Cosmological Solutions of Some String Inspired Modified Theories of Gravity

by

Neslihan Oflaz

A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Doctor of Philosophy

in

Physics

Koç University

2015

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a doctoral dissertation by

Neslihan Oflaz

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Prof. Dr. Tekin Dereli

Prof. Dr. Ali Mostafazadeh

Prof. Dr. Tonguç Rador

Assoc. Prof. Dr. Kaan Güven

Assoc. Prof. Dr. Özgür Akarsu

Date:

" ...Now you put water into a cup, it becomes the cup, put it in a teapot, it becomes the teapot..."

Bruce Lee

ABSTRACT

Some modified theories of gravity in four and higher dimensions that are motivated by the low energy effective string field theories are considered. The corresponding variational field equations are derived using the method of Lagrange multipliers in the language of exterior differential forms over the space-time manifold. Cosmological solutions are discussed, paying special attention to the anisotropy and the late-time acceleration of the expansion of our three dimensional space. Comments on recent observational data are made.

ÖZETÇE

Bu tezde düşük enerjilerde etkin sicim alan kuramlarından esinlenen dört ve/veya daha yüksek boyutlarda tanımlı bazi kütleçekim teorileri incelenmektedir. Bu teorilerin varyasyonel alan deklemleri uzay-zaman manifoldu üzerindeki dış diferansiyel formlar cinsinden verilen bir Lagrange çarpanları yöntemiyle belirlenmektedir. Bu denklemlerin kozmolojik çözümleri, özellikle içinde bulunduğumuz üç boyutlu uzayın anizotropisi ve geç zamanlarında ivmelenerek genişlemesi üzerinde durularak incelenmektedir. En son gözlemsel veriler tartışılmaktadır.

ACKNOWLEDGMENTS

I would like to express my sincere gratitute to my advisor Prof. Dr. Tekin Dereli for his guidance and sharing his knowledge and expertise. His support, encouragement and invaluable advices helped me through my progress.

I would also like to thank Dr. Özgür Akarsu for his suggestions, readiness to help and kindness.

I acknowledge that my Ph.D. study was supported by the scholarship TÜBİTAK-BİDEB 2211.

I cannot go without mentioning my dear friends from Koç University, especially residents of SCI-131. I feel always lucky to have such lovely and colorful personalities around. Their existence has been always an important source of spiritual support.

Last but not the least, I would like to thank to my parents for their support, and for having faith in my decisions.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

Einstein's theory of general relativity (GR) has completed a century as a successful theory of gravity. Alongside its successes for being a tested gravitational theory of our solar system, it is also worthwhile to note that GR initiated the use of differential geometry in physics, opening up the way to new physical theories. However, due to inherent nature of science, it also needs to be renovated as the scales of our interest change. As we investigate the larger scaled structure of the universe, this need becomes more apparent. One of the shortcomings of Einstein's GR is that it cannot predict the observationally confirmed late time accelerated expansion of the universe [1, 2], given the observed matter distribution. Thus, the need arises to modify either GR or the matter distribution to get a theory consistent with cosmological observations.

The standard cosmological model based on GR is the ΛCDM model[3]. It assumes that the universe is described by GR on a spatially flat, isotropic and homogeneous spacetime whose metric is given by Robertson-Walker (RW) metric and the material content of the universe consists of a positive cosmological constant Λ together with distributions of cold dark matter (CDM) and baryonic matter. The latest observational data from the Planck Cosmic Microwave Background (CMB) experiment indicates that the ΛCDM model provides a reasonably well description for the observed features of the universe [4, 5, 6]. However, some of the observational data brings the assumptions of the ΛCDM model into question. Foremost, the isotropy of the metric assumptions need to be reconsidered due to anomalies in the cosmic microwave background (CMB), obtained first from the WMAP experiment [7] and confirmed also by Planck data [5, 6]. Also, the cosmological constant assumption may need to be amended due to some theoretical and observational problems. A well known problem is called the cosmological constant problem. Taken as a perfect fluid, the value of Λ differs from the theoretical value of the vacuum energy density significantly ($\sim$120 orders of magnitude) [8, 9]. Another problem is mentioned as the coincidence problem. As the universe expands the decrease in density of matter is inversely proportional to the volume of the universe, while the density related to the cosmological constant does not change. However, today nearly 13.8 billion years after Big Bang, these densities are nearly the same, which suggest that there should be an explanation for such a coincidence. Furthermore, observationally it seems that the equation of state parameter of dynamical dark energy models fits the data slightly better than a cosmological constant when taken into account in the analysis [4], [10].

The Brans-Dicke gravity is a well-known extension of GR that is obtained by coupling of a scalar field. Scalar fields are natural to consider not just because of their string origins but also because they can provide a dynamical model of dark energy[11], such as in the inflationary models. The inflationary models, driven by scalar fields, predict an isotropic universe. Prompted by anomalies in CMB that are discovered recently, several models inducing a small anisotropy are considered. A way of introducing anisotropy is to consider inflationary scenario driven by a vector field [12, 13, 14, 15, 16]. However, as it is shown in [17] these models may suffer from ghost instabilities. Another way is to keep inflation scenario as it is and introduce an anisotropic dark energy source like a vector field so that the late time universe accelerates anisotropically [14, 18, 19, 20, 21, 22, 23, 24, 25].

Motivated by the arguments above, we believe that the modified theories of gravity may provide some answers to the problems that the cosmology is facing. Lovelock's theorem states that the unique theory with second-order gravitational field equations derivable from an action containing the four dimensional metric tensor alone is the Einstein's GR with a cosmological constant. This indicates that we may modify Einstein's theory of GR in the following ways: (i) by adding extra fields, (ii) by accepting higher dimensions, (iii) by adding terms with higher derivatives of metric in the field equations. Indeed, it would be more preferable if these modifications arise from a unified theory of fundamental interactions of nature such as the string theories, or M-theory. There is wide literature of gravity theories motivated by the low energy effective string field theories [26, 27, 28, 29, 30, 31, 32, 33, 34]. As the universe we observe is 3-dimensional, the reductions of these theories also play an important role for physical interpretation. The standard Kaluza-Klein reduction of gravitational actions that include dimensionally continued Euler-Poincaré forms in dimensions higher than four dimensions has been analysed [35, 36, 37]. Their various cosmological solutions have also been studied [38, 39, 40, 41, 42, 43].With the relatively recent notion of dark energy the interest in the cosmological solutions was sparked again [44, 45, 46, 47, 48, 49, 50].

In this thesis, we study the cosmological implications of some gravity models with string theory inspired modifications. In chapter 2, we review and set the notation for some geometrical concepts that will be used throughout this thesis. Many concepts can be approached and defined in more than one way, usually depending on the generalization needed. Here, we try to take the most practical approach, since we use them as mathematical tools to work on our subject. If needed, details can be found in [51], [52], [53].

In chapter 3 we consider the Brans-Dicke theory of gravity coupled to a mass varying vector field [54]. The anisotropy is induced by the an electric field along a chosen direction. In order to discuss cosmological solutions, we take a spatially homogeneous and flat but not necessarily isotropic locally rotationally symmetric Bianchi I type space-time metric. The accelerated expansion of the universe in our model turns out not to be driven by the vector field contrary to other vector field models in the literature , but rather by the scalar field.

In chapter 4, we analyse a higher dimensional gravity that includes a second order Euler-Poincaré form[55]. Euler-Poincaré forms appear in the expansion of superstring/supergravity theories as mentioned in [30] and references therein. In fact, long before the string models, they were suggested as higher dimensional generalizations of Einstein's gravity, because in four dimensions they reduce to GR, while for higher dimensions they contribute to the field equation by tensors which are covariantly constant and contain at most second derivatives of the metric components [56, 57]. We describe the cosmological space-time geometry by a spatially homogeneous and flat but not necessarily isotropic $(1+3+n)$-dimensional synchronous space-time metric. We specify the matter content of the universe by a higher dimensional fluid with distinct pressures for external and internal dimensions which are in accordance with the anisotropy of the metric. The field equations can be further reduced by the assumptions that the dynamics of the external space in a higher dimensional steady state universe as that was characterized by [58, 59]: (i) The higher dimensional universe has a constant volume as a whole but both the internal and external spaces are dynamical. (ii) The energy density is constant in the higher dimensional universe. Introduction of a higher dimensional anisotropic fluid provides us with new degrees of freedom that in return allow us to deal with the dimensional dichotomy of our higher dimensional steady state universe without over-determining the system of field equations. We are interested in a more general case, where the second assumption is relaxed. We take the parameters of the higher dimensional fluid along the external and internal dimensions as dynamical but keep the difference between them constant.

In chapter 5, we take an action of a bosonic part of a six dimensional supergravity with a curvature square correction. An important feature of the model is its well-established M-theory origin [60]. Written in the string frame, the action contains a scalar field corresponding to the dilaton, an antisymmetric closed 3-form related to the axion in four dimensions, an R^2 term that includes also the so called Lorentz-Chern-Simons term, which is crucial for getting a supersymmetric theory [61]. We find an equivalent action on a space-time with torsion. The particular space-time torsion is imposed by the Palatini-type variations of the action using the method of Lagrange multipliers. We discuss equations of motions on (i) $AdS_3 \times S^3$ and (ii)warped $AdS_3 \times S^3$ backgrounds. We find a solution for the first case, and we only list the reduced field equations for the second case.

Chapter 2

PRELIMINARIES

2.1 Exterior Algebra of Differential Forms on a Smooth Manifold

Let M be a n-dimensional smooth manifold. We denote the tangent space of M at x by T_xM and its dual space called the cotangent space at x by T_x^*M. A basis of T_xM is called a frame and a basis of T_x^*M is called a coframe at x. Taking a local coordinate system $(x^1, ..., x^n)$ in a neighborhood of x, we get the standard basis or the so called coordinate basis of $T_x(M)$ given by the set of vectors $\{\partial_1, ..., \partial_n\}$ and the corresponding dual basis given by the co-vectors or basis 1-forms $\{dx^1, ..., dx^n\}$.

A disjoint union of tangent spaces of M,

$$\bigcup_{x\in M} \{(x, y)|y \in T_xM\}$$

determines the tangent bundle TM of M, with a natural projection $\pi : TM \to M$ mapping each tangent space T_xM to x. Similiarly, the cotangent bundle $T*M$ of M is the disjoint union of all cotangent spaces on M. A vector field is an assignment of each point $x \in M$ to an element of T_xM and a 1-form is similiarly an assignment of a point $x \in M$ to an element of T_x^*M. More precisely, TM and T^*M are vector bundles with base space M, projection map $\pi : TM(T^*M) \to M$ with fibers $\pi^{-1}(x) = T_xM(T_x^*M)$. We can define a vector field as a smooth section of TM and a 1-form is as a smooth section of T^*M.

A tensor at $x \in M$ of type (r, s) can be defined as a multilinear map

$$\underbrace{T_x^*(M) \times T_x^*(M) \times ... \times T_x^*(M)}_{r\ times} \times \underbrace{T_x(M) \times ... \times T_x(M)}_{s\ times} \to \mathbb{R}.$$

This tensor is said to be contravariant of degree r and covariant of degree s. Given a co-ordinate transformation provided by the transition functions $(x^1, ..., x^n) \to (\tilde{x}^1, ..., \tilde{x}^n)$, we have

$$\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j}$$

and

$$d\tilde{x} = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j.$$

Thus vectors transform covariantly, while the covectors transform contravariantly.

The tensor product of two tensors T and S of type (p, q) and (r, s), respectively, gives a tensor of type $(p + r, q + s)$ denoted by $T \otimes S$, where

$$T \otimes S(\omega_{i_1}, ..., \omega_{i_p}, \omega_{j_1}, ..., \omega_{j_s}; v_{i_1}, ..., v_{i_q}, v_{j_1}, ..., v_{j_s})$$
$$= T(\omega_{i_1}, ..., \omega_{i_p}; v_{i_1}, ..., v_{i_q}) S(\omega_{j_1}, ..., \omega_{j_s}; v_{j_1}, ..., v_{j_s})$$

The indexed ω are elements of T_x^*M and the indexed v are elements of T_xM.

A *tensor field* is a map that assigns each point $x \in M$ to a tensor defined at x. Given a coordinate system $(x^1, ..., x^n)$, a tensor field of type (r, s) can be locally written as

$$T = T^{i_1...i^r}_{j_1...j_s} \partial_{i_1} \otimes ... \otimes \partial_{i_r} \otimes dx^{j^1} \otimes ... \otimes dx^{j_s} \tag{2.1}$$

A *differential form* on M is a completely antisymmetric tensor field of type $(0, p)$ for some non-negative integer p. It is also called p-form, that can be written locally as

$$\omega = \frac{1}{p!} \omega_{i_1 i_2 ... i_p} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge e^{i_p} \tag{2.2}$$

or in terms of a noncoordinate basis $\{e^a\}$ as

$$\omega = \frac{1}{p!} \omega_{a_1 a_2 ... a_p} e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_p}. \tag{2.3}$$

A 0-form is by definition a smooth function on M. The *wedge product* of a p-form α and a q-form β can be defined as

$$(\alpha \wedge \beta)(X_1, ..., X_{p+q}) = \frac{1}{(p+q)!} \sum_{P \in S_{p+q}} sgn(P) \alpha(X_{P(1)} ... X_{P(p)}) \beta(X_{P(p+1)}, ..., X_{P(p+q)}) \tag{2.4}$$

where S_{p+q} is symmetric group of order $p+q$.

Suppose we denote the space of p-forms on M by $\Lambda^p M$. Then the space of all differential forms will be

$$\Lambda M := \Lambda^0 M \oplus \Lambda^1 M \oplus \Lambda^2 M \oplus \cdots \oplus \Lambda^n M.$$

Given $\alpha \in \Lambda^p M$, $\beta \in \Lambda^q M$ and $\gamma \in \Lambda^s M$, the wedge product satisfies the following properties:

(i) (*associativity*) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

(ii) (*graded commutativity*) $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

(iii) (*bilinearity*) $(a\alpha + b\beta) \wedge \gamma = a(\alpha \wedge \gamma) + b(\beta \wedge \gamma)$ for $a, b \in \mathbb{R}$.

Therefore, ΛM with the wedge product $\wedge$ is a graded algebra and this algebra is called the *exterior algebra* on M.

Exterior derivative of a differentiable p-form on M is a map $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ satisfying

(i) For a smooth function $f : M \to \mathbb{R}$, the differential df is a 1-form given in terms of local coordinates as $df = \frac{\partial f}{\partial x^i} dx^i$.

(ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ where α is a p-form and β is a q-form.

(iii) $d^2 \alpha = 0$.

Therefore, the exterior derivative of a p-form ω given in (2.2) in local coordinates we have

$$d\omega = \frac{1}{p!} \frac{\partial}{\partial x^a} \omega_{i_1 i_2 \ldots i_p} dx^a \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}. \tag{2.5}$$

Let $X \in TM$. The *interior product* with respect to X is a linear map i_X : $\Lambda^p(T^*M) \to \Lambda^{p-1}(T^*M)$ such that given a p-form ω and vector fields $X_1, ..., X_{p-1}$

$$i_X \omega(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_{p-1})$$

Using (2.2), in terms of local coordinates

$$i_X\omega = \frac{1}{p!}\sum_{k=1}^{p} X^{i_k}\omega_{i_1...i_k...i_p}dx^{i_1}... \wedge dx^{i_{k-1}} \wedge ...dx^{i_{k+1}} \wedge ... \wedge dx^{i_p}$$

The interior product satisfies

(i) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^p\alpha \wedge i_X\beta$ for a p-form α and a q-form β ,

(ii) $(i_X)^2\alpha = 0$.

Both the interior product and exterior derivative satisfy the properties of a graded derivation.

2.2 Metric, Connection and Torsion

A metric tensor g on a smooth manifold M is a symmetric, nondegenerate $(0,2)$ tensor field on M. In other words, it is a way of assigning to each point x, an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$. We say the metric has signature (p,q) when for all $x \in M$, dimension of maximal subspace W of T_xM where $g_x|_W$ is positive (negative) is $p(q)$. Furthermore, in a coordinate system at $x \in M$, the metric can be prescribed in terms of a real symmetric matrix $[g_{ij}](x)$ as $g = g_{ij}(x)dx^i \otimes dx^j$. A symmetric matrix $[g_{ij}]$ can be diagonalized. Using eigenvectors of $[g_{ij}]$, we can choose a suitable orthonormal basis for T_xM such that in terms of the dual basis, i.e. orthonormal coframe 1-forms $\{e^0, ..., e^n\}$, the metric can be written locally as

$$g = -e^0 \otimes e^0 - ... - e^p \otimes e^p + e^{p+1} + ... + e^{p+q}.$$

A smooth n-dimensional manifold equipped with a metric of signature (p,q) for some positive integers p,q such that $p+q=n$ is called a pseudo-Riemannian or semi-Riemannian manifold. A Riemannian manifold, has a metric with definite signature, either $(n,0)$ or $(0,n)$. A Lorentzian metric with signature $(n-1,1)$ is essential for the theory of relativity.

Any given metric tensor g (pointwise) induces a canonical isomorphism between the tangent space and its cotangent space. In a local coordinate system this corresponds to index raising/lowering using the metric components g_{ij} or the inverse matrix g^{ij}.

Another structure we need to put on a manifold for our work is a connection, which is a way to relate the vectors at different points of the manifold. *Connection* can be realized as a kind of differential operator ∇ called the covariant derivative. Let TM denote the space of smooth vector fields on M. *Covariant derivative* $\nabla : TM \times TM \to TM$ on M is a bilinear map ,where $\nabla(v, w)$ is denoted by $\nabla_v w$, satisfying

(i) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$

(ii) $\nabla_X(fY) = df(X)Y + f\nabla_X Y$

where $f, g \in C^\infty(M)$ are smooth functions on M and $X, X_i, Y \in TM$ are smooth vector fields on M. The above definition of covariant derivative can be extended such that ∇_v takes tensor fields to tensor fields. Further information can be found in [52].

On a pseudo-Riemannian manifold, we say that a connection is metric compatible if $\nabla g = 0$. The torsion of a connection defined ny

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is a type(1,2) tensor field. A connection is said to be torsion-free if

$$\nabla_X Y - \nabla_X Y = [X, Y]$$

The unique metric compatible torsion-free connection is called the Levi-Civita connection. Finally, the *curvature* R can be defined as an operator on the connection such that for $X, Y \in TM$, $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and can be expressed as a (3,1) type curvature tensor [51].

2.3 Cartan Formalism

In this thesis, we express our work using differential forms rather than ordinary tensors. Let S be a tensor field of type $p, q+r$, S can induce an r-form as follows

$$S^{a_1...a_q}{}_{b_1...b_p}(X_1, ..., X_r) = S(X_1, ..., X_r, X_{b_1}, ..., X_{b_q}, e^{a_1}, ..., e_{a_p})$$

where indexed X_a are frame elements and indexed e^b elements of the corresponding coframe elements.

We can employ the connection using a operator called *exterior covariant derivative* acting on the space of differential forms. Linearity of connection suggests that a connection can also be described entirely using its action on basis elements. Let $\{e_1, ..., e_n\}$ be a frame and X be a vector field in TM. The image of the connection can be written as a linear combination of frame elements with coefficients depending on X:

$$\nabla_X e_a = \omega^b{}_a(X) e_a$$

The 1-forms $\omega^b{}_a$ are called the *connection 1-forms*.

With the use of connection 1-forms $\{\omega^a{}_b\}$, the exterior covariant derivative of the above r-form can be expressed as

$$\begin{aligned} DS^{a_1..a_q}{}_{b_1..b_p} = dS^{a_1..a_q}{}_{b_1..b_p} &+ \omega^{a_1}_{\ a} \wedge S^{a..a_q}{}_{b_1..b_p} + ... + \omega^{a_q}_{\ a} \wedge S^{a_1..a}{}_{b_1..b_p} \\ &- \omega^b{}_{b_1} \wedge S^{a_1..a_q}{}_{b..b_p} - ... - \omega^b{}_{b_p} \wedge S^{a_1..a_q}{}_{b_1..b}. \end{aligned} \tag{2.6}$$

Cartan's Lemma: Let $\{e^a\}$ be a coframe on M. If there are 1-forms $\omega^a{}_b$ satisfying

$$\omega^a{}_b \wedge e^b = 0, \quad \omega_{ab} + \omega_{ba} = 0,$$

then $\omega^a{}_b = 0$.

Using Cartan's Lemma we can prove that metric compatible torsion free connection of 1-forms $\omega^a{}_b$ are determined uniquely by

$$de^a + \hat{\omega}^a{}_b \wedge e^b = 0 \ , \quad \hat{\omega}_{ab} + \hat{\omega}_{ba} = 0$$

and called the Levi-Civita connection 1-forms.

For an arbitrary metric compatible connection, we have the following Cartan structure equations:

$$de^a + \omega^a{}_b \wedge e^b = T^a, \tag{2.7}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b \tag{2.8}$$

that define the torsion and the curvature 2-forms, respectively. The above equations are the main equations we use to describe the geometry of space-time.

If the manifold M is a Lie group, then things get much simpler. We know that for any Lie group G the vector space of all left invariant vector fields on the Lie group is isomorphic to its tangent space at identity as a vector space, called the Lie algebra g of G. If we take a frame $\{X_a\}$ on M satisfying

$$[X_a, X_b] = C^c_{ab} e_c, \tag{2.9}$$

where C^c_{ab} are structure constants of the Lie algebra of M, then

$$de^a = -C^a_{bc} e^b \wedge e^c. \tag{2.10}$$

The above equation is called as *Cartan-Maurer equation* and valid for any Lie group.

We also introduce the Hodge star map and integration on a pseudo-Riemannian manifold. In order to understand these two operations we first need to specify the orientability of M. Orientability of a n-dimensional smooth manifold can be determined by the existence of a nowhere vanishing n-form called the volume form.

Let M be an orientable smooth pseudo-Riemannian manifold and the metric g have signature $(k, n-k)$. Given an orthonormal coframe $\{e^0, ..., e^n\}$ and a p-form ω, *Hodge star map* is a linear bijection $* : \Lambda^p(T^*M) \to \Lambda^{n-p}(T^*M)$ such that for a p-form ω we have

$$*\omega = \frac{1}{p!(n-p)!} \omega_{i_1 \ldots i_p} \epsilon^{i_1 \ldots i_p}{}_{i_{p+1} \ldots i_n} e^{p+1} \wedge \ldots \wedge e^n.$$

Here , the Levi-Civita symbol is given by

$$\epsilon_{i_1...i_n} = \begin{cases} +1 & \text{if } (i_1...i_n) \text{ is an even permutation} \\ -1 & \text{if } (i_1...i_n) \text{ is an odd permutation} \\ 0 & \text{if any two indices are the same} \end{cases} \tag{2.11}$$

The Hodge star map given above satisfies the following properties:

(i) $**\omega = (-1)^{k+p(n-p)}\omega$,

(ii) $*1 = \frac{1}{n!}\varepsilon_{abc...}e^a \wedge ...e^n$,

(iii) For any p-forms α and β , $\alpha \wedge *\beta = \beta \wedge *\alpha$.

Let M be an orientable smooth manifold and $\omega = f(p)dx^1 \wedge ... \wedge dx^n$ in a neighbourhood U of p. For the coordinate chart (U, φ) with $x = \varphi(p)$. Then for a smooth function g on M

$$\int_U g\omega = \int_{\varphi(U_i)} g(\varphi^{-1}(x))f(\varphi^{-1}(x))dx^1 \wedge ... \wedge dx^n \tag{2.12}$$

Let $\{U_i\}$ be a covering of M and $\{\epsilon_i\}$ be a partition of unity subordinate to U_i. Then we can define *integration* on this manifold as

$$\int_M g\omega = \sum_i \int_{U_i} g_i\omega \tag{2.13}$$

where $g_i(p) = \epsilon_i(p)g(p)$ for a point $p \in U_i$.

Stokes' Theorem: Let ω be an $(n-1)$-form on a smooth oriented n-dimensional manifold M such that the subset where ω is nonzero is a compact subset of M. If ∂M denotes the boundary of M then

$$\int_M d\omega = \oint_{\partial M} \omega.$$

2.4 Homogeneous Spaces

An action of a group G on a set X is a binary operation $G \times X \to X$ such that for all $x \in X$ satisfying the following:

(i) (*Compatibility*)$(gh)x = g(hx)$ $\forall g, h \in G$

(ii) (*Identity*) $1x = x$, where 1 denotes identity of G.

For $g, h \in G$ and $x \in X$, if $gx = hx$ implies $g = h$, then the group action is said to be free. We can define an equivalence relation on X using the group action of G. Two elements $x, y \in G$ are in the same equivalence class, $x \sim y$, iff there exists $g \in G$ such that $y = gx$. The equivalence class $Gx = \{gx : g \in G\}$ is called the orbit of x in G. If there is only one orbit at x, in other words if for any $x, y \in X$ there exists $g \in G$ such that $y = gx$ the the group action is called transitive. We say the action is effective, if for all $x \in X$ $gx = x$ implies that $g = 1$. One can easily show that for $x \in X$, the set $G_x = \{g \in G : gx = x\}$ is a subgroup of G. This subgroup is called the isotropy group(or stabilizer) of x. The isotropy groups of two elements of the space are conjugate if they are in the same orbit.

A homogeneous space is a manifold with a transitive group action by a Lie group G. We know that for a set X with a group action G, there is a one to one correspondence between G/G_x and an orbit Gx given by the map $gG_x \to gx$. Transitivity of the group implies that there is only one orbit which is the set itself and the isotropy groups of all points are conjugate subgroups. Hence, a homogeneous space X is 'isomorphic' to G/H where H is the isotropy group.

2.4.1 Maximally Symmetric Spaces

Maximally symmetric spaces are important examples of homogeneous spaces, which are widely used in physics, especially in cosmology.

Let M be a pseudo-Riemannian manifold and g be the metric defined on M. Isometry group $Iso(M, g)$ of M is a subgroup of all diffeomorphisms $\phi : M \to M$ which preserves the metric, i.e. $\phi^* g = g$ and ϕ is called local a isometry

If we think in terms of local coordinates, the diffeomorphisms are coordinate transformations $x \to y$. We can express the isometry condition as

$$g'_{ab} := (\phi^* g)_{ab} = \frac{\partial x^m}{\partial y^a} \frac{\partial x^n}{\partial y^b} g_{mn} \tag{2.14}$$

One can define a Killing vector as the generator of an infinitesimal isometry. More precisely, a Killing vector X is a vector field on M satisfying $L_X g = 0$, where L stands for the Lie derivative. For the definition of Lie derivative and details related to maximally symmetric spaces see [52].

M can admit at most $\frac{n(n+1)}{2}$ independent Killing vectors, where $n=$ dim M. If M admits $\frac{n(n+1)}{2}$ independent Killing vectors, then it is called a maximally symmetric space.

We say M is isotropic if for any point $x \in M$ and unit vectors $u, v \in T_x M$ there is an isometry $\phi : M \to M$ such that $\phi_*(u) = v$. If a space is isotropic at a point $x \in M$ then there exist $\frac{N(N-1)}{2}$ independent Killing vector fields fixing the point x.

For a pseudo-Riemannian manifold M the following are equivalent:

(i) M is maximally symmetric.

(ii) M is isotropic at all points of M.

(iii) M has a constant curvature.

(iv) M is homogeneous and isotropic about some point.

Note that the cosmological principle assumes that there is no preferred point or direction in our space, so it should be isotropic and homogeneous and the above proposition tells us the motivation of a physicist to study maximally symmetric spaces.

The homogeneity of maximally symmetric spaces implies there is another important aspect of the Killing vectors in a maximally symmetric space. The isometry group of a homogeneous space is a Lie group. Therefore, the corresponding Lie algebra is spanned by the Killing vectors.

Riemannian examples of maximally symmetric spaces are Euclidean spaces and spheres. Pseudo-Riemannian examples are Minkowski, de Sitter and anti de Sitter spaces.

Given a constant curvature K and the signature of the metric a maximally symmetric space can be uniquely determined.

While studying cosmology, we usually deal with the spaces whose spatial parts are maximally symmetric. Robertson-Walker(RW) spacetimes are good examples of this type of maximally symmetric subspaces. We can identify an $(n+1)$ dimensional spacetime by a metric given in terms of coordinates (t, x^i) by

$$g = -dt^2 + a^2(t)\frac{\delta_{ij}dx^i \otimes dx^j}{(1 + K/4\sum_i^n(x^i)^2)^2}$$

where $a(t)$ is a scalar function called the scale factor. K is related to the curvature of the maximally symmetric spatial part. The choices $K = -1, 0, 1$ give negative , zero and positive curvature spaces, respectively.

2.5 Euler-Poincaré Densities

Characteristic classes provide a measure of deviation of topological spaces from the local product structure from a global one. Their importance in gravitational theories arises from their relation to the notion of curvature.

We are especially interested in Euler classes. For a detailed explanation on characteristic classes and their relation to curvatures see appendix of [62]. For a finite cell complex X, its Euler characteristic is defined as

$$\chi(X) = \sum_k (-1)^k \alpha_k \tag{2.15}$$

where α_k is the number of k-cells. Euler characteristic of a manifold M can be related to the cohomology classes by the Euler-Poincaré formula

$$\chi(X) = \sum_k \beta_k. \tag{2.16}$$

where β_k is the kth Betti number that can be identified with the dimension of kth cohomology class $H^k(M)$. For the definition of cohomology classes and other details see [53].

Generalized Gauss-Bonnet theorem: Let M be an $2n$-dimensional oriented and compact manifold, then

$$\chi(M) = \int_M e(TM) \tag{2.17}$$

where $e(M)$ denotes the Euler class of forms $\in H^{2n}(M)$. Note that for odd dimensions the above equality trivially holds, since both sides would be equal to zero.

For 2-dimensional, 4-dimensional and in general 2n-dimensional manifolds

$$L^{(0)} = *1, \tag{2.18}$$

$$L^{(1)} = \frac{1}{2} R_{ab} \wedge *(e^a \wedge e^b), \tag{2.19}$$

$$L^{(2)} = \frac{1}{4} R_{ab} \wedge R_{cd} \wedge *(e^a \wedge e^b \wedge e^c \wedge e^d), \tag{2.20}$$

$$L^{(n)} \frac{1}{2^n} R_{a_1 a_2} \wedge ... \wedge R_{a_{2n-1} a_{2n}} \wedge *(e^{a_1} \wedge ... \wedge e^{a_{2n}}. \tag{2.21}$$

are elements of the corresponding Euler class and called the Euler-Poincaré forms. The zeroth order Euler-Poincaré form is set equal to the volume form and by definition gives the cosmological constant. The first order Euler-Poincaré form is linear in the curvature. In a 2-dimensional space the Ricci scalar curvature R is the Gaussian curvature whose integral over a 2-manifold M with boundary leads to the famous Gauss-Bonnet theorem. In $D = 4$ dimensions the integral of $L^{(1)}$ gives the Einstein-Hilbert action density that yields the Einstein field equations. The second order Euler-Poincaré form that is quadratic in the curvature components is a closed form in $D = 4$ dimensions and its integral over a 4-manifold M gives the topological Euler number of M. In $D < 4$, the second order Euler-Poincaré form vanishes, while for $D > 4$ its variations lead to second order dynamical field equations. The higher dimensional modified gravitational action obtained by the integration of the second order Euler-Poincaré form over M is sometimes called the Gauss-Bonnet gravity in current literature. Similarly, the n-th order Euler-Poincaré form is a closed form in $D = 2n$ dimensions and vanishes for $D < 2n$. As Lovelock shows in [56], the action obtained by the integration of the n-th order Euler-Poincaré form on a $D > 2n$ dimensional manifold contributes to the highly non-linear variational field equations only in second order.

Chapter 3

BRANS-DICKE GRAVITY COUPLED TO A MASS-VARYING VECTOR FIELD

3.1 Field equations

We derive the non-minimally coupled Brans-Dicke-vector field equations from the infinitesimal variations of the action density

$$\mathcal{L}[e,\phi,A] = \frac{\phi}{2}\mathcal{R}*1 - \frac{\omega}{2\phi}d\phi\wedge *d\phi - \frac{1}{2}F\wedge *F - \frac{m^2[\phi]}{2}A\wedge *A, \tag{3.1}$$

where A is the vector potential 1-form of Proca with $F = dA$. ω is the Brans-Dicke parameter and $m[\phi]$ is the variable vector boson mass that is given as a function of the scalar field ϕ. We find it convenient to re-express the action density in terms of a new scalar field $\alpha^2 = \phi$ and consider the variations of

$$\mathcal{L} = \frac{\alpha^2}{2}\mathcal{R}*1 - 2\omega d\alpha\wedge *d\alpha - \frac{1}{2}F\wedge *F - \frac{m^2[\alpha]}{2}A\wedge *A, \tag{3.2}$$

subject to the constraint that the space-time torsion vanishes. The co-frame variations of (eq3.2) give the Einstein field equations

$$-\frac{\alpha^2}{2}R^{bc}\wedge *(e_a\wedge e_b\wedge e_c) = D(\iota_a * d\alpha^2) + 4\omega\tau_a[\alpha] + \tau_a[F] + m^2[\alpha]\tau_a[A], \tag{3.3}$$

where the stress-energy 3-forms on the right hand side are given by

$$\tau_c[\alpha] = \frac{1}{2}\left(\iota_a d\alpha\wedge *d\alpha + d\alpha\wedge\iota_a * d\alpha\right), \tag{3.4}$$

$$\tau_c[F] = \frac{1}{2}\left(\iota_a F\wedge *F - F\wedge\iota_a * F\right), \tag{3.5}$$

$$\tau_a[A] = \frac{1}{2}\left(\iota_a A\wedge *A + A\wedge\iota_a * A\right), \tag{3.6}$$

respectively. Here ι_a's denote the interior product operators dual to the co-frames, that is, $\iota_a(e^b) = \delta^a_b$. The scalar field equation is given by

$$(\omega + \frac{3}{2})d * d\alpha^2 + (m^2 - \frac{\alpha}{2}\frac{dm^2}{d\alpha})A\wedge *A = 0, \tag{3.7}$$

while the variations of (3.2) with respect to A yield the vector field equation

$$d * F + m^2 * A = 0. \tag{3.8}$$

3.2 The cosmological ansatz

We start by assuming a time-dependent scalar field

$$\alpha = \alpha(t) \tag{3.9}$$

and a time-dependent, spatially homogeneous potential 1-form

$$A = \beta(t)dz. \tag{3.10}$$

Since $F = \dot{\beta}dt \wedge dz$ is an electric field along the z-direction, the rotational symmetry of the space is broken. Accordingly, we introduce two metric scale factors; $a(t)$ in the z-direction that is pointed by the vector field and $b(t)$ for the transverse x- and y-directions and work with a locally rotationally symmetric (LRS) Bianchi type-I metric tensor

$$g = -dt \otimes dt + a^2(t)dz \otimes dz + b^2(t)\left(dx \otimes dx + dy \otimes dy\right). \tag{3.11}$$

With the choice of an orthonormal co-frame as

$$e^0 = dt,\ e^1 = a(t)dz,\ e^2 = b(t)dx,\ e^3 = b(t)dy; \tag{3.12}$$

the Levi-Civita connection 1-forms $\{\omega^a_{\ b}\}$ can be solved from the first Cartan structure equations

$$de^a + \omega^a_{\ b} \wedge e^b = 0. \tag{3.13}$$

We find

$$\omega^0_{\ 1} = \frac{\dot{a}}{a}e^1, \quad \omega^0_{\ 2} = \frac{\dot{b}}{b}e^2, \quad \omega^0_{\ 3} = \frac{\dot{b}}{b}e^3, \quad \omega^1_{\ 2} = \omega^2_{\ 3} = \omega^1_{\ 3} = 0, \tag{3.14}$$

in terms of which the curvature 2-forms are calculated from the second Cartan structure equations

$$R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}. \tag{3.15}$$

The resulting curvature 2-forms are

$$R^0{}_1 = \frac{\ddot{a}}{a} e^0 \wedge e^1, \quad R^0{}_2 = \frac{\ddot{b}}{b} e^0 \wedge e^2, \quad R^0{}_3 = \frac{\ddot{b}}{b} e^0 \wedge e^3,$$
$$R^1{}_2 = \frac{\dot{a}\dot{b}}{ab} e^1 \wedge e^2, \quad R^2{}_3 = \frac{\dot{b}^2}{b^2} e^1 \wedge e^2, \quad R^1{}_3 = \frac{\dot{a}\dot{b}}{ab} e^1 \wedge e^3, \tag{3.16}$$

and the corresponding Ricci scalar is

$$\mathcal{R} = 2\left(\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + 2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2}\right). \tag{3.17}$$

The set of coupled ordinary differential equations that describes our cosmological model can now be obtained by substituting the above expressions into the field equations (3.3), (3.7) and (3.8). We get the following vector field equation

$$\frac{\ddot{\beta}}{\beta} + \left(-\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\frac{\dot{\beta}}{\beta} + m^2[\alpha] = 0, \tag{3.18}$$

and the scalar field equation

$$(2\omega + 3)\left[\frac{\ddot{\alpha}}{\alpha} + \left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\frac{\dot{\alpha}}{\alpha}\right] = \frac{\beta^2}{2a^2\alpha^2}\left(m^2 - \frac{\alpha}{2}\frac{\mathrm{d}m^2}{\mathrm{d}\alpha}\right). \tag{3.19}$$

Three other independent equations are provided by the Einstein field equations:

$$2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} - 2\omega\frac{\dot{\alpha}^2}{\alpha^2} + 2\frac{\dot{\alpha}}{\alpha}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) = \frac{\beta^2}{2\alpha^2 a^2}\left(\frac{\dot{\beta}^2}{\beta^2} + m^2\right), \tag{3.20}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + (2\omega + 2)\frac{\dot{\alpha}^2}{\alpha^2} + 2\frac{\ddot{\alpha}}{\alpha} + 4\frac{\dot{\alpha}}{\alpha}\frac{\dot{b}}{b} = \frac{\beta^2}{2\alpha^2 a^2}\left(\frac{\dot{\beta}^2}{\beta^2} - m^2\right), \tag{3.21}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + (2\omega + 2)\frac{\dot{\alpha}^2}{\alpha^2} + 2\frac{\ddot{\alpha}}{\alpha} + 2\frac{\dot{\alpha}}{\alpha}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) = \frac{\beta^2}{2\alpha^2 a^2}\left(-\frac{\dot{\beta}^2}{\beta^2} + m^2\right). \tag{3.22}$$

We thus end up with a system of five ordinary differential equations (3.18)-(3.22) to be solved. It can be checked that this system is fully determined, yet they are far too complicated for us to be able to write down a general analytic solution. However, we can construct a family of exact solutions assuming a power-law behavior in cosmic time t for all the variables involved. In the next section, we first discuss some physical parameters related with anisotropic cosmological models that can be handled without making use of any explicit solution. In the section that follows we present the exact power-law solution and then discuss the corresponding cosmological model in detail.

3.3 Physical parameters of the model

It would be convenient at this stage to introduce some cosmological parameters that we shall later use. Namely, we define the average scale factor v, the average Hubble parameter H and the deceleration parameter of the volumetric expansion, respectively, as follows:

$$v = \left(ab^2\right)^{\frac{1}{3}}, \quad H = \frac{1}{3}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right), \quad q = -\frac{v\ddot{v}}{\dot{v}^2} = -1 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right). \tag{3.23}$$

In a similar way, the directional Hubble parameters and the directional deceleration parameters along the x, y and z axes will be given:

$$H_x = H_y = \frac{\dot{b}}{b}, \quad H_z = \frac{\dot{a}}{a}, \quad q_x = q_y = -1 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H_x}\right), \quad q_z = -1 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H_z}\right). \tag{3.24}$$

The deceleration parameters are the key parameters among the others, because for any expanding scale factor (viz. v, a or b), the negative values of the corresponding deceleration parameter imply acceleration, positive values deceleration and the special values -1 and 0 correspond either to exponential expansion or the constant-rate expansion, respectively. Two further cosmological parameters relevant to the discussion of anisotropic cosmological models are the shear scalar

$$\sigma^2 = \frac{1}{2}\sum_{i=1}^{3}\left(H_i - H\right)^2 \quad , \tag{3.25}$$

and the expansion anisotropy parameter

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \tag{3.26}$$

where the sums are on $(1,2,3) = (x,y,z)$. Δ is a measure of the deviation from isotropic expansion; the universe expands isotropically for $\Delta = 0$. The time-evolution of Δ is crucial for deciding whether the universe approaches isotropy at some stage or not. In particular, the spatial section of the metric approaches isotropy for $v \to \infty$ and $\Delta \to 0$ as $t \to \infty$ [63]. In order to determine the shear scalar and the expansion anisotropy, we consider the difference between the expansion rates of the z- and the

x- or y-axes. Then subtracting (3.22) from (3.21) and re-organizing the resultant equation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(H_z - H_x) + 3H(H_z - H_x) + 2\frac{\dot{\alpha}}{\alpha}(H_z - H_x) = -\frac{1}{\alpha^2 a^2}\left(\dot{\beta}^2 - m^2\beta^2\right). \qquad (3.27)$$

The integration of this equation determines the difference between the expansion rates along the z and the x- or y-axes as follows:

$$H_z - H_x = \frac{1}{\alpha^2 a b^2}\left[\lambda - \int \frac{b^2}{a}\,(\dot{\beta}^2 - m^2\,\beta^2)\mathrm{d}t\right] \qquad (3.28)$$

where λ is an integration constant. Using (3.28) and the average Hubble parameter defined in (3.23), we obtain the shear scalar

$$\sigma^2 = \frac{1}{3\alpha^4 a^2 b^4}\left[\lambda - \int \frac{b^2}{a}\,(\dot{\beta}^2 - m^2\,\beta^2)\mathrm{d}t\right]^2, \qquad (3.29)$$

and the expansion anisotropy

$$\Delta = \frac{2}{9H^2\alpha^4 a^2 b^4}\left[\lambda - \int \frac{b^2}{a}\,(\dot{\beta}^2 - m^2\,\beta^2)\mathrm{d}t\right]^2 \qquad (3.30)$$

as well. We note that the difference between the expansion rates along the x and z axes and the square root of the shear scalar are both inversely proportional to the volume of the universe and the square of the scalar field. In the case of a model based on GR alone, the square root of the shear scalar would have been simply inversely proportional to the volume of the universe. Here the electric vector field also contributes to the shear scalar, hence to the isotropization history of the universe in a non-trivial way through the integral term in the above expression. For our massive vector field, given the energy density and the pressures along the x, y and z-axes

$$\rho = \frac{1}{2a^2}(\dot{\beta}^2 + m^2\beta^2) \quad , \quad p_x = p_y = -p_z = \frac{1}{2a^2}(\dot{\beta}^2 - m^2\beta^2), \qquad (3.31)$$

we immediately observe that the corresponding EoS parameters along the x, y and z-axes are

$$w_x = w_y = -w_z = \left(\frac{\dot{\beta}^2}{\beta^2} - m^2\right) \Big/ \left(\frac{\dot{\beta}^2}{\beta^2} + m^2\right). \qquad (3.32)$$

Some limiting cases are as follows: i) $\frac{\dot{\beta}^2}{\beta^2} >> m^2 \rightarrow w_x = w_y = -w_z \sim 1$, the EoS is similar to those of the (static) electric/magnetic fields and of cosmic strings; ii) $\frac{\dot{\beta}^2}{\beta^2} << m^2 \rightarrow w_x = w_y = -w_z \sim -1$, the EoS is similar to those of cosmic domain walls; iii) $\frac{\dot{\beta}^2}{\beta^2} \sim m^2 \rightarrow w_x = w_y = -w_z \sim 0$ the EoS is similar to that of a dust. Hence, the electric vector field in our model may not only describe anisotropic EoS different than that in the electromagnetic theory but also provides a dynamical anisotropic EoS parameter that can even become isotropic in some cases.

3.4 The exact power-law solution

In this section, we present a particular solution of the system for which all the variable parameters are of the power-law form with respect to the cosmic time t, and then discuss the properties of this solution. Let

$$a := a_1 t^{p_a}, \quad b := b_1 t^{p_b}, \quad \alpha := \alpha_1 t^{-s}, \quad \beta := \beta_1 t^{-u} \quad \text{and} \quad m := m_1 t^r \tag{3.33}$$

where a_1, b_1, α_1, β_1, m_1 are the values of the variables at time $t = 1$ and p_a, p_b, s, u, r are the powers of t to be determined. As a side remark we note that under these assumptions the mass of the vector field should be related to the scalar field with a power-law relation as

$$m[\alpha] = m_1 \left(\frac{\alpha}{\alpha_1}\right)^{-\frac{r}{s}}. \tag{3.34}$$

Substituting (3.33) into the system of differential equations (3.18)-(3.22), we immediately see that a consistent solution of the system requires

$$r = -1, \quad p_a = s - u, \quad p_b = \frac{1}{2} + \frac{s}{2} + \frac{m_1^2}{2u}, \tag{3.35}$$

making both sides of the vector field equation (3.18) vanish identically and leading to the expressions

$$a = a_1 t^{s-u}, \quad b = b_1 t^{\frac{1}{2}+\frac{s}{2}+\frac{m_1^2}{2u}}, \quad \alpha = \alpha_1 t^{-s}, \quad \beta = \beta_1 t^{-u}, \quad m = m_1 t^{-1}. \tag{3.36}$$

Now substituting these into the remaining equations (3.19)-(3.22), we get a set of four algebraic relations:

$$
\begin{aligned}
4(s-1)u-(11+8\omega)s^2-2s+1-4m_1^2-2m_1^2(s-1)u^{-1}+m_1^4u^{-2} &= 2\kappa^2(u^2+m_1^2), \\
(11+8\omega)s^2+2s-1-2m_1^2(s-1)u^{-1}+3m_1^4u^{-2} &= 2\kappa^2(u^2-m_1^2), \\
4u^2-2(s-1)u+(11+8\omega)s^2+2s-1-2m_1^2+m_1^4u^{-2} &= 2\kappa^2(m_1^2-u^2), \\
2s^2\,(2\omega+3)\,(u^2-m_1^2) &= \kappa^2 m_1^2 u(s-1),
\end{aligned}
$$

where we set $\kappa^2 = \frac{\beta_1^2}{a_1^2\alpha_1^2}$. A solution to this algebraic system, parametrized in terms of u and m_1, is given by

$$s = \frac{2u^4-u^3-m_1^2(u^2-2u)+2m_1^4}{u(-u^2+2m_1^2)} \tag{3.37}$$

provided

$$\kappa^2 = \frac{-2u^2+2m_1^2}{u^2-2m_1^2} \quad \text{and} \quad \omega = -\frac{3}{2}+\frac{1}{2}\frac{m_1^2u^2(2m_1^4-m_1^2u^2+2u^4)}{(2m_1^4-(u^2-2u)m_1^2+2u^4-u^3)^2}. \tag{3.38}$$

However, u and m_1 cannot take arbitrary values: $\beta_1 \neq 0$, a_1 and α_1 are real parameters so that the positivity of $\kappa^2 > 0$ imposes a restriction on the possible values of the real parameters u and $m_1 > 0$ according to

$$m_1 < |u| < \sqrt{2}m_1. \tag{3.39}$$

This restriction on the parameter values is important since it in turn implies a constraint on the dynamics of the model as we shall see below.

3.5 Discussion

We first of all summarize our power-law solution by explicitly giving the metric

$$g = -dt^2 + \frac{\beta_1^2}{\kappa^2\alpha_1^2}t^{2s-2u}dz^2 + t^{1+s+m_1^2/u}(dx^2+dy^2) \tag{3.40}$$

and the scalar and vector fields

$$\alpha = \alpha_1 t^{-s} \quad , \quad A = \beta_1 t^{-u} dz \tag{3.41}$$

where $s = \frac{2u^4 - u^3 - m_1^2(u^2-2u) + 2m_1^4}{u(-u^2+2m_1^2)}$. We have set $m[\alpha] = m_1(\frac{\alpha}{\alpha_1})^{-\frac{1}{s}}$.

It can be quickly verified that the average Hubble and the deceleration parameters of the universe turn out to be

$$H = \frac{5u^4 - 3u^3 - 5m_1^2u^2 + 6m_1^4u + 6m_1^4}{3u(2m_1^2 - u^2)}\, t^{-1}, \tag{3.42}$$

$$q = -1 + \frac{3u(2m_1^2 - u^2)}{5u^4 - 3u^3 - 5m_1^2u^2 + 6m_1^2u + 6m_1^4}. \tag{3.43}$$

It follows from (3.42) that the condition for getting an expanding universe $Ht > 0$ is satisfied only in case the vector potential β decreases as t increases, i.e. only for $u > 0$. This further reduces the inequality (3.39) as follows:

$$m_1 < u < \sqrt{2}m_1. \tag{3.44}$$

The conditions $u > 0$ and $m_1 > 0$ put the deceleration parameter in the range $-1 < q < 0$, which means that an expanding universe is necessarily be accelerating in our model. Using the inequalities in (3.44), we find that the deceleration parameter reaches its minimum value

$$q_{\min} \to -1 \quad \text{as} \quad u \to \sqrt{2}m_1 \tag{3.45}$$

and its maximum value

$$q_{\max} \to -\frac{2m_1}{2m_1+1} \quad \text{as} \quad u \to m_1. \tag{3.46}$$

We note that the larger m_1 becomes, the narrower is the range for the allowed values of the deceleration parameter. Namely, one may check that $q_{\max} \to q_{\min} \to -1$ for $m_1 >> 1$ so that only the exponential expansion is allowed in this limit.

In an anisotropic universe model we should also check the expansion rates along different spatial axes. We obtain the directional Hubble parameters as

$$H_x = H_y = \frac{u^4 - u^3 - m_1^2u^2 + 2m_1^2u + 2m_1^4}{u(2m_1^2 - u^2)}\, t^{-1}, \quad H_z = \frac{3u^4 - u^3 - 3m_1^2u^2 + 2m_1^2u + 2m_1^4}{u(2m_1^2 - u^2)}\, t^{-1}.$$

It is easily verified that the universe expands in all directions since both H_x and H_z are positive for $u > 0$ and $m_1 > 0$. The difference between the directional Hubble parameters will be*

$$H_z - H_x = -\frac{2m_1^2 - 2u^2}{2m_1^2 - u^2}\,\frac{u}{t} = \kappa^2\frac{u}{t}, \tag{3.47}$$

and hence the expansion rate of the universe along the z-axis is always greater than the expansion rate along the x- and y-axes (since $u > 0$). On the other hand, the expansion rates along different axes approach each other inversely in time t and become identical as $t \to \infty$. Accordingly the shear scalar should be decreasing with the square of the cosmic time t as follows:

$$\sigma^2 = \frac{4}{3}u^2\left(\frac{u^2 - m_1^2}{u^2 - 2m_1^2}\right)^2 t^{-2}. \tag{3.48}$$

On the other hand, since the average Hubble parameter also decreases inversely with the cosmic time t, the decay of the shear scalar leads neither to isotropization nor to anisotropization. In fact we have a constant expansion anisotropy

$$\Delta = \left(\frac{2\sqrt{2}u^2(u^2 - m_1^2)}{5u^4 - 3u^3 - 5m_1^2u^2 + 6m_1^2u + 6m_1^4}\right)^2 \tag{3.49}$$

that takes values in the range $0 < \Delta < \frac{1}{8}$ such that $\Delta \to 0$ as $u \to m_1$ while $\Delta \to \frac{1}{8}$ as $u \to \sqrt{2}m_1$.

In an accelerating anisotropic universe, not only the anisotropy of the expansion rate but also the anisotropy of the deceleration parameter is of interest. We obtain the directional deceleration parameters as follows:

$$q_x = q_y = -1+\frac{u(2m_1^2 - u^2)}{u^4 - u^3 - m_1^2u^2 + 2m_1^2u + 2m_1^4} \quad , \quad q_z = -1+\frac{u(2m_1^2 - u^2)}{3u^4 - u^3 - 3m_1^2u^2 + 2m_1^2u + 2m_1^4}. \tag{3.50}$$

The level of the anisotropy of the deceleration parameter can be quantified using the normalized difference according to

$$\frac{\Delta q}{\bar{q}} = 2\frac{q_z - q_x}{q_z + q_x} = \frac{-2u^3(2m_1^2 - u^2)(m_1^2 - u^2)}{4m_1^8 + 3u^8 - 2u^7 + 11m_1^4u^4 - 8m_1^6u^2 - 6m_1^2u^6 + 6m_1^2u^5 - 6m_1^4u^3 + 4m_1^6u}$$

*The expression that follows is consistent with (28) provided the integration constant $\lambda = 0$.

which again takes a constant value.

Regarding the dynamics of the scalar and vector fields, we first note that the Brans-Dicke coupling parameter ω can only take negative values of the order of -1 and exhibit some interesting limits:

$$u \to m_1 \Longrightarrow \omega \to -\frac{3}{2}\,\frac{8m_1^2+6m_1+1}{9m_1^2+6m_1+1} \Longrightarrow \begin{cases} \omega \to -\frac{3}{2} \quad \text{as} \quad m_1 \to 0 \\ \omega \to -\frac{4}{3} \quad \text{as} \quad m_1 \to \infty \end{cases} \tag{3.51}$$

and

$$u \to \sqrt{2}m_1 \Longrightarrow \omega \to -\frac{11}{8}. \tag{3.52}$$

The above limiting values of ω are remarkable: $-\frac{3}{2} < \omega < -\frac{4}{3}$ is the range for which accelerated power-law expansion exists within Brans-Dicke gravity in the presence of dust [64]. We also find it interesting to note that the negative values of ω of order of -1 can be motivated from the string theories. For instance, the case $\omega = -1$ appears in the low energy limit of effective string field theories and the case $\omega = -\frac{4}{3}$ appears in four-dimensional 0-brane space-times ($d = 1$) in string models with $(d-1)$-branes [65, 66].

Next we consider the energy density and the directional pressures of the vector field given by

$$\rho = \frac{\beta_1^2}{2a_1^2}(u^2+m_1^2)\,t^{-2s-2} \quad , \quad p_x = p_y = -p_z = \frac{\beta_1^2}{2a_1^2}(u^2-m_1^2)\,t^{-2s-2}. \tag{3.53}$$

Then the directional EoS parameters for the vector field will be constant:

$$w_x = w_y = -w_z = \frac{u^2-m_1^2}{u^2+m_1^2}. \tag{3.54}$$

Using the allowed range for the values of u with respect to m_1 (3.44), we find that the EoS can get values in the range

$$0 < w_x = w_y = -w_z < \frac{1}{3}. \tag{3.55}$$

It is interesting to note that for the choice $m_1 \sim u$, the above distribution would be indistinguishable from a dust but with a small anisotropic pressure in contrast to the conventional dust fluid that is described with an isotropic EoS parameter $w = 0$. Hence, in the present case, the vector field may be regarded as a dark matter fluid yielding a slightly anisotropic pressure. Moreover, defining the average EoS by $\tilde{w} = (w_x + w_y + w_z)/3$, we see that it only takes positive values $0 < \tilde{w} < \frac{1}{9}$. This is another indication that the vector field in our model cannot be interpreted as a possible dark energy source since a negative EoS parameter is characteristic of such sources.

Before we conclude, we wish to consider some observationally meaningful cosmological parameter values and evaluate other physically relevant quantities to check if our model gives consistent results or not. Using the mean deceleration parameter value as $q = -0.7$ in (3.43) and the anisotropy of the expansion as $\Delta = 10^{-4}$ in (3.49) [24], we find a solution with $m_1 = 1.119$ and $u = 1.136$ that satisfies (3.44). Using these two values in (3.54), we find the directional EoS parameters of the vector field to be given by $w_x = w_y = -w_z = 0.013$ and the average EoS $\tilde{w} = 0.004$. We note that such a fluid is indistinguishable from dust/CDM but yields a slightly anisotropic pressure. We find the values of the directional deceleration parameters as $q_x = q_y = -0.698$ along the x- and y-axes and $q_z = -0.704$ along the z-axis. Accordingly, the difference between the directional deceleration parameters along the z and x- or y-axes will be $q_z - q_x = -0.006$. These imply a level of the anisotropy of the deceleration parameter $\frac{\Delta q}{\bar{q}} = 0.017$ that corresponds to a % 1.7 level of difference. Finally the Brans-Dicke coupling parameter value is determined to be $\omega = -1.377$.

Chapter 4

HIGHER DIMENSIONAL MODIFIED GRAVITY WITH EULER-POINCARÉ TERMS

The most general gravitational Lagrangian density in D-dimensions that yields second order variational field equations can be written as a linear combination of all possible dimensionally continued Euler-Poincaré densities:

$$L_g = \sum_{n=0}^{[\frac{D-1}{2}]} \Lambda_n L^{(n)} \tag{4.1}$$

where we introduced (dimension-full) coupling constants Λ_n and the Euler-Poincaré forms are given by

$$L^{(n)} = \frac{1}{2^n} R_{a_1 b_1} \wedge \cdots \wedge R_{a_n b_n} \wedge *e^{a_1 b_1 \ldots a_n b_n}. \tag{4.2}$$

$R^a{}_b$ are the curvature 2-forms of (torsion-free) Levi-Civita connection 1-forms $\{\omega^a{}_b\}$. They are uniquely determined by a Lorentzian signatured metric tensor $g = \eta_{ab} e^a \otimes e^b$ where $\eta_{ab} = diag(-+++)$, given in terms of an orthonormal co-frame $\{e^a\}$, through the Cartan structure equations $de^a + \omega^a{}_b \wedge e^b = 0$. We further introduced the notation $e^{ab\ldots c} = e^a \wedge e^b \wedge \ldots \wedge e^c$ and the Hodge $*$-map that fixes the space-time orientation with the volume D-form $*1 = e^{012\ldots D-1}$.

Here we vary the total action (in $D \geq 5$ dimensions)

$$I[e, \omega, \mu, \ldots] = \int_M (L_g + L_c + \kappa L_m) \tag{4.3}$$

where we take a truncated gravitational Lagrangian density given by

$$L_g = \lambda_0 * 1 + \frac{1}{2} R_{ab} \wedge *e^{ab} + \frac{\lambda_2}{4} R_{ab} \wedge R_{cd} \wedge *e^{abcd}, \tag{4.4}$$

where λ_0 stands as a higher dimensional negative cosmological constant. The constraint Lagrangian

$$L_c = \left(de^a + \omega^a{}_b \wedge e^b\right) \wedge \mu_a \tag{4.5}$$

imposes the zero-torsion constraint on the connection through the variations of the Lagrange multiplier $(D-2)$-forms μ_a. $\kappa > 0$ denotes the Newton's constant that is scaled appropriately in D-dimensions. Then the co-frame variations of the action (4.3) give

$$\lambda_0 * e_a + \frac{1}{2} R^{bc} \wedge *e_{abc} + \frac{\lambda_2}{4} R^{bc} \wedge R^{df} \wedge *e_{abcdf} + D\mu_a + \kappa \tau_a(m) = 0, \tag{4.6}$$

while the connection variations of (4.3) give

$$* e_{abc} \wedge T^c + \frac{\lambda_2}{2} * e_{abcdf} \wedge R^{cd} \wedge T^f + \kappa \Sigma_{ab}(m) = e_a \wedge \mu_b - e_b \wedge \mu_a. \tag{4.7}$$

In these expressions we introduced through the variations of the matter Lagrangian, the stress-energy-momentum tensor $T_{ab}(m)$ and the angular momentum tensor $S_{ab,c}(m)$ to be read from

$$\left(\frac{\delta L_m}{\delta e^a}\right) = \tau_a(m) \equiv T_{ab}(m) * e^b \tag{4.8}$$

and

$$\left(\frac{\delta L_m}{\delta \omega^{ab}}\right) = \Sigma_{ab}(m) \equiv S_{ab,c}(m) * e^c, \tag{4.9}$$

respectively. The variations of the action with respect to the Lagrange multipliers impose the zero-torsion constraint $T^a = 0$ and the field equations (4.6) and (4.7) above are to be solved subject to this constraint. Then the connection-variation equations simplify and an algebraic solution for the Lagrange multiplier forms can be found as follows:

$$\mu_a = \kappa \left(\iota_{X_b} \Sigma_a{}^b - \frac{1}{2} e_a \wedge \iota_{X_c} \iota_{X_b} \Sigma^{bc} \right). \tag{4.10}$$

We use interior products such that $\iota_{X_a}(e^b) = \delta^b{}_a$ with respect to orthonormal frame vectors $\{X_a\}$. Then the Lagrange multipliers are substituted into the remaining field equations and we arrive at the final form of our modified Einstein field equations

$$\begin{aligned} \lambda_0 * e_a &+ \frac{1}{2} R^{bc} \wedge *e_{abc} + \frac{\lambda_2}{4} R^{bc} \wedge R^{df} \wedge *e_{abcdf} \\ &= -\kappa \tau_a(m) + \kappa D(\iota_{X_b} \Sigma_a{}^b) + \frac{\kappa}{2} e_a \wedge D(\iota_{X_b} \iota_{X_c} \Sigma^{bc}). \end{aligned} \tag{4.11}$$

As a last remark we note that the integrability of the Einstein field equations (4.11) yields the conservation law of matter:

$$D\tau_a(m) = R_{ab} \wedge \iota_{X_c}\Sigma^{cb}(m). \tag{4.12}$$

4.1 Cosmological Ansatz

The geometry of a cosmological space-time will be given by the metric

$$g = -dt^2 + R(t)^2 \sum_{i=1}^{3}(dx^i)^2 + S(t)^2 \sum_{\alpha=1}^{n}(dy^\alpha)^2 \tag{4.13}$$

where $\{x^i\}$ are the Cartesian coordinates of a flat(external) 3-space and $\{y^\alpha\}$ are the coordinates of a (locally) flat, compact (internal) n-space. $R(t)$ and $S(t)$ are the corresponding scale factors of cosmic time t. Let us choose the following orthonormal basis 1-forms

$$e^0 = dt, \quad e^i = R(t)dx^i, \quad e^\alpha = S(t)dy^\alpha, \tag{4.14}$$

so that the non-zero connection 1-forms are determined to be

$$\omega^0{}_i = \frac{\dot{R}}{R}e^i, \quad \omega^0{}_\alpha = \frac{\dot{S}}{S}e^\alpha, \tag{4.15}$$

where $\dot{} = \frac{d}{dt}$ denotes time derivative. Therefore the curvature 2-forms are given by

$$R^0{}_i = \left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) e^0 \wedge e^i \quad , \quad R^0{}_\alpha = \left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2}\right) e^0 \wedge e^\alpha,$$
$$R^i{}_j = \frac{\dot{R}^2}{R^2}e^i \wedge e^j \quad , \quad R^\alpha{}_\beta = \frac{\dot{S}^2}{S^2}e^\alpha \wedge e^\beta \quad , \quad R^i{}_\alpha = \frac{\dot{R}}{R}\frac{\dot{S}}{S}e^i \wedge e^\alpha. \tag{4.16}$$

We will model the matter distribution in a D-dimensional space-time by a homogeneous but anisotropic fluid with no rotation. Thus we take $\Sigma_{ab}(m) = 0$ and the stress-energy-momentum tensor of the fluid will be specified by

$$T = T_{ab}e^a \otimes e^b = \rho(t)dt^2 + p_{ext}(t) \sum_{i=1}^{3}(dx^i)^2 + p_{int}(t) \sum_{\alpha=1}^{n}(dy^\alpha)^2 \tag{4.17}$$

where we assume that the mass density $\rho > 0$, but the pressures $p_{ext}(t)$ and $p_{int}(t)$ are arbitrary except for the conservation law they should satisfy:

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p_{ext}) + n\frac{\dot{S}}{S}(\rho + p_{int}) = 0. \tag{4.18}$$

At this point, we skip calculational details and give the independent equations of motion we obtained as follows:

$$
\begin{aligned}
\lambda_0 &+ 3\left\{\frac{\dot{R}^2}{R^2}+3n\frac{\dot{R}}{R}\frac{\dot{S}}{S}+\frac{n(n-1)}{2}\frac{\dot{S}^2}{S^2}\right\}\\
&+ \lambda_2\left\{9n(n-1)\frac{\dot{R}^2}{R^2}\frac{\dot{S}^2}{S^2}\right.\\
&+ 6n\frac{\dot{R}^3}{R^3}\frac{\dot{S}}{S}+3n(n-1)(n-2)\frac{\dot{R}}{R}\frac{\dot{S}^3}{S^3}\\
&+ \left.\frac{n(n-1)(n-2)(n-3)}{4}\frac{\dot{S}^4}{S^4}\right\}=\kappa\rho,
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
\lambda_0 &+ \left\{2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+n\frac{\ddot{S}}{S}+2n\frac{\dot{R}}{R}\frac{\dot{S}}{S}+\frac{n(n-1)}{2}\frac{\dot{S}^2}{S^2}\right\}\\
&+ \lambda_2\left\{\frac{\ddot{R}}{R}\left(2n(n-1)\frac{\dot{S}^2}{S^2}+4n\frac{\dot{R}}{R}\frac{\dot{S}}{S}\right)\right.\\
&+ \frac{\ddot{S}}{S}\left(2n\frac{\dot{R}^2}{R^2}+4n(n-1)\frac{\dot{R}}{R}\frac{\dot{S}}{S}+n(n-1)(n-2)\frac{\dot{S}^2}{S^2}\right)\\
&+3n(n-1)\frac{\dot{R}^2}{R^2}\frac{\dot{S}^2}{S^2}+2n(n-1)(n-2)\frac{\dot{R}}{R}\frac{\dot{S}^3}{S^3}\\
&+ \left.\frac{n(n-1)(n-2)(n-3)}{4}\frac{\dot{S}^4}{S^4}\right\}=-\kappa p_{ext},
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
\lambda_0 &+ \left\{3\frac{\ddot{R}}{R}+(n-1)\frac{\ddot{S}}{S}+3(n-1)\frac{\dot{R}}{R}\frac{\dot{S}}{S}+3\frac{\dot{R}^2}{R^2}+\frac{(n-1)(n-2)}{2}\frac{\dot{S}^2}{S^2}\right\}\\
&+ \lambda_2\left\{\frac{\ddot{R}}{R}\left(6(\frac{\dot{R}}{R})^2+12(n-1)\frac{\dot{R}}{R}\frac{\dot{S}}{S}+3(n-1)(n-2)\frac{\dot{S}^2}{S^2}\right)\right.\\
&+ \frac{\ddot{S}}{S}\left(6(n-1)\frac{\dot{R}^2}{R^2}+6(n-1)(n-2)\frac{\dot{R}}{R}\frac{\dot{S}}{S}+(n-1)(n-2)(n-3)\frac{\dot{S}^2}{S^2}\right)\\
&+9(n-1)(n-2)\frac{\dot{R}^2}{R^2}\frac{\dot{S}^2}{S^2}+6(n-1)\frac{\dot{R}^3}{R^3}\frac{\dot{S}}{S}\\
&+3(n-1)(n-2)(n-3)\frac{\dot{R}}{R}\frac{\dot{S}^3}{S^3}\\
&+ \left.\frac{(n-1)(n-2)(n-3)(n-4)}{4}\frac{\dot{S}^4}{S^4}\right\}=-\kappa p_{int}.
\end{aligned}
\tag{4.21}
$$

The coupled equations (4.19), (4.20) and (4.21) are to be satisfied by five functions R, S, ρ, p_{ext} and p_{int} and therefore the system is not fully determined as it is. Two extra constraints must be provided by some extra assumptions in order to get fully determined solutions.

4.2 Exact Solutions with Constant Volume

We are particularly interested in higher dimensional steady state cosmologies characterized by the following two properties: (i) The higher dimensional universe has a constant volume as a whole but the internal and external spaces are dynamical. (ii) The mass density is constant in the higher dimensional universe. However, in this study we are going to allow the energy to be dynamical in a particular way, so that a constant mass density can be studied as a particular case. Therefore, firstly we assume that the $(3+n)$-dimensional *volume scale factor* of the universe is constant:

$$R^3 S^n = V_0. \tag{4.22}$$

It will be convenient to introduce the Hubble functions of external and internal spaces given by $H = \frac{\dot{R}}{R}$ and $H' = \frac{\dot{S}}{S}$, respectively. Then it follows from (4.22) that $3H = -nH'$, assuring the dynamical contraction (hence reduction) of the internal space ($H' < 0$) for an expanding external space ($H > 0$). The field equations in terms of H read:

$$\lambda_0 - \frac{3n+9}{2n} H^2 + 3\lambda_2 b_n H^4 = \kappa\rho, \tag{4.23}$$

$$\lambda_0 - \dot{H} + \frac{3n+9}{2n} H^2 + \lambda_2 H^2 \left(a_n \dot{H} - b_n H^2 \right) = -\kappa p_{ext}, \tag{4.24}$$

$$\lambda_0 + \frac{3}{n}\dot{H} + \frac{3n+9}{2n} H^2 + \lambda_2 H^2 \left((a_n - 4b_n)\dot{H} - b_n H^2 \right) = -\kappa p_{int} \tag{4.25}$$

where we set*

$$a_n = \frac{9\,(n^2+3n-6)}{n^2}, \quad b_n = \frac{3(n+3)(n^2+15n-18)}{4n^3}. \tag{4.26}$$

*Note that a_n and b_n are both negative for $n = 1$ and both positive for all $n \geq 2$. For the sake of simplicity we will keep the second case in what follows.

Furthermore our conservation law of matter reduces to

$$\dot{\rho} + 3H\left(p_{ext} - p_{int}\right) = 0. \tag{4.27}$$

Secondly we assume a simple equation of state

$$\frac{p_{ext} - p_{int}}{\rho} = \Delta w \tag{4.28}$$

where Δw is a constant that we propose to call dimensional dichotomy parameter. It follows that $\rho \propto R^{-3\Delta w}$ in general. A higher dimensional isotropic distribution of matter with constant mass density follows in the case $\Delta w = 0$. If there is a dimensional dichotomy i.e. $\Delta w \neq 0$, on the other hand, then, as the external space expands, the mass density decreases if $p_{ext} > p_{int}$ and increases if $p_{ext} < p_{int}$.

In order to solve this system we proceed in the following way. We differentiate both sides of (4.23) and on the RHS use the conservation relation (4.27) together with our equation of state (4.28) to replace $\dot{\rho}$. Then we use the first equation (4.27) again to eliminate ρ itself, arriving at a first order master equation satisfied by H. We will solve this master equation and classify its solutions according to the signs and magnitudes of the cosmological coupling constants λ_0 and λ_2. Once an expression for H is fixed, the external and internal pressure functions can be found by substituting this H into the equations (4.24) and (4.25), respectively. Finally we integrate H and determine the scale factors R and S, thus completely specifying the cosmology.

From now on we will be looking at the master equation for H that reads

$$\begin{aligned}\left[\left(\frac{3n+9}{2n}\right)H^2\right. &- \left.\frac{n(n+3)}{2(n^2+15n-18)}\frac{1}{\lambda_2}\right]\left(\frac{3n+9}{2n}\right)\dot{H} \\ &= -\frac{3\Delta w}{4}\left[\left(\frac{3n+9}{2n}\right)^2 H^4 - \frac{n(n+3)}{(n^2+15n-18)}\frac{1}{\lambda_2}\left(\frac{3n+9}{2n}\right)H^2\right. \\ &+ \left.\frac{n(n+3)}{(n^2+15n-18)}\frac{\lambda_0}{\lambda_2}\right].\end{aligned} \tag{4.29}$$

We re-write it as a simple differential equation

$$\dot{x} = \omega\left[(x^2 - \lambda) - \frac{Q}{(x^2 - \lambda)}\right] \tag{4.30}$$

where we defined

$$x \equiv \sqrt{\frac{3n+9}{2n}}H, \quad \omega \equiv -\sqrt{\frac{2n}{3n+9}}\frac{3\Delta w}{4} \tag{4.31}$$

with two free parameters

$$\lambda \equiv \frac{n(n+3)}{(n^2+15n-18)}\frac{1}{2\lambda_2}, \quad Q \equiv \lambda^2 - 2\lambda_0\lambda. \tag{4.32}$$

It is worth to note that the field equation (4.23) with the above re-parametrization gives for the energy density

$$\kappa\rho = \frac{1}{2\lambda}\left[(x^2-\lambda)^2 - Q\right]. \tag{4.33}$$

We first consider briefly the case of an isotropic distribution of matter. Suppose $\lambda \neq 0$. We take $\omega = 0$ and the master equation becomes $\dot{x} = 0$. This leads to an exponentially expanding external scale factor $R(t) = R(0)e^{H_0 t}$ and an exponentially contracting internal scale factor $S(t) = S(0)e^{-\frac{3}{n}H_0 t}$. For isotropic solutions with $\lambda \neq 0$, the energy density may be positive definite or negative definite or zero depending on the roots of the algebraic equation

$$\lambda_0 - \frac{3n+9}{2n}H_0^2 + \frac{1}{2\lambda}\left(\frac{3n+9}{2n}\right)^2 H_0^4 = 0.$$

Next we consider cases with $\omega \neq 0$ and $\lambda \neq 0$:

1. Suppose $Q \neq 0$. The generic solution is implicitly given by

$$\left|\frac{\sqrt{\lambda+\sqrt{Q}}-x}{\sqrt{\lambda+\sqrt{Q}}+x}\right|^{\frac{1}{2\sqrt{\lambda+\sqrt{Q}}}} \left|\frac{\sqrt{\lambda-\sqrt{Q}}-x}{\sqrt{\lambda-\sqrt{Q}}+x}\right|^{\frac{1}{2\sqrt{\lambda-\sqrt{Q}}}} = Ce^{2\omega t} \tag{4.34}$$

where C is an integration constant. We cannot further develop such solutions and obtain explicit expression for $R(t)$ and $S(t)$ in general.

2. Suppose $Q = 0$, that is $\lambda = 2\lambda_0$. Now the generic solution reads

$$\left|\frac{\sqrt{\lambda}-x}{\sqrt{\lambda}+x}\right|^{\frac{1}{\sqrt{\lambda}}} = Ce^{2\omega t}, \tag{4.35}$$

and can be inverted to write down explicit expressions. For $\lambda > 0$ we have

$$x(t) = \begin{cases} -\sqrt{\lambda}\coth(\sqrt{\lambda}\omega t), & x^2 > \lambda \\ -\sqrt{\lambda}\tanh(\sqrt{\lambda}\omega t), & x^2 < \lambda \end{cases}. \tag{4.36}$$

These expressions can be integrated for the scale factor of the external space so that

$$R(t) = \begin{cases} |\sinh(\sqrt{\lambda}\omega t)|^{4/(3\Delta w)}, & x^2 > \lambda \\ |\cosh(\sqrt{\lambda}\omega t)|^{4/(3\Delta w)}, & x^2 < \lambda \end{cases}. \tag{4.37}$$

The corresponding mass density function turns out to be positive definite:

$$\kappa\rho(t) = \begin{cases} \frac{\lambda}{2}\sinh^{-4}(\sqrt{\lambda}\omega t), & x^2 > \lambda \\ \frac{\lambda}{2}\cosh^{-4}(\sqrt{\lambda}\omega t), & x^2 < \lambda \end{cases}. \tag{4.38}$$

For $\lambda < 0$ we have

$$x(t) = -\sqrt{|\lambda|}\cot(\sqrt{|\lambda|}\omega t). \tag{4.39}$$

Again we integrate the above expression and find that

$$R(t) = |\sin(\sqrt{|\lambda|}\omega t)|^{4/(3\Delta w)}. \tag{4.40}$$

The corresponding mass density function for this solution is negative definite:

$$\kappa\rho(t) = -\frac{|\lambda|}{2}\sin^{-4}(\sqrt{\lambda}\omega t). \tag{4.41}$$

We also wish to point out some special classes of solutions which cannot be obtained from above as limiting cases.

A. First consider the case of Einstein-Hilbert action with a cosmological constant where the Euler-Poincaré term is absent. We set $\lambda_2 = 0$ accordingly but still keep $\omega \neq 0$. Then the master equation reads

$$\dot{x} = 2\omega\left(x^2 - \lambda_0\right) \tag{4.42}$$

For $\lambda_0 > 0$ the solutions are given by

$$x(t) = \begin{cases} -\sqrt{\lambda_0}\coth(2\sqrt{\lambda_0}\omega t), & x^2 > \lambda_0 \\ -\sqrt{\lambda_0}\tanh(2\sqrt{\lambda_0}\omega t), & x^2 < \lambda_0 \end{cases}. \tag{4.43}$$

The external scale factor can be calculated as

$$R(t) = \begin{cases} |\cosh(2\sqrt{\lambda_0}\omega t)|^{2/(3\Delta w)}, & x^2 > \lambda_0 \\ |\sinh(2\sqrt{\lambda_0}\omega t)|^{2/(3\Delta w)}, & x^2 < \lambda_0 \end{cases}, \tag{4.44}$$

which give a positive definite the energy density for $0 < \lambda_0 < x^2$

$$\kappa\rho(t) = \frac{\lambda_0}{\cosh^2(2\sqrt{\lambda_0}\omega t)}, \tag{4.45}$$

and a negative definite energy density for $x^2 < \lambda_0$

$$\kappa\rho(t) = -\frac{\lambda_0}{\sinh^2(2\sqrt{\lambda_0}\omega t)}. \tag{4.46}$$

For $\lambda_0 < 0$ we have

$$x(t) = \sqrt{|\lambda_0|}\tan(2\sqrt{|\lambda_0|}\omega t) \tag{4.47}$$

and

$$R(t) = |\cos(2\sqrt{|\lambda_0|}\omega t)|^{2/(3\Delta w)}. \tag{4.48}$$

In this case, the corresponding energy density is always negative definite:

$$\kappa\rho = -\frac{|\lambda_0|}{\cos^2(2\sqrt{|\lambda_0|}\omega t)}. \tag{4.49}$$

For the particular case $\lambda_0 = 0$ we have the following external scale factor

$$R(t) \propto t^{2/(3\Delta w)}, \tag{4.50}$$

which leads to the following negative definite mass density

$$\kappa\rho = -\frac{1}{4\omega^2 t^2}. \tag{4.51}$$

It is easy to see that, in this particular case $\lambda_0 = 0$, the external space exhibits accelerating expansion provided that $0 < \Delta w < \frac{2}{3}$. However, we note that it suffers from the negativity of mass density.

B. Suppose the gravitational action consists of just the second order Euler-Poincaré term on its own. This theory is sometimes called the Gauss-Bonnet gravity. Let us consider the cases where $\omega \neq 0$. Then the master equation becomes

$$\dot{x} = \omega x^2, \tag{4.52}$$

which gives the following a power-law expansion

$$R(t) \propto t^{4/(3\Delta w)}, \tag{4.53}$$

and the following mass density

$$\rho = \frac{1}{2\lambda\omega^4}\frac{1}{t^4}. \tag{4.54}$$

It is easy to see that in this solution the external exhibits accelerating expansion if $0 < \Delta w < \frac{3}{4}$. Additionally the mass density takes positive values properly provided that λ is positive, i.e., λ_2 is positive in line with string theories.

Among the explicit solutions we presented above, the case for which the external space expands as $R \propto |\sinh(\sqrt{\lambda}\omega t)|^{4/(3\Delta w)}$ given in (4.37) is the most promising solution from the cosmological point of view. We note that in this solution the external space behaves as $R \sim t^{4/(3\Delta w)}$ for $t \sim 0$ and $R \sim e^{\frac{4\sqrt{\lambda}\omega}{3\Delta w}t}$ for $t \gg \frac{1}{\sqrt{\lambda}\omega}$. Accordingly, provided that $\Delta w > 3/4$, the external space evolves from a decelerating expansion phase to the de Sitter expansion as the cosmic time t runs. Moreover, if we choose $\Delta w = 2$ then the external space exhibits exactly the same behavior with the standard ΛCDM model [67], namely, $R_{\Lambda\mathrm{CDM}} \propto \sinh^{\frac{2}{3}}\left(\sqrt{\frac{3\Lambda}{4}}\,t\right)$, where Λ is the conventional positive cosmological constant. We note that similar results could be reached in the solution (4.44) where the second order Euler-Poincaré term is omitted. However, we notice an important difference between these two solutions when we consider the corresponding matter densities. The solution (4.44) for which the second order Euler-Poincaré term is omitted suffers from negativity of mass density (4.46), while the solution (4.37) where all the Einstein-Hilbert and the second order Euler-Poincaré terms are considered leads properly to positive definite mass density (4.38).

Chapter 5

SIX DIMENSIONAL R^2-GRAVITY INDUCED FROM A GAUGED CHIRAL SUPERGRAVITY

We start with the following action density:

$$\begin{aligned} \mathcal{L} &= e^{-2\phi}\left[R(\hat{\omega}) * 1 + 4d\phi \wedge *d\phi - \frac{1}{2}H \wedge *H + e^{2\phi}V(\phi) * 1\right] \\ &+ \frac{1}{2}\sigma\left[R_{ab}(\omega) \wedge *R^{ab}(\omega) + R_{ab}(\omega) \wedge R^{ab}(\omega) \wedge B\right] \end{aligned} \tag{5.1}$$

given in six dimensions with an R^2 invariant [68]. The above action density consists of two parts. The first part defines the low energy axi-dilatonic sector of the heterotic string field theory in the string frame. It contains the usual Einstein-Hilbert action density, kinetic term for 0-form dilaton field ϕ, dilaton potential $V(\phi)$ that generalizes a cosmological constant Λ in such a way $V(0) = \Lambda$ and an antisymmetric 3-form field H. In four dimensions H is related to the axion scalar $\mathcal{A}$ by the relation $d\mathcal{A} = e^{-2\phi} * H$ [69]. We want to note again that the first part is written on the string frame. By conformal redefinitions of the metric and the corresponding connection; we may rewrite the related Lagrangian density 6-form in either i) the Einstein frame or ii) the Jordan (Brans-Dicke) frame. Eventhough such "frames" are conformally-scale related, the physics they describe are distinct.

The second part consists of the quadratic curvature term $R^{ab} \wedge *R_{ab}$ together with a Lorentz-Chern-Simons term $R_{ab} \wedge R^{ab} \wedge B$ whose supersymmetrization is done in [61]. B is the potential of the antisymmetric 3-form field H such that $H = dB$. Here, $\{e^a\}_{a=0,1,..,5}$ are the coframe 1-forms in terms of which the metric can be written as $g = \eta_{ab}e^a \otimes e^b$, where $\eta_{ab} = diag(-1, 1, .., 1)$. The volume form follows as $*1 = e^0 \wedge e^1 \wedge ... \wedge e^5$. As it is seen from the above Lagrangian density, there are two different kinds of "ω" introduced in the action. $\hat{\omega}_{ab}$ denotes the Levi-Civita connection 1-forms, for which the corresponding torsion 2-forms are all zero. ω_{ab} are the connection 1-forms used to write the effective curvature term above:

$$\omega_{ab} := \omega_{\pm ab} = \hat{\omega}_{ab} \pm \frac{1}{2} i_b i_a H. \tag{5.2}$$

We checked that the field equations do not depend on the choice of ω_+ or ω_-. Although they do not differ in the context of gravity, their supersymmetry transformations distinguishes between them. * In this work, we take $\omega := \omega_+$ which is consistent with the model discussed in [68]. It follows from the first Cartan structure equations (2.7) and the definition (5.2) that the torsion of the spacetime equipped with the connection ω_{ab} will be given by

$$\begin{aligned} T^a &:= de^a + \hat{\omega}^a{}_b \wedge e^b + \frac{1}{2} i_b i_a H \wedge e^b && (5.3) \\ &= \frac{1}{2} i_b i_a H \wedge e^b. && (5.4) \end{aligned}$$

Furthermore, since for any p-form σ, $e^a \wedge i_a \sigma = p\sigma$, we have

$$T^a = -i^a H. \tag{5.5}$$

The curvature 2-forms follows from (2.8) as

$$R^a{}_b(\omega) = R^a{}_b(\hat{\omega}) + \frac{1}{2} \hat{D}(i_b i^a H) + \frac{1}{4} i_c i^a H \wedge i_b i^c H,$$

where we used $\hat{D}$ to indicate that the connection used for the covariant exterior derivative is the Levi-Civita connection. It follows from the definition of the curvature scalar

$$R(\omega) * 1 = R_{ab}(\omega) \wedge *(e^a \wedge e^b)$$

that

$$R(\omega) * 1 = R(\hat{\omega}) * 1 + \frac{1}{2} \hat{D}(i_b i_a H) \wedge *(e^a \wedge e^b) + \frac{1}{4} i_c i^a H \wedge i_b i^c H * (e^a \wedge e^b).$$

*Here, to prevent confusion we note that in [61] "R_{ab}" in the second part of the action density is taken as $R_{ab}(\omega_-)$, however in [68] a redefinition is made by sending $H \to -H$. Therefore,$R_{ab}(\omega_+)$ is used.

Since $i_a * (e^a \wedge e^b)$, $i_b i_a * (e^a \wedge e^b)$ and $\hat{D} * (e^a \wedge e^b)$ are zero, by the Leibniz rules for the derivations

$$\hat{D}(i_b i_a H) \wedge *(e^a \wedge e^b) = \hat{D}(i_b i_a (H \wedge *(e^a \wedge e^b))).$$

This term is the covariant derivative of a 7-form and also vanishes.

On the other hand, by a similiar reasoning, together with the fact that $i_b i_a = -i_a i_b$,

$$\begin{aligned}
i_c i^a H \wedge i_b i^c H * (e^a \wedge e^b) &= i_c H \wedge i_a i_b i^c H * (e^a \wedge e^b) \\
&= i_c H \wedge *(i_a i_b i^c H e^a \wedge e^b) \\
&= -i_c H \wedge *(2 i^c H) \\
&= -2H \wedge i_c * (i_c H) \\
&= -2H \wedge *(e_c \wedge i^c H) \\
&= -6H \wedge *H.
\end{aligned}$$

Therefore, we write the first order curvature term as

$$R(\hat{\omega}) * 1 = R(\omega) * 1 + \frac{3}{2} H \wedge *H. \tag{5.6}$$

We can also eliminate B from the action by partial differentiation and use only the corresponding field strength H:

$$R_{ab} \wedge R^{ab} \wedge B = d\omega_{ab} \wedge d\omega^{ab} \wedge B + 2d\omega_{ab} \wedge \omega^a{}_d \wedge \omega^{db} \wedge B + \omega_{ac} \wedge \omega^c{}_b \wedge \omega^a{}_d \wedge \omega^{db} \wedge B.$$

It can be seen that last term is zero by simply interchanging the indices. So,

$$R_{ab} \wedge R^{ab} \wedge B = d(\omega_{ab} \wedge d\omega^{ab} \wedge B) + \left(\omega_{ab} \wedge d\omega^{ab} + \frac{2}{3}\omega_{ab} \wedge \omega^a{}_d \wedge \omega^{db}\right) \wedge H.$$

By Stokes' theorem $d(\omega_{ab} \wedge d\omega^{ab} \wedge B)$ contributes to the action only as a surface term, hence can be neglected. We employ Lagrange multiplier forms μ and μ_a to constrain the torsion $T^a = -i^a H$ and the fact that $H = dB$ in the action density.

Finally, we rewrite (5.1) equivalently as

$$\begin{aligned}
\mathcal{L} &= e^{-2\phi}\left(R * 1 + \alpha d\phi \wedge *d\phi + \beta H \wedge *H\right) + V(\phi) * 1 + \gamma R_{ab} \wedge *R^{ab} \qquad (5.7) \\
&+ \delta\left(\omega_{ab} \wedge d\omega^{ab} + \frac{2}{3}\omega_{ab} \wedge \omega^a{}_d \wedge \omega^{db}\right) \wedge H \\
&+ (de^a + \omega^a{}_b \wedge e^b + i^a H) \wedge \mu_a + dH \wedge \mu,
\end{aligned}$$

with real coupling constants α, β, γ_1, γ_2. We keep the constants $\alpha, \beta, \gamma, \delta$ to track the contributions of each term to the field equations. We note that for our model the values of the coupling constants should be fixed by

$$\alpha = 4, \quad \beta = 1, \quad \gamma = \delta = \frac{1}{2}\sigma. \tag{5.8}$$

With all the construction and simplifications done above, the action we consider can be written as

$$I[e^a, \omega^a{}_b, \phi, H, \mu, \mu_a] = \int_{M_6} \mathcal{L} \tag{5.9}$$

where $\mathcal{L}$ is given by (5.7).

5.1 Variational equations

We first consider the coframe variations of the action (5.9):

$$\begin{aligned} e^{-2\phi} \quad & \{R_{ab} \wedge *(e^a \wedge e^b \wedge e_c) - \alpha(i_c d\phi \wedge *d\phi + d\phi \wedge i_c * d\phi) \\ - \quad & \beta(i_c H \wedge *H + H \wedge i_c * H)\} + V(\phi) * e_c - \gamma(i_c R_{ab} \wedge *R^{ab} - R_{ab} \wedge i_c * R^{ab}) \\ + \quad & D\mu_c - i^a(i_c H \wedge \mu_a) = 0. \end{aligned} \tag{5.10}$$

Then, we vary the action with respect to the connection 1-forms ω_{ab}. The resulting field equations

$$D * (e^{-2\phi}(e_a \wedge e_b)) + 2\gamma D * R_{ab} + 2\delta R_{ab} \wedge H + e_b \wedge \mu_a = 0 \tag{5.11}$$

can be used for the multiplier 4-forms μ_a. At this point, it is convenient to introduce an antisymmetric 5-form

$$\Pi_{ab} := e_a \wedge \mu_b - e_b \wedge \mu_a, \tag{5.12}$$

so that by (5.11),

$$\Pi_{ab} = 2D * (e^{-2\phi} e_a \wedge e_b) + 4\gamma D * R_{ab} + 4\delta R_{ab} \wedge H. \tag{5.13}$$

With the use of the identities:

$$i^a \Pi_{ab} = \mu_b + e_b \wedge i^c i^a \Pi_{ac},$$

and

$$i^b i^a \Pi_{ab} = 4 i^a \mu_a,$$

we can reexpress the multiplier forms μ_a as

$$\mu_a = i^b \Pi_{ba} - \frac{1}{4} e_a \wedge i^c i^b \Pi_{bc}. \tag{5.14}$$

We insert this expression for μ_a in (5.10). The first two variational equations accordingly reduce to the following single equation:

$$\begin{aligned} e^{-2\phi} \quad & \{R_{ab} \wedge *(e^a \wedge e^b \wedge e_c) - \alpha(i_c d\phi \wedge *d\phi + d\phi \wedge i_c * d\phi) \\ - \quad & \beta(i_c H \wedge *H + H \wedge i_c * H)\} + V(\phi) * e_c - \gamma(i_c R_{ab} \wedge *R^{ab} - R_{ab} \wedge i_c * R^{ab}) \\ + \quad & D i^a \Pi_{ac} + \frac{1}{4} e_c \wedge d(i^a i^b \Pi_{ba}) - \frac{1}{2} i_c H \wedge i^a i^b \Pi_{ba} - i^a i_c H \wedge i^b \Pi_{ba} = 0, \end{aligned} \tag{5.15}$$

where Π_{ab} given in (5.13).

Obviously, the variations of the action density with respect to the Lagrange multipliers impose the following constraints

$$T^a = -i^a H \quad \text{and} \quad H = dB \tag{5.16}$$

as we assumed earlier.

We obtain a remaining field equation by $H-$variations:

$$2\beta e^{-2\phi} * H - \delta \left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3} \omega^a{}_c \wedge \omega^c{}_b \wedge \omega^b{}_a \right) + \; i^a \mu_a + d\mu = 0. \tag{5.17}$$

We eliminate μ by taking the exterior derivative of the above equation and substituting in the expression for μ_a and obtain

$$2\beta d(e^{-2\phi} * H) - \delta R_{ab} \wedge R^{ab} + \frac{1}{4} d(i^a i^b \Pi_{ba}) = 0. \tag{5.18}$$

The final variations we perform are with respect to the scalar field ϕ:

$$R * 1 + \alpha(d\phi \wedge *d\phi + d * d\phi) + \beta H \wedge *H - \frac{e^{2\phi}}{2} V'(\phi) * 1 = 0. \tag{5.19}$$

The combination of the (5.19) with the trace of the equation (5.15) gives us

$$4e^{-2\phi}\{R*1+d\phi\wedge *d\phi\}+2\gamma R_{ab}\wedge *R^{ab}+e^c\wedge Di^a\Pi_{ac}-H\wedge i^b i^a\Pi_{ab} \\ +6V(\phi)*1=0 \tag{5.20}$$

implies the dilaton field equation

$$4e^{-2\phi}(\alpha d*d\phi+\beta H\wedge *H)-2\gamma R_{ab}\wedge *R^{ab}-e^c\wedge Di^a\Pi_{ac}+H\wedge i^b i^a\Pi_{ab} \\ -6V(\phi)*1-2V'(\phi)*1=0. \tag{5.21}$$

To sum up, we have at the end three variational field equations given by (5.15), (5.18), (5.20) subject to the constraint (5.16) follow from the action (5.9).

5.2 $AdS_3\times S^3$ background vacuum solutions

In this section, we will solve the variational field equations found in the previous section with the metric ansatz:

$$g=k_1^2 g_{AdS_3}+k_2^2 g_{S^3} \tag{5.22}$$

where g_{AdS_3} and g_{S^3} are the metrics of the spaces AdS_3 and S^3 , respectively. The generators of the associated Lie algebras for AdS_3 and S^3 are the same as the generators of the $su(1,1)$ and $su(2)$ Lie algebras, respectively, given explicitly as

$$\begin{aligned}
[X_0,X_1] &= X_2, \quad [X_1,X_2]=-X_0, \quad [X_0,X_2]=-X_1; \\
[X_4,X_5] &= X_6, \quad [X_5,X_6]=X_4, \qquad [X_6,X_4]=X_5.
\end{aligned} \tag{5.23}$$

$\{X_a\}$ constitutes a frame on $AdS_3\times S^3$. X_a correspond to the generators of $su(1,1)$ for a=0,1,2 and generators of $su(2)$ for a=4,5,6. The coframe 1-forms $\{e^a\}$ satisy $e^b(X_a)=\delta_a^b$ by duality. Using the structure constants ${C_{ab}}^c$ determined by the commutators $[X_a,X_b]={C_{ab}}^c X_c$, we can write the Maurer-Cartan equation as $de^a=-{C_{bc}}^a e^b\wedge e^c$.

We take the coframe 1-forms introduced here and scale them as $e^a\to k_1 e^a$ for $a\in\{0,1,2\}$ and $e^b\to k_2 e^a$ for $a\in\{4,5,6,\}$ and write the 6-metric (5.22) as

$$g=-e^0\otimes e^0+e^1\otimes e^1+e^2\otimes e^2+e^4\otimes e^4+e^5\otimes e^5+e^6\otimes e^6. \tag{5.24}$$

We determine the exterior derivatives of the coframe 1-forms e^a without effort thanks to the Maurer-Cartan equations:

$$
\begin{aligned}
de^0 &= \frac{2}{k_1} e^1 \wedge e^2, \quad de^1 = \frac{2}{k_1} e^0 \wedge e^2, \quad de^2 = -\frac{2}{k_1} e^0 \wedge e^1; \\
de^4 &= \frac{2}{k_2} e^5 \wedge e^6, \quad de^5 = \frac{2}{k_2} e^6 \wedge e^4, \quad de^6 = \frac{2}{k_2} e^4 \wedge e^5.
\end{aligned} \tag{5.25}
$$

The non-vanishing components of the Levi-Civita connection 1-forms are solved from the Cartan's first structure equations (2.7):

$$
\begin{aligned}
\hat{\omega}^0{}_1 &= \frac{1}{k_1} e^2, \quad \hat{\omega}^0{}_2 = -\frac{1}{k_1} e^1, \quad \hat{\omega}^1{}_2 = -\frac{1}{k_1} e^0; \\
\hat{\omega}^4{}_5 &= \frac{1}{k_2} e^6, \quad \hat{\omega}^5{}_6 = \frac{1}{k_2} e^4, \quad \hat{\omega}^6{}_4 = \frac{1}{k_2} e^5.
\end{aligned} \tag{5.26}
$$

These are the connection 1-forms without torsion. In order to find the connection 1-forms $\{\omega^a{}_b\}$ with torsion we again employ Cartan's first structure equation. We see that $\omega^a{}_b$ can be written as the sum of the Levi-Civita connection 1-forms $\hat{\omega}^a{}_b$ and the 1-forms , which we denote by K_{ab}, called the contortion 1-forms, that are related to the torsion 2-forms according to

$$
T_a = K_{ab} \wedge e^b. \tag{5.27}
$$

Solving the above equations with the torsion 2-forms $T_a = -i_a H$, the contortion forms can be written as

$$
K_{ab} = -\frac{1}{2} i_a i_b H. \tag{5.28}
$$

At this point we make the following ansatz for the antisymmetric 3-form field H:

$$
H = c_1 e^0 \wedge e^1 \wedge e^2 + c_2 e^4 \wedge e^5 \wedge e^6, \tag{5.29}
$$

whose Hodge dual is given by

$$
*H = -c_2 e^0 \wedge e^1 \wedge e^2 - c_1 e^4 \wedge e^5 \wedge e^6. \tag{5.30}
$$

Here c_1 and c_2 are two constant parameters.

It follows that all the nonvanishing contortion 1-forms are the following:

$$\begin{aligned} K_{01} &= \frac{c_1}{2}e^2, \quad K_{02} = -\frac{c_1}{2}e^1, \quad K_{12} = \frac{c_1}{2}e^0; \\ K_{45} &= \frac{c_2}{2}e^6, \quad K_{56} = \frac{c_2}{2}e^4, \qquad K_{64} = \frac{c_2}{2}e^5. \end{aligned} \tag{5.31}$$

Therefore the full connection 1-forms are explicitly given by

$$\begin{aligned} \omega^0{}_1 &= \left(\frac{1}{k_1} - \frac{c_1}{2}\right)e^2, \quad \omega^0{}_2 = -\left(\frac{1}{k_1} - \frac{c_1}{2}\right)e^1, \quad \omega^1{}_2 = -\left(\frac{1}{k_1} - \frac{c_1}{2}\right)e^0; \\ \omega^4{}_5 &= \left(\frac{1}{k_2} - \frac{c_2}{2}\right)e^6, \quad \omega^5{}_6 = \left(\frac{1}{k_2} - \frac{c_2}{2}\right)e^4, \quad \omega^6{}_4 = \left(\frac{1}{k_2} - \frac{c_2}{2}\right)e^5. \end{aligned} \tag{5.32}$$

The corresponding curvature 2-forms follow from the Cartan's second structure equations (2.8):

$$R_{ab} = \begin{cases} \left(\frac{c_1^2}{4} - \frac{1}{k_1^2}\right) e_a \wedge e_b, & \text{for } a,\ b \in \{0,1,2\}, \\ \\ \left(\frac{1}{k_2^2} - \frac{c_2^2}{4}\right) e_a \wedge e_b, & \text{for } a,\ b \in \{4,5,6\}. \end{cases} \tag{5.33}$$

At this point it is better to make some remarks about the resulting curvature. Since torsion is directly related with the 3-form H, setting $c_1 = c_2 = 0$ gives the torsion-free curvature 2-forms. We see that for $a,\ b \in \{0,1,2\}$ the curvature forms would be all negative, while $a,\ b \in \{4,5,6\}$ would be all positive. This shows again that the space with metric $-e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2$ is a maximally symmetric Lorentzian space with a constant negative curvature, which is AdS_3. On the other hand the space with metric $e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6$ is a maximally symmetric Euclidean space with positive curvature, which is S^3.

Let us look at the solutions with a vanishing dilaton field ϕ and zero potential $V(\phi)$. Starting from the last variational equations, we write them term by term. Of course, the terms involving the derivatives of the dilaton field ϕ are all zero. The total curvature scalar of the $AdS_3 \times S^3$ spacetime is found to be

$$R = i^b i^a R_{ab} = \frac{3}{2}\left(\frac{4}{k_2^2} - \frac{4}{k_1^2} + c_1^2 - c_2^2\right), \tag{5.34}$$

and we have

$$H \wedge *H \;=\; (c_2^2 - c_1^2) * 1. \tag{5.35}$$

Hence, (5.19) gives

$$6\left(\frac{1}{k_2^2} - \frac{1}{k_1^2}\right) + \left(\beta - \frac{3}{2}\right)(c_2^2 - c_1^2) = 0 \tag{5.36}$$

Next, we proceed with the H-variation (5.18). Since

$$dH = 0, \tag{5.37}$$

$$d * H = 0, \tag{5.38}$$

$$R_{ab} \wedge R^{ab} = 0, \tag{5.39}$$

$$d(i^a i^b \Pi_{ba}) = 0, \tag{5.40}$$

the H-field equation is satisfied with the given metric g and H.

Finally, we come to the Einstein field equations (5.15). With the use of the following identity:

$$i_a * e_b = *(e_b \wedge e_a). \tag{5.41}$$

We find

$$R_{ab} \wedge (e^a \wedge e^b \wedge e_c) = \begin{cases} \frac{3}{2}\left(\frac{4}{k_2^2} - c_2^2\right) + \frac{1}{2}\left(c_1^2 - \frac{4}{k_1^2}\right) * e_c, & \text{for } c \in \{0,1,2\}, \\ \\ \frac{3}{2}\left(c_1^2 - \frac{4}{k_1^2}\right) + \frac{1}{2}\left(\frac{4}{k_2^2} - c_2^2\right) * e_c, & \text{for } c \in \{4,5,6\} \end{cases} \tag{5.42}$$

The remaining terms are as follows:

$$i_c R_{ab} \wedge *R^{ab} - R_{ab} \wedge *R^{ab} = \begin{cases} 2\left(\frac{c_1^2}{4} - \frac{1}{k_1^2}\right)^2 - 6\left(\frac{1}{k_2^2} - \frac{c_2^2}{4}\right)^2 * e_c, & \text{for } c \in \{0,1,2\}, \\ \\ 2\left(\frac{1}{k_2^2} - \frac{c_2^2}{4}\right)^2 - 6\left(\frac{c_1^2}{4} - \frac{1}{k_1^2}\right)^2 * e_c, & \text{for } a,\ b \in \{4,5,6\} \end{cases} \tag{5.43}$$

$$i_c H \wedge *H + H \wedge i_c H = \begin{cases} -(c_1^2 + c_2^2) * e_c, & \text{for } c \in \{0,1,2\}, \\ (c_1^2 + c_2^2) * e_c, & \text{for } c \in \{4,5,6\} \end{cases} \tag{5.44}$$

It takes a little more effort to write the terms Π_{ab} explicitly:

$$\begin{aligned} D*(e^{-2\phi}e_a\wedge e_b) &= e^{-2\phi}D*(e_a\wedge e_b) \\ &= e^{-2\phi}De^c\wedge *(e_a\wedge e_b\wedge e_c) \\ &= -e^{-2\phi}i^cH\wedge *(e_a\wedge e_b\wedge e_c). \end{aligned} \tag{5.45}$$

Hence,

$$D*(e^{-2\phi}e_a\wedge e_b)=\begin{cases} -e^{-2\phi}c_1\ \varepsilon_{ab}{}^{c}*e_c, & \text{for } a,\ b,\ c\in\{0,1,2\}, \\ -e^{-2\phi}c_2\ \varepsilon_{ab}{}^{c}*e_c,, & \text{for } a,\ b,\ c\in\{4,5,6\} \end{cases} \tag{5.46}$$

The covariant derivative of $*R_{ab}$ is given by

$$D*R_{ab}=d*R_{ab}+\omega_{ac}\wedge *R^c{}_b-\omega^d{}_b\wedge *R_{ad} \tag{5.47}$$

and yields

$$D*R_{ab}=\begin{cases} \left(\frac{1}{k_1^2}-\frac{c_1^2}{4}\right)c_1\ \varepsilon_{ab}{}^{c}*e_c,, & \text{for } a,\ b,\ c\in\{0,1,2\}, \\ \\ \left(\frac{c_2^2}{4}-\frac{1}{k_2^2}\right)c_2\ \varepsilon_{ab}{}^{c}*e_c,, & \text{for } a,\ b,\ c\in\{4,5,6\}. \end{cases} \tag{5.48}$$

We also have

$$R_{ab}\wedge H=\begin{cases} \left(\frac{1}{k_1^2}-\frac{c_1^2}{4}\right)c_2\ \varepsilon_{ab}{}^{c}*e_c,, & \text{for } a,\ b,\ c\in\{0,1,2\}, \\ \\ \left(\frac{c_2^2}{4}-\frac{1}{k_2^2}\right)c_1\ \varepsilon_{ab}{}^{c}*e_c,, & \text{for } a,\ b,\ c\in\{4,5,6\}. \end{cases} \tag{5.49}$$

We put all these together and finally calculate

$$\Pi_{ab}=\begin{cases} \left(-2c_1+\left(\frac{4}{k_1^2}-c_1^2\right)(\gamma c_1+\delta c_2)\right)\ \varepsilon_{ab}{}^{c}*e_c, & \text{for } a,\ b,\ c\in\{0,1,2\}, \\ \\ \left(-2c_2+\left(c_2^2-\frac{4}{k_2^2}\right)(\gamma c_2+\delta c_1)\right)\ \varepsilon_{ab}{}^{c}*e_c, & \text{for } a,\ b,\ c\in\{4,5,6\} \end{cases} \tag{5.50}$$

Then, it follows that

$$Di^a\Pi_{ac} = \begin{cases} \left(-2c_1 + \left(\frac{4}{k_1^2} - c_1^2\right)(\gamma c_1 + \delta c_2)\right) 2c_1 * e_c, & \text{for } c \in \{0,1,2\}, \\ \\ \left(-2c_2 - \left(c_2^2 - \frac{4}{k_2^2}\right)(\gamma c_2 + \delta c_1)\right) 2c_2 * e_c, & \text{for } c \in \{4,5,6\} \end{cases} \tag{5.51}$$

and

$$\begin{aligned} i^b i^a \Pi_{ab} = &\left(12c_1 - 6\left(\frac{4}{k_1^2} - c_1^2\right)(\gamma c_1 + \delta c_2)\right) e^4 \wedge e^5 \wedge e^6 \\ &+ \left(12c_2 - 6\left(c_2^2 - \frac{4}{k_2^2}\right)(\gamma c_2 + \delta c_1)\right) e^0 \wedge e^1 \wedge e^2 \end{aligned} \tag{5.52}$$

Therefore the coframe variational equations (5.15) amount to two equations. For c = 0,1,2, we get:

$$\begin{aligned} &\frac{3}{2}\left(\frac{4}{k_2^2} - c_2^2\right) + \frac{1}{2}\left(c_1^2 - \frac{4}{k_1^2}\right) + \beta(c_1^2 + c_2^2) - 2\gamma\left(\frac{1}{k_1^2} - \frac{c_1^2}{4}\right)^2 \\ &+6\gamma\left(\frac{1}{k_2^2} - \frac{c_2^2}{4}\right)^2 - 2c_1^2 + \left(\frac{4}{k_1^2} - c_1^2\right)(\gamma c_1^2 + \delta c_2 c_1) = 0, \end{aligned} \tag{5.53}$$

and for c= 4,5,6 we get:

$$\begin{aligned} &\frac{3}{2}\left(c_1^2 - \frac{4}{k_1^2}\right) + \frac{1}{2}\left(\frac{4}{k_2^2} - c_2^2\right) - \beta(c_1^2 + c_2^2) - 2\gamma\left(\frac{1}{k_2^2} - \frac{c_2^2}{4}\right)^2 \\ &+6\gamma\left(\frac{1}{k_1^2} - \frac{c_1^2}{4}\right)^2 + 2c_2^2 - \left(c_2^2 - \frac{4}{k_2^2}\right)(\gamma c_2^2 + \delta c_1 c_2) = 0. \end{aligned} \tag{5.54}$$

Finally, we should solve the three algebraic equations (5.36), (5.53) and (5.54) obtained from the variational equations (5.15), (5.18), (5.19). As we mentioned at the beginning of this chapter we take $\delta = \gamma$ and $\beta = 1$. A natural solution can be written simply as:

$$c_1^2 = c_2^2 = \frac{4}{k_1^2} = \frac{4}{k_2^2}. \tag{5.55}$$

While the solution with $c_1 = -c_2$ corresponds to a self-dual 3-form H described by

$$H = *H, \tag{5.56}$$

the solution $c_1 = c_2$ corresponds to an anti-self dual 3-form H given by

$$H = - * H. \tag{5.57}$$

According to the solution above, the negative curvature space AdS_3 and positive curvature space S_3 after introducing the torsion to the connection, imply that the total curvature is zero. A similiar solution is obtained in a recent paper [70] where solutions on the all possible direct product of two constant curvature spaces are listed.

5.3 AdS_3 and S^3 in coordinate systems

In the previous section, we make use of the properties of the left invariant vector fields to determine the coframe 1-forms which satisfy (5.2). We did not choose a spesific coordinate system through the whole section. In this section, we will explicitly write the left invariant 1-forms in terms of a given coordinate system. This will play an important role in defining time-warped $AdS_3 \times S^3$ backgrounds.

The AdS_3 space can be thought as a hypersurface $X^2 + Y^2 - U^2 - V^2 = -1$ embedded in a flat space with metric $g = dX^2 + dY^2 - dU^2 - dV^2$. This condition can be given by the group of matrices

$$\mathbf{g} = \begin{pmatrix} V+X & Y+U \\ Y-U & V-X \end{pmatrix}. \tag{5.58}$$

with $det\mathbf{g} = U^2 + V^2 - X^2 - Y^2 = 1$, as well. AdS_3 as Lie group can be seen as isomorphic to $SL(2,\mathbb{R})$. AdS_3 is also an homogeneous space $SO(2,2)/SO(2,1)$ with the action of its isometry group $SO(2,2)$ and isotropy group $SO(2,1)$. Note that, $SO(2,2)$ is isomorphic to $SO(2,1) \times SO(2,1)$ and $SO(2,1)$ is locally to isomorphic to $SL(2,\mathbb{R})$. The Lie algebra of the isometry group will be $so(2,2) = sl_L(2,\mathbb{R}) \oplus sl_R(2,\mathbb{R})$, which gives two copies of the Lie algebra of AdS_3, with one copy corresponding to right and one copy to left translations†. Hence, we can make use of the Killing vectors on AdS_3 to find the generators of the Lie algebra of AdS_3. With the metric choosen

†Also, note that $sl(2,\mathbb{R}) \simeq su(1,1)$.

above, the Killing vectors can be given as

$$J_{X^a X^b} = X_a \frac{\partial}{\partial X^b} - X_b \frac{\partial}{\partial X^a}, \tag{5.59}$$

where $X_a = (-U, -V, X, Y)$ They satisfy the following commutation relations:

$$[J_{X^a X^b}, J_{X^b X^c}] = \begin{cases} -J_{X^a X^c}, & \text{for } \; b \in \{0,1\} \text{ and } a \neq b \neq c, \\ J_{X^a X^c}, & \text{for } \; b \in \{2,3\} \text{ and } a \neq b \neq c, \end{cases} \tag{5.60}$$

with $J_{X^a X^b} = -J_{X^b X^a}$.

Using the commutation relations above, the Killing vectors of $so(2,2)$ can be grouped into two commuting sets

$$\begin{aligned} \xi_0 &= -J_{UV} - J_{XY}, \qquad \eta_0 = -J_{UV} + J_{XY}, \\ \xi_1 &= J_{XU} + J_{YV}, \qquad \eta_1 = J_{XU} - J_{YV}, \\ \xi_2 &= J_{YU} - J_{XV}, \qquad \eta_2 = -J_{YU} - J_{XV}, \end{aligned} \tag{5.61}$$

corresponding to the generators of two copies of $sl(2,\mathbb{R})$ [71],[72]. We can identify the set $\{\xi_0, \xi_1, \xi_2\}$ as the generators of the Lie algebra of left invariant vector fields on AdS_3. By the relations (5.60), we can show that the generators satisfy

$$[\xi_0, \xi_1] = \xi_2, \quad [\xi_1, \xi_2] = -\xi_0, \quad [\xi_0, \xi_2] = -\xi_1. \tag{5.62}$$

First, we specify a local coordinate system on AdS_3

$$\begin{aligned} U &= \cos t \qquad\qquad X = \sin t \sinh\chi \cos\theta \\ V &= \sin t \cosh\chi \qquad Y = \sin t \sinh\chi \sin\theta. \end{aligned} \tag{5.63}$$

In the above coordinate system, the metric g reads

$$g = -dt^2 + \sin^2 t d\chi^2 + \sin^2 t \sinh^2 \chi d\theta^2 \tag{5.64}$$

By the relations (5.59) and (5.61), the generators $\{\xi^\mu\}$ are calculated as

$$\begin{aligned} \xi_0 &= \cosh\chi \partial_t - \cot t \sinh\chi \partial_\chi - \partial_\theta \\ \xi_1 &= -\sinh\chi\cos\theta\partial_t + (\cot t\cosh\chi\cos\theta + \sin\theta)\partial_\chi + (coth\chi\cos\theta - \cot t \mathrm{cosech}\chi\sin\theta)\partial_\theta \\ \xi_2 &= -sinh\chi\sin\theta\partial_t + (\cot t\cosh\chi\sin\theta - \cos\theta)\partial_\chi + (coth\chi\sin\theta + \cot t\mathrm{cosech}\chi\cos\theta)\partial_\theta \end{aligned} \tag{5.65}$$

Finally, we can invert these for the left-invariant 1-forms $\{e^0, e^1, e^2\}$ using the duality $e^\mu(\xi_\nu) = \delta^\mu_\nu$ for $\mu, \nu \in \{0, 1, 2\}$ as follows:

$$\begin{aligned} e^0 &= \cosh\chi dt + \cos t \sin t sinh\chi d\chi + \sin^2 t \sinh^2\chi d\theta, \\ e^1 &= \sinh\chi\cos\theta dt + (\cos t \sin t \cosh\chi\cos\theta + \sin^2 t \sin\theta) d\chi \\ &+ \sin^2 t \sinh\chi(\cosh\chi\cos\theta - \cot t \sin\theta) d\theta, \\ e^2 &= \sinh\chi\sin\theta dt + (\cos t \sin t \cosh\chi\sin\theta - \sin^2 t \cos\theta) d\chi \\ &+ \sin^2 t \sinh\chi(\cosh\chi\sin\theta + \cot t \cos\theta) d\theta. \end{aligned} \tag{5.66}$$

It can be checked that the metric (5.64) can be written in the following form:

$$g_{AdS_3} = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \tag{5.67}$$

A similiar method can be applied to the 3-sphere S^3. S^3 is also a homogeneous space, which can be identified with $SO(4)/SO(3)$. However, there is an easier way that we may reach to S^3 Lie algebra, because we already have results regarding AdS_3. As AdS_3 is defined by the embedding relation $X^2 + Y^2 - U^2 - V^2 = -1$, in a similiar way S^3 can be defined as an embedding by replacing X with $-iX$ and Y with $-iY$, and letting $\chi = i\psi$. Accordingly, we choose a coordinate system such that

$$\begin{aligned} U &= \cos\phi & X &= \sin\phi\sin\varphi\cos\psi \\ V &= \sin\phi\cos\varphi & Y &= \sin\phi\sin\varphi\sin\psi. \end{aligned} \tag{5.68}$$

results in the following generators of the Lie algebra $su(2)$ of S^3:

$$\begin{aligned} e^4 &= \cos\varphi d\phi + \cos\phi\sin\phi sin\varphi d\varphi + \sin^2\phi\sin^2\varphi d\psi, \\ e^5 &= \sin\varphi\cos\psi d\phi + (\cos\phi\sin\phi\cos\varphi cos\psi + sin^2\phi sin\psi) d\varphi \\ &+ \sin^2\phi\sin\varphi(\cos\varphi\cos\psi - \cot\phi\sin\psi) d\psi, \\ e^6 &= \sin\varphi\sin\psi d\phi + (cos\phi\sin\phi\cos\varphi\sin\psi - \sin^2\phi cos\psi) d\varphi \\ &+ \sin^2\phi\sin\varphi(\cos\varphi\sin\psi + \cot\phi\cos\psi) d\psi. \end{aligned} \tag{5.69}$$

We also note that the corresponding metric is

$$g_{S^3} = e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6$$

which is written in the given coordinate system as

$$g_{S^3} = d\phi^2 + \sin^2\phi(d\varphi^2 + \sin^2\varphi d\psi^2).$$

5.4 Equations of motion on a warped $AdS_3 \times S^3$ background

In this section, we start with a time-warped $AdS_3 \times S^3$ metric described by

$$g = f(t)^2 g_{AdS_3} + h^2(t) g_{S^3}, \tag{5.70}$$

where g_{AdS_3} and g_{S^3} denote the metrics of AdS_3 and S^3, that are derived in the previous section. $f, h : \mathbb{R} \to \mathbb{R}$ are smooth functions of time t identified from the metric:

$$g_{AdS_3} = -dt^2 + \sin^2 t d\chi^2 + \sin^2 t \sinh^2\chi d\theta^2. \tag{5.71}$$

Since the total metric is time warped, we do not need to specify a coordinate system for S^3. We again make use of the generators of the Lie algebra $su(2)$ and take the S^3 metric

$$g_{S^3} = \tilde{e}^4 \otimes \tilde{e}^4 + \tilde{e}^5 \otimes \tilde{e}^5 + \tilde{e}^6 \otimes \tilde{e}^6. \tag{5.72}$$

where the coframe 1-forms $d\tilde{e}^i = -\epsilon^i{}_{jk}\tilde{e}^j \wedge \tilde{e}^k$.

Hence the full metric can be given in the following form:

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \otimes e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6, \tag{5.73}$$

where the coframes are given as

$$e^0 = g(t)dt, \quad e^1 = f(t)\sin t d\chi, \quad e^2 = f(t)\sin t \sinh\chi, \quad e^i = h(t)\tilde{e}^i \tag{5.74}$$

We will later take $g(t) = f(t)$. However let us keep it as $g(t)$ in order to single out the time component of the metric. To simplify the relations, we make the following changes in the notation:

$$F := f(t)\sin t, \quad G := g(t), \quad \text{and} \quad H := h(t). \tag{5.75}$$

We first calculate the exterior derivatives of the coframe 1-forms:

$$\begin{aligned} de^0 &= 0, & de^2 &= \frac{F'}{FG}e^0\wedge e^2 + \frac{coth\chi}{F}e^1\wedge e^2 \\ de^1 &= \frac{F'}{FG}e^0\wedge e^1, & de^i &= \frac{H'}{HG}e^0\wedge e^i - \frac{1}{H}\epsilon_{ijk}e^j\wedge e^k. \end{aligned} \tag{5.76}$$

We solve the Levi-Civita connection 1-forms using the first Cartan-Maurer equations (2.7)

$$\begin{aligned} \hat{\omega}^0{}_1 &= \frac{F'}{FG}e^1, & \hat{\omega}^0{}_i &= \frac{H'}{HG}e^i, \\ \hat{\omega}^0{}_2 &= \frac{F'}{FG}e^2, & \hat{\omega}^i{}_j &= -\frac{1}{H}\epsilon_{ijk}e^k, \\ \hat{\omega}^1{}_2 &= -\frac{\coth\chi}{F}e^2, & \hat{\omega}^1{}_i &= \hat{\omega}^2{}_i = 0. \end{aligned} \tag{5.77}$$

Then the corresponding curvature 2-forms follow from the second Cartan equations (2.8):

$$\begin{aligned} \hat{R}^0{}_1 &= \left(\frac{1}{G}\left[\frac{F'}{FG}\right]' + \left[\frac{F'}{FG}\right]^2\right)e^0\wedge e^1, & \hat{R}^0{}_i &= \left(\frac{1}{G}\left[\frac{H'}{HG}\right]' + \left[\frac{H'}{HG}\right]^2\right)e^0\wedge e^i, \\ \hat{R}^0{}_2 &= \left(\frac{1}{G}\left[\frac{F'}{FG}\right]' + \left[\frac{F'}{FG}\right]^2\right)e^0\wedge e^2, & \hat{R}^1{}_i &= \left(\frac{F'}{FG}\right)\left(\frac{H'}{HG}\right)e^1\wedge e^i, \\ \hat{R}^1{}_2 &= \left(\left[\frac{F'}{FG}\right]^2 - \frac{1}{F^2}\right)e^1\wedge e^2, & \hat{R}^2{}_i &= \left(\frac{F'}{FG}\right)\left(\frac{H'}{HG}\right)e^2\wedge e^i \\ \hat{R}^i{}_j &= \left(\left[\frac{H'}{HG}\right]^2 + \frac{1}{H^2}\right)e^i\wedge e^j. \end{aligned}$$

We recall that the action (5.9) is written in terms of the connection $\omega_{ab} = \hat{\omega}_{ab} + \frac{1}{2}i_a i_b H$ that induces a torsion 1-form $T^a = -i^a H$. Let us choose the 3-form field H as

$$H = c_1(t)e^0\wedge e^1\wedge e^2 + c_2(t)e^4\wedge e^5\wedge e^6, \tag{5.78}$$

where contrary to the ansatz (5.29), we take the coefficients $c_1(t)$ and $c_2(t)$ as functions of time t. Also note that the coframe is different. Thus, (5.29) reduces to (5.78) if $f(t) = g(t) = k_1$ and $h(t) = k_2$. It follows from $dH = 0$ that c_2 is a constant. With the H-ansatz above the contortion 1-forms $K_{ab} = \frac{1}{2} i_b i_a H$ are same as the ones listed in (5.31), just with the replacement $c_1 \to c_1(t)$.

The curvature 2-forms of the spacetime with the ω connection can be written as

$$R^a{}_b = \hat{R}^a{}_b + dK^a{}_b + K^a{}_c \wedge K^c{}_b + K^a{}_c \wedge \hat{\omega}^c{}_b + \hat{\omega}^a{}_c \wedge K^c{}_b. \tag{5.79}$$

Therefore, we get

$$\begin{aligned}
R^0{}_1 &= \left(\frac{1}{G}\left[\frac{F'}{FG}\right]' + \left[\frac{F'}{FG}\right]^2 + \frac{c_1^2}{4} \right) e^0 \wedge e^1 - \frac{c_1'}{2G} e^0 \wedge e^2, \\
R^0{}_2 &= \left(\frac{1}{G}\left[\frac{F'}{FG}\right]' + \left[\frac{F'}{FG}\right]^2 + \frac{c_1^2}{4} \right) e^0 \wedge e^2 + \frac{c_1'}{2G} e^0 \wedge e^1, \\
R^1{}_2 &= \left(\left[\frac{F'}{FG}\right]^2 - \frac{1}{F^2} + \frac{c_1^2}{4} \right) e^1 \wedge e^2, \\
R^0{}_i &= \left(\frac{1}{G}\left[\frac{H'}{HG}\right]' + \left[\frac{H'}{HG}\right]^2 \right) e^0 \wedge e^i - \left(\frac{c_2}{2}\right)\left(\frac{H'}{HG}\right) \epsilon_{ijk} e^j \wedge e^k, \\
R^1{}_i &= \left(\frac{F'}{FG}\right)\left(\frac{H'}{HG}\right) e^1 \wedge e^i - \left(\frac{c_1}{2}\right)\left(\frac{H'}{HG}\right) e^2 \wedge e^i, \\
R^2{}_i &= \left(\frac{F'}{FG}\right)\left(\frac{H'}{HG}\right) e^2 \wedge e^i + \left(\frac{c_1}{2}\right)\left(\frac{H'}{HG}\right) e^1 \wedge e^i, \\
R^i{}_j &= \left(\left[\frac{H'}{HG}\right]^2 + \frac{1}{H^2} - \frac{c_2^2}{4} \right) e^i \wedge e^j + \left(\frac{c_2}{2}\right)\left(\frac{H'}{HG}\right) \epsilon_{ijk} e^0 \wedge e^k.
\end{aligned}$$

To make the notation short, we define a set of functions :

$$\begin{aligned}
A_1 &:= \frac{1}{G}\left[\frac{F'}{FG}\right]' + \left[\frac{F'}{FG}\right]^2 + \frac{c_1^2}{4}, & A_2 &:= \left[\frac{F'}{FG}\right]^2 - \frac{1}{F^2} + \frac{c_1^2}{4} \\
A_0 &:= \frac{c_1'}{2G}, & S_1 &:= \frac{1}{G}\left[\frac{H'}{HG}\right]' + \left[\frac{H'}{HG}\right]^2, \\
S_2 &:= \left[\frac{H'}{HG}\right]^2 + \frac{1}{H^2} - \frac{c_2^2}{4}, & S_3 &:= \left(\frac{F'}{FG}\right)\left(\frac{H'}{HG}\right), \\
S_4 &:= \left(\frac{c_1}{2}\right)\left(\frac{H'}{HG}\right), & S_0 &:= \left(\frac{c_2}{2}\right)\left(\frac{H'}{HG}\right).
\end{aligned}$$

Therefore, the curvature forms can be expressed equivalently as follows:

$$\begin{aligned} R^0{}_1 &= A_1 e^0 \wedge e^1 - A_0 e^0 \wedge e^2 \\ R^0{}_2 &= A_1 e^0 \wedge e^2 + A_0 e^0 \wedge e^1 \\ R^1{}_2 &= A_2 e_1 \wedge e^2 \\ R^0{}_i &= S_1 e^0 \wedge e^i - S_0 \epsilon_{ijk} e^j \wedge e^k \\ R^1{}_i &= S_3 e^1 \wedge e^i - S_4 e^2 \wedge e^i \\ R^2{}_i &= S_3 e^2 \wedge e^i + S_4 e^1 \wedge e^i \\ R^i{}_j &= S_2 e^i \wedge e^j + S_0 \epsilon_{ijk} e^0 \wedge e^k \end{aligned} \tag{5.80}$$

Finally, the curvature scalar will be

$$R = 4A_1 + 2A_2 + 6S_1 + 6S_2 + 12S_3. \tag{5.81}$$

One can check that this is equal to the scalar curvature of the previous section, if we take $g(t) = f(t) = k_1$ and $h(t) = k_2$. We set ϕ as a function of time $\phi := \phi(t)$. Now, we will write the variational field equations (5.15), (5.19) and (5.18) in the time warped $AdS_3 \times S^3$ background.

Starting from the (5.19), first we get the following equation:

$$\begin{aligned} & 4A_1 + 2A_2 + 6S_1 + 6S_2 + 12S_3 + \beta(c_2^2 - c_1^2) \\ - \ & \alpha \left\{ \left(\frac{\phi'}{G}\right)^2 + \frac{1}{G}\left(\frac{\phi'}{G}\right)' + \left(\frac{\phi'}{G}\right)\left(2\frac{F'}{FG} + 3\frac{H'}{HG}\right)\right\} - \frac{1}{2}e^{2\phi}V'(\phi) = 0 \end{aligned} \tag{5.82}$$

(5.18) give the following equation:

$$\begin{aligned} & 2\beta e^{-2\phi}\left\{2\frac{\phi'}{G}c_1 - \left(\frac{c_1'}{G} + 6\frac{S_0}{c_2}c_1\right)\right\} - 12\delta S_0(2S_1 + S_2) + \frac{1}{G}\Big[e^{-2\phi}c_1 \\ + \ & \gamma\left(4\frac{A_0'}{G} - 8A_0\frac{F'}{FG} + 12(A_0 - S_4)\frac{H'}{HG} + 2c_1(2A_1 - A_2)\right) + \delta c_2(-4A_1 + 6A_2 c_2\Big]' \\ + \ & 3\frac{H'}{HG}\left\{e^{-2\phi}c_1 + \gamma\left(4\frac{A_0'}{G} - 8A_0\frac{F'}{FG} + 12(A_0 - S_4)\frac{H'}{HG} + 2c_1(2A_1 - A_2)\right)\right. \\ + \ & \left.\delta c_2(-4A_1 + 6A_2 c_2\right\} = 0. \end{aligned} \tag{5.83}$$

The Einstein equations, obtained from the (5.15) give four independent equations. In (5.15), for c=0 we get

$$\begin{aligned}
&e^{-2\phi}\left\{2A_2+6S_2+12S_3+\alpha\left(\frac{\phi'}{G}\right)^2+\beta(c_1^2+c_2^2)\right\}+V(\phi) \\
&-\gamma\left\{4A_1^2-2A_2^2+4A_0^2+6S_1^2-6S_2^2-12S_3^2-12S_4^2+18S_0^2\right\} \\
&-2\frac{F'}{FG}\left\{4e^{-2\phi}\frac{\phi'}{G}+4\gamma\left(-\frac{A_1'}{G}+(A_2-A_1)\frac{F'}{FG}+3(S_3-A_1)\frac{H'}{HG}-\frac{c_1}{2}A_0\right)-4\delta A_0c_2\right\} \\
&+c_1\left\{-2e^{-2\phi}c_1+\gamma\left(4\frac{A_0'}{G}+4A_0\frac{F'}{FG}+12(A_0-S_4)\frac{H'}{HG}-2c_1(A_1+A_2)\right)-4\delta A_1c_2\right\} \\
&=0.
\end{aligned} \tag{5.84}$$

$c=1$ and $c=2$ cases give the following same set of equations:

$$\begin{aligned}
&e^{-2\phi}\left\{2A_1+6S_1+6S_2+6S_3-\alpha\left(\frac{\phi'}{G}\right)^2+\beta(c_1^2+c_2^2)\right\}+V(\phi) \\
&-\gamma\left\{2A_2^2-6S_1^2-6S_2^2+30S_0^2\right\}-\frac{1}{G}\left[4e^{-2\phi}\frac{\phi'}{G}+4\gamma\left(-\frac{A_1'}{G}+(A_2-A_1)\frac{F'}{FG}\right.\right. \\
&\left.\left.+3(S_3-A_1)\frac{H'}{HG}-\frac{c_1}{2}A_0\right)-4\delta A_0c_2\right]'+\left(\frac{F'}{FG}+3\frac{H'}{HG}\right)\left\{4e^{-2\phi}\frac{\phi'}{G}\right. \\
&\left.+4\gamma\left(-\frac{A_1'}{G}+(A_2-A_1)\frac{F'}{FG}+3(S_3-A_1)\frac{H'}{HG}-\frac{c_1}{2}A_0\right)-4\delta A_0c_2\right\} \\
&+\frac{c_1}{2}\left\{4e^{-2\phi}c_1+\gamma\left(-4\frac{A_0'}{G}-12A_0\left(\frac{F'}{FG}+\frac{H'}{HG}\right)+12S_4\frac{H'}{HG}+2c_1(A_2+3A_1)\right)\right. \\
&\left.+4\delta c_2(A_2-A_1)\right\}=0.
\end{aligned} \tag{5.85}$$

$$\begin{aligned}
&-2e^{-2\phi}(A_0+3S_4)+\frac{1}{G}\left[e^{-2\phi}c_1+\gamma\left(-4A_0\frac{F'}{FG}+2c_1A_1\right)+2\delta c_2A_2\right]' \\
&+\left(\frac{F'}{FG}+3\frac{H'}{HG}\right)\left\{e^{-2\phi}c_1+\gamma\left(-4A_0\frac{F'}{FG}+2c_1A_1\right)+2\delta c_2A_2\right\} \\
&+\frac{F'}{FG}\left\{-e^{-2\phi}c_1+\gamma\left(4\frac{A_0'}{G}+12(A_0-S_4)\frac{H'}{HG}-2c_1A_2\right)+\delta(-4A_1+2A_2)c_2\right\} \\
&+\frac{c_1}{2}\left\{4e^{-2\phi}\frac{\phi'}{G}+4\gamma\left(-\frac{A_1'}{G}+(A_2-A_1)\frac{F'}{FG}+3(S_3-A_1)\frac{H'}{HG}-\frac{c_1}{2}A_0\right)-4\delta A_0c_2\right\} \\
&=0
\end{aligned} \tag{5.86}$$

For $c \in \{4,5,6\}$, we get one independent equation:

$$
\begin{aligned}
&e^{-2\phi}\left\{4A_1 + 2A_2 + 4S_1 + 2S_2 + 8S_3 - \alpha\left(\frac{\phi'}{G}\right)^2 - \beta(c_1^2 + c_2^2)\right\} + V(\phi) \\
&+\gamma\left\{4A_1^2 + 2A_2^2 + 4A_0^2 + 2S_1^2 - 2S_2^2 + 4S_3^2 + 4S_4^2 + 6S_0^2\right\} - \frac{1}{G}\Bigg[4e^{-2\phi}\frac{\phi'}{G} \\
&+4\gamma\left(-\frac{S_1'}{G} + 2(S_3 - S_2)\frac{F'}{FG} + 2(S_2 - S_1)\frac{H'}{HG} + c_1S_4 - 2S_0c_2\right) - 8\delta S_0c_1\Bigg]' \\
&-2\left(\frac{F'}{FG} + \frac{H'}{HG}\right)\Bigg[4e^{-2\phi}\frac{\phi'}{G} + 4\gamma\Bigg(-\frac{S_1'}{G} + 2(S_3 - S_2)\frac{F'}{FG} + 2(S_2 - S_1)\frac{H'}{HG} \\
&+c_1S_4 - 2S_0c_2\Bigg) - 8\delta S_0c_1\Bigg]' - c_2\Bigg\{-c_2e^{-2\phi} - 2\gamma\Bigg(-\frac{S_0}{G} - 2S_0\frac{F'}{FG} + 2S_0\frac{H'}{HG} \\
&+c_2S_2\Bigg) - 2\delta c_1S_2\Bigg\} = 0
\end{aligned}
\tag{5.87}
$$

There are two assumptions that we can make here: (i) $f = g$ and (ii) $g = 1$, that define different types of spacetimes. However, we can not obtain an explicit solution other than the solution we obtained in section 5.2. Therefore, the equations above are left for further investigation.

Chapter 6

CONCLUSION

In this work, we consider gravity theories alternative to Einstein's theory of gravity, where the field content is usually inspired by effective string theories and discuss their cosmological implications. First, we study the Brans-Dicke gravity coupled to a mass-varying vector field, where the mass depends on the scalar field. We derive the variational field equations. Assuming a Bianchi type I metric and homogeneous time dependent scalar and vector fields we find a set of solutions parametrized by two variables. We also discussed the special cases where we can compare the cosmological parameters with the observational parameters. We show in our model that the accelerated expansion is driven by the scalar field. The vector field acts as a cold dark matter source. A constant expansion anisotropy is found to be due to both the scalar and vector fields that interact nontrivially through the mass of the vector field.

Secondly, we work on a higher dimensional model of gravity with a second order Euler-Poincaré form. The metric employs two scale factors to distinguish between the internal and external dimensions. In accordance with the metric, we take a matter distribution modelled by a cosmic fluid which exerts different pressures along the external and internal spaces. With the constant volume assumption extra space dimensions(internal dimensions) contract, while the three space dimensions(external dimensions) we can observe, expands. It is speculated that this picture would be driven by mass that is being continuously transferred from the contracting n-dimensional internal space into the expanding 3-dimensional external space [58]. We further introduce the assumption that this anisotropic fluid satisfies an equation of state with dimensional dichotomy. We show that the expansion of the external space in our model provides compatible results with the standard ΛCDM model. Even if we omit the second order Euler-Poincaré term, the solution still shows the same behaviour. However, the absence of Euler-Poincaré term results in a negative mass density, while the solution where all the Einstein-Hilbert and the second order Euler-Poincaré terms are considered can maintain the positivity of the mass density properly.

Finally, we consider the bosonic part of the six dimensional chiral gauged supergravity action[68] with a R^2 term as our modified gravity model. We first transformed the action by fixing a connection with a particular torsion tensor. Then we found the variational equations under the constraints put on the torsion and the antisymmetric 3-form field H. In order to determine the field equations on $AdS_3 \times S^3$ background, we benefit from the fact that AdS_3 and S^3 are both Lie group manifolds. We write the metric according to the corresponding Lie algebras without specifying a coordinate system and obtain the reduced field equations. We find a zero curvature solution. We also investigate the model in a time warped $AdS_3 \times S^3$ spacetime. Now, in order to specify the time coordinate, we give the AdS_3 metric in terms of a coordinate system for which we introduce the corresponding left invariant forms used in the previous solution. Unfortunately, we can not get an exact solution of the reduced system of equations on the warped background, which would have provided us with some cosmological results.

BIBLIOGRAPHY

[1] A. G. Riess et al., *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, Astron. J. **116** (1998) 1009.

[2] S. Perlmutter et al., *Measurements of Omega and Lambda from 42 High-Redshift Supernovae*, Astrophys. J. **517** (1999) 565.

[3] T. Padmanabhan, *Cosmological constant-the weight of the vacuum*, Phys. Rept. **380** (2003) 235-320.

[4] P.A.R. Ade et al. (Planck Collaboration), *Planck 2013 results. XVI. Cosmological parameters*, Astron. Astrophys. **571** (2014).

[5] P.A.R. Ade et al. (Planck Collaboration), *Planck 2013 results. XXIII. Isotropy and Statistics of the CMB*, Astron. Astrophys. **571** (2014).

[6] P.A.R. Ade et al. (Planck Collaboration), *Planck 2013 results. XXVI. Background geometry and topology of the Universe*, Astron.Astrophys. **571** (2014).

[7] C.L. Bennett et al. (WMAP collaboration), *Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Are There Cosmic Microwave Background Anomalies?*, Astrophys. J. Suppl. **192** (2011) 17.

[8] Y.B. Zeldovich,*The cosmological constant and the theory of elementary particles*, Sov. Phys. Usp. 11 (1968) 381-393.

[9] S. Weinberg, *The cosmological constant problem*, Rev. Mod. Phys. **61** (1989) 1-23.

[10] Ö. Akarsu, T. Dereli and J.A. Vazquez, *A divergence-free parametrization for dynamical dark energy*, JCAP, **06** (2015) 49.

[11] E.J. Copeland, , M. Sami, and S. Tsujikawa, *Dynamics of dark energy* ,International Journal of Modern Physics D **15** (2006) 1753-1935.

[12] L.H. Ford, *Inflation driven by a vector field*, Phys. Rev. D **40** (1989) 967-972.

[13] A. Golovnev, V. Mukhanov, V. Vanchurin, *Vector Inflation*, JCAP **06** (2008) 009.

[14] T. Koivisto, D.F. Mota, *Vector field models of inflation and dark energy*, JCAP **08** (2008) 021.

[15] S. Kanno, M. Kimura, J. Soda, S. Yokoyama, *Anisotropic inflation from vector impurity*, JCAP **08** (2008) 034.

[16] S. Kanno, J. Soda, M. Watanabe, *Anisotropic power-law inflation*,JCAP **12** (2010) 024.

[17] B. Himmetoglu, C.R. Contaldi, M. Peloso, *Instability of Anisotropic Cosmological Solutions Supported by Vector Fields*, Phys. Rev. Lett. **102** (2009) 111301.

[18] T. Koivisto, D.F. Mota, *Dark energy anisotropic stress and large scale structure*, Phys. Rev. D **73** (2006) 083502.

[19] T. Koivisto, D.F. Mota, *Accelerating cosmologies with an anisotropic equation of state*, Astrophys. J. **679** (2008) 1-5.

[20] D.C. Rodrigues, *Anisotropic cosmological constant and the CMB quadrupole anomly*, Phys. Rev. D **77** (2008) 023534.

[21] R. Battye, A. Moss, *Anisotropic dark energy and CMB anomalies*, Phys. Rev. D **80** (2009) 023531.

[22] Ö. Akarsu, C.B. Kılınç, *LRS Bianchi type I models with anisotropic dark energy and constant deceleration parameter*, Gen. Rel. Grav. **42** (2010) 119-140.

[23] Ö. Akarsu, C.B. Kılınç, *de Sitter expansion with anisotropic fluid in Bianchi type-I space-time*, Astrophys. Space Sci. **326** (2010) 315-322.

[24] L. Campanelli, P. Cea, G. L. Fogli, L. Tedesco, *Anisotropic dark energy and ellipsoidal universe*, Int. J. Mod. Phys. D **20** (2011) 1153-1166.

[25] M. Thorsrud, D.F. Mota, S. Hervik, *Cosmology of a Scalar Field Coupled to Matter and an Isotropy-Violating Maxwell Field*, J. High Energ. Phys. **110** (2012) 066.

[26] E. S. Fradkin, A. A. Tseytlin, *Effective field theory from quantized strings*, Phys. Lett. **158B** (1985) 316

[27] E. S. Fradkin, A. A. Tseytlin, *Quantum string field theory effective action*, Nucl. Phys. **B261** (1985) 1.

[28] C. G. Callan, I. Klebanov, M. Perry, *String theory effective actions*, Nucl. Phys. **B278** (1986) 78.

[29] D. J. Gross, J. H. Sloan, *The quartic effective action for the heterotic string*, Nucl. Phys. **B291** (1987) 41.

[30] D. G. Boulware, S. Deser, *String generated gravity models*, Phys. Rev. Lett.**55** (1985) 2656.

[31] B. Zwiebach, *Curvature squared terms and string theories*, Phys. Lett. **156B** (1985) 315.

[32] B. Zumino, *Gravity theories in more than four dimensions*, Phys. Rep. **137** (1986) 109.

[33] S. Deser, A. N. Redlich, *String-induced gravity and ghost freedom*, Phys.Lett. **176B** (1986) 350.

[34] T. Dereli, R. W. Tucker, *Variational methods and effective actions in string models*, Class.Q.Grav. **4** (1987) 791.

[35] M. Arık, T. Dereli, *Euler-Poincaré Lagrangians and Kaluza-Klein theory*, Phys. Lett. **189B** (1987) 96.

[36] M. Arık, T. Dereli, *Euler-form actions and the vanishing of the cosmological constant*, Phys. Rev. Lett. **62** (1989) 5.

[37] T. Dereli, G. Üçoluk, *Direct-curvature Yang-Mills field couplings induced by the Kaluza-Klein reduction of Euler-Poincaré actions in seven dimensions*, Class. Q. Grav. **7** (1990) 533.

[38] J. Madore, *Kaluza-Klein cosmology with the Lanczos Lagrangian*, Phys. Lett. **110A** (1985) 289

[39] E. Müller-Hoissen, *Spontaneous compactification with quadratic and cubic curvature terms*, Phys. Lett. **163B** (1985) 106.

[40] H. Ishihara, *Cosmological solutions of the extended Einstein gravity with the Gauss-Bonnet term*, Phys. Lett. **179B** (1986) 217.

[41] D. Lorentz-Petzold, *Higher-dimensional Gauss-Bonnet cosmologies*, Mod. Phys. Lett. **A3** (1988) 827.

[42] N. Deruelle, L. Fariňa-Busto, *Lovelock gravitational field equations in cosmology*, Phys. Rev. **D41** (1990) 3696

[43] T. Dereli, Ş. Özkurt, *Dynamical reduction of first-order effective string field equations in the early universe*, Class. Q. Grav.**8** (1991) 889.

[44] G. Calcagni, S. Tsujikawa, M. Sami, *Dark energy and cosmological solutions in second-order string gravity*, Class. Q. Grav. **22** (2005) 3977.

[45] S. Nojiri, S. D. Odintsov, M. Sasaki, *Gauss-Bonnet dark energy*, Phys. Rev. **D71** (2005) 123509.

[46] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, S. Zerbini, *String-inspired Gauss-Bonnet gravity reconstructed from the universe expansion history and yielding the transition from matter dominance to dark energy*, Phys. Rev. D **75** (2007) 086002.

[47] K. Andrew, B. Bolen, C. A. Middleton, *Solutions of higher dimensional Gauss-Bonnet FRW cosmology*, Gen. Rel. Grav. **39** (2007) 2061.

[48] B. M. Leith, I. P. Neupane, *Gauss-Bonnet cosmologies: Crossing the phantom divide and the transition from matter dominance to dark energy*, JCAP **0705** (2007) 019.

[49] K. Bamba, Z.-K. Guo, N. Ohta, *Accelerating cosmologies in the Einstein-Gauss-Bonnet theory with dilaton*, Prog. Theor. Phys. **118** (2007) 879.

[50] R. Chingangbam, M. Sami, P. V. Tretyakov, A. V. Toporensky, *A note on the viability of Gauss-Bonnet cosmology*, Phys. Lett. B **661** (2008) 162.

[51] I. Benn and R.Tucker, *An introduction to spinors and geometry with applications in physics*, Bristol: A. Hilger (1987).

[52] W., Thirring, *A Course in Mathematical Physics 1 and 2: Classical Dynamical Systems and Classical Field Theory*, Springer Verlag (2012).

[53] M. Nakahara, *Geometry, topology, and physics*, CRC Press (2003).

[54] Ö. Akarsu , T. Dereli, and N. Oflaz, *Accelerating anisotropic cosmologies in BransDicke gravity coupled to a mass-varying vector field*, Classical and Quantum Gravity **31** (2014) 045020.

[55] Ö. Akarsu , T. Dereli, and N. Oflaz, *Accelerated Expansion of the Universe in a Higher Dimensional Modified Gravity with Euler-Poincaré Terms*, arXiv preprint arXiv:1507.08225 (2015).

[56] D. Lovelock, *The Einstein tensor and its generalizations*, J. Math. Phys. 12 (1971) 498.

[57] D. Lovelock, *The four-dimensionality of space and the Einstein tensor*, J. Math. Phys. 13 (1972) 874.

[58] Ö. Akarsu, T. Dereli, *Late time acceleration of the 3-space in a higher dimensional steady state universe in dilaton gravity*, JCAP **02** (2013) 050.

[59] Ö. Akarsu, T. Dereli, The dynamical evolution of 3-space in a higher dimensional steady state universe, Gen. Relat. Gravit. **45** (2013) 959-986.

[60] M. Cvetic, G.W. Gibbons, C.N. Pope, *A string and M-theory origin for the Salam–Sezgin model*, Nuclear Physics B **677** (2004) 164–180.

[61] Bergshoeff, E. and Salam, A. and Sezgin, E., *Supersymmetric R2 actions, conformal invariance and the Lorentz Chern-Simons term in 6 and 10 dimensions*, Nuclear Physics B **279** (1987) 659–683.

[62] S.S. Chern, *Complex manifolds without potential theory*, Van Nostrand, Princeton (1967).

[63] C.B. Collins, S.W. Hawking, *Why is the Universe Isotropic?*, Astrophys. J. **180** (1973) 317-334.

[64] S. Weinberg, *Gravitation and cosmology*, Wiley, New York (1972).

[65] M.J. Duff, R.R. Khuri, J.X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213-326.

[66] J.E. Lidsey, D. Wands, E.J. Copeland, *Superstring cosmology*, Phys. Rept. **337** (2000) 343-492.

[67] V. Sahni, A. A. Starobinsky,*The case for a positive cosmological* Λ*-term*, Int. J. Mod. Phys. D **9** (2000) 373.

[68] H. Lü, C.N. Pope, E. Sezgin, *Massive three-dimensional supergravity from* $\mathcal{R}+\mathcal{R}^2$ *action in six dimensions*, JHEP **10** (2010) 1–27.

[69] T. Dereli, R. W. Tucker, *An Einstein-Hilbert action for axi-dilaton gravity in four dimensions*, Class. Quantum Grav. **12** (1995) 31.

[70] E. Bergshoeff, , F. Coomans, E. Sezgin, and A. Van Proeyen, *Higher derivative extension of 6D chiral gauged supergravity*, JHEP , **7** (2012) 1-30.

[71] Coussaert, O., Henneaux M., *Self-dual solutions of 2+ 1 Einstein gravity with a negative cosmological constant*, The Black Hole, 25 Years After **10** (1998).

[72] Bengtsson, I., Sandin, P., *Anti-de Sitter space, squashed and stretched*, Classical and Quantum Gravity, **23** (2006) 971.