CLOSED-FORM GREEN'S FUNCTIONS IN CYLINDRICALLY STRATIFIED MEDIA FOR METHOD OF MOMENTS APPLICATIONS

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND

ELECTRONICS ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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August, 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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August, 2006

Closed-form Green's function representations for cylindrically stratified media, which can be used in conjunction with a Method of Moments, are developed. The closed-form expressions are valid for all possible source and field points including the axial line ($\rho = \rho'$ and $\phi = \phi'$), which has not been available before.

Closed-form expressions in cylindrically stratified media are first reported by Tokgöz (M.S. thesis, 1997). However, the given expressions have accurate results only when the field point (ρ,ϕ,z) is far away from the source point (ρ',ϕ',z'), in particular along the ρ-axis. Therefore, the expressions given by Tokgöz, can not be used in a Method of Moments solution procedure for antenna problems.

The spectral domain Green's function components are modified in order to perform calculations for all possible source and field points. In the course of obtaining closed-form expressions, the modified spectral domain expressions are calculated efficiently and transformed to the spatial domain in the form of discrete complex images with the help of the generalized pencil of function method. Numerical results are presented in the form of mutual coupling between two tangential current modes which lie on the same air-dielectric interface, and compared with the eigenfunction results to assess the accuracy of the developed closed-form Green's function representations. Excellent agreement with the eigenfunction solution is obtained for arbitrary source and field locations.

Keywords: Closed-form Green's function representations, discrete complex image method, generalized pencil of function method, mutual coupling, cylindrically stratified media.

ÖZET

SİLİNDİRİK KATMANLI ORTAMDA MOMENTLER METODU UYGULAMALARI İÇİN KAPALI FORMDA GREEN FONKSİYONLAR

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Silindirik katmanlı ortamlar için Momentler Metodu ile kullanılabilecek kapalı formda Green fonksiyonları elde edilmiştir. Kapalı formdaki ifadeler, daha önce verilmeyen bir durum olan bütün kaynak ve gözlem noktaları için geçerlidir.

Silindirik katmanlı ortamlarda kapalı formdaki ifadeler ilk olarak Tokgöz tarafından verilmiştir (Yüksek Lisans Tezi, 1997). Fakat verilen ifadeler sadece gözlem noktası (ρ,ϕ,z) özellikle ρ-ekseni boyunca kaynak noktasından (ρ',ϕ',z') uzak olduğunda geçerlidir. Bu yüzden Tokgöz tarafından verilen ifadeler anten problemleri için Momentler Metodu çözümlerinde kullanılamaz.

İzgel düzlemdeki Green fonksiyonları, hesaplamaları bütün kaynak ve gözlem noktalarında yapabilmek için değiştirilmiştir. Kapalı formdaki ifadeleri hesaplama esnasında, izgel düzlemdeki ifadeler hesaplanmış, kalem fonksiyonu metodu kullanılarak uzamsal düzlemde kompleks imgeler şeklinde yazılmıştır. Nümerik sonuçlar aynı hava-dielektrik yüzeyinde yer alan yüzeye teğet iki akım modelinin karşılıklı etkileşimi şeklinde verilmiş ve kapalı formdaki ifadelerin doğruluğu özişlev sonuçları ile karşılaştırılmıştır. Değişik kaynak ve gözlem noktaları için özişlev sonuçları ile uyum mükemmeldir.

Anahtar sözcükler: Kapalı-form Green fonksiyon gösterimleri, ayrık kompleks imge metodu, genelleştirilmiş kalem fonksiyonu metodu, karşılıklı etkileşim, silindirik katmanlı ortamlar.

To My Father, the angel in heaven
To My Mother, the angel in earth
who are
the answers to all my how's and why's

Acknowledgement

I would like to express my gratitude to my advisor Asst. Prof. Dr. Vakur B. Ertürk for his instructive comments and continuing support in the supervision of the thesis.

I would like to express my special thanks and gratitude to Prof. Dr. Ayhan Altıntaş and Prof. Dr. Gülbin Dural for showing keen interest to the subject matter and accepting to read and review the thesis.

Finally I would like to thank Aselsan Inc. for letting me to involve in this thesis study and my family for their support.

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Chapter 1

Introduction

Many military and commercial applications require microstrip antennas and/or arrays, or other types of printed structures that conform to the host platform due to aerodynamic constraints, reduced radar cross section (compared to antennas and arrays on planar platforms), wider scan range and aesthetic reasons. Although the method of moments (MoM) based solutions are widely used for the rigorous analysis of printed structures on planar layered media [1]-[4], available tools for their curved counterparts are limited, and may suffer from the efficiency and accuracy problems. Among the curved geometries, cylindrical ones are very important since they sufficiently model the geometry of many complicated applications. Besides, they may be treated as the canonical geometries, whose solutions are regarded as the building blocks toward the development of analysis tools for arbitrary convex curved bodies.

A number of studies regarding the Green's functions in cylindrically stratified media have been reported before [5]-[13]. However, a vast majority of them are not in closed-form. Besides, majority of them either can not be used in conjunction with MoM or become extremely inefficient when used for antenna problems. Among them, closed-form results are given in [10]-[12] but they are not valid for arbitrary source and field points. In [13], a closed-form solution for cylindrically conformal microstrip antennas is given. However, the provided closed-from expressions are for the impedance matrix and the voltage vector (using entire domain basis functions) rather than the Green's functions, which can not be easily extended to a multilayer problem. Therefore, in this thesis we present closed-form Green's function representations in cylindrically stratified media which are valid for all possible source and field points. Consequently, the developed expressions can be used in conjunction with MoM to investigate antennas and arrays embedded in cylindrically layered media.

Spectral domain Green's function expression involves a summation over n that represents the cylindrical eigenmodes and a Fourier integral over k_z. The summation becomes a slowly convergent summation when both the field and source points lie on the same interface, which is the case for all MoM applications. As the first step special cylindrical functions, like Bessel and Hankel functions, are expressed in the form of ratios and their Debye representations [14] are used for large n values to avoid possible numerical overflow/underflow problems. The summation of cylindrical eigenmodes n is performed in the spectral domain and several techniques provided in [12] are implemented to improve the efficiency of this summation. Once the summation of the cylindrical eigenmodes is performed to have an integrable expression of the Fourier integral over k_z for all possible source and field points, the technique given in [12] is successfully implemented.

Finally after the summation and integration problems are solved, the closed-form spatial domain Green's function expressions are obtained using the discrete complex image method (DCIM) with the help of the generalized pencil of function (GPOF) method on a deformed integration path. Unlike previous studies, the large argument approximation of the Hankel function is not used. This way the singularity problem at $\phi = \phi'$ is easily handled, which has not been reported in literature before to the best of our knowledge.

In this thesis, the tangential electric field Green's functions (G^E_{zz}, $G^E_{\phi z}$, $G^E_{z\phi}$, $G^E_{\phi\phi}$) due to $\hat{z}$ and $\hat{\phi}$-directed tangential electric sources are analyzed. G is used for the spatial domain Green's function, whereas $\tilde{G}$ denotes its spectral domain counterpart. $\bar{X}$ is used for a 2×2 matrix. The organization of this thesis is as follows: In Chapter II, a brief summary of [10] is given. Chapter III deals with the $\rho = \rho'$ problem and gives the details of the computation technique. The other problem, $\phi = \phi'$ case is solved in chapter IV. Chapter V gives the closed-form Green's function representations in the spatial domain. Numerical results are presented in Chapter VI, which show the accuracy of the closed-form Green's function representations. Concluding remarks are in Chapter VII. Finally, three Appendices are provided. In Appendix A, a fairly detailed explanation about the GPOF method is given. In Appendix B, Debye representations of the Hankel and Bessel functions are given. Finally in Appendix C, current modes and their spectral domain counterparts are presented. Furthermore throughout this thesis, the time dependence of $e^{j\omega t}$ is used, where $\omega = 2\pi f$ and f is the operating frequency.

Chapter 2

Green's Function Representations

A general geometry of a cylindrically stratified media is given in Fig. 2.1. The point source is located at (ρ', ϕ', z') in the source layer $i = j$ and the field point is located at (ρ, ϕ, z) in the field layer $i = m$. As shown in Fig. 2.1, layers may vary in their electric and magnetic properties (ϵ_i,μ_i) as well as their thicknesses. Moreover, a perfect electric conductor (PEC) or perfect magnetic conductor (PMC) can be considered as the innermost or the outermost layer.

In this chapter, the spectral domain Green's function expressions given by [10]-[11] for $\rho \neq \rho'$ are summarized since the same expressions will be used for $\rho = \rho'$ case after they are modified. The details of the formulation given in this chapter can be found in [8] and [10]-[11].

2.1 Spectral Domain Field Expressions

The current of a dipole can be written in the spectral domain with respect to k_z as

$$\vec{J}(\vec{r}) = I\ell\hat{\alpha}e^{jk_z z'}\frac{\delta(\rho - \rho')}{\rho}\delta(\phi - \phi') \tag{2.1}$$

where $I\ell$ is the current moment, $\hat{\alpha}$ is the unit direction which shows the direction of current and z' is the location of the dipole along the z-axis. The field expression

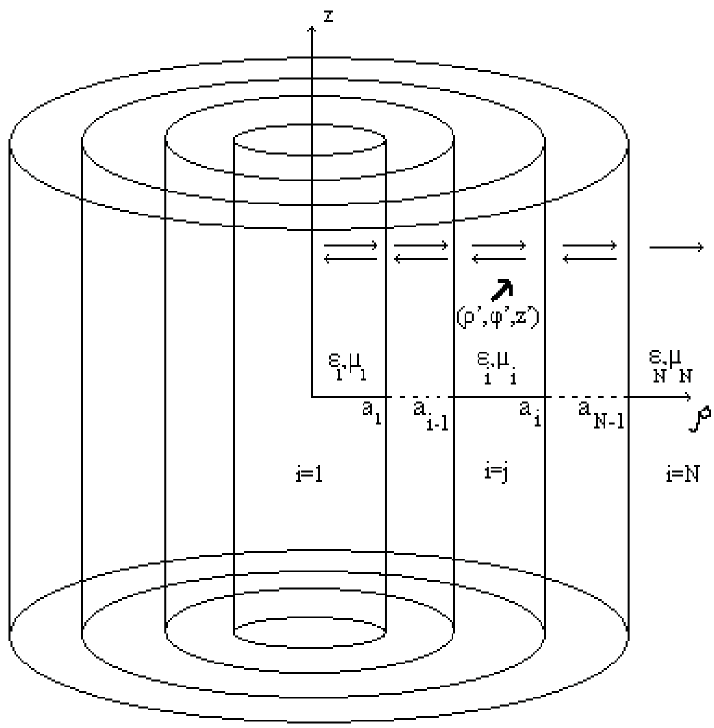

Figure 2.1: Propagation of waves due to a point source in a stratified media

in the observation point is the sum of the standing and outgoing waves formed by the multiple reflections from the inner and outer boundaries. The standing and outgoing waves are represented by the first-kind Bessel and second-kind Hankel functions, respectively. The z components of the fields are given in [10] as

$$\begin{bmatrix} \tilde{E}_z \\ \tilde{H}_z \end{bmatrix} = -\frac{I\ell e^{jk_z z'}}{4w} \sum_{n=-\infty}^{\infty} e^{jn(\phi-\phi')} \bar{F}_n \vec{S}_{n_j}. \tag{2.2}$$

For an electric source, $\vec{S}_{n_j}$ is a 2×1 matrix operator given by

$$\vec{S}_{n_j} = \begin{bmatrix} \frac{1}{\epsilon_j}(k_j^2 \hat{a}_z + jk_z \vec{\nabla}')\hat{\alpha} \\ -jw\hat{\alpha}(\hat{a}_z \times \vec{\nabla}') \end{bmatrix} \tag{2.3}$$

and acts to its left-hand side. In (2.3) $\vec{\nabla}'$ is defined to be

$$\vec{\nabla}' = \hat{a}_\rho \frac{\partial}{\partial \rho'} - \hat{a}_\phi \frac{jn}{\rho'} + \hat{a}_z jk_z. \tag{2.4}$$

When the field and source layers are the same $(m = j)$ $\bar{F}_n$, which is a 2×2 matrix, is defined as

$$\begin{aligned}
\bar{F}_n &= [J_n(k_{\rho_j}\rho)\bar{I} + H_n^{(2)}(k_{\rho_j}\rho)\tilde{\bar{R}}_{j,j-1}]\tilde{\bar{M}}_{j-}[H_n^{(2)}(k_{\rho_j}\rho')\bar{I} + J_n(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j+1}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad for\ \rho < \rho' \\
\bar{F}_n &= [H_n^{(2)}(k_{\rho_j}\rho)\bar{I} + J_n(k_{\rho_j}\rho)\tilde{\bar{R}}_{j,j+1}]\tilde{\bar{M}}_{j+}[J_n(k_{\rho_j}\rho')\bar{I} + H_n^{(2)}(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j-1}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad for\ \rho > \rho'
\end{aligned} \tag{2.5}$$

and when the field and source layers are different $(m \neq j)$ $\bar{F}_n$ becomes

$$\begin{aligned}
\bar{F}_n &= [J_n(k_{\rho_m}\rho)\bar{I} + H_n^{(2)}(k_{\rho_m}\rho)\tilde{\bar{R}}_{m,m-1}]\tilde{\bar{T}}_{j,m}\tilde{\bar{M}}_{j-}[H_n^{(2)}(k_{\rho_j}\rho')\bar{I} + J_n(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j+1}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad for\ m < j \\
\bar{F}_n &= [H_n^{(2)}(k_{\rho_m}\rho)\bar{I} + J_n(k_{\rho_m}\rho)\tilde{\bar{R}}_{m,m+1}]\tilde{\bar{T}}_{j,m}\tilde{\bar{M}}_{j+}[J_n(k_{\rho_j}\rho')\bar{I} + H_n^{(2)}(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j-1}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad for\ m > j.
\end{aligned} \tag{2.6}$$

In (2.5) and (2.6) $\tilde{\bar{M}}_j\pm$ is defined as

$$\tilde{\bar{M}}_j\pm = (\bar{I} - \tilde{\bar{R}}_{j,j\mp1}\tilde{\bar{R}}_{j,j\pm1})^{-1} \tag{2.7}$$

where, $\bar{I}$ is the unity matrix and $\tilde{\bar{R}}_{j,j\mp1}$ is the generalized reflection matrix. The generalized reflection matrix $\tilde{\bar{R}}_{j,j-1}$ contains multiple reflections from the inner layers with respect to j, while $\tilde{\bar{R}}_{j,j+1}$ contains multiple reflections from the outer layers. The subscript j denotes that $\tilde{\bar{R}}_{j,j\mp1}$ is the generalized reflection matrix for layer j. The generalized reflection matrix can be defined as

$$\tilde{\bar{R}}_{i,i\pm1} = \bar{R}_{i,i\pm1} + \bar{T}_{i\pm1,i}\tilde{\bar{R}}_{i\pm1,i\pm2}\tilde{\bar{T}}_{i,i\pm1} \tag{2.8}$$

where i denotes an arbitrary layer between 1 and N. Similar to $\tilde{\bar{R}}$, $\tilde{\bar{T}}$ is the generalized transmission matrix, which is defined as

$$\tilde{\bar{T}}_{i,i\pm1} = (\bar{I} - \bar{R}_{i\pm1,i}\tilde{\bar{R}}_{i\pm1,i\pm2})^{-1}\bar{T}_{i,i\pm1}. \tag{2.9}$$

In (2.8) and (2.9), $\bar{R}$ and $\bar{T}$ denote the local reflection and transmission matrices. They contain the interactions between only the two layers which are given in their subscripts. Consequently, the local reflection and transmission matrices $\bar{R}$ and $\bar{T}$, respectively, are given by

$$\bar{R}_{i,i+1} = \bar{D}_i^{-1}[H_n^{(2)}(k_{\rho_i}a_i)\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i) - H_n^{(2)}(k_{\rho_{i+1}}a_i)\bar{H}_n^{(2)}(k_{\rho_i}a_i)] \tag{2.10}$$

$$\bar{T}_{i,i+1} = \frac{2\omega}{\pi k_{\rho_i}^2 a_i}\bar{D}_i^{-1}\begin{bmatrix} \epsilon_i & 0 \\ 0 & -\mu_i \end{bmatrix} \tag{2.11}$$

$$\bar{R}_{i+1,i} = \bar{D}_i^{-1}[J_n(k_{\rho_i}a_i)\bar{J}_n(k_{\rho_{i+1}}a_i) - J_n(k_{\rho_{i+1}}a_i)\bar{J}_n(k_{\rho_i}a_i)] \tag{2.12}$$

$$\bar{T}_{i+1,i} = \frac{2\omega}{\pi k_{\rho_{i+1}}^2 a_i}\bar{D}_i^{-1}\begin{bmatrix} \epsilon_{i+1} & 0 \\ 0 & -\mu_{i+1} \end{bmatrix} \tag{2.13}$$

with

$$\bar{D}_i = H_n^{(2)}(k_{\rho_{i+1}}a_i)\bar{J}_n(k_{\rho_i}a_i) - J_n(k_{\rho_i}a_i)\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i). \tag{2.14}$$

Note that all the reflection and transmission matrices are 2×2 matrices, since the TE and TM modes are coupled in cylindrically stratified media. In the aforementioned equations (2.10)-(2.14), we used $H_n^{(2)}(x)$, $\bar{H}_n^{(2)}(x)$, $J_n(x)$ and $\bar{J}_n(x)$. To write the expressions in a more compact form we will use $B_n(x)$ and $\bar{B}_n(x)$ such that

$$\bar{B}_n(k_{\rho_i}a_i) = \frac{1}{k_{\rho_i}^2 a_i}\begin{bmatrix} -j\omega\epsilon_i k_{\rho_i}a_i B_n'(k_{\rho_i}a_i) & nk_z B_n(k_{\rho_i}a_i) \\ nk_z B_n(k_{\rho_i}a_i) & j\omega\mu_i k_{\rho_i}a_i B_n'(k_{\rho_i}a_i) \end{bmatrix} \tag{2.15}$$

where $\bar{B}_n(x)$ can be $\bar{J}_n(x)$ or $\bar{H}_n^{(2)}(x)$ and hence, the corresponding $B_n(x)$ will be $J_n(x)$ or $H_n^{(2)}(x)$, respectively. Finally, in all the previous equations $'$ is used for the derivative with respect to $k_{\rho_i}a_i$ product such that $\frac{\partial}{\partial(k_{\rho_i}a_i)}$ with $k_{\rho_i} = \sqrt{k_i^2 - k_z^2}$ being the transverse propagation constant of the i^{th} layer and $k_i = \omega\sqrt{\epsilon_i\mu_i}$ being the wave number of i^{th} layer. In these formulations, layer $(i+1)$ is the outer layer and layer $(i-1)$ is the inner layer with respect to layer i. Besides, the reflection matrix $\bar{R}_{i,i-1}$ can be obtained from (2.12) by writing $(i-1)$ instead of i. Similarly, $\bar{R}_{i-1,i}$ term is obtained from (2.10) by writing $(i-1)$ instead of i.

The innermost or outermost layers in Fig. 2.1 can be a perfect electric conductor (PEC) or a perfect magnetic conductor (PMC) layer. The local reflection matrices from these layers are also given in [10]. However, PEC is mostly used as the innermost layer for many cylindrical structures. Therefore, the reflection matrix for an innermost PEC layer is given by

$$\bar{R}_{2,1} = \begin{bmatrix} -\frac{J_n(k_{\rho_2}a_1)}{H_n^{(2)}(k_{\rho_2}a_1)} & 0 \\ 0 & -\frac{J_n'(k_{\rho_2}a_1)}{H_n'^{(2)}(k_{\rho_2}a_1)} \end{bmatrix} \tag{2.16}$$

where a_1 is the radius of the PEC layer, which is denoted by i=1 in Fig. 2.1.

2.2 Spectral Domain Green's Function Representations Due to an Electric Source

The ϕ components of the fields in the field layer m are obtained from (2.2) (with the help of (2.3)-(2.16)) as ([11])

$$\begin{bmatrix} \tilde{H}_\phi \\ \tilde{E}_\phi \end{bmatrix} = \begin{bmatrix} -\frac{jw\epsilon_m}{k_{\rho m}}\frac{\partial}{\partial(k_{\rho m}\rho)} & \frac{nk_z}{k_{\rho m}^2\rho} \\ \frac{nk_z}{k_{\rho m}^2\rho} & \frac{jw\mu_m}{k_{\rho m}}\frac{\partial}{\partial(k_{\rho m}\rho)} \end{bmatrix} \begin{bmatrix} \tilde{E}_z \\ \tilde{H}_z \end{bmatrix}. \tag{2.17}$$

The spectral domain Green's functions, which relate the $\hat{\phi}$ and $\hat{z}$ directed electric fields with the $\hat{\phi}$ and $\hat{z}$ directed electric sources, are defined as

$$\begin{bmatrix} \tilde{E}_z \\ \tilde{E}_\phi \end{bmatrix} = \begin{bmatrix} \tilde{G}_{zz}^E & \tilde{G}_{z\phi}^E \\ \tilde{G}_{\phi z}^E & \tilde{G}_{\phi\phi}^E \end{bmatrix} \begin{bmatrix} \tilde{J}_z \\ \tilde{J}_\phi \end{bmatrix} \tag{2.18}$$

and are given in [10] as

$$\tilde{G}_{zz}^E = -\frac{1}{2\omega}\sum_{n=0}^{\infty} \epsilon_n \cos[n(\phi-\phi')]\frac{k_{\rho_j}^2}{\epsilon_j}F_n(1,1) \tag{2.19}$$

$$\frac{\tilde{G}_{\phi z}^E}{k_z} = -\frac{j}{2\omega}\sum_{n=1}^{\infty}\sin[n(\phi-\phi')]\left[\frac{nk_{\rho_j}^2}{\epsilon_j\rho k_{\rho_m}^2}F_n(1,1) + \frac{k_{\rho_j}^2 j\omega\mu_m}{k_{\rho_m}\epsilon_j k_z}\frac{\partial F_n(2,1)}{\partial(k_{\rho_m}\rho)}\right] \tag{2.20}$$

$$\frac{\tilde{G}_{z\phi}^E}{k_z} = -\frac{j}{2\omega}\sum_{n=1}^{\infty}\sin[n(\phi-\phi')]\left[\frac{n}{\epsilon_j\rho'}F_n(1,1) - \frac{j\omega k_{\rho_j}}{k_z}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')}\right] \tag{2.21}$$

$$\begin{aligned}\tilde{G}_{\phi\phi}^E = & -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n\cos[n(\phi-\phi')]\left[\frac{n^2k_z^2}{\epsilon_j\rho\rho' k_{\rho_m}^2}F_n(1,1) + \frac{j\omega\mu_m nk_z}{k_{\rho_m}\epsilon_j\rho'}\frac{\partial F_n(2,1)}{\partial(k_{\rho_m}\rho)}\right. \\ & \left. -\frac{j\omega nk_zk_{\rho_j}}{k_{\rho_m}^2\rho}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')} + \omega^2\mu_m\frac{k_{\rho_j}}{k_{\rho_m}}\frac{\partial^2 F_n(2,2)}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_m}\rho)}\right]\end{aligned} \tag{2.22}$$

where $\epsilon_n = 0.5$ for $n = 0$ and 1, otherwise and $F_n(1,1)$, $F_n(1,2)$, $F_n(2,1)$ and $F_n(2,2)$ are the entries of $\bar{F}_n$. In (2.19)-(2.22), to increase the efficiency in the computation of the Green's function components, the odd and even properties of these components are used. For instance, using the odd and even properties with respect to cylindrical eigenmodes n, the summations that range from $-\infty$ to ∞ are folded over all n to range from 0 to ∞. Similarly, to speed up the integration with respect to k_z, if the Green's function component is an even function of k_z, the k_z integral is converted to a 0 to ∞ integral. However, if the Green's function component is an odd function of k_z, it is divided by k_z so the integrand becomes an even function of k_z. Then, the resultant integral is converted to a 0 to ∞ integral.

However, the Green's function representations given by (2.19)-(2.22) (and also given in [10]) can be used only when ρ is far away from ρ'. They can not be used when $\rho = \rho'$. The first problem for the calculation of spectral domain Green's functions when $\rho = \rho'$ is the slowly convergent summations. In order to make the expressions valid when $\rho = \rho'$ and to speed up their computations, the spectral domain Green's function expressions given by (2.19)-(2.22) are modified as explained in a detailed way in the next chapter.

Chapter 3

Spectral Domain Green's Function Representations and Their Computation when $(\rho = \rho')$

When $\rho = \rho'$, it is not possible to compute the spectral domain Green's function representations using (2.19)-(2.22). Therefore, in this chapter (2.19)-(2.22) are modified to be valid at $\rho = \rho'$.

The first problem when $\rho = \rho'$ is the slowly convergent behaviour of 0 to ∞ summations used in (2.19)-(2.22). Since these summations are slowly convergent, computation of Hankel and Bessel functions for large n values becomes mandatory. However, for large n values, evaluation of the Hankel (Bessel) functions shows overflow/underflow problems. Therefore, instead of evaluating each Hankel (Bessel) function one at a time, they are written in the form of ratios (i.e., each Hankel (Bessel) function is written in ratio form with another Hankel (Bessel) function) and these ratios are directly evaluated. In this thesis, an expression in the form of ratios means that all the Hankel (Bessel) functions in the expression are written in ratio form with another Hankel (Bessel) function.

3.1 Spectral Domain Expressions In The Form of Ratios

Although the case of $\rho = \rho'$ is being analyzed, ρ is not written instead of ρ' or vice versa, in order not to create any confusion in the Green's function expressions. This way, the developed expressions can easily be made valid for multilayer problems. Besides, there are derivatives with respect ρ and ρ' that should be distinguished. Therefore, only at the final expressions when a simplification is required, ρ is equated to ρ'.

In order to write the spectral domain Green's function representations in the form of ratios, we start with $\bar{B}_n(k_{\rho_i}a_i)$ term given by (2.15), and rewrite it as

$$\bar{B}_n(k_{\rho_i}a_i) = \frac{B_n(k_{\rho_i}a_i)}{k_{\rho_i}^2 a_i} \begin{bmatrix} -j\omega\epsilon_i k_{\rho_i} a_i \frac{B_n'(k_{\rho_i}a_i)}{B_n(k_{\rho_i}a_i)} & nk_z \\ nk_z & j\omega\mu_i k_{\rho_i} a_i \frac{B_n'(k_{\rho_i}a_i)}{B_n(k_{\rho_i}a_i)} \end{bmatrix}. \tag{3.1}$$

Then, the $\bar{D}_i$ expression in (2.14) is rewritten as

$$\bar{D}_i = H_n^{(2)}(k_{\rho_{i+1}}a_i) J_n(k_{\rho_i}a_i) \left(\frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} \right). \tag{3.2}$$

If we define

$$\bar{D}_i = H_n^{(2)}(k_{\rho_{i+1}}a_i) J_n(k_{\rho_i}a_i) \bar{D}_{in} \tag{3.3}$$

we obtain

$$\bar{D}_{in} = \frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} \tag{3.4}$$

and substituting (3.1) into (3.4), it is seen that (3.4) is obtained in the form of ratios. In a similar manner the local reflection matrix given by (2.10) can be written as

$$\bar{R}_{i,i+1} = \bar{D}_i^{-1} [H_n^{(2)}(k_{\rho_i}a_i) H_n^{(2)}(k_{\rho_{i+1}}a_i)] \left(\frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)} \right) \tag{3.5}$$

or in terms of $\bar{D}_{in}$, (3.5) is rewritten as

$$\bar{R}_{i,i+1} = \bar{D}_{in}^{-1} \left[\frac{H_n^{(2)}(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)} \frac{H_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} \right] \left(\frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)} \right). \tag{3.6}$$

When the certain simplifications are performed, $\bar{R}_{i,i+1}$ is given by

$$\bar{R}_{i,i+1} = \frac{H_n^{(2)}(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\bar{D}_{in}^{-1}\left(\frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)}\right). \tag{3.7}$$

In (B.12) and (B.16) we proved that

$$\lim_{n\to\infty}\frac{B_n'(x)}{nB_n(x)} = C(k_z) \tag{3.8}$$

where B_n is an Hankel or Bessel function and $C(k_z)$ is a constant with respect to n. Using this information, it is clear that $\bar{D}_{in}^{-1}$ decays with $1/n$ for large n values where $(\frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)})$ term grows with n. This means that if we define

$$\bar{R}n_{i,i+1} = \bar{D}_{in}^{-1}\left(\frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)}\right) \tag{3.9}$$

we obtain

$$\bar{R}_{i,i+1} = \frac{H_n^{(2)}(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\bar{R}n_{i,i+1} \tag{3.10}$$

where $\bar{R}n_{i,i+1}$ term becomes constant with respect to n for large n values.

Similar to $\bar{R}_{i,i+1}$, the $\bar{R}_{i+1,i}$ term given by (2.12) can be written as

$$\bar{R}_{i+1,i} = \bar{D}_i^{-1}[J_n(k_{\rho_i}a_i)J_n(k_{\rho_{i+1}}a_i)]\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\right) \tag{3.11}$$

or in terms of $\bar{D}_{in}$, (3.11) is rewritten as

$$\bar{R}_{i+1,i} = \left[\frac{J_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\frac{J_n(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\right]\bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\right). \tag{3.12}$$

When the simplifications are done, $\bar{R}_{i+1,i}$ is expressed as

$$\bar{R}_{i+1,i} = \frac{J_n(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\right). \tag{3.13}$$

Similar to $\bar{R}n_{i,i+1}$, we define

$$\bar{R}n_{i+1,i} = \bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\right) \tag{3.14}$$

and $\bar{R}n_{i+1,i}$ becomes constant with respect to n for large n values. Then, we obtain

$$\bar{R}_{i+1,i} = \frac{J_n(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\bar{R}n_{i+1,i}. \tag{3.15}$$

In [10], the simplified expressions given by (2.11) and (2.13) are used for the local transmission matrices $\bar{T}_{i,i+1}$ and $\bar{T}_{i+1,i}$, respectively. However, when they are used in the $\bar{F}_n$ expression given by (2.5), $\bar{F}_n$ can not be expressed in the form of ratios. Therefore, the actual transmission matrix definitions, which are not simplified, are used in this thesis. These actual transmission matrix definitions are

$$\bar{T}_{i,i+1} = \bar{D}_i^{-1}[H_n^{(2)}(k_{\rho_i}a_i)\bar{J}_n(k_{\rho_i}a_i) - J_n(k_{\rho_i}a_i)\bar{H}_n^{(2)}(k_{\rho_i}a_i)] \tag{3.16}$$

$$\bar{T}_{i+1,i} = \bar{D}_i^{-1}[H_n^{(2)}(k_{\rho_{i+1}}a_i)\bar{J}_n(k_{\rho_{i+1}}a_i) - J_n(k_{\rho_{i+1}}a_i)\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)]. \tag{3.17}$$

As a result similar to reflection matrices, the transmission matrix $\bar{T}_{i,i+1}$ is expressed as

$$\bar{T}_{i,i+1} = \bar{D}_i^{-1}\left[H_n^{(2)}(k_{\rho_i}a_i)J_n(k_{\rho_i}a_i)\right]\left(\frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)}\right) \tag{3.18}$$

or in terms of $\bar{D}_{in}$, (3.18) is rewritten as

$$\bar{T}_{i,i+1} = \left[\frac{H_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\frac{J_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\right]\bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)}\right), \tag{3.19}$$

and when the simplifications are performed, we obtain

$$\bar{T}_{i,i+1} = \frac{H_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_i}a_i)}{H_n^{(2)}(k_{\rho_i}a_i)}\right). \tag{3.20}$$

Now, $\bar{T}_{i,i+1}$ term is in the form of ratios and constant with respect to n for large n values.

Similarly, $\bar{T}_{i+1,i}$ is written as

$$\bar{T}_{i+1,i} = \bar{D}_i^{-1}[H_n^{(2)}(k_{\rho_{i+1}}a_i)J_n(k_{\rho_{i+1}}a_i)]\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\right) \tag{3.21}$$

or in terms of $\bar{D}_{in}$, (3.21) is rewritten as

$$\bar{T}_{i+1,i} = \left[\frac{H_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\frac{J_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_i}a_i)}\right]\bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\right) \tag{3.22}$$

and after the simplifications it becomes

$$\bar{T}_{i+1,i} = \frac{J_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_i}a_i)}\bar{D}_{in}^{-1}\left(\frac{\bar{J}_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_i)} - \frac{\bar{H}_n^{(2)}(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\right). \tag{3.23}$$

Similar to $\bar{T}_{i,i+1}$ expression, $\bar{T}_{i+1,i}$ term is now in the form of ratios and again constant with respect to n for large n values.

So far, we have tried to write the local reflection and transmission matrices in the form of ratios. As the next step, we substitute these matrices into the generalized reflection and transmission matrices and try to write them in the form of ratios. The $\tilde{\bar{R}}_{i,i+1}$ term is given by

$$\tilde{\bar{R}}_{i,i+1} = \bar{R}_{i,i+1} + \bar{T}_{i+1,i}\tilde{\bar{R}}_{i+1,i+2}\tilde{\bar{T}}_{i,i+1}. \tag{3.24}$$

In a structure with N layers, the first nonzero generalized reflection matrix for the outermost layers is $\tilde{\bar{R}}_{N-1,N}$. $\tilde{\bar{R}}_{N-1,N}$ term is in fact a local reflection matrix and is equal to $\bar{R}_{N-1,N}$, since we have only N layer and $\tilde{\bar{R}}_{N,N+1}$ is zero in (3.24). Thus, it is possible to write a general expression for $\tilde{\bar{R}}_{i+1,i+2}$ using the $\bar{R}_{i,i+1}$ given by (3.10)

$$\tilde{\bar{R}}_{i+1,i+2} = \frac{H_n^{(2)}(k_{\rho_{i+1}}a_{i+1})}{J_n(k_{\rho_{i+1}}a_{i+1})}\tilde{\bar{R}}n_{i+1,i+2} \tag{3.25}$$

where again $\tilde{\bar{R}}n_{i+1,i+2}$ term is in the from of ratios and constant with respect to n for large n values.

Since $\tilde{\bar{T}}_{i,i+1}$ term is given by

$$\tilde{\bar{T}}_{i,i+1} = (\bar{I} - \bar{R}_{i+1,i}\tilde{\bar{R}}_{i+1,i+2})^{-1}\bar{T}_{i,i+1} \tag{3.26}$$

using (3.15) and (3.25) it can be written as

$$\tilde{\bar{T}}_{i,i+1} = \left(\bar{I} - \frac{J_n(k_{\rho_{i+1}}a_i)}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\bar{R}n_{i+1,i}\frac{H_n^{(2)}(k_{\rho_{i+1}}a_{i+1})}{J_n(k_{\rho_{i+1}}a_{i+1})}\tilde{\bar{R}}n_{i+1,i+2}\right)^{-1}\bar{T}_{i,i+1} \tag{3.27}$$

or

$$\tilde{\bar{T}}_{i,i+1} = \left(\bar{I} - \frac{J_n(k_{\rho_{i+1}}a_i)}{J_n(k_{\rho_{i+1}}a_{i+1})}\frac{H_n^{(2)}(k_{\rho_{i+1}}a_{i+1})}{H_n^{(2)}(k_{\rho_{i+1}}a_i)}\bar{R}n_{i+1,i}\tilde{\bar{R}}n_{i+1,i+2}\right)^{-1}\bar{T}_{i,i+1}. \tag{3.28}$$

Since all the functions in $\tilde{\bar{T}}_{i,i+1}$ are in the form of ratios, $\tilde{\bar{T}}_{i,i+1}$ is constant with respect to n for large n values. Substituting all the terms which are in the form of ratios into (3.24), the generalized reflection matrix $\tilde{\bar{R}}_{i,i+1}$ becomes

$$\tilde{\bar{R}}_{i,i+1} = \frac{H_n^{(2)}(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\bar{R}n_{i,i+1} + \bar{T}_{i+1,i}\frac{H_n^{(2)}(k_{\rho_{i+1}}a_{i+1})}{J_n(k_{\rho_{i+1}}a_{i+1})}\tilde{\bar{R}}n_{i+1,i+2}\tilde{\bar{T}}_{i,i+1} \tag{3.29}$$

or

$$\tilde{\bar{R}}_{i,i+1} = \frac{H_n^{(2)}(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\left(\bar{R}n_{i,i+1} + \bar{T}_{i+1,i}\frac{H_n^{(2)}(k_{\rho_{i+1}}a_{i+1})}{H_n^{(2)}(k_{\rho_i}a_i)}\frac{J_n(k_{\rho_i}a_i)}{J_n(k_{\rho_{i+1}}a_{i+1})}\tilde{\bar{R}}n_{i+1,i+2}\tilde{\bar{T}}_{i,i+1}\right). \tag{3.30}$$

When we define $\tilde{\bar{R}}n_{i,i+1}$ as

$$\tilde{\bar{R}}n_{i,i+1} = \bar{R}n_{i,i+1} + \bar{T}_{i+1,i}\frac{H_n^{(2)}(k_{\rho_{i+1}}a_{i+1})}{H_n^{(2)}(k_{\rho_i}a_i)}\frac{J_n(k_{\rho_i}a_i)}{J_n(k_{\rho_{i+1}}a_{i+1})}\tilde{\bar{R}}n_{i+1,i+2}\tilde{\bar{T}}_{i,i+1}, \tag{3.31}$$

it is clear that $\tilde{\bar{R}}n_{i,i+1}$ term is in the form of ratios and constant with respect to n for large n values. So we can write

$$\tilde{\bar{R}}_{i,i+1} = \frac{H_n^{(2)}(k_{\rho_i}a_i)}{J_n(k_{\rho_i}a_i)}\tilde{\bar{R}}n_{i,i+1}. \tag{3.32}$$

Equation (3.32) is the desired representation for $\tilde{\bar{R}}_{i,i+1}$. However in a similar fashion, we must define the $\tilde{\bar{R}}_{i,i-1}$ term which is given by

$$\tilde{\bar{R}}_{i,i-1} = \bar{R}_{i,i-1} + \bar{T}_{i-1,i}\tilde{\bar{R}}_{i-1,i-2}\tilde{\bar{T}}_{i,i-1}. \tag{3.33}$$

Similar to $\tilde{\bar{R}}_{i+1,i+2}$, $\tilde{\bar{R}}_{i-1,i-2}$ is zero for the innermost layer, so the first nonzero $\tilde{\bar{R}}_{i-1,i-2}$ term behaves in the same manner as $\bar{R}_{i,i-1}$. Hence, it is written as

$$\tilde{\bar{R}}_{i-1,i-2} = \frac{J_n(k_{\rho_{i-1}}a_{i-2})}{H_n^{(2)}(k_{\rho_{i-1}}a_{i-2})}\tilde{\bar{R}}n_{i-1,i-2} \tag{3.34}$$

where $\tilde{\bar{R}}n_{i-1,i-2}$ is in the form of ratios and constant with respect to n for large n values. The $\tilde{\bar{T}}_{i,i-1}$ term is

$$\tilde{\bar{T}}_{i,i-1} = (\bar{I} - \bar{R}_{i-1,i}\tilde{\bar{R}}_{i-1,i-2})^{-1}\bar{T}_{i,i-1} \tag{3.35}$$

where the $\bar{R}_{i-1,i}$ term is obtained as

$$\bar{R}_{i-1,i} = \frac{H_n^{(2)}(k_{\rho_{i-1}}a_{i-1})}{J_n(k_{\rho_{i-1}}a_{i-1})}\bar{R}n_{i-1,i} \tag{3.36}$$

by writing $i-1$ instead of i in (3.10), and where $\bar{R}n_{i-1,i}$ term is in the form of ratios. The $\tilde{\bar{T}}_{i,i-1}$ term is then expressed as

$$\tilde{\bar{T}}_{i,i-1} = \left(\bar{I} - \frac{H_n^{(2)}(k_{\rho_{i-1}} a_{i-1})}{J_n(k_{\rho_{i-1}} a_{i-1})} \bar{R}n_{i-1,i} \frac{J_n(k_{\rho_{i-1}} a_{i-2})}{H_n^{(2)}(k_{\rho_{i-1}} a_{i-2})} \tilde{\bar{R}}n_{i-1,i-2} \right)^{-1} \bar{T}_{i,i-1} \quad (3.37)$$

or

$$\tilde{\bar{T}}_{i,i-1} = \left(\bar{I} - \frac{H_n^{(2)}(k_{\rho_{i-1}} a_{i-1})}{H_n^{(2)}(k_{\rho_{i-1}} a_{i-2})} \frac{J_n(k_{\rho_{i-1}} a_{i-2})}{J_n(k_{\rho_{i-1}} a_{i-1})} \bar{R}n_{i-1,i} \tilde{\bar{R}}n_{i-1,i-2} \right)^{-1} \bar{T}_{i,i-1} \quad (3.38)$$

where $\tilde{\bar{T}}_{i,i-1}$ is in the form of ratios.

The $\bar{R}_{i,i-1}$ term is obtained from (3.15) as

$$\bar{R}_{i,i-1} = \frac{J_n(k_{\rho_i} a_{i-1})}{H_n^{(2)}(k_{\rho_i} a_{i-1})} \bar{R}n_{i,i-1} \quad (3.39)$$

by writing $i-1$ instead of i. Finally, substituting all terms which are in the form of ratios into (3.33), $\tilde{\bar{R}}_{i,i-1}$ expression is obtained as

$$\tilde{\bar{R}}_{i,i-1} = \frac{J_n(k_{\rho_i} a_{i-1})}{H_n^{(2)}(k_{\rho_i} a_{i-1})} \bar{R}n_{i,i-1} + \bar{T}_{i-1,i} \frac{J_n(k_{\rho_{i-1}} a_{i-2})}{H_n^{(2)}(k_{\rho_{i-1}} a_{i-2})} \tilde{\bar{R}}n_{i-1,i-2} \tilde{\bar{T}}_{i,i-1}. \quad (3.40)$$

or

$$\tilde{\bar{R}}_{i,i-1} = \frac{J_n(k_{\rho_i} a_{i-1})}{H_n^{(2)}(k_{\rho_i} a_{i-1})} \left(\bar{R}n_{i,i-1} + \bar{T}_{i-1,i} \frac{H_n^{(2)}(k_{\rho_i} a_{i-1})}{H_n^{(2)}(k_{\rho_{i-1}} a_{i-2})} \frac{J_n(k_{\rho_{i-1}} a_{i-2})}{J_n(k_{\rho_i} a_{i-1})} \tilde{\bar{R}}n_{i-1,i-2} \tilde{\bar{T}}_{i,i-1} \right). \quad (3.41)$$

When we define the $\tilde{\bar{R}}n_{i,i-1}$ term as

$$\tilde{\bar{R}}n_{i,i-1} = \bar{R}n_{i,i-1} + \bar{T}_{i-1,i} \frac{H_n^{(2)}(k_{\rho_i} a_{i-1})}{H_n^{(2)}(k_{\rho_{i-1}} a_{i-2})} \frac{J_n(k_{\rho_{i-1}} a_{i-2})}{J_n(k_{\rho_i} a_{i-1})} \tilde{\bar{R}}n_{i-1,i-2} \tilde{\bar{T}}_{i,i-1} \quad (3.42)$$

we obtain

$$\tilde{\bar{R}}_{i,i-1} = \frac{J_n(k_{\rho_i} a_{i-1})}{H_n^{(2)}(k_{\rho_i} a_{i-1})} \tilde{\bar{R}}n_{i,i-1} \quad (3.43)$$

where again $\tilde{\bar{R}}n_{i,i-1}$ is constant with respect to n for large n values.

When $\rho = \rho'$, it means the source and field layers become the same layer. Let's denote it by the index j. Therefore, the expression in (2.5) must be used for the $\bar{F}_n$ term. There are two expressions in (2.5) which are equal to each other when $\rho = \rho'$. Let us take the second expression given by

$$\bar{F}_n = [H_n^{(2)}(k_{\rho_j}\rho)\bar{I} + J_n(k_{\rho_j}\rho)\tilde{\bar{R}}_{j,j+1}]\tilde{\bar{M}}_{j+}[J_n(k_{\rho_j}\rho')\bar{I} + H_n^{(2)}(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j-1}]. \quad (3.44)$$

In the Green's function expressions given by (2.19)-(2.22), there are also derivatives of $\bar{F}_n$ with respect to $k_{\rho_j}\rho$ and $k_{\rho_j}\rho'$. Since $\bar{I}$, $\tilde{\bar{R}}_{j,j-1}$, $\tilde{\bar{M}}_{j+}$ and $\tilde{\bar{R}}_{j,j+1}$ do not contain any ρ or ρ', these derivatives are given by

$$\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho)} = [H_n^{'(2)}(k_{\rho_j}\rho)\bar{I} + J_n^{'}(k_{\rho_j}\rho)\tilde{\bar{R}}_{j,j+1}]\tilde{\bar{M}}_{j+}[J_n(k_{\rho_j}\rho')\bar{I} + H_n^{(2)}(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j-1}] \quad (3.45)$$

$$\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho')} = [H_n^{(2)}(k_{\rho_j}\rho)\bar{I} + J_n(k_{\rho_j}\rho)\tilde{\bar{R}}_{j,j+1}]\tilde{\bar{M}}_{j+}[J_n^{'}(k_{\rho_j}\rho')\bar{I} + H_n^{'(2)}(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j-1}] \quad (3.46)$$

$$\begin{aligned}\frac{\partial^2 \bar{F}_n}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} = {} & [H_n^{'(2)}(k_{\rho_j}\rho)\bar{I} + J_n^{'}(k_{\rho_j}\rho)\tilde{\bar{R}}_{j,j+1}]\tilde{\bar{M}}_{j+} \\ & .[J_n^{'}(k_{\rho_j}\rho')\bar{I} + H_n^{'(2)}(k_{\rho_j}\rho')\tilde{\bar{R}}_{j,j-1}].\end{aligned} \quad (3.47)$$

The $\tilde{\bar{M}}_j+$ expression appearing in (3.44)-(3.47) is expressed as

$$\tilde{\bar{M}}_j+ = (\bar{I} - \tilde{\bar{R}}_{j,j-1}\tilde{\bar{R}}_{j,j+1})^{-1} \quad (3.48)$$

where $\tilde{\bar{R}}_{j,j-1}$ is obtained putting j instead of i in (3.43) such that

$$\tilde{\bar{R}}_{j,j-1} = \frac{J_n(k_{\rho_j}a_{j-1})}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1} \quad (3.49)$$

and similarly, $\tilde{\bar{R}}_{j,j+1}$ is written using (3.32) as

$$\tilde{\bar{R}}_{j,j+1} = \frac{H_n^{(2)}(k_{\rho_j}a_j)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1}. \quad (3.50)$$

Substituting the $\tilde{\bar{R}}_{j,j+1}$ and $\tilde{\bar{R}}_{j,j-1}$ terms into the $\tilde{\bar{M}}_j+$ expression given by (3.48), we obtain

$$\tilde{\bar{M}}_j+ = \left(\bar{I} - \frac{J_n(k_{\rho_j}a_{j-1})}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1}\frac{H_n^{(2)}(k_{\rho_j}a_j)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1}\right)^{-1} \quad (3.51)$$

or

$$\tilde{\bar{M}}_j + = \left(\bar{I} - \frac{H_n^{(2)}(k_{\rho_j} a_j)}{H_n^{(2)}(k_{\rho_j} a_{j-1})} \tilde{\bar{R}} n_{j,j-1} \frac{J_n(k_{\rho_j} a_{j-1})}{J_n(k_{\rho_j} a_j)} \tilde{\bar{R}} n_{j,j+1} \right)^{-1}. \tag{3.52}$$

It is seen that $\tilde{\bar{M}}_j+$ term is in the form of ratios and constant with respect to n for large n values.

Finally, we can write the $\bar{F}_n$ expression in (3.44) as

$$\bar{F}_n = H_n^{(2)}(k_{\rho_j}\rho) J_n(k_{\rho_j}\rho') \left(\left[\bar{I} + \frac{J_n(k_{\rho_j}\rho)}{H_n^{(2)}(k_{\rho_j}\rho)} \tilde{\bar{R}}_{j,j+1} \right] \tilde{\bar{M}}_{j+} \left[\bar{I} + \frac{H_n^{(2)}(k_{\rho_j}\rho')}{J_n(k_{\rho_j}\rho')} \tilde{\bar{R}}_{j,j-1} \right] \right), \tag{3.53}$$

and substituting the $\tilde{\bar{R}}_{j,j-1}$ and $\tilde{\bar{R}}_{j,j+1}$ terms, we obtain

$$\begin{aligned} \bar{F}_n &= H_n^{(2)}(k_{\rho_j}\rho) J_n(k_{\rho_j}\rho') \left(\left[\bar{I} + \frac{J_n(k_{\rho_j}\rho)}{H_n^{(2)}(k_{\rho_j}\rho)} \frac{H_n^{(2)}(k_{\rho_j} a_j)}{J_n(k_{\rho_j} a_j)} \tilde{\bar{R}} n_{j,j+1} \right] \tilde{\bar{M}}_{j+} \right. \\ & \quad \left. \cdot \left[\bar{I} + \frac{H_n^{(2)}(k_{\rho_j}\rho')}{J_n(k_{\rho_j}\rho')} \frac{J_n(k_{\rho_j} a_{j-1})}{H_n^{(2)}(k_{\rho_j} a_{j-1})} \tilde{\bar{R}} n_{j,j-1} \right] \right) \end{aligned} \tag{3.54}$$

or

$$\begin{aligned} \bar{F}_n &= H_n^{(2)}(k_{\rho_j}\rho) J_n(k_{\rho_j}\rho') \left(\left[\bar{I} + \frac{H_n^{(2)}(k_{\rho_j} a_j)}{H_n^{(2)}(k_{\rho_j}\rho)} \frac{J_n(k_{\rho_j}\rho)}{J_n(k_{\rho_j} a_j)} \tilde{\bar{R}} n_{j,j+1} \right] \tilde{\bar{M}}_{j+} \right. \\ & \quad \left. \cdot \left[\bar{I} + \frac{J_n(k_{\rho_j} a_{j-1})}{J_n(k_{\rho_j}\rho')} \frac{H_n^{(2)}(k_{\rho_j}\rho')}{H_n^{(2)}(k_{\rho_j} a_{j-1})} \tilde{\bar{R}} n_{j,j-1} \right] \right). \end{aligned} \tag{3.55}$$

When we define the terms in the form of ratios as $\bar{F}n_n$

$$\begin{aligned} \bar{F}n_n &= \left(\left[\bar{I} + \frac{H_n^{(2)}(k_{\rho_j} a_j)}{H_n^{(2)}(k_{\rho_j}\rho)} \frac{J_n(k_{\rho_j}\rho)}{J_n(k_{\rho_j} a_j)} \tilde{\bar{R}} n_{j,j+1} \right] \tilde{\bar{M}}_{j+} \right. \\ & \quad \left. \cdot \left[\bar{I} + \frac{J_n(k_{\rho_j} a_{j-1})}{J_n(k_{\rho_j}\rho')} \frac{H_n^{(2)}(k_{\rho_j}\rho')}{H_n^{(2)}(k_{\rho_j} a_{j-1})} \tilde{\bar{R}} n_{j,j-1} \right] \right) \end{aligned} \tag{3.56}$$

which is constant with respect to n for large n values, we obtain

$$\bar{F}_n = H_n^{(2)}(k_{\rho_j}\rho) J_n(k_{\rho_j}\rho') \bar{F} n_n. \tag{3.57}$$

Similarly, (3.45) can be written as

$$\frac{\partial \bar{F}_n}{\partial (k_{\rho_j}\rho)} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho') \left(\left[\frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\bar{I} + \frac{J_n'(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\tilde{\bar{R}}_{j,j+1} \right] \tilde{\bar{M}}_{j+} \right.$$
$$\left. . \left[\bar{I} + \frac{H_n^{(2)}(k_{\rho_j}\rho')}{J_n(k_{\rho_j}\rho')}\tilde{\bar{R}}_{j,j-1} \right] \right), \quad (3.58)$$

and substituting the $\tilde{\bar{R}}_{j,j-1}$ and $\tilde{\bar{R}}_{j,j+1}$ terms into (3.58), we obtain

$$\frac{\partial \bar{F}_n}{\partial (k_{\rho_j}\rho)} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho') \left(\left[\frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\bar{I} + \frac{J_n'(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)} \frac{H_n^{(2)}(k_{\rho_j}a_j)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1} \right] \right.$$
$$\left. . \tilde{\bar{M}}_{j+} \left[\bar{I} + \frac{H_n^{(2)}(k_{\rho_j}\rho')}{J_n(k_{\rho_j}\rho')} \frac{J_n(k_{\rho_j}a_{j-1})}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1} \right] \right) \quad (3.59)$$

or

$$\frac{\partial \bar{F}_n}{\partial (k_{\rho_j}\rho)} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho') \left(\left[\frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\bar{I} + \frac{H_n^{(2)}(k_{\rho_j}a_j)}{H_n^{(2)}(k_{\rho_j}\rho)} \frac{J_n'(k_{\rho_j}\rho)}{nJ_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1} \right] \right.$$
$$\left. . \tilde{\bar{M}}_{j+} \left[\bar{I} + \frac{J_n(k_{\rho_j}a_{j-1})}{J_n(k_{\rho_j}\rho')} \frac{H_n^{(2)}(k_{\rho_j}\rho')}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1} \right] \right). \quad (3.60)$$

Defining an expression $\bar{F}n_{nd\rho}$ as

$$\bar{F}n_{nd\rho} = \left(\left[\frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\bar{I} + \frac{H_n^{(2)}(k_{\rho_j}a_j)}{H_n^{(2)}(k_{\rho_j}\rho)} \frac{J_n'(k_{\rho_j}\rho)}{nJ_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1} \right] \tilde{\bar{M}}_{j+} \right.$$
$$\left. . \left[\bar{I} + \frac{J_n(k_{\rho_j}a_{j-1})}{J_n(k_{\rho_j}\rho')} \frac{H_n^{(2)}(k_{\rho_j}\rho')}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1} \right] \right), \quad (3.61)$$

which is in the form of ratios and constant with respect to n for large n values, (3.45) is written in terms of $\bar{F}n_{nd\rho}$ as

$$\frac{\partial \bar{F}_n}{\partial (k_{\rho_j}\rho)} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\bar{F}n_{nd\rho}. \quad (3.62)$$

The same methodology is also valid for (3.46) such that it is written as

$$\frac{\partial \bar{F}_n}{\partial (k_{\rho_j}\rho')} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho') \left(\left[\bar{I} + \frac{J_n(k_{\rho_j}\rho)}{H_n^{(2)}(k_{\rho_j}\rho)}\tilde{\bar{R}}_{j,j+1} \right] \right.$$
$$\left. . \tilde{\bar{M}}_{j+} \left[\frac{J_n'(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\bar{I} + \frac{H_n^{'(2)}(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\tilde{\bar{R}}_{j,j-1} \right] \right) \quad (3.63)$$

and by substituting $\tilde{\bar{\bar{R}}}_{j,j-1}$ and $\tilde{\bar{\bar{R}}}_{j,j+1}$ terms into (3.63), (3.46) becomes

$$\begin{aligned}\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho')} &= nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\left(\left[\bar{I}+\frac{J_n(k_{\rho_j}\rho)}{H_n^{(2)}(k_{\rho_j}\rho)}\frac{H_n^{(2)}(k_{\rho_j}a_j)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{\bar{R}}}n_{j,j+1}\right]\right.\\ &\left. .\tilde{\bar{\bar{M}}}_{j+}\left[\frac{J_n'(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\bar{I}+\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\frac{J_n(k_{\rho_j}a_{j-1})}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{\bar{R}}}n_{j,j-1}\right]\right)\end{aligned} \tag{3.64}$$

or

$$\begin{aligned}\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho')} &= nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\left(\left[\bar{I}+\frac{J_n(k_{\rho_j}\rho)}{J_n(k_{\rho_j}a_j)}\frac{H_n^{(2)}(k_{\rho_j}a_j)}{H_n^{(2)}(k_{\rho_j}\rho)}\tilde{\bar{\bar{R}}}n_{j,j+1}\right]\right.\\ &\left. .\ \tilde{\bar{\bar{M}}}_{j+}\left[\frac{J_n'(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\bar{I}+\frac{J_n(k_{\rho_j}a_{j-1})}{J_n(k_{\rho_j}\rho')}\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{nH_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{\bar{R}}}n_{j,j-1}\right]\right).\end{aligned} \tag{3.65}$$

Defining the constant term with respect to n for large n values as $\bar{F}n_{nd\rho'}$ and expressing it as

$$\begin{aligned}\bar{F}n_{nd\rho'} &= \left(\left[\bar{I}+\frac{J_n(k_{\rho_j}\rho)}{J_n(k_{\rho_j}a_j)}\frac{H_n^{(2)}(k_{\rho_j}a_j)}{H_n^{(2)}(k_{\rho_j}\rho)}\tilde{\bar{\bar{R}}}n_{j,j+1}\right]\tilde{\bar{\bar{M}}}_{j+}\right.\\ &\left. .\left[\frac{J_n'(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\bar{I}+\frac{J_n(k_{\rho_j}a_{j-1})}{J_n(k_{\rho_j}\rho')}\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{nH_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{\bar{R}}}n_{j,j-1}\right]\right),\end{aligned} \tag{3.66}$$

we obtain

$$\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho')} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\bar{F}n_{nd\rho'}. \tag{3.67}$$

Finally, (3.47) can be written as

$$\begin{aligned}\frac{\partial^2 \bar{F}_n}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} &= H_n^{'(2)}(k_{\rho_j}\rho)J_n'(k_{\rho_j}\rho')\left(\left[\bar{I}+\frac{J_n'(k_{\rho_j}\rho)}{H_n^{'(2)}(k_{\rho_j}\rho)}\tilde{\bar{\bar{R}}}_{j,j+1}\right]\tilde{\bar{\bar{M}}}_{j+}\right.\\ &\left. .\left[\bar{I}+\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{J_n'(k_{\rho_j}\rho')}\tilde{\bar{\bar{R}}}_{j,j-1}\right]\right)\end{aligned} \tag{3.68}$$

and substituting the $\tilde{\bar{\bar{R}}}_{j,j-1}$ and $\tilde{\bar{\bar{R}}}_{j,j+1}$ terms, we obtain

$$\begin{aligned}\frac{\partial^2 \bar{F}_n}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} &= H_n^{'(2)}(k_{\rho_j}\rho)J_n'(k_{\rho_j}\rho')\left(\left[\bar{I}+\frac{J_n'(k_{\rho_j}\rho)}{H_n^{'(2)}(k_{\rho_j}\rho)}\frac{H_n^{(2)}(k_{\rho_j}a_j)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{\bar{R}}}n_{j,j+1}\right]\tilde{\bar{\bar{M}}}_{j+}\right.\\ &\left. .\left[\bar{I}+\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{J_n'(k_{\rho_j}\rho')}\frac{J_n(k_{\rho_j}a_{j-1})}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{\bar{R}}}n_{j,j-1}\right]\right)\end{aligned} \tag{3.69}$$

or

$$
\begin{aligned}
\frac{\partial^2 \bar{F}_n}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} &= H_n^{'(2)}(k_{\rho_j}\rho)J_n^{'}(k_{\rho_j}\rho')\left(\left[\bar{I}+\frac{H_n^{(2)}(k_{\rho_j}a_j)}{H_n^{'(2)}(k_{\rho_j}\rho)}\frac{J_n^{'}(k_{\rho_j}\rho)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1}\right]\tilde{\bar{M}}_{j+}\right. \\
&\quad \left. \cdot\left[\bar{I}+\frac{J_n(k_{\rho_j}a_{j-1})}{J_n^{'}(k_{\rho_j}\rho')}\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1}\right]\right). \qquad (3.70)
\end{aligned}
$$

When we define the constant term with respect to n for large n values as

$$
\begin{aligned}
\bar{F}n_{nd\rho d\rho'} &= \left(\left[\bar{I}+\frac{H_n^{(2)}(k_{\rho_j}a_j)}{H_n^{'(2)}(k_{\rho_j}\rho)}\frac{J_n^{'}(k_{\rho_j}\rho)}{J_n(k_{\rho_j}a_j)}\tilde{\bar{R}}n_{j,j+1}\right]\tilde{\bar{M}}_{j+}\right. \\
&\quad \left. \cdot\left[\bar{I}+\frac{J_n(k_{\rho_j}a_{j-1})}{J_n^{'}(k_{\rho_j}\rho')}\frac{H_n^{'(2)}(k_{\rho_j}\rho')}{H_n^{(2)}(k_{\rho_j}a_{j-1})}\tilde{\bar{R}}n_{j,j-1}\right]\right) \qquad (3.71)
\end{aligned}
$$

we obtain (3.47) as

$$
\frac{\partial^2 \bar{F}_n}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} = H_n^{'(2)}(k_{\rho_j}\rho)J_n^{'}(k_{\rho_j}\rho')\bar{F}n_{nd\rho d\rho'}. \qquad (3.72)
$$

Consequently, $\bar{F}_n$, $\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho)}$, $\frac{\partial \bar{F}_n}{\partial(k_{\rho_j}\rho')}$ and $\frac{\partial\partial \bar{F}_n}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)}$ terms are obtained in the form of ratios. Furthermore, it is shown that they are constant with respect to n for large n values. The terms which can not be written in the form of ratios are also given as a multiplicant in the final expressions (3.57), (3.62), (3.67) and (3.72).

The main reasons to write the spectral domain Green's function components in such a form are (i) to improve the efficiency of the summation over the cylindrical eigenmodes n, which becomes important especially when $\rho = \rho'$; and (ii) to treat the axial line problem ($\rho = \rho'$ and $\phi = \phi'$), which will be addressed in the next chapter.

The Bessel and Hankel functions, which are written in the form of ratios, are evaluated using Matlab in the following way: For small n values, each function is evaluated separately. For large n values, where these functions can be replaced by their Debye representations, the Debye representations given in Appendix B are used in the ratio terms and instead of evaluating each function separately, the ratios are evaluated directly.

Although, the idea of the computation technique is the same for the $\tilde{G}^E_{zz}$, $\tilde{G}^E_{\phi z}$, $\tilde{G}^E_{z\phi}$ and $\tilde{G}^E_{\phi\phi}$ cases, since the expressions are different, the spectral domain Green's function components are investigated separately.

3.2 $\tilde{G}^E_{zz}$ Expression when $\rho = \rho'$

The $\tilde{G}^E_{zz}$ expression used for $\rho \neq \rho'$ is

$$\tilde{G}^E_{zz} = -\frac{1}{2\omega}\frac{k^2_{\rho_j}}{\epsilon_j}\sum_{n=0}^{\infty}\epsilon_n \cos[n(\phi-\phi')]F_n(1,1). \tag{3.73}$$

Using (3.57) it is possible to write the following expression for $F_n(1,1)$

$$F_n(1,1) = H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')Fn_n(1,1) \tag{3.74}$$

where $Fn_n(1,1)$ is the first entry of $\bar{F}n_n$ given by (3.56) and is constant with respect to n for large n values. Substituting the $F_n(1,1)$ term, the $\tilde{G}^E_{zz}$ expression becomes

$$\tilde{G}^E_{zz} = -\frac{1}{2\omega}\frac{k^2_{\rho_j}}{\epsilon_j}\sum_{n=0}^{\infty}\epsilon_n H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')]Fn_n(1,1). \tag{3.75}$$

To improve the summation over n, first, the limiting value of $Fn_n(1,1)$ for very large n value is numerically determined,

$$\lim_{n\to\infty} Fn_n(1,1) \approx C_{zz}(k_z) \tag{3.76}$$

which is actually constant with respect to n. Then, recognizing the series expansion of $H_0^{(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)$ given by

$$\sum_{n=-\infty}^{\infty} H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')e^{jn(\phi-\phi')} = H_0^2(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) = S_1, \tag{3.77}$$

an envelope extraction method with respect to n is applied to (3.75) such that this constant term $C_{zz}(k_z)$ is subtracted from the summation and added as a

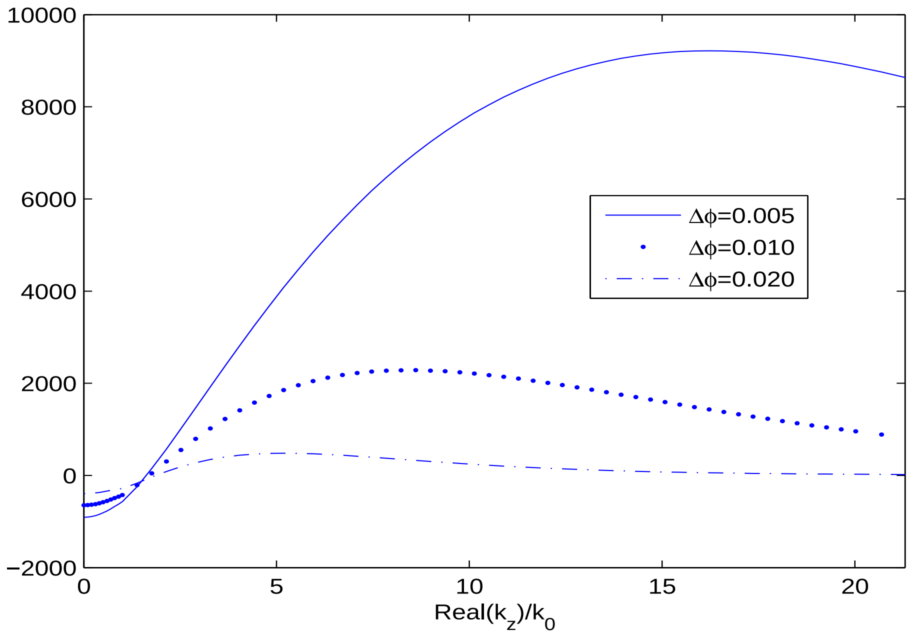

Figure 3.1: The imaginary part of $\tilde{G}^E_{zz}$ in Eq.(3.78) where $\Delta\phi = (\phi - \phi')rad$.

function of $H_0^{(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|)$ with the aid of (3.77). As a result, (3.75) is obtained as

$$
\begin{aligned}
\tilde{G}^E_{zz} &= -\frac{1}{2\omega}\frac{k^2_{\rho_j}}{\epsilon_j}\sum_{n=0}^{\infty}\epsilon_n H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')][Fn_n(1,1) - C_{zz}(k_z)] \\
&\quad -\frac{1}{4\omega}\frac{k^2_{\rho_j}}{\epsilon_j}H_0^{(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)C_{zz}(k_z)
\end{aligned}
\tag{3.78}
$$

where $|\bar{\rho} - \bar{\rho}'| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi')}$.

The modified summation given by (3.78) is now more rapidly convergent. However, for small $(\phi - \phi')$ values, the imaginary part of (3.78) yields problems as $k_z \to \infty$ which is the case shown in Fig. 3.1. Therefore, another envelope extraction method with respect to k_z is applied to (3.78). First, for the last k_z value, denoted as $k_{z\infty}$, on the k_z integration path (the details about this integration path and its parameters are given in Chapter 5), the value of $C_{zz}(k_z)$ denoted as $C_{zz}(k_{z\infty})$ is found. Then, the $C_{zz}(k_{z\infty})S_1$ product is subtracted in the spectral domain from (3.78) and its Fourier transform is added to the final spatial domain Green's function representation as a function of $\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|}$ using

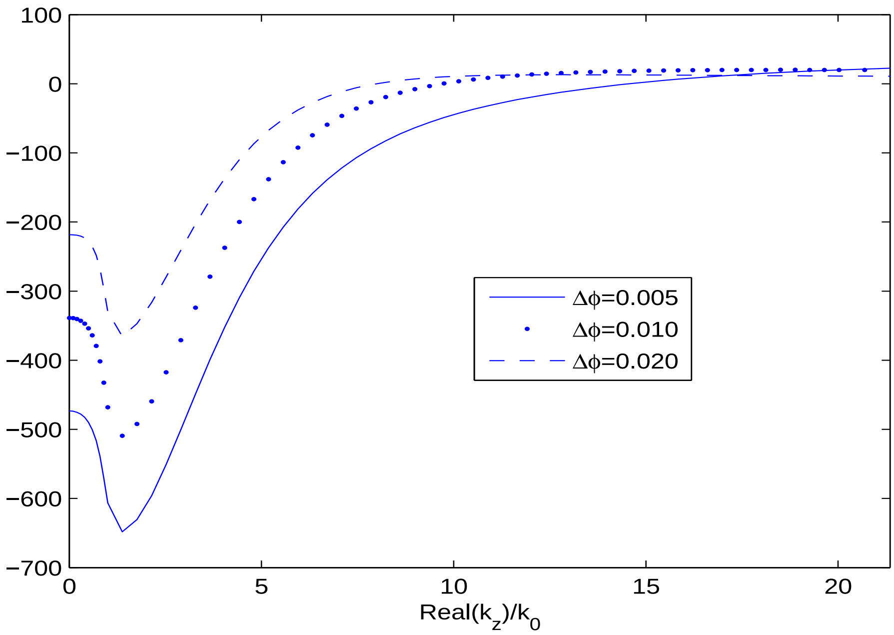

Figure 3.2: The imaginary part of $\tilde{G}^E_{zzf}$ in Eq.(3.81) where $\Delta\phi = (\phi - \phi')rad$.

the fact that

$$I_1 = \frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|} = \frac{-j}{2}\int_{-\infty}^{\infty} H_0^2(k_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|)e^{-jk_z(z-z')}dk_z \tag{3.79}$$

where

$$\left|\bar{r}-\bar{r}'\right| = \sqrt{\left|\bar{\rho}-\bar{\rho}'\right|^2 + (z-z')^2}. \tag{3.80}$$

The final expression in the spectral domain is obtained as

$$\begin{aligned}\tilde{G}^E_{zzf} &= -\frac{1}{2\omega}\frac{k^2_{\rho_j}}{\epsilon_j}\sum_{n=0}^{\infty}\epsilon_n H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')][Fn_n(1,1)-C_{zz}(k_z)] \\ &\quad -\frac{1}{4\omega}\frac{k^2_{\rho_j}}{\epsilon_j}H_0^{(2)}(k_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|)[C_{zz}(k_z)-C_{zz}(k_{z\infty})]. \end{aligned} \tag{3.81}$$

This $\tilde{G}^E_{zzf}$ expression given by (3.81) is now integrable even for small $(\phi - \phi')$ values. When the computations whose results are depicted in Fig. 3.1 are made again using the $\tilde{G}^E_{zzf}$ expression, Fig. 3.2 is obtained, which is well-suited for GPOF applications.

I_1 is used as the spatial domain representation of S_1 which is in spectral domain. Since $k_{\rho_j}^2 = k_j^2 - k_z^2$, and multiplication with k_z^2 in the spectral domain requires two derivatives with respect to z in the spatial domain, the Green's function component in the spatial domain is expressed as

$$G_{zz}^E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}_{zzf}^E e^{-jk_z(z-z')} dk_z - j\frac{C_{zz}(k_{z\infty})}{4\pi w\epsilon_j}\left(k_j^2 I_1 + \frac{\partial^2 I_1}{\partial z^2}\right) \tag{3.82}$$

which is written explicitly as

$$\begin{aligned} G_{zz}^E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}_{zzf}^E e^{-jk_z(z-z')} dk_z - j\frac{C_{zz}(k_{z\infty})}{4\pi w\epsilon_j}\Bigg[k_j^2 I_1 - jk_j\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|^2} \\ -k_j^2(z-z')^2\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|^3} - \frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|^3} + 3jk_j(z-z')^2\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|^4} \\ +3(z-z')^2\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|^5}\Bigg] \end{aligned} \tag{3.83}$$

and where $\tilde{G}_{zzf}^E$ is given in (3.81).

3.3 $\tilde{G}_{\phi z}^E$ Expression when $\rho = \rho'$

The method applied to $\tilde{G}_{zz}^E$, can be applied to $\tilde{G}_{\phi z}^E$. The $\tilde{G}_{\phi z}^E$ expression for $(\rho \neq \rho')$ is given as

$$\frac{\tilde{G}_{\phi z}^E}{k_z} = -\frac{j}{2\omega}\sum_{n=1}^{\infty}\sin[n(\phi-\phi')]\left[\frac{nk_{\rho_j}^2}{\epsilon_j\rho k_{\rho_m}^2}F_n(1,1) + \frac{k_{\rho_j}^2 j\omega\mu_m}{\epsilon_j k_z k_{\rho_m}}\frac{\partial F_n(2,1)}{\partial(k_{\rho_j}\rho)}\right]. \tag{3.84}$$

When $\rho = \rho'$, j is also equal to m so we can write

$$\frac{\tilde{G}_{\phi z}^E}{k_z} = -\frac{j}{2\omega}\sum_{n=1}^{\infty}\sin[n(\phi-\phi')]\left[\frac{n}{\epsilon_j\rho}F_n(1,1) + \frac{k_{\rho_j} j\omega\mu_j}{\epsilon_j k_z}\frac{\partial F_n(2,1)}{\partial(k_{\rho_j}\rho)}\right]. \tag{3.85}$$

Using (3.57) and (3.62), the following expressions are obtained

$$\frac{n}{\epsilon_j\rho}F_n(1,1) = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\frac{1}{\epsilon_j\rho}Fn_n(1,1) \tag{3.86}$$

$$\frac{k_{\rho_j} j\omega\mu_j}{\epsilon_j k_z}\frac{\partial F_n(2,1)}{\partial(k_{\rho_j}\rho)} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\frac{k_{\rho_j} j\omega\mu_j}{\epsilon_j k_z}F_{nd\rho}(2,1). \tag{3.87}$$

Let us define

$$F_{\phi z}(n,k_z) = \frac{1}{\epsilon_j\rho}Fn_n(1,1) + \frac{k_{\rho_j} j\omega\mu_j}{\epsilon_j k_z}F_{nd\rho}(2,1) \tag{3.88}$$

and we know from (3.56) and (3.61) that

$$\lim_{n\to\infty} F_{\phi z}(n,k_z) = C_{\phi z}(k_z). \tag{3.89}$$

Therefore, $\frac{\tilde{G}^E_{\phi z}}{k_z}$ is rewritten as

$$\frac{\tilde{G}^E_{\phi z}}{k_z} = -\frac{j}{2\omega}\sum_{n=1}^{\infty} nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\sin[n(\phi-\phi')]F_{\phi z}(n,k_z). \tag{3.90}$$

In order to use envelope extraction method with respect to n, we use the result of the following summation

$$S_2 = \sum_{n=-\infty}^{\infty} nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')e^{jn(\phi-\phi')} = -j\frac{\partial S_1}{\partial\phi} \tag{3.91}$$

or

$$S_2 = -jk_{\rho_j}\rho\rho'\sin(\phi-\phi')\frac{H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{|\bar{\rho}-\bar{\rho}'|}. \tag{3.92}$$

The modified spectral domain expression is then obtained as

$$\begin{aligned}\frac{\tilde{G}^E_{\phi z}}{k_z} &= -\frac{j}{2\omega}\sum_{n=1}^{\infty} nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\sin[n(\phi-\phi')][F_{\phi z}(n,k_z)-C_{\phi z}(k_z)] \\ &\quad -\frac{j}{4\omega}S_2 C_{\phi z}(k_z)\end{aligned} \tag{3.93}$$

where it is now fast convergent. However, during the k_z integration the problem that we experience for small $(\phi-\phi')$ values in the $\tilde{G}^E_{zz}$ component also occurs for this component. Therefore, similar to the $\tilde{G}^E_{zz}$ component, an envelope extraction method with respect to k_z is applied to (3.93) by first finding $C_{\phi z}(k_{z\infty})$, and then subtracting $S_2 C_{\phi z}(k_{z\infty})$ term from (3.93) (which is in the spectral domain) and adding its Fourier transform to the final spatial domain representation. Note that the Fourier transform of $S_2 C_{\phi z}(k_{z\infty})$ is expressed in terms of $k_{\rho_j} H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)$ where $'$ denotes the derivative with respect to argument. Therefore, we define

$$I_2 = \frac{-j}{2}\int_{-\infty}^{\infty} k_{\rho_j} H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)e^{-jk_z(z-z')}dk_z = \frac{\partial I_1}{\partial|\bar{\rho}-\bar{\rho}'|} \tag{3.94}$$

which is actually

$$I_2 = \frac{\partial}{\partial|\bar{\rho}-\bar{\rho}'|}\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|}. \tag{3.95}$$

Then, the final spectral domain Green's function expression is

$$\begin{aligned}\frac{\tilde{G}^E_{\phi zf}}{k_z} &= -\frac{j}{2\omega}\sum_{n=1}^{\infty} nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\sin[n(\phi-\phi')][F_{\phi z}(n,k_z)-C_{\phi z}(k_z)] \\ &\quad -\frac{1}{4\omega}S_2[C_{\phi z}(k_z)-C_{\phi z}(k_{z\infty})]. \end{aligned} \tag{3.96}$$

When we try to obtain the spatial domain representation of the $G^E_{\phi z}$ component, we should also note that we are working with $\frac{\tilde{G}^E_{\phi zf}}{k_z}$ rather than $\tilde{G}^E_{\phi zf}$. Recognizing that a division with $-jk_z$ in the spectral domain corresponds to an integration with respect to z in the spatial domain and making use of (3.95), the $G^E_{\phi z}$ component in the spatial domain can be expressed as

$$\begin{aligned}-j\int G^E_{\phi z}dz &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{G}^E_{\phi zf}}{k_z}e^{-jk_z(z-z')}dk_z \\ &\quad -\frac{j}{4\pi\omega}\left(-j\rho\rho'\sin(\phi-\phi')\frac{C_{\phi z}(k_{z\infty})}{|\bar{\rho}-\bar{\rho}'|}\right)I_2. \end{aligned} \tag{3.97}$$

Taking the derivatives with respect to z, we obtain

$$\begin{aligned}G^E_{\phi z} &= j\frac{\partial}{\partial z}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{G}^E_{\phi zf}}{k_z}e^{-jk_z(z-z')}dk_z\right) \\ &\quad +\frac{1}{4\pi\omega}\left(-j\rho\rho'\sin(\phi-\phi')\frac{C_{\phi z}(k_{z\infty})}{|\bar{\rho}-\bar{\rho}'|}\right)\left(\frac{\partial I_2}{\partial z}\right) \end{aligned} \tag{3.98}$$

where $\frac{\tilde{G}^E_{\phi zf}}{k_z}$ is given in (3.96) and

$$\begin{aligned}\frac{\partial I_2}{\partial z} &= -k_j^2\left|\bar{\rho}-\bar{\rho}'\right|(z-z')\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^3} + 3jk_j\left|\bar{\rho}-\bar{\rho}'\right|(z-z')\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^4} \\ &\quad +3\left|\bar{\rho}-\bar{\rho}'\right|(z-z')\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^5}. \end{aligned} \tag{3.99}$$

As seen in (3.98), after the closed-form expressions are obtained by applying GPOF to the integral term in (3.98), a derivative with respect to z is also taken in order to find $G^E_{\phi z}$.

3.4 $\tilde{G}^E_{z\phi}$ Expression when $\rho = \rho'$

The treatment of $\tilde{G}^E_{z\phi}$ case is very similar to $\tilde{G}^E_{\phi z}$ case. The expression for $\tilde{G}^E_{z\phi}$ is given by

$$\frac{\tilde{G}^E_{z\phi}}{k_z} = -\frac{j}{2\omega}\sum_{n=1}^{\infty}\sin[n(\phi-\phi')]\left[\frac{n}{\epsilon_j\rho'}F_n(1,1) - \frac{j\omega k_{\rho_j}}{k_z}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')}\right]. \tag{3.100}$$

Using the expressions (3.57) and (3.67), the terms in the bracket in (3.100) is written as

$$\frac{n}{\epsilon_j\rho'}F_n(1,1) = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\frac{1}{\epsilon_j\rho'}Fn_n(1,1) \tag{3.101}$$

$$-\frac{j\omega k_{\rho_j}}{k_z}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')} = nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\left(-\frac{j\omega k_{\rho_j}}{k_z}Fn_{nd\rho'}(1,2)\right). \tag{3.102}$$

We define a function, $F_{z\phi}(n,k_z)$, as

$$F_{z\phi}(n,k_z) = \frac{1}{\epsilon_j\rho'}Fn_n(1,1) - \frac{j\omega k_{\rho_j}}{k_z}Fn_{nd\rho'}(1,2) \tag{3.103}$$

and making use of (3.56) and (3.66) $F_{z\phi}(n,k_z)$ is found to be constant with respect to n for large n values such that

$$\lim_{n\to\infty} F_{z\phi}(n,k_z) = C_{z\phi}(k_z). \tag{3.104}$$

Similar to $\tilde{G}^E_{\phi z}$, we obtain the expression for $\frac{\tilde{G}^E_{z\phi}}{k_z}$ which is fast convergent as

$$\begin{aligned}\frac{\tilde{G}^E_{z\phi}}{k_z} &= -\frac{j}{2\omega}\sum_{n=1}^{\infty}\left[nH_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\sin[n(\phi-\phi')]\right]\left[F_{z\phi}(n,k_z) - C_{z\phi}(k_z)\right] \\ &\quad -\frac{1}{4\omega}S_2C_{z\phi}(k_z),\end{aligned} \tag{3.105}$$

where an envelope extraction with respect to n is performed. Finally performing another envelope extraction method to (3.105) similar to $\frac{\tilde{G}^E_{\phi z}}{k_z}$ case, the final spectral domain expression is obtained as

$$\begin{aligned}\frac{\tilde{G}^E_{z\phi f}}{k_z} &= -\frac{j}{2\omega}\sum_{n=1}^{\infty}nH_n^{(2)}(k_{\rho 3}\rho)J_n(k_{\rho 3}\rho')\sin[n(\phi-\phi')][F_{z\phi}(n,k_z) - C_{z\phi}(k_z)] \\ &\quad -\frac{1}{4\omega}S_2[C_{z\phi}(k_z) - C_{z\phi}(k_{z\infty})].\end{aligned} \tag{3.106}$$

As a result using the $\frac{\tilde{G}^E_{z\phi f}}{k_z}$ expression given by (3.106) the $G^E_{z\phi}$ expression in the spatial domain becomes

$$\begin{aligned} G^E_{z\phi} &= j\frac{\partial}{\partial z}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{G}^E_{z\phi f}}{k_z}e^{-jk_z(z-z')}dk_z\right) \\ &+\frac{1}{4\pi\omega}\left(-j\rho\rho'\sin(\phi-\phi')\frac{C_{z\phi}(k_{z\infty})}{|\bar{\rho}-\bar{\rho}'|}\right)\frac{\partial I_2}{\partial z}. \end{aligned} \tag{3.107}$$

3.5 $\tilde{G}^E_{\phi\phi}$ Expression when $\rho = \rho'$

The method used for $\tilde{G}^E_{\phi\phi}$ is not different from the others. The $\tilde{G}^E_{\phi\phi}$ expression for $(\rho \neq \rho')$ is

$$\begin{aligned} \tilde{G}^E_{\phi\phi} &= -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n\cos[n(\phi-\phi')]\left[\frac{n^2k_z^2}{\epsilon_j\rho\rho' k^2_{\rho_m}}F_n(1,1)+\frac{j\omega\mu_m nk_z}{k_{\rho_m}\epsilon_j\rho'}\frac{\partial F_n(2,1)}{\partial(k_{\rho_m}\rho)}\right. \\ &\left.-\frac{j\omega nk_zk_{\rho_j}}{k^2_{\rho_m}\rho}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')}+\omega^2\mu_m\frac{k_{\rho_j}}{k_{\rho_m}}\frac{\partial^2F_n(2,2)}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_m}\rho)}\right] \end{aligned} \tag{3.108}$$

since in our case $j = m$ when $\rho = \rho'$, we obtain

$$\begin{aligned} \tilde{G}^E_{\phi\phi} &= -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n\cos[n(\phi-\phi')]\left[\frac{n^2k_z^2}{\epsilon_j\rho\rho' k^2_{\rho_j}}F_n(1,1)+\frac{j\omega\mu_j nk_z}{k_{\rho_j}\epsilon_j\rho'}\frac{\partial F_n(2,1)}{\partial(k_{\rho_j}\rho)}\right. \\ &\left.-\frac{j\omega nk_z}{k_{\rho_j}\rho}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')}+\omega^2\mu_j\frac{\partial^2F_n(2,2)}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)}\right]. \end{aligned} \tag{3.109}$$

Using (3.57), (3.62), (3.67) and (3.72) in (3.109), the $\bar{F}_n$ related terms inside the bracket are written as

$$\frac{n^2k_z^2}{\epsilon_j\rho\rho' k^2_{\rho_j}}F_n(1,1) = n^2H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\left(\frac{k_z^2}{\epsilon_j\rho\rho' k^2_{\rho_j}}Fn_n(1,1)\right) \tag{3.110}$$

$$\frac{j\omega\mu_j nk_z}{k_{\rho_j}\epsilon_j\rho'}\frac{\partial F_n(2,1)}{\partial(k_{\rho_j}\rho)} = n^2H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\left(\frac{j\omega\mu_jk_z}{k_{\rho_j}\epsilon_j\rho'}Fn_{nd\rho}(2,1)\right) \tag{3.111}$$

$$-\frac{j\omega nk_z}{k_{\rho_j}\rho}\frac{\partial F_n(1,2)}{\partial(k_{\rho_j}\rho')} = n^2H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\left(-\frac{j\omega nk_z}{k_{\rho_j}\rho}Fn_{nd\rho'}(1,2)\right) \tag{3.112}$$

$$\omega^2\mu_j\frac{\partial^2F_n(2,2)}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} = H_n^{'(2)}(k_{\rho_j}\rho)J'_n(k_{\rho_j}\rho')(\omega^2\mu_jFn_{nd\rho d\rho'}(2,2)). \tag{3.113}$$

As the next step we define two functions, namely $F_{\phi\phi 1}$ and $F_{\phi\phi 2}$ as

$$F_{\phi\phi 1}(n,k_z)=\frac{k_z^2}{\epsilon_j\rho\rho' k_{\rho_j}^2}Fn_n(1,1)+\frac{j\omega\mu_j k_z}{k_{\rho_j}\epsilon_j\rho'}Fn_{nd\rho}(2,1)-\frac{j\omega n k_z}{k_{\rho_j}\rho}Fn_{nd\rho'}(1,2) \tag{3.114}$$

and

$$F_{\phi\phi 2}(n,k_z)=\omega^2\mu_j Fn_{nd\rho d\rho'}(2,2) \tag{3.115}$$

where their limiting values for large n values are constant and given by $C_{\phi\phi 1}(k_z)$ and $C_{\phi\phi 2}(k_z)$ such that

$$\begin{aligned}\lim_{n\to\infty}F_{\phi\phi 1}(n,k_z)&=C_{\phi\phi 1}(k_z)\\ \lim_{n\to\infty}F_{\phi\phi 2}(n,k_z)&=C_{\phi\phi 2}(k_z).\end{aligned} \tag{3.116}$$

Then, (3.109) can be written in terms of $F_{\phi\phi 1}$ and $F_{\phi\phi 2}$ as

$$\begin{aligned}\tilde{G}^E_{\phi\phi} &= -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n n^2 H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')]F_{\phi\phi 1}(n,k_z)\\ &\quad -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n H_n^{'(2)}(k_{\rho_j}\rho)J_n'(k_{\rho_j}\rho')\cos[n(\phi-\phi')]F_{\phi\phi 2}(n,k_z).\end{aligned} \tag{3.117}$$

At this point, similar to the previously treated components, we want to increase the convergence rate of the summation. Therefore, we define two summations S_3 and S_4 as

$$S_3=\sum_{n=-\infty}^{\infty}n^2 H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')e^{jn(\phi-\phi')} \tag{3.118}$$

$$S_4=\sum_{n=-\infty}^{\infty}H_n^{'(2)}(k_{\rho_j}\rho)J_n'(k_{\rho_j}\rho')e^{jn(\phi-\phi')} \tag{3.119}$$

which can be expressed in terms of the derivatives of S_1 as

$$S_3=\frac{\partial^2 S_1}{\partial\phi\partial\phi'}=\frac{\partial^2 H_0^{(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{\partial\phi\partial\phi'} \tag{3.120}$$

and

$$S_4=\frac{\partial^2 S_1}{\partial(k_{\rho_j}\rho)\partial(k_{\rho_j}\rho')}=\frac{\partial^2 H_0^{(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{\partial(k_{\rho_j}\rho)(\partial k_{\rho_j}\rho')}. \tag{3.121}$$

The explicit expressions for S_3 and S_4 are given by

$$\begin{aligned} S_3 &= -\frac{\rho\rho'}{|\bar{\rho}-\bar{\rho}'|}\cos(\phi-\phi')k_{\rho_j}H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) \\ &\quad -\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^2}k_{\rho_j}^2H_0^{''(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) \\ &\quad +\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^3}k_{\rho_j}H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) \end{aligned} \tag{3.122}$$

$$\begin{aligned} S_4 &= \frac{H_0^{''(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{|\bar{\rho}-\bar{\rho}'|^2}[\rho'-\rho\cos(\phi-\phi')][\rho-\rho'\cos(\phi-\phi')] \\ &\quad -\frac{H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{k_{\rho_j}}\frac{\cos(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|} \\ &\quad -\frac{H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{k_{\rho_j}|\bar{\rho}-\bar{\rho}'|^3}[\rho'-\rho\cos(\phi-\phi')][\rho-\rho'\cos(\phi-\phi')]. \end{aligned} \tag{3.123}$$

Using the envelope extraction method with respect to n and making use of (3.118)-(3.119), the fast convergent summation for $\tilde{G}^E_{\phi\phi}$ can be obtained as

$$\begin{aligned} \tilde{G}^E_{\phi\phi} &= -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n n^2 H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')][F_{\phi\phi 1}(n,k_z)-C_{\phi\phi 1}(k_z)] \\ &\quad -\frac{1}{4\omega}S_3C_{\phi\phi 1}(k_z) \\ &\quad -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n H_n^{'(2)}(k_{\rho_j}\rho)J_n'(k_{\rho_j}\rho')\cos[n(\phi-\phi')][F_{\phi\phi 2}(n,k_z)-C_{\phi\phi 2}(k_z)] \\ &\quad -\frac{1}{4\omega}S_4C_{\phi\phi 2}(k_z). \end{aligned} \tag{3.124}$$

In order to make this modified expression (3.124) also integrable for small $(\phi-\phi')$ values, we must find the spatial domain representations for the spectral domain terms given in S_3 and S_4. Recall that in the evaluation of $\frac{\tilde{G}^E_{\phi z}}{k_z}$ we used the expression

$$I_2 = \frac{-j}{2}\int_{-\infty}^{\infty}k_{\rho_j}H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)e^{-jk_z(z-z')}dk_z \tag{3.125}$$

which was written in terms of I_1 ($I_1 = \frac{e^{-jk_j|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}$) as

$$I_2 = \frac{\partial}{\partial|\bar{\rho}-\bar{\rho}'|}\frac{e^{-jk_j|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} = -jk_j|\bar{\rho}-\bar{\rho}'|\frac{e^{-jk_j|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|^2}-|\bar{\rho}-\bar{\rho}'|\frac{e^{-jk_j|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|^3}. \tag{3.126}$$

Similarly, if we define another integral, I_3, as

$$I_3 = \frac{-j}{2}\int_{-\infty}^{\infty} k_{\rho_j}^2 H_0^{''(2)}(k_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|)e^{-jk_z(z-z')}dk_z \tag{3.127}$$

and express it in terms of I_1, we obtain

$$I_3 = \frac{\partial^2 I_1}{\partial|\bar{\rho}-\bar{\rho}'|^2} = \frac{\partial^2}{\partial|\bar{\rho}-\bar{\rho}'|^2}\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{|\bar{r}-\bar{r}'|} \tag{3.128}$$

or explicitly

$$\begin{aligned} I_3 &= -jk_j\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^2} - k_j^2\left|\bar{\rho}-\bar{\rho}'\right|^2\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^3} - \frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^3} \\ &\quad +3jk_j\left|\bar{\rho}-\bar{\rho}'\right|^2\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^4} + 3\left|\bar{\rho}-\bar{\rho}'\right|^2\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|^5}. \end{aligned} \tag{3.129}$$

Similarly, we define I_4 as

$$I_4 = -\frac{j}{2}\int_{-\infty}^{\infty}\frac{H_0^{'(2)}(k_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|)}{k_{\rho_j}}e^{-jk_z(z-z')}dk_z. \tag{3.130}$$

I_4 can be expressed in terms of the known expression I_1 as

$$I_4 = \frac{1}{k_j\left|\bar{\rho}-\bar{\rho}'\right|}\frac{\partial I_1}{\partial k_j} = \frac{1}{k_j\left|\bar{\rho}-\bar{\rho}'\right|}\frac{\partial}{\partial k_j}\left(\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|}\right) \tag{3.131}$$

or in a simplified from I_4 becomes

$$I_4 = -\frac{je^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{k_j\left|\bar{\rho}-\bar{\rho}'\right|}. \tag{3.132}$$

Finally, the last integral we define is I_5 given by

$$I_5 = -\frac{j}{2}\int_{-\infty}^{\infty} H_0^{''(2)}(k_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|)e^{-jk_z(z-z')}dk_z, \tag{3.133}$$

and can be obtained in terms of I_4 as

$$I_5 = \frac{\partial I_4}{\partial|\bar{\rho}-\bar{\rho}'|} = \frac{\partial}{\partial|\bar{\rho}-\bar{\rho}'|}\left(-\frac{je^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{k_j\left|\bar{\rho}-\bar{\rho}'\right|}\right) \tag{3.134}$$

or

$$I_5 = -\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{r}-\bar{r}'\right|} + j\frac{e^{-jk_j\left|\bar{r}-\bar{r}'\right|}}{\left|\bar{\rho}-\bar{\rho}'\right|^2 k_j}. \tag{3.135}$$

Finding the limiting values of $C_{\phi\phi 1}(k_z)$ and $C_{\phi\phi 2}(k_z)$ at $k_z = k_{z\infty}$ and performing an envelope extraction method with respect to k_z, the final expression in the spectral domain is obtained as

$$\begin{aligned}
\tilde{G}^E_{\phi\phi f} \;=\;& -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n n^2 H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')][F_{\phi\phi 1}(n,k_z)-C_{\phi\phi 1}(k_z)] \\
& -\frac{1}{4\omega}S_3[C_{\phi\phi 1}(k_z)-C_{\phi\phi 1}(k_{z\infty})] \\
& -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n H_n^{'(2)}(k_{\rho_j}\rho)J'_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')][F_{\phi\phi 2}(n,k_z)-C_{\phi\phi 2}(k_z)] \\
& -\frac{1}{4\omega}S_4[C_{\phi\phi 2}(k_z)-C_{\phi\phi 2}(k_{z\infty})] \qquad (3.136)
\end{aligned}$$

where using the $\tilde{G}^E_{\phi\phi f}$ expression given in (3.136), the final spatial domain expression is given by

$$\begin{aligned}
G^E_{\phi\phi} \;=\;& \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}^E_{\phi\phi f}e^{-jk_z(z-z')}dk_z + \frac{-j}{4\pi\omega}\left[-I_2\frac{\rho\rho'}{|\bar{\rho}-\bar{\rho}'|}\cos(\phi-\phi')\right. \\
& \left. -I_3\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^2} + I_2\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^3}\right] C_{\phi\phi 1}(k_{z\infty}) \\
& +\frac{-j}{4\pi\omega}\left[I_5\frac{1}{|\bar{\rho}-\bar{\rho}'|^2}[\rho'-\rho\cos(\phi-\phi')][\rho-\rho'\cos(\phi-\phi')] - I_4\frac{\cos(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|}\right. \\
& \left. -I_4\frac{1}{|\bar{\rho}-\bar{\rho}'|^3}[\rho'-\rho\cos(\phi-\phi')][\rho-\rho'\cos(\phi-\phi')]\right] C_{\phi\phi 2}(k_{z\infty}). \qquad (3.137)
\end{aligned}$$

At this point, the spatial domain Green's function representations given by (3.83), (3.98), (3.107) and (3.137), which are still not in closed-form due to the Fourier integration with respect to k_z, work well everywhere except the axial line (i.e., $\phi = \phi'$ and $\rho = \rho'$). Therefore, in the next chapter we treat this problem.

Chapter 4

Green's Function Representations Valid Along the Axial Line

The Green's function expressions given in Chapter 3 are fast convergent and integrable expressions when $\rho = \rho'$. Besides, the S_1, S_2, S_3 and S_4 terms, derived from $H_0^{(2)}(k_{\rho_j} |\bar{\rho} - \bar{\rho}'|)$ (and functions of $H_0^{(2)}(k_{\rho_j} |\bar{\rho} - \bar{\rho}'|)$) are used in Chapter 3 to make the summations fast convergent.

However, the $H_0^{(2)}(k_{\rho_j} |\bar{\rho} - \bar{\rho}'|)$ related terms yield singularity problems along the axial line since the argument of the Hankel function becomes zero. This problem manifests itself in particular for the G_{zz}^E and $G_{\phi\phi}^E$ components, which has not been addressed in the literature before to the best of our knowledge. In this chapter we provide a solution for this problem so that the final closed-form Green's function representations become valid for arbitrary source and field point locations and can be used in conjunction with the MoM. Note that $G_{\phi z}^E = G_{z\phi}^E = 0$ along the axial line.

First, we use the small argument approximation of $H_0^{(2)}(k_{\rho_j} |\bar{\rho} - \bar{\rho}'|)$ for the problematic terms, and then we manipulate with the resultant expression such that the singular term will appear as a k_z independent term. This way once the singular term is omitted, DCIM method can be safely applied to the remaining expressions, which are k_z dependent.

4.1 Solution for $\tilde{G}^E_{zz}$ case

The problematic term in $\tilde{G}^E_{zzf}$ expression is S_1 and the small argument approximation of S_1 is

$$S_1 = H_0^{(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) = 1 - j\frac{2}{\pi}\log\left(\frac{\gamma k_{\rho_j}|\bar{\rho}-\bar{\rho}'|}{2}\right) \tag{4.1}$$

where $\gamma = 1.781$. Using the fact that $\rho = \rho'$, we can write

$$|\bar{\rho}-\bar{\rho}'| = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')} \tag{4.2}$$

$$|\bar{\rho}-\bar{\rho}'| = \sqrt{2\rho^2-2\rho^2\cos(\phi-\phi')} \tag{4.3}$$

$$|\bar{\rho}-\bar{\rho}'| = \sqrt{2}\rho\sqrt{1-\cos(\phi-\phi')} \tag{4.4}$$

and S_1 becomes

$$S_1 = 1 - j\frac{2}{\pi}\log\left(\frac{\gamma k_{\rho_j}\rho\sqrt{1-\cos(\phi-\phi')}}{\sqrt{2}}\right). \tag{4.5}$$

Making use of the properties of the log function, we rewrite S_1 as

$$S_1 = 1 - j\frac{2}{\pi}\log\left(\frac{\gamma k_{\rho_j}\rho}{\sqrt{2}}\right) - j\frac{2}{\pi}\log\left(\sqrt{1-\cos(\phi-\phi')}\right). \tag{4.6}$$

As seen in (4.6), the last term $-j\frac{2}{\pi}\log\left(\sqrt{1-\cos(\phi-\phi')}\right)$ is responsible from the singularity and it is k_z independent. Therefore, if we omit this term for $\phi = \phi'$ case, we can use S_1 with the final expression

$$S_1 \approx 1 - j\frac{2}{\pi}\log\left(\frac{\gamma k_{\rho_j}\rho}{\sqrt{2}}\right). \tag{4.7}$$

4.2 Solution for $\tilde{G}^E_{\phi\phi}$ Case

In order to see the behaviour of $\tilde{G}^E_{\phi\phi f}$ expression when ϕ goes to ϕ', the S_3 and S_4 expressions must be analyzed carefully. S_3 expression is given by

$$\begin{aligned} S_3 \quad = \quad & -\frac{\rho\rho'}{|\bar{\rho}-\bar{\rho}'|}\cos(\phi-\phi')k_{\rho_j}H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) \\ & -\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^2}k_{\rho_j}^2H_0^{''(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) \\ & +\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^3}k_{\rho_j}H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|). \end{aligned} \tag{4.8}$$

Since $\rho = \rho'$, let us replace ρ' with ρ and when ϕ goes to ϕ', we obtain

$$\cos(\phi - \phi') \approx 1 \tag{4.9}$$
$$\frac{\sin^2(\phi - \phi')}{|\bar{\rho} - \bar{\rho}'|^2} = \frac{1 - \cos^2(\phi - \phi')}{2\rho^2(1 - \cos(\phi - \phi'))} = \frac{1 + \cos(\phi - \phi')}{2\rho^2} \approx \frac{1}{\rho^2}. \tag{4.10}$$

As a result S_3 becomes

$$\begin{aligned} S_3 &= -\frac{\rho^2}{|\bar{\rho} - \bar{\rho}'|} k_{\rho_j} H_0^{'(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|) - \rho^2 k_{\rho_j}^2 H_0^{''(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|) \\ &\quad + \frac{\rho^2}{|\bar{\rho} - \bar{\rho}'|} k_{\rho_j} H_0^{'(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|) \end{aligned} \tag{4.11}$$

and can simply be expressed as

$$S_3 = -\rho^2 k_{\rho_j}^2 H_0^{''(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|). \tag{4.12}$$

Since the small argument approximation of $H_0^{''(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|)$ is

$$H_0^{''(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|) = -\frac{1}{2} + \frac{k_{\rho_j}^2 |\bar{\rho} - \bar{\rho}'|^2}{8} + \frac{2j}{\pi k_{\rho_j}^2 |\bar{\rho} - \bar{\rho}'|^2} \tag{4.13}$$

for small $(\phi - \phi')$ values, the S_3 expression is obtained as

$$S_3 = \frac{\rho^2 k_{\rho_j}^2}{2} - \frac{\rho^2 k_{\rho_j}^4 |\bar{\rho} - \bar{\rho}'|^2}{8} - \frac{2j\rho^2}{\pi |\bar{\rho} - \bar{\rho}'|^2}. \tag{4.14}$$

In this S_3 expression given by (4.14), the singular term is $\frac{2j\rho^2}{\pi|\bar{\rho}-\bar{\rho}'|^2}$ and like $\tilde{G}_{zz}^E$ case, the singular term is k_z independent. By omitting this singular term, we can use S_3 for $\phi = \phi'$.

The other problematic term S_4 is analyzed similarly as follows:

$$\begin{aligned} S_4 &= \frac{H_0^{''(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|)}{|\bar{\rho} - \bar{\rho}'|^2}[\rho' - \rho\cos(\phi - \phi')][\rho - \rho'\cos(\phi - \phi')] \\ &\quad - \frac{H_0^{'(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|)}{k_{\rho_j}} \frac{\cos(\phi - \phi')}{|\bar{\rho} - \bar{\rho}'|} \\ &\quad - \frac{H_0^{'(2)}(k_{\rho_j}|\bar{\rho} - \bar{\rho}'|)}{k_{\rho_j}|\bar{\rho} - \bar{\rho}'|^3}[\rho' - \rho\cos(\phi - \phi')][\rho - \rho'\cos(\phi - \phi')]. \end{aligned} \tag{4.15}$$

Replacing ρ' by ρ and using the fact that $(\phi-\phi')$ is very small (i.e., goes to zero), we obtain

$$\cos(\phi-\phi') \approx 1 \tag{4.16}$$
$$|\bar{\rho}-\bar{\rho}'| = \sqrt{2}\rho\sqrt{1-\cos(\phi-\phi)} \tag{4.17}$$
$$[\rho'-\rho\cos(\phi-\phi')]\,[\rho-\rho'\cos(\phi-\phi')] = \rho^2(1-\cos(\phi-\phi'))^2 \tag{4.18}$$
$$[\rho'-\rho\cos(\phi-\phi')]\,[\rho-\rho'\cos(\phi-\phi')] = \frac{|\bar{\rho}-\bar{\rho}'|^4}{4\rho^2}. \tag{4.19}$$

Hence, S_4 can be written as

$$\begin{aligned} S_4 &= H_0^{''(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)\frac{|\bar{\rho}-\bar{\rho}'|^2}{4\rho^2} - \frac{H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)}{k_{\rho_j}|\bar{\rho}-\bar{\rho}'|} \\ &\quad - H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)\frac{|\bar{\rho}-\bar{\rho}'|}{4\rho^2 k_{\rho_j}} \end{aligned} \tag{4.20}$$

and using the small argument approximations of both $H_0^{''(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)$ and $H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|)$ given by

$$H_0^{'(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) = -\frac{k_{\rho_j}|\bar{\rho}-\bar{\rho}'|}{2} - \frac{2j}{\pi k_{\rho_j}|\bar{\rho}-\bar{\rho}'|} \tag{4.21}$$
$$H_0^{''(2)}(k_{\rho_j}|\bar{\rho}-\bar{\rho}'|) = -\frac{1}{2} + \frac{k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^2}{8} + \frac{2j}{\pi k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^2}, \tag{4.22}$$

respectively, for small $(\phi-\phi')$ values, S_4 becomes

$$\begin{aligned} S_4 &= \left(-\frac{|\bar{\rho}-\bar{\rho}'|^2}{8\rho^2} + \frac{k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^4}{32\rho^2} + \frac{j}{\pi 2\rho^2 k_{\rho_j}^2}\right) \\ &\quad - \left(-\frac{1}{2} - \frac{2j}{\pi k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^2}\right) \\ &\quad - \left(-\frac{|\bar{\rho}-\bar{\rho}'|^2}{8\rho^2} - \frac{j}{2\pi\rho^2 k_{\rho_j}^2}\right) \end{aligned} \tag{4.23}$$

or in a simplified form S_4 is given by

$$S_4 = \frac{1}{2} + \frac{k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^4}{32\rho^2} + \frac{j}{\pi\rho^2 k_{\rho_j}^2} + \frac{2j}{\pi k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^2}. \tag{4.24}$$

In this S_4 expression given by (4.24), the singular term is $\frac{2j}{\pi k_{\rho_j}^2|\bar{\rho}-\bar{\rho}'|^2}$, but now it is k_z dependent and it can not be omitted. Therefore, in order to make the calculations for $\phi = \phi'$, the $\tilde{G}^E_{\phi\phi f}$ expression must be rewritten in a different form to change this singular term in S_4.

In fact, we can change the formulation given in (3.113) as follows:

$$\begin{aligned} \omega^2\mu_j \frac{\partial^2 F_n(2,2)}{\partial(k_{\rho_j}\rho')\partial(k_{\rho_j}\rho)} &= n^2 H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho') \\ &\cdot \left[\frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\frac{J_n^{'}(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\omega^2\mu_j Fn_{nd\rho d\rho'}(2,2)\right] \end{aligned} \quad (4.25)$$

We proved that

$$\lim_{n\to\infty} \frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)} = C_1 \quad (4.26)$$

and

$$\lim_{n\to\infty} \frac{J_n^{'}(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')} = C_2. \quad (4.27)$$

where C_1 and C_2 are just constants with respect to n. We redefine the $F_{\phi\phi 1}(n, k_z)$ term as

$$\begin{aligned} F_{\phi\phi 1}(n,k_z) &= \frac{k_z^2}{\epsilon_j\rho\rho' k_{\rho_j}^2}Fn_n(1,1) + \frac{j\omega\mu_j k_z}{k_{\rho_j}\epsilon_j\rho'}Fn_{nd\rho}(2,1) \\ &-\frac{j\omega n k_z}{k_{\rho_j}\rho}Fn_{nd\rho'}(1,2) + \frac{H_n^{'(2)}(k_{\rho_j}\rho)}{nH_n^{(2)}(k_{\rho_j}\rho)}\frac{J_n^{'}(k_{\rho_j}\rho')}{nJ_n(k_{\rho_j}\rho')}\omega^2\mu_j Fn_{nd\rho d\rho'}(2,2) \end{aligned} \quad (4.28)$$

where

$$\lim_{n\to\infty} F_{\phi\phi 1}(n,k_z) = C_{\phi\phi 1n}(k_z). \quad (4.29)$$

We obtain the new expression for $\tilde{G}^E_{\phi\phi f}$, using $F_{\phi\phi 1}(n, k_z)$ given by (4.28), as

$$\begin{aligned} \tilde{G}^E_{\phi\phi f} &= -\frac{1}{2\omega}\sum_{n=0}^{\infty}\epsilon_n n^2 H_n^{(2)}(k_{\rho_j}\rho)J_n(k_{\rho_j}\rho')\cos[n(\phi-\phi')][F_{\phi\phi 1}(n,k_z) - C_{\phi\phi 1n}(k_z)] \\ &-\frac{1}{4\omega}S_3[C_{\phi\phi 1n}(k_z) - C_{\phi\phi 1n}(k_{z\infty})] \end{aligned} \quad (4.30)$$

and using the $\tilde{G}^E_{\phi\phi f}$ expression given by (4.30), the spatial domain expression for $G^E_{\phi\phi}$ becomes

$$
\begin{aligned}
G^E_{\phi\phi} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}^E_{\phi\phi f}e^{-jk_z(z-z')}dk_z - \frac{j}{4\pi\omega}\left[-I_2\frac{\rho\rho'}{|\bar{\rho}-\bar{\rho}'|}\cos(\phi-\phi')\right.\\
&\left.-I_3\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^2}+I_2\rho^2\rho'^2\frac{\sin^2(\phi-\phi')}{|\bar{\rho}-\bar{\rho}'|^3}\right]C_{\phi\phi 1n}(k_{z\infty}). \qquad (4.31)
\end{aligned}
$$

As seen in (4.30), in the new spectral domain expression there is only S_3 term (S_4 term is eliminated) and when ϕ goes to ϕ' S_3 term is given by (4.14) as

$$
S_3 = \frac{\rho^2 k^2_{\rho_j}}{2} - \frac{\rho^2 k^4_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|^2}{8} - \frac{2j\rho^2}{\pi\left|\bar{\rho}-\bar{\rho}'\right|^2} \qquad (4.32)
$$

or after we omit the singular term which is k_z independent, S_3 becomes

$$
S_3 = \frac{\rho^2 k^2_{\rho_j}}{2} - \frac{\rho^2 k^4_{\rho_j}\left|\bar{\rho}-\bar{\rho}'\right|^2}{8} \qquad (4.33)
$$

which is used in (4.30) when $\phi = \phi'$.

We have now two different representations for $G^E_{\phi\phi}$. When the performance of the summation is considered, the expression (3.137) has a faster convergent summation than (4.31) since the limiting value used in (3.137) is simpler. However, since the singular term of (3.137) is k_z dependent, evaluating the k_z integration of this expression is difficult for small $(\phi-\phi')$ values. So for an accurate integration, (4.31) is used for small $(\phi-\phi')$ values and for a fast summation (3.137) is used for large $(\phi-\phi')$ values.

Chapter 5

Closed-Form Representations in the Spatial Domain

The modified representation of the spectral domain Green's function which has a Fourier integral with respect to k_z are transformed into the spatial domain by the inverse Fourier transformation given by

$$G^E_{uv} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z e^{-jk_z(z-z')}\tilde{G}^E_{uv} \tag{5.1}$$

where G^E_{uv} is the spatial domain Green's function representation. This formula is a time consuming numerical integration of spectral domain Green's function representation such that it ranges from $-\infty$ to ∞ along the real axis of complex k_z plane. Therefore, in this chapter we provide closed-form representations for the spatial domain Green's functions using the DCIM with the aid of GPOF in a similar fashion to [10] and [11].

The final form of the modified spectral domain Green's function representations, which are given by (3.81), (3.96), (3.106), (3.136) and (4.30), are even functions of k_z. Therefore, the original Fourier k_z integral is folded to a 0 to ∞ integral and the original path is deformed as shown in Fig. 5.1 to overcome the effects of the pole and branch-point singularities. It should be noted that the new path is an approximation of the original one. Hence, the value of T_1 should be kept small to minimize the deviation from the original path. Moreover, T_2 should

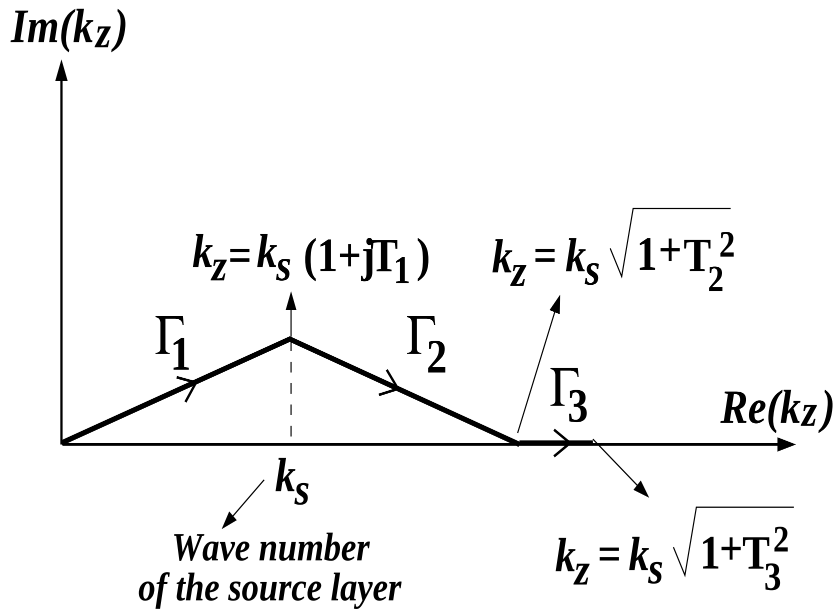

Figure 5.1: Deformed integration path

be large enough so that the value of k_z is larger than those of the wave numbers of all layers ensuring that none of the singularities lies on the deformed path. For a suitable choice of T_2, the value of T_3 should be around $1.1T_2$. Note that unlike [10], in this thesis the large argument approximation of the Hankel function on path Γ_3 is not used in order to handle the singularity problem along the axial line. The parameters that define the deformed integration path are as follows:

$$\Gamma_1: \quad k_z = k_s(1 + jT1)\frac{t_1}{T_1}, \quad 0 \le t_1 < T_1 \tag{5.2}$$

$$\Gamma_2: \quad k_z = k_s\left[1 + jT_1 + (\sqrt{1+T_2^2} - 1 - jT_1)\frac{t_2}{T_2 - T_1}\right], \quad 0 \le t_2 < T_2 - T_1 \tag{5.3}$$

$$\Gamma_3: \quad k_z = k_s\left[\sqrt{1+T_2^2} + (\sqrt{1+T_3^2} - \sqrt{1+T_2^2})\frac{t_3}{T_3 - T_2}\right], \quad 0 \le t_3 < T_3 - T_2 \tag{5.4}$$

where k_s is the wave number of the source layer.

The modified spectral domain Green's function representations are sampled on paths Γ_1, Γ_2 and Γ_3 by taking N_1, N_2 and N_3 samples, respectively. Using GPOF, the samples are expressed in terms of M_1, M_2 and M_3 complex exponentials of k_z on each path. Let the approximated spectral domain Green's function representation (or the outputs of GPOF on each path) be

$$\tilde{G}_{uv}^{E'} \cong \sum_{m=1}^{M_1} b_{m_t} e^{s_{m_t} t_1} \tag{5.5}$$

$$\tilde{G}_{uv}^{E''} \cong \sum_{n=1}^{M_2} b_{n_t} e^{s_{n_t} t_2} \tag{5.6}$$

$$\tilde{G}_{uv}^{E'''} \cong \sum_{l=1}^{M_3} b_{l_t} e^{s_{l_t} t_3} \tag{5.7}$$

where the spectral domain Green's function representation is approximated as

$$\tilde{G}_{uv}^{E} \cong \tilde{G}_{uv}^{E'} + \tilde{G}_{uv}^{E''} + \tilde{G}_{uv}^{E'''}. \tag{5.8}$$

These outputs of GPOF are than transformed to complex exponentials of k_z by using the following technique:

$$\tilde{G}_{uv}^{E'} \cong \sum_{m=1}^{M_1} b_{m_t} e^{s_{m_t} t_1} = \sum_{m=1}^{M_1} b_{m_k} e^{k_z s_{m_k}} \tag{5.9}$$

$$\tilde{G}_{uv}^{E''} \cong \sum_{n=1}^{M_2} b_{n_t} e^{s_{n_t} t_2} = \sum_{n=1}^{M_2} b_{n_k} e^{k_z s_{n_k}} \tag{5.10}$$

$$\tilde{G}_{uv}^{E'''} \cong \sum_{l=1}^{M_3} b_{l_t} e^{s_{l_t} t_3} = \sum_{l=1}^{M_3} b_{l_k} e^{k_z s_{l_k}} \tag{5.11}$$

such that

$$b_{m_k} = b_{m_t} \tag{5.12}$$

$$s_{m_k} = \frac{s_{m_t}}{k_s} \frac{T_1}{1 + jT_1} \tag{5.13}$$

$$b_{n_k} = b_{n_t} e^{-s_{n_t} \frac{(T_2 - T_1)(1 + jT_1)}{\sqrt{1 + T_2^2} - 1 - jT_1}} \tag{5.14}$$

$$s_{n_k} = \frac{s_{n_t}}{k_s} \frac{T_2 - T_1}{\sqrt{1 + T_2^2} - 1 - jT_1} \tag{5.15}$$

$$b_{l_k} = b_{l_t} e^{-s_{l_t} \frac{(T_3 - T_2)\sqrt{1 + T_2^2}}{\sqrt{1 + T_3^2} - \sqrt{1 + T_2^2}}} \tag{5.16}$$

$$s_{l_k} = \frac{s_{l_t}}{k_s} \frac{T_3 - T_2}{\sqrt{1 + T_3^2} - \sqrt{1 + T_2^2}}. \tag{5.17}$$

Note that as explained in Appendix A inputs of the GPOF method are the spectral domain samples, the number of samples N, the number of complex exponentials M and the sampling interval δt. The sampling interval δt is chosen to be $\delta t = T_1/N_1$, $\delta t = T_2/N_2$ and $\delta t = T_3/N_3$ on paths Γ_1, Γ_2 and Γ_3, respectively. Finally, since the spectral domain Green's function representations are now in the form of complex exponentials of k_z, the integral given by

$$G_{uv}^{E}(z-z') = \frac{1}{\pi}\int_{\Gamma_1+\Gamma_2+\Gamma_3} dk_z cos[k_z(z-z')]\tilde{G}_{uv}^{E} \tag{5.18}$$

is taken analytically. Consequently, by adding the results of the three paths, the following expression is obtained for the spatial domain Green's function representations:

$$\begin{aligned} G_{uv}^{E}(z-z') \approx\ & \frac{1}{2\pi}\sum_{m=1}^{M_1} b_{m_k}\left(\frac{e^{k_s(1+jT_1)[s_{m_k}+j(z-z')]}-1}{s_{m_k}+j(z-z')} + \frac{e^{k_s(1+jT_1)[s_{m_k}-j(z-z')]}-1}{s_{m_k}-j(z-z')}\right) \\ & + \frac{1}{2\pi}\sum_{n=1}^{M_2} b_{n_k}\left(\frac{e^{k_z\sqrt{1+T_2^2}[s_{n_k}+j(z-z')]}-e^{k_s(1+jT_1)[s_{n_k}+j(z-z')]}}{s_{n_k}+j(z-z')}\right. \\ & \left. + \frac{e^{k_z\sqrt{1+T_2^2}[s_{n_k}-j(z-z')]}-e^{k_s(1+jT_1)[s_{n_k}-j(z-z')]}}{s_{n_k}-j(z-z')}\right) \\ & + \frac{1}{2\pi}\sum_{l=1}^{M_3} b_{l_k}\left(\frac{e^{k_z\sqrt{1+T_3^2}[s_{l_k}+j(z-z')]}-e^{k_z\sqrt{1+T_2^2}[s_{l_k}+j(z-z')]}}{s_{l_k}+j(z-z')}\right. \\ & \left. + \frac{e^{k_z\sqrt{1+T_3^2}[s_{l_k}-j(z-z')]}-e^{k_z\sqrt{1+T_2^2}[s_{l_k}-j(z-z')]}}{s_{l_k}-j(z-z')}\right). \end{aligned} \tag{5.19}$$

As seen in (5.19), the closed-form spatial domain expression is in the form of a sum of discrete complex images. Hence, the name DCIM is used.

Chapter 6

Numerical Results

To assess the accuracy of the method developed in this thesis, several numerical examples in the form of Green's function representations for $\rho \neq \rho'$, and in the form of mutual coupling between two tangential electric sources when $\rho = \rho'$ are given and compared with the previously published as well as available eigenfunction solutions given by [14] and [15].

6.1 Closed-Form Green's Function Results for $\rho \neq \rho'$

The first set of numerical results belong to a case where $\rho \neq \rho'$, which was presented in [10]. Here, we duplicate the same results to verify the accuracy of our codes. Besides, unlike [10] we are not using the large argument approximation of the Hankel function on path Γ_3. However, as will be seen from the numerical results, we do not sacrifice anything from the accuracy.

Fig. 6.1 depicts the geometry for a cylindrically stratified media with a perfect electric conductor (PEC) as the innermost layer. The radius of the PEC cylinder is a and the thickness of the dielectric coating, which has a relative dielectric constant $\epsilon_r > 1$, is t_h (and hence, $b = a + t_h$). Let a tangential electric source be

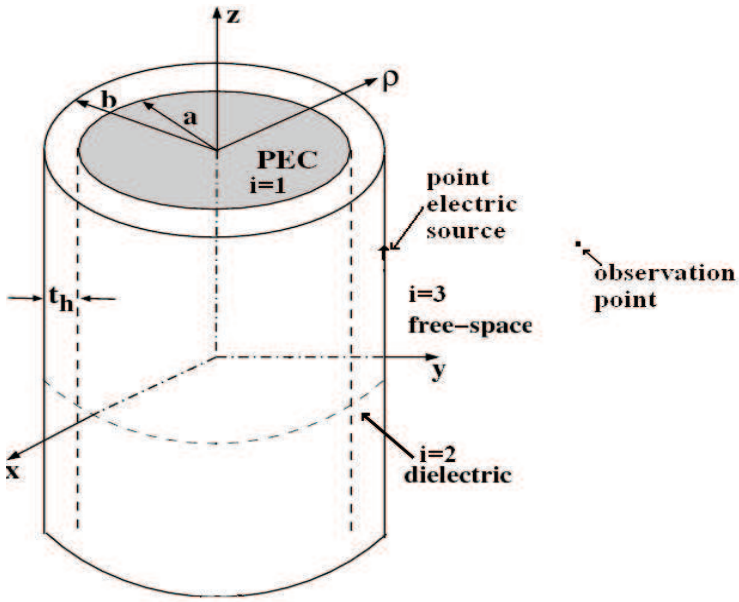

Figure 6.1: Dielectric-coated conducting cylinder: Layer 1:PEC, Layer 2: ϵ_{r1}=2.3, μ_{r1}=1, Layer 3: free space, a=20mm, b=21mm, ρ'=21mm, ϕ'=0, ρ=40mm, $\phi = \frac{\pi}{6} rad$, f=4.7Ghz

located at the air-dielectric interface $\rho' = b = 21mm$ and the observation point is at $\rho = 40mm > b$.

The results obtained from the closed-form spatial domain Green's function representations, which are approximated in terms of M_1, M_2 and M_3 complex exponentials, are compared with the results obtained from the direct numerical integration of the following inverse Fourier transform integral given by

$$G^E_{uv} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z e^{-jk_z(z-z')} \tilde{G}^E_{uv}. \tag{6.1}$$

The results are given in Figs. 6.2-6.5. In this example since we have $\rho >> \rho'$ case, there is not any summation or integration problem for the spectral domain Green's function components and hence, the expressions given in (2.19)-(2.22) can be used directly as the inputs for the GPOF method. To generate the results depicted in Figs. 6.2-6.5 we sum 40 terms for the summations. Besides, N_1=50, N_2=100 and N_3=10 samples are taken on the deformed path given by T_1=0.1, T_2=4, T_3=4.4. M_1=4, M_2=5, M_3=1 complex exponentials are used for DCIM. As seen in Figs. 6.2-6.5, the agreement between the DCIM results and the numerical

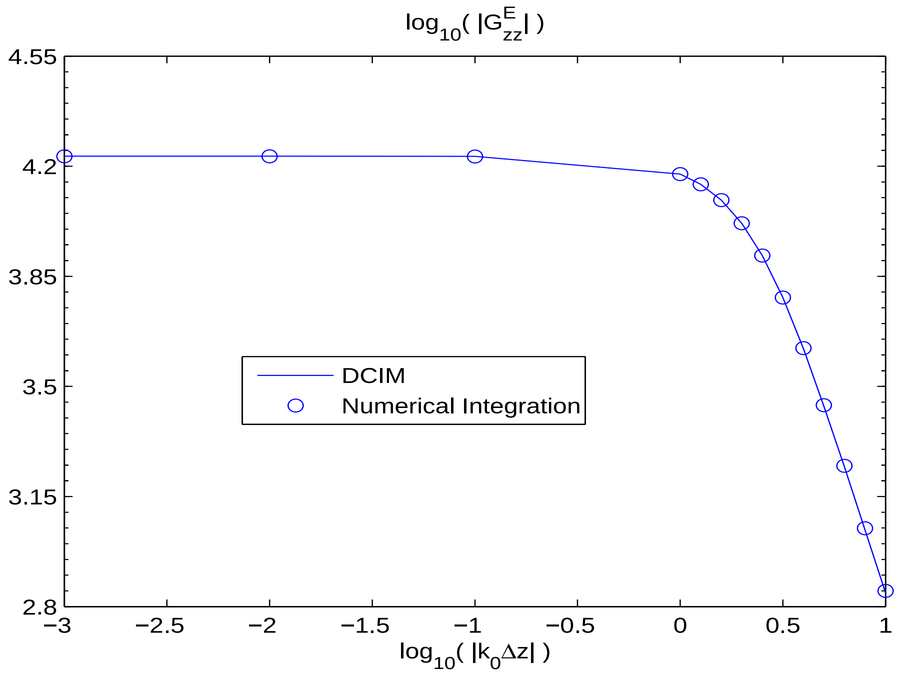

Figure 6.2: $\log(|G^E_{zz}|)$ for the geometry shown in Fig. 6.1 (M_1=4, M_2=5, M_3=1, T_1=0.1, T_2=4, T_3=4.4)

integration results is excellent. Besides, the same results were obtained in [10] which uses the large argument approximation of the Hankel function.

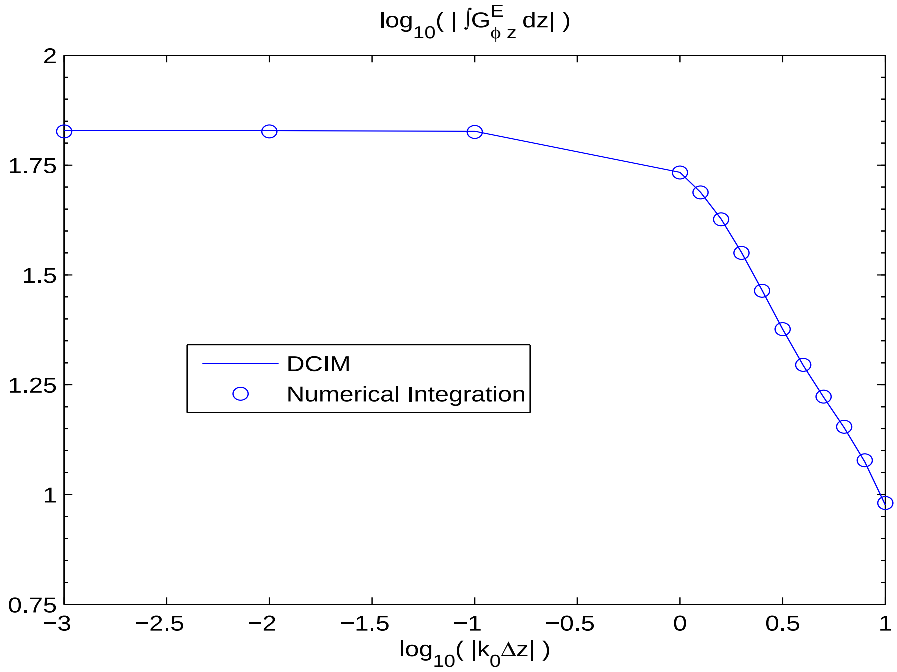

Figure 6.3: $\log(|\int G^E_{\phi z} dz|)$ for the geometry shown in Fig. 6.1 (M_1=4, M_2=5, M_3=1, T_1=0.1, T_2=4, T_3=4.4)

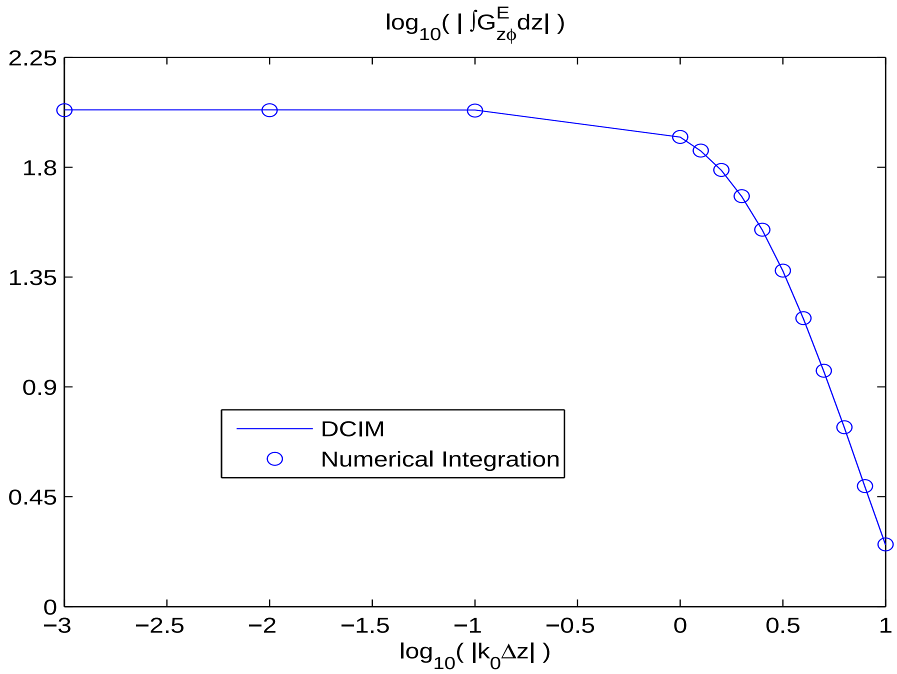

Figure 6.4: $\log(|\int G^E_{z\phi} dz|)$ for the geometry shown in Fig. 6.1 (M_1=4, M_2=5, M_3=1, T_1=0.1, T_2=4, T_3=4.4)

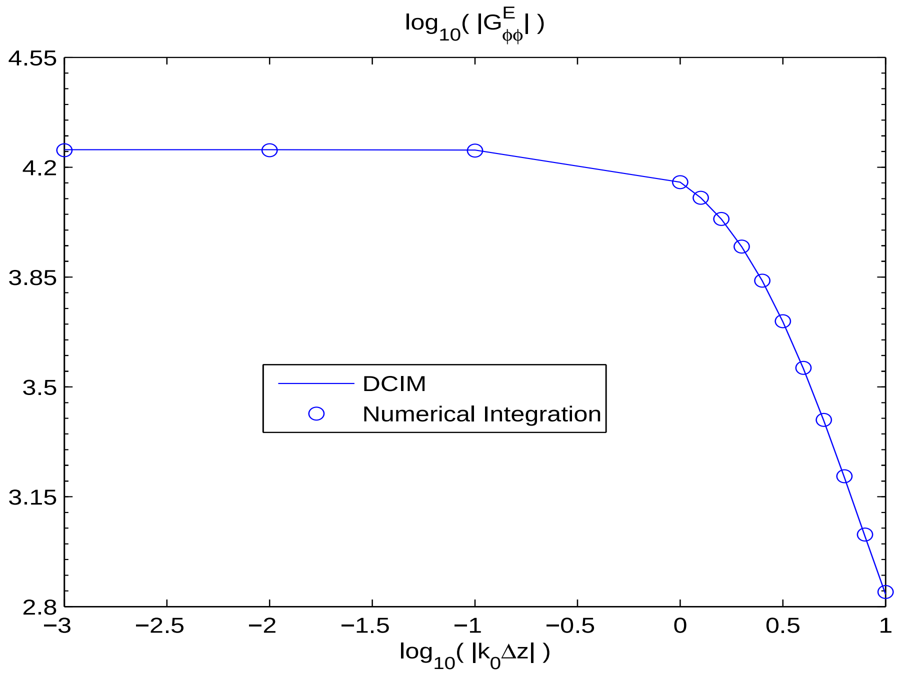

Figure 6.5: $\log(|G^E_{\phi\phi}|)$ for the geometry shown in Fig. 6.1 (M_1=4, M_2=5, M_3=1, T_1=0.1, T_2=4, T_3=4.4)

6.2 Mutual Coupling Results when $\rho = \rho'$

In this section numerical results in the form of mutual coupling between two tangential current modes on the air-dielectric interface of a coated PEC cylinder are given. The closed-form Green's function representations are used to find the mutual coupling results, and the obtained results are compared with an eigenfunction solution given by [14] and [15]. For the mutual coupling calculations we used Fig. 6.6, which is a slightly modified version of Fig. 6.1. In Fig. 6.6 two tangential current modes P and P' are located at the air-dielectric interface where $\rho = \rho' = b$.

To assess the validity and accuracy of the given method for $\rho = \rho'$, numerical results for the mutual impedance between two tangential electric current modes, J_{1_u} and J_{2_v}, are obtained using the closed-form expressions for the dielectric coated circular PEC cylinder with $a = 3\lambda_0$, $t_h = 0.06\lambda_0$ (λ_0 = free-space wavelength), $\epsilon_r = 3.25$, $\mu_r = 1$, as shown in Fig. 6.6. The mutual impedance $Z12_{uv}$

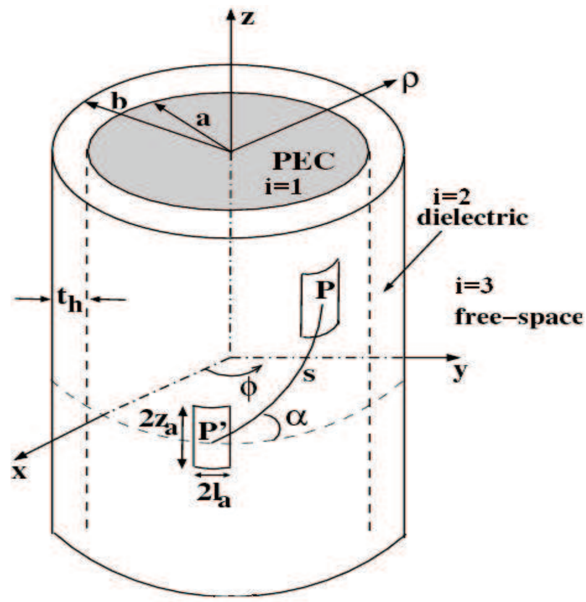

Figure 6.6: Two current modes P and P' on a coated conducting cylinder

between the current modes is simply given by

$$Z12_{uv} = \int_{S_2} E_{1_u} J_{2_v} \, ds \qquad (6.2)$$

where E_{1_u} is the field due to source (current mode) J_{1_u} and S_2 is the area occupied by source (current mode) J_{2_v}, where u and v are the tangential current modes ($\hat{z}$ or $\hat{\phi}$ directed).

The current modes, which are also given in Appendix C, are defined by a piecewise sinusoid along the direction of the current and by a constant along the direction perpendicular to the current [14]. Each element has dimensions of $0.1\lambda_0$ (along the direction of the current) by $0.04\lambda_0$. This particular choice of current modes alleviates the convergence of the reference spectral domain solution for relatively large cylinders.

When s is fixed, for different α values, the mutual coupling $Z12_{zz}$, $Z12_{\phi z}$, and $Z12_{\phi\phi}$ results are given in Figs. 6.7-6.12. Also for fixed α values, the mutual couplings are plotted and given in Figs. 6.13-6.30 when s is between $0.2\lambda_0$ and $3\lambda_0$. Figs. 6.18 and 6.30 depict the real and imaginary parts of the mutual impedance between two $\hat{z}$- and $\hat{\phi}$-directed sources versus separation for $\alpha = 90°$, respectively. This angle corresponds to the axial line, which has been remaining as a problematic region in the previous studies. In a similar fashion, Figs. 6.19-6.24 illustrate the mutual impedance between a $\hat{z}$- and a $\hat{\phi}$-directed current sources versus separation. Since the mutual impedance is zero along the axial direction for this case, it is not given.

In obtaining the closed-form Green's function representations, $N_1 = 150$, $N_2 = 250$ and $N_3 = 10$ spectral domain samples are used on the path shown in Fig. 5.1 defined by $T_1 = 0.2$, $T_2 = 20$ and $T_3 = 22$. The samples are written in terms of $M_1 = 15$, $M_2 = 25$ and $M_3 = 2$ complex exponentials in the spatial domain. Besides, for $G^E_{\phi\phi}$, (4.31) is used when $(\phi - \phi') < 0.1rad$, and (3.137) is used for the rest. 300 terms are used for the summation over the cylindrical eigenmodes in the spectral domain. However, this number is actually quite big since the cylinder that we use is electrically large. This number will definitely decrease if smaller cylinders are used.

Note that the mutual coupling between $\hat{\phi}$ and $\hat{z}$ directed current modes can be calculated by using the spatial domain Green's functions $G_{\phi z}$ or $G_{z\phi}$. The $G_{\phi z}$ and $G_{z\phi}$ expressions yield the same coupling results as expected. In order to show that they give the same coupling results, we defined $Z12_{\phi z}$ as the coupling calculated by using $G_{\phi z}$ and $Z12_{z\phi}$ as the coupling calculated by using $G_{z\phi}$.

Figs. 6.7-6.30 depict that the agreement between the results obtained with the closed-form Green's function representations and the eigenfunction results obtained from [14] and [15] is excellent.

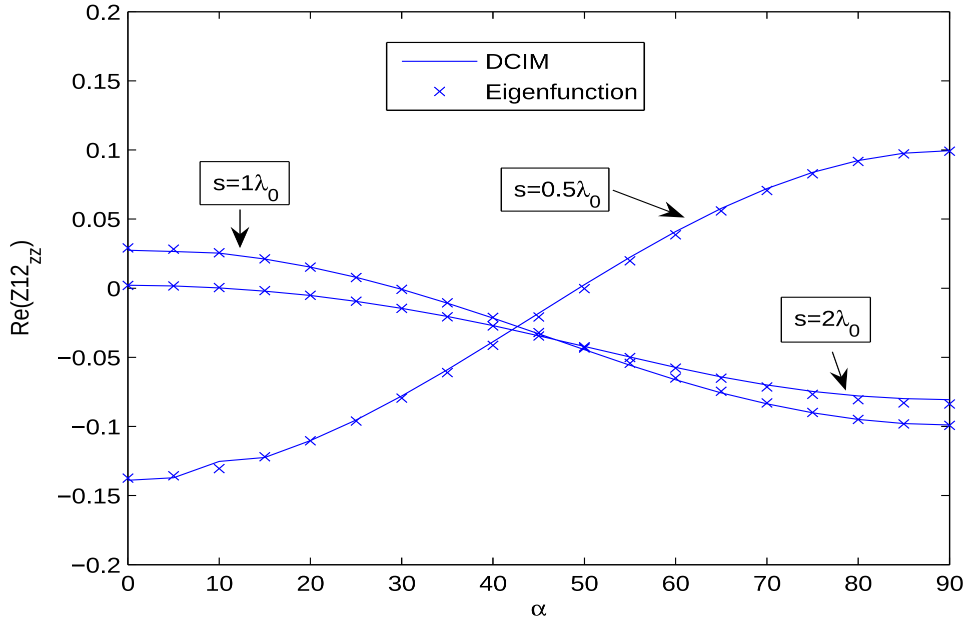

Figure 6.7: Real parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus α when s is fixed.

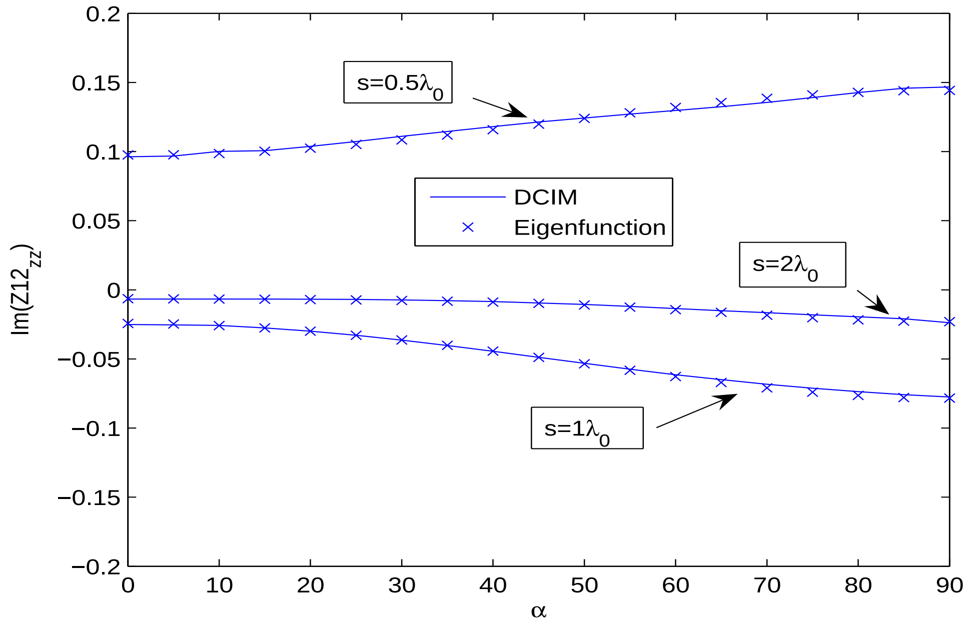

Figure 6.8: Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus α when s is fixed.

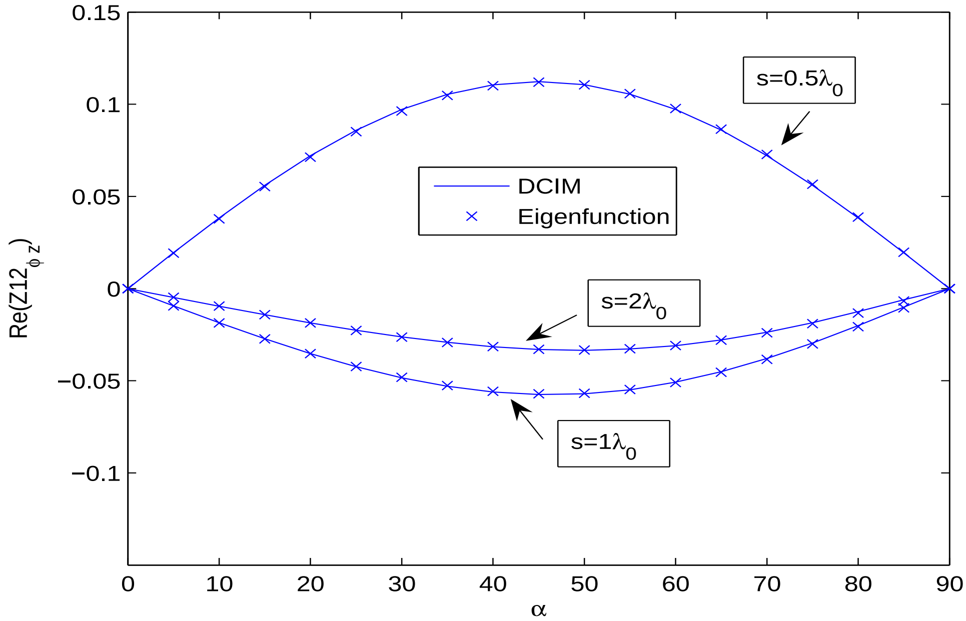

Figure 6.9: Real parts of the mutual impedance ($Z12_{\phi z}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus α when s is fixed.

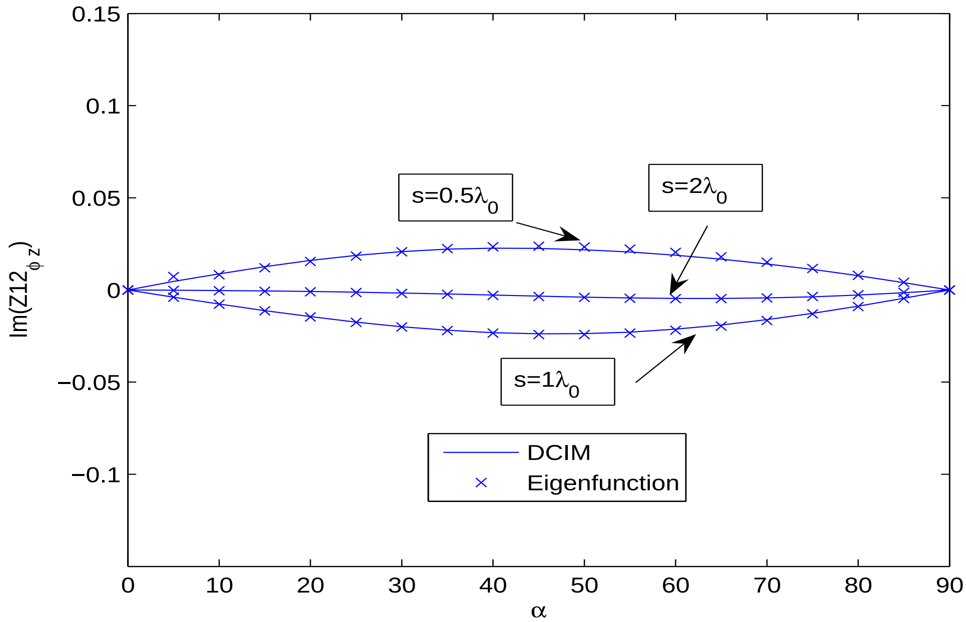

Figure 6.10: Imaginary parts of the mutual impedance ($Z12_{\phi z}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus α when s is fixed.

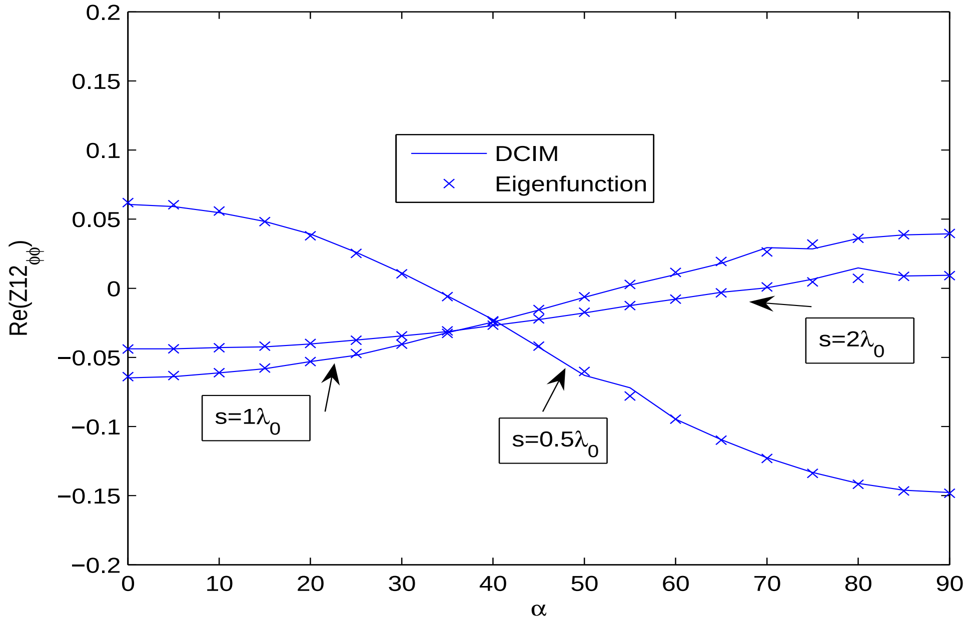

Figure 6.11: Real parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus α when s is fixed.

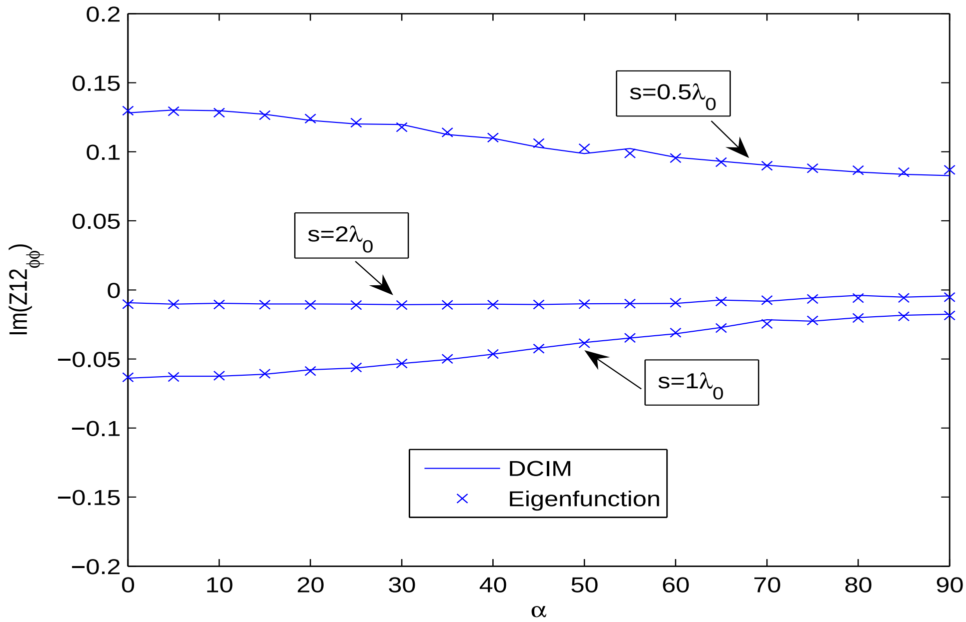

Figure 6.12: Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus α when s is fixed.

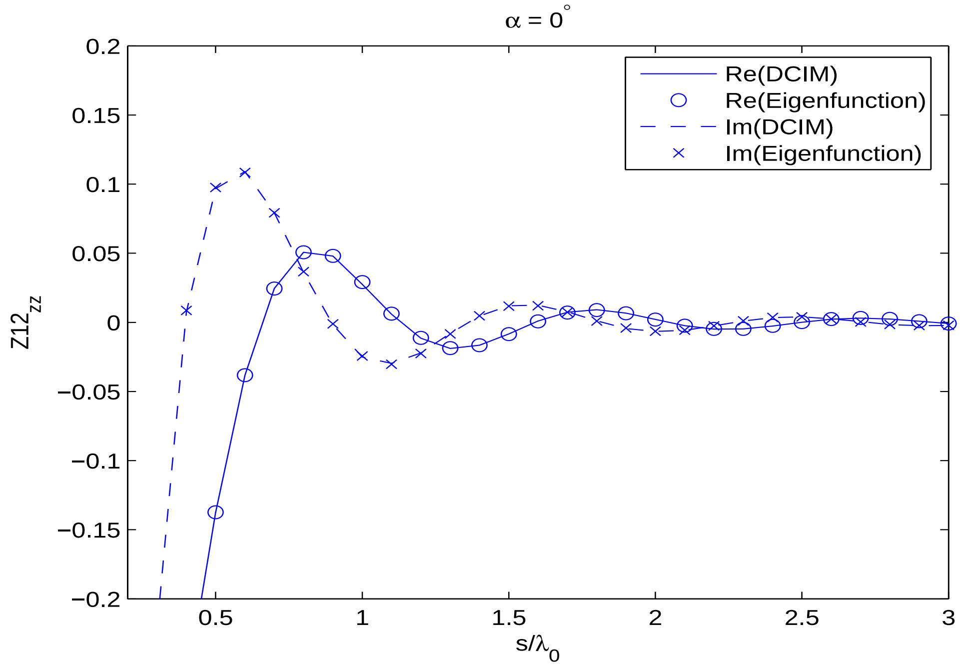

Figure 6.13: Real and Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus separation when $\alpha = 0°$.

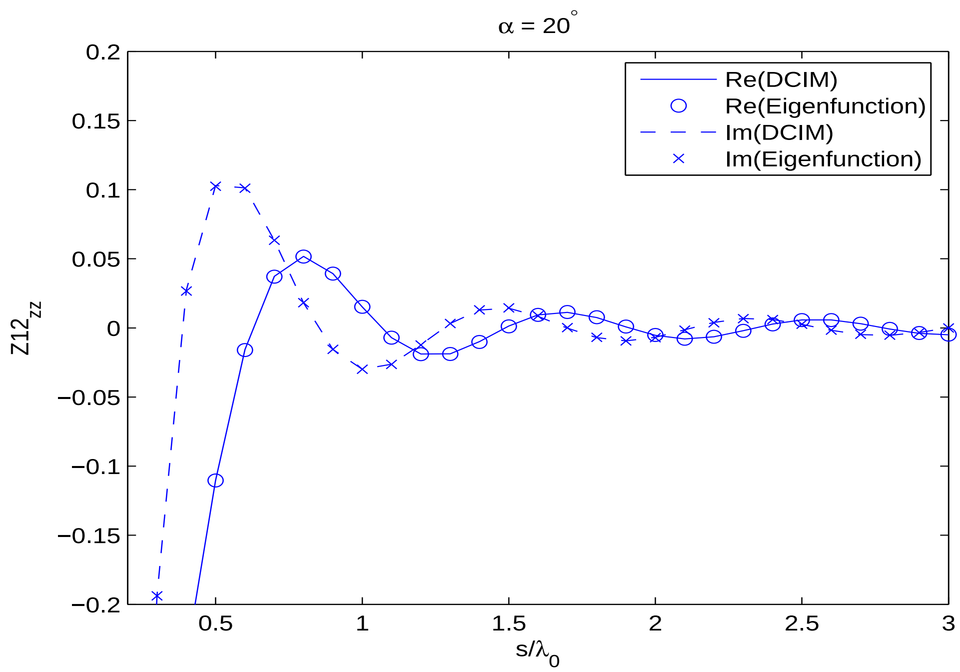

Figure 6.14: Real and Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus separation when $\alpha = 20°$.

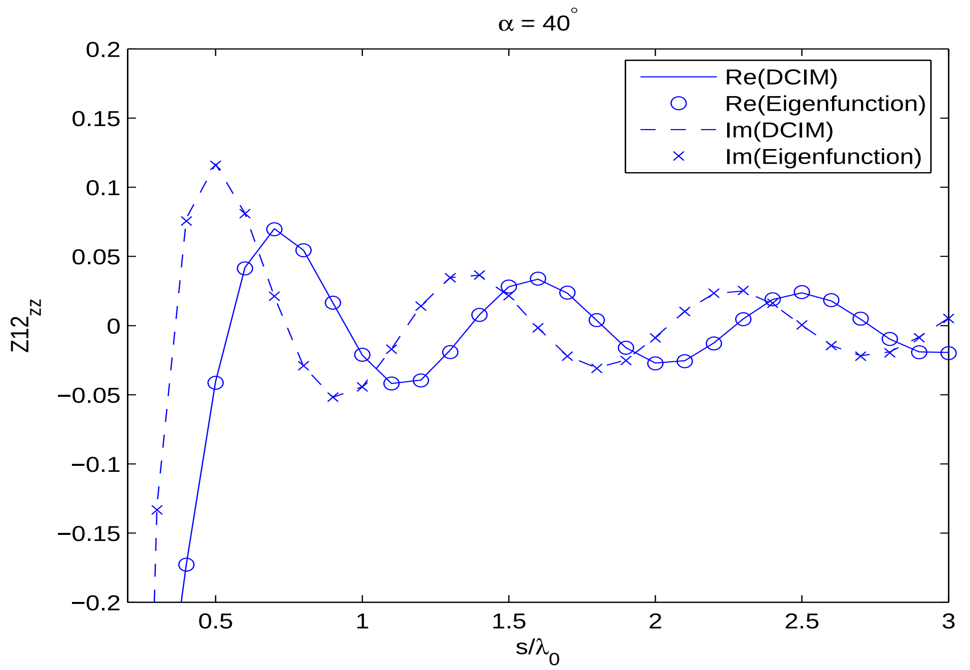

Figure 6.15: Real and Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus separation when $\alpha = 40°$.

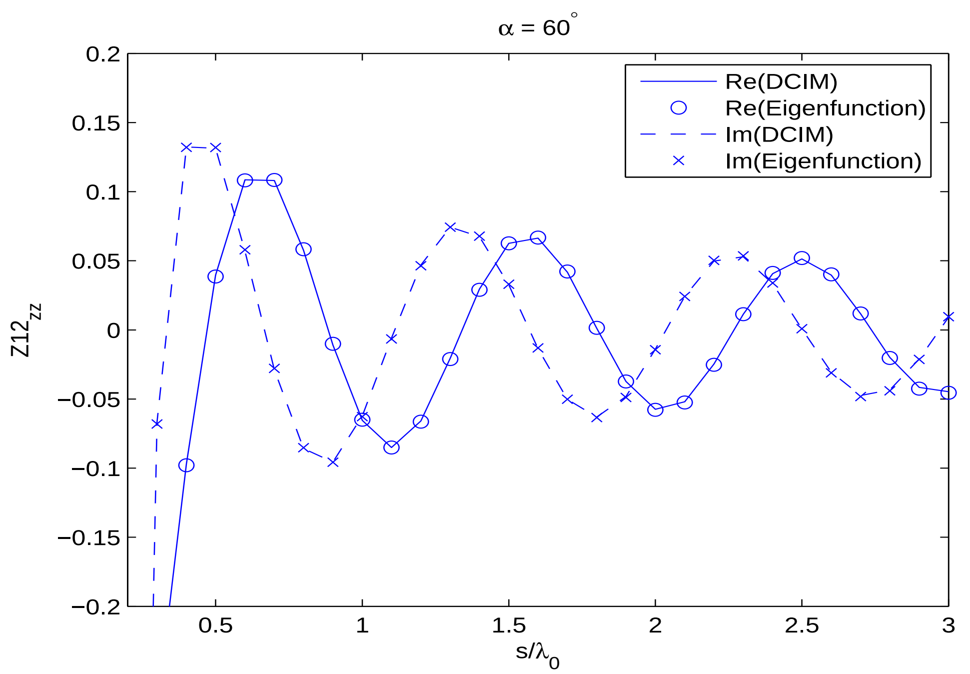

Figure 6.16: Real and Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus separation when $\alpha = 60°$.

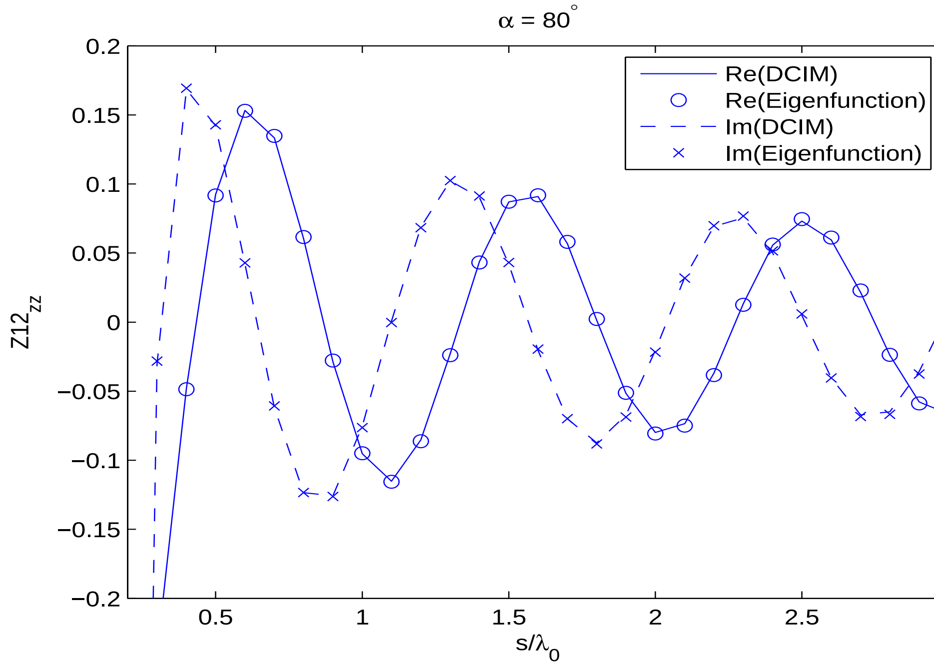

Figure 6.17: Real and Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus separation when $\alpha = 80°$.

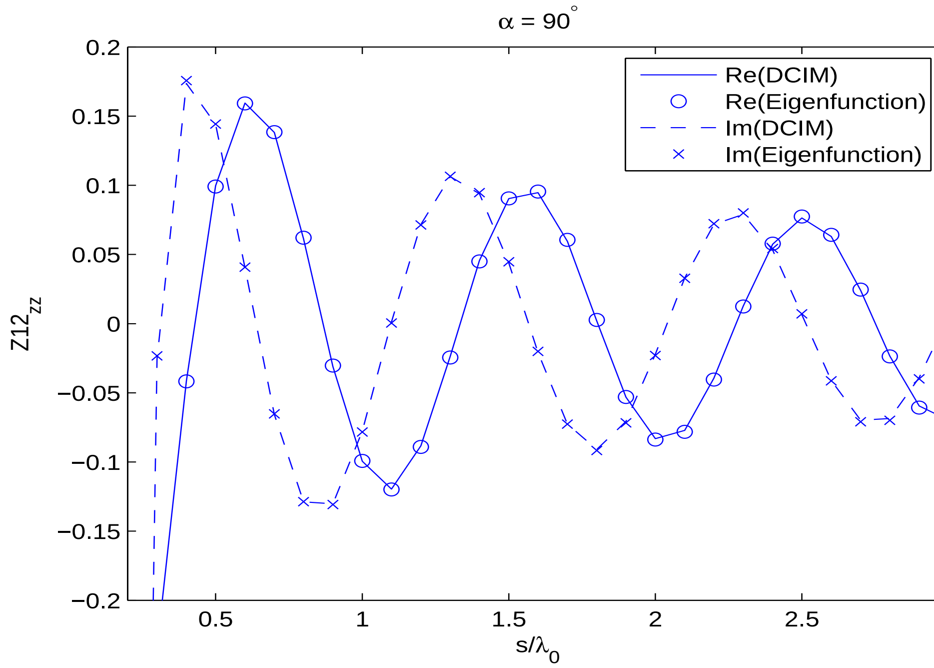

Figure 6.18: Real and Imaginary parts of the mutual impedance ($Z12_{zz}$) between two identical $\hat{z}$-directed current sources versus separation when $\alpha = 90°$.

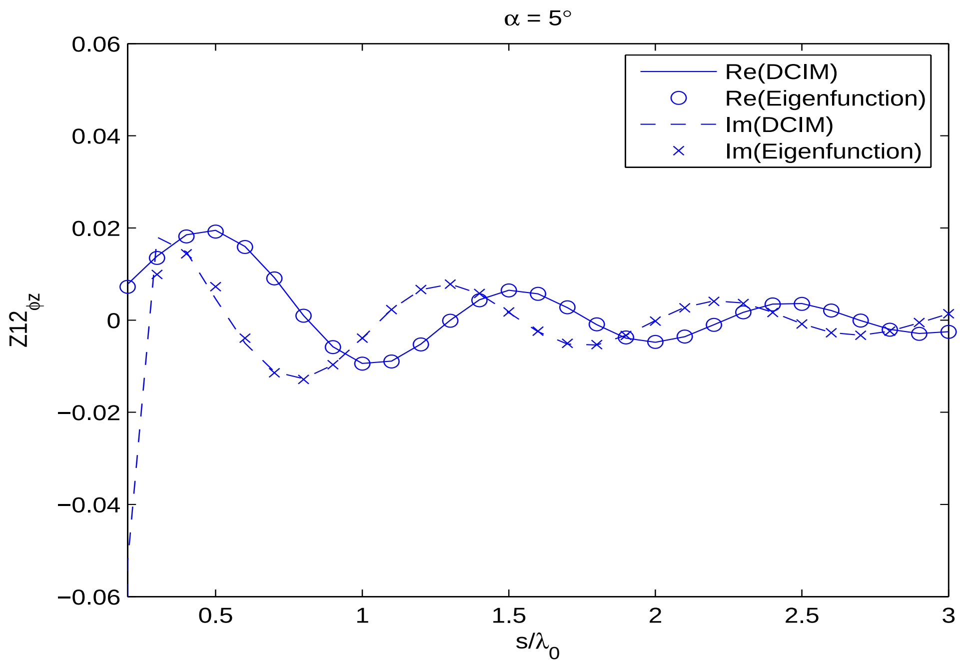

Figure 6.19: Real and Imaginary parts of the mutual impedance ($Z12_{\phi z}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus separation when $\alpha = 5°$ and $G_{\phi z}$ is used.

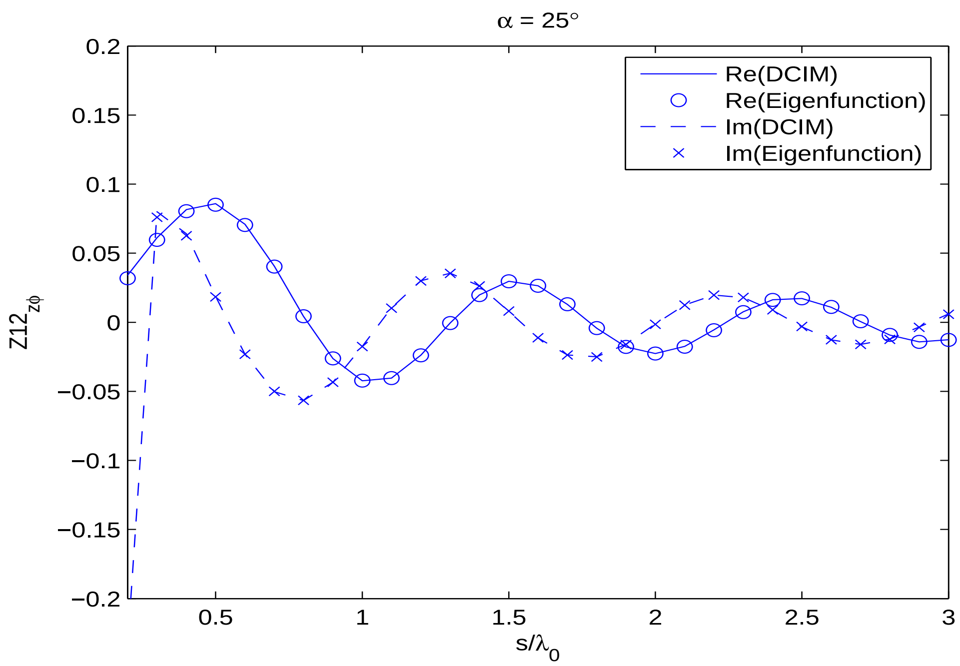

Figure 6.20: Real and Imaginary parts of the mutual impedance ($Z12_{z\phi}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus separation when $\alpha = 25°$ and $G_{z\phi}$ is used.

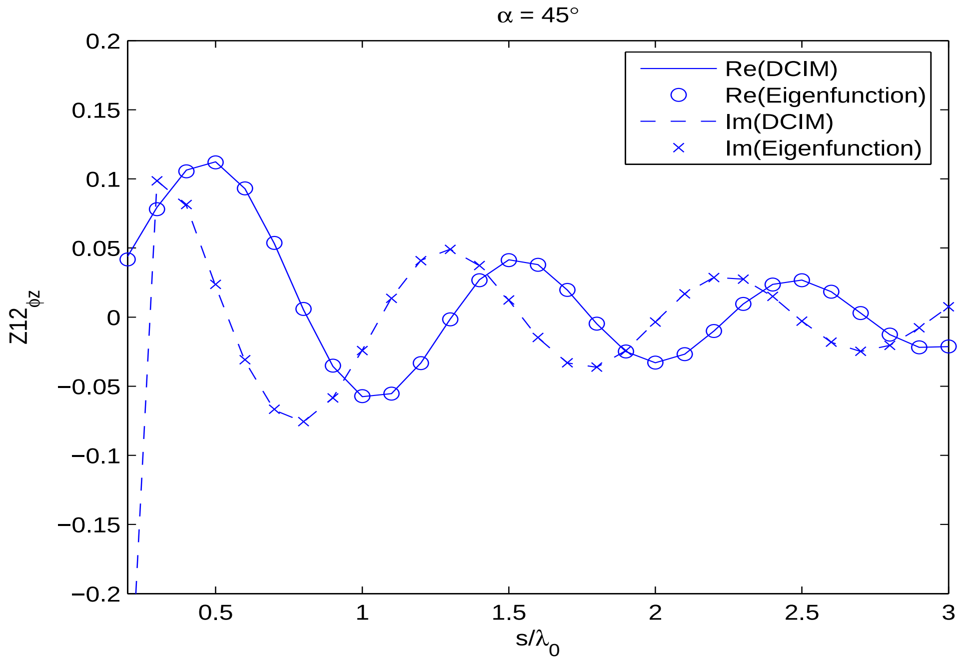

Figure 6.21: Real and Imaginary parts of the mutual impedance ($Z12_{\phi z}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus separation when $\alpha = 45°$ and $G_{\phi z}$ is used.

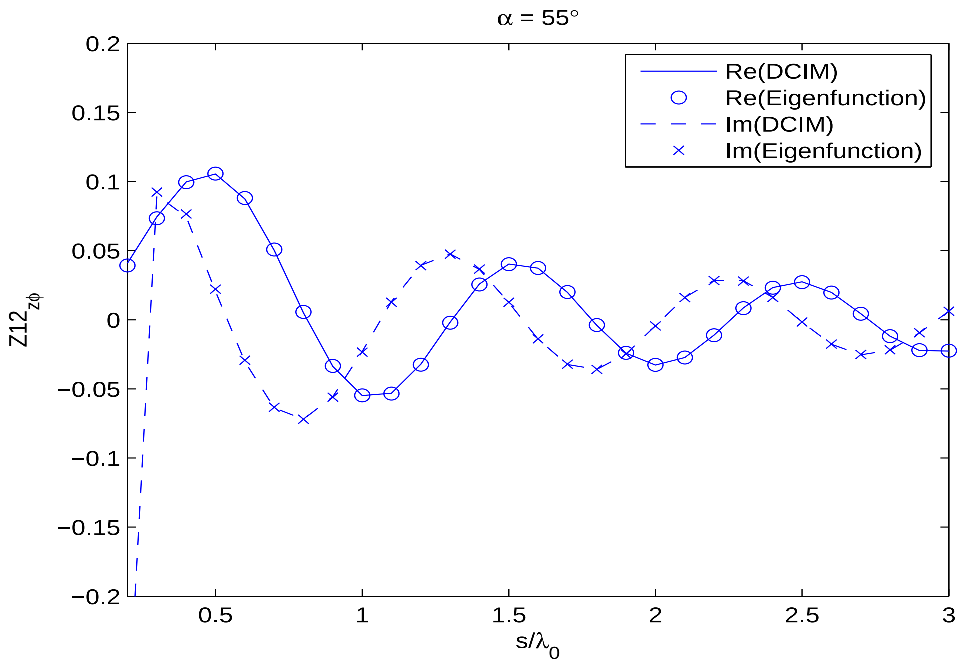

Figure 6.22: Real and Imaginary parts of the mutual impedance ($Z12_{z\phi}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus separation when $\alpha = 55°$ and $G_{z\phi}$ is used.

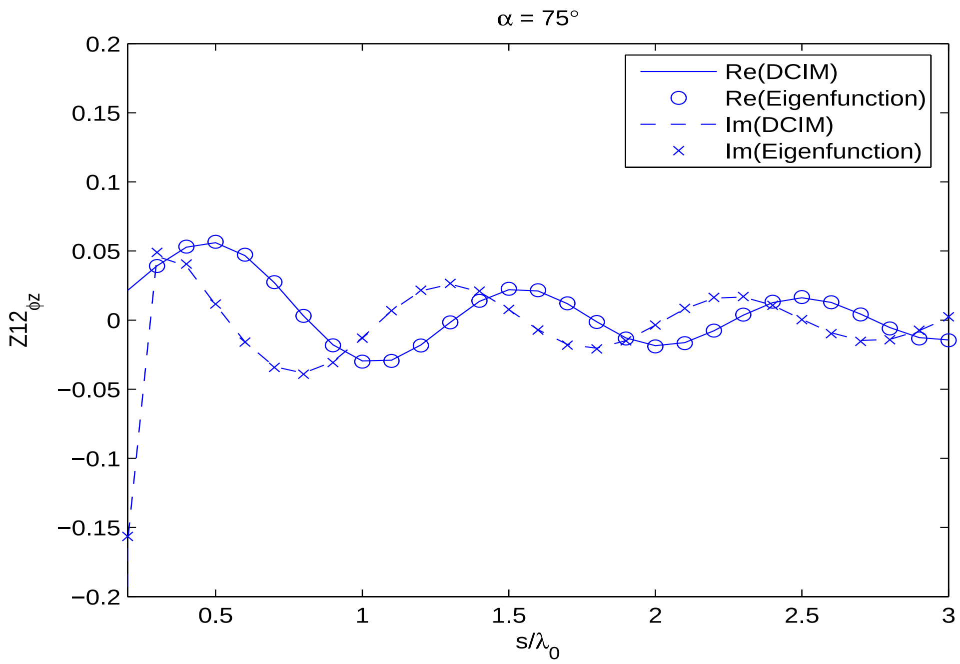

Figure 6.23: Real and Imaginary parts of the mutual impedance ($Z12_{\phi z}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus separation when $\alpha = 75^\circ$ and $G_{\phi z}$ is used.

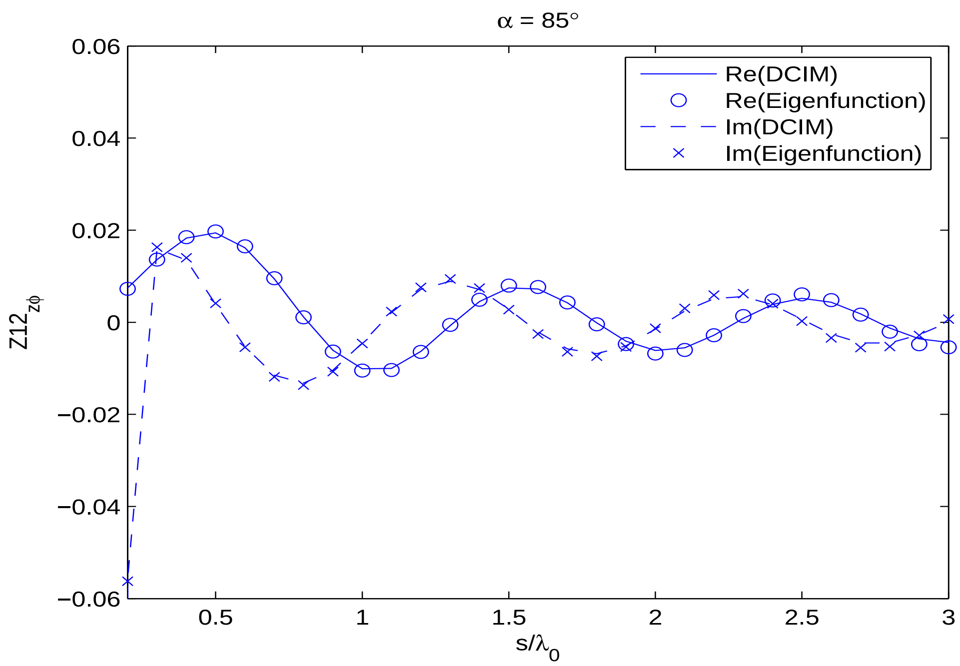

Figure 6.24: Real and Imaginary parts of the mutual impedance ($Z12_{z\phi}$) between a $\hat{\phi}$ and a $\hat{z}$-directed current source versus separation when $\alpha = 85^\circ$ and $G_{z\phi}$ is used.

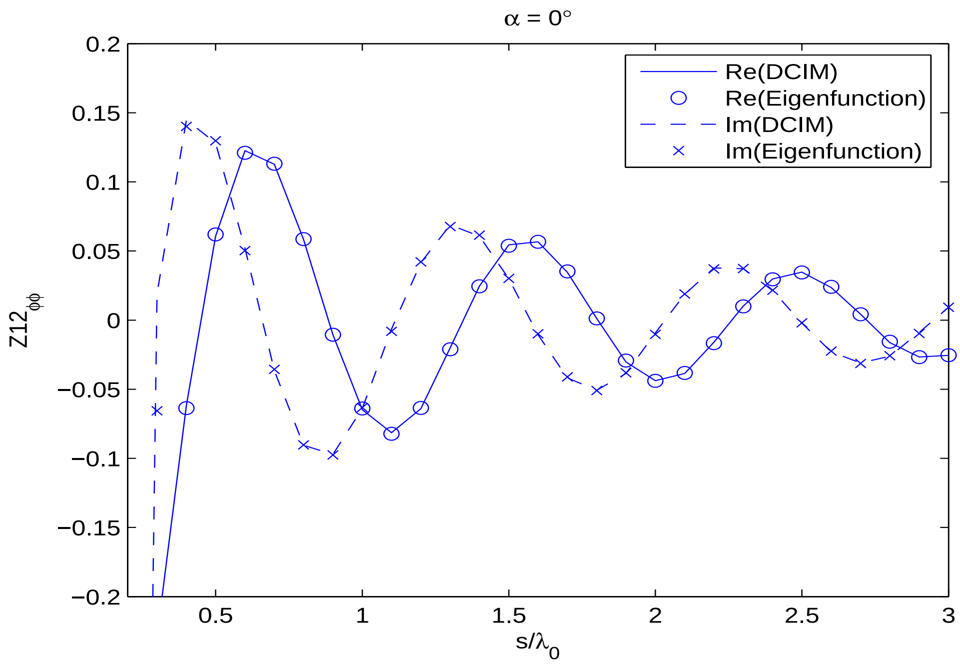

Figure 6.25: Real and Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus separation when $\alpha = 0°$.

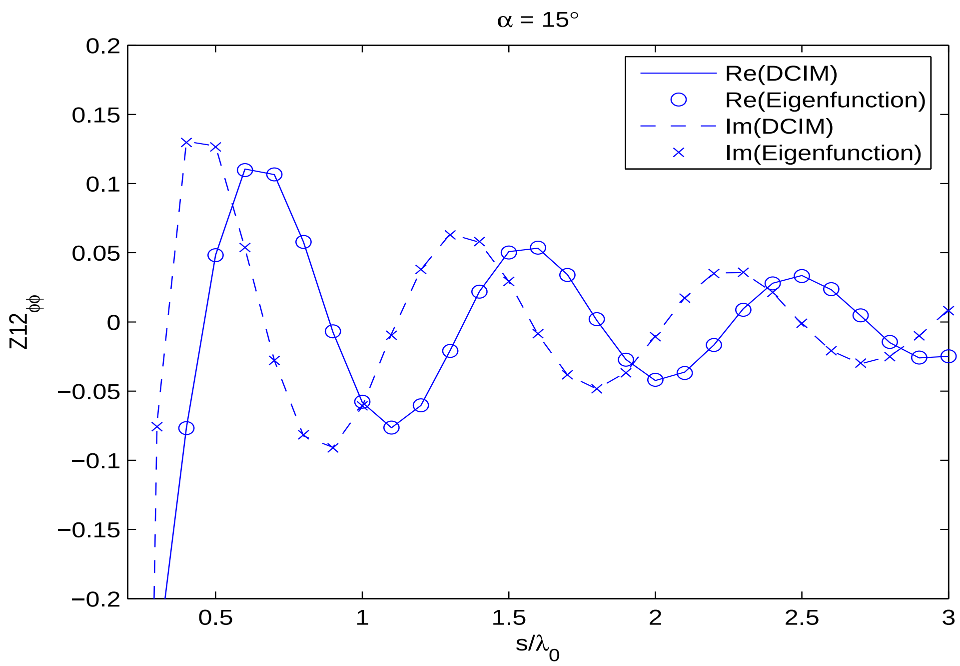

Figure 6.26: Real and Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus separation when $\alpha = 15°$.

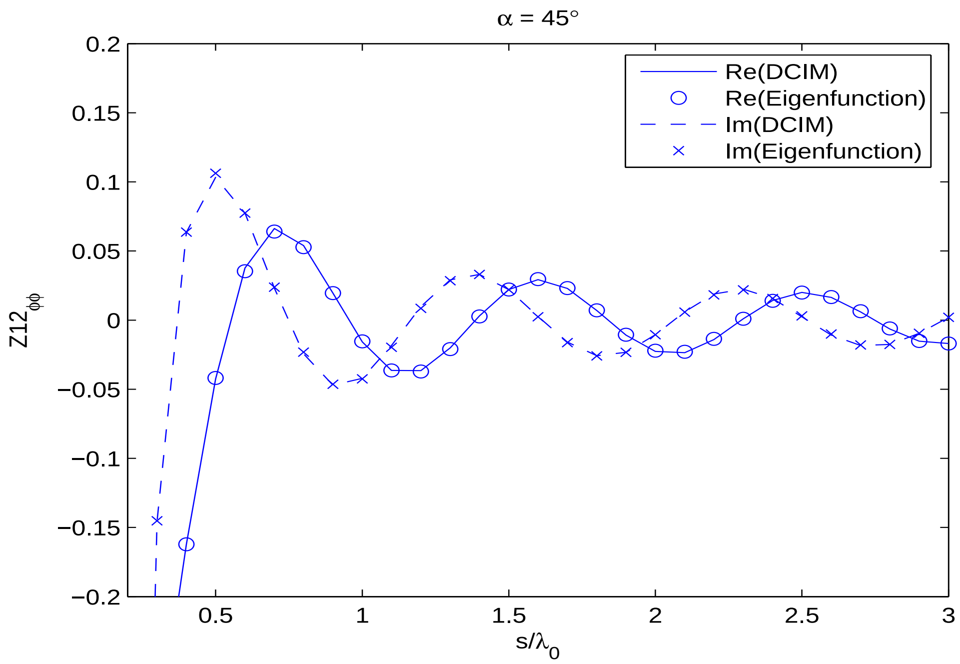

Figure 6.27: Real and Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus separation when $\alpha = 45°$.

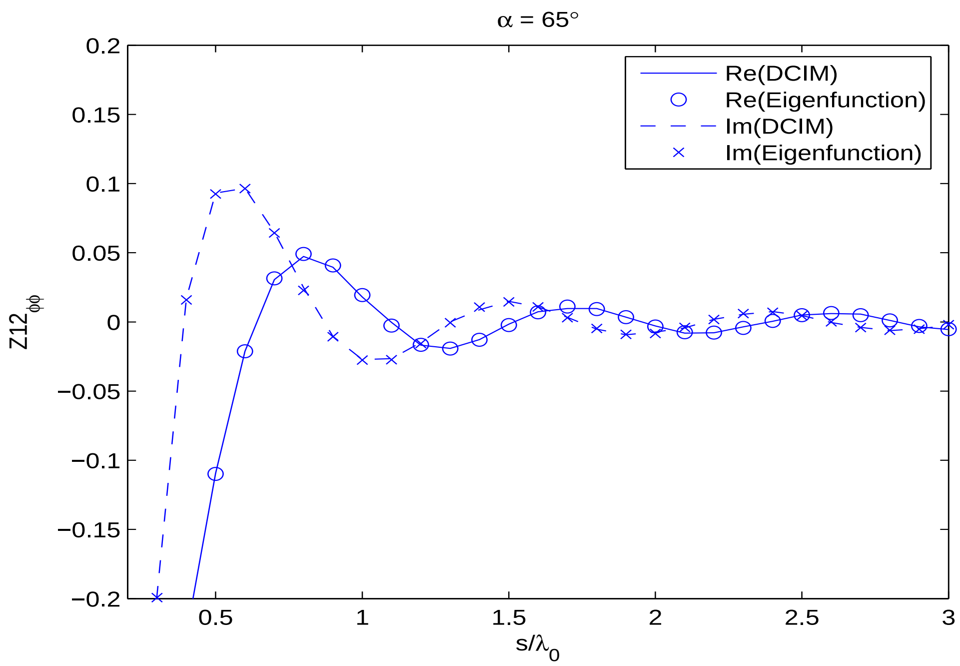

Figure 6.28: Real and Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus separation when $\alpha = 65°$.

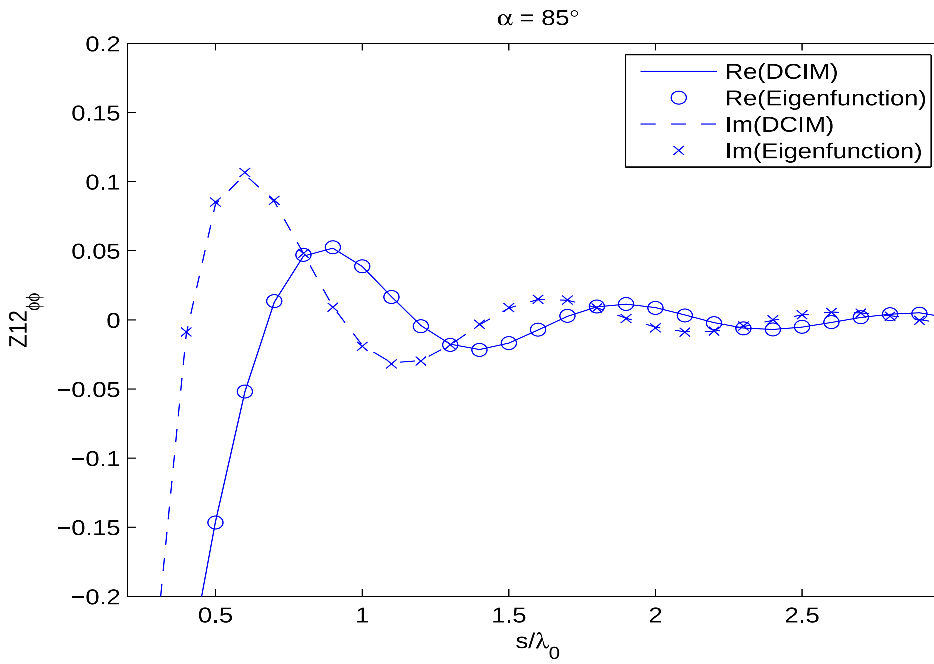

Figure 6.29: Real and Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus separation when $\alpha = 85°$.

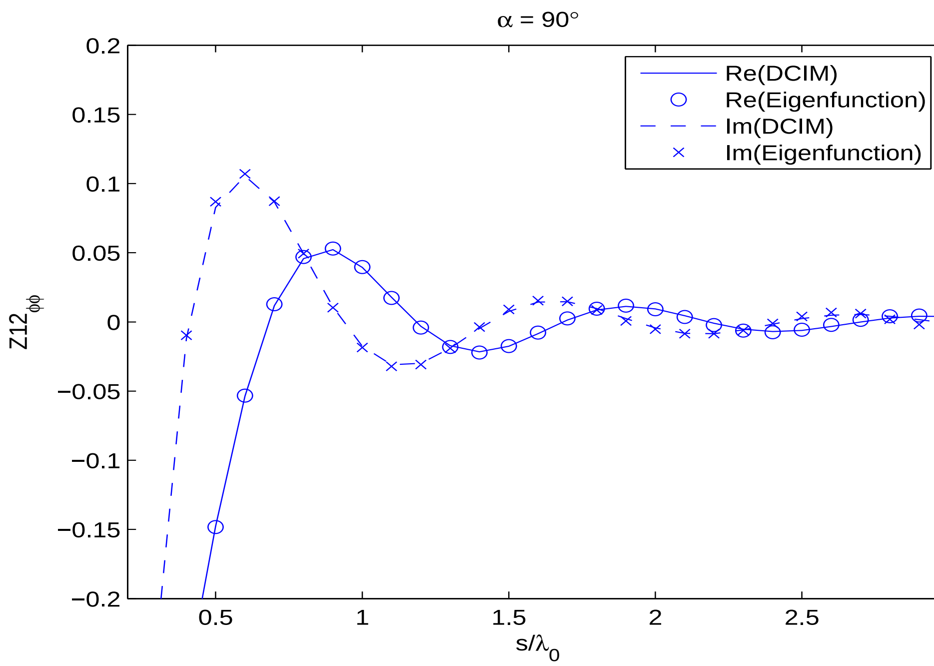

Figure 6.30: Real and Imaginary parts of the mutual impedance ($Z12_{\phi\phi}$) between two identical $\hat{\phi}$-directed current sources versus separation when $\alpha = 90°$.

Chapter 7

Conclusions

Closed-form Green's function representations in cylindrically stratified media which are valid for all possible source and field points are derived to be used in conjunction with MoM to investigate printed antennas and arrays embedded in cylindrically layered media. The original representations presented in the literature are modified to handle the axial line problem which is due to the argument of the Hankel function, which becomes zero along the axial line. Besides, several numerical and analytical techniques are used to improve the efficiency of the spectral summation over the cylindrical eigenmodes (n) and to avoid possible overflow/underflow problems.

In the course of obtaining closed-form Green's function representations, the large argument approximation of Hankel functions is not used on the last integration path. This way the singularity problem along the axial line is handled.

After the modified Green's function expressions are developed for all possible values of $(\phi - \phi')$ when $\rho = \rho'$, the mutual couplings between $\hat{\phi}$ and $\hat{z}$ directed current modes are calculated and compared with the eigenfunction results. The results obtained from closed-form expressions are in good agreement with the eigenfunction results [14]-[15].

As a result, the closed-form approximation procedure (DCIM) is successfully applied to $\rho = \rho'$ case in order to use it in a Method of Moment (MoM) application and also a method is offered for the ($\rho = \rho'$ and $\phi = \phi'$) problem which has not been reported in literature.

Appendix A

Generalized Pencil of Function Method

The generalized pencil of function method is used to approximate the spectral domain Green's functions with complex exponentials. Since this method is an important step in approximating the Green's functions, it is given in this appendix.

The Prony method and its variants can be used to extract the poles [16]-[17] of an EM system. The pencil of function (POF) method [18] is an alternative method to the Prony method to find the system poles. In POF, the poles are found from the solution of a generalized eigenvalue problem, whereas the Prony method contains two-step process where the first step involves the solution of a matrix equation and the second step involves finding the roots of a polynomial. The generalized pencil of function method is a generalization to the POF method and it is used to estimate the poles of EM system from its transient response [19]. The GPOF method is more robust and less noise sensitive, compared to the Prony method.

An EM transient signal with the samples y_k , can be approximated as,

$$y_k = \sum_{i=1}^{M} b_i e^{s_i \delta t k} = \sum_{i=1}^{M} b_i z_i^k \qquad k = 0, 1, ..., N-1 \tag{A.1}$$

where b_i are the complex residues, s_i are the complex poles, and δt is the sampling interval. The method can be given as follows:

1. The following matrices are constructed,

$$\bar{Y}_1 = \left[\begin{array}{c} \bar{y}_0, \bar{y}_1, ..., \bar{y}_{L-1} \end{array} \right] \tag{A.2}$$

$$\bar{Y}_2 = \left[\begin{array}{c} \bar{y}_1, \bar{y}_2, ..., \bar{y}_L \end{array} \right] \tag{A.3}$$

where

$$\bar{y}_i = \left[\begin{array}{c} y_i, y_{i+1}, ..., y_{i+N-L-1} \end{array} \right]^T \tag{A.4}$$

and L is the pencil parameter, and its optimal choice is around $L = N/2$ [19].

2. Find a $\bar{Z}$ matrix as,

$$\bar{V}\bar{D}^{-1}\bar{U}^H = SVD(\bar{Y}_1) \tag{A.5}$$

where $\bar{V}$,$\bar{D}^{-1}$ and $\bar{U}^H$ are $(N-L)$x$(N-L)$, $(N-L)$xL and LxL matrices. $SVD(.)$ is the singular value decomposition process and the superscript H is the complex conjugate transpose of a matrix. The $\bar{Z}$ matrix is

$$\bar{Z} = \bar{D}^{-1}\bar{U}^H\bar{Y}_2\bar{V} \tag{A.6}$$

3. The poles of the system are obtained as

$$s_i = \frac{\log z_i}{\delta t} \qquad i = 1, 2, \cdots, M \tag{A.7}$$

where z_i's are the eigenvalues of the $\bar{Z}$ matrix.

4. The residues are found from the least-squares solution of the following system.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_M^{N-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_M \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{N-1} \end{bmatrix} \tag{A.8}$$

or

$$\bar{A}\bar{B} = \bar{Y} \tag{A.9}$$

so the b_i's are found by using the pseudo-inverse of the $\bar{A}$ matrix as $\bar{B} = \bar{A}^{+}\bar{Y}$

Appendix B

Debye Approximations

The Debye approximations used for $J_n(z)$ and $Y_n(z)$ for large n values are given by

$$J_n(z) \approx \frac{e^{(\sqrt{n^2-z^2}-n\cosh^{-1}(\frac{n}{z}))}}{\sqrt{2\pi}(n^2-z^2)^{\frac{1}{4}}} \tag{B.1}$$

$$Y_n(z) \approx -\frac{e^{(n\cosh^{-1}(\frac{n}{z})-\sqrt{n^2-z^2})}}{\sqrt{\frac{\pi}{2}}(n^2-z^2)^{\frac{1}{4}}}. \tag{B.2}$$

In this thesis $\frac{J'_{n_1}(z_1)}{J_{n_2}(z_2)}$, $\frac{H^{(2)}_{n_1}(z_1)}{H^{(2)}_{n_2}(z_2)}$, $\frac{H^{'(2)}_{n_1}(z_1)}{H^{(2)}_{n_2}(z_2)}$, $H^{(2)}_{n_1}(z_1)J_{n_2}(z_2)$ and $H^{'(2)}_{n_1}(z_1)J'_{n_2}(z_2)$ terms are frequently used. For large n values these terms can be calculated by writing them in terms of $J_n(z)$ and $Y_n(z)$ as

$$\frac{J'_{n_1}(z_1)}{J_{n_2}(z_2)} = \frac{1}{2}(\frac{J_{n_1-1}(z_1)}{J_{n_2}(z_2)} - \frac{J_{n_1+1}(z_1)}{J_{n_2}(z_2)}) \tag{B.3}$$

$$\frac{H^{(2)}_{n_1}(z_1)}{H^{(2)}_{n_2}(z_2)} = \frac{Y_{n_1}(z_1)}{Y_{n_2}(z_2)} \frac{\frac{J_{n_1}(z_1)}{Y_{n_1}(z_1)} - j}{\frac{J_{n_2}(z_2)}{Y_{n_2}(z_2)} - j} \tag{B.4}$$

$$\frac{H^{'(2)}_{n_1}(z_1)}{H^{(2)}_{n_2}(z_2)} = \frac{1}{2}(\frac{H^{(2)}_{n_1-1}(z_1)}{H^{(2)}_{n_2}(z_2)} - \frac{H^{(2)}_{n_1+1}(z_1)}{H^{(2)}_{n_2}(z_2)}) \tag{B.5}$$

$$H^{(2)}_{n_1}(z_1)J_{n_2}(z_2) = J_{n_1}(z_1)J_{n_2}(z_2) - jY_{n_1}(z_1)J_{n_2}(z_2) \tag{B.6}$$

$$H_{n_1}^{'(2)}(z_1)J_{n_2}^{'}(z_2) = \frac{1}{4}(H_{n_1-1}^{(2)}(z_1)J_{n_2-1}(z_2) - H_{n_1-1}^{(2)}(z_1)J_{n_2+1}(z_2)$$
$$-H_{n_1+1}^{(2)}(z_1)J_{n_2-1}(z_2) + H_{n_1+1}^{(2)}(z_1)J_{n_2+1}(z_2)) \quad \text{(B.7)}$$

where $B_n^{'}(x)$ is used for the derivative $\frac{\partial B_n(x)}{\partial x}$, and denotes a Hankel or a Bessel function. Using the $J_n(z)$ and $Y_n(z)$ expressions given by (B.1) and (B.2), respectively and making the multiplications and divisions analytically the terms given by (B.3)-(B.7) are calculated for large n values.

Besides, the Debye approximations for $J_n^{'}(z)$ and $Y_n^{'}(z)$ are given by

$$J_n^{'}(z) \approx \frac{(n^2 - z^2)^{\frac{1}{4}}}{\sqrt{2\pi}z} e^{(\sqrt{n^2-z^2} - n\cosh^{-1}(\frac{n}{z}))} \quad \text{(B.8)}$$

$$Y_n^{'}(z) \approx \frac{(n^2 - z^2)^{\frac{1}{4}}}{\sqrt{\frac{\pi}{2}}z} e^{(n\cosh^{-1}(\frac{n}{z}) - \sqrt{n^2-z^2})} \quad \text{(B.9)}$$

and these $J_n^{'}(z)$ and $Y_n^{'}(z)$ expressions will be used to show that

$$\lim_{n\to\infty} \frac{B_n^{'}(z)}{nB_n(z)} = C(k_z) \quad \text{(B.10)}$$

where $B_n(z)$ denotes a Hankel or a Bessel function. Using (B.8) and (B.1) we obtain

$$\frac{J_n^{'}(z)}{J_n(z)} \approx \frac{(n^2 - z^2)^{\frac{1}{2}}}{z} \quad \text{(B.11)}$$

where it is clear that

$$\lim_{n\to\infty} \frac{J_n^{'}(z)}{nJ_n(z)} \approx \frac{(n^2 - z^2)^{\frac{1}{2}}}{nz} \approx \frac{1}{z}. \quad \text{(B.12)}$$

Similarly, we can write

$$\frac{H_n^{'(2)}(z)}{H_n^{(2)}(z)} = \frac{Y_n^{'}(z)}{Y_n(z)} \frac{\frac{J_n^{'}(z)}{Y_n^{'}(z)} - j}{\frac{J_n(z)}{Y_n(z)} - j} \quad \text{(B.13)}$$

and since the $n\cosh^{-1}(\frac{n}{z})$ term grows faster than the $\sqrt{n^2 - z^2}$ term, the $\frac{J_n^{'}(z)}{Y_n^{'}(z)}$ and the $\frac{J_n(z)}{Y_n(z)}$ ratios decays to zero for large n values. Hence we can write

$$\frac{H_n^{'(2)}(z)}{H_n^{(2)}(z)} \approx \frac{Y_n^{'}(z)}{Y_n(z)}. \quad \text{(B.14)}$$

Using (B.2) and (B.9) we obtain

$$\frac{H_n^{'(2)}(z)}{H_n^{(2)}(z)} \approx \frac{Y_n'(z)}{Y_n(z)} \approx -\frac{(n^2 - z^2)^{\frac{1}{2}}}{z}. \tag{B.15}$$

Finally, it is obvious that

$$\lim_{n\to\infty} \frac{H_n^{'(2)}(z)}{nH_n^{(2)}(z)} \approx -\frac{(n^2 - z^2)^{\frac{1}{2}}}{nz} \approx -\frac{1}{z}. \tag{B.16}$$

Appendix C

Piecewise Sinusoid Current Modes

A piecewise sinusoid $\hat{z}$-directed current mode J_z is given in the spatial domain as

$$J_z(\phi, z) = rect(\frac{b\phi}{l_a})\frac{1}{2l_a}PWS(k_a, z_a, z) \tag{C.1}$$

where $2l_a$ is the dimension along the $\hat{\phi}$-direction and $2z_a$ is the dimension along the $\hat{z}$-direction and

$$rect(\frac{x}{a}) = \begin{cases} 1, & |x| < a \\ 0, & otherwise \end{cases} \tag{C.2}$$

$$PWS(k_a, z_a, z) = \begin{cases} \frac{sin[k_a(z_a - |z|)]}{sin(k_a z_a)}, & |z| < z_a \\ 0, & otherwise \end{cases} \tag{C.3}$$

with $k_a = k_0\sqrt{\frac{1+\epsilon_r}{2}}$.

Similarly, a piecewise sinusoid $\hat{\phi}$-directed current mode J_ϕ is expressed in the spatial domain as

$$J_\phi(\phi, z) = rect(\frac{z}{z_a})\frac{1}{2z_a}PWS(k_a, l_a, b\phi). \tag{C.4}$$

When we denote the $\tilde{J}_z(n, k_z)$ and $\tilde{J}_\phi(n, k_z)$ as the spectral domain representations of $J_z(\phi, z)$ and $J_\phi(\phi, z)$, respectively we obtain

$$\tilde{J}_z(n, k_z) = \frac{1}{2\pi} \frac{sin(nl_a/b)}{nl_a} \frac{2k_a}{sin(k_a z_a)} \frac{cos(k_z z_a) - cos(k_a z_a)}{k_a^2 - k_z^2} \tag{C.5}$$

$$\tilde{J}_\phi(n, k_z) = \frac{1}{2\pi} \frac{sin(k_z z_a)}{k_z z_a} \frac{2k_a}{bsin(k_a l_a)} \frac{cos(nl_a/b) - cos(k_a l_a)}{k_a^2 - (n/b)^2}. \tag{C.6}$$

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