DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SYMMETRIC CHAIN DECOMPOSITIONS

by
Eren CANAN

August, 2021
İZMİR

SYMMETRIC CHAIN DECOMPOSITIONS

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**SYMMETRIC CHAIN DECOMPOSITIONS**" completed by **EREN CANAN** under supervision of **ASST. PROF. DR. ASLI GÜÇLÜKAN İLHAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Eren CANAN

SYMMETRIC CHAIN DECOMPOSITIONS

ABSTRACT

Let P_n be the quotient poset of the action of the dihedral group D_n on the Boolean lattice 2^n. In Duffus & Thayer (2015), Duffus and Thayer ask whether the poset P_n has a symmetric saturated chain decomposition. In this thesis, we give an affirmative answer to this problem for small dimensions. For this, we first introduce the notion of a symmetric saturated chain decomposition of a partially ordered set. Then we discuss some properties of partially ordered sets with symmetric saturated chain decompositions. More precisely, we show that if the posets P and Q have a symmetric saturated chain decomposition, so does the direct product $P \times Q$. Then we recall the Sperner property and show that if a poset has a symmetric saturated chain decomposition, then it has the Sperner property. We show that each division poset has a symmetric saturated chain decomposition. We also give symmetric saturated chain decompositions for P_n when $n = 5, 6, 7, 8$. We conclude the thesis by giving an isomorphism between the poset P_n and the quotient poset $\tilde{Q}_n/D_n$ where $\tilde{Q}_n$ is the poset obtained by adding a minimal element to a poset of ordered partitions of a positive integer n. This isomorphism might be useful to show that P_n has a symmetric saturated chain decomposition in general.

Keywords: Symmetric saturated chain decompositions, quotient posets, the Sperner property

SİMETRİK ZİNCİR AYRIŞMALARI

ÖZ

P_n, Boolean kısmi sıralı kümesinin dihedral grup D_n altındaki etkisiyle oluşan kısmi sıralı küme olsun. Duffus & Thayer (2015)'de Duffus ve Thayer, P_n kısmi sıralı kümesinin bir simetrik doymuş zincir ayrışmasına sahip olup olmadığı sorusunu ortaya attı. Bu tezde, küçük boyutlar için bu soruya olumlu bir cevap vereceğiz. Bunun için, ilk olarak kısmi sıralı bir kümenin simetrik doymuş zincir ayrışmasını tanıtacağız. Sonrasında P ve Q kısmi sıralı kümelerinin birer simetrik doymuş zincir ayrışmasi varsa, direkt çarpım $P \times Q$'nun da bir simetrik doymuş zincir ayrışması olduğunu göstereceğiz. Ardından Sperner özelliğini tanıtacağız ve bir kısmi sıralı kümenin simetrik doymuş zincir ayrışması varsa, o zaman Sperner özelliğine sahip olduğunu göstereceğiz. Her bölüm kısmi sıralı kümesinin simetrik doymuş zincir ayrışmasına sahip olduğunu göstereceğiz. Elemanları, toplamları pozitif bir n tamsayısına eşit olan pozitif tam sayılardan oluşan kısmi sıralı kümesine bir minimal eleman ekleyerek oluşturulan kısmi sıralı küme $\tilde{Q}_n$ olsun. P_n kısmi sıralı kümesi ile $\tilde{Q}_n/D_n$ kısmi sıralı kümesi arasında bir izomorfizm vererek tezi sonlandıracağız. Ayrıca, $n = 5, 6, 7, 8$ için P_n'in simetrik doymuş zincir ayrışmalarına örnekler vereceğiz.

Anahtar kelimeler: Simetrik doymuş zincir ayrışmaları, bölüm kısmi sıralı kümeleri, Sperner özelliği

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CHAPTER ONE
INTRODUCTION

The Boolean poset 2^n is the poset obtained by ordering the powerset of $[n] := \{1, 2, \ldots, n\}$ by inclusion. In Duffus & Thayer (2015), Duffus and Thayer give a method which lists a family of subgroups G of S_n for which the quotient $2^n/G$ has a symmetric saturated chain decomposition. It turns out that these families does not cover dihedral groups. In this thesis, we discuss symmetric saturated chain decompositions and we give a symmetric saturated chain decomposition for $2^n/D_n$ when $n \leq 8$ where D_n denotes the dihedral group of order $2n$.

Symmetric saturated chain decompositions were first introduced by Bruin, Tengbergen, and Kruyswijk (De Bruijn et al. (1951)) and it was used to prove Sperner's Theorem. This shows that symmetric saturated chain decompositions are useful for different problems. So the existence of symmetric saturated chain decompositions in a partially ordered set is a significant question for combinatorics. Here the Sperner's Theorem gives an upper bound, which is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, for the size of a family of subsets of $[n]$ in which none of the elements is a subset of the another one. In terms of partially ordered sets, this is equivalent to he statement that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is an upper bound for the maximal antichain of 2^n. Therefore, one can generalize this theorem to other kinds of posets. In a ranked poset, the number of elements which has the same rank, is called the Whitney number of the corresponding layer. We say that a ranked poset has the Sperner property if the length of any antichain is less than or equal to the largest Whitney Number.

Let P be a poset and let $G \leq \operatorname{Aut} P$ be a group acting on P. The quotient P/G is the set of orbits with the relation "$\leq$" defined by $[x] \leq [y]$ in P/G if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \leq y'$ in P. In Dhand (2011), Dhand proved that if a poset P has a symmetric saturated chain decomposition and $\mathbb{Z}_n$ acts on P^n by permuting coordinates then $P^n/\mathbb{Z}_n$ has a symmetric saturated chain decomposition. In Griggs et al. (2004), Griggs, Killian and Savage built a symmetric saturated chain decomposition of the quotient poset $2^n/G$ provided that n is prime number and G is generated by a single n-cycle. In Duffus et al. (2012), Duffus, Thayer and McKibben-Sanders found more general version of the Dhand's result, $2^n/G$ has a symmetric saturated chain decomposition provided that G is generated by powers of disjoint cycles. In Canfield & Mason (2006), Canfield and Mason conjectured that $2^n/G$ has a symmetric saturated chain decomposition for every subgroup G of the symmetric group S_n. In Duffus & Thayer (2015), Duffus and Thayer prove that the conjecture holds when G is the wreath product of the group $K \leq S_k$ by the group $T \leq S_t$ where both K and T are generated by powers of disjoint cycles with $n = kt$. One can not use their result to show that $2^n/D_n$ has a symmetric saturated chain decomposition for $n \geq 5$, since D_n can not be written as a wreath product. As a result, Duffus and Thayer asked to show that $2^n/D_n$ has a symmetric saturated chain decomposition for all $n \geq 1$. In this thesis, we give an affirmative answer to this problem for $n = 5, 6, 7, 8$ by explicitly giving the corresponding symmetric saturated chain decomposition. We observe that there is an isomorphism between the poset $2^n/D_n$ and the poset $\tilde{Q}_n/D_n$ where $\tilde{Q}$ is obtained by extending the poset of ordered partitions of n by adding a minimal element. It might be easier to show that the poset $\tilde{Q}_n/D_n$ has a symmetric saturated chain decomposition.

This thesis is organized as follows. We give a necessary background on posets in Chapter 2. In Chapter 3, we discuss symmetric saturated chain decompositions and their properties. We also give the definition of the quotient poset. In Chapter 4, we give some results on the quotient poset $2^n/D_n$.

CHAPTER TWO

PRELIMINARIES

In this section, we give some preliminary definitions and results on partially ordered sets. We refer reader to Anderson (2002), Harzheim (2006) and Zjevik (2014) for more details.

2.1 Posets

A partial order is a binary relation $\leq$ on a set P satisfying the properties given below.

- Every element related to itself $(a \leq a)$ **(reflexivity)**,
- If $a \leq b$ and $b \leq c$, then $a \leq c$ **(transitivity)**,
- If $a \leq b$ and $b \leq a$, we have $a = b$ **(antisymmetry)**.

If $a \leq b$, we say that a is related to b. Moreover, a set P equipped with a partial order relation is called a **partially ordered set (poset)**. In addition, if we can find a relation between any two elements in a poset P, we say that P is a **totally ordered set**. The set of natural numbers equipped with the relation of divisibility is an infinite poset. However it is not a totally ordered set since neither 2 divides 3 nor 3 divides 2. The set of natural numbers dividing the natural number n equipped with the relation of divisibility is called the **division poset** and it is denoted by $\mathcal{D}_n$. In Figure 2.1, we give the Hasse diagram of the division poset of 100.

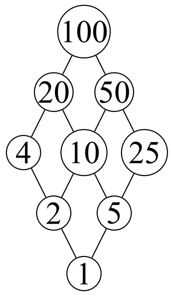

Figure 2.1 The Hasse diagram of the division poset $\mathcal{D}_{100}$.

Another standard example of a poset is the Boolean poset that is the set of subsets of the set $[n] := \{1, 2, \dots, n\}$ equipped with the relation of inclusion. In Figure 2.2, we give the Hasse diagram of the Boolean poset 2^3.

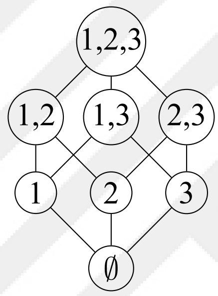

Figure 2.2 The Hasse diagram of the Boolean Poset 2^3

Definition 2.1.1. *An ordered n-tuple* $(a_1, a_2, \dots, a_{t+1})$ *is called a* ***chain*** *of length t in the poset* P *if* $a_i < a_{i+1}$ *for* $1 \le i \le t$.

Example 2.1.1. *In the division poset* $\mathcal{D}_{100}$*, the tuples* $(2, 4)$, $(5, 100)$, $(1, 5, 10, 20, 100)$ *are all chains.*

2.2 Some Properties of Posets

Definition 2.2.1. *The elements* a *and* b *in* P *are called* ***comparable*** *if* $a \le b$ *or* $b \le a$. *Otherwise,* a *and* b *are called* ***incomparable***.

Definition 2.2.2. *An element* a ***covers*** b*, denoted as* $b \lessdot a$*, if* $b < a$ *and* $b < x \le a$ *implies* $x = a$.

Example 2.2.1. *In Figure 2.3, the integers 2 and 3 are incomparable and similarly 7 and 8 are incomparable. Here, 2 covers 1. Similarly, 4 covers 1. 12 covers 9, 10 and 11, but 6 does not cover 1.*

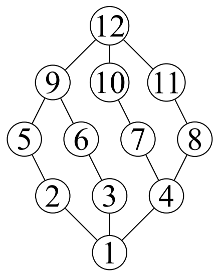

Figure 2.3 Example

Informally, a partially ordered set is called **ranked** if elements of its Hasse diagram can be decomposed into horizontal layer sets so that line segments must be only between the nearest set. There are no line segments between the elements which lie in the same layer or between the layers that are separated by one or more other layers. Hence all directed paths from one element to an another element in a ranked partially ordered set must be of the same length.

Example 2.2.2. *The diagram given in Figure 2.1 is ranked but the diagram given in Figure 2.4 is not ranked. Indeed, the paths 2-5-8 and 2-3-6-8 from 2 to 8 have different lengths.*

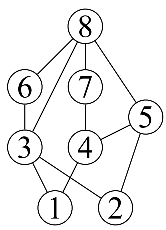

Figure 2.4 Example of non-ranked diagram.

Now we give the formal definition of a ranked poset.

Definition 2.2.3. *A **rank function** $r : P \to \mathbb{Z}$ is a function that satisfies the following properties.*

1. $r(a) = r(b) + 1$ *if* $b \lessdot a$,
2. *rank of a minimal element is* 0.

*A poset P equipped with a rank function is called **ranked poset**. The **rank of a ranked poset** is the largest one among the rank of all elements in that poset.*

By definition, a poset P is ranked if and only if all maximal chains in P have the same length. Moreover the rank of P is equal to the length of a maximal chain.

Example 2.2.3. *The rank of $\mathcal{D}_{100}$ is 4 since*

$$\begin{aligned} r(1) &= 0, & (2.1)\\ r(2) &= r(5) = 1, & (2.2)\\ r(4) &= r(10) = r(25) = 2, & (2.3)\\ r(20) &= r(50) = 3, & (2.4)\\ r(100) &= 4. & (2.5) \end{aligned}$$

Definition 2.2.4. *A subset $Q \subset P$ is an **antichain** if every pair of elements in Q is incomparable.*

Example 2.2.4. *In $\mathcal{D}_{100}$ (Figure 2.1), the subsets $\{2, 5\}$ and $\{4, 10, 25\}$ are antichains.*

Definition 2.2.5. *The **direct product** of two poset P and Q, is a poset whose underlying set is the cartesian product of P and Q, where the order relation of $P \times Q$ is given by $(p, q) \leq (p', q')$ if $p \leq_P p'$ and $q \leq_Q q'$. It is denoted by $P \times Q$.*

Clearly, the direct product of two ranked poset is also ranked poset by

$$r((p, q)) = r_P(p) + r_Q(q). \tag{2.6}$$

Definition 2.2.6. *The number of the elements in the i^{th} layer of a ranked poset P is called the i^{th} Whitney number, denoted by W_i. Here W_k=0 if $k < 0$ or $k > r(P)$.*

Definition 2.2.7. *A **generating function** of a ranked poset P is given by*

$$GF(P) = \sum_{w \in P} q^{r(w)} = \sum_{i=0}^{r(P)} W_i q^i. \tag{2.7}$$

Example 2.2.5. *The generating function of $\mathcal{D}_{100}$ (Figure 2.5) is*

$$GF(\mathcal{D}_{100}) = 1 + 2q + 3q^2 + 2q^3 + q^4. \tag{2.8}$$

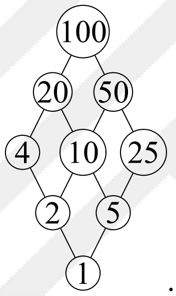

Figure 2.5 The Hasse diagram of the division poset $\mathcal{D}_{100}$.

Example 2.2.6. *The generating function of the Boolean Poset 2^4 (Figure 2.6) is*

$$GF(2^4) = 1 + 4q + 6q^2 + 4q^3 + q^4. \tag{2.9}$$

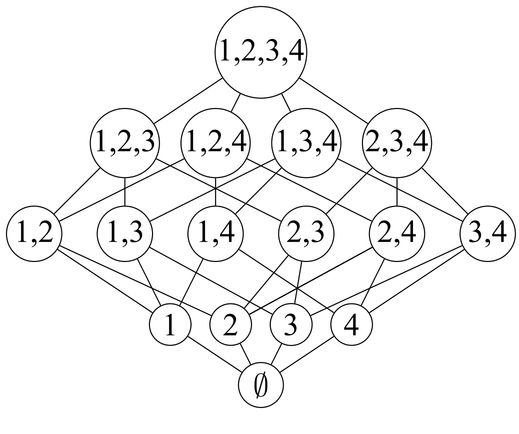

Figure 2.6 The Hasse diagram of the Boolean poset 2^4

Since the number of subsets of $[n]$ of size i is $\binom{n}{i}$, we have the following theorem.

Theorem 2.2.1. *The generating function of the Boolean poset 2^n is*

$$GF(2^n) = \sum_{i=0}^{n} \binom{n}{i} q^i. \tag{2.10}$$

Theorem 2.2.2 (Engel (1997))**.** *Let P and Q be posets. Then the generating function of the poset $P \times Q$ is the multiplication of the generating function of P and the generating function of Q.*

Proof. Let P and Q be two posets. Then

$$\begin{aligned}
GF(P \times Q) &= \sum_{(a,b) \in (P\text{x}Q)} q^{r((a,b))} && (2.11)\\
&= \sum_{(a,b) \in (P\text{x}Q)} q^{r(a)+r(b)} && (2.12)\\
&= \sum_{(a,b) \in (P\text{x}Q)} q^{r(a)} q^{r(b)} && (2.13)\\
&= \Big(\sum_{a \in P} q^{r(a)}\Big)\Big(\sum_{b \in Q} q^{r(b)}\Big) && (2.14)\\
&= GF(P).GF(Q). && (2.15)
\end{aligned}$$

□

2.3 Sperner's Property

Theorem 2.3.1 (Sperner's Theorem (Sperner (1928)))**.** *Let n be a positive integer and K be an antichain at 2^n. Then the length of K is less than or equal to the greatest Whitney number in 2^n.*

It is easy to see that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the greatest Whitney Number in 2^n. One can generalize Sperner's Theorem to other kinds of posets. For the ranked posets, we have the following generalization.

Definition 2.3.1. *We say that a ranked poset has the **Sperner Property** if the length of any antichain is less than or equal to the largest Whitney Number.*

Since every level of a ranked poset contains only incomparable elements, the largest Whitney number gives a lower bound for the length of a maximal antichain. Hence a poset P has the Sperner property if and only if the length of a maximal antichain is equal to the largest Whitney number.

Example 2.3.1. *The poset $\mathcal{D}_{100}$ (Figure 2.5) has the Sperner property since the length of the maximal antichain is* 3*. But the poset on the Figure 2.7 does not have the Sperner property since* $(3, 4, 5, 6, 7)$ *is an antichain of length* 5 *while the largest Whitney number is* 4.

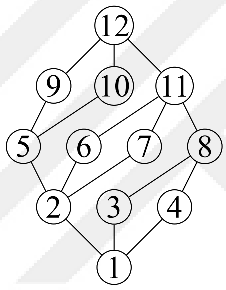

Figure 2.7 Example

In the next section, we show that the division poset $\mathcal{D}_n$ has the Sperner property for general n.

Note that, the product of two poset with Sperner's property does not necessarily have Sperner's property.

Example 2.3.2. *Clearly the posets P and Q in Figure 2.8 have the Sperner property.*

Since the largest Whitney number is 4 *in Figure 2.9 and the antichain* $(b1, c1, a3, b2, c2)$ *has the length* 5*, $P \times Q$ does not have the Sperner property.*

Definition 2.3.2. *Suppose that $\phi = (t_1, t_2, \cdots, t_k)$ is an element of a symmetric group S_k. The **inversion sequence** of ϕ is a sequence $(m_1, m_2, \cdots, m_k)$ where m_j equals the number of elements in ϕ that lies on the left of ϕ_j and greater than ϕ_j.*

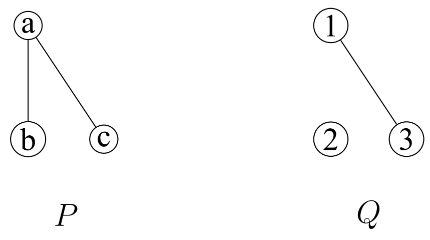

Figure 2.8 Example

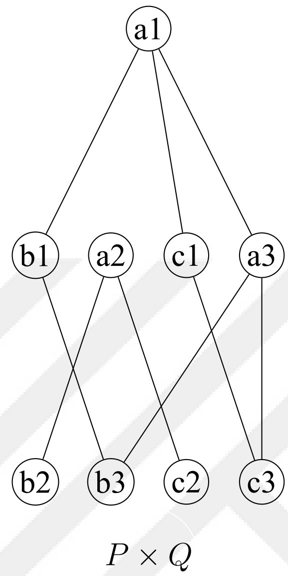

Figure 2.9 The Hasse Diagram of $P \times Q$

For instance, the inversion sequence of the sequence $(9, 1, 5, 6, 4, 2, 8, 7, 3)$ is $(0, 1, 1, 1, 3, 4, 1, 2, 6)$.

Definition 2.3.3. *The* ***inversion number*** *of a permutation ϕ is the sum of elements of its inversion sequence, i.e.,*

$$\text{inv}(\phi) = \sum_{j=1}^{k} m_j. \tag{2.16}$$

Definition 2.3.4. *Define a relation on S_k by $\phi \lessdot \phi^*$ if ϕ^* is created from ϕ by interchanging t_j with t_{j+1} when $t_j < t_{j+1}$. The poset S_k with this relation is called the* ***inversion poset*** *and denoted by I_k.*

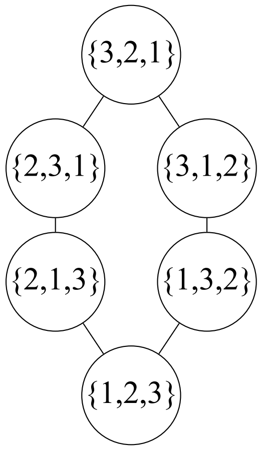

Figure 2.10 The inversion poset I_3

Example 2.3.3. *The inversion poset I_3 is given in Figure 2.10. Hence the generating function of I_3 is*

$$GF(I_3) = 1 + 2q + 2q^2 + q^3. \tag{2.17}$$

Theorem 2.3.2 (Wei (2010))**.** *The generating function of the inversion poset is given by*

$$GF(I_n) = \prod_{i=1}^{n-1} (1 + q + q^2 + ... + q^i). \tag{2.18}$$

It is still an open problem to find whether the inversion poset has the Sperner's property or not. Zjevik (2014), Zjevik introduces an algorithm for finding SSCD's for inversion posets.

2.4 Poset Isomorphisms

Definition 2.4.1. *Let P and Q be posets. A function $f : P \to Q$ is said to be a* ***poset homomorphism*** *if for any $a, b \in P$*

$$a \leq_P b \Rightarrow f(a) \leq_Q f(b). \tag{2.19}$$

Definition 2.4.2. *A function $f : P \to Q$ is called a* ***poset isomorphism*** *if f is a bijection and both f and f^{-1} are poset homomorphisms.*

Equivalently, we say that f is a poset isomorphism if f is bijective and $a \leq b$ if and only if $f(a) \leq f(b)$ for any $a, b \in P$. The inverse of a bijective poset homomorphism is not necessarily a poset homomorphism as shown in the following example.

Example 2.4.1. *Let $f : P_1 \to P_2$ be defined by $f(a) = x$, $f(b) = y$, $f(c) = z$ where the posets P_1 and P_2 are as given in Figure 2.11. Then f is a bijective poset homomorphism but f^{-1} is not a poset homomorphism since $y \leq z$ even $f^{-1}(y)$ and $f^{-1}(z)$ are incomparable. Therefore f is not a poset isomorphism.*

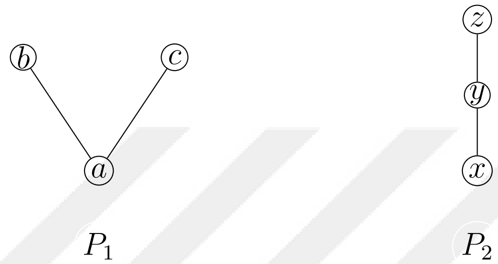

Figure 2.11 Example

Theorem 2.4.1 (Zjevik (2014)). *Let $n = {p_1}^{r_1}{p_2}^{r_2}\cdots{p_k}^{r_k}$ be the prime factorization of a positive integer n. Then a division poset $\mathcal{D}_n$ is isomorphic to a direct product of chains*

$$C = (1 \lessdot p_1 \lessdot p_1^2 \cdots \lessdot {p_1}^{r_1}) \times (1 \lessdot p_2 \lessdot p_2^2 \cdots \lessdot {p_2}^{r_2}) \times \cdots \times (1 \lessdot p_k \lessdot p_k^2 \cdots \lessdot {p_k}^{r_k}).$$

Proof. Let f be a function from $\mathcal{D}_n$ to C given by

$$f({p_1}^{i_1}{p_2}^{i_2}\cdots{p_k}^{i_k}) = ({p_1}^{i_1}, {p_2}^{i_2}, \cdots, {p_k}^{i_k}). \tag{2.20}$$

Clearly f is bijective. Our aim is to show that $a \leq b$ if and only if $f(a) \leq f(b)$ for any $a, b \in \mathcal{D}_n$. Let $a \leq b$ where $a = {p_1}^{a_1}{p_2}^{a_2}\cdots{p_k}^{a_k}$ and $b = {p_1}^{b_1}{p_2}^{b_2}\cdots{p_k}^{b_k}$. Therefore, $a_i \leq b_i$ for $i = 1, 2, \ldots, k$. Since

$$({p_1}^{a_1}, {p_2}^{a_2}, {p_3}^{a_3}, \cdots, {p_k}^{a_k}) \leq ({p_1}^{b_1}, {p_2}^{b_2}, {p_3}^{b_3}, \cdots, {p_k}^{b_k}), \tag{2.21}$$

we have $f(a) \leq f(b)$.

Conversely, let $f(a) \leq f(b)$ where

$$f(a) = ({p_1}^{a_1}, {p_2}^{a_2}, \cdots, {p_k}^{a_k}) \text{ and } f(b) = ({p_1}^{b_1}, {p_2}^{b_2}, \cdots, {p_k}^{b_k}). \tag{2.22}$$

Then $a_i \leq b_i$ for all $1 \leq i \leq k$. Since f is one to one, a divides b. So, $a \leq b$. Hence f is a poset isomorphism and $\mathcal{D}_n$ is isomorphic to a direct product of chains $C = (1 \lessdot p_1 \lessdot p_1^2 \cdots \lessdot {p_1}^{r_1}) \times (1 \lessdot p_2 \lessdot p_2^2 \cdots \lessdot {p_2}^{r_2}) \times \cdots \times (1 \lessdot p_k \lessdot p_k^2 \cdots \lessdot {p_k}^{r_k})$ where $n = {p_1}^{r_1}{p_2}^{r_2} \cdots {p_k}^{r_k}$. □

Theorem 2.4.2 (Zjevik (2014)). *A Boolean poset 2^k is isomorphic to a division poset $\mathcal{D}_n$ where n is equal to product of distinct k primes.*

Proof. Let $n = p_{x_1}p_{x_2}\ldots p_{x_t}$ and f be a function from 2^k to $D_{\mathcal{D}}$ given by

$$f(\{x_1, x_2, \ldots x_t\}) = p_{x_1}p_{x_2}\ldots p_{x_t}. \tag{2.23}$$

Clearly, f is a bijective function. Let A and B two elements of 2^k such that $A \subseteq B$. Then $A = \{x_1, x_2, \ldots, x_t\}$ and $B = \{x_1, x_2, \ldots, x_r\}$ with $t \leq r$. Therefore, $p_{x_1}p_{x_2}\ldots p_{x_t} \mid p_{x_1}p_{x_2}\ldots p_{x_r}$ and hence $f(A) \mid f(B)$. Since f is one to one, $f(A)|f(B)$. Clearly this implies $A \subseteq B$.

For the other direction, let $f(A)|f(B)$ for $A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$ and $B = \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}$. Then $t \leq r$ and $\{i_1, i_2, \ldots, i_t\} \subseteq \{j_1, j_2, \ldots, j_r\}$. Hence $A \subseteq B$. □

CHAPTER THREE
SYMMETRIC CHAIN DECOMPOSITIONS

In this chapter, we discuss symmetric chain decompositions. The main references for this chapter are Dixon & Mortimer (1996), Duffus & Thayer (2015) and Zjevik (2014). We define the quotient poset and discuss the quotients obtained by group actions. We also discuss a result given by Duffus & Thayer (2015) which can be used to show that $2^n/G$ has a SSCD for particular subgroups G of the symmetric group S_n.

3.1 Symmetric Chain Decomposition of a Poset

Definition 3.1.1. *For a poset P, a set of disjoint subsets $\{P_1, P_2, \ldots P_m\}$ such that $\bigcup_{i=1}^{m} P_i$ is called a **decomposition** of P.*

For instance, $\{\{1, 2, 5, 25\}, \{4, 20, 50\}, \{10, 100\}\}$ gives a decomposition of $\mathcal{D}_{100}$ (Figure 2.1).

Definition 3.1.2. *We say that a chain $(a_1, a_2, ..., a_k)$ is **saturated** if $a_1 \lessdot a_2 \lessdot ... \lessdot a_k$. Moreover, if we cannot get a valid chain when we add an element to the far left or right of this chain, we call it **maximal**, that is, there is no $a_0 \in P$ such that $(a_0, a_1, a_2, ..., a_k)$ or $(a_1, a_2, ..., a_k, a_0)$ is a valid chain.*

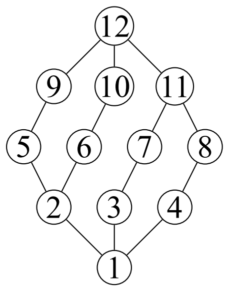

Figure 3.1 Example

Example 3.1.1. *In Figure 3.1, $(2, 6)$ and $(6, 10)$ are saturated chains, but $(2, 10)$ is not saturated. Moreover, the chains $(1, 2, 5, 9, 12)$, $(1, 2, 6, 10, 12)$, $(1, 3, 7, 11, 12)$ and $(1, 4, 8, 11, 12)$ are all maximal.*

Definition 3.1.3. *Let P be a ranked poset. We say that a saturated chain $(a_1, a_2, ..., a_k)$ in P is **symmetric** if*

$$r(a_1) + r(a_k) = r(P). \tag{3.1}$$

For instance, in Figure 2.1, the chain $(1, 2, 4, 20, 100)$ is symmetric since

$$r(1) + r(100) \quad = \quad 0 + 4 = 4 \tag{3.2}$$

Similarly the chains $(2, 10, 50)$ and (25) are symmetric too. However, the chain $(1, 5, 10, 20)$ is not symmetric since $r(1) + r(20) = 0 + 3 \neq 4$

Definition 3.1.4. *A poset P has **symmetric saturated chain decomposition** if P can be decomposed into symmetric saturated chains.*

For instance, the chains $(1, 2, 4, 20, 100)$, $(5, 25, 50)$ and (10) gives a symmetric saturated chain decomposition of $\mathcal{D}_{100}$.

Example 3.1.2. *Consider the posets given in Figure 3.2. When we want to decompose the poset on the left into symmetric saturated chains, we need to write both 9 and 10 in symmetric saturated chains. Since 9 and 10 are incomparable, they must be elements of different chains. We can clearly see that the chain containing 9 must also contain 2. Likewise, the chain containing 10 has to contain 2. Since these chains are not disjoint, we cannot decompose this poset into symmetric saturated chains. Similarly, writing 9 and 10 into symmetric saturated chains on the poset on the right breaks disjointness, so we cannot divide this poset into symmetric saturated chains.*

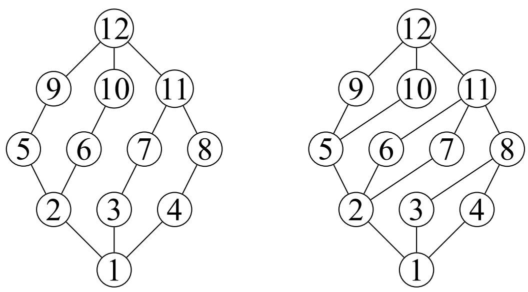

Figure 3.2 Example

3.2 Properties of SSCD's

Theorem 3.2.1 (Anderson (2002))**.** *If P and Q are posets with symmetric saturated chain decompositions, then $P \times Q$ also has a symmetric saturated chain decomposition.*

Proof. Let $P_0, P_1, \ldots, P_k$ and $Q_0, Q_1, \ldots, Q_r$ be symmetric saturated chain decompositions for the posets P and Q, respectively. Let (P_j, Q_i) be a pair of chains. Suppose that

$$P_j = a_0 \lessdot a_1 \lessdot \cdots \lessdot a_m \text{ and } Q_i = b_0 \lessdot b_1 \lessdot \cdots \lessdot b_n. \tag{3.3}$$

We can obtain a new chain N_t by

$$N_t = ((a_0, b_t), (a_1, b_t), \ldots, (a_{m-t}, b_t), (a_{m-t}, b_{t+1}), \ldots, (a_{k-t}, b_n)) \tag{3.4}$$

for $t = 0, 1, \ldots, \min\{m, n\}$.

One can easily see that the change in N_t is only in the first or the second term. Since P_i and Q_j are saturated chains in P_j and Q_i, respectively, N_t is saturated in $P \times Q$. Now, we show that N_t is symmetric in $P \times Q$. Since P_j is symmetric in P, $r(P) = r(a_0) + r(a_m)$ and similarly, $r(Q) = r(q_0) + r(q_n)$. Therefore

$$\begin{aligned}
r((a_0, q_t)) + r((a_{m-t}, q_n)) &= r(a_0) + r(b_t) + r(a_{m-t}) + r(b_n) && (3.5)\\
&= r(a_0) + r(b_0) + t + r(a_m) - t + r(b_n) && (3.6)\\
&= r(a_0) + r(a_m) + r(b_0) + r(b_n) && (3.7)\\
&= r(P) + r(Q) && (3.8)\\
&= r(P \times Q). && (3.9)
\end{aligned}$$

Thus, N_t is symmetric in $P \times Q$.

It remains to show that every element in $P \times Q$ is contained in exactly one of N_t's. For this, it is suffices to show that $\sum_{t=0}^{min\{m,n\}} |N_t| = |P_j||Q_i|$.

If $m \leq n$, we have

$$\begin{aligned} \sum_{t=0}^{m} |N_t| &= \sum_{t=0}^{m} (m+n+1-2t) && (3.10) \\ &= (m+1)(m+n+1) - 2\sum_{t=0}^{m} t && (3.11) \\ &= (m+1)(n+1) && (3.12) \\ &= |P_j||Q_i|. && (3.13) \end{aligned}$$

Similarly, one can verify the formula for the case $n < m$. Therefore the number of elements in all N_t's is

$$\begin{aligned} \sum_{i=0}^{k}\sum_{j=0}^{r}\sum_{t=0}^{min\{m,n\}} |N_t| &= \sum_{i=0}^{k}\sum_{j=0}^{r} |P_i||Q_j| && (3.14) \\ &= \sum_{i=0}^{k} |P_i| \cdot \sum_{j=0}^{r} |Q_j| && (3.15) \\ &= |P| \cdot |Q| && (3.16) \end{aligned}$$

□

By combining Theorem 2.4.1 and Theorem 3.2.1, one can obtain the following result.

Corollary 3.2.1. *The division poset $\mathcal{D}_n$ has a symmetric saturated chain decomposition.*

The following result follows from Corollary 3.2.1 and Theorem 2.4.2.

Corollary 3.2.2. *The Boolean poset 2^n has a symmetric saturated chain decomposition.*

The following is a well-known theorem. We give a proof of it for completeness.

Theorem 3.2.2. *If a poset P has a symmetric saturated chain decomposition, then it has the Sperner property.*

Proof. In any symmetric saturated chain decomposition of P, every chain must contain at least one element from the middle layer(s). If $r(P)$ is even, then every chain contains exactly one element from the middle layer since there is only one middle layer, otherwise there are two middle layers and every chain contains exactly one element from each middle layer. Therefore if a poset P has a symmetric saturated chain decomposition, then the largest whitney number in P is $W_{\lfloor \frac{r(P)}{2} \rfloor}(P)$. Now, we show that the length of any antichain in P is less than or equal to $W_{\lfloor \frac{r(P)}{2} \rfloor}(P)$, say k. Let $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ be a symmetric saturated chain decomposition for P and F be an antichain in P. Let $a_1 \in F$. Since α contains every element in P, a_1 must be contained in α_j for some $j \in \{1, 2, ..., k\}$. Without loss of generality, let $a_1 \in \alpha_1$. Let a_2 be another element of F. Since F is an antichain, a_1 and a_2 must be incomparable. In this case, we have $a_2 \notin \alpha_1$. Without loss of generality, let $a_2 \in \alpha_2$. Let a_3 be another element of F. Since a_3 must be incomparable with a_1 and a_2, it can not be in α_1 or α_2. Without loss of generality, let $a_3 \in \alpha_3$. By repeating the same procedure, we get $|F| \leq k$. $\square$

The converse of the above theorem does not hold. For instance;

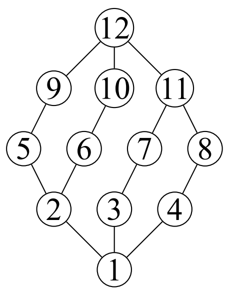

Figure 3.3 Example

This poset has the Sperner's property but has no SSCD.

Corollary 3.2.3. *The posets $\mathcal{D}_n$ and 2^n both have the Sperner's property.*

Remark 3.2.1. *Note that, the Sperner's theorem is equal to the statement that 2^n has the Sperner's property.*

3.3 Quotient Posets of 2^n by Subgroups of S_n

A poset isomorphism from P to P is called poset automorpshism. For a given poset P, let Aut P be the set of poset automorphisms. Now we'll give a proof of a well-known result on the set of automorphsism of the poset 2^n.

Proposition 3.3.1. $\operatorname{Aut} 2^n \cong S_n$

Proof. For any $\sigma \in S_n$ define

$$f_\sigma : 2^n \to 2^n \text{ by } \{a_1, a_2, \ldots, a_k\} \mapsto \{\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)\}.$$

We first show that $f_\sigma \in \operatorname{Aut}(2^n)$ for any $\sigma \in S_n$. To show that f_σ is one-to-one, suppose that $f_\sigma(A) = f_\sigma(B)$ where $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Then we have

$$\{\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)\} = \{\sigma(b_1), \sigma(b_2), \ldots, \sigma(b_m)\}. \tag{3.17}$$

Since σ is injective, $k = m$. Then for all $1 \leq i \leq k$, there exist $1 \leq j \leq k$ such that $\sigma(a_i) = \sigma(b_j)$. Since σ is a bijection, $a_i = b_j$. This give us $A = B$. Therefore f_σ is one-to-one. Now we show that f_σ is onto. For this let B be an element of 2^n. Then $f_\sigma(A) = B$ where

$$A := \{\sigma^{-1}(b) : b \in B\}. \tag{3.18}$$

Clearly $A \subseteq B$ if and only if $f_\sigma(A) \subseteq f_\sigma(B)$. Hence f_σ is a poset isomorphism. Consequently, $f_\sigma \in \operatorname{Aut}(2^n)$ for all $\sigma \in S_n$.

For the other direction, note that any $f \in \operatorname{Aut}(2^n)$ sends a subset of size k to a subset of size k. Hence f permutes the subsets of size 1. Therefore σ_f defined by $\sigma_f(a) = b$ where $f(\{a\}) = \{b\}$ is an element of S_n. Clearly $f_{\sigma_f} = f$ and $\sigma_{f_\sigma} = \sigma$. Moreover, for any $\sigma, \beta \in S_n$, we have

$$f_{\sigma\circ\beta}(\{i_1, i_2, \ldots, i_k\}) = \{(\sigma\circ\beta)(i_1), (\sigma\circ\beta)(i_2), \ldots, (\sigma\circ\beta)(i_k)\} \quad (3.19)$$
$$= f_\sigma(\{\beta(i_1), \beta(i_2), \ldots, \beta(i_k)\}) \quad (3.20)$$
$$= (f_\sigma \circ f_\beta)(\{i_1, i_2, \ldots, i_k\}) \quad (3.21)$$

Therefore the map $\phi : S_n \to \operatorname{Aut} 2^n$ defined by $\phi(\sigma) = f_\sigma$ is an group isomorphism. □

Definition 3.3.1. *Let P be a poset and let $G \leq \operatorname{Aut} P$. The quotient P/G is the set of orbits with the relation "$\leq$" defined by $[x] \leq [y]$ in P/G if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \leq y'$ in P.*

Here it is convenient to ask whether P/G actually has a poset structure. For this, let us show that $(P/G, \leq)$ is a partial order relation. Suppose that P is a poset and $a, b, c \in P$. Since P is reflexive we have $a \leq a$. Then $[a] \leq [a]$ and hence P/G is reflexive. Now suppose that $[a] \leq [b]$ and $[b] \leq [a]$. Since $[a] \leq [b]$, there exist $g, h \in G$ such that $ga \leq hb$. Then we have $a \leq g^{-1}hb$. Since $[b] \leq [a]$, there exist $g', h' \in G$ such that $h'b \leq g'a$. Thus we have

$$a \leq g^{-1}hb = g^{-1}h(h')^{-1}h'b \quad (3.22)$$
$$\leq g^{-1}h(h')^{-1}g'a \quad (3.23)$$

Set $\gamma = g^{-1}h(h')^{-1}g' \in \operatorname{Aut} P$. Since P is finite, $\operatorname{Aut} P$ is finite. Thus

$$a \leq \gamma a \leq \gamma^2 a \leq \cdots \leq \gamma^r a = a \quad (3.24)$$

for some positive integer r. Therefore $a \leq g^{-1}hb \leq a$. So $a = g^{-1}hb$ and hence $a \in [b]$. Similarly, one can show that $b \in [a]$. Consequently $[a] = [b]$ and P/G is antisymmetric. Now suppose $[a] \leq [b]$ and $[b] \leq [c]$. We have $ga \leq g'b$ and $hb \leq h'c$ for some $g, g', h, h' \in G$ from the first and the second relation, respectively. Then

$$ga \leq g'b = g'h^{-1}hb \leq g'h^{-1}h'c. \quad (3.25)$$

Thus, $[a] \leq [c]$ and so P/G is transitive. Consequently, $(P/G, \leq)$ is a partial order.

When $G = S_n$ and $P = 2^n$, then P/G has a SSCD for all $n \geq 1$ since each layer has exactly one element. (see Figure 3.4)

$$\begin{tikzcd}
\boxed{[\{1,2,\ldots,n\}]} \arrow[d, no head] \\
\boxed{\cdots} \arrow[d, no head] \\
\boxed{[\{1,2\}]} \arrow[d, no head] \\
\boxed{[\{1\}]} \arrow[d, no head] \\
\boxed{[\{\emptyset\}]}
\end{tikzcd}$$

Figure 3.4 The Hasse diagram of the poset $2^n/S_n$

It is natural to ask, for which $G \leq S_n$, $2^n/G$ have a SSCD. In Duffus & Thayer (2015), Duffus and Thayer partially answer this question by providing a theorem which shows that for certain wreath products G, $2^n/G$ have a SSCD. Before stating this result, we give the definition of the wreath product.

Definition 3.3.2. *Given subgroups K of S_k and T of S_t, the wreath product $G = K \wr T$ of K by T is the set of all $\psi \in S_n$ defined as follows: given $\rho = (\rho_1, \rho_2, \ldots, \rho_t) \in K^t$ and $\tau \in T$, for all $(i, r) \in [k] \times [t]$,*

$$\psi(i, r) = (\rho_r(i), \tau(r)).$$

Theorem 3.3.1 (Theorem 1, Duffus & Thayer (2015))**.** *Let n, k, t be positive integers with $n = kt$. The poset $2^n/G$ has a SSCD for every subgroup G of* $\mathrm{Aut}(2^n)$ *defined as follows: the wreath product of K by T, where K is a subgroup of S_k, T is a subgroup of S_t, and both K and T are generated powers of disjoint cycles.*

In Duffus & Thayer (2015), the table of subgroups (up to conjugation) of S_6 that gives quotients of 2^6 with SSCD is given using the following results.

- if $[n] = X_1 \cup X_2 \cup \cdots \cup X_m$ is a partition and $G = G_1 \times G_2 \times \cdots \times G_m$ where G_i is the restriction of G into X_i, then $2^n/G$ has an SSCD provided that each $2^{X_i}/G_i$ has an SSCD.
- if $G = \langle \sigma^r \rangle$ for some cycle in $\sigma \in S_n$ then $2^n/G$ has an SSCD.
- Theorem 3.3.1.

Now we give the list for n = 5 and 7 in Table 3.1, Table 3.2 and Table 3.3.

Table 3.1 List of subgroups of S_5 that generate quotients of 2^5 with SSCDs.

Partition of 5	**Generators**	**Description**	**Order**
$1+1+1+1+1$	(1)	Id.	1
$1+1+1+2$	(12)	$\mathbb{Z}_2$	2
$1+1+3$	(123)	$\mathbb{Z}_3$	3
	$(123)(12)$	S_3	6
$1+4$	(1234)	$\mathbb{Z}_4$	4
	$(1234), (12)$	S_4	24
	$(123), (234)$	A_4	12
	$(1234), (24)$	D_4	8
5	(12345)	$\mathbb{Z}_5$	5
	$(12345), (12)$	S_5	120
	$(12345), (123)$	A_5	60
$1+2+2$	$(12), (34)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
	$(12)(34)$	$\mathbb{Z}_2$	2
$2+3$	$(12), (345), (34)$	$\mathbb{Z}_2 \times S_3$	12
	$(12), (345)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	6

Table 3.2 List of subgroups of S_7 that generate quotients of 2^7 with SSCDs.

Partition of 7	**Generators**	**Description**	**Order**
$1+1+1+1+1+1+1$	(1)	Id.	1
$1+1+1+1+1+2$	(12)	$\mathbb{Z}_2$	2
$1+1+1+1+3$	(123)	$\mathbb{Z}_3$	3
	$(123)(12)$	S_3	6

Table 3.3 List of subgroups of S_7 that generate quotients of 2^7 with SSCDs.

$1+1+1+4$	(1234)	$\mathbb{Z}_4$	4
	$(1234),(12)$	S_4	24
	$(123),(234)$	A_4	12
	$(1234),(24)$	D_4	8
$1+1+5$	(12345)	$\mathbb{Z}_5$	5
	$(12345),(12)$	S_5	120
	$(12345),(123)$	A_5	60
$1+6$	(123456)	$\mathbb{Z}_6$	6
	$(12345),(123),(12)(56)$	A_6	360
	$(123456),(12)$	S_6	720
	$(123)(456),(14)$	$\mathbb{Z}_2 \wr \mathbb{Z}_3$	24
	$(123),(14)(25)(36)$	$\mathbb{Z}_3 \wr \mathbb{Z}_2$	18
	$(123),(12),(14)(25)(36)$	$S_3 \wr \mathbb{Z}_2$	72
7	(1234567)	$\mathbb{Z}_7$	7
	$(1234567),(12)$	S_7	5040
	$(1234567),(123)$	A_7	2520
$1+1+1+2+2$	$(12),(34)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
	$(12)(34)$	$\mathbb{Z}_2$	2
$1+1+2+3$	$(12),(345),(34)$	$\mathbb{Z}_2 \times S_3$	12
	$(12),(345)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	6
$1+2+2+2$	$(12),(34),(56)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8
	$(12)(34),(56)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
	$(12)(34)(56)$	$\mathbb{Z}_2$	2
$1+2+4$	$(12),(3456)$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8
	$(12),(3456)(34)$	$\mathbb{Z}_2 \times S_4$	48
	$(12),(345)(456)$	$\mathbb{Z}_2 \times A_4$	24
	$(12),(3456)(46)$	$\mathbb{Z}_2 \times D_4$	16
$1+3+3$	$(123),(456)$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	9
	$(123),(12),(456)$	$S_3 \times \mathbb{Z}_3$	18
	$(123),(12),(456),(45)$	$S_3 \times S_3$	36
$2+5$	$(12),(34567)$	$\mathbb{Z}_2 \times \mathbb{Z}_5$	10
	$(12),(34567),(34)$	$\mathbb{Z}_2 \times S_5$	240
	$(12),(34567),(345)$	$\mathbb{Z}_2 \times A_5$	120
$2+2+3$	$(12),(34),(456)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	12
	$(12),(34),(456),(45)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$	24
$3+4$	$(123),(4567)$	$\mathbb{Z}_3 \times \mathbb{Z}_4$	12
	$(123),(4567),(45)$	$\mathbb{Z}_3 \times S_4$	72
	$(123),(456),(567)$	$\mathbb{Z}_3 \times A_4$	36
	$(123),(4567),(57)$	$\mathbb{Z}_3 \times D_4$	24
	$(123),(12),(4567)$	$S_3 \times \mathbb{Z}_4$	24
	$(123),(12),(4567),(45)$	$S_3 \times S_4$	144
	$(123),(12),(456),(567)$	$S_3 \times A_4$	72
	$(123),(12),(4567),(57)$	$S_3 \times D_4$	48

Consider the **dihedral group** D_n defined by

$$D_n := \{r^k s^j : r^n = s^2 = 1 \text{ with } srs = r^{-1} \text{ and } i \in \{0, 1, \ldots, n-1\}, j \in \{0, 1\}\}.$$

Duffus and Thayer, show that $2^n/D_n$ has a SSCD for $n \leq 4$. However for $n \geq 5$, D_n can not be written as a wreath product. Hence we can not use theorem 3.3.1 to prove that $2^n/D_n$ has a SSCD for $n \geq 5$. As a result, Duffus and Thayer pose the following problem.

Problem: Duffus & Thayer (2015) Show that $2^n/D_n$ has a SSCD for all $n \geq 5$.
We discuss this problem in the next chapter.

CHAPTER FOUR

QUOTIENT POSET OF 2^N BY DIHEDRAL GROUPS

In this chapter, we consider D_n as a subgroup of S_n where

$$r = (12 \dots n) \text{ and } s = [(2\ n)(3\ (n-1)) \dots (i\ (n+2-i))]. \tag{4.1}$$

We show that $2^n/D_n$ has a SSCD when $n = 5, 6, 7, 8$. Here the action of D_n on 2^n is given by

$$r \cdot \{b_1, b_2, \dots, b_k\} = \begin{cases} \{1, (b_1+1), (b_2+1) \dots, (b_{k-1}+1)\} & b_k = n \\ \{(b_1+1), (b_2+1) \dots, (b_k+1)\} & b_k \neq n \end{cases} \tag{4.2}$$

$$s \cdot \{b_1, b_2, \dots, b_k\} = \begin{cases} \{1, n+2-b_k, \dots, n+2-b_2\} & b_1 = 1 \\ \{n+2-b_k, \dots, n+2-b_1\} & b_1 \neq 1, \end{cases} \tag{4.3}$$

4.1 Some results of $2^n/D_n$

Proposition 4.1.1. *The poset $2^n/D_n$ has a SSCD when $n = 5$.*

Proof. The Hasse diagram of $2^5/D_5$ is given in the Figure 4.1.

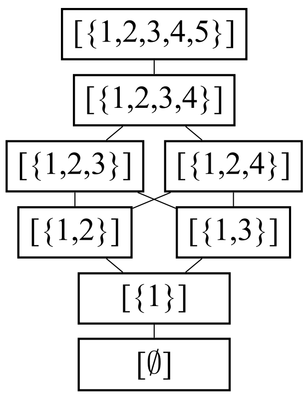

Figure 4.1 The Hasse diagram of $2^5/D_5$

Here,

- $[\emptyset] = \{\emptyset\}$
- $[\{1\}] = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$
- $[\{1,2\}] = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}$
- $[\{1,3\}] = \{\{1,3\}, \{2,4\}, \{3,5\}, \{4,1\}, \{5,2\}\}$
- $[\{1,2,3\}] = \{\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,1\}, \{5,1,2\}\}$
- $[\{1,2,4\}] = \{\{1,2,4\}, \{2,3,5\}, \{3,4,1\}, \{4,5,2\}, \{5,1,3\}\}$
- $[\{1,2,3,4\}] = \{\{1,2,3,4\}, \{2,3,4,5\}, \{3,4,5,1\}, \{4,5,1,2\}, \{5,1,2,3\}\}$
- $[\{1,2,3,4,5\}] = \{\{1,2,3,4,5\}\}$.

Then

$$\begin{aligned} C_1 = & ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}]), \\ C_2 = & ([\{1,3\}] \lessdot [\{1,2,4\}]), \end{aligned}$$

gives a symmetric saturated chain decomposition for $2^5/D_5$. $\square$

Remark 4.1.1. *There are indeed four different SSCD's for $2^5/D_5$. The other three are given as below.*

1. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}])$,

$C_2 = ([\{1,3\}] \lessdot [\{1,2,3\}])$.

2. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,3\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}])$,

$C_2 = ([\{1,2\}] \lessdot [\{1,2,4\}])$.

3. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,3\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}])$,

$C_2 = ([\{1,2\}] \lessdot [\{1,2,3\}])$.

Proposition 4.1.2. *The poset $D_n/2^n$ has a SSCD when $n = 6$.*

Proof. Let $n = 6$. The Hasse diagram of $2^6/D_6$ is given in the Figure 4.2. Then

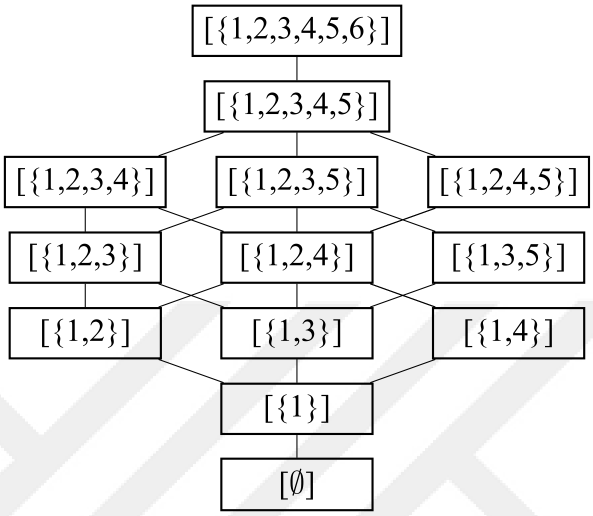

Figure 4.2 The Hasse diagram of $2^6/D_6$

$C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}])$,

$C_2 = ([\{1,3\}] \lessdot [\{1,3,5\}] \lessdot [\{1,2,3,5\}])$,

$C_3 = ([\{1,4\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,4,5\}])$,

gives a symmetric saturated chain decomposition for $2^6/D_6$. $\square$

Remark 4.1.2. *There are indeed three different SSCD's for $2^6/D_6$. The other two are given as below.*

1. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,3\}] \lessdot [\{1,3,5\}] \lessdot [\{1,2,3,5\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}])$,

$C_2 = ([\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}])$,

$C_3 = ([\{1,4\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,4,5\}])$.

2. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,4\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,4,5\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}])$,

$C_2 = ([\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}])$,

$C_3 = ([\{1,3\}] \lessdot [\{1,3,5\}] \lessdot [\{1,2,3,5\}])$.

Proposition 4.1.3. *The poset $D_n/2^n$ has a SSCD when $n = 7$.*

Proof. When $n = 7$, the Hasse diagram of $2^7/D_7$ is given in the figure Figure 4.3.

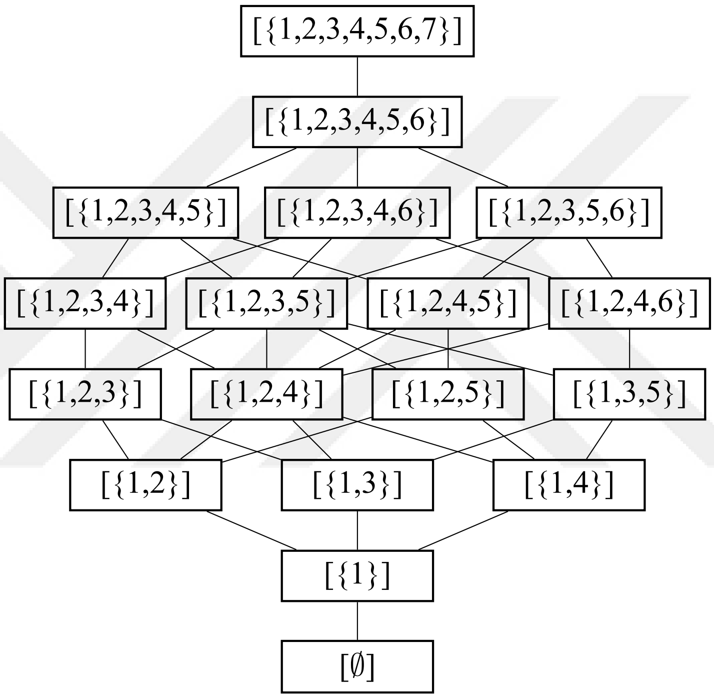

Figure 4.3 The Hasse diagram of $2^7/D_7$

Then

$C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}] \lessdot [\{1,2,3,4,5,6,7\}])$,

$C_2 = ([\{1,3\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,3,5\}] \lessdot [\{1,2,3,4,6\}])$,

$C_3 = ([\{1,4\}] \lessdot [\{1,2,5\}] \lessdot [\{1,2,4,5\}] \lessdot [\{1,2,3,5,6\}])$,

$C_4 = ([\{1,3,5\}] \lessdot [\{1,2,4,6\}])$, gives a symmetric saturated chain decomposition for $2^7/D_7$.

Remark 4.1.3. *The poset $2^7/D_7$ has many different SSCD's. Some of the others are given as below.*

1. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}] \lessdot [\{1,2,3,4,5,6,7\}])$,

   $C_2 = ([\{1,3\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,4,6\}] \lessdot [\{1,2,3,4,6\}])$,

   $C_3 = ([\{1,4\}] \lessdot [\{1,2,5\}] \lessdot [\{1,2,4,5\}] \lessdot [\{1,2,3,5,6\}])$,

   $C_4 = ([\{1,3,5\}] \lessdot [\{1,2,3,5\}])$.

2. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}] \lessdot [\{1,2,3,4,5,6,7\}])$,

   $C_2 = ([\{1,3\}] \lessdot [\{1,3,5\}] \lessdot [\{1,2,3,5\}] \lessdot [\{1,2,3,4,6\}])$,

   $C_3 = ([\{1,4\}] \lessdot [\{1,2,5\}] \lessdot [\{1,2,4,5\}] \lessdot [\{1,2,3,5,6\}])$,

   $C_4 = ([\{1,2,4\}] \lessdot [\{1,2,4,6\}])$.

3. $C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}] \lessdot [\{1,2,3,4,5,6,7\}])$,

   $C_2 = ([\{1,3\}] \lessdot [\{1,3,5\}] \lessdot [\{1,2,4,6\}] \lessdot [\{1,2,3,4,6\}])$,

   $C_3 = ([\{1,4\}] \lessdot [\{1,2,5\}] \lessdot [\{1,2,4,5\}] \lessdot [\{1,2,3,5,6\}])$,

   $C_4 = ([\{1,2,4\}] \lessdot [\{1,2,3,5\}])$.

□

Proposition 4.1.4. *The poset $D_n/2^n$ has a SSCD when $n = 8$.*

Proof. When $n = 8$, the Hasse diagram of $2^8/D_8$ is given in the figure Figure 4.4 (Since there is not enough space, we will not use the "[ ]" notation in the Hasse diagram of $2^8/D_8$).

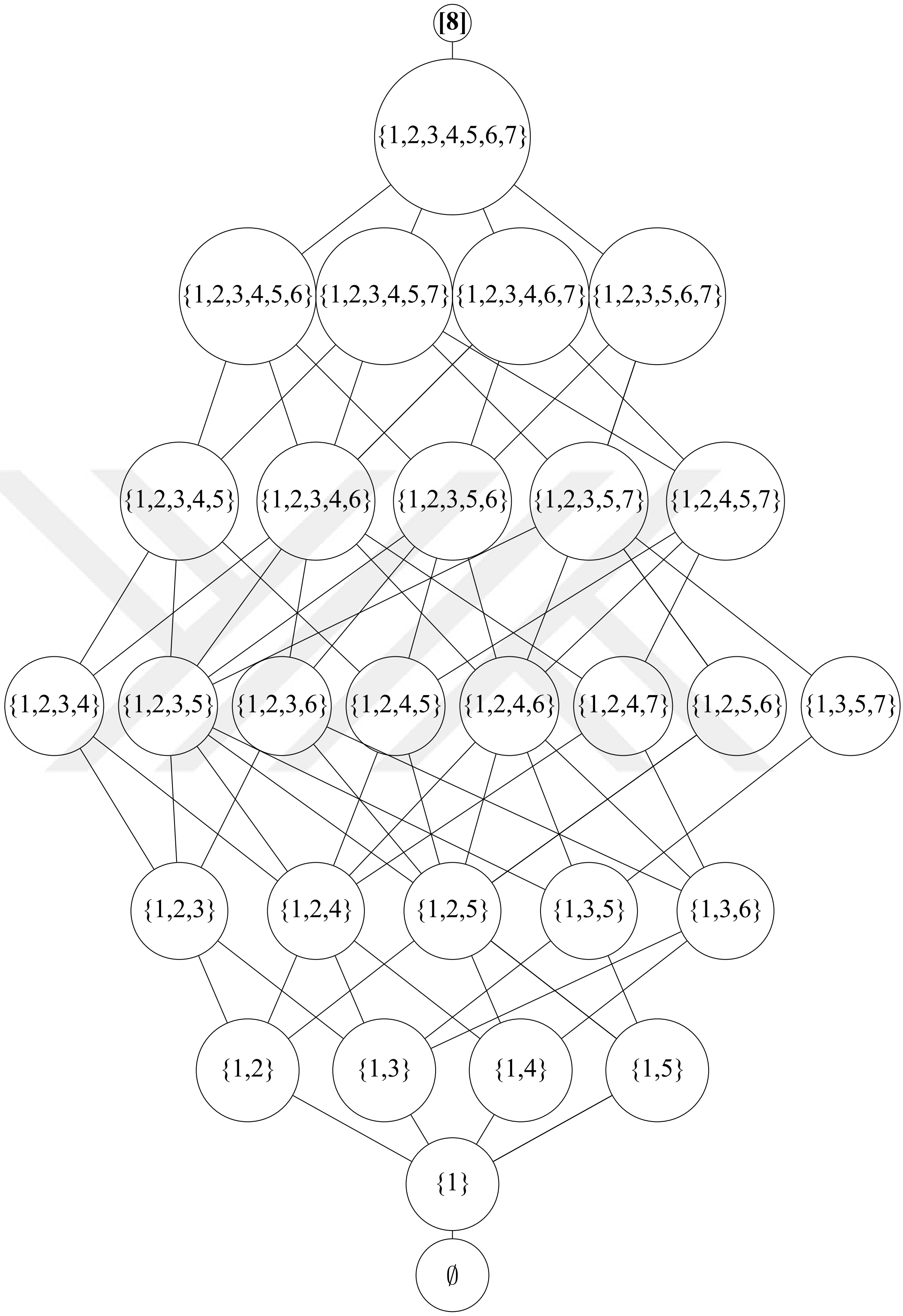

Figure 4.4 The Hasse diagram of $2^8/D_8$

Then

$$C_1 = ([\emptyset] \lessdot [\{1\}] \lessdot [\{1,5\}] \lessdot [\{1,3,5\}] \lessdot [\{1,3,5,7\}] \lessdot [\{1,2,3,5,7\}] \lessdot [\{1,2,3,5,6,7\}] \lessdot [\{1,2,3,4,5,6,7\}] \lessdot [\{1,2,3,4,5,6,7,8\}]),$$

$$C_2 = ([\{1,2\}] \lessdot [\{1,2,3\}] \lessdot [\{1,2,3,4\}] \lessdot [\{1,2,3,4,5\}] \lessdot [\{1,2,3,4,5,6\}]),$$

$$C_3 = ([\{1,3\}] \lessdot [\{1,2,4\}] \lessdot [\{1,2,3,5\}] \lessdot [\{1,2,3,4,6\}] \lessdot [\{1,2,3,4,5,7\}]),$$

$$C_4 = ([\{1,4\}] \lessdot [\{1,3,6\}] \lessdot [\{1,2,4,7\}] \lessdot [\{1,2,4,5,7\}] \lessdot [\{1,2,3,4,6,7\}]),$$

$$C_5 = ([\{1,2,5\}] \lessdot [\{1,3,6\}] \lessdot [\{1,2,3,5,6\}]),$$

$$C_6 = ([\{1,2,4,5\}]),$$

$$C_7 = ([\{1,2,4,6\}]),$$

$$C_8 = ([\{1,2,5,6\}]),$$

gives a symmetric saturated chain decomposition for $2^8/D_8$. □

Corollary 4.1.1. *The poset $2^n/D_n$ has the Sperner's property when n=5,6,7,8.*

Now, we will realize the poset $2^n/D_n$ as a quotient of the poset of ordered partitions of n.

Let Q_n be the poset of ordered partitions of a positive integer n. An ordered partition of an integer $n \geq 1$ is the succession of k different positive integers whose sum is n where $0 \leq k \leq n$. For instance, we can write ordered partitions of 4 into 8 different ways: $1+1+1+1$, $2+1+1$, $1+2+1$, $1+1+2$, $3+1$, $1+3$, $2+2$, 4. Here the relation on Q_n is given by

$$x = a_1 + a_2 + \cdots + a_k \lessdot b_1 + b_2 + \cdots + b_t = y$$

if $t = k+1$ there exist $1 \leq j \leq k+1$ such that $a_i = b_i$ for all $i < j$ and $a_j = b_j + b_{j+1}$ and $a_t = b_{t+1}$ for all $t > j$. The group D_n acts on Q_n by

$$r(a_1 + a_2 + \cdots + a_k) = a_k + a_1 + a_2 + \cdots + a_{k-1} \tag{4.4}$$

$$s(a_1 + a_2 + \cdots + a_k) = a_k + a_{k-1} + \cdots + a_1. \tag{4.5}$$

Consider the quotient poset Q_n/D_n when $n = 4$. Indeed, the Hasse diagram of Q_4/D_4 given as Figure 4.5. Q_4/D_4 represented by the elements $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, 4.

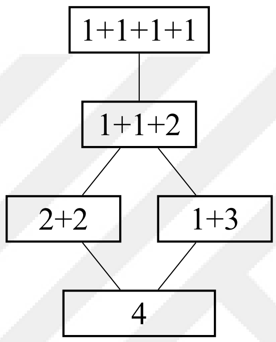

Figure 4.5 The Hasse diagram of Q_4/D_4

This poset isomorphic to the poset of partitions of 4. However, this is not true in general. For example when $n = 6$, $1+1+2+2$ and $1+2+1+2$ represents different elements in Q_6/D_6. In fact we show that the poset Q_n/D_n is closely related to the quotient poset $2^n/D_n$. Note that, only difference between Q_4/D_4 is the element $\{\emptyset\}$. For this reason we extend Q_n by adding an element $\emptyset$ and letting $\emptyset < a$ for all $a \in Q_n$. We denote the new poset by $\tilde{Q}_n$. We extend the action of D_n on Q_n to $\tilde{Q}_n$ by letting $g \cdot \emptyset = \emptyset$ for all $g \in D_n$.

Example 4.1.1. *We can observe that from the hasse diagrams of the posets $2^5/D_5$ and $\tilde{Q}_5/D_5$ are the same.*

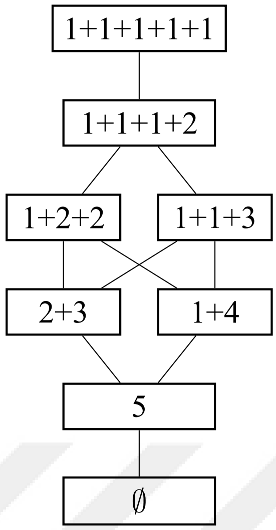

Figure 4.6 The Hasse diagram of $\tilde{Q}_5/D_5$

Example 4.1.2. *Hasse diagrams of the posets $2^6/D_6$ and $\tilde{Q}_6/D_6$ are the same.*

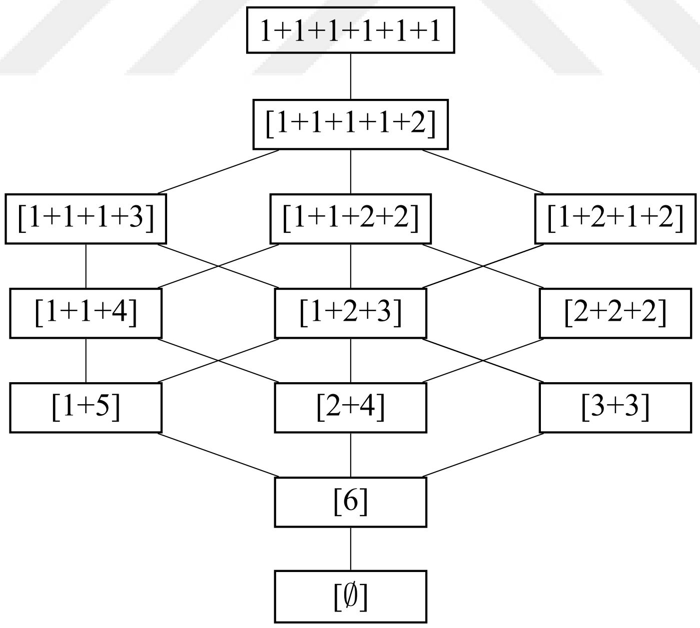

Figure 4.7 The Hasse diagram of $\tilde{Q}_6/D_6$

Example 4.1.3. *Similarly, Hasse diagrams of the posets* $2^7/D_7$ *and* $\tilde{Q}_7/D_7$ *are the same.*

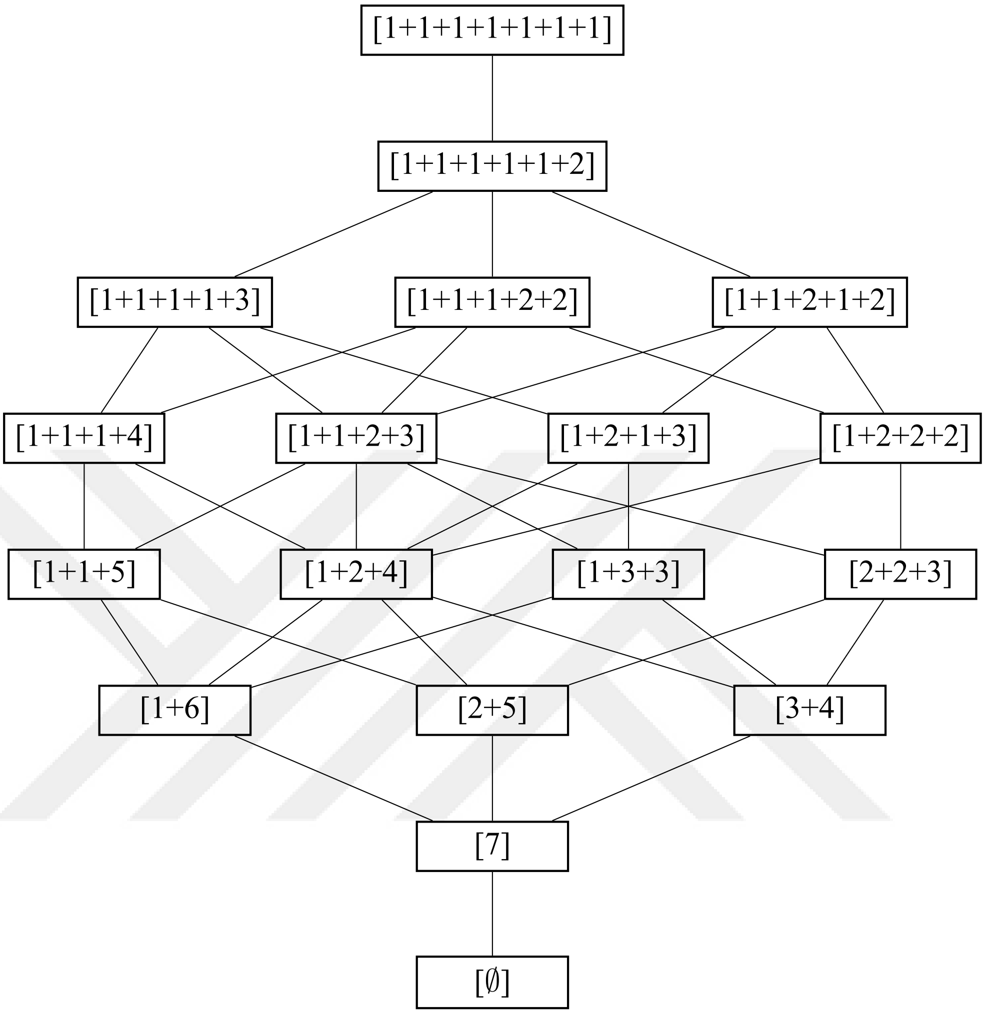

Figure 4.8 The Hasse diagram of $\tilde{Q}_7/D_7$

Example 4.1.4. *As observed in Figure 4.9, the Hasse diagrams of the posets* $2^8/D_8$ *and* $\tilde{Q}_8/D_8$ *are the same.*

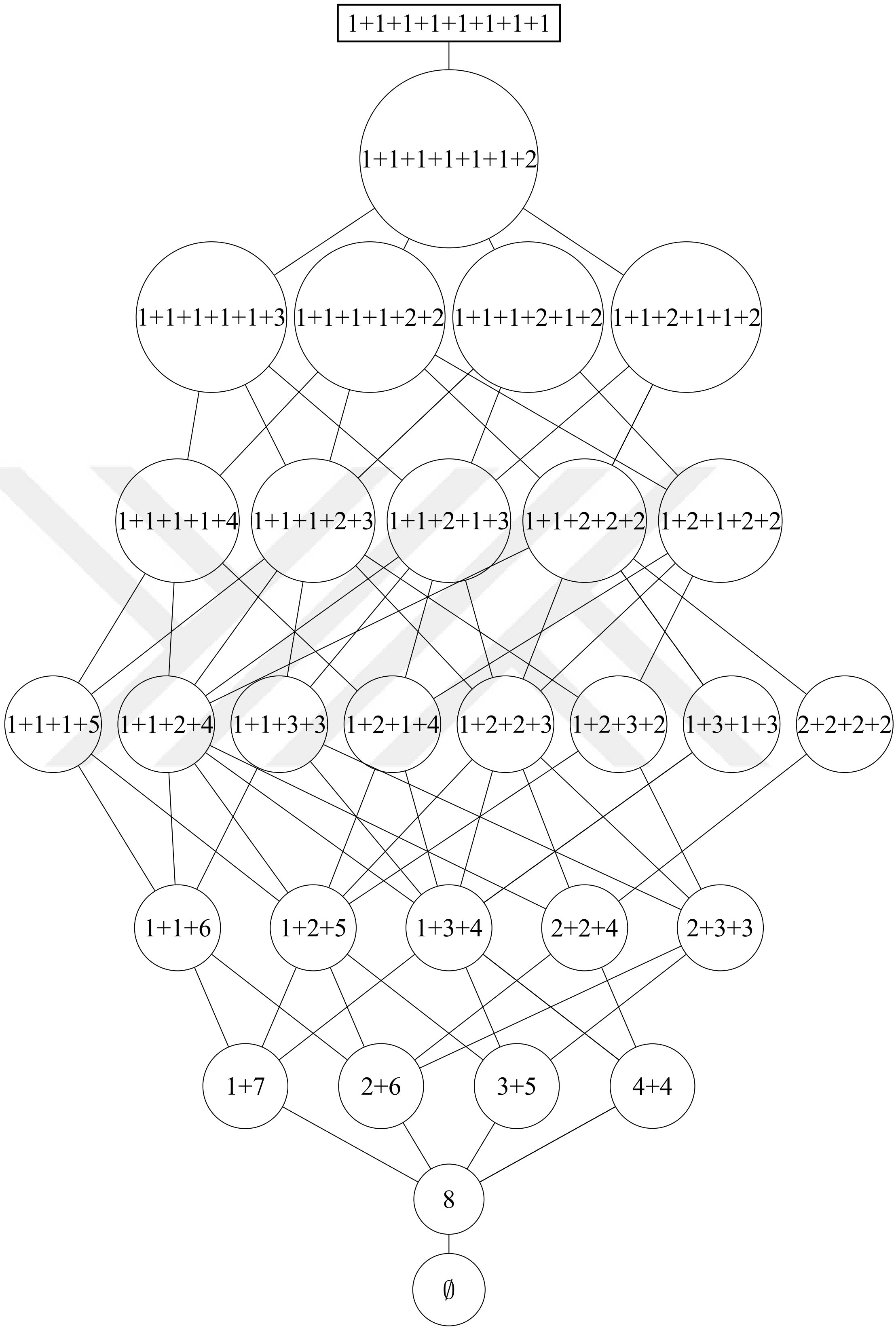

Figure 4.9 The Hasse diagram of $\tilde{Q}_8/D_8$

Corollary 4.1.2. *The poset $\tilde{Q}_n/D_n$ has a symmetric saturated chain decomposition for $n = 5, 6, 7, 8$. In particular, $\tilde{Q}_n/D_n$ has the Sperner's property for $n = 5, 6, 7, 8$.*

Theorem 4.1.1. *The poset $2^n/D_n$ is isomorphic to the poset $\tilde{Q}_n/D_n$*

Proof. For this, let $(2^n)_k$ be the set of subsets of $[n]$ of size k and $(\tilde{Q}_n)_k$ be the ordered partitions of n into k parts. Define a function $f : (2^n)_k \to (\tilde{Q}_n)_k$ by

$$\{a_1, a_2, \ldots, a_k\} \mapsto (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_k - a_{k-1}) + (n - a_k + a_1).$$

where $a_1 < a_2 < \cdots < a_k$. Let $a_1 + a_2 + \cdots + a_k \in (\tilde{Q}_n)_k$. Suppose that

$$A := \{1, a_1 + 1, a_2 + a_1 + 1, \ldots, \sum_{i=0}^{k-1} a_i\} \in (2^n)_k \tag{4.6}$$

where $a_0 = 1$. Thus $f(A) = a_1 + a_2 + \cdots + a_k$. This shows that f is surjective.

Define a function $\phi : (2^n/D_n)_k \to (\tilde{Q}_n/D_n)_k$ by $\phi([B]) = [f(B)]$. First, we need to show that ϕ is well defined. For this, it suffices to show that $[f(r \cdot B] = [f(B)]$ and $[f(s \cdot B] = [f(B)]$ for any $B = \{b_1, b_2, \ldots, b_k\}$. By definition

$$r \cdot \{b_1, b_2, \ldots, b_k\} \quad = \quad \begin{cases} \{1, (b_1 + 1), (b_2 + 1) \ldots, (b_{k-1} + 1)\} & b_k = n \\ \{(b_1 + 1), (b_2 + 1) \ldots, (b_k + 1)\} & b_k \neq n \end{cases} \tag{4.7}$$

If $b_k = n$, $f(r \cdot \{b_1, \ldots, n\})$ is equal to

$$= \quad (b_1) + (b_2 - b_1) + \cdots + (b_{k-1} - b_{k-2}) + (n - b_{k-1}) \tag{4.8}$$
$$= \quad r \cdot f(\{b_1, b_2, \ldots, b_{k-1}, n\}). \tag{4.9}$$

If $b_k \neq n$,

$$f(r \cdot \{b_1, b_2, \ldots, b_k\}) \quad = \quad (b_2 - b_1) + (b_3 - b_2) + \cdots + (n - b_k + b_1) \tag{4.10}$$
$$= \quad f(\{b_1, b_2, \ldots, b_k\}) \tag{4.11}$$

Therefore, in both cases, we have $[f(r \cdot B)] = [f(B)]$. Since

$$s \cdot \{b_1, b_2, \ldots, b_k\} = \begin{cases} \{1, n+2-b_k, \ldots, n+2-b_2\} & b_1 = 1 \\ \{n+2-b_k, \ldots, n+2-b_1\} & b_1 \neq 1, \end{cases} \tag{4.12}$$

$f(s \cdot \{b_1, b_2, \ldots, b_k\})$ is equal to

$$\begin{cases} (n+1-b_k) + (b_k - b_{k-1}) + \cdots + (b_2 - 1) & b_1 = 1 \\ (b_k - b_{k-1}) + \cdots + (b_2 - b_1) + (n + b_1 - b_k) & b_1 \neq 1. \end{cases} \tag{4.13}$$

On the other hand, $f(\{b_1, b_2, \ldots, b_k\}) = (b_2 - b_1) + (b_3 - b_2) + \cdots + (n - b_k + b_1)$ and hence

$$s \cdot f(\{b_1, b_2, \ldots, b_k\}) = (n - b_k + b_1) + (b_k - b_{k-1}) + \cdots + (b_2 - b_1).$$

Therefore, we have

$$f(s \cdot B) = \begin{cases} s \cdot f(B) & b_1 = 1 \\ r^{-1} s \cdot f(B) & b_1 \neq 1. \end{cases} \tag{4.14}$$

Thus, in both cases, we obtain $[f(s \cdot B)] = [f(B)]$. Consequently, ϕ is well-defined. Since f is onto, so does ϕ

Now, we show that ϕ is one-to-one. For this, it suffices to show that if $f(A) = s \cdot f(B)$ or $f(A) = r \cdot f(B)$, then $[A] = [B]$ since f is onto. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$. Assume first that $f(A) = s \cdot f(B)$. Then $(a_2 - a_1) + \cdots + (n - a_k + a_1)$ is equal to

$$\begin{aligned} &= s \cdot ((b_2 - b_1) + (b_3 - b_2) + \cdots + (n - b_k + b_1)) && (4.15) \\ &= (n - b_k + b_1) + (b_k - b_{k-1}) + \cdots + (b_2 - b_1). && (4.16) \end{aligned}$$

Thus we have $a_i = n - b_{k-i+2} + b_1 + a_1$ for $2 \leq i \leq k$. Since

$A = \{a_1, n - b_k + b_1 + a_1, \dots, n - b_2 + b_1 + a_1\}$, we have

$$r^{1-a_1} \cdot A = \{1, n - b_k + b_1 + 1, \dots, n - b_2 + b_1 + 1\}.$$

Therefore

$$sr^{1-a_1} \cdot A = \{1, 1 + b_2 - b_1, 1 + b_3 - b_1, \dots, 1 + b_k - b_1\}$$

and hence

$$r^{b_1-1}sr^{1-a_1} \cdot A = \{b_1, b_2, \dots, b_k\} = B.$$

This proves that if $s \cdot f(A) = f(B)$, we have $[A] = [B]$.

Now assume that $f(A) = r \cdot f(B)$. Then $(a_2 - a_1) + \cdots + (n - a_k + a_1)$ is equal to

$$\begin{aligned} &= r \cdot ((b_2 - b_1) + (b_3 - b_2) + \cdots + (n - b_k + b_1)) && (4.17) \\ &= (n - b_k + b_1) + (b_2 - b_1) + \cdots + (b_k - b_{k-1}) && (4.18) \end{aligned}$$

Thus we have $a_i = n - b_k + a_1 + b_{i-1}$ for $2 \leq i \leq k$.
Since

$$\begin{aligned} A &= \{a_1, a_2, \dots, a_k\} && (4.19) \\ &= \{a_1, n - b_k + a_1 + b_1, \dots, n - b_k + a_1 + b_{k-1}\}, && (4.20) \end{aligned}$$

we have $r^{b_k - a_1} \cdot A = \{b_1, b_2, \dots, b_k\} = B$ and hence $[A] = [B]$.

Now we want to show that $[A] \leq [B]$ if and only if $[f(A)] \leq [f(B)]$. Suppose that $[A] \leq [B]$. It is enough to show that $[f(s \cdot A] \leq [f(B)]$ and $[f(r \cdot A] \leq [f(B)]$. Let $A := \{a_1, a_2, \dots, a_k\}$ and $B := \{b_1, b_2, \dots, b_t\}$ for some $k, t \in \mathbb{Z}^+$ such that $k < t$. Now suppose that $C = s \cdot A \subseteq B$ where $C := \{c_1, c_2, \dots, c_k\}$. Then for all $1 \leq l \leq k$, there exist $1 \leq t_l \leq t$ such that $c_l = b_{t_l}$. Suppose $1 \leq l < l + 1 \leq k$. Since $c_l = b_{t_l}$ and $c_{l+1} = b_{t_{l+1}}$, we have $c_{l+1} - c_l = b_{t_{l+1}} - b_{t_l}$. Then we can easily observe that $c_{l+1} - c_l = b_{t_{l+1}} - b_{t_l} = (b_{t_{l+1}} - b_{t_{l+1}-1}) + (b_{t_{l+1}-1} - b_{t_{l+1}-2}) + \cdots + (b_{t_l+1} - b_{t_l})$.

Thus $f(C) \leq f(B)$ and this yields $[f(s \cdot A)] = [f(C)] \leq [f(B)]$. Similarly, one can obtain $[f(r \cdot A] \leq [f(B)]$ by assuming $C = r \cdot A$. Then we have $[f(A)] \leq [f(B)]$. Conversely, suppose that $[f(A)] \leq [f(B)]$. We want to show that $[A] \leq [B]$. For this, it suffices to show that if $f(A) \leq s \cdot f(B)$ or $f(A) \leq r \cdot f(B)$, then $[A] \leq [B]$. Assume first that $f(A) \leq s \cdot f(B)$. Then we have $(a_2 - a_1) + \cdots + (n - a_k + a_1)$ is

$$\begin{aligned} &\leq s \cdot ((b_2 - b_1) + (b_3 - b_2) + \cdots + (n - b_t + b_1)) && (4.21)\\ &= (n - b_t + b_1) + (b_t - b_{t-1}) + \cdots + (b_2 - b_1). && (4.22)\end{aligned}$$

Thus we have $a_i = n + b_1 + a_1 - b_{l_{i-1}}$ for $2 \leq i \leq k$, for some $1 \leq l_1 < \cdots < l_{k-1} \leq t$. Since

$$\begin{aligned} A &= \{a_1, a_2, \ldots, a_k\} && (4.23)\\ &= \{a_1, n + b_1 + a_1 - b_{l_1}, \ldots, n + b_1 + a_1 - b_{l_{k-1}}\}, && (4.24)\end{aligned}$$

we have

$$s \cdot A = \begin{cases} \{1, 1 - b_1 + b_{l_{k-1}}, \ldots, 1 - b_1 + b_{l_1}\} & a_1 = 1 \\ \{2 - b_1 - a_1 + b_{l_{k-1}}, \ldots, n + 2 - a_1\} & a_1 \neq 1 \end{cases}$$

So if $a_1 = 1$ then $r^{b_1 - 1} s \cdot A \subseteq B$ and if $a_1 \neq 1$ then $r^{b_1 + a_1 - 2} s \cdot A \subseteq B$. Therefore if $f(A) \leq s \cdot f(B)$ then $[A] \leq [B]$, as desired.

Now suppose that $f(A) \subseteq r \cdot f(B)$. Then

$$\begin{aligned} (a_2 - a_1) + \cdots + (n - a_k + a_1) &\leq r \cdot ((b_2 - b_1) + (b_3 - b_2) + && (4.25)\\ &\qquad \cdots + (n - b_t + b_1)) && (4.26)\\ &= (n - b_t + b_1) + (b_2 - b_1) + && (4.27)\\ &\qquad \cdots + (b_t + b_{t-1}) && (4.28)\end{aligned}$$

and hence $a_i = n - b_k + a_1 + b_{l_{i-1}}$ for $2 \leq i \leq k$. Since

$$A = \{a_1, a_2, \ldots, a_k\} \tag{4.29}$$

$$= \{a_1, n - b_k + a_1 + b_{l_1}, \ldots, n - b_k + a_1 + b_{l_{k-1}}\}. \tag{4.30}$$

$r^{b_k - a_1} \cdot A \subseteq B$. Therefore if $f(A) \subseteq r \cdot f(B)$ then $[A] \leq [B]$ and consequently ϕ is an poset isomorphism. □

Corollary 4.1.3. *The poset $2^n/D_n$ has a symmetric saturated chain decomposition if the poset $\tilde{Q}_n/D_n$ has a symmetric saturated chain decomposition.*

Corollary 4.1.4. *The poset $2^n/D_n$ has a symmetric saturated chain decomposition if the poset $\tilde{Q}_n/D_n$ has the Sperner's property.*

CHAPTER FIVE
CONCLUSION

In this thesis, we give symmetric saturated chain decompositions for the poset $2^n/D_n$ when $n = 5, 6, 7, 8$. We believe this result is true for all integes n. For this reason, we have shown that the poset $2^n/D_n$ is isomorphic to the poset $\tilde{Q}_n/D_n$ where $\tilde{Q}$ is obtained by extending the poset of ordered partitions of n by adding a minimal element. Since a family of representatives of elements of the poset $\tilde{Q}_n/D_n$ can be easily chosen, it might be easier to work with $\tilde{Q}_n/D_n$.

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