ON CALCULATION OF BİLATERAL GENERATING FUNCTIONS OF SPECIAL FUNCTIONS USING SL(2) REPRESENTATIONS

by

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THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE ABANT İZZET BAYSAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS

FEBRUARY 2007

Approval of the Graduate School of Natural and Applied Sciences.

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

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ABSTRACT

ON CALCULATION OF BİLATERAL GENERATING FUNCTIONS OF SPECIAL FUNCTIONS

USING SL(2) REPRESENTATIONS

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FEBRUARY 2007, 81 pages

This thesis is using of theory of matrix representation of Lie groups and Lie algebras, particularly, representations of SL_2 using differential operators on calculations of bilateral generating functions of generalized Laguarre and some hypergeometric functions.

The thesis consists of the details of calculations of linear generating functions of special functions, with along applications of the theory of Lie groups and algebras, their representations to calculations of generating functions.

Keywords: Generalized Laguerre functions, Lie group, Lie algebra,Hypergeometric function, Casimir element.

ÖZ

SL(2) TEMSİLİNİN KULLANILMASIYLA ÖZEL FONKSİYONLARIN

İKİLİ DOĞURUCU FONKSİYONLARININ HESAPLANMASI

Atak, Öznur

Master Tezi, Matematik Bölümü

Tez Yöneticisi: Doçent Doktor Cenap ÖZEL

ŞUBAT 2007, 81 sayfa

Bu tez genelleştirilmiş Laguerre ve bazı hipergeometrik fonksiyonların çiftli üreteçlerinin hesaplanmasında Lie grup ve Lie cebirlerinin matris temsillerinin özellikle SL_2 'nin diferansiyel operatörlerle temsili kuramının kullanılması üzerinedir.

Tez, yukarıda belirtilen özel fonksiyonların doğrusal üreteç fonksiyonlarının hesaplanması ile ilgili detaylar ile beraber Lie grup ve Lie cebirlerinin ve temsillerinin kuramı ve bunların uygulamasını içermektedir.

Anahtar Kelimeler: Genelleştirilmiş Laguerre fonksiyonlar, Lie grup, Lie cebiri, Hipergeometrik fonksiyon, Casimir eleman.

ACKNOWLEDGEMENTS

I would like to acknowledge my indebtedness to my supervisor Associate Prof. Cenap Özel who has guided, encouraged and believed in me throughout this thesis. I am grateful to him for a critical reading of my studies and valuable suggestions in correcting my errors. I also like to thank to the other members of my Committee, Prof. Dr. Mahide Küçük and Assistant Prof. Erol Yılmaz for their assistance.

It is also a pleasure to acknowledge a special dept of my gratitude to all the professors who gave me help in my master studies.

Finally, I would like to thank to my family for their unvanishing support and encouragement.

To My Family

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CHAPTER 1

INTRODUCTION.

In [18] Weisner has devised a method for obtaining bilateral generating functions for sets of functions which satisfies certain conditions. Among the functions which do satisfy these conditions are special functions such as Hermite, Bessel, generalized Laguerre, and Gegenbauer. In this thesis we will study on only Laguerre and generalized Laguerre functions which are solutions of Laguerre and generalized Laguerre differential equations respectively.

From the ordinary differential equation which is satisfied by the set of functions under consideration a partial differential equation is constructed. The method is based on finding a nontrivial continuous group of transformations under which this partial differential equation is invariant. In this thesis we consider only the special linear group $SL(2)$ with determinant 1.

Some comments about the structure of this thesis are in order. It is written for a reader with a first course in matrix groups and some understanding of the structure of compact semi-simple Lie groups and their representations, plus some mathematical maturity. Some good general references are Rainville [14] and Srivistava & Monacha [16] for special functions and their generating functions, Humpreys [9] for Lie algebras and representations, Baker [2] for matrix groups.

The organization of this thesis is as follows.

In Chapter 2, we define some special functions like simple Laguerre, generalized Laguerre and hypergeometric after then we obtain their generating functions using some manipulations on series.

Chapter 3 is about the theory of Lie algebras and their representations on vector spaces.

Chapter 4 includes descriptions of manifolds, tangent space, Lie groups, one-parameter groups, adjoint derivative and differentials.

Chapter 5 is using of theory of matrix representation of Lie groups and Lie algebras, particularly, representations of SL_2 using differential operators on calculations of bilateral generating functions of generalized Laguerre and some hypergeometric functions.

CHAPTER 2

A BACKGROUND ON GENERATING FUNCTIONS

In this chapter, we define some special functions like simple Laguerre, generalized Laguerre and hypergeometric after then we obtain their generating functions using some manipulations on series.

Definition 2.1 (***Linear Generating Functions***) *If a two-variable function $F(x,t)$ possesses a formal power series expansion in t such that*

$$F(x,t)=\sum_{n=0}^{\infty} C_n f_n(x) t^n$$

where C_n is a function of n that may contain the parameters of the set $\{f_n(x)\}$ but is independent of x and t, then $F(x,t)$ is called a generating function *for the set $\{f_n(x) : n \in \mathbb{N}\}$.*

Definition 2.2 (***Bilinear Generating Functions***) *If a three-variable function $F(x,y,t)$ possesses a formal power series expansion in t such that*

$$F(x,y,t)=\sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n$$

where the sequence $\{\gamma_n\}$ is independent of x, y and t, then $F(x,y,t)$ is called a bilinear generating function *for the set $\{f_n(x) : n \in \mathbb{N}\}$.*

Definition 2.3 (***Bilateral Generating Functions***) *Suppose that a three-variable function $H(x,y,t)$ possesses a formal power series expansion in t such that*

$$H(x,y,t)=\sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n$$

where h_n is independent of x, y and t, and the set of functions $\{f_n(x) : n \in \mathbb{N}\}$ and $\{g_n(x) : n \in \mathbb{N}\}$ are different. Then $H(x,y,t)$ is called a bilateral generating function *for the set $\{f_n(x) : n \in \mathbb{N}\}$ or $\{g_n(x) : n \in \mathbb{N}\}$.*

Definition 2.4 (***Multivariable Generating Functions***) *If $G(x_1, \ldots, x_r; t)$ is a function of $r+1$ variables, possesses a formal expansion in powers of t such that*

$$G(x_1, \ldots, x_r; t) = \sum_{n=0}^{\infty} C_n g_n(x_1, \ldots, x_r) t^n$$

where C_n is independent of the variables $x_1, \ldots, x_r$ and t, then we will say that $G(x_1, \ldots, x_r; t)$ is a multivariable generating function *for the set $\{g_n(x_1, \ldots, x_r) : n \in \mathbb{N}\}$ corresponding to the non-zero coefficients C_n.*

Definition 2.5 (***Multilinear and Multilateral Functions***) *A multi-variable generating function $G(x_1, \ldots, x_r; t)$ is said to be a* multilinear generating function *if*

$$g_n(x_1, \ldots, x_r) = f_{\alpha_{1(n)}}(x_1), \ldots, f_{\alpha_{r(n)}}(x_r)$$

where $\alpha_{1(n)}, \ldots, \alpha_{r(n)}$ are functions of n which are not necessarily equal. More generally, if the functions $f_{\alpha_{i(n)}}, i = 1, 2, \ldots, r$ are all different, in this case $G(x_1, \ldots, x_r; t)$ will be called a multilateral generating function.

2.1 On Pochhammer symbol and Hypergeometric functions.

Definition 2.6 *For any number a,* Pochhammer symbol *$(a)_n$, is defined by*

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1), n \geq 1$$

and $(a)_0 = 1$ for $a \neq 0$.

The Pochhammer symbol is an extension of the ordinary factorial since $(1)_n = n!$. By using the definition of the gamma function, we have the following equality

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, n \in \mathbb{N}$$

where a is a non-negative integer.

Lemma 2.7 (***Duplication formula***) *For $n \in \mathbb{N}$, we have*

$$(a)_{2n} = 2^{2n}(\frac{a}{2})_n(\frac{a+1}{2})_n.$$

Proof. By definition we have

$$
\begin{aligned}
(a)_{2n} &= a(a+1)(a+2)\dots(a+2n-2)(a+2n-1)\\
&= [a(a+2)\dots(a+2n-2)][(a+1)(a+3)\dots(a+2n-1)]\\
&= 2^n[(\frac{a}{2})(\frac{a}{2}+1)(\frac{a}{2}+2)\dots(\frac{a}{2}+n-1)]\\
&2^n[(\frac{a+1}{2})(\frac{a+1}{2}+1)(\frac{a+1}{2}+2)\dots(\frac{a+1}{2}+n-1)]\\
&= 2^{2n}(\frac{a}{2})_n(\frac{a+1}{2})_n.
\end{aligned}
$$

□

Lemma 2.8 *The Pochhammer symbol $(a)_n$ satisfies the identity*

$$
(a)_{n-k} = \frac{(a)_n}{(-1)^k(1-a-n)_k}.
$$

Proof. By definition we have

$$
\begin{aligned}
(a)_{n-k} &= \frac{[a(a+1)...(a+n-k-1)][(a+n-k)...(a+n-1)]}{(a+n-k)...(a+n-1)},\ k \leq n\\
&= \frac{(a)_n}{[-(k-a-n)][-(k-a-n-1)]...[-(2-a-n)][-(1-a-n)]}\\
&= \frac{(a)_n}{(-1)^k[(1-a-n)(1-a-n+1)...(1-a-n+k+1)]}\\
&= 2^{2n}(\frac{a}{2})_n(\frac{a+1}{2})_n.
\end{aligned}
$$

□

Lemma 2.9 *Let c be a reel or complex number, and let n and k be positive integers. Then we have*

$$
\frac{(c)_{n+k}}{(c)_n} = (c+n)_k.
$$

Proof. By a simple calculation we have

$$\begin{aligned}\frac{(c)_{n+k}}{(c)_n} &= \frac{c(c+1)...(c+n-1)(c+n)(c+n+1)...(c+n+k-1)}{c(c+1)...(c+n-1)} \\ &= (c+n)[(c+n)+1]...[(c+n)+k-1] \\ &= (c+n)_k.\end{aligned}$$

□

Definition 2.10 *Suppose that* $\alpha_1, \dots, \alpha_r$ *and* $\gamma_1, \dots, \gamma_s$ *are reel or complex constants. Then the function*

$$F(\alpha_1, \dots, \alpha_r; \gamma_1, \dots, \gamma_s; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdot \dots \cdot (\alpha_r)_n}{(\gamma_1)_n \cdot \dots \cdot (\gamma_s)_n} \frac{x^n}{n!}$$

is called hypergeometric function *and it is denoted by symbol* $F = {}_rF_s$ *where the* r *and* s *refer to the number of parameters in numerator and denominator, respectively.*

2.2 Manipulations on series.

One of the most effective methods for obtaining generating function is series manipulation. In this section by using this method we will obtain generating functions for some special functions. First we will give some useful lemmas involving power series.

Lemma 2.11 *Let* m *be arbitrary positive integer. Then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m \rfloor} A(k, n-mk)$$

where the symbol $\lfloor n/m \rfloor$ *denotes the greatest integer less than or equal to* n/m.

Proof. Let consider the following double series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)t^{n+mk}.$$

In the last double series let us define $k = j$, $n + mk = N$. Then we have $n = N - mj$. Since $k \geq 0$ and $n \geq 0$, we can take $j \geq 0$. Therefore we have $N - mj \geq 0 \Rightarrow N \geq mj \geq 0 \Rightarrow N \geq 0$ and

$N - mj \geq 0 \Rightarrow N \geq mj \geq 0 \Rightarrow 0 \leqq j \leqq N/m.$

Since $0 \leq j \geq N/m$ and j is an integer, the new summation index j can run from 0 to the greatest integer in N/m. Thus we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)t^{n+mk} = \sum_{N=0}^{\infty}\sum_{j=0}^{\lfloor N/m \rfloor} A(j, N-mj)t^{N}.$$

In this equation, if we put $t = 1$ and replace the indices j and N on the right by the indices K and n, respectively, then we arrive at the desired result. □

Now we can give the following results as the applications of the last lemma just taking $m = 1$ and $m = 2$ respectively.

Corollary 2.12 *For $n \in \mathbb{N}$ we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k).$$

Corollary 2.13

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2 \rfloor} A(k, n-2k).$$

Lemma 2.14

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+mk) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m \rfloor} B(k,n).$$

Proof. In the last corollary let us take $k = j$, $n = N - mk$. Then $n = N - mj$ and we have $0 \leq k < \infty$ and $0 \leq n < \infty$ from Corollary 2.13. If we replace new indices $0 \leq j < \infty$ and $0 \leq N - mj < \infty$, we get $mj \leq N$ and $0 \leq j \leq N/m$. Thus we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+mk) = \sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor n/m \rfloor} B(j, N-mj+mj) = \sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor n/m \rfloor} B(j,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m \rfloor} B(k,n).$$

If we are interchanging the indices j and N on the right by the dummy indices k and n, respectively then we arrive the desired result in the following

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+mk) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m \rfloor} B(k,n).$$

□

If a set of functions $\{f_n(x) : n \in \mathbb{N}\}$ is defined by a series representation of the form

$$f_n(x) = \sum_{k=0}^{n} G(k, x)$$

then it is generally possible to use series manipulation method to find a generating function $F(x, t)$ such that

$$\sum_{n=0}^{\infty} c_n f_n(x) t^n = \sum_{n=0}^{\infty} c_n \left[\sum_{k=0}^{n} G(k, x) \right] t^n = F(x, t).$$

As an example let us consider simple Laguerre polynomials $L_n(x)$ so that it has the series representation

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!}.$$

Since n is independent of k, the series representation can be written as

$$\frac{L_n(x)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k x^k}{(k!)^2 (n-k)!}.$$

Now we will find a generating function for the sets $\{\frac{L_n(x)}{n!} : n \in \mathbb{N}\}$ and $\{L_n(x) : n \in \mathbb{N}\}$ respectively.

By definition, the generating function for the set of Laguerre polynomials $\{L_n(x) : n \in \mathbb{N}\}$ is

$$\sum_{n=0}^{\infty} L_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!} t^n.$$

By Lemma 2.14, we obtain

$$\sum_{n=0}^{\infty} L_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)! x^k}{(k!)^2 (n-k)!} t^{n+k}.$$

By interchanging the order of summation, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n(x) t^n &= \sum_{k=0}^{\infty} \frac{(-1)^k (xt)^k}{k!} \sum_{n=0}^{\infty} \frac{(1+k)_n t^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!} (1-t)^{-1-k} \\
&= (1-t)^{-1} \exp\{\frac{-xt}{1-t}\}
\end{aligned}$$

because $(1-t)^{-1}$ is independent of k, and $\frac{(n+k)}{k!} = (1+k)_n$.

Next we seek generating function for the set $\{\frac{L_n(x)}{n!} : n \in \mathbb{N}\}$.

Since

$$\begin{aligned} L_n{}^{\alpha}(x) &= \sum_{k=0}^{n} \frac{(-1)^k(1+\alpha)_n x^k}{(k!)(n-k)!(1+\alpha)_k} \\ &= \frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n, \alpha+1, x), \end{aligned}$$

if we take $\alpha = 0$, we obtain a series representation of $L_n(x)$,

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k}{(k!)(n-k)!} t^n.$$

By Lemma 2.14 we obtain

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k}{(k!)^2 n!} t^{n+k}.$$

Since the factors depending on n can be separated from those depending on k, we have

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(k!)^2}.$$

So we have established the generating relation

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n = e^t F_1(-;1;-xt)$$

which is exponential generating function of $L_n(x)$.

Let $G(x,t)$ be a generating function as follows:

$$G(x,t) = \sum_{n=0}^{\infty} a_n f_n(x) t^n.$$

For a given generating function $G(x,t)$, we can generalize it as

$$K(x,t,k) = \sum_{n=0}^{\infty} b(n,k) f_{n+k}(x) t^n$$

where $K(x,t,0) = G(x,t)$ and $b(n,0) = a_n$.

If we take $t+v$ in place of t in the last equation then we have

$$G(x, t+v) = \sum_{n=0}^{\infty} a_n f_n(x)(t+v)^n.$$

By binomial expansion

$$(t+v)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} v^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} t^{n-k} v^k,$$

$$\begin{aligned}
G(x,t) &= \sum_{n=0}^{\infty} a_n f_n(x)(t+v)^n \\
&= \sum_{n=0}^{\infty} a_n f_n(x) \sum_{k=0}^{\infty} \frac{n!}{(n-k)!k!} t^{n-k} v^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n f_n(x) \frac{n!}{(n-k)!k!} t^{n-k} v^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n+k} f_{n+k}(x) \frac{(n+k)!}{n!k!} t^n v^k \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+k)! a_{n+k} f_{n+k}(x)}{n!k!} t^n v^k \\
&= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(n+k)! a_{n+k} f_{n+k}(x)}{n!k!} \right] t^n v^k.
\end{aligned}$$

We will illustrate this technique in the following example. First we begin with a lemma which gives generating function of the set of generalized Laguerre functions $\{{L_n}^{\alpha}(x) : n \in \mathbb{N}\}$.

Lemma 2.15

$$\sum_{n=0}^{\infty} {L_n}^{\alpha}(x) t^n = \frac{1}{(1-t)^{1+\alpha}} \exp\left\{ \frac{-xt}{1-t} \right\}.$$

Lemma 2.16

$$\sum_{k=0}^{\infty} \left[(1-t)^{-1-\alpha-k} \exp\left\{ \frac{-xt}{1-t} \right\} {L_k}^{\alpha}\left(\frac{x}{1-t} \right) \right] v^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} {L_n}^{\alpha}(x) t^{n-k} v^k.$$

Proof. In the following equation

$$\sum_{n=0}^{\infty} {L_n}^{\alpha}(x) t^n = \frac{1}{(1-t)^{1+\alpha}} \exp\left\{ \frac{-xt}{1-t} \right\} \tag{2.1}$$

if we replace x by $\frac{x}{1-t}$ and t by $\frac{v}{1-t}$ respectively, then we have

$$\sum_{k=0}^{\infty} L_k{}^{\alpha}\left(\frac{x}{1-t}\right)(1-t)^{-k}v^k = \frac{1}{\left(1-\frac{v}{1-t}\right)^{1+\alpha}}\exp\left\{\frac{\frac{-x}{1-t}\frac{v}{1-t}}{1-\frac{v}{1-t}}\right\}$$
$$= (\frac{1-t-v}{1-t})^{-1-\alpha}\exp\left\{\frac{-xv}{(1-t)(1-t-v)}\right\}$$
$$= \frac{(1-t-v)^{-1-\alpha}}{(1-t)^{-1-\alpha}}\exp\left\{\frac{-xv}{(1-t)(1-t-v)}\right\}.$$

Multiplying two-sides of the last equation with $(1-t)^{-1-\alpha}\exp\left\{\frac{-xt}{(1-t)}\right\}$, we obtain

$$\sum_{k=0}^{\infty}\left[(1-t)^{-1-\alpha-k}\exp\left\{\frac{-xt}{1-t}\right\}L_k{}^{\alpha}\left(\frac{x}{1-t}\right)\right]v^k = (1-t-v)^{-1-\alpha}\exp\left\{\frac{-x(v+t)}{1-t-v}\right\}.$$

If we replace t by $v+t$ in the right side of the last equation, we obtain right-side of 2.1

$$\sum_{k=0}^{\infty}\left[(1-t)^{-1-\alpha-k}\exp\left\{\frac{-xt}{1-t}\right\}L_k{}^{\alpha}\left(\frac{x}{1-t}\right)\right]v^k = \sum_{n=0}^{\infty}L_n{}^{\alpha}(x)(v+t)^n.$$

By binomial expansion,

$$\sum_{k=0}^{\infty}\left[(1-t)^{-1-\alpha-k}\exp\left\{\frac{-xt}{1-t}\right\}L_k^{\alpha}\left(\frac{x}{1-t}\right)\right]v^k = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}L_n{}^{\alpha}(x)t^{n-k}v^k$$

which is the desired result. □

Lemma 2.17 *Generalized Laguerre polynomials satisfy the following equation.*

$$\sum_{n=0}^{\infty}\frac{L_n^{\alpha}(x)}{(1+\alpha)_n}t^n = e^t F_1(-;1+\alpha;-xt)$$

where $e^t F_1(-;1+\alpha;-xt)$ *is called exponential generating function for the generalized Laguerre polynomials.*

Proof. By Taylor series explanation of e^t, we have

$$
\begin{aligned}
e^t F_1(-;1+\alpha;-xt) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{n!(1+\alpha)_k} x^k t^k \right] \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{n!k!(1+\alpha)_k} x^k t^{n+k}.
\end{aligned}
$$

By Lemma 2.14, we have

$$
\begin{aligned}
e^t F_1(-;1+\alpha;-xt) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k t^n}{(n-k)!k!(1+\alpha)_k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{(n-k)!k!(1+\alpha)_k} \frac{t^n}{(1+\alpha)_n} \\
&= \sum_{n=0}^{\infty} \frac{{L_n}^{(\alpha)}(x)}{(1+\alpha)_n} t^n.
\end{aligned}
$$

□

2.3 Calculation of bilinear generating functions of generalized Laguerre polynomials.

In this section we will show by an example how a linear generating functions can be extended to a bilinear generating function.

By Lemma 2.16, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} {L_n}^{\alpha}(x) {L_n}^{\alpha}(y) t^n &= \sum_{k=0}^{\infty} \left[(1-t)^{-1-\alpha-k} \exp\left\{ \frac{-xt}{1-t} \right\} L_k^{\alpha}\left(\frac{x}{1-t} \right) \right] \frac{(-1)^k y^k t^n}{(1+\alpha)_k} \\
&= (1-t)^{-1-\alpha} \exp\left\{ \frac{-xt}{1-t} \right\} \sum_{k=0}^{\infty} \frac{L_k^{\alpha}\left(\frac{x}{1-t} \right)}{(1+\alpha)_k} \left(\frac{-yk}{1-t} \right)^k.
\end{aligned}
$$

By Lemma 2.17 and replacing x by $\dfrac{x}{1-t}$ and t by $\dfrac{-yk}{1-t}$, we get

$$
\sum_{k=0}^{\infty} \frac{{L_k}^{\alpha}\left(\frac{x}{1-t} \right)}{(1+\alpha)_k} \left(\frac{-yk}{1-t} \right)^k = \exp\left\{ \frac{-yk}{1-t} \right\} F_1(-;1+\alpha; \frac{x}{1-t} \frac{yk}{1-t}).
$$

Hence we obtain

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} {L_n}^{\alpha}(x) {L_n}^{\alpha}(y) t^n = (1-t)^{-1-\alpha} \exp\left\{\frac{-xt}{1-t}\right\} \exp\left\{\frac{-yt}{1-t}\right\} F_1(-; 1+\alpha; \frac{xyt}{(1-t)^2})$$
$$= (1-t)^{-1-\alpha} \exp\left\{\frac{-t(x+y)}{1-t}\right\} F_1(-; 1+\alpha; \frac{xyt}{(1-t)^2})$$

which is a bilinear generating function for the generalized Laguerre polynomials.

2.4 Calculation of bilateral generating functions of some special functions.

To find a bilateral generating function for some set $\{f_n(x) : n \in \mathbb{N}\}$ we need a linear generating function of the following type

$$K(x,t,k) = \sum_{n=0}^{\infty} a_{(n,k)} f_{n+k}(x) t^n.$$

We also need another generating function

$$H(x,t) = \sum_{k=0}^{\infty} b_k f_k(x) t^k$$

such that $H(x,t) \neq K(x,t,0)$. We can transform $H(x,t)$ into $J(x,t,y)$ where

$$H(y,t) = \sum_{k=0}^{\infty} b_k f_k(x) t^k$$
$$= \sum_{k=0}^{\infty} c_k y^k t^k$$

and

$$
\begin{aligned}
J(x,t,y) &= \sum_{k=0}^{\infty} c_k y t^k K(x,t,k) \\
&= \sum_{k=0}^{\infty} c_k y t^k \sum_{n=0}^{\infty} a_{(n,k)} f_{n+k}(x) t^n \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{(n,k)} c_k f_{n+k}(x) y^k t^{n+k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{(n-k,k)} c_k f_n(x) y^k t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{(n-k,k)} c_k y^k f_n(x) t^n \\
&= \sum_{n=0}^{\infty} d_n f_n(x) g_n(y) t^n.
\end{aligned}
$$

Now we will give a result involving calculation of bilateral generating function of a given hypergeometric function ${}_1F_1(-n,c;1+\alpha;y)$ with generalized Laguerre polynomials.

Proposition 2.18 *For a given hypergeometric function ${}_1F_1(-n,c;1+\alpha;y)$, we have*

$$
(1-t)^{-1+c-\alpha}(1-t+yt)^{-c} \exp\left\{\frac{-xt}{1-t}\right\} {}_1F_1\left(c;1+\alpha;\frac{xyt}{(1-t)(1-t+yt)}\right) = \sum_{n=0}^{\infty} {}_1F_1(-n,c;1+\alpha;y) {L_n}^{\alpha}(x) t^n.
$$

Proof. We know that the generalized Laguerre polynomial is

$$
L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n}{k!(n-k)!(1+\alpha)_k} x^k.
$$

For an arbitrary fixed number c, first we multiply both sides of the last equation with $\dfrac{c_n}{(1+\alpha)_n} t^n$ and after then let us take summation on $n \in \mathbb{N}$. Hence we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{c_n L_n^{\alpha}(x)}{(1+\alpha)_n} t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{c_n}{k!(n-k)!(1+\alpha)_k} (-x)^k t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{c_{n+k}}{k!n!(1+\alpha)_k} (-x)^k t^{n+k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c+k)_n t^n}{n!} \frac{c_k (-xt)^k}{k!(1+\alpha)_k}
\end{aligned}
$$

by Lemma 2.14 and 2.9.

Using binomial expansion $\displaystyle\sum_{n=0}^{\infty} \frac{(c+k)_n t^n}{n!} = (1-t)^{-c-k}$, we obtain

$$\sum_{n=0}^{\infty} \frac{c_n {L_n}^{\alpha}(x)}{(1+\alpha)_n} t^n = \sum_{k=0}^{\infty} \frac{c_k(-xt)^k}{k!(1+\alpha)_k(1-t)^{c+k}}$$
$$= \frac{1}{(1-t)^c} {}_1F_1\left(c; 1+\alpha; \frac{-xt}{1-t}\right).$$

By replacing x by $\dfrac{x}{1-t}$ and t by $\dfrac{-yt}{1-t}$, we find

$$\sum_{k=0}^{\infty} \frac{c_k {L_k}^{\alpha}\left(\frac{x}{1-t}\right)\left(\frac{-yt}{1-t}\right)^k}{(1+\alpha)_k} = \frac{1}{(1+\frac{yt}{1-t})^c} {}_1F_1\left(c; 1+\alpha; \frac{xyt(1-t)^{-2}}{1+yt(1-t)^{-1}}\right).$$

If we multiply both sides with $(1-t)^{-1-\alpha} \exp\left\{\dfrac{-xt}{1-t}\right\}$, we find

$$(1-t)^{-1-\alpha}(1+\frac{yt}{1-t})^{-c} \exp\left\{\frac{-xt}{1-t}\right\} {}_1F_1\left(c; 1+\alpha; \frac{xyt(1-t)^{-2}}{1+yt(1-t)^{-1}}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{c_k(1-t)^{-1-\alpha} \exp\left\{\frac{-xt}{1-t}\right\} {L_k}^{\alpha}\left(\frac{x}{1-t}\right)(-1)^k y^k t^k}{(1+\alpha)_k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{c_k(n+k)! {L_{n+k}}^{\alpha}(x) t^n (-1)^k y^k t^k}{k!n!(1+\alpha)_k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_k n! L_n^{\alpha}(x) t^{n-k}(-1)^k y^k t^k}{k!(n-k)!(1+\alpha)_k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k c_k y^k L_n^{\alpha}(x) t^n}{k!(1+\alpha)_k}$$
$$= \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n \sum_{k=0}^{n} \frac{(-n)_k c_k y^k}{k!(1+\alpha)_k}$$
$$= \sum_{n=0}^{\infty} F_1(-n, c; 1+\alpha; y) L_n^{\alpha}(x) t^n$$

which is a bilateral generating function for the set ${}_1F_1(-n, c; 1+\alpha; y)$. $\square$

Now we will find a bilateral generating function for the simple Laguerre polynomials $L_n(x)$. In Lemma 2.17 if we replace t by $\dfrac{yt}{1-t}$ and x by $\dfrac{x}{1-t}$, we obtain

$$\exp\left\{\frac{yt}{1-t}\right\} F_1\left(-; 1; \frac{-xyt}{(1-t)^2}\right) = \sum_{k=0}^{\infty} \frac{L_k\left(\frac{x}{1-t}\right)}{k!}\left(\frac{yt}{1-t}\right)^k.$$

Let us multiply both sides of the last equality with $(1-t)^{-1}\exp\left\{\frac{-xt}{1-t}\right\}$, we find

$$\begin{aligned}(1-t)^{-1}\exp\left\{\frac{-xt}{1-t}\right\}\exp\left\{\frac{yt}{1-t}\right\}F_1\left(-;1;\frac{-xyt}{(1-t)^2}\right)&\\ =\sum_{k=0}^{\infty}[(1-t)^{-1}\exp\left\{\frac{-xt}{1-t}\right\}\frac{L_k(\frac{x}{1-t})}{k!}\frac{(yt)^k}{(1-t)^k}&\\ =\sum_{k=0}^{\infty}\left[(1-t)^{-1-k}\exp\left\{\frac{-xt}{1-t}\right\}L_k\left(\frac{x}{1-t}\right)\frac{(yt)^k}{k!}\right]&\\ =\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+k)!L_{n+k}(x)t^n}{k!n!}\frac{y^kt^k}{k!}.&\end{aligned}$$

By replacing y by $-y$, we have

$$\begin{aligned}(1-t)^{-1}\exp\left\{\frac{-t(x+y)}{1-t}\right\}F_1(-;1;\frac{xyt}{(1-t)^2})&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+k)!L_{n+k}(x)t^{n-k}}{k!n!}\frac{(-y)^kt^k}{k!}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!L_n(x)}{k!^2(n-k)!}(-1)^ky^k\\ &=\sum_{n=0}^{\infty}L_n(x)\left(\sum_{k=0}^{n}\frac{n!(-1)^k}{k!^2(n-k)!}y^k\right)t^n\\ &=\sum_{n=0}^{\infty}L_n(x)L_n(y)t^n\end{aligned}$$

which is a special case of the bilinear generating function for set of Laguerre functions $\{L_n(x):n\in\mathbb{N}\}$.

If we use some technique to transform

$$(1-t)^{-c}F_1\left(c;1;\frac{-xt}{1-t}\right)=\sum_{k=0}^{\infty}\frac{c_kL_k(x)t^k}{k!},$$

we obtain bilateral generating relation

$$(1-t)^{-1+c}(1-t+yt)^{-c}\exp\left\{\frac{-xt}{1-t}\right\}F_1\left(c;1;\frac{xyt}{(1-t)(1-t+yt)}\right)=\sum_{n=0}^{\infty}F_1(-n,c;1;y)L_n(x)t^n.$$

CHAPTER 3
SEMI-SIMPLE LIE ALGEBRAS AND THEIR REPRESENTATIONS

The general reference for this chapter is [9].

Lie algebras arise "in nature " as vector spaces of linear transformations endowed with a new operation bracket $[x, y] = xy - yx$ which is in general neither commutative nor associative where the operations on the right side are the usual ones.

Definition 3.1 *A vector space L over a field $\mathbb{F}$, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ and called* bracket or commutator *of x and y, is called a* Lie algebra *over $\mathbb{F}$ if the following axioms are satisfied.*

(L1) The bracket operation is bilinear.

(L2) $\forall x \in L, [x, x] = 0.$

(L3) $\forall x, y, z \in L, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity).*

We note that if $(L1)$ and $(L2)$ are applied to $[x + y, x + y]$, then we have anti-commutativity $[x, y] = -[y, x]$. Conversely, if $\operatorname{char} \mathbb{F} \neq 2$, it is clear that the anti-commutativity will imply $(L2)$.

We say that two Lie algebras L, L' over $\mathbb{F}$ are *isomorphic* if there exists a vector space isomorphism $\phi : L \to L'$ satisfying $\phi([xy]) = [\phi(x), \phi(y)]$ for all $x, y \in L$ where ϕ is called an *isomorphism of Lie algebra.*

A subspace K of L is called a *subalgebra* if $[x, y] \in K$ whenever $x, y \in K$; in particular, K is a Lie algebra in its own right relative to the inherited operations. Note that any nonzero element $x \in L$ defines a one dimensional subalgebra $\mathbb{F}x$, with trivial multiplication, because of $(L2)$.

Let V be a finite dimensional vector space over $\mathbb{F}$, and let $\hom(V)$ be the set of linear transformations $V \to V$. As a vector space over $\mathbb{F}$, $\hom(V)$ has dimension n^2 where $n = \dim V$, and $\hom(V)$ is a ring relative to the usual product operation *composition.* We define a new operation $[x, y] = xy - yx$, called the bracket of x and y. With this operation $\hom(V)$

becomes a Lie algebra over $\mathbb{F}$. In order to distinguish this new algebra structure from the old associative one, we write $\mathfrak{g}l$ for $\hom(V)$ viewed as a Lie algebra and call it *general linear algebra* because it is closely associated with the general linear group $GL(V)$ consisting of all invertible endomorphisms of V.

Definition 3.2 *Any subalgebra of a Lie algebra $\mathfrak{g}l(V)$ is called a linear Lie algebra.*

If we fix a basis for V, then we are identifying $\mathfrak{gl}(V)$ with the set of all nxn matrices over $\mathbb{F}$, denoted $\mathfrak{gl}(n, \mathbb{F})$. For the standard basis consisting of the matrices $e_{ij} = [E_{\mu\nu}]$ where

$$E_{\mu\nu} = \begin{cases} 1 & \mu = i, \nu = j, \\ 0 & \text{elsewhere,} \end{cases}$$

we can write down the multiplication table for $\mathfrak{g}l(n, \mathbb{F})$. Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$$

where the coefficients are all ± 1 or 0.

They fall into four families A_l, B_l, C_l, D_l, $l \geq 1$ and they are called the *classical algebras* because they correspond to certain classical linear Lie groups. In the thesis we will be interested to only classical linear Lie algebra A_l, $l \geq 1$.

Let $\dim V = l + 1$ and Let $\mathfrak{sl}(V)$, or $\mathfrak{sl}(l + 1, \mathbb{F})$ be the set of endomorphisms of V having trace zero. Since $\text{trace}(xy) = \text{trace}(yx)$, and $\text{trace}(x + y) = \text{trace}(x) + \text{trace}(y)$, then $\text{sl}(V)$is a subalgebra of $\text{gl}(V)$ which is called *special linear algebra* because of its connection with the special linear group $SL(V)$ of endomorphisms of $\det 1$. On the one hand $\text{sl}(V)$ is a proper subalgebra of $\text{gl}(V)$, hence the dimension is at most $(l + 1)^2 - 1$. So, we can exhibit this number of linearly independent matrices of trace zero. Let us take all e_{ij} $(i \neq j)$, along with all $h_i = e_{ii} - e_{i+1,i+1}$ $(1 \leq i \leq l)$, for a total of $l + (l + 1)^2 - (l + 1) = l(l + 2)$ matrices. We shall always view this as the standard basis for $\text{sl}(l + 1, \mathbb{F})$.

Now we introduce other subalgebras of $\mathfrak{gl}(n, \mathbb{F})$. Let $\mathfrak{t}(n, \mathbb{F})$ be the set of upper triangular matrices $[a_{ij}]$ where $a_{ij} = 0$ if $i > j$. Let $\mathfrak{n}(n, \mathbb{F})$ be the strictly upper triangular matrices $[a_{ij}]$ where $a_{ij} = 0$ if $i \geq j$). Finally, let $\mathfrak{d}(n, \mathbb{F})$ be the set of all diagonal matrices. It is trivial to check that each of these is closed under the bracket. Notice also that $\mathfrak{t}(n, \mathbb{F}) = \mathfrak{d}(n, \mathbb{F}) \bigoplus \mathfrak{n}(n, \mathbb{F})$ with $[\mathfrak{d}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] = \mathfrak{n}(n, \mathbb{F})$, hence $[\mathfrak{t}(n, \mathbb{F}), \mathfrak{t}(n, \mathbb{F})] = \mathfrak{n}(n, \mathbb{F})$.

Some Lie algebras of linear transformations arise most naturally as derivations of algebras.

Definition 3.3 (***Lie algebras of derivations***) *Let U be an $\mathbb{F}$-algebra. By a derivation of U we mean a linear map $\delta : U \to U$ satisfying the familiar product rule $\delta(ab) = a\delta(b) + \delta(a)b$.*

It is easily checked that the collection Der$U$ is a subalgebra of $\mathfrak{gl}(U)$.

Since a Lie algebra L is an $\mathbb{F}$-algebra, Der$L$ can be defined. Certain derivations arise quite naturally, as follows. If $x \in L$, by bilinearity of the bracket operation, the map $y \mapsto [x, y]$ is an endomorphism of L which is denoted by adj$x$. Using Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] = [y, [x, z] + [[x, y], z],$$

hence adj$x \in DerL$. The map $L \to DerL$ sending x to adj$x$ is called *adjoint representation* of L. Derivations of this form are called inner, all others outer.

We will use $\mathrm{adj}_L x$ or $\mathrm{ad}_K x$ to indicate that x is acting on L and K respectively. For example, if x is a diagonal matrix, then $\mathrm{adj}_{\mathfrak{n}(n,\mathbb{F})}(x) = 0$, whereas $\mathrm{adj}_{\mathfrak{gl}(n,\mathbb{F})}(x)$ need not be zero.

3.1 Abstract Lie algebras.

So far we have looked at some natural examples of linear Lie algebras. In fact, every finite dimensional Lie algebra is isomorphic to some linear Lie algebra. Later this claim will be proved using matrix representation.

For example, if L is an arbitrary finite dimensional vector space over $\mathbb{F}$, we can view L as a Lie algebra by $[x, y] = 0$ for all $x, y \in L$. Such an algebra having trivial Lie multiplication is called abelian. If L is any Lie algebra with basis $x_1, \ldots, x_n$ it is clear that the entire multiplication table of L can be obtained from the structure constants ${a^k}_{ij}$ which occur in the expressions

$$[x_i, x_j] = \sum_{k=1}^{n} {a^k}_{ij} x_k.$$

By anti-commutativity these constant structures for which $i \geq j$ can be deduced from the others. So, it is possible to define an abstract Lie algebra by specifying a set of structure constants. Of course, just any set of scalars $\{a_{ij}^k\}$ will not do, it is enough to require the following identities which are implied by (L2) and (L3)

$$a_{ii}^k = 0 = a_{ij}^k + a_{ji}^k$$

and

$$\sum_k \left(a_{ij}^k a_{kl}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m\right) = 0.$$

In practice, we will not construct Lie algebras in this artificial way. But, as an application of the abstract point of view, we can determine all Lie algebras of dimension ≤ 2 up to isomorphism. In dimension 1 there is a single basis vector x, with multiplication table $[x, x] = 0$ by $(L2)$. In dimension 2, we start with a basis x, y of L. Clearly, all products in L yield scalar multiples of $[x, y]$. If these are all 0, then L is abelian. Otherwise, we can replace x in the basis by a vector spanning one dimensional space of multiplies of the original $[x, y]$, and take y to be any other vector independent of the new x. Abstractly, therefore, at most one non-abelian L with $\dim L = 2$ exists.

3.2 Ideals and homomorphisms.

Definition 3.4 *A subspace I of a Lie algebra L is called an* ideal *of L if for $x \in L, y \in I, [x, y] \in I$.*

Since $[x, y] = -[y, x]$, the condition $\forall x \in L, \forall y \in I, [x, y] \in I$ implies $[y, x] \in I$. Ideals play the role in Lie algebra theory which is played by normal subgroups in group theory and by two sided ideals in ring theory. They arise as kernels of homomorphisms.

Obviously $\{0\}$ and L itself are ideals of L. A less trivial example is the center

$$Z(L) = \{z \in L \mid [xz] = 0 \,\forall\, x \in L\}.$$

Another important example of ideals is the *derived algebra* of L, denoted $[L, L]$, which is analogous to the commutator subgroup of a group. It consists of all linear combinations of commutators $[x, y]$, and is clearly an ideal.

Evidently L is abelian if and only if $[L, L] = 0$. If I, J are two ideals of a lie algebra L, $I + J = \{x + y \mid x \in I, y \in J\}$ is also an ideal. Similarly, $[I, J] = \{\sum a_{ij}[x_i, y_j] \mid x_i \in I, y_j \in J\}$ is an ideal; the derived algebra $[L, L]$ is just a special case of this construction.

It is natural to analyze the structure of a Lie algebra by looking at its ideals.

Definition 3.5 *L is called* simple *If L has no ideals except itself and* $\{0\}$.

Clearly, if L is simple then $Z(L) = 0$ and $L = [L, L]$.

Let $L = \mathfrak{sl}(2, \mathbb{F})$ with char$\mathbb{F} \neq 2$. Let us consider as standard basis for L the three matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The multiplication table is then completely determined by the equations $[x, y] = h, [h, x] = 2x, [h, y] = -2y$. Notice that x, y, h are eigenvectors for $\operatorname{adj} h$, corresponding to the eigenvalues $2, -2, 0$ respectively. Let $I \neq 0$ be an ideal of L, and let $ax + by + ch$ be an arbitrary non-zero element of I. Applying $\operatorname{adj} x$ twice, we get $-2ay \in I$. Therefore, if a or b is nonzero, I contains either x or y. Then $I = L$ follows. We conclude that L is simple.

Definition 3.6 (*Homomorphisms and representations*) *Let L and L' be two Lie algebras over $\mathbb{F}$. A linear transformation $\phi : L \to L'$ is called a* homomorphism *from L to L' if it preserves bracket operations. i.e. ϕ is a homomorphism from L to L' if it is a linear map satisfying $\phi[x, y]_L = [\phi(x), \phi(y)]_{L'}$.*

A representation of a Lie algebra L on V *is a homomorphism $\phi : L \to \mathfrak{gl}(V)$ where V is a vector space over $\mathbb{F}$.*

Since a Lie algebra L is a vector space and $DerL \subset \mathfrak{gl}(L)$, the only important example to keep in mind is the adjoint representation $\operatorname{adj} : L \to \mathfrak{gl}(L)$ which sends x to $\operatorname{adj} x$, where $\operatorname{adj} x(y) = [x, y]$. By bilinearity of the bracket operation $[,]$, adj is a linear transformation. To see that it preserves the bracket, we calculate as follows

$$\begin{aligned}
[\operatorname{adj} x, \operatorname{adj} y](z) &= \operatorname{adj} x \operatorname{adj} y(z) - \operatorname{adj} y \operatorname{adj} x(z) \\
&= \operatorname{adj} x([y, z]) - \operatorname{adj} y([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [[x, z], y] \\
&= [[x, y], z] \\
&= \operatorname{adj}\, [x, y]\,(z).
\end{aligned}$$

What is the kernel of adj? It consists of all $x \in L$ for which $\operatorname{adj} x = 0$, for which $[x, y] = 0, \forall y \in L$. This has an interesting consequence. If L is simple, then $Z(L) = 0$, so that $\operatorname{adj} : L \to \mathfrak{gl}(L)$ is a monomorphism. It means that *any simple Lie algebra is isomorphic to a linear Lie algebra.*

Definition 3.7 (***Automorphism***) *An automorphism of L is an isomorphism of L onto itself.*

Let Aut$L$ be the group of all such isomorphisms. Important examples occur when L is a linear Lie algebra $\subset \mathfrak{gl}(V)$. If $g \in GL(V)$ is any invertible endomorphism of V, and if moreover $gLg^{-1} = L$, then it is immediate that the map $x \mapsto gxg^{-1}$ is an automorphism of L. For instance, if $L = \mathfrak{gl}(V)$ or even $\mathfrak{sl}(V)$, the second condition is automatic, so we obtain in this way a large collection of automorphisms.

3.3 Solvable and Nnilpotent Lie algebras.

Definition 3.8 *Let char$\mathbb{F} = 0$. If $x \in L$ is an element for which $adjx$ is nilpotent, or $(adjx)^k = 0$ for some $k > 0$ we say that x is* ad-nilpotent*.*

It is natural to study a Lie algebra L via its ideals. In this section we exploit the formation of derived algebras.

Definition 3.9 (***Solvability***) *First we define a sequence of ideals of L (* derived series*) by $L^{(0)} = L, L^{(1)} = [LL], L^{(2)} = [L^{(1)}L^{(1)}], ..., L^{(i)} = [L^{(i-1)}L^{(i-1)}]$. We say that L is* solvable *if $L^{(n)} = 0$ for some n.*

For example, abelian implies solvable, whereas simple algebras are definitely non-solvable.

An example which turns out to be rather general is the algebra $\mathfrak{t}(n, \mathbb{F})$ of upper triangular matrices. The obvious basis for $\mathfrak{t}(n, \mathbb{F})$ consists of the matrix units e_{ij} for which $i \leq j$; the dimension is $1 + 2 + ... + n = n(n+1)/2$. To show that $L = \mathfrak{t}(n, F)$ is solvable we compute explicitly its derived series, using the formula for commutators. In particular, we have $[e_{ii}, e_{i\ell}] = e_{i\ell}$ for $i < \ell$, which shows that $\mathfrak{n}(n, \mathbb{F}) \subset [L, L]$, where $\mathfrak{n}(n, \mathbb{F})$ is the subalgebra of upper triangular nilpotent matrices. Since $\mathfrak{t}(n, \mathbb{F}) = \mathfrak{d}(n, \mathbb{F}) \bigoplus \mathfrak{n}(n, \mathbb{F})$, and $\mathfrak{d}(n, \mathbb{F})$ is abelian, we conclude that $\mathfrak{n}(n, \mathbb{F})$ is equal to the derived algebra of L. Working next inside algebra $\mathfrak{n}(n, \mathbb{F})$, we have a natural notion of "level" for e_{ij}, namely $j - i$. In the formula for commutators, assume that $i < j, k < \ell$. Without losing any products we may also require $i \neq l$. Then $[e_{ij}, e_{kl}] = e_{il}$ if $j = k$ or 0 otherwise. In particular, each e_{il} is commutator of two matrices whose levels add up to that of e_{il}. We conclude that $L^{(2)}$ is spanned by those e_{ij} of level $\geq 2, L^{(i)}$ by those of level $\geq 2^{(i-1)}$. Finally, it is clear that $L^{(i)} = 0$ whenever $2^{i-1} > n - 1$.

Next we give a few simple observations about solvability.

Proposition 3.10 *Let L be a Lie algebra.*

(i) If L is solvable, then so are all subalgebras and homomorphic images of L.

(ii) If I is a solvable ideal of L such that L/I is solvable, then L itself is solvable.

(iii) If I, J are solvable ideals of L, then so is $I + J$.

Let L be an arbitrary Lie algebra and let M be a maximal solvable ideal. If I is any other solvable ideal of L, by part (c) of Proposition 3.10, then $S + I = S$ or $I \subset S$. This proves the existence of a unique maximal solvable ideal, called *radical* of L and denoted $\operatorname{Rad} L$. In case $\operatorname{Rad} L = 0$, L is called *semi-simple.* For example, a simple algebra is semi-simple. If L is simple then it is non-solvable. Also $L = 0$ is semi-simple. Notice that for arbitrary L, $L/\operatorname{Rad} L$ is semi-simple.

The definition of solvability imitates the corresponding notion in group theory, which goes back to Abel and Galois. By contrast, the notion of nilpotent group is more recent, and is modelled on the corresponding notion for Lie algebras.

Definition 3.11 (*Nilpotency*) *Define a sequence of ideals of L (*descending central series, also called the lower central series*) by $L^0 = L, L^1 = [L, L](= L^{(1)}), L^2 = [L, L^1], \ldots, L^i = [L, L^{i-1}]$. We say that L is called* nilpotent *if $L^n = 0$ for some n.*

For example, any abelian algebra is nilpotent. Clearly, $L^{(i)} \subset L^i$ for all i, so nilpotent algebras are solvable but the converse is false. Consider again $L = \mathfrak{t}(n, \mathbb{F})$. So we have $L^{(1)} = L^1$ is $\mathfrak{n}(n, \mathbb{F})$, and also that $L^2 = [LL^1] = L^1$, so $L^i = L^1$ for all $i \geq 1$. On the other hand, it is easy to see that $M = \mathfrak{n}(n, \mathbb{F})$ is nilpotent. M^1 is spanned by those e_{ij} of level $\geq 2, M^2$ by those of level $\geq 3, ..., M^i$ by those of level $\geq i + 1$.

Proposition 3.12 *Let L be a Lie algebra.*

(i) If L is nilpotent, then so are all subalgebras and homomorphic images of L.

(ii) If $L/Z(L)$ is nilpotent, then so is L.

(iii) If L is nilpotent and non-zero, then $Z(L) \neq 0$.

Theorem 3.13 *If all elements of L are ad-nilpotent, then L is nilpotent.*

Lemma 3.14 *: Let $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphism. Then $adjx$ is also nilpotent.*

A single nilpotent linear transformation always has at least one eigenvector, corresponding to its unique eigenvalue 0. This is just the case $\dim L = 1$ of the following theorem.

Theorem 3.15 *Let L be a subalgebra of $\mathfrak{gl}(V)$ where V is finite dimensional. If L consists of nilpotent endomorphism and $V \neq 0$, then there exists a non-zero vector $v \in V$ such that $L.v = 0$.*

Definition 3.16 *If V is a n dimensional vector space, a* flag *in V is a chain of subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V, \dim V_i = i$. If $x \in \hom(V)$, we say that x* stabilizes (or leaves invariant) *this flag provided $x.V_i \subset V_i$ for all i.*

Corollary 3.17 *Under the hypotheses of Theorem 4.1, there exists a flag $0 = V_0 \subset V_1 \subset \ldots \subset V_i \subset \ldots \subset V_n = V, \dim V_i = i$ in V stable under L, with $x.V_i \subset V_{i-1}$ for all i. In other words, there exists a basis of V relative to which the matrices of L are all in $\mathfrak{n}(n, \mathbb{F})$.*

Lemma 3.18 *: Let L be nilpotent, K an ideal of L. Then if $K \neq 0, K \cap Z(L) \neq 0$.*

3.4 Theorems of Lie and Cartan.

Theorem 3.19 *Let L be a solvable subalgebra of $\mathfrak{gl}(V)$ where V is finite dimensional. If $V \neq 0$, then V contains a common eigenvector for all the endomorphisms in L.*

Corollary 3.20 (*Lie's Theorem*) *Let L be a solvable subalgebra of $\mathfrak{gl}(V), \dim V = n < \infty$. Then L stabilizes some flag in V. In other words, the matrices of L relative to a suitable basis of V are upper triangular.*

Corollary 3.21 *Let L be solvable. Then there exists a chain of ideals of L, $0 = L_0 \subset L_1 \subset ... \subset L_n = L$, such that $\dim L_i = i$.*

Theorem 3.22 (*Cartan's Criterion*) *Let L be a subalgebra of $\mathfrak{gl}(V)$ where V is finite dimensional. Suppose that trace$(xy) = 0$ for all $x \in [L, L], y \in L$. Then L is solvable.*

3.5 Killing Form.

Definition 3.23 (*Killing form*) *Let L be any Lie algebra. For $x, y \in L$, we define $\kappa(x, y) = trace\,(adj x \cdot adj y)$. Then κ is a symmetric bilinear form on L, which is called* Killing form. *It is also associative in the sense that $\kappa\,([x, y], z) = \kappa\,(x, [y, z])$.*

This follows from the identity trace$([x, y]z) =$ trace$(x[y, z])$, for endomorphism x, y, z of a finite dimensional vector space.

Lemma 3.24 *Let I be an ideal of L. If κ is Killing form on L and κ_I is Killing form on the restriction of I, then $\kappa_I = \kappa|_{(IxI)}$.*

Proof. First we need a simple fact from linear algebra. If W is a subspace of a finite dimensional vector space V, and ϕ is an endomorphism of V mapping V into W, then trace $\phi = \text{trace}(\phi \mid_w)$. If $x, y \in I$, then $(\text{adj}\, x)(\text{adj}\, y)$ is an endomorphism of L, mapping L into I, so trace of $\kappa(x, y)$ coincides with the trace of $\kappa_I(x, y)$ of $(\text{adj}\, x)(\text{adj}\, y) \mid_I = (\text{adj}_I\, x)(\text{adj}_I\, y)$. □

In general a symmetric bilinear form $\beta(x, y)$ is called *non-degenerate* if *its radical or annihilator*

$$S = \{x \in L \mid \beta(x, y) = 0, \forall y \in L\} = 0.$$

Because the Killing form is associative, its radical is more than just a subspace. It is an ideal of L. From linear algebra, a practical way to test non-degeneracy is as follows. Fix a basis $x_1, ..., x_n$ of L. Then κ is non-degenerate if and only if the $n \times n$ matrix whose (ij)th entry is $\kappa(x_i, x_j)$ has non-zero determinant.

As an example, we can compute the Killing form on $\mathfrak{sl}(2, \mathbb{F})$, using the ordered standard basis $\{x, h, y\}$. Then we have

$$\text{adj}\, h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \text{adj}\, x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{adj}\, y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Therefore κ has matrix $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$ with determinant -128, and κ is non degenerate.

Recall that a Lie algebra L is called semi-simple if Rad$L = 0$. This is equivalent to requiring that L have no non-zero abelian ideals. Indeed any such ideal must be in the radical, and conversely, the radical includes such an ideal of L, the last nonzero term in the derived series or Rad$L$.

Theorem 3.25 *Let L be a Lie algebra. Then L is semi-simple if and only if its Killing form is non-degenerate.*

Definition 3.26 (*Direct sum*) *A Lie algebra L is said to be the direct sum of ideals $I_1, ..., I_t$ provided $L = I_1 \oplus ... \oplus I_t$.*

Theorem 3.27 *Let L be semi-simple. Then there exist simple ideals $L_1, ..., L_t$ of L such that $L = L_1 \oplus ... \oplus L_t$. Every simple ideal of L coincides with one of the L_i. Moreover, the Killing form of L_i is the restriction of κ to $L_i \times L_i$.*

Proof. Let I be an arbitrary ideal of L. Then $I^\perp = \{x \in L \mid \kappa(x, y) = 0 \text{ for all } y \in I\}$ is also an ideal, by the associativity of κ. By Cartan's Criterion, the intersection ideal $I \cap I^\perp$ of L is solvable. Therefore, since $\dim I + \dim I^\perp = \dim L$, we must have $L = I \oplus I^\perp$.

Now proceed by induction on $\dim L$ to obtain the desired decomposition into direct sum of simple ideals. If L has no nonzero proper ideal, then L is simple already and we're done. Otherwise let L_1 be a minimal nonzero ideal; by the preceding paragraph, $L = L_1 \oplus L_1^\perp$. In particular, any ideal of L_1 is also an ideal of L, so L_1 is semi-simple because it is simple by minimality. For the same reason, $L_1^\perp$ is semi-simple; by induction, it splits into a direct sum of simple ideals, which are also ideals of L. The decomposition of L is established.

Next we have to prove that these simple ideals are unique. If I is any simple ideal of L, then [I,L] is also an ideal of I, which is non-zero because $Z(L) = 0$; this forces $[I, L] = I$. On the other hand,

$$[I, L] = [I, L_1 \oplus ... \oplus L_t] = [I, L_1] \oplus \ldots \oplus [I, L_t],$$

so all but one summand must be 0. Let say $[I, L_i] = I$. Then $I \subset L_i$ and $I = L_i$ because L_i is simple. $\square$

Corollary 3.28 *If L is semi-simple, then $L = [L, L]$, and all ideals and homomorphic images of L are semi-simple. Moreover, each ideal of L is a direct sum of certain simple ideals of L.*

3.6 On Completely reducible representations.

It is often convenient to use the language of modules along with the equivalent language of representations. As in other algebraic theories, there is a natural definition.

Definition 3.29 (*L-Modules*) *Let L be a Lie algebra. A vector space V, endowed with an operation $L \times V \to V$ sending $(x, v) \mapsto x \cdot v$ is called an* L-module *if the following conditions are satisfied:*

(i) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,

(ii) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$,

(iii) $[x,y] \cdot v = xy \cdot v - yx \cdot v \, \forall x, y \in L; v, w \in V; a, b \in \mathbb{F}$.

For example, if $\phi : L \to \mathfrak{gl}(V)$ is a representation of L, then V may be viewed as an L-module via the action $x \cdot v = \phi(x)(v)$. Conversely, given an L-module V, this action defines a representation $\phi : L \to \mathfrak{gl}(V)$.

A homomorphism of L-modules is a linear map $\phi : V \to W$ such that $\phi(x \cdot v) = x \cdot \phi(v)$. The kernel of such a homomorphism is then an L-submodule of V. When ϕ is an isomorphism of vector spaces, we call it an *isomorphism of L-modules*; in this case, the two modules are said to be *equivalent representations of L.*

Definition 3.30 *An L-module V is called* irreducible *if it has precisely two L-submodules itself and* 0*.*

We should note that we do not regard a zero dimensional vector space as an irreducible L-module. However we do allow a one dimensional space on which L acts to be called irreducible.

Definition 3.31 *An L-module V is called* completely reducible *if V is a direct sum of irreducible L-submodules or equivalently if each L-submodule W of V has a complement W' as an L-submodule such that $V = W \oplus W'$.*

Given a representation $\phi : L \to \mathfrak{gl}(V)$, the associative algebra generated by $\phi(L)$ in $\hom(V)$ leaves invariant precisely the same subspaces. Therefore, all the usual result for modules over associative rings hold for L as well.

Theorem 3.32 (*Schur's Lemma*) *Let $\phi : L \to gI(V)$ be an irreducible representation. Then the only endomorphisms of V commuting with all $\phi(x)$, $\forall x \in L$ are scalars.*

W4e should notify that L itself is an L-module as the adjoint representation space. An L-submodule is just an ideal, so it follows that a simple algebra L is irreducible as L-module, while a semi-simple algebra is completely reducible.

3.7 Casimir element of a representation.

Let L be a semi-simple Lie algebra and let $\phi : L \to \mathfrak{gl}(V)$ be a faithful(1 to 1) representation of L over V. Let us define a symmetric bilinear form $\beta(x,y) = \text{trace}(\phi(x) \cdot \phi(y))$ on L. The form β is associative, so in particular its annihilator (or radical) S is an ideal of L. Moreover, β is non-degenerate because S=0. The formβ is just a generalization of the Killing form.

Now let L be semi-simple, and let β be any non-degenerate symmetric associative bilinear form on L. If $\{x_1, ..., x_n\}$ is an ordered basis of L, there is a uniquely determined dual basis $\{y_1, ..., y_n\}$ relative to β satisfying $\beta(x_i, y_j) = \delta_{ij}$. If $x \in L$, we can write

$$[x, x_i] = \sum_j a_{ij} x_j \text{ and } [x, y_i] = \sum_j b_{ij} y_j.$$

Using the associativity of β, we can compute that

$$\begin{aligned} a_{ik} &= \sum_j a_{ij} \beta(x_j, y_k) = \beta([x, x_i], y_k \\ &= \beta(-[x_i, x], y_k) = \beta(x_i, -[xy_k]) \\ &= -\sum_j b_{kj} \beta(x_i, y_i) \\ &= -b_{ki} \end{aligned}$$

holds.

If $\phi : L \to \mathfrak{gl}(V)$ is any representation of L, define $c_\phi(\beta) = \sum_i \phi(x_i)\phi(y_i) \in \hom(V)$. Using the identity $[x, yz] = [x, y]z + y[x, z]$ and the fact that $a_{ik} = -b_{ki}$, we obtain that

$$\begin{aligned} [\phi(x), c_\phi(\beta)] &= \sum_i [\phi(x), \phi(x_i)]\phi(y_i) + \sum_i \phi(x_i)[\phi(x), \phi(y_i)] \\ &= \sum_{i,j} a_{ij} \phi(x_j)\phi(y_i) + \sum_{i,j} b_{ij} \phi(x_i)\phi(y_j) \\ &= 0 \end{aligned}$$

holds.

In other words, $c_\phi(\beta)$ is an endomorphism of V commuting with $\phi(L)$.

Let $\phi : L \to \mathfrak{gl}(V)$ be a faithful representation with trace form $\beta(x, y) = \text{trace}(\phi(x) \cdot \phi(y))$. In this case, having fixed a basis $(x_1, ..., x_n)$ of L, we write simply c_ϕ for $c_\phi(\beta)$ and call this *Casimir element*of ϕ. Its trace is $\sum_i \text{trace}(\phi(x_i)\phi(y_i)) = \sum_i \beta(x_i, y_i) = \dim L$. In case ϕ is also irreducible, Schur's lemma Theorem implies that c_ϕ is a scalar which is equal to $\dim L / \dim V$. In this case we see that c_ϕ is independent of the basis of L.

As an example let consider $L = \mathfrak{sl}(2, \mathbb{F}), V = \mathbb{F}^2$ and let ϕ be the identity representation

$1_L : L \to \mathfrak{gl}(V)$. Let $\{x, h, y\}$ be the ordered standard basis of L. It is quickly seen that the dual basis relative to the trace form is $\{y, h/2, x\}$, so

$$c_\phi = xy + (1/2)h^2 + yx = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = 3/2 = \dim L / \dim V.$$

When ϕ is not faithful, a slight modification is needed. We know that $\ker \phi$ is an ideal of L, hence it is a sum of certain simple ideals. Let L' denote the sum of the remaining simple ideals. Then the restriction of ϕ to L' is a faithful representation of L' by the fundamental isomorphism theorem, and we make the preceding construction using dual bases of L' then the resulting element of $\hom(V)$ is again called the Casimir element of ϕ and denote c_ϕ. Evidently it commutes with $\phi(L) = \phi(L')$.

We have one last remark. It is often convenient to assume that we are dealing with a faithful representation of L, which amounts to studying the representation of certain ideals of L. If L is simple, only the one dimensional module will fail to be faithful, among all irreducible modules.

Lemma 3.33 *Let $\phi : L \to \mathfrak{gl}(V)$ be a representation of a semi-simple Lie algebra L. Then $\phi(L) \subset \mathfrak{sl}(V)$. In particular, L acts trivially on any one dimensional L-module.*

Theorem 3.34 (*Weyl's Theorem*) *Let $\phi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of a semi-simple Lie algebra. Then ϕ is completely reducible.*

3.8 On Classification of Representations of sl(2, F).

In this section all modules will be assumed to be finite dimensional over $\mathbb{F}$. Let L denotes $\mathfrak{sl}(2, \mathbb{F})$, whose ordered standard basis consists of

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By a simple calculation, then $[h, x] = 2x, [h, y] = -2y, [x, y] = h$.

Let V be an arbitrary L-module. Since h is semi-simple, then h acts diagonally on V. The assumption that $\mathbb{F}$ is algebraically closed insures that all the required eigenvalues already lie in $\mathbb{F}$. This gives a decomposition of V as direct sum of eigen-spaces $V_\lambda = \{v \in V | h \cdot v = \lambda \cdot v, \, \lambda \in \mathbb{F}\}$.

Definition 3.35 *Whenever $V_\lambda \neq 0$, λ is called a* weight *of h in V and we call V_λ a* weight space.

Lemma 3.36 *: If $v \in V_\lambda$, then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$.*

CHAPTER 4

LIE GROUPS AND THEIR LIE ALGEBRAS

The general reference for this chapter is [2].

4.1 Groups of matrices.

Let $M_{m,n}(\mathbb{F})$ be the set of $m \times n$ matrices whose entries are in $\mathbb{F}$. We will denote the (i, j) entry of an $m \times n$ matrix A by A_{ij} or a_{ij} and also write

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

We will use the special notations

$$M_n(\mathbb{F}) = M_{n,n}(\mathbb{F}), \mathbb{F}^n = M_{n,1}(\mathbb{F}).$$

$M_{m,n}(\mathbb{F})$ is a $\mathbb{F}$-vector space with the operations of matrix addition and scalar multiplication. The zero vector is the $m \times n$ zero matrix $O_{m,n}$ which we will often denote O when the size is clear from the context. The matrices E^{rs} with $r = 1, ..., m$,$s = 1, ..., n$ and

$$(E^{rs})_{ij} = \delta_{ir}\delta_{js} = \begin{cases} 1 & \text{if } i = r \text{ and } j = s, \\ 0 & \text{otherwise} \end{cases}$$

form a basis of $M_{m,n}(\mathbb{F})$, hence its dimension as a $\mathbb{F}$-vector space is

$$\dim_{\mathbb{F}} M_{m,n}(\mathbb{F}) = mn.$$

When n=1 we will denote the standard basis vectors of $(\mathbb{F})^n = M_{n,1}(\mathbb{F})$ by

$$e_r = E^{r1}(r = 1, ..., m).$$

As well as being a $\mathbb{F}$-vector space of dimension n^2, $M_n(\mathbb{F})$ is also a ring with the usual addition and multiplication of square matrices, with zero $O_n = O_{n,n}$ and the $n \times n$ identity matrix I_n as its unity; $M_n(\mathbb{F})$is not commutative except when n=1.

Proposition 4.1 *The determinant function* $\det : M_n(\mathbb{F}) \to \mathbb{F}$ *has the following properties.*

(i) For $A, B \in M_n(\mathbb{F})$, $\det(AB) = \det A \det B$

(ii) $\det I_n = 1$.

(iii) $A \in M_n(\mathbb{F})$ *is invertible if and only if* $\det A \neq 0$.

We will use the notation

$$GL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det A \neq 0\}$$

for the set of invertible $n \times n$ matrices, and

$$SL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det A = 1\} \subseteq GL_n(\mathbb{F})$$

for the set of $n \times n$ *unimodular* matrices.

Theorem 4.2 *The sets* $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$ *are groups under matrix multiplication. Furthermore,* $SL_n(\mathbb{F}) \leq GL_n(\mathbb{F})$*,i.e,* $SL_n(\mathbb{F})$ *is a subgroup of* $GL_n(\mathbb{F})$.

Because of these group structures, $GL_n(\mathbb{F})$ is called $n \times n$ *general linear group.* When $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we will refer to $GL_n(\mathbb{R}), SL_n(\mathbb{R})$ or $GL_n(\mathbb{C}), SL_n(\mathbb{C})$ as *real or complex general linear or special linear groups.*

4.2 The matrix exponential and logarithm.

Throughout this section, we will assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The power series

$$\operatorname{Exp}(X) = \sum_{n \geq 0} \frac{1}{n!} X^n, \ \log(X) = \sum_{n \geq 0} \frac{-1^{n-1}}{n} X^n$$

have radii of convergence (r.o.c) ∞ and 1 respectively. If $z \in \mathbb{C}$, the series $\operatorname{Exp}(z), \log(z)$ converge absolutely whenever $|z| <$ (r.o.c).

Let $A \in M_n(\mathbb{F})$. The matrix-valued series

$$\text{Exp}(A) = \sum_{n\geq 0} \frac{1}{n!}A^n = I + A + \frac{1}{2n!}A^2 + \frac{1}{3n!}A^3 + ...,$$

$$\log(A) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}A^n = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \dots,$$

converge whenever $\parallel A \parallel <$ (r.o.c). So $\text{Exp}(A)$ makes sense for every $A \in M_n(\mathbb{F})$ while $\log(A)$ only exists if $\parallel A \parallel < 1$.

Proposition 4.3 *Let $A \in M_n(\mathbb{F})$.*

(i) For $u, v \in \mathbb{C}$, $Exp\,((u+v)A) = Exp\,(uA) Exp\,(vA)$.

(ii) $ExpA \in GL_n(\mathbb{F})$ and $Exp\,(A)^{-1} = Exp\,(-A)$.

Proof.

(i)By expanding the first series we obtain

$$\begin{aligned}\text{Exp}((u+v)A) &= \sum_{n\geq 0} \frac{1}{n!}(u+v)^n A^n \\ &= \sum_{n\geq 0} \frac{(u+v)^n}{n!}A^n.\end{aligned}$$

By a sequence of obvious manipulations that are justified since these series are all absolutely convergent,

$$\begin{aligned}\text{Exp}(uA)\text{Exp}(vA) &= \left(\sum_{r\geq 0} \frac{u^r}{r!}A^r\right)\left(\sum_{s\geq 0} \frac{v^s}{s!}A^s\right) \\ &= \sum_{r\geq 0, s\geq 0} \frac{u^r v^s}{r!s!}A^{r+s} \\ &= \sum_{n\geq 0}(\sum_{r=0}^{n} \frac{u^r v^{n-r}}{r!(n-r)!A^n} \\ &= \sum_{n\geq 0} \frac{1}{n!}\left(\sum_{r=0}^{n} \binom{n}{r} u^r v^{n-r}\right)A^n \\ &= \sum_{n\geq 0} \frac{(u+v)^n}{n!}A^n \\ &= \text{Exp}((u+v)A).\end{aligned}$$

(ii) From part (i),

$$I = \mathrm{Exp}(0) = \mathrm{Exp}((1 + (-1))A) = \mathrm{Exp}(A)\mathrm{Exp}(-A)$$

so $\mathrm{Exp}(A)$ is invertible with inverse $\mathrm{Exp}(-A)$. $\square$

Using these series we can define the matrix version of the exponential function

$$\exp : M_n(\mathbb{F}) \to GL_n(\mathbb{F}) \text{ by } \exp(A) = \mathrm{Exp}(A).$$

Proposition 4.4 *If* $A, B \in M_n(\mathbb{F})$ *commute then* $\exp(A + B) = \exp(A)\exp(B)$.

Proof. Again we expand the series and perform a sequence of manipulations, all of which are justified because these series are absolutely convergent. Then we have

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_{r\geq 0} \frac{1}{n!} A^r\right) \left(\sum_{s\geq 0} \frac{1}{s!} B^s\right) \\
&= \sum_{r\geq 0, s\geq 0} \frac{1}{r!s!} A^r B^s \\
&= \sum_{n\geq 0} \left(\sum_{r=0}^{n} \frac{1}{r!(n-r)!} A^r B^{n-r}\right) \\
&= \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{r=0}^{n} \binom{n}{r} A^r B^{n-r}\right) \\
&= \sum_{n\geq 0} \frac{1}{n!} (A + B)^n \\
&= \mathrm{Exp}(A + B) \\
&= \exp(A + B).
\end{aligned}$$

Of course the identity

$$\sum_{r=0}^{n} \binom{n}{r} A^r B^{n-r} = (A + B)^n$$

holds whenever A and B commute. $\square$

We define the logarithm function by

$$\log : N_{M_n(\mathbb{F})}(I; 1) \to M_n(\mathbb{F}) \text{ by } \log(A) = \log(A - I).$$

Then for $\| A - I \| < 1$,

$$\log(A) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (A - I)^n.$$

Proposition 4.5 *The functions* exp *and* log *are inverses each other.*

(i) If $\| A - I \| < 1$*, then* $\exp(\log(A)) = A$*;*

(ii) If $\| \exp(B) - I \| < 1$*, then* $\log(\exp(B)) = B$*.*

Proof. These results follow from the formal identities between power series

$$\sum_{m \geq 0} \frac{1}{m!} \left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (A - 1)^n \right)^m = A,$$

and

$$\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \left(\sum_{m \geq 1} \frac{1}{m!} B^m \right)^n = B$$

which are powered by comparing coefficients. □

The functions exp and log are continuous and in fact infinitely differentiable on their domains. By continuity of exp at 0, there is a $\delta_1 > 0$ such that

$$N_{M_n(\mathbb{F})}(0; \delta_1) \subseteq \exp^{-1} N_{GL_n(\mathbb{F})}(I; 1)).$$

In fact we can actually take $\delta_1 = \log 2$ since

$$\exp N_{M_n(\mathbb{F})}(0; r) \subseteq N_{M_n(\mathbb{F})}(I; e^r - 1))$$

Hence we have the following results.

Proposition 4.6 (*inverse mapping theorem*) *The exponential function* exp *is injective when restricted to the open subset* $N_{M_n(\mathbb{F})}(0; \ln 2) \subseteq N_{M_n(\mathbb{F})}$*, hence it is locally a diffeomorphism at* 0 *with local inverse* log*.*

It will sometimes be useful to have a formula for derivative of exp at an arbitrary $A \in M_n(\mathbb{F})$.

When $B \in M_n(\mathbb{F})$ commutes with A,

$$\begin{aligned}\frac{d}{dt_{|t=0}} \exp(A+tB) &= \lim_{h\to 0} \frac{1}{h} \left(\exp(A+hB) - \exp(A)\right) \\ &= \exp(A)B = B\exp(A).\end{aligned}$$

The general situation is more complicated.

For a variable X consider the series

$$F(X) = \sum_{k\geq 0} \frac{1}{(k+1)!} X^k = \frac{\exp(X)-1}{X}$$

which has infinite radius of convergence. If we have a linear operator Φ on $M_n(\mathbb{C})$ we can apply the convergent series of operators

$$F(\Phi) = \sum_{k\geq 0} \frac{1}{(k+1)!} \Phi^k$$

to elements of $M_n(\mathbb{C})$. In particular we can consider the adjoint operator $\mathrm{adj}_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\mathrm{adj}_A(C) = [A, C] = AC - CA.$$

Then

$$F(\mathrm{adj}_A)(C) = \sum_{k\geq 0} \frac{1}{(k+1)!} (\mathrm{adj}_A)^k(C).$$

Proposition 4.7 *For $A, B \in M_n(\mathbb{C})$ we have*

$$\frac{d}{dt_{|t=0}} \exp(A+tB) = F(adj_A)(B)\exp(A).$$

In particular, if $A = 0$ or more generally if $AB = BA$,

$$\frac{d}{dt_{|t=0}} \exp(A+tB) = B\exp(A).$$

Proof. We begin by observing that if $D = \frac{d}{ds}$ and $f(s)$ is a smooth function of the real variable s, then

$$F(D)_{|s=0} f(s) = \int_0^1 f(s)ds \tag{4.1}$$

holds since the Taylor expansion of a smooth function g satisfies

$$\sum_{k\geq 1}\frac{1}{k!}D^k g(s) = g(s+1) - g(s).$$

Taking $g(s) = \int f(s)ds$ to be an indefinite integral of f we obtain

$$\sum_{k\geq 0}\frac{1}{(k+1)!}D^k f(s) = g(s+1) + g(s).$$

Evaluating at s=0 gives Equation 4.1.

The matrix-valued function

$$\varphi(s) = \exp(sA)B\exp((1-s)A)$$

satisfies

$$\begin{aligned}\varphi(s) &= \exp(sA)B\exp(A)\exp(-sA)\\ &= \exp(s \operatorname{adj} A)(B\exp(A))\\ &= \exp(s \operatorname{adj} A)(B)\exp(A),\end{aligned}$$

since for $m, n \geq 1$,

$$(\operatorname{adj} A)^m(BA^n) = (\operatorname{adj} A)^m(B)A^n.$$

So

$$F(D)(\varphi(s)) = \left(\sum_{k\geq 0}\frac{(s+1)^{k+1} - s^{k+1}}{(k+1)!}(\operatorname{adj} A)^k\right)(B)\exp(A)$$

is giving

$$\begin{aligned}F(D)(\varphi(s))_{|s=0} &= \left(\sum_{k\geq 0}\frac{1}{(k+1)!}(\operatorname{adj} A)^k\right)(B)\exp(A)\\ &= F(\operatorname{adj} A)(B)\exp(A),\end{aligned}$$

which is the desired formula. $\square$

4.3 One-parameter subgroups in matrix groups.

Let $G \leq GL_n(\mathbb{F})$ be a matrix group. Since $G \subseteq M_n(\mathbb{F})$, in particular the derivative is defined by

$$\gamma'(t) = \lim_{s \to t} \frac{1}{(s-t)} (\gamma(s) - \gamma(t)) \in M_n(\mathbb{F})$$

provided this limit exists.

Definition 4.8 *A differentiable curve in G is a function $\gamma : (a,b) \to G$ for which the derivative $\gamma'(t)$ exists at each $t \in (a,b)$.*

Since $G \subseteq M_n(\mathbb{F})$, such a curve is also a curve in $M_n(\mathbb{F})$.

Definition 4.9 *Let $\varepsilon > 0$.* A one parameter semigroup in G *is a continuous function $\gamma : (-\varepsilon, \varepsilon) \to G$ which is differentiable at* 0 *and also satisfies*

$$\gamma(s+t) = \gamma(s) \cdot \gamma(t)$$

whenever $s, t, (s+t) \in (-\varepsilon, \varepsilon)$.

We will refer to the last condition as the homomorphism property.

If $\varepsilon = \infty$ then $\gamma : \mathbb{R} \to G$ is called *a one-parameter group in G or one-parameter subgroup of G.*

Notice that for a one-parameter semigroup in G, $\gamma(0) = I_G \in G$.

Proposition 4.10 *Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a one-parameter semigroup in G. Then for every $t \in (-\varepsilon, \varepsilon)$, γ is differentiable at t and*

$$\gamma'(t) = \gamma'(0)\gamma(t) = \gamma(t)\gamma'(0).$$

Proof. For a small $h \in \mathbb{R}$,

$$\gamma(h)\gamma(t) = \gamma(h+t) = \gamma(t+h) = \gamma(t)\gamma(h).$$

Hence

$$\begin{aligned}\gamma'(t) &= \lim_{h\to 0}\frac{1}{h}(\gamma(t+h)-\gamma(t))\\ &= \lim_{h\to 0}\frac{1}{h}(\gamma(h)-I)\gamma(t)\\ &= \gamma'(0)\gamma(t),\end{aligned}$$

and similarly $\gamma'(t) = \gamma(t)\gamma'(0)$.

□

Of course, part of the importance of this result is the implication that γ is differentiable curve in G even though we only assumed it was continuous.

Proposition 4.11 *Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a one-parameter semigroup in G. Then there is a unique extension to a one-parameter group $\tilde{\gamma} : \mathbb{R} \to G$,i.e, a function $\tilde{\gamma}$ for which $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in (-\varepsilon, \varepsilon)$.*

Proof. Let $t \in \mathbb{R}$. Then for a large enough natural number $m, t/m \in (-\varepsilon, \varepsilon)$, hence $\gamma(t/m), (\gamma(t/m))^m \in G$. Similarly, for a second such natural number n, $\gamma(t/n), (\gamma(t/n))^n \in G$. Since $mn \geq m$ and $mn \geq n$, we also have $t/mn \in (-\varepsilon, \varepsilon)$ therefore

$$\begin{aligned}\gamma(t/n)^n &= \gamma(mt/mn)^n\\ &= \gamma(t/mn)^{mn}\\ &= \gamma(nt/mn)^m\\ &= \gamma(t/m)^m.\end{aligned}$$

Thus $\gamma(t/n)^n = \gamma(t/m)^m$, which shows that we obtain a well-defined element of G for every real number t. This defines a function $\tilde{\gamma} : \mathbb{R} \to G$ by $\tilde{\gamma}(t) = \lim_{n\to\infty} \gamma(t/n)^n$. It is easy to see that $\tilde{\gamma}$ is one-parameter group in G. □

We can now determine the form of all one-parameter groups in G.

Theorem 4.12 *Let $\gamma : \mathbb{R} \to G$ be a one parameter group in G. Then it has the form*

$$\gamma(t) = \exp(tA)$$

for some $A \in M_n(\mathbb{F})$.

Proof. Let $A = \gamma'(0)$. By Proposition 4.10, $\gamma(t) = \exp(tA)$ satisfies the following differential equation

$$\gamma'(t) = \gamma'(0)\gamma(t),\ \gamma(0) = I.$$

□

We can not yet reverse this process and decide for which $A \in M_n(\mathbb{F})$ the one-parameter group

$$\gamma : \mathbb{R} \to GL_n(\mathbb{F});\ \gamma(t) = \exp(tA)$$

actually takes values in G. Notice that we also have the curious phenomenon that although the definition of a one-parameter group only involves first order differentiability, the general form $\exp(tA)$ is always infinitely differentiable and indeed analytic as a function of t.

4.4 Curves, tangent spaces and Lie algebras.

Let $G \leq GL_n(\mathbb{F})$ be a matrix group.

Definition 4.13 *: The tangent space to G at $U \in G$ is*

$$T_U G = \{\gamma'(0) \in M_n(\mathbb{F}) : \gamma \text{ is a differentiable curve in} G \text{ with} \gamma(0) = U\}.$$

Proposition 4.14 *$T_U G$ is a real vector subspace of $M_n(\mathbb{F})$*

Proof. Suppose that α, β are differentiable curves in G for which $\alpha(0) = \beta(0) = U$. Then

$$\gamma : \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) \to G$$

defined by $\gamma(t) = \alpha(t)U^{-1}\beta(t)$ is also differentiable curve in G with $\gamma(0) = U$. The product Rule gives

$$\gamma'(t) = \alpha'(t)U^{-1}\beta(t) + \alpha(t)U^{-1}\beta'(t),$$

hence

$$\gamma'(0) = \alpha'(0)U^{-1}\beta(0) + \alpha(0)U^{-1}\beta'(0) = \alpha'(0) + \beta'(0),$$

which shows that T_U is closed under addition.

Similarly, if $r \in \mathbb{R}$ and α is a differentiable curve in G with $\alpha(0) = U$, then $\eta(t) = \alpha(rt)$ defines another such curve. Since $\eta'(0) = r\alpha'(0)$, we see that $T_U G$ is closed under real scalar multiplication. $\square$

Definition 4.15 *The dimension of the real matrix group G is $\dim G = \dim_{\mathbb{R}} T_I G$ If G is complex then its complex dimension is $\dim_{\mathbb{C}} G = \dim_{\mathbb{C}} T_I G$.*

We will use the notation $\mathfrak{g} = T_I G$ for this real vector subspace of $M_n(\mathbb{F})$. In fact, $\mathfrak{g}$ has the algebraic structure of a real Lie algebra as we now show.

Theorem 4.16

(i) If $G \leq GL_n(\mathbb{R})$ is a matrix subgroup, then $\mathfrak{g}$ is a real Lie subalgebra of $M_n(\mathbb{R})$.

(ii) If $G \leq GL_n(\mathbb{C})$ is a matrix subgroup then $\mathfrak{g}$ is a complex Lie subalgebra of $M_n(\mathbb{C})$.

Proof.

(i) We will show that for two differentiable curves α and β in G which satisfy $\alpha(0) = \beta(0) = I_n$, there is another such curve γ with $\gamma'(0) = [\alpha'(0), \beta'(0)]$.

Consider a function

$$F : \mathrm{dom}(\alpha) \times \mathrm{dom}(\beta) \to G; F(s,t) = \alpha(s)\beta(t)\alpha(s)^{-1}.$$

This is clearly continuous and differentiable with respect to each of the variables s, t. For each $s \in \mathrm{dom}\alpha$, the function $F(s, -) : \mathrm{dom}(\beta) \to G$ is a differentiable curve in G with $F(s, 0) = I_n$. By differentiating we find

$$\frac{dF(s,t)}{dt}_{|t=o} = \alpha(s)\beta'(0)\alpha(s)^{-1},$$

and so

$$\alpha(0)\beta'(0)\alpha(s)^{-1} \in g.$$

Since $\mathfrak{g}$ is a closed subspace of $M_n(\mathbb{F})$, whenever this limits exists we also have

$$\lim_{s \to 0} \frac{1}{s}(\alpha(0)\beta'(0)\alpha(s)^{-1} - \beta'(0))) \in g.$$

We will use following easily verified matrix version of the usual rule for differentiating inverse:

$$\frac{d}{dt}(\alpha(t)^{-1}) = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1}.$$

We have

$$\begin{aligned}
\lim_{s\to 0}\frac{1}{s}(\alpha(0)\beta'(0)\alpha(s)^{-1} - \beta'(0))) &= \frac{d}{ds_{|s=o}}\alpha(s)\beta'(0)\alpha(s)^{-1} \\
&= \alpha'(0)\beta'(0)\alpha(0) - \alpha(0)\beta'(0)\alpha(0)^{-1}\alpha'(0)\alpha(0)^{-1} \\
&= \alpha'(0)\beta'(0)\alpha(0) - \alpha(0)\beta'(0)\alpha'(0) \\
&= \alpha'(0)\beta'(0) - \beta'(0)\alpha'(0) \\
&= [\alpha'(0), \beta'(0)].
\end{aligned}$$

This shows that $[\alpha'(0), \beta'(0)] \in \mathfrak{g}$, hence it must be of the form $\gamma'(0)$ for some differentiable curve.

So for each matrix group G there is a Lie algebra $\mathfrak{g} = T_I G$. $\square$

A suitable type of homomorphism $G \to H$ between matrix group gives rice to a linear transformation $g \to h$ which is a homomorphism of Lie algebras.

Definition 4.17 : *Let $G \leq GL_n(\mathbb{F}), H \leq GL_m(\mathbb{F})$ be matrix group and $\varphi : G \to H$ be a continuous map. Then φ is said to be a* differentiable map *if it satisfies the following two conditions:*

- *for every differentiable curve $\gamma : (a, b) \to G$, the curve $\varphi o\gamma : (a, b) \to H$ is differentiable and has derivative*

$$(\varphi o\gamma)'(t) = \frac{d}{dt}\varphi(\gamma(t));$$

- *if two differentiable curves $\alpha, \beta : (a, b) \to G$ satisfies*

$$\alpha(0) = \beta(0), \alpha'(0) = \beta'(0),$$

then

$$(\varphi o\alpha)'(0) = (\varphi o\beta)'(0).$$

A differentiable map which also a group homomorphism is called a differentiable homomorphism. *A continuous of matrix groups that is also a differentiable map is called a* Lie homomorphism.

Proposition 4.18 *Let G, H, K be matrix groups with $\varphi : G \to H$ and $\theta : H \to K$ differentiable homomorphisms. Then the following are true.*

(i) For each $A \in G$ there is an $\mathbb{R}$-linear transformation $d_\varphi : T_A G \to T_{\varphi(A)} H$ given by

$$d_{\varphi A}(\gamma'(0)) = (\varphi o\gamma)'(0),$$

for every differentiable curve $\gamma : (a, b) \to G$ with $\gamma(0) = A$

(ii) We have

$$d\theta_{\varphi(A)} \ o \ d_{\varphi(A)} = d(\theta o\varphi)_A.$$

(iii) For the identity map $dId_G = Id_{T_A} G$.

If $\varphi : G \to H$ is a differentiable homomorphism then since $\varphi(I) = I$, $d_{\varphi I} : T_I G \to T_I H$ is a linear transformation called *derivative of* φ which is usually denoted $d_\varphi : \mathfrak{g} \to \mathfrak{h}$.

Theorem 4.19 *Let G, H be matrix groups and $\varphi : G \to H$ be a differentiable homomorphism. Then the derivative $d_\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras.*

Proof. Following ideas and notation in the proof of Theorem 4.16, for differentiable curves α, β in G with $\alpha(0) = \beta(0) = I$, we can use the composite function φoF given by

$$\varphi oF(s, t) = \varphi(F(s, t)) = \varphi(\alpha(s))\varphi(\beta(t))\varphi(\alpha(s))^{-1},$$

to deduce that

$$d_\varphi([\alpha'(0), \beta'(0)]) = [d_\varphi(\alpha'(0)), d_\varphi(\beta'(0))].$$

□

As a special case, for each $A \in G$ the *conjugation map*

$$\chi A : G \to G;\ \chi A(U) = AUA^{-1}$$

has *as its derivative at identity I the adjoint action of A on $\mathfrak{g}$,* namely the linear transformation

$$\mathrm{adj}_A = d_\chi : g \to g;\ \ \chi_A(X) = AXA^{-1}.$$

4.5 The Lie algebra of some matrix groups.

4.5.1 General and special linear groups.

We will start with the real matrix group $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$. For $A \in M_n(\mathbb{R})$ and $\varepsilon > 0$ there is a differentiable curve

$$\alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R});\ \alpha(t) = I + tA.$$

For $t \neq 0$, the roots of the equation $\det(A - tI+) = 0$ are of the form $t = -1/\lambda$ where λ is a non-zero eigenvalue of A. Hence if we choose ε as $\varepsilon < min\{\frac{1}{|\lambda|} : \lambda\,\text{a non-zero eigenvalue of}\,A\}$, then $\mathrm{Im}(\alpha) \subseteq GL_n(\mathbb{R})$, so we will view α as a function $\alpha : (-\varepsilon, \varepsilon) \to GL_n(\mathbb{R})$. Calculating the derivative we find that $\alpha'(t) = A$, hence $\alpha'(0) = A$. This shows that $A \in T_I GL_n(\mathbb{R})$. Since $A \in M_n(\mathbb{R})$ was arbitrary, we have

$$\mathfrak{gl}_n(\mathbb{R}) = T_I GL_n(\mathbb{R}) = M_n(\mathbb{R}),$$

$$\dim GL_n(\mathbb{R}) = n^2.$$

Similarly,

$$\mathfrak{gl}_n(\mathbb{C}) = T_I GL_n(\mathbb{C}) = M_n(\mathbb{C}),$$

$$\dim_{\mathbb{C}} GL_n(\mathbb{C}) = n^2.$$

For $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$, suppose that $\alpha : (a, b) \to SL_n(\mathbb{R})$ is a curve lying in $SL_n(\mathbb{R})$ and satisfying $\alpha(0) = I$. For $t \in (a, b)$ we have $\det \alpha(t) = 1$, so

$$\frac{d(\det \alpha(t))}{dt} = 0.$$

Lemma 4.20 *We have*

$$\frac{d\,(\det \alpha(t))}{dt_{|t=0}} = trace\,\alpha'(0).$$

Proof. Recall that for $A \in M_n(\mathbb{F})$,

$$\operatorname{trace} A = \sum_{i=1}^{n} A_{ii}.$$

It is easy to verify that the operation $\partial = \frac{d}{dt_{|t=0}}$ on functions has the derivation property

$$\partial(\gamma_1\gamma_2) = (\partial\gamma_1)\gamma_2(0) + \gamma_1(0)\partial\gamma_2.$$

Put $a_{ij} = \alpha(t)_{ij}$ and notice that when $t = 0$, $a_{ij} = \delta_{ij}$. Write C_{ij} for the cofactor matrix obtained from $\alpha(t)$ by deleting the ith row and jth column. By expanding along the nth row we obtain

$$\det \alpha(t) = \sum_{j=1}^{n} (-1)^{n+j} a_{nj} \det C_{nj}.$$

Then

$$\begin{aligned}\partial \det \alpha(t) &= \sum_{j=1}^{n} (-1)^{n+j} \left((\partial_{a_{nj}}) \det C_{nj} + a_{nj}(\partial \det C_{nj}) \right) \\ &= \sum_{j=1}^{n} (-1)^{n+j} (\partial_{a_{nj}}) \det C_{nj} + \partial \det C_{nn}.\end{aligned}$$

For $t = 0$, $\det C_{nj} = \delta_{nj}$ since $\alpha(0) = I$, hence

$$\partial \det \alpha(t) = \partial a_{nn} + \partial \det C_{nn}.$$

We can repeat this calculation with the $(n-1) \times (n-1)$ matrix C_{nn} and so on. This yields

$$\begin{aligned}\partial \det \alpha(t) = \partial a_{nn} + \partial a_{(n-1)(n-1)} + \partial \det C_{(n-1)(n-1)} \\ &= \partial a_{nn} + \partial a_{(n-1)(n-1)} + \ldots + \partial a_{11} \\ &= \operatorname{trace}(\alpha'(0)),\end{aligned}$$

giving the result. $\square$

So we have $\mathrm{trace}(\alpha'(0)) = 0$ and hence

$$\mathfrak{sl}_n(\mathbb{R}) = T_I SL_n(\mathbb{R}) \subseteq \ker(\mathrm{trace}) \subseteq M_n(\mathbb{R}).$$

If $A \in \ker(\mathrm{trace}) \subseteq M_n(\mathbb{R})$, the function

$$\alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R}); \alpha(t) = exp(tA) = \sum_{k \geq 0} \frac{t^k}{k!} A^k,$$

is defined for every $\varepsilon > 0$ and satisfy the boundary conditions

$$\alpha(0) = I, \alpha'(0) = A.$$

We will make use of the following lemma.

Lemma 4.21 *For $A \in M_n(\mathbb{C})$ we have*

$$\det \exp(A) = e^{trace(A)}.$$

Proof. Approach using Jordan form. If $S \in GL_n(\mathbb{C})$,

$$\det(\exp(SAS^{-1})) = \det(S \exp(A) S^{-1})$$

$$= \det S \det(\exp(A)) \det S^{-1}$$

$$= \det \exp A,$$

and

$$e^{\mathrm{trace}(SAS^{-1})} = e^{\mathrm{trace}(A)}.$$

So it suffices to prove the identity for SAS^{-1} for a suitably chosen invertible matrix S. Using the theory of Jordan forms,there is a suitable choice of such an S for which

$$B = SAS^{-1} = D + N,$$

where D is diagonal, N is strictly upper triangular,i.e. $N_{ij} = 0$ whenever $i \geq j$) and with D, N commuting. Then N is nilpotent, $N^k = 0_n$ for k large.

We have

$$\begin{aligned} exp(B) = \sum_{k \geq 0} \frac{1}{k!}(D+N)^k \\ &= \sum_{k \geq 0} \frac{1}{k!}(D)^k + \sum_{k \geq 0} \frac{1}{(k+1)!} \left(\sum_{j=0}^{k} \binom{k+1}{j} D^j N^{k+1-j} \right) \\ &= exp(D) + \sum_{k \geq 0} \frac{1}{(k+1)!} N \left(\sum_{j=0}^{k} \binom{k+1}{j} D^j N^{k-j} \right). \end{aligned}$$

For each $k \geq 0$, the matrix

$$N \sum_{j=0}^{k} \binom{k+1}{j} D^j N^{k-j}$$

is strictly upper triangular, hence

$$\exp(B) = \exp(D) + N',$$

where N' is strictly upper triangular. If $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, by calculating the determinant we find that

$$\begin{aligned} \det \exp(A) &= det \exp(B) \\ &= \det \exp(D) = \det(\operatorname{diag}(e^{\lambda_1}, ..., e^{\lambda_n})) \\ &= e^{\lambda_1} \ldots e^{\lambda_n} = e^{(\lambda_1 + \ldots + \lambda_n)}. \end{aligned}$$

Since $\operatorname{trace}(D) = (\lambda_1 + ... + \lambda_n)$, this implies that

$$\det \exp(A) = e^{\operatorname{trace}(D)}.$$

$\square$

Using this lemma and the function α, we obtain

$$\mathfrak{sl}_n(\mathbb{R}) = T_I SL_n(\mathbb{R}) = \ker(\text{trace}) \subseteq M_n(\mathbb{R})$$

$$\dim SL_n(\mathbb{R}) = n^2 - 1.$$

Working over $\mathbb{C}$ we also have

$$\mathfrak{sl}_n(\mathbb{C}) = T_I SL_n(\mathbb{C}) = \ker \text{trace} \subseteq M_n(\mathbb{C}).$$

Hence

$$\dim_{\mathbb{C}} SL_n(\mathbb{C}) = n^2 - 1.$$

4.6 Some observations on the exponential function of a matrix group.

The function

$$\exp_G : \mathfrak{g} \to GL_n(\mathbb{R});\ \exp_G(X) = \exp(X),$$

has image contained in G, $\exp_G g \subseteq G$, so we will usually write $\exp_G : \mathfrak{g} \to G$ for the exponential of G and sometimes even just exp.

If G is compact and connected then $\exp_G(\mathfrak{g}) = G$. y inverse mapping theorem, there is an open disc $N_{\mathfrak{g}}(0; r) \subseteq \mathfrak{g}$ on which exp is injective and gives a homeomorphism

$$\exp : N_{\mathfrak{g}}(0; r) \to \exp(N_g(0; r)),$$

where $\exp(N_{\mathfrak{g}}(0; r)) \subseteq G$ is in fact an open subset.

Here is an example which shows that the exponential map need not be surjective for a connected matrix group.

Example 4.22 *Consider the exponential* $\exp_{SL_2(\mathbb{R})} : \mathfrak{sl}(2, \mathbb{R}) \to SL_2(\mathbb{R})$. *Then for* $\delta \geq 0$, *the matrix*

$$\begin{pmatrix} -(2+\delta) & 0 \\ 0 & \frac{-1}{(2+\delta)} \end{pmatrix} \in SL_2(\mathbb{R})$$

is not in the image of $\exp_{SL_2(\mathbb{R})}$.

Proof. Let

$$A \in \mathfrak{sl}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \mathrm{trace} A = 0\}.$$

By Cayley-Hamilton Theorem, we have

$$A^2 + (\det A) I_2 = 0.$$

If $\det A = 0$ then the only eigenvalue of A is 0 and for any $t \in \mathbb{R}$ we have

$$\exp(tA) = I_2 + tA,$$

from which we obtain

$$\mathrm{trace} \exp(tA) = 2.$$

If $\det A \neq 0$, then A has two distinct non-zero eigenvalues, namely

$$\begin{cases} \pm\sqrt{\det A i} & \text{if } \det A > 0, \\ \pm\sqrt{-\det A} & \text{if } \det A < 0. \end{cases}$$

□

4.7 Smooth manifolds.

Definition 4.23 *:A continuous map $g : V_1 \to V_2$ where each $V_k \subseteq \mathbb{R}^{m_k}$ is open is* smooth *if it is infinitely differentiable. A smooth bijection g is a* diffeomorphism *if its inverse $g^{-1} : V_2 \to V_1$ is also smooth.*

Definition 4.24 *: A topological space X is* separable *if it has a countable open covering, an open of the form $\{U_j\}_{j=1}^{\infty} = \{U_1, U_2, U_3, ...\}$.*

Every compact topological space is separable, as is any subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ for $n \geq 1$, so all matrix groups are separable. A quotient space of a separable space is also separable.

From now on, let M be a separable Hausdorff topological space.

Definition 4.25 *If $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open subsets, a homomorphism $f : U \to V$ is called an* n-chart *for U.*

If U $= \{U_\alpha \in A\}$ *is an open covering of M and* F $= \{f_\alpha : U_\alpha \to V_\alpha\}$ *is a collection of charts, then F is called an* atlas *for M if, whenever $U_\alpha \cap U_\beta \neq \phi$,*

$$f_\beta o f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.

*Sometimes we will denote an atlas by (M, U, F) and refer to it as a s*mooth manifold of dimension n *or* smooth n-manifold.

Let (M, U, F) and (M', U', F') be atlases on topological spaces M and M'. A *smooth map* $h : (M, U, F) \to (M', U', F')$ is a continuous map $h : M \to M'$ such that for each pair α, α' with $h(U_\alpha) \cap U'_{\alpha'} \neq \phi$, the composite

$$f'_{\alpha'} \ o \ h \ o \ f_\alpha^{-1} : f_\alpha(h^{-1}U'_{\alpha'}) \to V'_{\alpha'}$$

is smooth.

4.8 Tangent spaces and derivatives.

Let (M, U, F) be a smooth n-manifold and $p \in M$. Let $\gamma : (a, b) \to M$ be a continuous curve with $a < 0 < b$.

Definition 4.26 *γ is* differentiable *at $t \in (a, b)$ if for every chart $f : U \to V$ with $\gamma(t) \in U$, the curve $f o\gamma : (a, b) \to V$, is differentiable at $t \in (a, b)$, i.e., $(f o\gamma)'(t)$ exists.*

γ is smooth *at $t \in (a, b)$ if all the derivatives of $(f o\gamma)$ exists at t.*

The curve γ is differentiable if it is differentiable at all points in (a, b). Similarly γ is smooth at all points in (a, b).

From now on we will often write $fg = fog$ for the composition of functions f and g when no confusion seems likely to result.

Lemma 4.27 *Let $f_0 : U_0 \to V_0$ be a chart with $\gamma(t) \in U_0$ and suppose that*

$$f_0 \; o \; \gamma : (a, b) \cap f_0^{-1} V_0 \to V_0$$

is differentiable at t. Then for any chart $f : U \to V$ with $\gamma(t) \in U$,

$$f o \gamma : (a, b) \cap f^{-1} V \to V$$

is differentiable at t.

Proof. The composition $f o\gamma$ is defined on a subinterval of (a, b) containing t and is smooth. There is the usual chain rule or function of a function rule for the derivative

$$(f\gamma)'(t) = \mathrm{Jac}_{ff_0^{-1}}(f o\gamma(t))(f o\gamma)'(t) \tag{4.2}$$

Hence, for a differentiable function

$$h : W_1 \to W_2;\; h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_{m_2}(x) \end{pmatrix}$$

where $W_1 \subseteq \mathbb{R}^{m_1}$ and $W_2 \subseteq \mathbb{R}^{m_2}$ are open subsets, $x \in W_1$ and

$$\mathrm{Jac}_h(x) = \left[\frac{\partial h_i}{\partial x_j}(x)\right] \in M_{m_2,m_1}(\mathbb{R})$$

is the Jacobian matrix of h at x.

If $\gamma(0) = p$ and γ is differentiable at 0, then for any chart $f_0 : U_0 \to V_0$ with $\gamma(0) \in U_0$, there is a derivative vector $v_0 = (f\gamma)'(0) \in \mathbb{R}^n$. In passing to another chart $f : U \to V$ with $\gamma(0) \in U$ by equation 4.2, we have

$$(f\gamma)'(0) = \mathrm{Jac}_{ff_0^{-1}}(f o\gamma(0))(f o\gamma)'(0)$$

□

In order to define the notion of the tangent space T_pM to the manifold M at p, we consider all pairs of the form

$$(f\gamma)'(0), f : U \to V$$

where $\gamma(0) = p \in U$, and then impose an equivalence relation $\sim$ under which

$$((f_1\gamma)'(0), f_1 : U_1 \to V_1) \sim ((f_2\gamma)'(0), f_2 : U_2 \to V_2).$$

Since

$$(f_2\gamma)'(0) = Jac_{f_2 f_1^{-1}}((f_1\gamma)(0)(f_1\gamma)'(0),$$

whenever there is a curve α in M for which $\gamma(0) = p$, $(f_1\gamma)'(0) = v$. The set of equivalence classes is TpM and we will sometimes denote the equivalence class of $(v, f : U \to V)$ by $[v, f : U \to V]$.

Proposition 4.28 *: For $p \in M, TpM$ is an $\mathbb{R}$-vector space of dimension n.*

Proof. For any chart $f : U \to V$ with $p \in U$, we can identify the elements of TpM with objects of the form $(v, f : U \to V)$. Every vector $v \in \mathbb{R}^n$ arises as the derivative of of a curve $\bar{\gamma} : (-\varepsilon, \varepsilon) \to V$ for which $\bar{\gamma}(0) = f(p)$. For example, for small enough ε, we would take

$$\bar{\gamma}(t) = f(p) + tv.$$

There is an associated curve in M,

$$\gamma : (-\varepsilon, \varepsilon) \to M; \gamma(t) = f^{-1}\bar{\gamma}(t),$$

for which $\gamma(0) = p$. So using such a chart we can identify TpM with $\mathbb{R}^n$ by

$$[v, f : U \to V] \leftrightarrow v.$$

The same argument as used the prove Proposition 4.14 shows that TpM is a vector space and that the above correspondence is a linear isomorphism. □

Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimensions n, n'. We will use the notation of Definition 4.25. For $p \in M$, consider a pair of charts with $p \in U_\alpha$ and

$h(p) \in U'_{\alpha'}$. Since $h_{\alpha',\alpha} = f'_{\alpha'} o h o f_{\alpha}^{-1}$ is differentiable, the Jacobian matrix $\mathrm{Jac}_{h_{(\alpha',\alpha)}}(f_\alpha(p))$ has an associated $\mathbb{R}$-linear transformation

$$dh_{(\alpha',\alpha)} : \mathbb{R}^n \to \mathbb{R}^{n'}; dh_{(\alpha',\alpha)}(x) = Jac_{h_{(\alpha',\alpha)}}(f_\alpha(p))x.$$

It can be verified that this passes to equivalence classes to give a well-defined $\mathbb{R}$-linear transformation

$$dh_p : TpM \to T_{h(p)}M'.$$

Proposition 4.29 *: Let $h : (M, U, F) \to (M', U', F')$ and $g : (M', U', F') \to (M'', U'', F'')$ be smooth maps between manifolds M, M', M'' of dimensions n, n', n''. Then for each $p \in M$ we have*

(i) $dg_{h_{(p)}} o d_{h_p} = d(goh)_p$;

(ii) $dId_p = Id_{TpM}$, where $Id : M \to M$ is the identity map.

Definition 4.30 *Let (M, U, F) be a manifold of dimension n. A subset $N \subseteq M$ is a* submanifold *of dimension k if for every $p \in N$ there is an open neighbourhood $U \subseteq M$ of p and an n-chart $f : U \to V$ such that*

$$p \in f^{-1}(V \cap \mathbb{R}^k) = N \cap U.$$

For such an N we can form k-charts of the form

$$f_0 : N \cap U \to V \cap \mathbb{R}^k; f_0(x) = f(x).$$

We will denote this manifold by (N, U_N, F_N).

The following result is immediate.

Proposition 4.31 *For a submanifold $N \subseteq M$ of dimension k, the inclution function incl : $N \to M$ is smooth and for every $p \in N$, $dincl_p : T_pN \to T_pM$ is an injection.*

The next result allows us to recognise submanifolds as inverse image of points under smooth mappings.

Theorem 4.32 (*Implicit Function Theorem for manifolds*) *Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimensions n, n'. Suppose that for some $q \in M', dh_p$:*

$T_pM \to T_{h(p)}M'$ is surjective for every $p \in N = h^{-1}q$. Then $N \subseteq M$ is submanifold of dimension $n - n'$ and the tangent space at $p \in N$ is given by $T_pN = \ker dh_p$.

Theorem 4.33 *: Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimensions n, n'. Suppose that for some $p \in M, dh_p : T_pM \to T_{h(p)}M'$ is an isomorphism.*

Then there is an open neighbourhood $U \subseteq M$ of p and an open neighbourhood $V \subseteq M'$ of $h(p)$ such that $hU = V$ and the restriction of h to the map $h_1 : U \to V$ is a diffeomorphism. In particular, the derivative $dh_p : T_p \to T_{h(p)}$ is an $\mathbb{R}$- linear isomorphism and $n = n'$.

When this occurs we say that h is *locally a diffeomorphism* at p.

4.9 Lie groups.

Definition 4.34 *Let G be a smooth manifold which is also a topological group with multiplication map $mult : G \times G \to G$ and inverse map $inv : G \to G$ and view $G \times G$ as a product manifold. Then G is a Lie group if $mult, inv$ are smooth maps.*

Definition 4.35 *: Let G be a Lie group. A closed subgroup $H \leq G$ that is also a submanifold is called a Lie group of G. It is then automatic that the restrictions to H of the multiplication and inverse maps on G are smooth, hence H is also a Lie group.*

For a Lie group G, let $g \in G$. The following three functions are of particular importance:

$L_g : G \to G; L_g(x) = gx,$ *Left translation*

$R_g : G \to G; R_g(x) = xg,$ *Right translation*

$\chi_g : G \to G; \chi_g(x) = gxg^{-1}$ *Conjugation*

Proposition 4.36 *For each $g \in G$, the maps L_g, R_g, χ_g are diffeomorphism with inverses*

$$L_g{}^{-1} = L_{g^{-1}}, R_g{}^{-1} = R_{g^{-1}}, \chi_g{}^{-1} = \chi_{g^{-1}}.$$

Proof. Charts for $G \times G$ have the form

$$\varphi_1 \times \varphi_2 : U_1 \times U_2 \to V_1 \times V_2,$$

where $\varphi_k : U_k \to V_k$ are charts of G. Now suppose that $\mu\,(U_1 \times U_2) \subseteq W \subseteq G$ where there is a chart $\theta : W \to Z$. By assumption, the composition

$$\theta o\mu o(\varphi_1 \times \varphi_2)^{-1} = \theta o\mu o(\varphi_1^{-1} \times \varphi_2^{-1}) : V_1 \times V_2 \to Z$$

is smooth. Then $L_g(x) = \mu(g, x)$ so if $g \in U_1$ and $x \in U_2$, we have

$$L_g(x) = \theta^{-1} o(\theta o L_g o \varphi_2^{-1}) o \varphi_2(x).$$

But then it is clear that

$$\theta o \varphi_2^{-1} : V_2 \to Z$$

is smooth since it is obtained from $\theta o\mu o(\varphi_1 \times \varphi_2)^{-1}$, but treating the first variable as a constant. A similar argument deals with R_g.

For χ_g, notice that

$$\chi_g = L_g o R_g = R_g o L_g,$$

and a composite of smooth maps is smooth. □

The derivatives of these maps at the identity $1 \in G$ are worth studying. Since L_g and R_g are diffeomorphism with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, the derivatives

$$d(L_g)_1, d(R_g)_1 : \mathfrak{g} = T_1 G \to T_g G$$

are $\mathbb{R}$-linear isomorphisms. We can use this to identify every tangent space of G with g. The conjugation map χ_g fixes 1, so it induces an $\mathbb{R}$- linear isomorphism

$$\mathrm{adj}_g = d(\chi_g)_1 : g \to g.$$

This is the adjoint action of $g \in G$.

Theorem 4.37 *Let G, H be Lie groups and $\varphi : G \to H$ be a Lie homomorphism. Then the derivative is a homomorphism of Lie algebras. In particular, if $G \leq H$ is a Lie subgroup, the inclusion map $i : G \to H$ induces an injection of Lie algebras $di : \mathfrak{g} \to \mathfrak{h}$.*

4.9.1 Some examples of Lie groups.

Example 4.38 *When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}, GL_n(\mathbb{F})$ is a Lie group.*

Proof. Since $\det : M_n(\mathbb{F}) \to \mathbb{F}$ is continuous and

$$GL_n(\mathbb{F}) = \det^{-1}(\mathbb{F} - \{0\}),$$

$GL_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ is an open subset, where as usual we identify $M_n(\mathbb{F})$ with $\mathbb{F}^{n^2}$. For charts we take the open sets $U \subseteq GL_n(\mathbb{F})$ and the identity function $Id : U \to U$. The tangent space at each point $A \in GL_n(\mathbb{F})$ is just $M_n(\mathbb{F})$. The multiplication and inverse maps are obviously smooth as they are defined by polynomial and rational functions between open subsets of $M_n(\mathbb{F})$. $\square$

Example 4.39 *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}, SL_n(\mathbb{F}) \leq GL_n(\mathbb{F})$ is a Lie subgroup.*

Proof. Since $\det : GL_n(\mathbb{F}) \to \mathbb{F}^* = \mathbb{F} - \{0\}$ is a group homomorphism,

$$SL_n(\mathbb{F}) = \ker(\det) \leq GL_n(\mathbb{F}).$$

Since $T_1\mathbb{F}^* = \mathbb{F}$ is a smooth manifold, by the implicit function theorem, $SL_n(\mathbb{F})$ is a submanifold of $GL_n(\mathbb{F})$. Hence $SL_n(\mathbb{F})$ is a Lie subgroup of $GL_n(\mathbb{F})$. $\square$

Proposition 4.40 (*Left Translation Trick*) *Let $F : GL_n(\mathbb{R}) \to M$ be a smooth function into a smooth manifold M and suppose that $B \in GL_n(\mathbb{R})$ satisfies $F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. Let $A \in GL_n(\mathbb{R})$ with dF_A is surjective. Then dF_{BA} is surjective.*

Proof. Left multiplication by $B \in G, L_B : GL_n(R) \to GL_n(R)$, is a diffeomorphism and its derivative at $A \in GL_n(R)$ is

$$d(L_B) : M_n(R) \to M_n(R); dL_B(X) = BX$$

By assumption $FoL_B = F$ as a function on $GL_n(R)$. Then

$$\begin{aligned} dF_{BA}(X) &= dF_{BA}(B(B^{-1}X)) \\ &= dF_{BA}od(L_B)_A(B^{-1}X) \\ &= d(FoL_B)_A(B^{-1}X) \\ &= dF_A(B^{-1}X). \end{aligned}$$

Since left multiplication by B^{-1} on $M_n(\mathbb{R})$ is surjective, the result follows. □

Proposition 4.41 *Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup and let M be a smooth manifold. Let $F : GL_n(\mathbb{R}) \to M$ be a smooth function with $F^{-1}q = G$ for some $q \in M$. Suppose that for every $B \in G, F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. If dF_I is surjective then dF_A is surjective for all $A \in G$ and $\ker dF_A = A\mathfrak{g}$.*

Theorem 4.42 *Let $G \leq GL_n(\mathbb{R})$ be a matrix group which is also a submanifold, hence a Lie subgroup. Then the tangent space to G at I agrees with the Lie algebra $\mathfrak{g}$ and the dimension of the smooth manifold G is $\dim G$; more generally, $T_AG = A\mathfrak{g}$.*

Theorem 4.43 *Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup. Then G is a Lie subgroup of $GL_n(\mathbb{R})$.*

Theorem 4.44 *Let $H \leq G$ be a closed subgroup of a Lie group G. Then H is a Lie subgroup of G.*

4.9.2 Some useful formula in matrix groups.

Theorem 4.45 (*Trotter Product Formula,Commutator Formula*) *For $U, V \in M_n(\mathbb{R})$ we have the following formulas.*

$$\exp(U+V) = \lim_{r\to\infty}(\exp((1/r)U)\exp((1/r)V))^r$$
$$\exp([U,V]) = \lim_{r\to\infty}(\exp((1/r)U)\exp((1/r)V))\exp(-(1/r)U)\exp(-(1/r)V))^2$$

Proof. For large r we may take $A = (1/r)U$ and $B = 1/r)V$ to give

$$(\exp((1/r)U)\exp((1/r)V)) = \exp(C_r)$$

with

$$\| C_r - (1/r)(U+V) \| \leq \frac{17(\| U \| + \| V \|)^2}{r^2}).$$

As $r \to \infty$

$$\| rC_r - (U+V) \| = \frac{17(\| U \| + \| V \|)^2}{r} \to 0,$$

hence $rC_r \to (U+V)$. Since $\exp(rC_r) = \exp(C_r)^r$, the Trotter formula follows by continuity of exp.

We also have

$$C_r = (1/r)(U+V) + \frac{1}{2r^2}[U,V] + S_r$$

where

$$\| S_r \| \leq 65 \frac{(\| U \| + \| V \|)^3}{r^3}$$

Similarly, replacing U, V with $-U, -V$ we obtain

$$\exp((-1/r)U)\exp((-1/r)V)) = \exp(C'_r),$$

where

$$C'_r = (1/r)(U+V) + \frac{1}{2r^2}[U,V] + S'_r$$

and

$$\| S'_r \| \leq 65 \frac{(\| U \| + \| V \|)^3}{r^3}.$$

Combining these we obtain

$\exp((1/r)U)\exp((1/r)V)\exp((-1/r)U)\exp((-1/r)V) = \exp(C_r)\exp(C'_r) = \exp(E_r),$

where

$$\begin{aligned} E_r &= C_r + C'_r + 1/2[C_r, C'_r] + T_r \\ &= \frac{1}{r^2} + 1/2[C_r, C'_r] + S_r + S'_r + T_r. \end{aligned}$$

Another tedious computation shows that

$[C_r, C'_r]$

$$\begin{aligned} &= [(1/r)(U+V) + \frac{1}{2r^2}[U,V] + S_r, (-1/r)(U+V) + \frac{1}{2r^2}[U,V] + S'_r] \\ &= \frac{1}{r^3}[U+V,[U,V]] + (1/r)[U+V, S_r + S'_r] + \frac{1}{2r^2}[[U,V], S'_r - S_r] + [S_r, S'_r]. \end{aligned}$$

By the estimation, all four of these terms has norm bounded by an expression of the form (constant)$/r^3$, so the same is true of $[C_r, C'_r]$. The estimation also implies that S_r, S'_r, T_r have similarly bounded norms. Setting

$$\theta_r = r^2 E_r - [U,V],$$

we obtain

$$\| \theta_r \| = \| E_r - \frac{1}{r^2}[U,V] \| \leq (\text{constant})/r^3 \to 0$$

as $r \to \infty$, so

$$\exp(E_r)^{r^2} = \exp\left([U,V] + \theta_r\right) \to \exp\left([U,V]\right).$$

The commutator formula now follows using continuity of exp. □

4.10 All matrix groups are Lie groups.

Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup. Recall that the Lie algebra $\mathfrak{g} = T_1 G$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$. Let

$$\tilde{\mathfrak{g}} = \{A \in M_n(\mathbb{R}) : \forall t \in \mathbb{R}, \exp(tA) \in G\}.$$

Theorem 4.46 *$\tilde{\mathfrak{g}}$ is a Lie subalgebra of $M_n(\mathbb{R})$.*

Proof. By defining, $\tilde{\mathfrak{g}}$ is closed under multiplication by real scalars. If $U, V \in \tilde{\mathfrak{g}}$ and $r \geq 1$, then following are in G:

$$\exp((1/r)U)\exp((1/r)V), \exp((1/r)U)\exp((1/r)V)^r,$$

$$\exp((1/r)U)\exp((1/r)V)\exp(-(1/r)U)\exp(-(1/r)V)^{r^2},$$

$$\exp((1/r)U)\exp((1/r)V)\exp(-(1/r)U)\exp(-(1/r)V)^{r^2}.$$

By Theorem 4.45, for $t \in \mathbb{R}$, we have

$\exp([tU, tV]) = \lim r \to \infty(\exp((1/r)tU)\exp((1/r)tV))^r$, and $\exp(t[U, V]) = \exp([tU, V]) = \lim r \to \infty(\exp((1/r)tU))\exp((1/r)V)\exp(-(1/r)tU)\exp(-(1/r)V)^{r^2}$.

As these are both limits of elements of the closed subgroup $G \leq GL_n(\mathbb{R})$ they are also in G. This shows that $\tilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$. $\square$

Proposition 4.47 *Let $U \in \tilde{\mathfrak{g}}$. Then the curve*

$$\gamma : \mathbb{R} \to G; \gamma(t) = \exp(tU),$$

has $\gamma(0) = I$ and $\gamma'(0) = U$, hence $U \in \mathfrak{g}$.

Eventually we will see that $\tilde{\mathfrak{g}} = \mathfrak{g}$.

Lemma 4.48 *Let $\{A_n \in (\exp)^{-1}G : n \geq 1\}$ and $\{s_n \in \mathbb{R} : n \geq 1\}$ be sequence for which $\| A_n \| \to 0$ and $s_n A_n \to A \in M_n(\mathbb{R})$ as $n \to \infty$. Then $A \in \tilde{\mathfrak{g}}$.*

Theorem 4.49 *A subgroup of $GL_n(\mathbb{R})$ is a closed Lie group if and only if it is a matrix subgroup, i.e, a closed subgroup.*

Notice that the dimension of a matrix group G as a manifold is $\dim_{\mathbb{R}} \tilde{\mathfrak{g}}$. By Proposition 4.47, $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}$ and $\dim_{\mathbb{R}} \tilde{\mathfrak{g}} \leq \dim_{\mathbb{R}} \mathfrak{g}$. By Theorem 4.46, these dimensions are in fact equal, giving $\tilde{\mathfrak{g}} = \mathfrak{g}$.

Theorem 4.50 *For a matrix group $G \leq GL_n(\mathbb{R})$, the exponential map* $\exp : g \to M_n(\mathbb{R})$ *has image in G, $Im(\exp) \leq G$. Moreover,* $\exp$ *is a local diffeomorphism at* 0*, mapping some open neighbourhood of* 0 *onto an open neighbourhood of I in G.*

CHAPTER 5

LINEAR DIFFERENTIAL OPERATORS, SL(2) MULTIPLIER REPRESENTATION AND WEISNER METHOD

The general reference for this chapter is [16].

5.1 Linear Differential Operators.

Let ${A^0}_x$ be the complex space of all functions which are analytic in some neighbourhood of $x = (x_1^0, ..., x_m^0) \in \mathbb{C}^m$. Let

$$L_k = \sum_{i=1}^{m} p(x)\frac{\partial}{\partial x_i} + q_k(x),\ k = 1, ..., n \tag{5.1}$$

be linearly independent linear operators on ${A^0}_x$.

We know from Chapter 3 that if there exist C^{ι}_{rs} such that

$$[L_r, L_s] = \sum_{\iota=1}^{n} C^{\iota}_{rs} L_{\iota},\ 1 \leq r, s \leq n,$$

then the operators L_k form a basis for a Lie algebra $\mathfrak{g}'$ which is isomorphic to some matrix Lie algebra $\mathfrak{g}$.

Furthermore, we know from Chapter 4 if $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of Lie groups G and G' respectively, then G and G' are isomorphic.

Let Θ denote the local isomorphism of G onto a neighbourhood of the identity element I in G'. Then Θ is a *multiplier representation* of G on the representation space ${A^0}_x$. In particular, if $L_\alpha \in \mathfrak{g}'$ is an isomorphic image of $\alpha \in \mathfrak{g}$, we have

$$\begin{aligned}\left((e^{\alpha t})f\right)(x) &= \exp(tL_\alpha)f(x)\\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}(L^n_\alpha f)(x),\end{aligned}$$

where t lies in a sufficiently small neighbourhood of $0 \in \mathbb{C}$ and $f \in {A^0}_x$.

The following theorem states the operation of $exp(tL_\alpha)$ on $f \in {A^0}_x$.

Theorem 5.1 (*multiplier representation*) *Let*

$$L_\alpha = \sum_{k=1}^{n} \alpha_k L_k$$

be a linear combination of the operators L_k in the equation 5.1.

Then we have the following equation

$$(exp(tL_\alpha)f)\,(x) = v(t)f(x(t)),$$

where $x_i(t)$, $i = 1, ..., m$ and $v(t)$ are solutions of integrable system of differential equations

$$\frac{d}{dt}x_i(t) = \sum_{k=1}^{n} \alpha_k p_{ki}(x(t)),$$
$$\frac{d}{dt}v(t) = v(t)\sum_{k=1}^{n} \alpha_k q_k(x(t)),$$

$x_i(0) = x_i, i = 1, ..., m; v(0) = 1$ *respectively.*

Now let us make an application of the theorem of multiplier representation.

Let consider the following linear operators on ${A^0}_x$, $z^0 = (x^0, y^0) \in \mathbb{C}^2$,

$$J^+ = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (1 + \alpha - x)y,$$
$$J^- = xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$
$$J^3 = y\frac{\partial}{\partial y} + \frac{1}{2}(\alpha + 1).$$

These operators satisfy Lie bracket relations

$[J^3, J^\pm] = \pm J^\pm$, $[J^+, J^-] = 2J^3$. We can easily check that these relations satisfy Lie algebra conditions. So these operators span a 3-dimensional Lie alegbra which is isomorphic to $\mathfrak{sl}_2$.

In order to determine the multiplier representation Θ of SL_2, we first compute the actions of $\exp(a'J^+), \exp(b'J^-)$ and $\exp(c'J^3)$ on $f \in {A^0}_z$.

To obtain $\exp(a'J^+)f$, we need to integrate the equations

$$
\begin{aligned}
&\frac{\partial}{\partial a'}x(a') = x(a')y(a'),\\
&\frac{\partial}{\partial a'}y(a') = \{y(a')\}^2,\\
&\frac{\partial}{\partial a'}v(a') = v(a')[1+\alpha - x(a')]y(a'), x(0) = x,\\
&\qquad y(0) = y, v(0) = 1,
\end{aligned}
$$

giving

$$
x(a') = \frac{x}{1-a'y},\ y(a') = \frac{y}{1-a'y},\ v(a') = (1-a'y)^{-\alpha-1}\exp\left(-\frac{a'xy}{1-a'y}\right), |a'y| < 1.
$$

Therefore we have by multiplier representation theorem

$$
\exp\,(a'J^+)f\ (x,y) = (1-a'y)^{-\alpha-1}(-\frac{a'xy}{1-a'y})f\left(\frac{x}{1-a'y}, \frac{y}{1-a'y}\right), |a'y| < 1.
$$

A similar computation yields

$$
\exp\,(b'J^-)f(x,y) = f\left(\frac{x}{1-b'/y}, y-b'\right), |\frac{b'}{y}| < 1,
$$

and

$$
\exp\,(c'J^3)f(x,y) = e^{(1/2)(\alpha+1)c'}f(x, ye^{c'}).
$$

Thus

$$
\begin{aligned}
[T(\exp(a'f^+)\exp(b'f^-)\exp(c'f^3))f](x,y) &= [T(\exp(a'f^+))T(\exp(b'f^-))T(\exp(c'f^3))f](x,y)\\
&= e^{(1/2)(\alpha+1)c'}(1-a'y)^{-\alpha-1}\\
&\cdot \exp\left(-\frac{a'xy}{1-a'y}\right)f\left(\frac{xy}{(1-a'y)(y-b'+a'b'y)}, \frac{(y-b'+a'b'y)e^{c'}}{1-a'y}\right).
\end{aligned}
$$

So we have

$$\exp(a'f^+)\exp(b'f^-)\exp(c'f^3) = \begin{pmatrix} (1+a'b')e^{c'/2} & -a'e^{-c'/2} \\ -b'e^{c'/2} & e^{-c'/2} \end{pmatrix}.$$

Therefore, setting $e^{(1/2)c'} = 1/d, a' = -b/d, b' = -cd$, and using the fact that $ad - bc = 1$, we get

$$[T(g)f](x,y) = (by+d)^{-\alpha-1} exp(\frac{bxy}{by+d}) f(\frac{xy}{(ay+c)(by+d)}, \frac{ay+c}{by+d}), \ \max\left\{|\frac{by}{d}|, |\frac{c}{ay}|\right\} < 1, \ |\arg(d)| < \pi,$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

lies sufficiently close to the identity

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

of SL_2.

5.2 Application of Weisner's Method to Laguerre Polynomials L_n.

We now make some preliminary observations which will serve useful tools while obtaining generating functions.

Observation 1. Let

$$L(x, \frac{d}{dx}, n) = xD^2 + (1-x)D + n$$

be a linear differential operator containing a parameter n where $D = \frac{d}{dx}$. Assuming that L is a polynomial in n, we construct a partial differential operator

$$L(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y})$$

by substituting $y\frac{\partial}{\partial y}$ for n. Then $z = y^n v_n(x)$ is a solution of

$$L(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y})z = 0$$

if and only if $u = v_n(x)$ is a solution of

$$L(x, \frac{\partial}{\partial x}, n)u = 0.$$

Observation 2. Let $G(x, y)$ have a convergent expansion of the form $G(x, y) = \sum_n g_n(x)y^n$, where n is not necessarily a non-negative integer. If

$$L(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y})G(x, y) = 0,$$

then, within the region of convergence of the series $G(x, y)$,

$$\left[xD^2 + (1 - x)D + n\right](g_n(x)) = 0.$$

In particular, if $G(x, y)$ is regular at $x = 0$, $u = g_n(x)$ is also regular at $x = 0$.

Observation 3. Let

$$S = \psi(x)L(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}) + \lambda, \lambda \in \mathbb{C}$$

be the differential operator commuting with all first order linear differential operator

$$J = A_1(x, y)\frac{\partial}{\partial x} + A_2(x, y)\frac{\partial}{\partial y} + A_3(x, y).$$

Actually S is the Casimir element of the multiplier representation of the simple Lie algebra which is spanned by J^+, J^-, J^3, and which is isomorphic to $\mathfrak{sl}_2$.

$$Sf(x, y) = \lambda f(x, y),$$

then

$$(S[e^{tj}f](x, y) = \lambda[e^{tj}f](x, y).$$

The groundwork having been completed, we can attend to the task of obtaining generating functions. The process, in short, involves introducing first order linear differential operators forming a Lie algebra isomorphic to $\mathfrak{sl}_2$, and then, based on these operators, determining a local representation $[T(g)f](x,y), g \in SL_2$. By choosing $f(x,y)$ in certain ways, this local representation leads us to generating functions.

We need the following results in our analysis:

$$(1-ay)^{-\alpha}(1-\frac{b}{y})^{\beta} = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}(ay)^n, \tag{5.2}$$

$$\{\Gamma(1+n)\}^{-1}{}_2F_1\left[\begin{array}{ll} \alpha+n,\beta; & \\ & ab \\ 1+n; & \end{array}\right], |b| < |y| < |a|^{-1}; \tag{5.3}$$

$$(1-ay)^{-\beta}{}_1F_1\left[\begin{array}{ll} \alpha & \\ & \frac{b}{y}(1-ay) \\ 1-\beta & \end{array}\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\beta)}(ay)^n \{\Gamma(1+n)\}^{-1}{}_1F_1\left[\begin{array}{ll} \alpha & \\ & -ab \\ 1+n & \end{array}\right], \tag{5.4}$$

$$0 < |y| < |a|^{-1}.$$

The terms of the series in 5.2 and 5.4, corresponding to n=-1, -2, -3, ..., are well defined in view of the relations

$$\lim_{\gamma\to -n}\{\Gamma(\gamma\})^{-1}{}_2F_1\left[\begin{array}{ll} \alpha,\beta; & \\ & x \\ \gamma & \end{array}\right] = \frac{(\alpha)_{n+1}(\beta)_{n+1}}{(n+1)!}x^{n+1}{}_2F_1\left[\begin{array}{ll} \alpha+n+1,\beta; & \\ & x \\ n+2; & \end{array}\right] \tag{5.5}$$

and

$$\lim_{\gamma\to -n}\{\Gamma(\gamma\})^{-1}{}_1F_1\left[\begin{array}{ll} \alpha; & \\ & x \\ \gamma & \end{array}\right] = \frac{(\alpha)_{n+1}}{(n+1)!}x^{n+1}{}_1F_1\left[\begin{array}{ll} \alpha+n+1; & \\ & x \\ n+2 & \end{array}\right], \tag{5.6}$$

$n = 0, 1, 2...$

More generally, we have

$$\lim_{\gamma \to -n} \{\Gamma(\gamma)\}^{-1} {}_pF_{q+1} \begin{bmatrix} a_1, ..., a_p; & \\ & x \\ \gamma, b_1, ..., b_q & \end{bmatrix} = \frac{\prod_{j=1}^{p}(a_j)_{n+1}}{\prod_{j=1}^{q}(b_j)_{n+1}} \frac{x^{n+1}}{(n+1)!}$$

$$\cdot {}_pF_{q+1} \begin{bmatrix} a_1 + n + 1, ..., a_p + n + 1; & \\ & x \\ n + 2, b_1 + n + 1, ..., b_q + n + 1 & \end{bmatrix},$$

where as also in 5.5 and 5.5, $n = 0, 1, 2, \ldots$.

5.3 Application of Weisner's method to generalized Laguerre polynomials $L_n^{(\alpha)}$.

In this section we appeal to the theoretic concepts presented in the preceding sections and derive generating functions involving series of the Laguerre function $L_n^{(\alpha)}(x)$, where n is not necessarily a non-negative integer. We begin with the Laguerre differential equation

$$[x\frac{d^2}{dx^2} + (1 + \alpha - x)\frac{d}{dx} + n]u(x) = 0, \tag{5.7}$$

whose two solutions are given by

$$u_1(x) = L_n^{(\alpha)}(x) = \frac{\Gamma(1 + \alpha + n)}{\Gamma(1 + +n)\Gamma(1 + \alpha)} {}_1F_1 \begin{bmatrix} -n; & \\ & x \\ 1 + \alpha; & \end{bmatrix} \tag{5.8}$$

and

$$u_2(x) = x^{-\alpha} {}_1F_1 \begin{bmatrix} -\alpha - n; & \\ & x \\ 1 - \alpha; & \end{bmatrix} \tag{5.9}$$

Of these solutions of 5.7, the solution $u_1(x)$ given by 5.8 is regular at $x = 0$.

Substituting $y\frac{\partial}{\partial y}$ for n in 5.7, we construct a partial differential equation in the form:

$$[x\frac{\partial^2}{\partial x^2} + (1+\alpha-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}]f(x,y) = 0, \tag{5.10}$$

so that, by Observation 1,

$$f_1(x,y) = y^n u_1(x) \tag{5.11}$$

and

$$f_2(x,y) = y^n u_2(x) \tag{5.12}$$

are solutions of 5.10.

We now seek linearly independent differential operators J^3, J^+, J^- each of the form

$$A_1(x,y)\frac{\partial}{\partial x} + A_2(x,y)\frac{\partial}{\partial y} + A_3(x,y),$$

such that

$$J^3[y^n L_n^{(\alpha)}(x)] = a_n y^n L_n^{(\alpha)}(x), \tag{5.13}$$

$$J^-[y^n L_n^{(\alpha)}(x)] = b_n y^{n-1} L_{n-1}^{(\alpha)}(x) \tag{5.14}$$

and

$$J^+[y^n L_n^{(\alpha)}(x)] = c_n y^{n+1} L_{n+1}^{(\alpha)}(x), \tag{5.15}$$

where a_n, b_n and c_n are expressions in n which are independent of x and y, but not necessarily of α. Each $A_i(x,y)$, $i = 1,2,3$ on the other hand, is an expression in x and y which is independent of n, but not necessarily of α.

$$\frac{\partial}{\partial x}L_n^{(\alpha)}(x) = \frac{1}{x}[nL_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x)] \tag{5.16}$$

and

$$\frac{\partial}{\partial x}L_n^{(\alpha)}(x) = \frac{1}{x}[(x-\alpha-n-1)L_n^{(\alpha)}(x) + (n+1)L_{n+1}^{(\alpha)}(x)] \tag{5.17}$$

with the help of 5.16 and 5.17, it follows from 5.13,5.14 and 5.15 that

$$J^3 = y\frac{\partial}{\partial y} + \lambda, J^- = xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y},\ J^+ = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (1+\alpha-x)y. \tag{5.18}$$

If we take $\lambda = 1/2(\alpha+1)$, we find that the operators J^3, J^+, J^- satisfy the commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, [J^+, J^-] = 2J^3. \tag{5.19}$$

Looking at the commutation relations also satisfied by the basis vectors of $s\ell_2$, we conclude that the J-operators generate a Lie algebra isomorphic to $s\ell_2$, Lie algebra of SL_2.

In terms of the J-operators, we introduce the Casimir operator

$$S = J^+J^- + J^3J^3 - J^3 \tag{5.20}$$

$$= x[x\frac{\partial^2}{\partial x^2} + (1+\alpha-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}]1/4(\alpha^2-1).$$

We can verify that S commutes with J^3, J^+, J^-. Equation 5.20 enables us to rewrite 5.13 as

$$Sf(x,y) = 1/4(\alpha^2-1)f(x,y). \tag{5.21}$$

Now we turn our attention to determining the multiplier representation $[T(g)f](x,y)$, where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2, \tag{5.22}$$

corresponding to the J-operators. This has been worked out in 5.24, and we have

$$[T(g)f](x,y) = (by+d)^{-\alpha-1}\exp(\frac{bxy}{by+d})f(\frac{xy}{(ay+c)(by+d)}, \frac{ay+c}{by+d}), \tag{5.23}$$

where $|\frac{by}{d}| < 1, |\frac{c}{ay}| < 1$,

and g giving by 5.22 with $ad - bc = 1$ lies in a sufficiently small neighbourhood of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$$

To accomplish our job of obtaining generating functions, we search for the functions $f(x, y)$ which satisfy 5.21. The following case arise.

Case 1. When $f(x, y)$ is a common eigenfunction of S and J^3.

Let $f(x, y)$ be a solution of the simultaneous equations

$$Sf(x, y) = 1/4(\alpha^2 - 1)f(x, y), \tag{5.24}$$

$$J^3 f(x, y) = (1/2\alpha + v + 1/2)f(x, y), \tag{5.25}$$

which may be rewritten as

$$[x\frac{\partial^2}{\partial x^2} + (1 + \alpha - x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}]f(x, y) = 0, \tag{5.26}$$

$$(y\frac{\partial}{\partial y} - v)f(x, y) = 0. \tag{5.27}$$

Equations 5.26 and 5.27 yield

$$f(x, y) = y^v {L_v}^{\alpha}(x) \tag{5.28}$$

so that, by observation 3, we have

$$[T(g)f](x, y) = (by + d)^{-\alpha - v - 1}(ay + c)^v \exp\frac{bxy}{by + d} \tag{5.29}$$

$${L_v}^{\alpha}(\frac{xy}{(ay + c)(by + d)}$$

satisfying the relation

$$S[T(g)f](x, y) = 1/4(\alpha^2 - 1)[T(g)f](x, y). \tag{5.30}$$

If v is not an integer, 5.29 has an expansion of the form:

$$[T(g)f](x,y) = \sum_{n=-\infty}^{\infty} h_n(g,x) y^{v+n}, \tag{5.31}$$

$$|\frac{by}{d}| < 1, |\frac{c}{ay}| < 1, -\pi < \arg(a), \arg(d) < \pi,$$

$ad - bc = 1.$

Considering that $[T(g)f](x,y)$ is regular at $x = 0$, by Observation 2 of the preceding section, we have

$$h_n(g,x) = j_n(g) L^{\alpha}_{v+n}(x). \tag{5.32}$$

To determine $j_n(g)$, we set $x = 0$ in 5.31, and we have

$$(by+d)^{-\alpha-v-1}(ay+c)^{v} \tag{5.33}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+\alpha+v+n)\Gamma(1+v)}{\Gamma(1+\alpha+v)\Gamma(1+v+n)} j_n(g) y^{v+n},$$

which by 5.20, yields

$$j_n(g) = (-1)^n \frac{\Gamma(1+v+n)}{\Gamma(1+v)} a^v a^n d^{-\alpha-v-n-1} \{\Gamma(1+n)\}^{-1} {}_2F_1 \left[\begin{matrix} -v, \alpha+v+n+1; & \\ & \frac{bc}{ad} \\ 1+n; & \end{matrix} \right]. \tag{5.34}$$

Thus the generating function 5.31 becomes

$$(1+\frac{by}{d})^{-\alpha-v-1}(1+\frac{c}{ay})^{v} \exp(\frac{bxy}{by+d} L_v{}^{(\alpha)}(\frac{xy}{(ay+c)(by+d)}) = \tag{5.35}$$

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(1+v+n)}{\Gamma(1+v)} (-\frac{by}{d})^{n} L_{n+v}{}^{(\alpha)}(x)$$

$$\cdot \Gamma(1+n)^{-1} {}_2F_1 \left[\begin{matrix} -v, \quad \alpha+v+n+1; & \\ & \frac{bc}{ad} \\ 1+n; & \end{matrix} \right],$$

where $|\frac{by}{d}| < 1$, $|\frac{c}{ay}| < 1$, $ad - bc = 1$. The following special cases of 5.35 are worthy of note.

I When $a = d = y = 1$ and $c = 0$, we have

$$(1-b)^{-\alpha-v-1}\exp(\frac{bx}{1-b})L_v^{(\alpha)}(\frac{x}{(1-b)}) = \sum_{n=0}^{\infty}\frac{(1+v)_n}{n!}L_{v+n}^{(\alpha)}b^n, |b| < 1, \tag{5.36}$$

which may be compared with the well-known result 5.16.

II When $a = d = y = 1$ and $b = 0$, we have

$$(1-c)^v L_v^{(\alpha)}(\frac{x}{(1-c)}) = \sum_{n=0}^{\infty}\frac{(-\alpha-v)_n}{n!}L_{v-n}^{(\alpha)}(x)c^n, |c| < 1. \tag{5.37}$$

The generating function 5.35 was obtained under the assumption that $v, \alpha + v \in \mathbb{C}$ are not integers. However, when v is a nonnegative integer, say $v = k$, the generating function following from 5.35 is

$$(1+\frac{by}{d})^{-\alpha-k-1}(1+\frac{c}{ay})^k\exp(\frac{bxy}{by+d}L_k^{(\alpha)}(\frac{xy}{(ay+c)(by+d)})$$

$$= \sum_{n=0}^{\infty}\frac{n!}{k!}(-\frac{by}{d})^{n-k}L_n^{(\alpha)}(x)\{\Gamma(n-k+1)\}^{-1}{}_2F_1\left[\begin{matrix} -k, \alpha+n+1; & \\ & \frac{bc}{ad} \\ n-k+1; & \end{matrix}\right], \tag{5.38}$$

$|\frac{by}{d}| < 1, ad - bc = 1,$

whose validity is guaranteed by 5.17.

Case 2. When $f(x, y)$ is a common eigenfunction of the operators S and J^-. Let $f(x, y)$ satisfy the equations

$$Sf(x,y) = 1/4(\alpha^2 - 1)f(x,y) \tag{5.39}$$

and

$$J^- f(x,y) = -f(x,y), \tag{5.40}$$

that is, the partial differential equations

$$[x\frac{\partial^2}{\partial x^2} + (1+\alpha-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}]f(x,y) = 0 \tag{5.41}$$

and

$$(xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 1)f(x,y) = 0. \tag{5.42}$$

Assuming the general solution of 5.41 and 5.42 in the form

$$f(x,y) = e^y k(xy), \tag{5.43}$$

and substituting this in 5.41, we get

$$[u\frac{d^2}{du^2} + (1+\alpha)\frac{d}{du} + 1]k(u) = 0, u = xy. \tag{5.44}$$

This is a modification of Bessel's differential equation and has for its solution

$$k(u) = \Gamma(\alpha+1)u^{-(1/2)\alpha}J_\alpha(2\sqrt{u}) = {}_0F_1\left[\begin{array}{cc} -; & \\ & -u \\ \alpha+1; & \end{array}\right]. \tag{5.45}$$

Thus

$$f(x,y) = e^y {}_0F_1\left[\begin{array}{cc} -; & \\ & -xy \\ \alpha+1; & \end{array}\right], \tag{5.46}$$

and therefore

$$[T(g)f](x,y) = (by+d)^{-\alpha-1}\exp(\frac{ay+bxy+c}{by+d}){}_0F_1\left[\begin{array}{cc} -; & \\ & -\frac{xy}{(by+d)^2} \\ \alpha+1; & \end{array}\right] \tag{5.47}$$

satisfies

$$S[T(g)f](x,y) = 1/4(\alpha^2-1)[T(g)f](x,y). \tag{5.48}$$

Since $[T(g)f](x,y)$ is analytic at $y=0$, it can be expanded in the form

$$[T(g)f](x,y) = \sum_{n=0}^{\infty} p_n(g) L_n^{(\alpha)}(x) y^n. \tag{5.49}$$

To compute the coefficients $p_n(g)$, we put x=0 in 5.49, and we have

$$(by+d)^{-\alpha-1} \exp(\frac{ay+c}{by+d}) = \sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!} p_n(g) y^n. \tag{5.50}$$

Upon comparing 5.50 with the familiar generating function 5.22 , we find that

$$p_n(g) = \frac{n!}{(1+\alpha)_n} e^{c/d} d^{-1-\alpha} (-b/d)^n L_n^{(\alpha)}(\frac{1}{bd}). \tag{5.51}$$

Thus the generating function 5.49 assumes the form:

$$(1+\frac{by}{d})^{-\alpha-1} exp\{\frac{y(1+bdx)}{d(by+d)}\}{}_0F_1 \begin{bmatrix} -; & \\ & -\frac{xy}{(by+d)^2} \\ \alpha+1; & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{n'}{(1+\alpha)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(\frac{1}{bd})(-\frac{by}{d})^n, |\frac{by}{d}| < 1. \tag{5.52}$$

For $b = i/\sqrt{w}$ and $d = -i/\sqrt{w}$, $i = \sqrt{-1}$, 5.52immediately yields the Hille Hardy formula 5.24 in the form

$$(1-y)^{-\alpha-1} exp(-\frac{(x+w)y}{1-y}){}_0F_1 \begin{bmatrix} -; & \\ & \frac{wxy}{(1-y)^2} \\ 1+\alpha; & \end{bmatrix} \tag{5.53}$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(w) y^n, |y| < 1.$$

5.4 The Hypergeometric Function ${}_0F_1[-n, \beta; \gamma; x]$

The function $u(x)$ defined by

$$u(x) = {}_2F_1 \begin{bmatrix} -n, \beta; & \\ & x \\ \gamma; & \end{bmatrix} \tag{5.54}$$

is a solution, regular at $x = 0$, of the hypergeometric differential equation:

$$x(1-x)\frac{d^2}{dx^2} + [\gamma - (1+\beta-n)x]\frac{d}{dx} + n\beta u = 0. \tag{5.55}$$

We construct a partial differential equation by substituting $y\frac{\partial}{\partial y}$ for n, and thus we have

$$\{x(1-x)\frac{\partial^2}{\partial x^2} + xy\frac{\partial^2}{\partial x\partial y} + [\gamma - (1+\beta)x]\frac{d}{dx} + \beta y\frac{\partial}{\partial y}\}f(x,y) = 0, \tag{5.56}$$

where

$$f(x,y) = y^n u(x) \tag{5.57}$$

is a solution of 5.56.

$$J^3 = y\frac{\partial}{\partial y} + 1/2\gamma, \tag{5.58}$$

$$J^- = xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$

$$J^+ = xy(1-x)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (\gamma - \beta x)y$$

obeying the commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, [J^+, J^-] = 2J^3 \tag{5.59}$$

Obviously, the J-operators form the basis of a Lie algebra isomorphic to the Lie algebra $s\ell_2$. Considering that

$$ß = J^+J^- + J^3J^3 - J^3 = x\{x(1-x)\frac{\partial^2}{\partial x^2} + xy\frac{\partial^2}{\partial x\partial y} + [\gamma - (1+\beta)x]\frac{d}{dx} + \beta y\frac{\partial}{\partial y}\} + 1/4\gamma(\gamma-2), \tag{5.60}$$

we rewrite 5.56 in the form

$$\mathbb{S}f(x,y) = 1/4\gamma(\gamma-2)f(x,y). \tag{5.61}$$

To compute $\exp(a'J^+)f(x,y)$, $\exp(b'J^-)f(x,y)$ and $\exp(c'J^3)f(x,y)$, we apply in last theorem. We find that

$$\exp(a'J^+)f(x,y) = (1-a'y)^{\beta-\gamma}\{1+(x-1)a'y\}^{-\beta}f(\frac{x}{1+(x-1)a'y}, \frac{y}{1-a'y}), \tag{5.62}$$

$$|a'y| < 1, |(x-1)a'y| < 1;$$

$$\exp(b'J^-)f(x,y) = f(\frac{x}{1-b'/y}, y-b'), |\frac{b'}{y}| < 1; \tag{5.63}$$

$$\exp(c'J^3)f(x,y) = e^{(1/2)\gamma c'}f(x, ye^{c'}). \tag{5.64}$$

This leads us to

$$\exp(a'J^+)\exp(b'J^-)\exp(c'J^3)f(x,y) \tag{5.65}$$

$$= 1-a'y)^{\beta-\gamma}\{1+a'(x-1)y\}^{-\beta}e^{(1/2)\gamma c'}f(\frac{xy}{\{1+a'(x-1)y\}\{(1+a'b')y-b'\}}, \frac{(1+a'b')y-b'}{1-a'y}e^{c'}).$$

We obtain the multiplier representation

$$[T(g)f](x,y), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$$

as in Example (4.2) by setting in 5.65

$$a' = -b/d, b' = -cd, e^{(1/2)c'} = 1/d. \tag{5.66}$$

We thus have

$$[T(g)f](x,y) = (by+d)^{\beta-\gamma}\{d+(1-x)by\}^{-\beta}f(\xi,\eta), \tag{5.67}$$

where

$$\xi = \frac{xy}{\{d + (1-x)by\}(ay+c)}, \eta = \frac{ay+c}{by+d}, \tag{5.68}$$

and, for the validity of 5.67, we require that

$$|\frac{by}{d}| < 1, |\frac{c}{ay}| < 1, |\frac{(1-x)by}{d}| < 1, ad - bc = 1, \tag{5.69}$$

g lying in a sufficiently small neighborhood of the identity element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2.$$

Case 1. When $f(x,y)$ is a common eigenfunction of S and J^3.

From the simultaneous equation

$$Sf(x,y) = 1/4\gamma(\gamma-2)f(x,y). \tag{5.70}$$

and

$$J^3 f(x,y) = (1/2\gamma + v)f(x,y), \tag{5.71}$$

it follows

$$f(x,y) = y^v {}_2F_1 \left[\begin{matrix} -v, \beta; & \\ & x \\ \gamma; & \end{matrix} \right], \tag{5.72}$$

so that

$$[T(g)f](x,y) = (ay+c)^v (by+d)^{\beta-\gamma-v} \{d + (1-x)by\}^{-\beta} {}_2F_1 \left[\begin{matrix} -v, \beta; & \\ & \frac{xy}{\{d+(1-x)by\}(ay+c)} \\ \gamma; & \end{matrix} \right] \tag{5.73}$$

satisfies

$$\mathbb{S}[T(g)f](x,y) = 1/4\gamma(\gamma-2)[T(g)f](x,y). \tag{5.74}$$

If v is not integer, 5.73 has a Laurent series expansion

$$[T(g)f](x,y) = \sum_{n=-\infty}^{\infty} j_n(g)_2F_1\left[\begin{matrix} -v-n,\beta; & \\ & x \\ \gamma; & \end{matrix}\right] y^{v+n}. \tag{5.75}$$

To find $j_n(g)$, we set $x=0$. This enables us to arrive at

$$j_n(g) = (-1)^n \frac{\Gamma(\gamma+v+n)}{\Gamma(\gamma+v)} a^v b^n d^{-\gamma-v-n} \tag{5.76}$$

$$\{\Gamma(1+n)\}^{-1}{}_2F_1\left[\begin{matrix} -v,\gamma+v+n; & \\ & \frac{bc}{ad} \\ 1+n; & \end{matrix}\right].$$

Thus the generating function that we obtain in this case is

$$(1+\frac{by}{d})^{(\beta-\gamma-v)}\{1+(1-x)\frac{by}{d}\}^{-\beta}(1+\frac{c}{ay})^{v}{}_2F_1\cdot\left[\begin{matrix} -v \quad \beta & \\ & \frac{xy}{d+(1-x)by(ay+c)} \\ \gamma & \end{matrix}\right] =$$

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(\gamma+v+n)}{\Gamma(\gamma+v)}(-\frac{by}{d})^n{}_2F_1\left[\begin{matrix} -v-n,\beta; & \\ & x \\ \gamma; & \end{matrix}\right]\{\Gamma(1+n)\}^{-1}{}_2F_1\left[\begin{matrix} -v,\gamma+v+n; & \\ & \frac{bc}{ad} \\ 1+n; & \end{matrix}\right], \tag{5.77}$$

where, with ξ given by 5.68,

$$max\{|\xi|,|x|,|\frac{by}{d}|,|\frac{c}{ay}|,|\frac{(1-x)by}{d}|\} < 1. \tag{5.78}$$

The following special cases of 5.77 can easily be deduced:

$$(1-b)^{\beta-\gamma-v}\{1+(x-1)b\}^{-\beta}{}_2F_1\left[\begin{array}{cc}-v,\beta; & \\ & \frac{x}{1+(x-1)b} \\ \gamma; & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(\gamma+v)_n}{n!}{}_2F_1\left[\begin{array}{cc}-v-n,\beta; & \\ & x \\ \gamma; & \end{array}\right]b^n, \tag{5.79}$$

$|x|<1, |b|<1, |(x-1)b|<1;$

$$(1-c)^v{}_2F_1\left[\begin{array}{cc}-v-n,\beta; & \\ & \frac{x}{1-c} \\ \gamma; & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(-v)_n}{n!}{}_2F_1\left[\begin{array}{cc}-v+n,\beta; & \\ & x \\ \gamma; & \end{array}\right]c^n, \tag{5.80}$$

$|x|<1, |c|<1-|x|.$

Case 2. When $f(x,y)$ is a common eigenfunction of $\mathcal{G}$ and J^-.

The simultaneous equations

$$\mathcal{G}f(x,y)=1/4\gamma(\gamma-2)f(x,y),$$

$$J^-f(x,y)=-f(x,y)$$

admit themselves of a solution

$$f(x,y)=e^y{}_1F_1\left[\begin{array}{cc}\beta; & \\ & -xy \\ \gamma; & \end{array}\right].$$

Therefore

$$[T(g)f](x,y)=(by+d)^{\beta-\gamma}\{d+(1-x)by\}^{-\beta}\exp(\frac{ay+c}{by+d}){}_1F_1\left[\begin{array}{cc}\beta; & \\ & \frac{xy}{\{d+(1-x)by\}(by+d)} \\ \gamma; & \end{array}\right]$$

satisfies

$$\mathcal{G}[T(g)f](x,y) = 1/2\gamma(\gamma-2)[T(g)f](x,y),$$

and we are led eventually to the bilateral generating function:

$$(1+\frac{by}{d})^{\beta-\gamma}\{1+(1-x)\frac{by}{d}\}^{-\beta}\exp(\frac{y}{(by+d)d})\,{}_1F_1\left[\begin{array}{ll}\beta; & \\ & \frac{xy}{\{d+(1-x)by\}(by+d)} \\ \gamma; & \end{array}\right]$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{array}{ll}-n,\beta; & \\ & x \\ \gamma; & \end{array}\right]L_n^{\gamma-1}(\frac{1}{bd})(-\frac{by}{d})^n,$$

$$|\frac{by}{d}|<1, |(1-x)\frac{by}{d}|<1.$$

Setting $b=i/\sqrt{w}$, $d=-i/\sqrt{w}$, $i=\sqrt{-1}$, $\mathcal{G}$ assumes the familiar form

$$(1-y)^{\beta-\gamma}\{1+(x-1)y\}^{-\beta}\exp(-\frac{wy}{1-y})$$

$$\cdot{}_1F_1\left[\begin{array}{ll}\beta; & \\ & \frac{wxy}{(1-y)(1-y+xy)} \\ \gamma; & \end{array}\right]$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{array}{ll}-n,\beta; & \\ & x \\ \gamma; & \end{array}\right]L_n^{\gamma-1}(w)(y)^n,$$

$$|y|<1, |(x-1)y|<1.$$

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