2}{\sqrt{2}}(\frac{2\theta_1 - \theta_{\max_{i,j}} - \theta_{\min_{i,j}} + \frac{\delta_2}{2\sqrt{2}}}{2(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}})(Y_{\max_{i,j}} - \epsilon_{i,j} + \frac{\delta_2}{\sqrt{2}}).
\end{aligned} \tag{C.14}$$

Using the definition of  $\delta_2$ , and the fact that  $\theta_{\max_{i,j}} - \zeta_{i,j} > 0$  and  $\theta_{\min_{i,j}} + \zeta_{i,j} < 0$ ,

an upper bound on (C.14) can be obtained as

$$\begin{aligned} & |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{h}(Y_{\max_{i,j}} - \epsilon_{i,j}))(1 - \hat{f}(\theta_1))| \\ & < \frac{\delta_2}{\sqrt{2}}((1 + \hat{f}(\theta_1)) + (\frac{2\theta_1 - \theta_{\max_{i,j}} - \theta_{\min_{i,j}} + \frac{\epsilon_{i,j}}{2}}{2(\theta_{\max_{i,j}} - \theta_{\min_{i,j}} - \zeta_{i,j})\zeta_{i,j}})Y_{\max_{i,j}}) \leq \gamma. \end{aligned} \quad (\text{C.15})$$

This shows that  $\lim_{(\theta_{v_{i,j}}, Y_{i,j}) \rightarrow (\theta_1, Y_{\max_{i,j}} - \epsilon_{i,j})} \text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) = (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1))$ , in set  $S_{3,1}$ . Now, consider taking the same limit in  $S_{1,1}$ . For  $Y_{i,j} \in (Y_{\max_{i,j}} - \epsilon_{i,j} - \frac{\delta_2}{\sqrt{2}}, Y_{\max_{i,j}} - \epsilon_{i,j})$  and  $\theta_{v_{i,j}} \in (\theta_1 - \frac{\delta_2}{2\sqrt{2}}, \theta_1 + \frac{\delta_2}{2\sqrt{2}})$ ,  $0 < \sqrt{(\theta_{i,j} - \theta_0)^2 + (Y_{i,j} - Y_{\max_{i,j}} + \epsilon_{i,j})^2} \leq \delta_2$ . Then, we have

$$\begin{aligned} & |\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{h}(Y_{\max_{i,j}} - \epsilon_{i,j}))(1 - \hat{f}(\theta_1))| \\ & = |Y_{i,j}(1 - \hat{f}_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1))| \\ & \leq |Y_{i,j}(1 - \hat{f}(\theta_1 - \frac{\delta_2}{2\sqrt{2}})) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{f}(\theta_1 + \frac{\delta_2}{2\sqrt{2}}))|. \end{aligned} \quad (\text{C.16})$$

Using the same procedure as (C.13)-(C.15), it can be shown that  $|\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j}) - (Y_{\max_{i,j}} - \epsilon_{i,j})(1 - \hat{h}(Y_{\max_{i,j}} - \epsilon_{i,j}))(1 - \hat{f}(\theta_1))| < \gamma$ . Therefore,  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  is continuous on the boundary of  $S_{1,1}$  and  $S_{3,1}$ .

Continuity of the proposed projection function on the other boundaries can be proved following the same procedure as above. Therefore,  $\text{Proj}_m(\theta_{v_{i,j}}, Y_{i,j})$  is continuous on  $S$ .

## C.4 Proof of Lemma 9

In order to prove that a function  $g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is locally Lipschitz, it must be shown that there exists a positive constant  $K$  such that  $\|g(x) - g(y)\| \leq K\|x - y\|$ , for any  $x, y \in D \subset \mathbb{R}^n$ . Let  $a^1 \equiv (Y_{i,j}^1, \theta_{v_{i,j}}^1) \in S \subset \mathbb{R}^2$  and  $a^0 \equiv (Y_{i,j}^0, \theta_{v_{i,j}}^0) \in S \subset \mathbb{R}^2$ , where  $S$  is given as  $S = S_\theta \times S_Y \subset \mathbb{R}^2$ . Furthermore, let  $a^\mu = (Y_{i,j}^\mu, \theta_{v_{i,j}}^\mu)$ ,  $\mu \in [0, 1]$ , be any point on the line connecting  $a^0$  and  $a^1$ , which satisfy

$$Y_{i,j}^\mu = \mu Y_{i,j}^1 + (1 - \mu) Y_{i,j}^0, \quad (\text{C.17})$$

$$\theta_{i,j}^\mu = \mu \theta_{v_{i,j}}^1 + (1 - \mu) \theta_{v_{i,j}}^0. \quad (\text{C.18})$$

The Lipschitz condition needs to be investigated for 4 different cases, which are given below. The subsets of  $S$ , defined in (C.5) and demonstrated in Figure C.1, are used throughout the proof.

Case 1: If for all  $\mu \in [0, 1]$ ,  $a^\mu$  lies in the set  $S_0$ , then, using (4.8), it can be shown that

$$\begin{aligned} |\text{Proj}_m(a^1) - \text{Proj}_m(a^0)| &= |Y_{i,j}^1 - Y_{i,j}^0| \\ &\leq |Y_{i,j}^1 - Y_{i,j}^0| + |\theta_{v_{i,j}}^1 - \theta_{v_{i,j}}^0| \\ &\leq k_0 \|a^1 - a^0\|, \end{aligned} \quad (\text{C.19})$$

where  $k_0$  is a positive constant. This satisfies the Lipschitz condition on  $S_0$ .

Case 2: If for all  $\mu \in [0, 1]$ ,  $a^\mu$  lies in the set  $S_{3,1}$ , then

$$\begin{aligned} |\text{Proj}_m(a^1) - \text{Proj}_m(a^0)| &= |Y_{i,j}^1(1 - \hat{f}_{i,j}^1)(1 - \hat{h}_{i,j}^1) \\ &\quad - Y_{i,j}^0(1 - \hat{f}_{i,j}^0)(1 - \hat{h}_{i,j}^0)|, \end{aligned} \quad (\text{C.20})$$

where  $\hat{f}_{i,j}^\ell = \hat{f}(\theta_{v_{i,j}}^\ell)$  and  $\hat{h}_{i,j}^\ell = \hat{h}(Y_{i,j}^\ell)$  for  $\ell = \{0, 1\}$ . Using (4.2) and (4.7), it can be shown that there exist positive constants  $k_{\theta 0}$  and  $k_{Y 0}$  such that

$$|\hat{f}_{i,j}^1 - \hat{f}_{i,j}^0| < k_{\theta 0} |\theta_{v_{i,j}}^1 - \theta_{v_{i,j}}^0| \quad (\text{C.21})$$

$$|\hat{h}_{i,j}^1 - \hat{h}_{i,j}^0| < k_{Y 0} |Y_{i,j}^1 - Y_{i,j}^0|. \quad (\text{C.22})$$

Using (C.21) and (C.22), an upper bound on (C.20) can be obtained as

$$\begin{aligned} |\text{Proj}_m(a^1) - \text{Proj}_m(a^0)| &\leq k_{Y 1} |Y_{i,j}^1 - Y_{i,j}^0| + k_{\theta 1} |\theta_{v_{i,j}}^1 - \theta_{v_{i,j}}^0| \\ &\leq k_1 \|a^1 - a^0\|, \end{aligned} \quad (\text{C.23})$$

where  $k_{\theta 1}$ ,  $k_{Y 1}$  and  $k_1$  are positive constants. The same procedure can be followed for each subsets of  $S_1$ ,  $S_2$  and  $S_3$ , and therefore the Lipschitz condition is satisfied on each subsets of  $S_1, S_2$  and  $S_3$ .

Case 3: If  $a^0$  and  $a^1$  are in two neighboring subsets of  $S$ , then the following analysis can be conducted: Let  $a^1$  belong to  $S_{3,1}$  and  $a^0$  to  $S_{1,1}$ . Then, the segment  $[a^0, a^1]$  can be divided into two segments  $[a^1, a^{\mu^*}] \in S_{3,1}$  and  $(a^{\mu^*}, a^0] \in S_{1,1}$ , where

$$\begin{aligned} \mu^* &= \min \mu, \\ \text{s.t. } \mu &\in [0, 1] \text{ and } a^\mu \in S_{3,1}. \end{aligned} \quad (\text{C.24})$$

Using the mean value theorem in  $S_{1,1} \setminus \partial S_{1,1}$ , where  $\partial S_{1,1}$  denotes the boundary of  $S_{1,1}$ , and using (C.17) and (C.18), we obtain that

$$\begin{aligned} |\text{Proj}_m(a^{\mu^*}) - \text{Proj}_m(a^0)| &\leq k'_2(|Y_{i,j}^{\mu^*} - Y_{i,j}^0| + |\theta_{v_{i,j}}^{\mu^*} - \theta_{v_{i,j}}^0|) \\ &\leq k''_2(|Y_{i,j}^1 - Y_{i,j}^0| + |\theta_{v_{i,j}}^1 - \theta_{v_{i,j}}^0|), \end{aligned} \quad (\text{C.25})$$

where  $k'_2$  and  $k''_2$  are positive constants. Also, following the procedure in Case 2, it can be shown that  $|\text{Proj}_m(a^1) - \text{Proj}_m(a^{\mu^*})| \leq k_1|a^1 - a^0|$ . Therefore, using the triangle inequality, we get

$$\begin{aligned} |\text{Proj}_m(a^1) - \text{Proj}_m(a^0)| &\leq |\text{Proj}_m(a^1) - \text{Proj}_m(a^{\mu^*})| \\ &\quad + |\text{Proj}_m(a^{\mu^*}) - \text{Proj}_m(a^0)| \\ &\leq k_2|a^1 - a^0|, \end{aligned} \quad (\text{C.26})$$

where  $k_2$  is a positive constant. The same procedure can be used for the other two neighboring subsets.

Case 4: If  $a^0$  and  $a^1$  are in two non-neighboring subsets of  $S$ , then the following analysis can be conducted: Let  $a^0$  belong to  $S_{1,1}$  and  $a^1$  to  $S_{2,1}$ . Then, the segment  $[a^0, a^1]$  can be divided into three segments  $[a^0, a^{\alpha^*}] \in S_{1,1}$ ,  $[a^{\alpha^*}, a^{\beta^*}] \in S_0$ , and  $[a^{\beta^*}, a^1] \in S_{2,1}$ , where  $\alpha^*$  and  $\beta^*$  are defined as

$$\begin{aligned} \alpha^* &= \min \mu \\ \text{s.t. } \mu &\in [0, 1] \text{ and } a^\mu \in S_0, \end{aligned} \quad (\text{C.27})$$

and

$$\begin{aligned} \beta^* &= \max \mu \\ \text{s.t. } \mu &\in [0, 1] \text{ and } a^\mu \in S_0. \end{aligned} \quad (\text{C.28})$$

Then, the same procedure used in Case 3 can be followed to obtain the Lipschitz condition.

Since the Lipschitz condition is satisfied for any two points  $a^0, a^1 \in S$ , the projection algorithm is locally Lipschitz on  $S$ .

# Appendix D

## Proofs of Chapter 5

### D.1 Proof of Lemma 10

The matrix  $G(t)^T G(t) - I_2$  can be written as

$$\begin{bmatrix} \left(\frac{2}{\pi} \tan^{-1}(\bar{\lambda} t)\right)^2 - 1 & 0 \\ 0 & \alpha_G - 1 \end{bmatrix}. \quad (\text{D.1})$$

To show that the matrix (D.1) is negative definite for all  $t > 0$ , we need to check the following two requirements:

$$\left(\frac{2}{\pi} \tan^{-1}(\bar{\lambda} t)\right)^2 - 1 < 0, \quad \forall t > 0, \quad (\text{D.2})$$

$$\det(G(t)^T G(t) - I_2) > 0, \quad \forall t > 0, \quad (\text{D.3})$$

where  $\det(\cdot)$  provides the determinant of the matrix. The inequality (D.2) is always satisfied for all  $t > 0$ . The left hand side of the inequality (D.3) is obtained as

$$\det(G(t)^T G(t) - I_2) = \left( \left(\frac{2}{\pi} \tan^{-1}(\bar{\lambda} t)\right)^2 - 1 \right) (\alpha_G - 1). \quad (\text{D.4})$$

With  $|\alpha_G| < 1$ , it can be concluded that (D.4) is negative.

## D.2 Proof of Lemma 11

We start with

$$(\epsilon_G^{\frac{1}{2}} X^T - \epsilon_G^{\frac{-1}{2}} G(t) Y)^T (\epsilon_G^{\frac{1}{2}} X^T - \epsilon_G^{\frac{-1}{2}} G(t) Y) \geq 0, \quad (\text{D.5})$$

which can be written as

$$\epsilon_G X X^T - X G(t) Y - Y^T G^T(t) X^T + \epsilon_G^{-1} Y^T G^T(t) G(t) Y \geq 0, \quad (\text{D.6})$$

Then, by using Lemma 10, (D.6) leads to (5.72).

# Appendix E

## Proofs of Chapter 6

### E.1 Proof of Theorem 17

Consider a Lyapunov-Krasovskii functional [134]

$$\begin{aligned}
 V(t) = & e^T P e + \text{tr}(\Phi^T(t)\Phi(t)) + \text{tr}(\tilde{\theta}_1^T(t)\tilde{\theta}_1(t)) \\
 & + \int_{-\tau}^0 \int_{t+v}^t \text{tr}(\dot{\tilde{\theta}}_1^T(\xi)\dot{\tilde{\theta}}_1(\xi))d\xi dv + \int_{-\tau}^0 \int_{t+v}^t \text{tr}(\dot{\Phi}^T(\xi)\dot{\Phi}(\xi))d\xi dv \\
 & + \int_{-\tau}^0 \text{tr}(\tilde{\lambda}_1^T(t, \eta)\tilde{\lambda}_1(t, \eta))d\eta + \int_{-\tau}^0 \int_{t+v}^t \int_{-\tau}^0 \text{tr}(\dot{\tilde{\lambda}}_1^T(\xi, \eta)\dot{\tilde{\lambda}}_1(\xi, \eta))d\eta d\xi dv.
 \end{aligned} \tag{E.1}$$

The derivative of  $V(t)$  can be calculated as

$$\begin{aligned}
 \dot{V}(t) = & \dot{e}^T(t)^T P e(t) + e^T(t) P \dot{e}(t) + 2\text{tr}(\dot{\tilde{\theta}}_1^T(t)\tilde{\theta}_1(t)) + 2\text{tr}(\dot{\Phi}^T(t)\Phi(t)) \\
 & + \int_{-\tau}^0 2\text{tr}(\dot{\tilde{\lambda}}_1^T(t, \eta)\tilde{\lambda}_1(t, \eta))d\eta + \tau\text{tr}(\dot{\tilde{\theta}}_1^T(t)\dot{\tilde{\theta}}_1(t)) - \int_{-\tau}^0 \text{tr}(\dot{\tilde{\theta}}_1^T(t+v)\dot{\tilde{\theta}}_1(t+v))dv \\
 & + \tau\text{tr}(\dot{\Phi}^T(t)\dot{\Phi}(t)) - \int_{-\tau}^0 \text{tr}(\dot{\Phi}^T(t+v)\dot{\Phi}(t+v))dv \\
 & + \tau \int_{-\tau}^0 \text{tr}(\dot{\tilde{\lambda}}_1^T(t, \eta)\dot{\tilde{\lambda}}_1(t, \eta))d\eta - \int_{-\tau}^0 \int_{-\tau}^0 \text{tr}(\dot{\tilde{\lambda}}_1^T(t+v, \eta)\dot{\tilde{\lambda}}_1(t+v, \eta))d\eta dv.
 \end{aligned} \tag{E.2}$$

Substituting (6.24) into (E.2) and using the Lyapunov equation  $A_m^T P + P A_m = -Q$ , it is obtained that

$$\begin{aligned}
\dot{V}(t) = & -e^T(t)Qe(t) + 2e^T(t)PB_m\tilde{\theta}_1(t-\tau)x_{hp}(t-\tau) \\
& + 2e^T(t)PB_m \int_{-\tau}^0 \tilde{\lambda}_1(t-\tau, \eta)u(t+\eta-\tau)d\eta + 2e^T(t)PB_m\Phi(t-\tau)u(t-\tau) \\
& + 2\text{tr}(\dot{\theta}_1^T(t)\dot{\theta}_1(t)) + 2\text{tr}(\dot{\Phi}^T(t)\Phi(t)) + \int_{-\tau}^0 2\text{tr}(\dot{\lambda}_1^T(t, \eta)\tilde{\lambda}_1(t, \eta))d\eta \\
& + \tau\text{tr}(\dot{\theta}_1^T(t)\dot{\theta}_1(t)) - \int_{-\tau}^0 \text{tr}(\dot{\theta}_1^T(t+v)\dot{\theta}_1(t+v))dv \\
& + \tau\text{tr}(\dot{\Phi}^T(t)\dot{\Phi}(t)) - \int_{-\tau}^0 \text{tr}(\dot{\Phi}^T(t+v)\dot{\Phi}(t+v))dv \\
& + \tau \int_{-\tau}^0 \text{tr}(\dot{\lambda}_1^T(t, \eta)\dot{\lambda}_1(t, \eta))d\eta - \int_{-\tau}^0 \int_{-\tau}^0 \text{tr}(\dot{\lambda}_1^T(t+v, \eta)\dot{\lambda}_1(t+v, \eta))d\eta dv.
\end{aligned} \tag{E.3}$$

Using  $g(t) - g(t-\tau) = \int_{-\tau}^0 \dot{g}(t+v)dv$ , (E.3) can be rewritten as

$$\begin{aligned}
\dot{V}(t) = & -e^T(t)Qe(t) + 2\text{tr}\left(x_{hp}(t-\tau)e^T(t)PB_m\tilde{\theta}_1(t) + \dot{\theta}_1^T(t)\tilde{\theta}_1(t)\right) \\
& + 2\text{tr}\left(u(t-\tau)e^T(t)PB_m\Phi(t) + \dot{\Phi}^T(t)\Phi(t)\right) \\
& + \int_{-\tau}^0 2\text{tr}\left(u(t+\eta-\tau)e^T(t)PB_m\tilde{\lambda}_1(t, \eta) + \dot{\lambda}_1^T(t, \eta)\tilde{\lambda}_1(t, \eta)\right)d\eta \\
& - 2e^T(t)PB_m\left(\int_{-\tau}^0 \dot{\theta}_1(t+v)dv\right)x_{hp}(t-\tau) \\
& - 2e^T(t)PB_m\left(\int_{-\tau}^0 \dot{\Phi}(t+v)dv\right)u(t-\tau) \\
& - 2e^T(t)PB_m\left(\int_{-\tau}^0 \left(\int_{-\tau}^0 \dot{\lambda}_1(t+v, \eta)dv\right)u(t+\eta-\tau)d\eta\right) \\
& + \tau\text{tr}(\dot{\theta}_1^T(t)\dot{\theta}_1(t)) - \int_{-\tau}^0 \text{tr}(\dot{\theta}_1^T(t+v)\dot{\theta}_1(t+v))dv \\
& + \tau\text{tr}(\dot{\Phi}^T(t)\dot{\Phi}(t)) - \int_{-\tau}^0 \text{tr}(\dot{\Phi}^T(t+v)\dot{\Phi}(t+v))dv \\
& + \tau \int_{-\tau}^0 \text{tr}(\dot{\lambda}_1^T(t, \eta)\dot{\lambda}_1(t, \eta))d\eta - \int_{-\tau}^0 \int_{-\tau}^0 \text{tr}(\dot{\lambda}_1^T(t+v, \eta)\dot{\lambda}_1(t+v, \eta))d\eta dv.
\end{aligned} \tag{E.4}$$

By substituting (6.29)-(6.31) into (E.4), it is obtained that

$$\begin{aligned}
\dot{V}(t) &= -e^T(t)Qe(t) - 2 \int_{-\tau}^0 \text{tr}(x_{hp}(t-\tau)e(t)^T P B_m \dot{\theta}_1(t+v))dv \\
&\quad - 2 \int_{-\tau}^0 \text{tr}(u(t-\tau)e(t)^T P B_m \dot{\Phi}(t+v))dv \\
&\quad - 2 \int_{-\tau}^0 \int_{-\tau}^0 \text{tr}(u(t+\eta-\tau)e(t)^T P B_m \dot{\lambda}_1(t+v, \eta))dv d\eta \\
&\quad + \tau \text{tr}(\dot{\theta}_1^T(t)\dot{\theta}_1(t)) - \int_{-\tau}^0 \text{tr}(\dot{\theta}_1^T(t+v)\dot{\theta}_1(t+v))dv \\
&\quad + \tau \text{tr}(\dot{\Phi}^T(t)\dot{\Phi}(t)) - \int_{-\tau}^0 \text{tr}(\dot{\Phi}^T(t+v)\dot{\Phi}(t+v))dv \\
&\quad + \tau \int_{-\tau}^0 \text{tr}(\dot{\lambda}_1^T(t, \eta)\dot{\lambda}_1(t, \eta))d\eta - \int_{-\tau}^0 \int_{-\tau}^0 \text{tr}(\dot{\lambda}_1^T(t+v, \eta)\dot{\lambda}_1(t+v, \eta))d\eta dv \\
&= -e^T(t)Qe(t) + \int_{-\tau}^0 \text{tr}\left(2\dot{\theta}_1^T(t)\dot{\theta}_1(t+v) + \dot{\theta}_1^T(t)\dot{\theta}_1(t) - \dot{\theta}_1^T(t+v)\dot{\theta}_1(t+v)\right)dv \\
&\quad + \int_{-\tau}^0 \text{tr}\left(2\dot{\Phi}^T(t)\dot{\Phi}(t+v) + \dot{\Phi}^T(t)\dot{\Phi}(t) - \dot{\Phi}^T(t+v)\dot{\Phi}(t+v)\right)dv \\
&\quad + \int_{-\tau}^0 \int_{-\tau}^0 \text{tr}\left(2\dot{\lambda}_1^T(t, \eta)\dot{\lambda}_1(t+v, \eta) \right. \\
&\quad \left. + \dot{\lambda}_1^T(t, \eta)\dot{\lambda}_1(t, \eta) - \dot{\lambda}_1^T(t+v, \eta)\dot{\lambda}_1(t+v, \eta)\right)d\eta dv.
\end{aligned} \tag{E.5}$$

Using the trace property  $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$ , and the algebraic inequality  $a^2 \geq 2ab - b^2$  for two scalars  $a$  and  $b$ , it can be shown that  $\text{tr}(2A^T B + A^T A - B^T B) \leq 2\text{tr}(A^T A)$ . Using these inequalities, (E.5) can be rewritten as

$$\begin{aligned}
\dot{V}(t) &\leq -e^T(t)Qe(t) + \int_{-\tau}^0 2\text{tr}(\dot{\theta}_1^T(t)\dot{\theta}_1(t))dv \\
&\quad + \int_{-\tau}^0 2\text{tr}(\dot{\Phi}^T(t)\dot{\Phi}(t))dv \\
&\quad + \int_{-\tau}^0 \int_{-\tau}^0 2\text{tr}(\dot{\lambda}_1^T(t, \eta)\dot{\lambda}_1(t, \eta))d\eta dv.
\end{aligned} \tag{E.6}$$

By substituting (6.29)-(6.31) into (E.6), and using the trace operator property  $\text{tr}(AB) = \text{tr}(BA)$  for two square matrices  $A$  and  $B$ , (E.6) can be rewritten as

$$\begin{aligned}\dot{V}(t) &\leq -e^T(t)Qe(t) + 2\tau \text{tr}(e(t)x_{hp}^T(t-\tau)x_{hp}(t-\tau)e(t)^T P B_m B_m^T P) \\ &\quad + 2\tau \text{tr}(e(t)u^T(t-\tau)u(t-\tau)e(t)^T P B_m B_m^T P) \\ &\quad + 2\tau \int_{-\tau}^0 \text{tr}(e(t)u^T(t+\eta-\tau)u(t+\eta-\tau)e(t)^T P B_m B_m^T P) d\eta.\end{aligned}\tag{E.7}$$

Using  $\text{tr}(AB) \leq \text{tr}(A)\text{tr}(B)$  for two positive semidefinite matrices  $A$  and  $B$ , and  $\text{tr}(X^T X) = \|X\|_F^2$  for a matrix  $X$ , an upper bound for (E.7) can be derived as

$$\begin{aligned}\dot{V}(t) &\leq -e^T(t)Qe(t) + 2\tau \text{tr}(e(t)x_{hp}^T(t-\tau)x_{hp}(t-\tau)e(t)^T) \text{tr}(P B_m B_m^T P) \\ &\quad + 2\tau \text{tr}(e(t)u^T(t-\tau)u(t-\tau)e(t)^T) \text{tr}(P B_m B_m^T P) \\ &\quad + 2\tau \int_{-\tau}^0 \text{tr}(e(t)u^T(t-\tau+\eta)u(t-\tau+\eta)e(t)^T) \text{tr}(P B_m B_m^T P) d\eta \\ &\leq -\lambda_{\min}(Q)\|e(t)\|^2 + 2\tau \|x_{hp}(t-\tau)e(t)^T\|_F^2 \|B_m^T P\|_F^2 \\ &\quad + 2\tau \|u(t-\tau)e(t)^T\|_F^2 \|B_m^T P\|_F^2 + 2\tau \int_{-\tau}^0 \|u(t+\eta-\tau)e(t)^T\|_F^2 \|B_m^T P\|_F^2 d\eta \\ &\leq -\lambda_{\min}(Q)\|e(t)\|^2 + 2\tau \|x_{hp}(t-\tau)\|^2 \|e(t)\|^2 \|B_m^T P\|_F^2 \\ &\quad + 2\tau \|u(t-\tau)\|^2 \|e(t)\|^2 \|B_m^T P\|_F^2 + 2\tau \int_{-\tau}^0 \|u(t+\eta-\tau)\|^2 \|e(t)\|^2 \|B_m^T P\|_F^2 d\eta \\ &= \|B_m^T P\|_F^2 \|e(t)\|^2 \left( -\frac{\lambda_{\min}(Q)}{\|B_m^T P\|_F^2} + 2\tau (\|x_{hp}(t-\tau)\|^2 + \|u(t-\tau)\|^2 \right. \\ &\quad \left. + \int_{-\tau}^0 \|u(t+\eta-\tau)\|^2 d\eta) \right).\end{aligned}$$

Defining  $q \equiv \frac{\lambda_{\min}(Q)}{\|B_m^T P\|_F^2}$ , the inequality

$$q - 2\tau (\|x_{hp}(t-\tau)\|^2 + \|u(t-\tau)\|^2 + \int_{-\tau}^0 \|u(t+\eta-\tau)\|^2 d\eta) > 0.\tag{E.8}$$

needs to be satisfied for the non-positiveness of  $\dot{V}$ . Assuming that  $x_{hp}$  and  $u$  are bounded in the interval  $[t_0 - 2\tau, t_0)$ , the rest of the proof is divided into the following four steps:

**Step 1** In this step, the negative semi-definiteness of the Lyapunov-Krasovskii functional's (E.1) time derivative in the interval  $[t_0 - \tau, t_0)$  is shown which leads to

the boundedness of the the signals in this interval. In addition, an upper bound for  $u$  in the interval  $[t_0 - 2\tau, t_0)$  is given.

Suppose that

$$\begin{aligned} \sup_{\xi \in [t_0 - \tau, t_0)} \|x_{hp}(\xi)\|^2 &\leq \gamma_1 \\ \sup_{\xi \in [t_0 - 2\tau, t_0)} \|u(\xi)\|^2 &\leq \gamma_2 \end{aligned} \quad (\text{E.9})$$

for some positive  $\gamma_1, \gamma_2$ , and a  $\tau_1$  is given such that

$$2\tau_1(\gamma_1 + \gamma_2 + \tau_1\gamma_2) < q. \quad (\text{E.10})$$

Then the following inequality is satisfied:

$$\begin{aligned} q - 2\tau(\|x_{hp}(\xi - \tau)\|^2 + \|u(\xi - \tau)\|^2 + \int_{-\tau}^0 \|u(\xi + \eta - \tau)\|^2 d\eta) &> 0, \\ \forall \xi \in [t_0, t_0 + \tau), \forall \tau \in [0, \tau_1]. \end{aligned} \quad (\text{E.11})$$

It follows that  $V(t)$ , defined in (E.1), is non-increasing for  $t \in [t_0, t_0 + \tau)$ . Thus, we have

$$\lambda_{\min}(P)\|e(\xi)\|^2 \leq e(\xi)^T P e(\xi) \leq V(t_0), \quad (\text{E.12})$$

which leads to

$$\|x_{hp}(\xi)\| - \|x_m(\xi)\| \leq \|e(\xi)\| \leq \sqrt{\frac{V(t_0)}{\lambda_{\min}(P)}}. \quad (\text{E.13})$$

Then, we have

$$\|x_{hp}(\xi)\| \leq \sqrt{\frac{V(t_0)}{\lambda_{\min}(P)}} + \|x_m(\xi)\|, \quad (\text{E.14})$$

for  $\forall \xi \in [t_0, t_0 + \tau)$ . We also have the inequality

$$\begin{aligned} \|\Phi(\xi)\|^2 \leq V(t_0) &\implies \|K_r^{*-1} - K_r^{-1}(\xi)\|^2 \leq V(t_0) \\ &\implies \|K_r^{-1}(\xi)\| \leq \sqrt{V(t_0)} + \|K_r^{*-1}\|. \end{aligned} \quad (\text{E.15})$$

for  $\forall \xi \in [t_0, t_0 + \tau)$ . It is noted that the boundedness of  $\Phi = K_r^{*-1} - K_r^{-1}$  does not guarantee the boundedness of  $\tilde{K}_r$ . In order to guarantee the boundedness of  $\tilde{K}_r$  independent of the boundedness of  $\Phi$ , the projection algorithm [9] is employed as

$$\dot{K}_r = \text{Proj}(K_r, -K_r B_m^T P e(t) u^T(t - \tau) K_r), \quad (\text{E.16})$$

with an upper bound  $K_{max}$ , that is  $\|K_r\| \leq K_{max}$ . Thus, a lower bound for  $\|K_r^{-1}(\xi)\|$  can be calculated using the following algebraic manipulations:

$$\begin{aligned} K_r(\xi)K_r^{-1}(\xi) &= I \Rightarrow \|K_r(\xi)K_r^{-1}(\xi)\| = 1 \\ \Rightarrow 1 &\leq \|K_r(\xi)\| \|K_r^{-1}(\xi)\| \leq K_{max} \|K_r^{-1}(\xi)\| \\ \Rightarrow \frac{1}{K_{max}} &\leq \|K_r^{-1}(\xi)\|. \end{aligned} \quad (E.17)$$

Defining  $k_1 = \sqrt{V(t_0)} + \|K_r^{*-1}\|$ , and using (E.15), it is obtained that

$$\frac{1}{K_{max}} \leq \|K_r^{-1}(\xi)\| \leq k_1, \quad \xi \in [t_0, t_0 + \tau]. \quad (E.18)$$

Therefore,  $K_r$  is always bounded and  $K_r^{-1}(\xi)$  is bounded for  $\forall \xi \in [t_0, t_0 + \tau]$ .

Furthermore, using the definitions of  $\theta_x, \theta_1, \lambda, \lambda_1$  given in Theorem 17, and the non-increasing Lyapunov functional (E.1), it can be concluded that

$$\|\tilde{\theta}_1(\xi)\|_F^2 \leq V(t_0) \implies \|\tilde{K}_r^{-1}(\xi)\tilde{\theta}_x(\xi)\|_F^2 \leq V(t_0), \quad (E.19)$$

$$\int_{-\tau}^0 \|\tilde{\lambda}_1(\xi, \eta)\|_F^2 d\eta \leq V(t_0) \implies \int_{-\tau}^0 \|K_r^{-1}(\xi)\tilde{\lambda}(\xi, \eta)\|_F^2 d\eta \leq V(t_0), \quad (E.20)$$

for  $\forall \xi \in [t_0, t_0 + \tau]$ . Using (E.19) and (E.20), it can be obtained that

$$\begin{aligned} \|\tilde{\theta}_x(\xi)\|_F^2 &\leq K_{max}^2 V(t_0), \\ \int_{-\tau}^0 \|\tilde{\lambda}(\xi, \eta)\|_F^2 d\eta &\leq K_{max}^2 V(t_0). \end{aligned} \quad (E.21)$$

for  $\forall \xi \in [t_0, t_0 + \tau]$ .

To simplify the notation, we define

$$I_0 \equiv \max\left(\sqrt{\frac{V(t_0)}{\lambda_{min}(P)}} + \sup_{[t_0, t_0 + \tau]} \|x_m(\xi)\|, K_{max}\sqrt{V(t_0)}, K_{max}^2 V(t_0)\right), \quad (E.22)$$

where  $R_{max}$  is the upper bound of the reference input  $r(t)$ .

An upper bound on the control signal  $u(t)$  for  $t \in [t_0, t_0 + \tau]$  can be derived by using Lemma 13 and (6.15). In particular, setting  $t'_i = t_0$ ,  $t'_j = t_0 + \tau$ ,  $c_0^2 = V(t_0)$ , we obtain that

$$|u(\xi)| \leq 2\left(\bar{f} + \left(\int_{-\tau}^0 u^2(t_0 + \eta) d\eta\right)^{1/2} I_0\right) e^{I_0 \tau}, \quad (E.23)$$

for  $\forall \xi \in [t_0, t_0 + \tau)$ , where  $\bar{f}$ , which is the upper bound of  $\theta_x(t)x_{hp}(t) + K_r(t)r(t)$ , depends only on  $I_0$ . Defining  $g(\gamma_2, I_0, \tau) \equiv 2(\bar{f} + \gamma_2 I_0 \sqrt{\tau})e^{I_0 \tau}$ , (E.23) can be rewritten as

$$|u(\xi)| \leq g(\gamma_2, I_0, \tau), \quad \forall \xi \in [t_0, t_0 + \tau). \quad (\text{E.24})$$

The rest of the proof is similar to the one given in [134]. Below, a summary of the next steps are given.

**Step 2** A delay range  $[0, \tau_2]$  is found that satisfies the condition (E.8) over the interval  $[t_0, t_0 + 2\tau)$  as

$$2\tau_2 (I_0^2 + (\max(\gamma_2, g(\gamma_2, I_0, \tau_2)))^2 (1 + \tau_2)) < q, \quad (\text{E.25})$$

which leads to  $\|x_{hp}(\xi)\| < I_0, \forall \xi \in [t_0, t_0 + 2\tau), \forall \tau \in [0, \bar{\tau}_2], \bar{\tau}_2 = \min\{\tau_1, \tau_2\}$ .

**Step 3** It is shown in this step that the bound on  $u$  over the interval  $[t_0, t_0 + \tau)$  depends only on  $A_{hp}, B_{hp}, T$  and  $\tau$ , where  $T$  is a value between  $t_0$  and  $\tau$ . Denoting this upper bound as  $U(I_0)$ , we have  $|u(t)| \leq U(I_0), t \in [t_0, t_0 + \tau)$ .

**Step 4** Using the calculated upper bound for  $u$  in the previous step, a delay range  $[0, \tau_2]$  is found that satisfies the condition

$$2\tau_3 (I_0^2 + (\max(U(I_0), g(U(I_0), I_0, \tau_3)))^2 (1 + \tau_3)) < q. \quad (\text{E.26})$$

For  $\tau^* = \min(\bar{\tau}_2, \tau_3)$ ,  $\|x_{hp}(\xi)\| \leq I_0$  and  $|u(\xi)| \leq U(I_0)$  for all  $\xi \in [t_0, t_0 + \tau]$ ,  $\forall \tau \in [0, \tau^*]$ .

The above four steps show that  $x_{hp}(\xi)$  and  $u(\xi)$  are bounded for  $\forall \xi \in [t_0, t_0 + k\tau]$ , for  $k = 1$  and  $\tau \in (0, \tau^*]$ . By assuming that  $x_{hp}$  and  $u$  are bounded for a given  $k$ , the rest of the proof consists of showing that the boundedness of these variables hold for  $k + 1$ . Using this assumption and repeating steps 1-4, which leads to satisfying (E.26), we conclude that the Lyapunov function is non-increasing and  $\|x_{hp}(\xi)\| \leq I_0$ , and  $|u(\xi)| \leq g(U(I_0), I_0, \tau)$  for  $\xi \in [t_0, t_0 + (k + 1)\tau]$ ,  $\tau \leq \tau^* \leq \tau_3$ . This completes the boundedness proof using induction. Then, using Barbalat's Lemma, it can be shown that the error between the human-in-the-loop system output  $x_{hp}$  and the reference model output  $x_m$  converges to zero.