DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

CLASSIFICATIONS OF SMALL COVERS OVER A PRODUCT OF SIMPLICES

by
Didem ÇİL

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İZMİR

CLASSIFICATIONS OF SMALL COVERS OVER A PRODUCT OF SIMPLICES

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We have read the thesis entitled "**CLASSIFICATIONS OF SMALL COVERS OVER A PRODUCT OF SIMPLICES**" completed by **DİDEM ÇİL** under supervision of **ASST. PROF. DR. ASLI GÜÇLÜKAN İLHAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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CLASSIFICATIONS OF SMALL COVERS OVER A PRODUCT OF SIMPLICES

ABSTRACT

In this study, our aim is to calculate the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over some product of simplices. First, we introduce the notion of a small cover over a simple convex polytope and discuss them up to some equivalences which are Davis-Januskiewicz equivalence, $\mathbb{Z}_2^n$-equivariant homeomorphism and weakly $\mathbb{Z}_2^n$-equivariant homeomorphism. Then we give some formulas that gives the number of small covers over a given simple convex polytope up to Davis-Januskiewicz equivalence and $\mathbb{Z}_2^n$-equivariant homeomorphism, respectively. Finally we do some calculations to find the number of small covers over some specific polytopes up to weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes.

Keywords: Small cover, Davis-Januskiewicz equivalence classes, $\mathbb{Z}_2^n$-equivariant homeomorphism classes, Weakly $\mathbb{Z}_2^n$-homeomorphism classes

SİMPLEKSLERİN ÇARPIMLARI ÜZERİNDEKİ DAR ÖRTÜLERİN SINIFLANDIRILMALARI

ÖZ

Bu çalışmanın amacı bazı simpleks çarpımları üzerindeki dar örtülerin, zayıf $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma sınıflarının sayılarını hesaplamaktır. İlk olarak, basit konveks politoplar üzerinde dar örtü kavramını verdik ve onları Davis-Januskiewicz denklik bağıntısı, $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma ve zayıf $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma gibi denklik bağıntıları altında tartıştık. Sonrasında verilen bir basit konveks politop üzerindeki dar örtülerin, Davis-Januskiewicz denklik bağıntısı ve $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma sınıflarının sayılarını hesaplayan formüller verdik. Son olarak, birkaç politop için bunlar üzerindeki dar örtülerin zayıf $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma sınıflarının sayıları ile ilgili hesaplamalar yaptık.

Anahtar kelimeler: Dar örtü, Davis-Januskiewicz denklik bağıntısı, $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma, zayıf $\mathbb{Z}_2^n$-ekuvaryant homeomorfizma

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CHAPTER ONE
INTRODUCTION

The notion of a small cover was introduced by Davis and Januskiewicz in Davis et al. (1991). They define a small cover as a generalization of real toric manifolds. A small cover over a simple convex polytope P^n is a smooth closed manifold M^n with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is P^n. For each small cover over P^n, there is an associated characteristic function (see Definition 3.1.3). In Davis et al. (1991) it was shown that every small cover over P^n can be obtained from a characteristic function up to Davis-Januskiewicz equivalence (see Definition 3.1.2). There is a free action of $GL(n, \mathbb{Z}_2)$ on the set of characteristic functions over P^n and there is a one-to-one correspondence between the orbit space of this action and the set of Davis-Januskiewicz equivalence classes of small covers over P^n. In Choi (2008), Choi shows that there is a one-to-one correspondence between the set of Davis-Januskiewicz equivalence classes over an n-cube and the set of acyclic directed graphs with n-labeled nodes. Using this correspondence, Choi finds a formula for the number of Davis-Januskiewicz equivalence classes of small covers over an n-cube. He also gives a formula that gives the number of Davis-Januskiewicz equivalence classes of small covers over a product of simplices. For this, Choi uses the relation between this set and the set of vector matrices given in Choi et al. (2010).

The automorphism group of the set of facets of a simple convex polytope P^n acts on the set of characteristic functions on P^n by composition. In Lü & Masuda (2008), Lü and Masuda construct a one-to-one correspondence between the orbit space of this action and the set of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a simple convex polytope P^n. By Burnside's Lemma, the number of orbits of this action is equal to the average number of the characteristic functions fixed by a given element of the group. Therefore one can obtain the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P^n by calculating the number of points fixed by g, for every g in the automorphism group. In Choi (2008) Choi obtaines a formula which gives the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over an n-cube. In

Altunbulak & Güçlükan İlhan (2018) İlhan and Altunbulak give a similar formula for the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a product of simplices.

There is a one-to-one correspondence between the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes and the double coset of the set of characteristic functions on P^n by $GL(n, \mathbb{Z}_2)$ and the automorphism group of the set of facets of P^n. In Choi (2008), Choi gives an upper bound for weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes over a cube using this identification. In Altunbulak & Güçlükan İlhan (2018), Altunbulak and İlhan give an upper bound for the set of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a given product of simplices.

This thesis is organized as follows. We give a necessary background on the simple convex polytopes, groups and graphs on Chapter 2. We give a survey on small covers in Chapter 3. Then we give some calculations related to the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes over a specific products of simplices in Chapter 4.

CHAPTER TWO
PRELIMINARIES

In the first section, we give a background on the simple convex polytopes. We refer reader to Brondsted (2012) for more details about simple convex polytopes. In the remaining of this chapter, we give a necessary definitions on group actions and graph theory. We also discuss the Burnside's Lemma.

2.1 Simple Convex Polytopes

Let C be a convex polytope. For every point $x \in C$, we can talk about the smallest face containing x. These faces are characterized as follows.

Definition 2.1.1. *The interior of a convex set C in the affine hull of C is called the* ***relative interior*** *of C in $\mathbb{R}^n$ and denoted by* $\mathrm{ri}C$.

Theorem 2.1.1. *(Brondsted (2012), Theorem 5.6) A face F is the smallest face of C containing an element x in C if and only if $x \in riF$.*

By the above theorem for every $x \in C$, there is a unique face F of C which contains x in its relative interior.

Definition 2.1.2. *The convex hull of k distinct points on the curve $m(t) = (t, t^2, \cdots, t^{n-1})$ is called a* ***cyclic polytope****, where $n + 1 \leq k$, and denoted by C^n_k.*

Definition 2.1.3. *Let $p_1, \cdots, p_n$ be affinely independent points in $\mathbb{R}^n$. Then the convex set*

$$S=\{\lambda_1 p_1 + \cdots \lambda_n p_n | \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i\}$$

is called the ***(n-1)-simplex*** *determined by $p_1, \cdots, p_n$. If these n points are the n standard basis vectors of $\mathbb{R}^n$, then the set S is called the $(n-1)$-dimensional* ***standard simplex*** *and denoted by $\triangle^{n-1}$.*

Definition 2.1.4. *Let P be a convex polytope of dimension n. Then P is called* ***simple*** *if the number of facets of P containing any k-face of P is equal to $n-k$. An n-dimensional convex polytope P is called* ***simplicial*** *if each proper face of P is a simplex.*

A standard example of a simple convex polytope is a k-cube. A k-cube is a convex set which includes 2^k corners, each of which is connected to nearest k adjacent corners with edges of the same length. This is a generalization of 3-dimensional cube to k dimension. A bipyramid which is a 3-dimensional figure consisting of two identical pyramids base to base, is an example of a simplicial convex polytope.

Now, we will discuss the duality of polytopes.

Definition 2.1.5. *Two polytopes are said to be* ***dual****, if their face-lattices are anti-isomorphic.*

Theorem 2.1.2. *(Brondsted (2012), Theorem 10.3) Let ϕ be an anti-isomorphism between P and its dual Q. If the dimension of P is n, then the dimension of Q is n. Moreover, for every face F of P, the dimension of $\phi(F)$ is $n-1-\dim(F)$.*

Theorem 2.1.3. *(Brondsted (2012), Theorem 12.10) Let P and Q be dual n-polytopes. Then P is simple if and only if Q is simplicial.*

Theorem 2.1.4. *(Brondsted (2012), Theorem 12.11) An n-polytope P is simple if and only if every vertex of P is the intersection of precisely n facets.*

Theorem 2.1.5. *(Brondsted (2012), Theorem 12.14) Let P be a simple n-polytope, and let $F_1, \ldots, F_{n-k}$ be facets of P, where $0 \leq k \leq n-1$. Let*

$$F := \bigcap_{i=1}^{n-k} F_i.$$

If $F \neq \emptyset$ then F is a k-face of P, and $F_1, \ldots, F_{n-k}$ are the only facets of P containing F.

Theorem 2.1.6. *(Brondsted (2012), Theorem 12.15) Every proper face of a simple n-polytope is simple.*

Theorem 2.1.7. *(Brondsted (2012), Theorem 12.16) Let P be a simple n-polytope. Then for $0 \leq j \leq k \leq n$ there are precisely $\binom{n-j}{n-k}$ k-faces of P containing the given j-face of P.*

Proof. When $k = n$, we have $\binom{n-j}{n-n} = \binom{n-j}{0} = 1$. Since there is precisely one n-face of P containing the given j-face for any $0 \leq j \leq n$, which is P itself, there is nothing to prove in this case. Let $k < n$, P and Q be dual polytopes and $\varphi : (\mathcal{F}(P), \subset) \rightarrow (\mathcal{F}(Q), \subset)$ be an anti-isomorphism between them. Suppose also that F is a j-face of P. Then, by Theorem 1.1.2, $\varphi(F)$ is an $(n-1-j)$-face of Q, and the number of k-faces of P containing F is the number of $(n-1-k)$-faces of $\varphi(F)$. Since P is simple and Q is a dual of P, Q is simplicial. So, $\varphi(F)$ is a simplex. Since calculating the k-faces of P containing F is the same as choosing $(n-1-k)+1$ vertices from $(n-1-j)+1$ vertices of $\varphi(F)$, the number of k-faces of P containing F is $\binom{(n-1-j)+1}{(n-1-k)+1} = \binom{n-j}{n-k}$. $\square$

2.2 Group Actions

In this subsection, we recall the Burnside's Lemma.

Theorem 2.2.1 (Burnside's Lemma). *Let G be a finite group, X be a G-set and X/G be the orbit space. Then,*

$$|X \setminus G| = \frac{1}{|G|} \sum_{g \in G} |X^g| \tag{2.1}$$

where $X^g = \{x \in X : g \cdot x = x\}$.

Proof. Let $A = \{(g, x) \in G \times X : g \cdot x = x\}$. We calculate the number of elements of A in two different ways. Firstly, we have

$$|A| = \sum_{g \in G} |\{x \in X : g \cdot x = x\}| = \sum_{g \in G} |X^g|. \tag{2.2}$$

Secondly,

$$|A| = \sum_{x \in X} |\{g \in G : g \cdot x = x\}| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|Gx|}. \tag{2.3}$$

Note that, if x and y are in the same orbit, then $|Gx| = |Gy|$. So if T is the set which contains only one element of each orbit, we have

$$\sum_{x \in Gt} \frac{|G|}{|Gx|} = \sum_{x \in Gt} \frac{|G|}{|Gt|} = |G| \tag{2.4}$$

for all $t \in T$. Then we have

$$|A| = \sum x \in X \frac{|G|}{|Gx|} = \sum_{t \in T} (\sum_{x \in Gt} \frac{|G|}{|Gx|}) = \sum_{t \in T} |G| = |T||G|. \tag{2.5}$$

Therefore,

$$\sum_{g \in G} |X^g| = |A| = |T||G|. \tag{2.6}$$

Since $|T| = |X \setminus G|$ we have

$$|X \setminus G| = \frac{1}{|G|} \sum_{g \in G} |X^g| \tag{2.7}$$

□

There is a natural action of the elementary abelian group $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ given by $(g_1, g_2, \cdots, g_n) \cdot (v_1, v_2, \cdots, v_n) = (g_1 v_1, g_2 v_2, \cdots, g_n v_n)$. The orbit space of this action can be identified with the set of all vectors in $\mathbb{R}^n$ whose coordinates are non-negative.

The real projective space is the space of lines passing through the origin in $\mathbb{R}^{n+1}$. It is denoted by $\mathbb{R}P^n$. Let I_v denote the line through origin in the direction of v in $\mathbb{R}^n$. The natural action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ induces an action of $\mathbb{Z}_2^n$ on $\mathbb{R}P^n$ by $g \cdot I_v = I_{g \cdot v}$. The orbit space $\mathbb{R}P^n / \mathbb{Z}_2^n$ of the action is homeomorphic to the standard simplex $\triangle^n$.

Definition 2.2.1. *Two spaces M_1 and M_2 are called* ***weakly G-homeomorphic*** *if* *there is an automorphism $\varphi : G \to G$ and a homeomorphism $f : M_1 \to M_2$ such that $f(tx) = \varphi(t)f(x)$ for all $t \in G$ and $x \in M_1$. Here φ is called a **weakly G-homeomorphism**.*

*If $\varphi = id$ then M_1 and M_2 are said to be **G-homeomorphic** and φ is called a **G-homeomorphism**.*

Definition 2.2.2. *The action of $\mathbb{Z}_2^n$ over a manifold M is said to be **locally standard**, if for all $x \in M$, there exists a $\mathbb{Z}_2^n$-invariant open neighborhood $x \in U \subseteq M$ and a $\mathbb{Z}_2^n$-invariant subset $V \subseteq \mathbb{R}^n$, under the natural action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ with a homeomorphism $f : U \to V$ such that $f(gu) = \theta(g)f(u)$, where θ is an automorphism of $\mathbb{Z}_2^n$.*

In this thesis, we work with manifolds on which there is a locally standard action of $\mathbb{Z}_2^n$ and we try to find the number of equivalence classes of such manifolds up to specific isomorphisms.

2.3 Graph Theory

Definition 2.3.1. *A **directed graph** or a **digraph** is a graph in which every edge has a direction. In this case an element (v_1, v_2) of an edge set represents a directed edge from v_1 to v_2.*

Definition 2.3.2. *A **directed path** of a directed graph G is a path in which all the edges directed in the same direction.*

Definition 2.3.3. *Let G be a digraph. A **cycle** in G is a directed path that begins and ends at the same vertex. If G contains a directed cycle, then G is called a **cyclic digraph**. Otherwise, G is called an **acyclic digraph**.*

Definition 2.3.4. *Let G be a digraph and v be a vertex of G. If there is an edge from another vertex to v, then such a vertex is called an **in-neighbour** of v. The number of in-neighbours of v is called an **inner degree** of V and is denoted by **indeg(v)**. Similarly, if there is an edge from v to another vertex then such a vertex is called an **out-neighbour***

of v. The number of out-neighbours of v is called an ***outer degree*** *of V and denoted by* ***outdeg(v)***.

Definition 2.3.5. *The* ***adjacency matrix*** *of a finite digraph G is a matrix $A = [a_{ij}]$ where a_{ij} is 1 if there is a directed edge from v_i to v_j in G and a_{ij} is 0, otherwise. If the digraph is acyclic, all diagonal entries of A is 0.*

Theorem 2.3.1. *A digraph G is acyclic if and only if adjacency matrix of G is conjugate to a strictly upper triangular matrix.*

CHAPTER THREE
SMALL COVERS

In the first part of this chapter, we introduce the notion of a small cover over a given polytope and the Davis-Januskiewicz equivalence between them. For more details about this part, we refer to reader Davis et al. (1991).

Let $\Lambda(P)$ be the set of all characteristic functions over P, (see Definition 3.1.3). It turns out that the set of Davis-Januskiewicz equivalence classes, $\mathbb{Z}_2^n$-equivariant homeomorphism classes and weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers are equivalent to the orbit spaces of $\Lambda(P)$ under certain actions. We also discuss these relations in the first section. In the second section, we give the classifications of small covers over a cube using the relations given in the first section. In the last section of this chapter, we give the classifications of small covers over a product of simplices and give some formulas to find the number of small covers over product of simplices. Our main references for the second part are Choi (2008) and Altunbulak & Güçlükan İlhan (2018).

3.1 Small Covers

Let P be an n-dimensional simple convex polytope.

Definition 3.1.1. *Let M^n be a smooth closed manifold of dimension n. If there is a locally standard $\mathbb{Z}_2^n$-action on M^n whose orbit space is homeomorphic to P, then M^n is called a* ***small cover*** *over P. We denote a small cover over P by $\pi : M \to P$.*

Definition 3.1.2. *Let $\pi_1 : M_1 \to P$ and $\pi_2 : M_2 \to P$ be small covers over P. If there is a weakly $\mathbb{Z}_2^n$-equivariant homeomorphism $f : M_1 \to M_2$ with $\pi_2 \circ f = \pi_1$, then M_1 and M_2 are said to be* ***Davis-Januskiewicz equivalent (D-J equivalent)****.*

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ P & \xrightarrow{id} & P \end{array}$$

It is standard to show that the D-J equivalence is an equivalence relation.

If F is a k-face of P, we will denote $\pi^{-1}(F)$ by M_F^k. Then we have the following observation.

Lemma 3.1.1. *(Davis et al. (1991), Lemma 1.3) Suppose that $\pi : M^n \to P$ is a small cover over P. Let F be a k-face of P. Then*

i) *M_F^k is a connected submanifold of dimension k.*

ii) *$M_F^k \to F$ is naturally a small cover over F.*

Lemma 3.1.2. *(Davis et al. (1991), Lemma 1.4) Let $\pi : M^n \to P$ be a small cover over P. There is a continuous map $f : \mathbb{Z}_2^n \times P \to M^n$ so that for each $p \in P$, f maps $\mathbb{Z}_2^n \times \{p\}$ onto $\pi^{-1}(p)$.*

Proof. Consider the restriction of π to the interior of P^n. So π^{-1} maps int(P^n) to int(P^n). Since int(P^n) is simply connected, then this is a trivial covering.

Let int(C) be a component of $\pi^{-1}(\text{int}(P^n))$ and let C be the closure of int(C). Then the restriction of π to C is a homeomorphism. Therefore the map $f : \mathbb{Z}_2^n \times P^n \to M^n$ can be defined by $f(g, p) = g(\pi|_C)^{-1}(p)$ where $g \in \mathbb{Z}_2^n$ and $p \in P^n$. □

Let $\pi : M \to P$ be a small cover over P and F be a k-face of P. Since for any $x \in \pi^{-1}(\text{int}(F))$, G_x is independent of the choice of x by Lemma 3.1.1 we can denote the isotropy group by G_F. If we take $k = n-1$, then F is a facet and G_F is a rank-one subgroup. Hence, G_F is determined by a non-zero vector v in $\mathbb{Z}_2^n$. Let $\mathcal{F}(P)$ be the set of facets of P. Therefore, there is a function λ from $\mathcal{F}(P)$ to non-zero vectors in $\mathbb{Z}_2^n$.

Definition 3.1.3. *Let $\mathcal{F}(P) = \{F_1, F_2, ..., F_m\}$ be the set of facets of a simple convex polytope. A function $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ that satisfies the non-singularity condition that*

*"$F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_n} \neq \varnothing \Rightarrow \{\lambda(F_{i_1}), ..., \lambda(F_{i_n})\}$ is linearly independent" is said to be a **characteristic function**.*

Lemma 3.1.3. *(Davis et al. (1991)) The non-singularity condition is equivalent to the following:*

"$F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_n} \neq \varnothing \Rightarrow \{\lambda(F_{i_1}), ..., \lambda(F_{i_n})\}$ is a basis for $\mathbb{Z}_2^n$"

Proof. Let $F_{i_1}, F_{i_2}, \cdots, F_{i_n}$ be facets of P such that $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_n} \neq \varnothing$. Then $\{\lambda(F_{i_1}), ..., \lambda(F_{i_n})\}$ is linearly independent in $\mathbb{Z}_2^n$ by the non-singularity condition. Since $\dim(\mathbb{Z}_2^n) = n$, $\{\lambda(F_{i_1}), ..., \lambda(F_{i_n})\}$ is a basis for $\mathbb{Z}_2^n$.

For the other direction, let $F_{i_1}, F_{i_2}, \cdots, F_{i_k}$ be facets of P such that $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k} \neq \varnothing$ and denote their intersection by F. Let v be a vertex of P which is also contained in F. Since P is simple, $\{v\} = \bigcap_{k=1}^{n} F'_{i_k}$ for some facets F'_{i_k} s.t. $1 \leq k \leq n$. Since $v \in F_{i_j}$ for each j, $F_{i_j} \in \{F'_{i_1}, \cdots, F'_{i_n}\}$ by Theorem 2.1.4. Without loss of generality, let $F'_{i_j} = F_{i_j}$ for $1 \leq j \leq k$. Since $\{\lambda(Fi_1), \cdots, \lambda(F_{i_k}), \lambda(F'_{i_{k+1}}), \cdots, \lambda(F'_{i_n})\}$ is a basis, $\{\lambda(F_{i_1}), ..., \lambda(F_{i_k})\}$ must be linearly independent. □

For a given point $p \in P$, let $\mathbb{Z}_2(p)$ be the subgroup of $\mathbb{Z}_2^n$ generated by $\lambda(F_{i_1}), ..., \lambda(F_{i_k})$ where $F_{i_1} \cap \cdots \cap F_{i_k}$ is the minimal face containing p in its relative interior. One can construct a manifold $M(\lambda) = P \times \mathbb{Z}_2^n / \sim$ where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(p)$. There is a free action of $\mathbb{Z}_2^n$ on $P \times \mathbb{Z}_2^n$ given by $g(x, a) = (x, ga)$ where $g, a \in \mathbb{Z}_2^n$ and $x \in P$. It induces an action of $\mathbb{Z}_2^n$ on $M(\lambda)$ by $g[x, a] = [x, ga]$.

Lemma 3.1.4. *(Davis et al. (1991)) The space $M(\lambda)$ is a manifold.*

Lemma 3.1.5. *(Davis et al. (1991), Lemma 1.6) Let $i : \mathbb{R}_+^n \to \mathbb{R}^n$ be the inclusion. Define an equivalence relation on $\mathbb{Z}_2^n \times \mathbb{R}_+^n$ by $(g, x) \sim (h, y)$ if and only if $x = y$ and $g^{-1}h \in G_{i(x)}$ where $G_{i(x)}$ is the isotropy subgroup of $\mathbb{Z}_2^n$ at $i(x)$. Then the natural map*

$\mathbb{Z}_2^n \times \mathbb{R}_+^n \to \mathbb{R}^n$ given by $(g, x) \mapsto gi(x)$ descends to a homeomorphism $(\mathbb{Z}_2^n \times \mathbb{R}_+^n)/_\sim \to \mathbb{R}^n$.

By the above lemma, the action of $\mathbb{Z}_2^n$ on $M(\lambda)$ is locally standard. Therefore we have the following result.

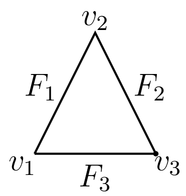
Proposition 3.1.1. *(Davis et al. (1991), Proposition 1.7) Let $\mathcal{F}(P)$ be the set of facets of P and $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ be a characteristic function. Then $\pi : M(\lambda) \to P$ is a small cover over P.*

Proposition 3.1.2. *(Davis et al. (1991), Proposition 1.7) Let $\pi : M \to P$ be a small cover over P. Let $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ be its characteristic function. Let $f : \mathbb{Z}_2^n \times P \to M$ be the map constructed in Lemma 3.1.2. Then f descends to $\mathbb{Z}_2^n$-homeomorphism $M(\lambda) \to M$ covering the identity on P.*

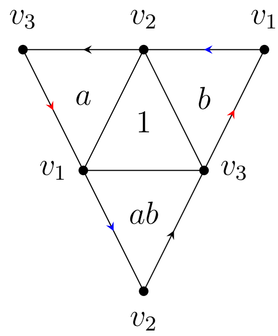
The following result immediately follows from the above proposition.

Theorem 3.1.1. *(Davis et al. (1991)) Let M be a small cover over P. There is a characteristic function λ with $\mathbb{Z}_2^n$-equivariant homeomorphism $M(\lambda) \to M$ covering the identity on P.*

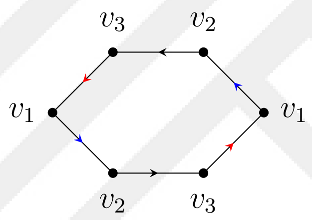
Example 3.1.1. *Let P be a triangle. The set of facets is $\mathcal{F}(P) = \{F_1, F_2, F_3\}$ as shown in the figure below.*



Let $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^2$ be a characteristic function of P and let $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2 =< a, b >$. Suppose that λ takes F_1 to a, F_2 to b and F_3 to ab. Then $M(\lambda) = P \times \mathbb{Z}_2^2/\sim$ where $(p, g) \sim (q, h) \Leftrightarrow p = q$ and $g^{-1}h \in \mathbb{Z}_2^2(p)$. Therefore we give 4-copies of Δ^2 as *follows:*



Hence it is homeomorphic to



which is $\mathbb{R}P^2$, that is, $M(\lambda) \cong \mathbb{R}P^2$.

Now we will give an example of a simple polytope which doesn't admit a small cover.

Example 3.1.2. *(Davis et al. (1991), Nonexamples 1.22)*

Let C_k^n be an n-dimensional cyclic polytope with k vertices. This is a simplicial polytope. Let Q_k^n be the dual polytope of C_k^n. It is a simple polytope by Theorem 2.1.3. Our claim is, Q_k^n has no small cover for $n \geq 4$ and large values of k.
Let $\mathcal{F}(Q_k^n)$ be the set of facets of Q_k^n. Consider a characteristic function $\lambda : \mathcal{F}(Q_k^n) \to \mathbb{Z}_2^n$. If we take any two facets $F_1, F_2 \in \mathcal{F}(Q_k^n)$, since $F_1 \cap F_2 \neq \emptyset$, $\lambda(F_1)$ and $\lambda(F_2)$ are distinct non-zero vectors in $\mathbb{Z}_2^n$. But there is no such function for $k \geq 2^n$. So Q_k^n admits no small cover for $n \geq 4$ and $k \geq 2^n$.

Let $\Lambda(P)$ be the set of all characteristic functions over P. There is a free action of $GL(n, \mathbb{Z}_2)$ on $\Lambda(P)$ defined by $\lambda \to \sigma \circ \lambda$. Two small covers $M(\lambda)$ and $M(\beta)$ are said to be D-J equivalent if there exists a $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda = \sigma \circ \beta$.
By Theorem 3.1.1, we have the following result.

Theorem 3.1.2. *The number of D-J equivalence classes of small covers over P is $|GL(n, \mathbb{Z}_2) \backslash \Lambda(P)|$.*

Proof. Let X be the set of D-J equivalence classes of small covers over P, Y be the orbit space $GL(n, \mathbb{Z}_2) \backslash \Lambda(P)$. Define $\phi : X \to Y$ to be a function which takes $[M] \mapsto [\lambda]$ where λ is the characteristic function associated to M^n. We want to show that ϕ is a bijection.

First of all we show that ϕ is well-defined. Let M_1 and M_2 be two small covers over P such that M_1 is D-J equivalent to M_2. So there is a weakly-equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P, and there is an automorphism $\theta : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ such that $f(gm) = \theta(g)f(m)$ where $g \in \mathbb{Z}_2^n$ and $m \in M_1$. Let F be a facet of P and λ_1 and λ_2 be characteristic functions corresponding small covers M_1 and M_2. Suppose that $\lambda_i(F) = v_i$ where $< v_i >= G_F^i$ for $i = 1, 2$. Recall that G_F^i is the isotropy group of $\pi_i^{-1}(\text{int}(F))$. We claim that $\theta(v_1) = v_2$. Let $y \in \pi_2^{-1}(\text{int}(F))$ and let $x \in M_1$ such that $f(x) = y$. Then $\pi_1(x) = \pi_2(f(x)) = \pi_2(y) \in \text{int}(F)$. This means that $x \in \pi_1^{-1}(\text{int}(F))$. Then we have $\theta(v_1) \cdot y = \theta(v_1) \cdot f(x) = f(v_1 \cdot x)$. So $\theta(v_1) \cdot y$ is equal to $f(x)$ since $x \in \pi_1^{-1}(\text{int}(F))$ and v_1 fixes relative interior of F. So we have $\theta(v_1) \cdot y = f(x) = y$, this means $\theta(v_1)$ is a fixed point of $\pi_2^{-1}(\text{int}(F))$. Since θ is an isomorphism, $\theta(v_1) \neq 1$. So, $\theta(v_1) = v_2$. Therefore we have $\lambda_1(F) = \theta \circ \lambda_2(F)$. So $[\lambda_1] = [\lambda_2]$.

Now we show that ϕ is injective. Let $\phi([M_1]) = \phi([M_2])$, in other words $[\lambda_1] = [\lambda_2]$. So there is an automorphism $\theta : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ such that $\lambda_1 = \theta \circ \lambda_2$. Recall that $M(\lambda_i) = P \times \mathbb{Z}_2^n / \sim$ where $(p, g) \sim (q, h)$ if and only if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(p) =< \lambda_i(F_1), \cdots, \lambda_i(F_k) >$ where $p \in \text{ri}(F_1 \cup \cdots \cup F_k)$ for $i = 1, 2$. Define a map $\psi : M(\lambda_1) \to M(\lambda_2)$ which takes $[p, g] \mapsto [p, \theta(g)]$. Clearly, ψ covers the identity on P.

We claim that ψ is a θ-equivariant homeomorphism covering the identity. Firstly we show that ψ is well defined. Let $[p, g] = [p, h]$, that is $g^{-1}h \in \mathbb{Z}_2^1(p)$. Let $x \in \pi_2^{-1}(\text{int}(F))$, i.e, there is an element $a \in M_1$ such that $f(a) = x$. Here $a \in \pi_1^{-1}(\text{int}(F))$ as shown above. Then $\theta(g^{-1}h)x = \theta(g^{-1}h)f(a) = f((g^{-1}h)\cdot a)$ Since $g^{-1}h \in \mathbb{Z}_2^1(p)$ we have that $f((g^{-1}h)\cdot a) = f(a) = x$. So $\theta(g^{-1}h) \in \mathbb{Z}_2^2(p)$ and hence $[p, \theta(g)] = [p, \theta(h)]$ as desired.

Now we show that ψ is θ-equivariant. By definition we have $\psi(h \cdot [p, g]) = \psi([p, hg]) = [p, \theta(hg)]$. Since θ is a homomorphism, $[p, \theta(hg)] = [p, \theta(h)\theta(g)] = \theta(h)[p, \theta(g)] = \theta(h)\psi[p, g])$. Hence we have that $\psi(h[p, g]) = \theta(h)\psi[p, g]$. This shows that ψ is θ-equivariant.

Next, we show that ψ is injective, let $[p, \theta(g)] = [p, \theta(h)]$, that is, $\theta^{-1}(g)\theta(h) \in \mathbb{Z}_2(p)$. Let $x \in \pi_1^{-1}(\text{int}(F))$. Then there is an element $y \in M_2$ such that $f(x) = y$. Since θ is a homomorphism $\theta^{-1}(g)\theta(h) \in \mathbb{Z}_2(p) = \theta(g^{-1}h)$ and by assumption, we have $\theta(g^{-1}h)\cdot y = y$. Since $f(x)$ is equal to y, $\theta(g^{-1}h)\cdot y = \theta(g^{-1}h)\cdot f(x) = f(g^{-1}hx)$. So we have $f((g^{-1}h)x) = y$ and also we have $f(x) = y$. Therefore $g^{-1}hx = x$ and hence $g^{-1}h \in \mathbb{Z}_2^1(p)$. Then $[p, g] = [p, h]$, as desired. So ψ is injective. Clearly ψ is onto. Indeed, $\psi([p, \theta^{-1}(g)]) = [p, g]$ for all $[p, g] \in M(\lambda_2)$. Hence ψ is a θ-equivariant homeomorphism. Therefore $M(\lambda_1)$ and $M(\lambda_2)$ are DJ-equivalent. This shows that ϕ is injective. For any $\lambda \in Y$ there exists $[M] \in X$ such that $\phi([M(\lambda)]) = [\lambda]$. So ϕ is onto. Consequently, ϕ is a bijection. So we have $|X| = |Y|$ as desired. □

Let Aut($\mathcal{F}(P)$) be the set of automorphisms of the face poset of P. There is a right action of Aut($\mathcal{F}(P)$) on $\Lambda(P)$ by $\lambda \to \lambda \circ h$.

Theorem 3.1.3. *There is a one-to-one correspondence between the set of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P and the coset $\Lambda(P)/\text{Aut}(\mathcal{F}(P))$.*

By the above theorem, in order to find the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P, one needs to calculate the number of orbits of $\Lambda(P)$ under the action of Aut$(\mathcal{F}(P))$.

Combining Theorem 3.1.2 and Theorem 3.1.3, we have the following.

Theorem 3.1.4. *The number of weakly $\mathbb{Z}_2^n$-equiavariant homeomorphism classes of small covers over P is the size of the double coset $GL(n, \mathbb{Z}_2)\backslash\Lambda(P)/Aut(\mathcal{F}(P))$*

3.2 Small Covers over an n-cube

Let P be an n-cube, λ be a characteristic function of P and Λ be an $(n \times m)$ matrix obtained by ordering the facets of P and choosing a basis for $\mathbb{Z}_2^n$. Then

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_m)) = (A|B) \tag{3.1}$$

where A is an $(n \times n)$ matrix and B is an $(n \times (m-n))$ matrix. By Theorem 3.1.2, there is a bijection between the set of D-J equivalence classes of small covers over P and the orbit space $GL(n, \mathbb{Z}_2\backslash\Lambda(P)$. A representative of the orbit class of λ can be chosen as $\Lambda = (E_n|A^{-1}B)$ where E_n is an identity matrix of size n. Here the $(n \times m)$ matrix Λ is called the **refined form** of λ. The matrix $A^{-1}B$ is called its **reduced submatrix** and denoted by Λ_*.

In our case an n-dimensional cube has $2n$ facets. So Λ is an $(n \times 2n)$ matrix and Λ_* is an $(n \times n)$ matrix. One can reorder the facets so that the facets F_j and F_{n+j} do not intersect for $1 \leq j \leq n$.

Lemma 3.2.1. *(Choi (2008)) Every principal minor of Λ_* is* 1.

Proof. Let λ be a characteristic function with reduced submatrix $\Lambda_* = (a_{i,j})$ and P be a principal minor of Λ_* given by

$$P = \begin{bmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{bmatrix}$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Without loss of generality we can assume that

$$\lambda(F_i) = \begin{cases} e_i, & 1 \leq i \leq n \\ \sum_{k=1}^{n} a_{k,i-n} e_k, & n+1 \leq i \leq 2n \end{cases} \tag{3.2}$$

Consider the $n \times n$ matrix $P' = (b_{i,j})$ where

$$b_{i,j} = \begin{cases} a_{i,j} & j = i_t \\ \delta_{i,j} & otherwise \end{cases} \tag{3.3}$$

Here $\delta_{i,j}$ is the Kronecker Delta. Hence we have $\det(P) = \det(P')$. Since $F'_1, F'_2, \cdots, F'_n$ intersects, where $F'_i = F_{n+i}$ if $i = i_t$, and $F'_i = F_i$, otherwise determinant of P is 1. □

We denote the set of $\mathbb{Z}_2$-matrices of size n whose all principal minors are 1 with $M(n)$. Then we have a bijection between $M(n)$ and the set of D-J equivalence classes of small covers over an n-cube I^n.

Let μ be a permutation of n elements and let $P(\mu)$ be the $n \times n$ permutation matrix of μ which has 1 in entries $(\mu(i), i)$ for $1 \leq i \leq n$, and 0, otherwise.

Lemma 3.2.2. *(Masuda & Panov (2008)) Let A be an $n \times n$ matrix with entries in R where R is a commutative integral domain with identity. Suppose also that every proper principal minor of A is 1. If $A \in M(n)$, A is conjugate to a unipotent upper triangular matrix by a permutation matrix. Otherwise A is conjugate to a matrix of the form*

$$\begin{bmatrix} 1 & b_1 & 0 & \cdots & 0 \\ 0 & 1 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-1} \\ b_n & 0 & \cdots & 0 & 1 \end{bmatrix}$$

where $b_i \neq 0$ *for all* $1 \leq i \leq n$*.*

Theorem 3.2.1. *(Choi (2008), Theorem 2.2) The number of D-J equivalence classes of all small covers over an* n*-cube is equal to the number of acyclic directed graphs with* n *labeled nodes.*

Proof. Let $A(G)$ be its adjacency matrix with entries in $\mathbb{Z}_2^n$ where G be a directed graph with n labeled vertices. Define $B(G)$ by $B(G) := A(G) + E_n$.

Note that relabelling vertices is a conjugation action at $B(G)$. Since G is acyclic, $A(G)$ is conjugate to a strictly upper triangular matrix. So $B(G)$ is conjugate to an upper triangular matrix with diagonal entries are 1. Hence we have det$(B(G)) = 1$.

Let $\mathcal{G}_n$ be the set of acyclic directed graphs with n labeled vertices. Define a function $\phi : \mathcal{G}_n \to M(n)$ by $G \mapsto B(G)$. We will show that ϕ is a bijection. Let $V = \{v_1, v_2, \cdots, v_n\}$ be a set of vertices of acyclic directed graph G. For any subset V' of V, we have the induced subraph G' of G and it is also an acyclic directed graph. Suppose that V' has k elements. Then $B(G')$ is $k \times k$ submatrix of $B(G)$ whose entries b_{ij} have indices i and j are the elements of the same k element subset of $\{1, \cdots, n\}$. Thus, det$(B(G')) = 1$ for any G'. Since every principal minor of $B(G)$ can be written as $B(G')$ for some subgraph G' of G, $B(G) \in M(n)$. Thus ϕ is well-defined. Since every graph G has a unique adjacency matrix up to graph isomorphism, $G \mapsto A(G)$ is injective, so ϕ is injective too.

Take an element B of $M(n)$. If $B \in M(n)$, every diagonal entry of B is 1. Hence, there is a directed graph G such that $B(G) = B$. Since $B \in M(n)$, every principal minor of B is 1 and it is conjugated to a unipotent upper triangular matrix by Lemma

3.2.2. So G is acyclic and ϕ is surjective. Therefore, ϕ is a bijection between $\mathcal{G}_n$ and $M(n)$. $\square$

We can count the acyclic directed graphs with n labeled nodes with the following theorem.

Theorem 3.2.2. *(Robinson (1970)) Let R_n denote the number of acyclic digraphs with n labeled nodes. Then*

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}. \tag{3.4}$$

Proof. Let P_i be the number of different labeled acyclic digraphs on labeled vertices $\{1, 2, \cdots, n\}$ whose i-th vertex is an outpoint for $1 \leq i \leq n$. Then

$$R_n = \sum_{1 \leq i \leq n} |P_i| - \sum_{1 \leq i < j \leq n} |P_i \cap P_j| + \cdots \tag{3.5}$$

by the inclusion-exclusion principle. Without loss of generality, let the first k points be outpoints in $\{1, 2, \cdots, n\}$ for $1 \leq k \leq n$. The remaining $n-k$ points can be outpoints or not. But we have a subgraph induced by these $n-k$ points. So there are R_{n-k} possible labeled acyclic digraphs. Since we have k outpoints, these k points can only be adjacent to the remaining $n-k$ points. There are $2^{n(n-k)}$ possible configurations of edges. So the cardinality of $P_1 \cap P_2 \cap \cdots \cap P_n$ is $2^{n(n-k)} \cdot R_{n-k}$. Also we have $\binom{n}{k}$ different choices of different outpoints. Therefore the sum of all $1 \leq k \leq n$ gives the formula for R_n. $\square$

Theorem 3.2.3. *(Choi (2008), Theorem 3.3) Let Q_n be the number of $\mathbb{Z}_2^n$-homeomorphism classes of small covers over I^n. Then*

$$Q_n = \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} R_k}{2^n n!} \cdot \prod_{i=0}^{n-1} (2^n - 2^i) \tag{3.6}$$

Proof. Let I^n be an n-cube. We can write all elements of Aut($\mathcal{F}(I^n)$) as a composition of permutations and reflections. Indeed, every elemet of Aut($\mathcal{F}(I^n)$) is of the form

$\mu.\chi_1^{e_1}\cdots\chi_n^{e_n}$ where $\mu \in S_n$ is a permutation of the pairs of opposite facets, χ_i's are reflections which interchange the i-th opposite facets for $1 \leq i \leq n$ and $e_j \in \mathbb{Z}_2$ for $1 \leq j \leq n$. So we have $|\mathrm{Aut}(\mathcal{F}(I^n))| = 2^n.n!$.

Take an element g of Aut($\mathcal{F}(I^n)$) such that $g = \mu\chi_1^{e_1}\cdots\chi_n^{e_n}$. Let $\Lambda(I^n)^g$ be the set of elements in $\Lambda(I^n)$ fixed by g. We will show that if $\Lambda(I^n) \neq \emptyset$ then $\mu = 1$. Let $\lambda \in \Lambda(I^n)^g$ and let $\mathcal{F}(I^n) = \{F_1, F_2, ..., F_{2n}\}$ be the set of facets of I^n such that for all $i = 1, \cdots, n$ $F_i \cap F_{n+i} = \emptyset$. If we construct a matrix such that $(\lambda(F_{\epsilon(1)}), \cdots, \lambda(F_{\epsilon(n)}))$ where $\epsilon(t)$ is t or $n+t$, then the determinant of this matrix is 1 because of the non-singularity condition. This means that there is no F_i and F_j such that $\lambda(F_i) = \lambda(F_j)$ where $n \nmid i-j$. Therefore, we have $\mu = 1$.

Now, we will find the number of elements of the set $\Lambda(I^n)^g$ when $\mu = 1$. Assume that $g = \chi_1\cdots\chi_k$ for some $1 \leq k \leq n$. Let $\lambda \in \Lambda(I^n)^g$ and let $\Lambda = (A|B)$ be the refined form of λ. Note that Λ is an $n \times 2n$ matrix and $\Lambda = (A|B) = A.(I_n|\Lambda_*)$ where $\Lambda_* = A^{-1}B$ is the reduced submatrix. Note that λ is fixed by an element g if and only if the first columns of A and B are equal. Therefore, Λ_* is of the form:

$$\begin{bmatrix} I_k & S \\ 0 & T \end{bmatrix}$$

where I_k is the identity matrix of size $k \times k$, T is an $(n-k)\times(n-k)$ and S is $k\times(n-k)$ matrix.

Here, $\Lambda_* \in M(n)$ if and only if $T \in M(k)$. Therefore we have that $|\Lambda(I^n)^g| = |GL(n,\mathbb{Z}_2)| \times 2^{k.(n-k)}$. Note that $|\Lambda(I^n)^g|$ is independent of the chocie of χ_i.

Recall that the number of elements of $GL(n,\mathbb{Z}_2)$ is equal to $\prod_{i=0}^{n-1}(2^n - 2^i)$. So we have

$$Q_n = \frac{\sum_{k=0}^{n}\binom{n}{k}2^{k(n-k)}R_k}{|Aut(I^n)|}.\prod_{i=0}^{n-1}(2^n - 2^i) \tag{3.7}$$

by the Burnside's Lemma (Theorem 2.2.1) □

3.3 Small Covers over a Product of Simplices

Let $P = \Delta^{n_1} \times \cdots \times \Delta^{n_l}$ with $\sum\limits_{i=1}^{l} n_i = n$ where Δ^{n_i} is an n_i-simplex for $1 \leq i \leq l$. Then the set of facets of P is

$$\{F^i_{k_i} = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f^i_{k_i} \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_l} | 0 \leq k_i \leq n_i, 1 \leq i \leq l\}$$

where $\{f^i_0, \cdots, f^i_{n_i}\}$ is the set of facets of the n_i-simplex Δ^{n_i}. Hence the number of facets of P is $n + l$. Then the matrix Λ_* of a characteristic function λ over P is an $(n \times l)$-matrix which is not a square matrix unless $n_1 = \cdots = n_l = 1$. However we can write Λ_* as an $l \times l$ vector matrix $(v_{i,j})$ whose entries in the j-th row are vectors in $\mathbb{Z}_2^{n_j}$. Let $\Lambda_{k_1 \cdots k_l}$ denote the $l \times l$ submatrix of Λ whose j-th row is the k_j-th row of $(v_{i,j})$. Then every principal minor of $\Lambda_{k_1 \cdots k_l}$ is 1 by the non-singularity condition of Λ as shown in Lemma 3.2.1.

Theorem 3.3.1. *(Choi (2008), Theorem 2.8) The number of D-J equivalence classes over* $P = \prod\limits_{i=1}^{l} \triangle^{n_i}$ *is,*

$$\sum_{G \in \mathcal{G}_l} \prod_{v_i \in V(G)} (2^{n_i} - 1)^{outdeg(v_i)} \tag{3.8}$$

where $\mathcal{G}_l$ *is the set of acyclic digraphs with labeled nodes and* $V(G) = \{v_1, v_2, ...v_l\}$ *is the labeled vertex set of* G*.*

Proof. Let λ be a characteristic function over P and let $\Lambda_* = (v_{i,j})$ be the reduced submatrix of λ with $v_{i,j} \in \mathbb{Z}_2^{n_i}$. We construct the matrix $B(\Lambda_*) = (b_{i,j})$ corresponding $l \times l$ matrix over $\mathbb{Z}_2$ such that if $v_{i,j} \neq 0$ then the matrix has 1 in (i,j)-th positions and zeros otherwise. Since the diagonals of $\Lambda_{1,1,\cdots,1}$ is 1 for each $1 \leq i \leq l$, $b_{i,i} = 1$. Moreover, every principal minor of length 2 of $\Lambda_{k_1,\cdots,k_l}$ has determinant 1. Hence $(v_{i,i})k_i \cdot (v_{j,j})_{k_j} + (v_{i,j})_{k_i} \cdot (v_{j,i})_{k_j} = 1$. But we know that $(v_{i,i})_{k_i} = (v_{j,j})_{k_j} = 1$ since they are diagonal. So we have that, $(v_{i,j})_{k_i} \cdot (v_{j,i})_{k_j} = 0$ for each k_i and k_j. Then either $(v_{i,j})_{k_i} = 0$ for all k_i or $(v_{j,i})_{k_j} = 0$ for all k_j. In other words we have $b_{i,j} = 0$ or $b_{j,i} = 0$. Therefore $B(\Lambda_*) - I_l$ is an adjacency matrix of a directed graph. As shown before, every principal minor of $B(\Lambda_*)$ is 1.

Define a map $\psi : GL(n, \mathbb{Z}_2)\backslash\Lambda(P) \to \mathcal{G}_l$ by $\Lambda_* \mapsto G$ such that the adjacency matrix of a graph G is $B(\Lambda_*) - I_l$. Now we need to find P such that $P \cdot (B(\Lambda_*) - I_l) \cdot P^{-1}$ is a strictly upper triangular matrix to show that G is acyclic. By Lemma 3.2.2 there exists a matrix P such that $P \cdot B(\Lambda_*) \cdot P^{-1}$ is a unipotent upper triangular matrix. Then $P \cdot (B(\Lambda_*) - I_l) \cdot P^{-1} = P \cdot B(\Lambda_*) \cdot P^{-1} - I_l$. This means that $P \cdot B(\Lambda_*) \cdot P^{-1}$ is strictly upper triangular matrix. So G is acyclic by Theorem 2.3.1. Therefore we have that $|\Lambda(P)| = \sum_{G \in \mathcal{G}_l} |\psi^{-1}(G)|$. Note that if there is a directed edge from i to j in G then $v_{i,j} \in \mathbb{Z}_2^{n_i}$ is non-zero. So the non-singularity condition holds for arbitrary non-zero vectors except in diagonal entries. In other words, for each non-zero vector $v_{i,j}$ we have $2^{n_i} - 1$ choices. Therefore, we have

$$|\psi^{-1}(G)| = \prod_{e \in E(G)} (2^{n_{i(e)}} - 1) = \prod_{v_i \in V(G)} (2^{n_i} - 1)^{outdeg(v_i)} \tag{3.9}$$

where $E(G)$ is the set of directed edges in G and $i(e)$ is the label of the initial vertex of a directed edge e in $E(G)$. $\square$

The automorphism group of the set of facets of Δ_n is isomorphic to S_{n+1} which is the symmetric group of degree $n+1$. Here, it is important to know how many copy the simplices of given degree are in the product since the automorphism group of $\mathcal{F}(P)$ is depend on it. For this reason, let $P = P_1 \times \cdots \times P_l$ where $P_i = \Delta_1^{n_i} \times \cdots \times \Delta_{m_i}^{n_i}$ with $\sum_{i=1}^{l} n_i m_i = n$ and $1 \le n_1 < n_2 < \cdots < n_l$. Then the set of facets of P_i is

$$\{f_{j,k}^i = \Delta_1^{n_i} \times \cdots \times \Delta_{j-1}^{n_i} \times \tilde{f}_{j,k}^i \times \Delta_{j+1}^{n_i} \times \cdots \times \Delta_{m_i}^{n_i} | 0 \le k \le n_i, 1 \le i \le m_i\}$$

where $\{\tilde{f}_{j,0}^i, \cdots, \tilde{f}_{j,n_i}^i\}$ is the set of facets of $\Delta_j^{n_i}$. So we can write the set of facets of P as

$$\mathcal{F}(P) = \{F_{j,k}^i = P_1 \times \cdots \times P_{i-1} \times f_{j,k}^i \times P_{i+1} \times \cdots \times P_l | 0 \le k \le n_i, 1 \le j \le m_i, 1 \le i \le l\}$$

Note that the number of facets of P is $(n+m)$ where $\sum_{i=1}^{l} m_i = m$.

Recall that the automorphism group of $\mathcal{F}(\Delta^n)$ is isomorphic to the symmetric group on $n+1$ elements whichis denoted by S_{n+1}. Then Aut($\mathcal{F}(P_i)$) is the wreath product of S_{n_i+1} and S_{m_i}. As a set, Aut($\mathcal{F}(P_i)$) is the product $S_{n_i+1} \times \cdots \times S_{n_i+1} \times S_{m_i}$ and the group multiplication is given by

$$(\sigma_1, \cdots, \sigma_{m_i}, \mu)(\sigma'_1, \cdots, \sigma'_{m_i}, \mu') = (\sigma_1 \sigma'_{\mu^{-1}(1)}, \cdots, \sigma_{m_i} \sigma'_{\mu^{-1}(m_i)}, \mu\mu') \tag{3.10}$$

for $\sigma_i, \sigma'_i \in S_{n_i+1}$ and $\mu, \mu' \in S_{m_i}$. Indeed, the group multiplication in an automorphism group is given by composition and hence for any $f^i_{j,k} \in \mathcal{F}(P_i)$ we have

$$\begin{aligned} (\sigma_1, \cdots, \sigma_{m_i}, \mu)(\sigma'_1, \cdots, \sigma'_{m_i}, \mu')(f^i_{j,k}) &= (\sigma_1, \cdots, \sigma_{m_i})(f^i_{\mu'(j), \sigma'_{\mu'(j)}(k)}) && (3.11) \\ &= f^i_{\mu\mu'(j), \sigma_{\mu\mu'(j)} \sigma'_{\mu'(j)}(k)}. && (3.12) \end{aligned}$$

On the other hand, the right side of the equality gives us

$$(\sigma_1 \sigma'_{\mu^{-1}(1)}, \cdots, \sigma_{m_i} \sigma'_{\mu^{-1}(m_i)}, \mu\mu')(f^i_{j,k}) = f^i_{\mu\mu'(j), \sigma_{\mu\mu'(j)} \sigma'_{\mu^{-1}(\mu\mu'(j))}(k)}. \tag{3.13}$$

Therefore we have the following lemma:

Lemma 3.3.1. *(Altunbulak & Güçlükan İlhan (2018), Lemma 3.2) The automorphism group of $\mathcal{F}(P)$ is isomorphic to $\prod_{i=1}^{l} (S_{n_i+1} \wr S_{m_i})$ where $n_1 < n_2 < \cdots < n_l$.*

Let λ be a characteristic function of P. Then λ sends any set obtained by taking n_i-many elements from the set $\{F^i_{j,k} | 0 \leq k \leq n_i | 1 \leq j \leq m_i, 1 \leq i \leq l\}$ to a basis of $\mathbb{Z}_2^n$ by the non-singularity condition. If $n_1 > 1$ then there is more than one element from every set. Let $g \neq$ id be an element of Aut($\mathcal{F}(P)$). There exists at least two elements in $\{F^i_{j,k} | 0 \leq k \leq n_i\}$ which are not fixed by g. Assume that $g = (g_1, \cdots, g_l\}$ fixes some arbitrary characteristic function λ where $g_i = (\sigma_1, \sigma_2, \cdots, \sigma_{m_i}, \mu\}$. Since g is non-trivial, without loss of generality, we can let g_i be non-trivial. We have $g_i \lambda(f^i_{j,k}) = \lambda(g_i f^i_{j,k})$. Since g fixes λ then

$g_i\lambda(f^i_{j,k}) = \lambda(f^i_{j,k})$ and hence $\lambda(g_i f^i_{j,k}) = \lambda(f^i_{\mu(j),\sigma_{\mu(j)}(k)})$. So we have

$$\lambda(f^i_{j,k}) = \lambda(f^i_{\mu(j),\sigma_{\mu(j)}(k)}) \tag{3.14}$$

Note that, by the non-singularity condition, $\{\lambda(f^i_{j,k}), \lambda(f^i_{\mu(j),\sigma_{\mu(j)}(k)})\}$ must be linearly independent, if $n_1 > 1$. So, $\mu(j) = j, \sigma_{\mu(j)}(k) = \sigma_j(k) = k$. Since this is true for all j, k we have a contradiction. This means that g cannot fix λ. Since λ and g are arbitrarily chosen, the action of Aut($\mathcal{F}(P)$) on $\Lambda(P)$ is free. So we have the following equality for the number of $\mathbb{Z}_2^n$-homeomorphism classes of small covers over P when $n_1 > 1$:

$$\frac{|\Lambda(P)|}{|\text{Aut}(\mathcal{F}(P))|} = \frac{|\Lambda(P)|}{\prod\limits_{i=1}^{l}[(n_i+1)!]^{m_i}(m_i)!}. \tag{3.15}$$

Now we have the following theorem by using the equality above, Theorem 3.3.1 and $|GL(n,\mathbb{Z}_2)| = \prod\limits_{k=1}^{n}(2^n - 2^{k-1})$.

Theorem 3.3.2. *(Altunbulak & Güçlükan İlhan (2018), Corollary 3.3) Let* $P = \prod\limits_{i=1}^{l} \triangle_1^{n_i} \times \cdots \times \triangle_{m_i}^{n_i}$ *with* $n_1 > 1$, $\sum\limits_{i=1}^{l} m_i = m$ *and* $\sum\limits_{i=1}^{l} n_i m_i = n$. *Then the number of* $\mathbb{Z}_2^n$*-homeomorphism classes of small covers over* P *is*

$$\frac{|\Lambda(P)|}{|\text{Aut}(\mathcal{F}(P))|} = \frac{(\prod\limits_{k=1}^{n}(2^n - 2^{k-1}))(\sum\limits_{G\in\mathcal{G}_m}\prod\limits_{v_s\in V(G)}(2^{n(s)}-1)^{outdeg(v_s)})}{\prod\limits_{i=1}^{l}[(n_i+1)!]^{m_i}(m_i)!} \tag{3.16}$$

where $n : \{1, ..., m\} \to \{n_1, ...n_l\}$ *defined by* $n(s) = n_i$ *when* $n_1 + \cdots + n_{i-1} + 1 \leq s \leq n_1 + \cdots + m_i$.

When $n_1 = 1$, by using the bijection between the set of small covers over P^n and the set of acyclic digraphs, Altunbulak and Güçlükan obtains the following theorem.

Theorem 3.3.3. *(Altunbulak & Güçlükan İlhan (2018), Theorem 3.4) The number of elements of* $\Lambda(P)_{\chi_1^{\epsilon_1}\cdots\chi_{m_1}^{\epsilon_{m_1}}}$ *is equal to*

$$(\prod_{k=1}^{n}(2^n - 2^{k-1}))(\sum_{G\in\mathcal{G}_m(\epsilon_1,\cdots,\epsilon_{m_1})}\prod_{v_i\in V(G)}(2^{n_i}-1)^{outdeg(v_i)}) \tag{3.17}$$

where $\mathcal{G}_m(\epsilon_1, \cdots, \epsilon_{m_1})$ *is the set of acyclic digraphs with m-labeled nodes* $\{v_1, \cdots, v_m\}$ *such that* $indeg(v_i) = 0$ *whenever* $\epsilon_i = 1$ *for* $1 \leq i \leq m_1$.

Therefore, we have the following result.

Theorem 3.3.4. *(Altunbulak & Güçlükan İlhan (2018), Theorem 3.5) If* $n_1 = 1$ *then the number of* $\mathbb{Z}_2^n$*-homeomorphism classes of small covers over* P *is*

$$\frac{\displaystyle\sum_{(\varepsilon_1,\ldots,\varepsilon_{m_1})\in\{0,1\}^{m_1}} \sum_{G\in\mathcal{G}_m(\varepsilon_1,\ldots\varepsilon_{m_1})} \prod_{v_i\in V(G)} (2^{n_i}-1)^{outdeg(v_i)}}{\displaystyle\prod_{i=1}^{l}[(n_i+1)!]^{m_i}(m_i)!} \cdot \prod_{k=1}^{n}(2^n-2^{k-1}) \tag{3.18}$$

CHAPTER FOUR

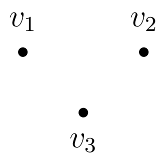
WEAKLY EQUIVARIANT CLASSIFICATIONS FOR SMALL COVERS

In this chapter, we give some calculations related to the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over some product of simplices. We also give the number of $\mathbb{Z}_2^n$-equivariant homeomorphism class of small covers over $P = I \times I \times \Delta^3$.

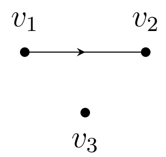
Example 4.0.1. *Let $P = I \times I \times \Delta^3$. We calculate the number of $\mathbb{Z}_2^n$-equivariant homeomorphism class of small covers over P using the formula in Theorem 3.3.4. Here, $m_1 = 2$, $m_2 = 1$, $m = 3$, $n_1 = 1$, $n_2 = 3$, $n = 5$ and $l = 2$ and hence the formula is*

$$\frac{\displaystyle\sum_{(\varepsilon_1,\varepsilon_2)\in\{0,1\}^2} \sum_{G\in\mathcal{G}_3(\varepsilon_1,\varepsilon_2)} \prod_{v_i\in V(G)} (2^{n_i}-1)^{outdeg(v_i)}}{\displaystyle\prod_{i=1}^{2}[(n_i+1)!]^{m_i}(m_i)!} \cdot \prod_{k=1}^{5}(2^5-2^{k-1}) \tag{4.1}$$

Therefore, to find the number, we need to understand the sets $\mathcal{G}_3(0,0)$, $\mathcal{G}_3(1,0)$, $\mathcal{G}_3(0,1)$ and $\mathcal{G}_3(1,1)$ first. Note that all acyclic digraphs with labeled vertices $\{v_1, v_2, v_3\}$ can be lists as below:



$Type\ 1$



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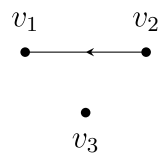

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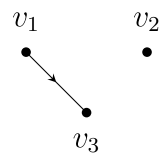

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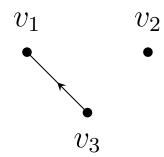
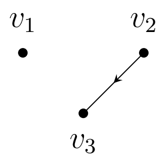


$Type\ 5$



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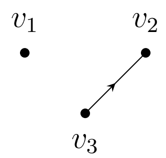
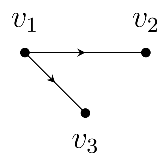


$Type\ 7$



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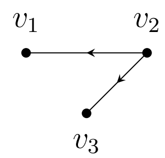

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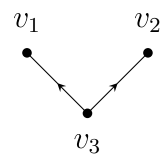

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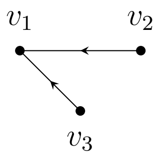

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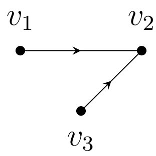

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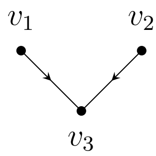

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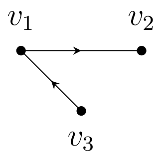

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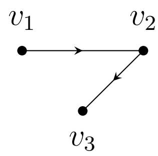

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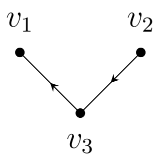

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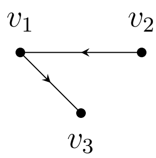

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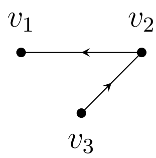

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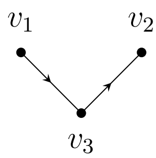

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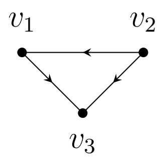

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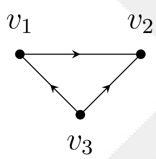
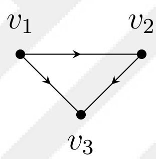


$Type\ 21$



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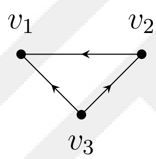

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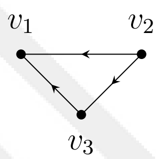
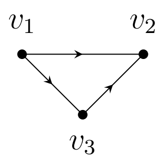


$Type\ 24$



$Type\ 25$

Let $f(G) = \prod_{v_i \in V(G)} (2^{n_i} - 1)^{outdeg(v_i)}$*. Then*

$$f(G) = \begin{cases} 1 & if\ G\ is\ of\ Type\ 1,2,3,4,6,7,8,9,13,15,17,20,22 \\ 7 & if\ G\ is\ of\ Type\ 5,11,12,14,16,18,19,24,25 \\ 49 & if\ G\ is\ of\ Type\ 10,21,23 \end{cases}$$

When $(\epsilon_1, \epsilon_2) = (0,0)$*,* $\mathcal{G}_3(\epsilon_1, \epsilon_2)$ *is indeed the set* $\mathcal{G}_3$ *of acyclic digraphs. So* $\mathcal{G}_3(0,0)$ *has 25 acycic digraph. By calculating the outdegree of each, we get*

$$\sum_{G \in \mathcal{G}_3(0,0)} \prod_{v_i \in V(G)} (2^{n_i} - 1)^{outdeg(v_i)} = 223. \tag{4.2}$$

The set $\mathcal{G}_3(1,0)$ *includes all digraphs in* $\mathcal{G}_3$ *with* $\operatorname{indeg}(v_1) = 0$ *and hence consists* *of types 1, 2, 4, 6, 7, 8, 12, 13, 15, 16, 19, 22 and 25. Therefore, we have*

$$\sum_{G\in\mathcal{G}_3(1,0)}\prod_{v_i\in V(G)}(2^{n_i}-1)^{outdeg(v_i)}=37. \tag{4.3}$$

The set $\mathcal{G}_3(0,1)$ consists of all digraphs in $\mathcal{G}_3$ with $\mathrm{indeg}(v_2)=0$. *These are the acyclic digraphs of types 1, 3, 4, 5, 6, 8, 9, 11, 13, 16, 17, 20 and 24. Therefore, we have*

$$\sum_{G\in\mathcal{G}_3(0,1)}\prod_{v_i\in V(G)}(2^{n_i}-1)^{outdeg(v_i)}=37. \tag{4.4}$$

The set $\mathcal{G}_3(1,1)$ consists of all digraphs in $\mathcal{G}_3$ with $\mathrm{indeg}(v_1)=0$ *and* $\mathrm{indeg}(v_2)=0$ *which are Types 1, 4, 6, 8, 13 and 16. Therefore, we have*

$$\sum_{G\in\mathcal{G}_3(1,1)}\prod_{v_i\in V(G)}(2^{n_i}-1)^{outdeg(v_i)}=12. \tag{4.5}$$

Therefore the number of $\mathbb{Z}_2^n$-homeomorphism class of small covers over P is

$$\frac{223+37+37+12}{192}.(31.30.28.24.16)=\frac{309}{192}.9999360=16092720 \tag{4.6}$$

Example 4.0.2. *Let $P=\Delta^2\times\Delta^2$. In this case $\mathrm{Aut}(P)=S_3\wr S_2$ and hence an arbitrary element of $\mathrm{Aut}(P)$ is of the form $g=(\sigma_1,\sigma_2,\chi)$ where $\sigma_1,\sigma_2\in S_3$ and $\chi\in S_2$. Moreover the set of facets of P is $\mathcal{F}(\Delta^2\times\Delta^2)=\{f_0^1,f_1^1,f_2^1,f_0^2,f_1^2,f_2^2\}$. As discussed before, we work with the representatives of $[\lambda]\in GL_4(\mathbb{Z}_2)\backslash\Lambda(P)$ that are of the form:*

$$\left[\begin{array}{cccc|cc} 1&0&0&0&1&c_{12}\\ 0&1&0&0&1&c_{22}\\ 0&0&1&0&c_{31}&1\\ 0&0&0&1&c_{41}&1 \end{array}\right]. \tag{4.7}$$

We first calculate the number of fixed points of $\sigma_1=(12),\sigma_2=id\in S_3$ and $\chi=id\in S_2$. In this case, we have

$$g_1 \cdot \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{12} \\ 0 & 1 & 0 & 0 & 1 & c_{22} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 1 & c_{12} \\ 1 & 0 & 0 & 0 & 1 & c_{22} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right] = \left[\begin{array}{c|c} A & B \end{array}\right]. \tag{4.8}$$

Since we want to choose a representative of the form $\left[\begin{array}{c|c} I_n & \Lambda_* \end{array}\right]$ *we need to find* A^{-1}*. Clearly we have*

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.9}$$

and hence

$$\left[\begin{array}{c|c} A & B \end{array}\right] = A \cdot \left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right] = A \cdot \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{22} \\ 0 & 1 & 0 & 0 & 1 & c_{12} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right]. \tag{4.10}$$

Therefore g fixes $[\Lambda]$ *if and only if*

$$\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{12} \\ 0 & 1 & 0 & 0 & 1 & c_{22} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right] = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{22} \\ 0 & 1 & 0 & 0 & 1 & c_{12} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right] \tag{4.11}$$

i.e, $c_{12} = c_{22}$ *. If* $c_{12} = c_{22} = 0$ *then there are 4 possibilities:* $c_{31} = 1, c_{41} = 0$*;* $c_{31} = 0, c_{41} = 1$*;* $c_{31} = 1, c_{41} = 1$ *and* $c_{31} = 0, c_{41} = 0$*. If* $c_{12} = c_{22} = 1$ *then* $c_{31} = c_{41} = 0$ *by the non-singularity condition. Hence number of fixed points of* λ *under* $g_1 = (12, id, id)$ *is* 5.

Let $g_2 = ((123), id, id)$*. Then for the representative of* $[\lambda] \in GL_4(\mathbb{Z}_2)\backslash\Lambda(P)$ *given* *above, we have*

$$g_2 \cdot \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{12} \\ 0 & 1 & 0 & 0 & 1 & c_{22} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 1 & c_{12} \\ 1 & 1 & 0 & 0 & 0 & c_{22} \\ 0 & c_{31} & 1 & 0 & 0 & 1 \\ 0 & c_{41} & 0 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A & B \end{array}\right]. \quad (4.12)$$

One can calculate A^{-1} *easily and gets*

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ c_{31} & 0 & 1 & 0 \\ c_{41} & 0 & 0 & 1 \end{bmatrix} \quad (4.13)$$

and hence

$$\left[\begin{array}{c|c} A & B \end{array}\right] = A \cdot \left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right] = A \cdot \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{12}+c_{22} \\ 0 & 1 & 0 & 0 & 1 & c_{12} \\ 0 & 0 & 1 & 0 & c_{31} & c_{31}c_{12}+1 \\ 0 & 0 & 0 & 1 & c_{41} & c_{41}c_{12}+1 \end{array}\right]. \quad (4.14)$$

So g_2 *fixes* $[\Lambda]$ *if and only if*

$$\begin{bmatrix} 1 & c_{12} \\ 1 & c_{22} \\ c_{31} & 1 \\ c_{41} & 1 \end{bmatrix} = \begin{bmatrix} 1 & c_{12}+c_{22} \\ 1 & c_{12} \\ c_{31} & c_{31}c_{12}+1 \\ c_{41} & c_{41}c_{12}+1 \end{bmatrix}. \quad (4.15)$$

The above equality holds only if $c_{12} = c_{22} = 0$*. By the non-singularity condition, there are 4 possibilities:* $c_{31} = c_{41} = 0$*;* $c_{31} = c_{41} = 1$*;* $c_{31} = 1$*,* $c_{41} = 0$*;* $c_{31} = 0$*,* $c_{41} = 1$*. Hence the number of fixed points of* λ *under* $g_2 = ((123), id, id)$ *is* 4*.*

Let $g_3 = ((123), id, (12))$*. Then we have*

$$g_3 \cdot \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & c_{12} \\ 0 & 1 & 0 & 0 & 1 & c_{22} \\ 0 & 0 & 1 & 0 & c_{31} & 1 \\ 0 & 0 & 0 & 1 & c_{41} & 1 \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 1 & c_{12} & 1 \\ 0 & 0 & 1 & 1 & c_{22} & 0 \\ 1 & 0 & 0 & c_{31} & 1 & 0 \\ 0 & 1 & 0 & c_{41} & 1 & 0 \end{array}\right] 0 = \left[\begin{array}{c|c} A & B \end{array}\right] \quad (4.16)$$

Here the inverse of A is

$$A^{-1} = \begin{bmatrix} c_{31} & 0 & 1 & 0 \\ c_{41} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (4.17)$$

and hence

$$\left[\begin{array}{c|c} A & B \end{array}\right] = A \cdot \left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right] = A \cdot \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & c_{31}c_{12}+1 & c_{31} \\ 0 & 1 & 0 & 0 & c_{41}c_{22}+1 & c_{41} \\ 0 & 0 & 1 & 0 & c_{12}+c_{22} & 1 \\ 0 & 0 & 0 & 1 & c_{12} & 1 \end{array}\right]. \quad (4.18)$$

So g_3 fixes such an element if and only if

$$\begin{bmatrix} 1 & c_{12} \\ 1 & c_{22} \\ c_{31} & 1 \\ c_{41} & 1 \end{bmatrix} = \begin{bmatrix} c_{31}c_{12}+1 & c_{31} \\ c_{41}c_{22}+1 & c_{41} \\ c_{12}+c_{22} & 1 \\ c_{12} & 1 \end{bmatrix} \quad (4.19)$$

Then we have $c_{31} = c_{41} = c_{12} = c_{22} = 0$ by the non-singularity condition. So we have only one case. This means that the number of fixed points of λ under $g_3 = ((123), id, (12))$ is 1.

One can calculate the number of fixed points of an arbitrary element of $\mathrm{Aut}(P)$ using the same idea. Indeed, the number of fixed points of g that is of the form $g = (\sigma_1, \sigma_2, \chi)$ where σ_1,$\sigma_2 \in S_3$ and $\chi \in S_2$ is given as below.

Table 4.1 Table of the number of fixed points of $\Delta^2 \times \Delta^2$

order of σ_1	order of σ_2	order of χ	the number of fixed points
1	1	1	7
2	1	1	5
1	2	1	5
3	1	1	4
1	3	1	4
2	2	1	3
3	3	1	1
2	3	1	2
3	2	1	2
1, 2, or, 3	1, 2, or, 3	2	1

We have $|S_3 \wr S_2| = 72$. Moreover, there are 3 elements of order 2 and 2 elements of order 3 in S_3. Hence the number of small covers over $\Delta^2 \times \Delta^2$ is

$$\frac{1{\cdot}7+3{\cdot}5+3{\cdot}5+2{\cdot}4+2{\cdot}4+3{\cdot}3{\cdot}3+2{\cdot}2{\cdot}1+2{\cdot}3{\cdot}2+3{\cdot}2{\cdot}2+6{\cdot}6}{72} = \frac{144}{72} = 2.$$

Example 4.0.3. *Let $P = \Delta^2 \times \Delta^2 \times \Delta^2$. In this example, we calculate the number of fixed points of some elements of* $\mathrm{Aut}(P)$. *An arbitrary element of* $\mathrm{Aut}(P) = S_3 \wr S_3$, *which is equal to $S_3 \times S_3 \times S_3 \times S_3$ as a set, is of the form $(\sigma_1, \sigma_2, \sigma_3, \chi)$ where $\sigma_1, \sigma_2, \sigma_3, \chi \in S_3$. Moreover the set of facets of P is*

$$\mathcal{F}(\Delta^2 \times \Delta^2 \times \Delta^2) = \{f_0^1, f_1^1, f_2^1, f_0^2, f_1^2, f_2^2, f_0^3, f_1^3, f_2^3\}.$$

As discussed before, an arbitrary representative of $[\lambda] \in GL_4(\mathbb{Z}_2)\backslash\Lambda(P)$ is of the form:

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & c_{12} & c_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & c_{22} & c_{23} \\ 0 & 0 & 1 & 0 & 0 & 0 & c_{31} & 1 & c_{33} \\ 0 & 0 & 0 & 1 & 0 & 0 & c_{41} & 1 & c_{43} \\ 0 & 0 & 0 & 0 & 1 & 0 & c_{51} & c_{52} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & c_{61} & c_{62} & 1 \end{array}\right] \tag{4.20}$$

To find the number of fixed points for $g_1 = ((12), (12), (123), id)$, note that g_1 sends this representative to

$$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 0 & 0 & c_{13} & 1 & c_{12} & 0 \\ 1 & 0 & 0 & 0 & 0 & c_{23} & 1 & c_{22} & 0 \\ 0 & 0 & 0 & 1 & 0 & c_{33} & c_{31} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & c_{43} & c_{41} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & c_{51} & c_{52} & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & c_{61} & c_{62} & 0 \end{array}\right] \tag{4.21}$$

which is equal to $\left[\begin{array}{c|c} A & B \end{array}\right]$. *Since we want to find a representative of this element of* *the form* $\left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right]$ *we need to find* A^{-1}. *Clearly*

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & c_{23} & 0 \\ 1 & 0 & 0 & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 1 & c_{43} & 0 \\ 0 & 0 & 1 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.22}$$

and hence $\left[\begin{array}{c|c} A & B \end{array}\right] = A \cdot \left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right]$ *is equal to*

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 + c_{51}c_{23} & c_{22} + c_{52}c_{23} & c_{23} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 + c_{51}c_{13} & c_{12} + c_{13}c_{52} & c_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 & c_{41} + c_{43}c_{51} & 1 + c_{52}c_{43} & c_{43} \\ 0 & 0 & 0 & 1 & 0 & 0 & c_{31} + c_{33}c_{51} & 1 + c_{52}c_{33} & c_{33} \\ 0 & 0 & 0 & 0 & 1 & 0 & c_{51} + c_{61} & c_{52} + c_{62} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & c_{51} & c_{52} & 1 \end{array}\right]. \tag{4.23}$$

So g_1 *fixes such an element of* $GL_n(\mathbb{Z}_2)\backslash\Lambda(P)$ *if and only if*

$$\begin{bmatrix} 1 & c_{12} & c_{13} \\ 1 & c_{22} & c_{23} \\ c_{31} & 1 & c_{33} \\ c_{41} & 1 & c_{43} \\ c_{51} & c_{52} & 1 \\ c_{61} & c_{62} & 1 \end{bmatrix} = \begin{bmatrix} 1 + c_{51}c_{23} & c_{22} + c_{52}c_{23} & c_{23} \\ 1 + c_{51}c_{13} & c_{12} + c_{13}c_{52} & c_{13} \\ c_{41} + c_{43}c_{51} & 1 + c_{52}c_{43} & c_{43} \\ c_{31} + c_{33}c_{51} & 1 + c_{52}c_{33} & c_{33} \\ c_{51} + c_{61} & c_{52} + c_{62} & 1 \\ c_{51} & c_{52} & 1 \end{bmatrix} \tag{4.24}$$

Then we have $c_{51} = c_{61} = c_{52} = c_{62} = 0$ *and* $c_{33} = c_{43}$, $c_{13} = c_{23}$, $c_{31} = c_{41}$, $c_{12} = c_{22}$ *by the non-singularity condition. So there are 12 possibilities:* $c_{12} = c_{22} = c_{31} = c_{41} = 0$, $c_{13} = c_{23} = c_{33} = c_{43} = 1$; $c_{12} = c_{22} = c_{31} = c_{41} = c_{13} = c_{23} = c_{33} = c_{43} = 0$; $c_{12} = c_{22} = c_{31} = c_{41} = c_{13} = c_{23} = 0$, $c_{33} = c_{43} = 1$; $c_{12} = c_{22} = c_{31} = c_{41} = c_{33} = c_{43} = 0$, $c_{13} = c_{23} = 1$; $c_{12} = c_{22} = c_{13} = c_{23} = c_{33} = c_{43} = 0$, $c_{31} = c_{41} = 1$; $c_{12} = c_{22} = c_{13} = c_{23} = 0$, $c_{31} = c_{41} = c_{33} = c_{43} = 1$; $c_{12} = c_{22} = c_{33} = c_{43} = 0$,

$c_{31} = c_{41} = c_{13} = c_{23} = 1$; $c_{12} = c_{22} = 0$, $c_{31} = c_{41} = c_{13} = c_{23} = c_{33} = c_{43} = 1$; $c_{12} = c_{22} = 1$, $c_{31} = c_{41} = c_{13} = c_{23} = c_{33} = c_{43} = 0$; $c_{12} = c_{22} = c_{13} = c_{23} = 1$, $c_{31} = c_{41} = c_{33} = c_{43} = 0$; $c_{12} = c_{22} = c_{33} = c_{43} = 1$, $c_{31} = c_{41} = c_{13} = c_{23} = 0$; $c_{12} = c_{22} = c_{13} = c_{23} = c_{33} = c_{43} = 1$, $c_{31} = c_{41} = 0$. *Hence the number of fixed points of* λ *under* $g_1 = ((12), (12), (123), id)$ *is 12.*

When $\chi \in S_3$ *is identity, the number of fixed points of* $g \in S_3 \wr S_3$ *is of the form* $g = (\sigma_1, \sigma_2, \sigma_3, id)$ *where* $\sigma_1, \sigma_2, \sigma_3 \in S_3$ *is given as below.*

Table 4.2 Table of the number of fixed points of $\Delta^2 \times \Delta^2 \times \Delta^2$ when $\chi = id$

the order of σ_1	the order of σ_2	the order of σ_3	the number of fixed points
1	1	1	289
2	1	1	157
1	2	1	157
1	1	2	157
3	1	1	112
1	3	1	112
1	1	3	112
2	2	1	69
2	1	2	69
1	2	2	69
3	3	1	16
3	1	3	16
1	3	3	16
2	3	1	40
3	2	1	40
3	1	2	40
2	1	3	40
3	2	1	40
2	3	1	40
2	2	2	25
3	3	3	1
2	2	3	12
2	3	2	12
3	2	2	12
2	3	3	4
3	2	3	4
3	3	2	4

Let $g_2 = (id, (12), (123), (13))$, *note that* g_2 *sends the representative to*

$$\left[\begin{array}{cccccc|ccc} 0 & c_{13} & 0 & 0 & 1 & 0 & 0 & c_{12} & 1 \\ 0 & c_{23} & 0 & 0 & 0 & 1 & 0 & c_{22} & 1 \\ 0 & c_{33} & 0 & 1 & 0 & 0 & 0 & 1 & c_{31} \\ 0 & c_{43} & 1 & 0 & 0 & 0 & 0 & 1 & c_{41} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & c_{52} & c_{51} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & c_{62} & c_{61} \end{array}\right] \tag{4.25}$$

which is equal to $\left[\begin{array}{c|c} A & B \end{array}\right]$. *Since we want to find a representative of this element of the form* $\left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right]$ *we need to find* A^{-1}. *Clearly*

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & c_{43} & 0 \\ 0 & 0 & 1 & 0 & c_{33} & 0 \\ 1 & 0 & 0 & 0 & c_{13} & 0 \\ 0 & 1 & 0 & 0 & c_{23} & 0 \end{bmatrix} \tag{4.26}$$

and hence $\left[\begin{array}{c|c} A & B \end{array}\right] = A \cdot \left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right]$ *is equal to*

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & c_{52}+c_{62} & c_{51}+c_{61} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & c_{52} & c_{51} \\ 0 & 0 & 1 & 0 & 0 & 0 & c_{43} & 1+c_{52}c_{43} & c_{41}+c_{51}c_{43} \\ 0 & 0 & 0 & 1 & 0 & 0 & c_{33} & 1+c_{52}c_{33} & c_{31}+c_{51}c_{33} \\ 0 & 0 & 0 & 0 & 1 & 0 & c_{13} & c_{12}+c_{13}c_{52} & 1+c_{13}c_{51} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1+c_{23} & c_{22}+c_{52}c_{23} & 1+c_{51}c_{23} \end{array}\right]. \tag{4.27}$$

So g_2 fixes such an element of $GL_n(\mathbb{Z}_2)\backslash\Lambda(P)$ if and only if

$$\begin{bmatrix} 1 & c_{12} & c_{13} \\ 1 & c_{22} & c_{23} \\ c_{31} & 1 & c_{33} \\ c_{41} & 1 & c_{43} \\ c_{51} & c_{52} & 1 \\ c_{61} & c_{62} & 1 \end{bmatrix} = \begin{bmatrix} 1 & c_{52}+c_{62} & c_{51}+c_{61} \\ 1 & c_{52} & c_{51} \\ c_{43} & 1+c_{52}c_{43} & c_{41}+c_{51}c_{43} \\ c_{33} & 1+c_{52}c_{33} & c_{31}+c_{51}c_{33} \\ c_{13} & c_{12}+c_{13}c_{52} & 1+c_{13}c_{51} \\ 1+c_{23} & c_{22}+c_{52}c_{23} & 1+c_{51}c_{23} \end{bmatrix} \tag{4.28}$$

The above equality holds only if $c_{31} = c_{43}$ *and all the other unknown entries are* 0 *by the non-singularity condition. So there are two possibilities:* $c_{31} = c_{43} = 0$ *and* $c_{31} = c_{43} = 1$*. Hence the number of fixed points of* λ *under* $g_2 = (1, (12), (123), (13))$ *is* 2*.*

Let $g_3 = ((12), (12), (12); (23))$*. First we note that* g_3 *sends the representative to*

$$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 1 & c_{13} & c_{12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & c_{23} & c_{22} \\ 0 & 0 & 0 & 0 & 0 & 1 & c_{31} & c_{33} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & c_{41} & c_{43} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & c_{51} & 1 & c_{52} \\ 0 & 0 & 1 & 0 & 0 & 0 & c_{61} & 1 & c_{62} \end{array}\right] \tag{4.29}$$

which is equal to $\left[\begin{array}{c|c} A & B \end{array}\right]$*. Since we work with a representative of the form* $\left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right]$*, we need to find* A^{-1}*. Clearly*

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{4.30}$$

and hence

$$\left[\begin{array}{c|c} A & B \end{array}\right] = A.\left[\begin{array}{c|c} I_n & A^{-1}B \end{array}\right] = \left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & c_{23} & c_{22} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & c_{13} & c_{12} \\ 0 & 0 & 1 & 0 & 0 & 0 & c_{61} & 1 & c_{62} \\ 0 & 0 & 0 & 1 & 0 & 0 & c_{51} & 1 & c_{52} \\ 0 & 0 & 0 & 0 & 1 & 0 & c_{41} & c_{43} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & c_{31} & c_{33} & 1 \end{array}\right]. \quad (4.31)$$

So g_3 fixes $[\Lambda]$ in $GL_n(\mathbb{Z}_2)\backslash\Lambda(P)$ if and only if

$$\begin{bmatrix} 1 & c_{12} & c_{13} \\ 1 & c_{22} & c_{23} \\ c_{31} & 1 & c_{33} \\ c_{41} & 1 & c_{43} \\ c_{51} & c_{52} & 1 \\ c_{61} & c_{62} & 1 \end{bmatrix} = \begin{bmatrix} 1 & c_{23} & c_{22} \\ 1 & c_{13} & c_{12} \\ c_{61} & 1 & c_{62} \\ c_{51} & 1 & c_{52} \\ c_{41} & c_{43} & 1 \\ c_{31} & c_{33} & 1 \end{bmatrix} \quad (4.32)$$

Then we have $c_{22} = c_{13}$, $c_{12} = c_{23}$,$c_{31} = c_{61}$, $c_{41} = c_{51}$ and $c_{62} = c_{33} = 0 = c_{43} = c_{52} = 0$ by the non-singularity condition. So there are 7 possibilities: $c_{31} = c_{41} = 0$, $c_{12} = c_{22} = 1$; $c_{31} = c_{41} = c_{22} = 0$, $c_{12} = 1$; $c_{31} = c_{41} = c_{12} = 0$, $c_{22} = 1$; $c_{31} = c_{41} = c_{12} = c_{22} = 0$; $c_{12} = c_{22} = 0$, $c_{31} = c_{41} = 1$; $c_{12} = c_{22} = c_{31} = 0$, $c_{41} = 1$; $c_{12} = c_{22} = c_{41} = 0$, $c_{31} = 1$. Hence the number of fixed points of λ under $g_3 = ((12), (12), (12); (23))$ is 7.

CHAPTER FIVE
CONCLUSION

In this thesis, we aimed to calculate the number of small covers over some specific product of simplices. After giving necessary background, we calculated the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $I \times I \times \Delta^3$. Then we calculated the number of fixed points of $GL(4,\mathbb{Z}_2)\backslash\Lambda(P)$ under different elements of Aut(P) where P is $\Delta^2 \times \Delta^2$ and we give a complete list of the number of fixed points. By using this numbers, we found the number of small covers over P. Finally we calculated the number of fixed points of $GL(6,\mathbb{Z}_2)\backslash\Lambda(P)$ under some elements of Aut(P) where P is $\Delta^2 \times \Delta^2 \times \Delta^2$.

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