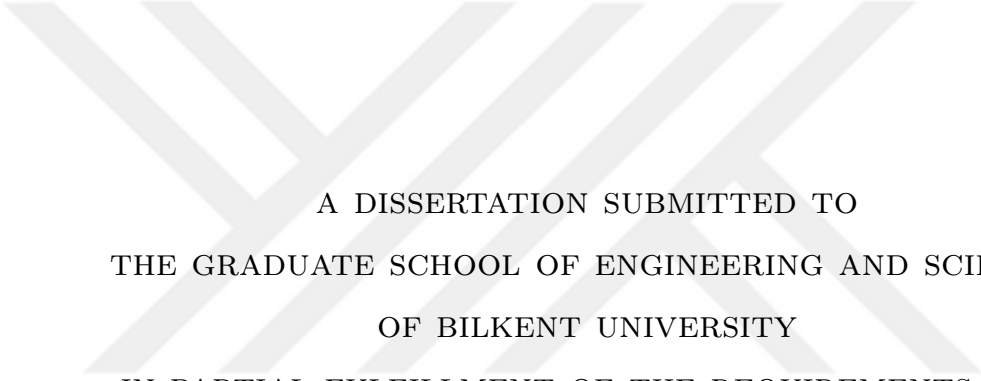


MODELING LUBRICANTS ENHANCED BY FINITE ELASTICITY POLYMERS



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Humayun Ahmed
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

MODELING LUBRICANTS ENHANCED BY FINITE ELASTICITY POLYMERS

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Lubrication is essential for the longevity of mechanical and biological surfaces in relative motion and susceptible to friction and wear. A well designed lubricant, for example a base oil enhanced with polymer additives, can effectively reduce both energetic and material losses. However, difficulty arises when modeling these lubricant mixtures exhibiting complex rheological behavior, in particular, a dependence of the viscosity on pressure (piezoviscosity), temperature (thermal thinning), shear rate (shear thinning) and the onset of viscoelasticity. Accurate estimates of the load carrying capacity of the thin lubricating film requires careful modeling of shear thinning. Available models such as the generalized Reynolds equation (GR) and the approximate shear distribution (ASD) have drawbacks such as large computational time and poor accuracy, respectively. In this work, we present a new approach, i.e. the modified viscosity (MV) model. We investigate, for both MV and GR, the load, the maximum pressure and the computational time, for (i) sliding (non-cavitating) contacts, (ii) cavitating and (iii) squeezing contacts. We observe that the computational time is reduced (i) considerably for non-cavitating sliding and rolling contacts and (ii) by several order of magnitudes for cavitating and squeezing contacts. For strongly elastic lubricants, the viscoelastic Reynolds (VR) approach (Ahmed & Biancofiore, Journal of Non-Newtonian Fluid Mechanics, 292, 104524, 2021.) has been shown to be effective in modeling (i) the pressure distribution and (ii) the load carrying capacity of a viscoelastic lubricating film for mechanical contacts for the Oldroyd-B constitutive relation. In this work, we have extended the VR approach to the non-linear finitely extensible non-linear elastic (FENE) type constitutive relations that account for the (i) finite extension of the polymer chains and (ii) shear thinning. We have validated the VR approach against DNS, showing an excellent agreement over a wide range of the Weissenberg number Wi , i.e. the ratio between the polymer relaxation time and the flow time scale, and finite extensibility parameter L , using FENE-CR and FENE-P. Following a thorough

validation, the pressure distribution and the load carrying capacity of a journal bearing, whose channel height is governed by the journal eccentricity ratio e , is considered. It is observed that the load carrying capacity of the film portrays a strongly non-linear dependence on Wi , L and e : while it increases for small values of Wi , limited greatly by the capacity of the polymer to stretch, a saturation and a subsequent decline is observed for highly viscoelastic regimes. Additionally, a weakly (strongly) eccentric configuration plays an important role in promoting (hindering) the growth in load versus both Wi and L . These effects are significant and have to be considered when modeling thin contacts lubricated with a strongly viscoelastic fluid. Additionally, we have extended the VR approach towards three-dimensional lubricated contacts (in cartesian and cylindrical coordinate systems) for several non-linear constitutive relations and have provided a linearized model in De . Owing to the increase in computational requirements, a globally fully-implicit numerical technique was adopted for the efficient solution of the equations. The load and friction response for an extruded journal bearing $e = 0.9$ (and parabolic slider) showed a strong variation versus the channel aspect ratio (otherwise zero for a Newtonian lubricant), i.e. $a = \ell_x/\ell_z$, the ratio between the channel streamwise and spanwise lengths. The effects of transient surface motion on the response of an elastic polymer have also been examined, with a specific focus on the load carrying capacity and the friction, via a second-order perturbation model and the VR approach. We find, the perturbed models only offer a matching prediction (i) once the motion has proceeded from some time and, (ii) the De is small. A simplified look into the influence of polymer elasticity on the temperature distribution of the film showed a weak dependency versus De . The film heating owing to the fluid dissipation remained largely unaffected unless the De was large.

Keywords: Viscoelasticity, thin film lubrication, reduced-order modeling, computational methods, polymers.

ÖZET

SINIRLI ELASTISITE POLİMERLERİ İLE GELİSTİRİLMİS YAĞLAYICILARIN MODELLENMESİ

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Yağlama, göreceli hareket halindeki ve sürtünme ve aşınmaya duyarlı mekanik ve biyolojik yüzeylerin uzun ömürlülüğü için gereklidir. İyi tasarlanmış bir yağlayıcı, örneğin polimer katkı maddeleri ile zenginleştirilmiş bir baz yağ, hem enerji hem de malzeme kayıplarını etkili bir şekilde azaltabilir. Bununla birlikte, karmaşık reolojik davranış sergileyen, özellikle viskozitenin basınca (piezo-viskozite), sıcaklığa (termal incelme), kayma hızına (kayma incelmesi) ve viskoelastisitenin başlangıcına bağımlılığını sergileyen bu yağlayıcı karışımlarının modellenmesinde zorluk ortaya çıkar.

İnce yağlayıcı filmin yük taşıma kapasitesinin doğru tahminleri, kayma incelmesinin dikkatli bir şekilde modellenmesini gerektirir. Genelleştirilmiş Reynolds denklemi (GR) ve yaklaşık kayma dağılımı (ASD) gibi mevcut modellerin sırasıyla büyük hesaplama süresi ve zayıf doğruluk gibi dezavantajları vardır. Bu çalışmada yeni bir yaklaşım, yani değiştirilmiş viskozite (MV) modelini sunuyoruz. Hem MV hem de GR için, (i) kayan (kavitasyon oluşturmeyan) kontaklar, (ii) kavitasyon yapan ve (iii) sıkıştıran kontaklar için yükü, maksimum basıncı ve hesaplama süresini araştırıyoruz. Hesaplama süresinin (i) kavitasyon oluşturmeyan kayan ve yuvarlanan kontaklar için önemli ölçüde ve (ii) kavitasyon yapan ve sıkıştıran kontaklar için birkaç büyüklük sırası kadar azaldığını gözlemledik.

Güçlü elastik yağlayıcılar için viskoelastik Reynolds (VR) yaklaşımı (Ahmed & Biancofiore, Journal of Non-Newtonian Fluid Mechanics, 292, 104524, 2021.) Oldroyd-B kurucu ilişki için mekanik temaslar için viskoelastik yağlama filminin (i) basınç dağılımını ve (ii) yük taşıma kapasitesini modellemede etkili olduğu gösterilmiştir. Bu çalışmada, VR yaklaşımını, (i) polimer zincirlerinin sonlu uzamasını ve (ii) kayma incelmesini açıklayan doğrusal olmayan, sonlu uzaya-bilir, doğrusal olmayan elastik (FENE) tipi kurucu ilişkilere kadar genişlettik. FENE- kullanarak, Weissenberg sayısı Wi , yani polimer gevşeme süresi ile akış

süresi ölçeği arasındaki oran ve sonlu genişletilebilirlik parametresi L arasındaki geniş bir aralıkta mükemmel bir uyum göstererek, DNS'ye karşı VR yaklaşımını doğruladık. CR ve FENE-P. Kapsamlı bir doğrulamanın ardından, kanal yüksekliği e muylu dışmerkezlik oranı tarafından yönetilen bir kaymalı yatağın basınç dağılımı ve yük taşıma kapasitesi dikkate alınır. Filmin yük taşıma kapasitesinin Wi , L ve e 'a oldukça doğrusal olmayan bir bağımlılık gösterdiği gözlemlenmiştir: küçük Wi değerleri için artarken, polimerin kapasitesiyle büyük ölçüde sınırlanmıştır. Yüksek derecede viskoelastik rejimler için gerilme, doygunluk ve ardından bir düşüş gözlemlenir. Ek olarak, zayıf (kuvvetli) bir eksantrik konfigürasyon, hem Wi hem de L 'a kıyasla yükteki büyümeyi desteklemede (engellemede) önemli bir rol oynar. Bu etkiler önemlidir ve güçlü bir viskoelastik sıvı ile yağlanan ince temas noktaları modellenirken dikkate alınmalıdır.

Ek olarak, çeşitli doğrusal olmayan kurucu ilişkiler için VR yaklaşımını üç boyutlu yağlanmış temaslara (kartezyen ve silindirik koordinat sistemlerinde) doğru genişlettik ve De cinsinden doğrusallaştırılmış bir model sağladık. Hesaplama gereksinimlerindeki artış nedeniyle, denklemlerin etkin çözümü için küresel olarak tam örtülü bir sayısal teknik benimsendi. $e = 0.9$ (ve parabolik kaydırıcı) taşıyan uzatılmış mil yatağı için yük ve sürtünme tepkisi, kanal en boy oranına (aksi takdirde Newton tipi bir yağlayıcı için sıfır) karşı güçlü bir değişiklik gösterdi; $a = \ell_x/\ell_z$, yani kanalın akış yönünde ve açıklık yönündeki uzunlukları arasındaki oran. Geçici yüzey hareketinin elastik bir polimerin tepkisi üzerindeki etkileri de, ikinci dereceden pertürbasyon modeli ve VR yaklaşımı aracılığıyla, yük taşıma kapasitesi ve sürtünmeye özel olarak odaklanılarak incelenmiştir. Pertürbasyon yapılan modellerin yalnızca (i) hareket belli bir süre ilerledikten sonra (ii) De 'nin küçük olduğunda geçerli olduğunu bulduk. Polimer esnekliğinin filmin sıcaklık dağılımı üzerindeki etkisine basitleştirilmiş bir bakış, De 'a karşı zayıf bir bağımlılık gösterdi. Sıvı yayılımı nedeniyle filmin ısınması, De büyük olmadığı sürece büyük ölçüde etkilenmeden kaldı.

Anahtar sözcükler: Viskoelastisite, ince film yağlama, azaltılmış dereceli modelleme, hesaplamalı yöntemler, polimerler.

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Chapter 1

Introduction

A large fraction of useful energy, and valuable material, is spent in overcoming losses due to friction and wear [6]. A simple but effective way to minimize these losses for machine elements, such as bearings and gears, is via oil based lubricants which form a continuous thin film that separates, the otherwise contacting, surfaces in relative motion. Therefore, the performance of not just rotary components, but artificial implants for biological joints, synthetic valves, can be significantly improved with the use of well designed lubricants. In hydrodynamic lubrication, a special case of fluid-based lubrication in which the moving surfaces are assumed to be rigid, friction and damage due to wear are minimized via the formation of a thin liquid film. This protective film generates a pressure field within the contact that supports the applied load and reduces friction at the boundaries. However, the oil mixture must be carefully designed to resist thermal degradation and shear, both of which can lead to a significant loss in viscosity and consequently a reduction in the load carrying capacity. For this reason, lubricants are enhanced via the addition of high molecular weight polymers [7, 8] which introduce several beneficial attributes such as, resistance to corrosion, thermal degradation, enhanced wear resistance [9].

Therefore, lubricant design has a direct impact on the performance of the thin film supporting the load. In fact, several additives of a polymeric type are added

to enhance its function, e.g. anti-freezing agents, viscosity index improver (VII) etc. [9]. In addition, they act as a viscosity thickener and impart shear stability, thereby further enhancing the pressure distribution within the film. For machine elements, visualizing the pressure distribution across the entire contact requires computational models because the contact point spans a few millimeters which prohibits the use of costly sensing apparatus. The addition of these polymers combined with the high speed ratios, whilst providing benefits, transforms the lubricant rheology from Newtonian and isoviscous to strongly non-Newtonian and pressure dependent.

In terms of the film pressure and the load carrying capacity the most important property of the lubricant is the viscosity. This is because, more effort is required in displacing a viscous fluid which means it offers greater resistance as viscosity increases. The viscosity remains constant for a Newtonian lubricant in a contact at low bearing numbers, i.e. the ratio between the viscous and the pressure forces. However, in a heavily loaded contact with large operating pressures, i.e. at high bearing numbers which is the ratio between the viscous forces and bearing forces, we find that the viscosity depends strongly on the pressure and the flow kinematics. The first important, and easily measurable, effect on the viscosity is its decrease at high shear rate, which is called shear thinning effect. In lubricated channels with high pressure (and pressure gradients), this is the first major effect.

Shear thinning commonly occurs in almost all polymer enhanced lubricants. This is due to (i) alignment of the polymer chains with the flow direction [10] and (ii) molecular scission, caused by the high strain rates, which leads to a permanent break down of the polymer coil itself [9]. The second major effect due to the pressure is piezoviscosity, which is the possibility of the lubricant viscosity to increase (exponentially) due to the large pressure [11], which causes a single polymer molecule to become more packed. In addition, the intertwining of neighboring polymer molecules as they are brought closer to each other due to the pressure leads to an increase in the flow resistance, which results in a higher viscosity [12]. This relationship is not linear and is generally explained by using empirical relations obtained from experimental data [13]. Furthermore, the high pressures can even lead to significant compression of the liquid lubricant resulting

in a substantial increase in the lubricant density.

As far as Newtonian, isoviscous and incompressible lubricants are concerned, the Reynolds equation suffices within the limits of lubrication theory [14]. However, it fails to account for (i) a more complex lubricant, oils with polymer additives for instance, and (ii) significant variations in the thermophysical properties of the lubricant. Therefore, in time the classical form of the Reynolds equation has been modified to account for variable viscosity and density [15, 16]. In particular shear thinning and the thermophysical variation in the viscosity must be considered to have accurate predictions of the load and friction. The latter may be overcome for certain cases by making appropriate assumptions, such as the temperature is constant or invariant across the thin film. Models based on these constraints need at most to take into account the temperature variation along the streamwise direction and ignore the cross-film variations [17]. However, such assumption does not work for a non-Newtonian lubricant. The change in lubricant viscosity due to shear thinning must be accounted for in a more robust manner to avoid numerical errors.

The introduction of piezoviscosity, compressibility and shear thinning leads to a complex non-linear model for the lubricant rheology. The most suitable model for these effects is the generalized Reynolds equation (GR) [18, 19]. GR is intended to account for any variation in the thermophysical properties of the lubricant, and non-Newtonian properties, such as shear thinning, can be implemented as well. However, the resulting differential equation is complex and computationally intensive. An alternative to the GR model is the use of approximate shear distribution methods (ASD) [20]. These methods assume that the shear strain rate required to model shear thinning is defined by a single function along the lubricated contact. The approximation of the shear strain rate could bring to an inaccurate estimate of the film pressure, especially if the pressure forces are high compared to the viscous forces. However, the methods are computationally appealing since the calculation of the shear strain rate is limited to evaluating a single function, which is usually constant [21].

Mathematical modeling, via a continuum approximation, of a lubricant film

containing finitely elastic polymers is challenging mainly due to the complex chemistry of the resulting mixture (solvent and polymer). For a Newtonian lubricant the mixture viscosity, strongly governing the film's characteristics, is seen to be a function of the pressure and temperature. Although the viscous response becomes non-linear, computational models are easily modified to take into account the additional non-linearity. On the other hand, non-Newtonian behavior can occur when the (i) viscosity decreases due to increasing shear stresses i.e., shear thinning (or generalized Newtonian) fluids [22], and/or the (ii) elasticity of the polymers becomes significant (viscoelasticity) [23, 4]. The additional complexity incurred, in constructing accurate mathematical models, for purely shear thinning lubricants has been dealt with in several ways over the past few decades. Whereas, computational techniques that can describe the effect of lubricant viscoelasticity on the film pressure have received comparatively little attention and is, therefore, the focus of this work.

The most accurate continuum model, for a viscoelastic flow, couples the fundamental conservation equations of mass and momentum with an equation describing the elastic polymer, i.e., a constitutive relation depending strongly on the polymer structure and topology. The strength of these flows is measured via the (i) Deborah number (De) which is the ratio of the polymer relaxation time, λ_p , and the residence time L/U^* (characteristic flow speed U^* and domain length L), and the (ii) Weissenberg number (Wi) which is the ratio of the polymer relaxation time and the flow time scale H^*/U^* (domain height H^*). For thin film flows, $De = \epsilon Wi$, where the thin film ratio $\epsilon = H^*/\ell \ll 1$. In this instance, the Deborah number may be small but the Weissenberg number is not limited and can acquire large values [24] because of the (i) highly elastic polymer additives (long polymer relaxation time) and the (ii) high shear rates (short deformation time scale) in lubrication problems [25].

The computational methods, to date, used to solve the continuum model become unstable for large values of the Weissenberg number, i.e., the high Weissenberg number problem (HWNP) [26]. Furthermore, the restriction on the maximum time step (Courant–Friedrichs–Lewy condition, i.e. the CFL number) due to the thin film approximation leads to a prohibitively large computational time.

Hence, the search for accurate, stable, computationally efficient and comparatively simpler models is becoming increasingly important.

A good portion of the literature aimed at understanding lubricant viscoelasticity, and developing simple models, relies on the second grade fluid [27, 28, 29, 30, 31] (or, more generally, a Rivlin fluid [32]) which expresses the polymer stress as a function of the Rivlin-Ericksen tensor, up to second order. The underlying approach involves linearizing the terms in the governing equations using in two different ways; first (i) a perturbation in either the second grade parameter or De and/or (ii) an assuming the velocity field to be quadratic along the flow [33, 34, 35]. In a similar manner, linearized models were also developed for Oldroyd-B which offers a rate-dependent and objective measure of the polymer stress tensor. In both cases, the constitutive relation is linear in the non-Newtonian stress tensor, which greatly aids the simplification of the governing equations. Interestingly, these methods are heavily reliant, and can differ significantly, on the scaling of the stress with respect to the film pressure. For a small deviation from the Newtonian case, i.e., very small polymer relaxation times, these linearized models work well and can lead to useful analytical expressions for the film pressure, and the load carrying capacity for simple geometries [36, 37]. In fact, it was demonstrated that the load carrying capacity increases as viscoelasticity strengthens [38, 4], in line with experimental observations [39]. However, if the relaxation time is large in comparison to the flow time scale the small-deviation approximation may not be suitable and higher orders need to be considered which raises the complexity of the problem.

While the linearized second-grade and Oldroyd-B fluid models are appealing due to their simplicity, they do not capture non-linear effects and shear thinning, both of which are commonly observed in tribological contacts under high operating pressures and shear rates [40]. The generalized Reynolds equation (GR) [18], which can accurately model the effects of shear thinning, was modified to also include polymer elasticity [41, 42, 43, 44] resulting in a system of non-linear equations capable of accounting for polymer viscoelasticity. However, this approach is based on a generalized Newtonian framework, where the polymer stress was assumed to scale with the strain rate. Additionally, the time rate of change of the

polymer shear stress was approximated via a reduced material derivative, keeping terms only along the streamwise and spanwise directions, since the stream line velocity u^* is larger than the cross-film velocity component v^* . The assumptions appear to be driven by a lack of experimental observations on the effect of polymer elasticity in the absence of shear thinning and thermal degradation, which can potentially dominate the flow regime [45]. It should be noted that the success of the GR in capturing non-linear effects of pure shear thinning, pressure and temperature dependence of viscosity and density [16] encouraged attempts to capture polymer elasticity as well. This further demonstrates the complex nature of the problem when deriving reduced order models.

Despite the limitations of the linearized models and the generalized Reynolds equations, they represent a significant step in obtaining reduced models that can capture viscoelastic effects, albeit in a limited range of Wi . As Wi increases, non-linear effects such as finite stretching of the polymer and shear thinning become significant. To accurately describe these effects, non-linear constitutive relations must be employed. One commonly used class of models are the finitely extensible non-linear elastic (FENE) models [46]. They provide an objective description of the stress tensor and place an upper limit on the maximum extension of the elastic polymer chain. The model introduced by Chilcott & Rallison [47] FENE-CR considers only the finite extension of the polymer, while the Peterlin approximation [48] FENE-P model also accounts for shear thinning. However, the FENE models are non-linear in the polymer stress and contain more than one parameter. As a result, it is difficult to simplify the FENE models without making significant assumptions [32] that can potentially affect their accuracy.

This thesis is organized as follows; the mathematical formulation for flow of a non-Newtonian lubricant is provided in chapter 2. In it, we describe the governing equations and the reduced-order models that follow from the application of lubrication theory, in particular the viscoelastic Reynolds approach of Ahmed and Biancofiore [23] which is extended towards three-dimensional cartesian and cylindrical contacts. We also construct and examine the use of perturbation models (up to the second-order) and compare it with the VR for time dependent surface motion.

In chapter 3, we introduce the numerical methodology employed for the solution of the equations. A special focus on the use of globally implicit methods for the solution of the polymer stress constitutive relations is provided along with a simplified numerical stability analysis.

Chapter 4 describes a new approach for modeling purely shear thinning (and viscoplastic) lubricants. It summarizes the problems pertaining to current modeling techniques and introduces a simple correction, allowing accurate estimates of the shear rate dependent viscosity in a computationally efficient manner. Finally, it concludes by offering comparisons with robust models and data available in literature.

In chapter 5, we apply the reduced-order models for lubricant films subjected to the vertical motion of the bounding channel. A comparison of the perturbation models in De with the VR approach, and a discussion on their applicability/viability, is made for the two cases; (i) a constant velocity and (ii) variable (oscillatory) surface motion.

We examine the effects of viscoelasticity for sliding (translating) two-dimensional contacts using the VR approach extended towards non-linear constitutive relations, specifically, the FENE models and also the PTT and Geisikus models. In particular, a discussion on the results obtained for a journal bearing (after a favorable comparison with the direct numerical simulations) is presented, including the variation in load versus the additional parameters introduced by these models. Finally, we conclude by addressing the possibility of lubricant

cavitation along the channel in the presence of strongly elastic polymers.

The study conducted for two-dimensional sliding contacts is extended to three-dimensional channels in chapter 7. A comparison of the load, average local force along the streamwise and spanwise directions, between the De -order linear models is made with the VR approach for an extruded parabolic slider and high eccentricity journal bearing; varying the ratio of the width to the length of the channels. The analyses are extended towards non-linear constitutive relations and comments on the suitability of the models are made. Finally, we look into the influence of polymer viscoelasticity in channels conforming to cylindrical coordinate systems via the VR, for a textured pad and a cone-and-plate rheometer configuration.

Chapter 8 presents a simplified approach for examining the effect of polymer elasticity on the temperature distribution of the channel. We examine for both an insulated and fixed wall temperature, how the polymers non-Newtonian traits affect the distribution of energy, and offer a discussion based on the simplified analysis.

Chapter 9.2 summarizes the findings of this thesis and derives conclusions based on the results obtained. Finally, some potential topics are presented that overcome the limitations of this study, offer avenues for further research and exploration.

Chapter 2

Problem formulation

In this chapter, we introduce the fundamental equations governing the passage of a non-Newtonian lubricant through a thin channel. The particular type of lubricant considered is one that comprises a base solvent, which is assumed Newtonian, and a polymer additive, exhibiting non-Newtonian behavior. The latter, depending strongly on the macro-molecular structure, e.g., branched, linear [49]. In special cases, where the molecular weight is high and the viscosity is large, the overall lubricant exhibits elastic properties, not captured via the standard Newtonian models or approximations thereof [50]. The equations that describe the overall behavior are consequently complex and challenging to solve from a mathematical perspective.

Firstly, we introduce governing system of equations in section 2.1, followed by an brief visitation of elementary lubrication theory in section 2.2 and its application to the viscoelastic case is given in section 2.3 for a cartesian and cylindrical configuration. The simplified models and modeling techniques, arising from it, namely the linearized model and the viscoelastic Reynolds are introduced in section 2.4 and section 2.5, respectively. We also introduce the problem of a film squeeze squeezed between two flat plates in section 2.6 and construct second-order perturbation models. Finally, some effort into modeling cavitating viscoelastic contacts is given in section 2.7

2.1 Governing equations

The flow of a fluid through a thin channel, with impermeable walls, is mathematically described via the fundamental conservation laws. We have for the conservation of mass

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{u}^*) = 0, \quad (2.1)$$

where, ρ^* is the fluid density, $\mathbf{u}^*$ is the velocity field and t^* is the time, and the conservation of momentum,

$$\frac{D(\rho^* \mathbf{u}^*)}{Dt^*} = -\nabla^* p^* + \nabla^* \cdot \boldsymbol{\sigma}^*, \quad (2.2)$$

where, p^* is the fluid pressure, $\boldsymbol{\sigma}^*$ is the Cauchy stress tensor. Eq. 2.2 is generally referred to as Cauchy's equation of motion.

Eq. 2.1 and Eq. 2.2 require an equation of state, expressing the density as some function of pressure. Moreover, an appropriate definition of the stress tensor is required in order to close the system. This is strongly dependent on lubricant rheology, concentration (dilute or concentrated), and flow conditions. The simplest definition exists for a Newtonian fluid

$$\boldsymbol{\sigma}^* = 2\eta^* \mathbf{D}^*, \quad (2.3)$$

where, η^* is the fluid viscosity, $\mathbf{D}^* = \nabla^* \mathbf{u}^* + \nabla^{*T} \mathbf{u}^{*T}$ is the deformation rate tensor and $\nabla^* \mathbf{u}^*$ is the velocity gradient tensor. Substituting Eq. 2.3 into Eq. 2.2, and assuming the fluid is incompressible ($\rho^* = \rho_0$) and isoviscous ($\eta^* = \eta_0$) gives the classical Navier-Stokes system of equations

$$\nabla^* \cdot \mathbf{u}^* = 0, \quad (2.4a)$$

$$\rho_0 \frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* p^* + \eta_0 \nabla^{*2} \mathbf{u}^*, \quad (2.4b)$$

where, ρ_0 is the reference density, and η_0 is the Newtonian reference viscosity.

The focus of this work is on the use of a non-Newtonian fluid mixture, acting as a lubricating medium, separating two surfaces in relative motion, with properties dependent on the pressure, and velocity. There are two major non-Newtonian

effects that can arise as a result of the combination of the relative surface speed and thin separation (or film height), namely the effect of shear thinning and emergence of elastic behavior. The former has been extensively studied in thin-film lubrication, via mathematical modeling and experimental techniques, but under the assumption of negligible elasticity and, therefore, constitutes a purely shear thinning or viscoplastic case. We use the latter term with care, since it implies that flow only occurs when a certain yield stress has been reached. It is also the case, when a highly viscous lubricant (example grease) flows weakly below a threshold value. The onset of viscoelastic effects are more complicated and have received particularly less attention (see chapter 1 for a detailed introduction).

For the viscoplastic case the stress

$$\boldsymbol{\sigma}^* = 2\eta^*(\mathbf{D}^* : \mathbf{D}^*)\mathbf{D}^*, \quad (2.5)$$

is a function of the velocity gradients, where $\mathbf{D}^* : \mathbf{D}^*$ is the equivalent scalar strain rate. Such a lubricant is also called a generalized Newtonian fluid, since the dependency is still on the kinematics of the flow.

For the viscoelastic case, the stress is assumed to be the sum of the Newtonian solvent and the elastic polymer additive

$$\boldsymbol{\sigma}^* = 2\eta_s\mathbf{D}^* + \boldsymbol{\tau}_p^*, \quad (2.6)$$

where, $\boldsymbol{\tau}_p^*$ is the polymer extra stress tensor and η_s is the solvent viscosity. Unlike the viscoplastic case, the polymer stress tensor can depend on both the fluid kinematics and also its own history. In other words, it requires an implicit definition which may or may not be linear. The expression describing the stress is the constitutive relation and several have been proposed for modeling the dilute concentration (by mass) of polymers. This is appropriate for a lubricant that comprises of a base solvent with a small quantity of dissolved polymers.

The simple understanding for most materials is that they can be characterized by their intrinsic properties, such as density, viscosity, material moduli and, in general, their state may comprise either a solid, liquid or gas at room temperature. However, given the thermodynamic properties of a material, it is sometimes

difficult to characterize its response strictly as that of solid, liquid or gas. Such is the case for viscoelastic materials that exist in between what we consider to be an elastic solid and viscous fluid. Interestingly, the ratio of time scale of observation (or the time over which flow process occurs), t_{flow}^* , to the materials intrinsic response time (relaxation time for polymers), λ , may cause the material, which may have been purely solid or liquid prior to the process, to exhibit strange dynamics. The ratio of the two quantities, as described in chapter 1, is the Deborah number, i.e. $De = \lambda/t_{flow}^*$.

From a modeling perspective, the simplest approach, introduced by Maxwell, involves a linear combination of the two limiting behaviors, namely the elastic solid ($\lambda \rightarrow \infty$) and the viscous fluid ($\lambda \rightarrow 0$). If we consider a Newtonian base fluid with a small concentration (by mass) of elastic polymer additives, then the objective stress evolution is described by the Oldroyd-B model,

$$\boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2\eta_p \mathbf{D}^*, \quad (2.7)$$

in which the polymers are treated as an ensemble of Hookean dumbbells (two masses connected via an ideal spring) with infinite extension.

The mathematical expressions of the different constitutive relations, used in this work, are provided in Tab. 2.1 and follow a general formulation (at least for the models adopted in this thesis)

$$F^* \boldsymbol{\tau}_p^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}_p^* = 2\eta_p G^* \mathbf{D}^*, \quad (2.8)$$

where, λ is the polymer relaxation time, $\overset{\nabla}{\boldsymbol{\tau}}_p^* = \frac{\partial \boldsymbol{\tau}_p^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \boldsymbol{\tau}_p^* - \mathbf{L}^{*T} \boldsymbol{\tau}_p^* - \boldsymbol{\tau}_p^* \mathbf{L}^*$ is the upper-convected rate, F^* and G^* are model specific functions that allow incorporating non-linear effects, such as shear thinning, finite extensibility, and phase-change.

2.2 Lubrication theory

The governing system of equations comprise a non-linear system of partial differential equations that, to this date and perhaps for the foreseeable future, elude

Model	Differential expression
Oldroyd-B	$\boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2\eta_p \mathbf{D}^*$
FENE-CR	$f(1 + \lambda \frac{Df^{-1}}{Dt}) \boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2f\eta_p \mathbf{D}^*,$
	$f = \frac{L^2 + \frac{\lambda}{\eta_p} \text{tr} \boldsymbol{\tau}^*}{L^2 - 3}$
FENE-P	$f(1 + \lambda \frac{Df^{-1}}{Dt}) \boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2a\eta_p \mathbf{D}^*,$
	$f = \frac{L^2 + \frac{\lambda}{a\eta_p} \text{tr} \boldsymbol{\tau}^*}{L^2 - 3}, a = \frac{L^2 - 3}{L^2}$
PTT-linear	$(1 + \frac{\kappa}{\eta_p} \lambda \text{tr}(\boldsymbol{\tau}^*)) \boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2\eta_p \mathbf{D}^*$
PTT-Exponential	$\exp(\frac{\kappa}{\eta_p} \lambda \text{tr}(\boldsymbol{\tau}^*)) \boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2\eta_p \mathbf{D}^*$
Giesekus	$\boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* + \alpha \boldsymbol{\tau}^* \boldsymbol{\tau}^* = 2\eta_p \mathbf{D}^*$

Table 2.1: A list of different viscoelastic constitutive relations used in this work.

any meaningful analytical treatment without making strong simplifications. One such method is the application of lubrication theory, which despite its name, pertains to problems involving fluid flow where the ratio of the characteristic length, ℓ_0 , to the characteristic height, H_0 , of the system (in the case of a channel flow) is small, i.e., the thin film ratio $\epsilon = \ell_0/H_0 \ll 1$. This assumption allows simplifying the original set of non-linear equations (i) significantly for a Newtonian lubricant, and (ii) to a minor extent the non-Newtonian case.

Although, general lubrication theory is independent of the coordinate system, the choice of the relevant length, height and time scales can strongly affect the reduced system of equations that follow. In this work, we present an analysis for contacts that are well suited for a cartesian coordinate system in both two and three-dimensions but also for scenarios that are ideal for a cylindrical coordinate system.

First, we shall present the classical case of a Newtonian fluid, a result that was first derived by Osborne Reynolds in 1886, using the standard Cartesian coordinates (x^*, y^*, z^*) . A typical lubricated contact can undergo sliding (streamwise, x^* , and/or spanwise z^*) motion and/or vertical motion (crossfilm, y^*). The latter also referred to as squeezing.

The governing system of equations for an incompressible, isoviscous and

isothermal lubricant film subjected to boundary motion comprise Eqs. 2.4a-2.4b. The equations are non-dimensionalized using the appropriate length scale

$$\mathbf{x}^* = (x \cdot \ell_x, y \cdot H_0, z \cdot \ell_z), \quad (2.9)$$

which, upon substituting into continuity equation (Eq. 2.4a and simplifying, yields the corresponding velocity scale

$$\mathbf{u}^* = (u \cdot U_0, v \cdot \epsilon U_0, z \cdot U_0). \quad (2.10)$$

Using Eq. 2.9 and Eq. 2.10, and substituting into the momentum equation (Eq. 2.4b), we obtain the non-dimensional system of (scalar) equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + a \frac{\partial w}{\partial z} = 0, \quad (2.11a)$$

$$\epsilon Re \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + aw \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \epsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \epsilon^2 a^2 \frac{\partial^2 u}{\partial z^2}, \quad (2.11b)$$

$$\epsilon^3 Re \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + aw \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \epsilon^4 \frac{\partial^2 v}{\partial x^2} + \epsilon^2 \frac{\partial^2 v}{\partial y^2} + \epsilon^4 a^2 \frac{\partial^2 v}{\partial z^2}, \quad (2.11c)$$

$$\epsilon Re \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + aw \frac{\partial w}{\partial z} \right) = -a \frac{\partial p}{\partial z} + \epsilon^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \epsilon^2 a^2 \frac{\partial^2 w}{\partial z^2}, \quad (2.11d)$$

where, $a = \frac{\ell_x}{\ell_z}$ is the channel aspect ratio, and $Re = \frac{\rho_0 U_0 H_0}{\eta_0}$ is the Reynolds number. The pressure scale, $p^* = p \cdot \frac{\eta_0 U_0 \ell_x}{H_0^2}$ obtained by assuming that the pressure gradients must be $O(1)$. This must be the case, since the thin lubricant film supports a load over a very short channel length. The pressure distribution, which eventually governs the maximum allowable load, F_N , must be significant. In fact, in most mechanical contacts, it can easily exceed several Giga Pascals over a few millimeters of the contact region. In addition, the time scale, $t^* = t \cdot \frac{\ell_x}{U_0}$, is chosen assuming a dominant sliding motion along the streamwise direction.

We enforce the assumption of a very thin film by asserting $\lim \epsilon^2 \rightarrow 0$ on Eqs.

2.11a-2.11d and obtain the thin-film reduced system of equations for the Newtonian case

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + a \frac{\partial w}{\partial z} = 0, \quad (2.12a)$$

$$\epsilon Re \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + aw \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad (2.12b)$$

$$\frac{\partial p}{\partial y} = 0, \quad (2.12c)$$

$$\epsilon Re \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + aw \frac{\partial w}{\partial z} \right) = -a \frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial y^2}. \quad (2.12d)$$

The most immediate outcome of Eqs. 2.12a-2.12d is the invariance of the pressure across the film. In addition, the gradients of the velocity across the film strongly dominate and all streamwise and spanwise variations in the velocity field, although not zero, are inconsequential in the balance of forces. It also demonstrates the significance of inertial effects, which in most cases can be ignored owing to $\epsilon \ll 1$. We obtained Eqs. 2.12a-2.12d for a cartesian coordinate system, but the outcome is the same across the cylindrical and spherical coordinate systems (or some curvilinear domain), namely that the thin film allows retaining the cross film gradients of the streamwise and spanwise velocity fields. In the case of a cylindrical system, which generally involve rotating surfaces, the streamwise direction is along the circumferential axis and the spanwise direction is along the radial axis.

Before proceeding we make some comments on the additional parameter, i.e., the aspect ratio a , which is a measure of the strength of the spanwise gradients. For small a , the gradients of the pressure and velocity along the spanwise direction are consequently small and the dominant terms are along the streamwise direction. For $a \ll 1$ ($a^2 \approx 0$), the system reduces to the two dimensional problem along x and y only. However, for large values of a , the argument is not straightforward, since we require that the film remain thin, along both the streamwise and spanwise directions, i.e., $\epsilon_x = \frac{H_0}{\ell_x} \ll 1$ and $\epsilon_z = \frac{H_0}{\ell_z} \ll 1$. Clearly, $\epsilon_z = a \cdot \epsilon_x$, and, if we arbitrarily increase the aspect ratio, such that $a \gg 1$ for a fixed ϵ_x , then ϵ_z approaches $O(1)$, violating the assumptions of lubrication theory. To ensure lubrication theory holds for situations where $a \gg 1$,

we must further enforce $\epsilon_x \ll \epsilon_z \ll 1$ [51].

Note, that in addition to the thin-film approximation, it is also common to state, for most practical problems involving creeping flow, that $Re \ll 1$. For several microfluidic applications and, in general, lubricated contacts, this assumption is not invalid. However, care must be taken in the event the time scale is chosen to reflect the squeezing motion instead. For example, a high frequency, low amplitude oscillation will yield a different scaling for the inertial terms, see Szeri for a detailed discussion [16].

The classical Reynolds equation is retrieved by solving the system Eqs. 2.11a-2.11d

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\frac{h^3}{12} \frac{\partial p}{\partial z} \right) = \frac{1}{2} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}, \quad (2.13a)$$

$$u = \frac{1}{2} \frac{\partial p}{\partial x} (y^2 - yh) + \left(1 - \frac{y}{h} \right), \quad (2.13b)$$

$$w = \frac{a}{2} \frac{\partial p}{\partial z} (y^2 - yh), \quad (2.13c)$$

and having used the boundary conditions

$$u(x, y = 0, z) = 1, \quad u(x, y = h, z) = 0, \quad (2.14a)$$

$$v(x, y = 0, z) = 0, \quad v(x, y = h, z) = \frac{\partial h}{\partial t}, \quad (2.14b)$$

$$w(x, y = 0, z) = 0, \quad w(x, y = h, z) = 0, \quad (2.14c)$$

$$p(x = 0, z) = 0, \quad p(x = 1, z) = 0, \text{ and} \quad (2.14d)$$

$$p(x, z = 0) = 0, \quad p(x, z = 1) = 0, \quad (2.14e)$$

where, $h = h^*/H_0$ is the dimensionless film thickness (or surface/channel height). These boundary conditions conform to a sliding/squeezing contact in which the bottom surface is moving and the top boundary approaches the bottom surface. There are several variations of these boundary conditions (spanwise velocity, deformable surface etc) and each will yield a slightly modified form for Eq. 2.13a. However, the fundamental premise of the Reynolds equation is to obtain the pressure distribution given the surface height. This holds true regardless of the specific boundary conditions. In this work, two specific scenarios are considered,

namely a (i) purely sliding contact (in two and three-dimensions), and (ii) a squeezing cylindrical flat contact.

2.3 Viscoelastic thin films

The application of lubrication theory to the Newtonian case offers insights into the order of magnitude of terms in the balance of momentum. Now, we apply lubrication theory, in the same manner as the Newtonian case, to the viscoelastic case. The governing system of equations for an incompressible, isoviscous, and isothermal thin film with a dilute concentration of polymers are

$$\nabla^* \cdot \mathbf{u}^* = 0, \quad (2.15a)$$

$$\rho_0 \frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* p^* + \eta_s \nabla^{*2} \mathbf{u}^* + \nabla^* \cdot \boldsymbol{\tau}_p^*, \quad (2.15b)$$

$$\boldsymbol{\tau}_p = f(\boldsymbol{\tau}_p, \mathbf{D}^*), \quad (2.15c)$$

where, η_s is the solvent viscosity and the polymer stress $\boldsymbol{\tau}_p^*$ is dictated by an implicit constitutive relation (see Tab. 2.1). Eq. 2.15b represents the Cauchy's equation of motion and takes into account the forces due to the polymer extra stress tensor.

If we straightforwardly apply lubrication theory to Eqs. 2.15a-2.15b, i.e., choosing the same length, time, velocity and pressure scaling, and substituting into Eqs. 2.15a-2.15b, we obtain the thin-film reduced Cauchy's equations of motion (assuming $Re = 0$)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + a \frac{\partial w}{\partial z} = 0, \quad (2.16a)$$

$$\frac{\partial p}{\partial x} = \beta_s \frac{\partial^2 u}{\partial y^2} + \epsilon^2 \frac{\partial \tau_{xx}}{\partial x} + \epsilon \frac{\partial \tau_{yx}}{\partial y} + a \epsilon^2 \frac{\partial \tau_{zx}}{\partial z}, \quad (2.16b)$$

$$\frac{\partial p}{\partial y} = 0, \quad (2.16c)$$

$$a \frac{\partial p}{\partial z} = \beta_s \frac{\partial^2 w}{\partial y^2} + \epsilon^2 \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + a \epsilon^2 \frac{\partial \tau_{zz}}{\partial z}, \quad (2.16d)$$

where, the thin-film ratio appears in the terms involving the gradients of the components of the polymer stress tensor. If we naively assume these terms vanish

as $\epsilon \rightarrow 0$, then the effect of the extra stress is lost and we obtain the Newtonian system of equations. Clearly, the scaling must be chosen in a manner that retains these additional terms. However, this is not a trivial process and requires some knowledge about the specific problem. Indeed, no definite choice of scaling exists and choices can be made, such that the normal stress, for example, is ignored and only the polymer shear stress is retained. Similarly, if one assumes all components of the stress tensor are on the same order of magnitude, i.e., $\boldsymbol{\tau}_p^* = \boldsymbol{\tau}_p \cdot \frac{\eta_0 U_0}{H_0}$ [52] or $\boldsymbol{\tau}_p^* = \boldsymbol{\tau}_p \cdot \frac{1}{\epsilon}$ [53], then different sets of equations are obtained.

Here, we make the strong statement that the polymer stress components, $\tau_{i,j}$, are significant, owing to the elasticity of the polymer allowing it to stretch significantly, and their gradients must, therefore, be retained in the governing equations. Furthermore, in order to obtain equations conforming completely to lubrication theory (see Tanner [54] for a system of lubrication-like equations) the pressure variation across the film must vanish as in the Newtonian case. These two assumptions provide the scaling

$$\tau_{xx} = \tau_{xx}^* \frac{h_0^2}{\eta_0 U_0 \ell_x}, \tau_{zx} = \tau_{zx}^* \frac{h_0^2}{\eta_0 U_0 \ell_x}, \tau_{xy} = \tau_{xy}^* \frac{h_0}{\eta_0 U_0} \quad (2.17a)$$

$$\tau_{yy} = \tau_{yy}^* \frac{\ell_x}{\eta_0 U_0}, \tau_{yz} = \tau_{yz}^* \frac{h_0^2}{\eta_0 U_0 \ell}, \tau_{zz} = \tau_{zz}^* \frac{h_0^2}{\eta_0 U_0 \ell_x}, \quad (2.17b)$$

where, $\eta_0 = \eta_s + \eta_p$, and η_p is the polymer viscosity. Now, we substitute Eqs. 2.17a-2.17b, into the reduced system of Eqs. 2.18a-2.18j, and consider, for simplicity, the Oldroyd-B constitutive relationship. Neglecting terms on the order of

ϵ^2 , we obtain a thin-film reduced system of equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + a \frac{\partial w}{\partial z} = 0, \quad (2.18a)$$

$$\frac{\partial p}{\partial x} = \beta_s \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + a \frac{\partial \tau_{zx}}{\partial z}, \quad (2.18b)$$

$$\frac{\partial p}{\partial y} = 0, \quad (2.18c)$$

$$a \frac{\partial p}{\partial z} = \beta_s \frac{\partial^2 w}{\partial y^2} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + a \frac{\partial \tau_{zz}}{\partial z}, \quad (2.18d)$$

$$\tau_{xx} + De \left(\frac{D\tau_{xx}}{Dt} - 2\tau_{xy} \frac{\partial u}{\partial y} - 2\tau_{xx} \frac{\partial u}{\partial x} - 2a \frac{\partial u}{\partial z} \tau_{xz} \right) = 0, \quad (2.18e)$$

$$\begin{aligned} \tau_{xy} + De \left(\frac{D\tau_{xy}}{Dt} - \tau_{yy} \frac{\partial u}{\partial y} - \tau_{xx} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \tau_{xy} - \frac{\partial v}{\partial y} \tau_{xy} - a \frac{\partial u}{\partial z} \tau_{yz} \right. \\ \left. - a \frac{\partial v}{\partial z} \tau_{xz} \right) = (1 - \beta) \frac{\partial u}{\partial y}, \end{aligned} \quad (2.18f)$$

$$\begin{aligned} \tau_{xz} + De \left(\frac{D\tau_{yz}}{Dt} - \tau_{xz} \frac{\partial u}{\partial x} - \tau_{xx} \frac{\partial w}{\partial x} - \tau_{yz} \frac{\partial u}{\partial y} - \tau_{xy} \frac{\partial w}{\partial y} \right. \\ \left. - a \tau_{zz} \frac{\partial u}{\partial z} - a \tau_{xz} \frac{\partial w}{\partial z} \right) = 0, \end{aligned} \quad (2.18g)$$

$$\tau_{yy} + De \left(\frac{D\tau_{yy}}{Dt} - 2\tau_{xy} \frac{\partial v}{\partial x} - 2\tau_{yy} \frac{\partial v}{\partial y} - 2a \frac{\partial v}{\partial z} \tau_{yz} \right) = 2(1 - \beta) \frac{\partial v}{\partial y}, \quad (2.18h)$$

$$\begin{aligned} \tau_{yz} + De \left(\frac{D\tau_{yy}}{Dt} - \tau_{xz} \frac{\partial v}{\partial x} - \tau_{xy} \frac{\partial w}{\partial x} - \tau_{yz} \frac{\partial v}{\partial y} - \tau_{yy} \frac{\partial w}{\partial y} - a \tau_{zz} \frac{\partial v}{\partial z} \right. \\ \left. - a \tau_{yz} \frac{\partial w}{\partial z} \right) = (1 - \beta) \frac{\partial w}{\partial y}, \end{aligned} \quad (2.18i)$$

$$\tau_{zz} + De \left(\frac{D\tau_{yy}}{Dt} - 2\tau_{xz} \frac{\partial w}{\partial x} - 2\tau_{xy} \frac{\partial w}{\partial y} - 2a \frac{\partial w}{\partial z} \tau_{zz} \right) = 0, \quad (2.18j)$$

where, $\beta = \frac{\eta_s}{\eta_s + \eta_p}$ is the solvent concentration ratio, and $\frac{D}{Dt} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + aw \frac{\partial}{\partial z}$ is the material derivative operator.

One immediately observes the absence of any helpful simplifications in the scalar constitutive equations governing the transport of stress in the lubricant film. The velocity gradients along the streamwise and spanwise directions vanish, whereas the crossfilm gradients are retained. However, those same gradients, when appearing together with the stress are not negligible and are interpreted as the stretch terms. Furthermore, by setting (i) $De = 0$, the equivalent Newtonian system, and (ii) $a = 0$, the two-dimensional reduced system is retrieved.

Eqs. 2.18a-2.18j comprise a non-linear coupled system of partial differential equations. In order to determine the pressure, velocity and stress distribution, they must be solved either via analytical or numerical techniques. However, it is easy to see the difficulty with the former, but the latter is also not without its constraints.

For the moment, we have considered the Oldroyd-B constitutive model which demonstrates the effect of polymer stretch and is linear in the stress components. However, the choice of the constitutive relation introduces additional parameters, e.g. the finite extensibility parameter, L^2 , for the FENE-type models, and κ for the PTT models, that can affect the scaling of the stress components. As an example, if we consider the linear PTT model, we will have for the non-linear term, $(1 + \frac{\kappa}{\eta_p} \lambda \text{tr}(\tau_{i,j}^*)) \tau_{i,j}^*$, using the scaling introduced earlier, we will get $(1 + \frac{\kappa}{(1-\beta)} \frac{De}{\epsilon^2} \tau_{xx}) \tau_{i,j}$. Clearly, a term on the ϵ^{-2} -order will dominate the remaining terms. However, the choice of κ can be chosen small enough ($\kappa \approx \epsilon^2$) to prevent the $O(1)$ terms from disappearing. Similar observations can be made for the FENE and Giesekus models.

2.3.1 Cylindrical coordinate system

A set of thin-film reduced equations can also be obtained for a lubricated channel modeled via the cylindrical coordinate system (r^*, θ, z^*) , following the same procedure as for the cartesian system. An example of this would be the rotational motion for two perfectly aligned circular flat plates (mimicking a cone-and-plate rheometer). The appropriate length scaling

$$r^* = \frac{r}{R_0}, z^* = \frac{z}{H_0}, \epsilon = \frac{H_0}{R_0}, \quad (2.19)$$

where, R_0 is the characteristic radius, is substituted into the continuity equation (Eq. 2.4a) yielding the velocity scaling

$$u_r^* = u_r \cdot U_0, u_\theta^* = u_\theta \cdot \frac{U_0}{R_0}, u_z^* = u_z \cdot \epsilon V_0, \quad (2.20)$$

where, V_0 is the reference vertical approach velocity, U_0 is the reference radial velocity. Note that we have assumed an arbitrary reference speed U_0 for the sake

of generality. In practice, the rotation rate Ω_0 is specified.

Substituting Eqs. 2.19 and Eqs. 2.20, into the governing system of Eqs. ??-?? gives the dimensionless system of equations for an incompressible, isoviscous and isothermal film,

$$\frac{1}{r} \frac{\partial(ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \quad (2.21a)$$

$$\begin{aligned} \epsilon^2 Re \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} \right) &= -\frac{\partial p}{\partial r} + \\ \epsilon^2 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \epsilon^2 \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} &- \epsilon^2 \frac{u_r}{r^2} - \epsilon^2 \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta}, \end{aligned} \quad (2.21b)$$

$$\begin{aligned} \epsilon^2 Re \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} \right) &= -\frac{\partial p}{\partial r} + \\ \epsilon^2 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \epsilon^2 \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} &- \epsilon^2 \frac{u_\theta}{r^2} - \epsilon^2 \frac{2}{r^2} \frac{\partial u_r}{\partial \theta}, \end{aligned} \quad (2.21c)$$

$$\begin{aligned} \epsilon^4 Re \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \\ \epsilon^4 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \epsilon^4 \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \epsilon^2 \frac{\partial^2 u_z}{\partial z^2}, \end{aligned} \quad (2.21d)$$

where, $Re = \frac{\rho_0 U_0 R_0}{\eta_0}$, and assuming $\epsilon^2 \rightarrow 0$, gives the set of reduced scalar equations

$$\frac{\partial p}{\partial r} = \frac{\partial^2 u_r}{\partial z^2} \quad (2.22a)$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{\partial^2 u_\theta}{\partial z^2} \quad (2.22b)$$

$$\frac{\partial p}{\partial z} = 0 \quad (2.22c)$$

for a Newtonian lubricant, where the pressure scaling $p^* = p \cdot \frac{\eta_0 U_0 R_0}{H_0^2}$ (or $p^* = p \cdot \frac{\eta_0 \Omega_0 R_0^2}{H_0^2}$) emerges. We observe, consistent with lubrication theory, that the pressure remains invariant across the film and only the velocity gradients across the film remain.

Applying the boundary conditions for a sliding, and squeezing lubricated contact,

$$u_r(r, \theta, z = 0) = 0, \quad u_r(r, \theta, z = h) = 0, \quad (2.23a)$$

$$u_\theta(r, \theta, z = 0) = 1, \quad u_\theta(r, \theta, z = h) = 0, \quad (2.23b)$$

$$u_z(r, \theta, z = 0) = 0, \quad u_z(r, \theta, z = h) = \frac{\partial h}{\partial t}, \quad (2.23c)$$

$$\frac{\partial p}{\partial r}(r = 0, z) = 0, \quad p(r = 1, \theta) = 0, \text{ and} \quad (2.23d)$$

$$p(r, \theta = 0) = p(r, \theta = 2\pi), \quad (2.23e)$$

the classical Reynolds equation is retrieved

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r h^3}{12} \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{h^3}{12r} \frac{\partial p}{\partial \theta} \right) = \frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial h}{\partial \theta}. \quad (2.24)$$

Note the absence of a parameter, such as the aspect ratio a in the cartesian case, that would, in its limits, reduce the three-dimensional problem to a two-dimensional one. In fact, one must enforce, based on the channel geometry, a symmetry argument about some axis, thereby allowing $\frac{\partial}{\partial \theta} \rightarrow 0$ for all quantities. This implies that the circumferential surface gradients must vanish, but in doing so we remove the effect of sliding motion, effectively eliminating fluid entrainment and pressure generation, yielding the trivial solution $p = 0$ (in the absence of squeezing motion).

Now, we assume the lubricant to comprise a mixture; a Newtonian solvent and viscoelastic polymer additive. Following the procedure demonstrated for the cartesian coordinate system, and using the polymer stress scaling

$$\tau_{rr}^* = \tau_{rr} \cdot \frac{\eta_0 U_0 R_0}{H_0^2}, \tau_{\theta r}^* = \tau_{\theta r} \cdot \frac{\eta_0 U_0}{H_0}, \tau_{zr}^* = \tau_{zr} \cdot \frac{\eta_0 U_0}{H_0}, \quad (2.25a)$$

$$\tau_{\theta\theta}^* = \tau_{\theta\theta} \cdot \frac{\eta_0 U_0 R_0}{H_0^2}, \tau_{z\theta}^* = \tau_{z\theta} \cdot \frac{\eta_0 U_0}{H_0}, \tau_{zz}^* = \tau_{zz} \cdot \frac{\eta_0 U_0}{R_0}, \quad (2.25b)$$

we arrive at a system of thin-film reduced system of equations (considering the

Oldroyd-B constitutive relation)

$$\frac{1}{r} \frac{\partial(ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0, \quad (2.26a)$$

$$\frac{\partial p}{\partial r} = \beta \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r}, \quad (2.26b)$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \beta \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}, \quad (2.26c)$$

$$\frac{\partial p}{\partial z} = 0 \quad (2.26d)$$

$$\tau_{rr} + De \left(\frac{\partial \tau_{rr}}{\partial t} + u_r \frac{\partial \tau_{rr}}{\partial r} + \frac{u_\theta}{r} \frac{\partial \tau_{rr}}{\partial \theta} + u_z \frac{\partial \tau_{rr}}{\partial z} - 2 \frac{u_\theta}{r} \tau_{\theta r} \right) \quad (2.26e)$$

$$- De \left(2\tau_{rr} \frac{\partial u_r}{\partial r} + \frac{2}{r} \tau_{\theta r} \frac{\partial u_r}{\partial \theta} + 2\tau_{zr} \frac{\partial u_r}{\partial z} - \frac{2}{r} u_\theta \right) = 0,$$

$$\tau_{\theta r} + De \left(\frac{\partial \tau_{\theta r}}{\partial t} + u_r \frac{\partial \tau_{\theta r}}{\partial r} + \frac{u_\theta}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + u_z \frac{\partial \tau_{\theta r}}{\partial z} - \frac{u_\theta}{r} (\tau_{\theta r} - \tau_{\theta\theta}) \right) \quad (2.26f)$$

$$- De \left(\tau_{rr} \frac{\partial u_\theta}{\partial r} + \tau_{zr} \frac{\partial u_\theta}{\partial z} + \tau_{\theta r} \frac{\partial u_r}{\partial r} + \tau_{z\theta} \frac{\partial u_r}{\partial z} + \frac{u_r \tau_{\theta r}}{r} - \frac{u_\theta \tau_{\theta\theta}}{r} \right. \\ \left. + \frac{\tau_{\theta r}}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\tau_{\theta\theta}}{r} \frac{\partial u_r}{\partial \theta} \right) = 0,$$

$$\tau_{zr} + De \left(\frac{\partial \tau_{zr}}{\partial t} + u_r \frac{\partial \tau_{zr}}{\partial r} + \frac{u_\theta}{r} \frac{\partial \tau_{zr}}{\partial \theta} + u_z \frac{\partial \tau_{zr}}{\partial z} - \frac{u_\theta}{r} \tau_{\theta z} \right) \quad (2.26g)$$

$$- De \left(\tau_{rr} \frac{\partial u_z}{\partial r} + \tau_{zr} \frac{\partial u_r}{\partial r} + \tau_{zr} \frac{\partial u_z}{\partial z} + \tau_{zz} \frac{\partial u_r}{\partial z} - \frac{u_\theta \tau_{\theta z}}{r} + \frac{\tau_{\theta r}}{r} \frac{\partial u_z}{\partial \theta} \right. \\ \left. + \frac{\tau_{\theta z}}{r} \frac{\partial u_r}{\partial \theta} \right) = (1 - \beta) \frac{\partial u_r}{\partial z},$$

$$\tau_{\theta\theta} + De \left(\frac{\partial \tau_{\theta\theta}}{\partial t} + u_r \frac{\partial \tau_{\theta\theta}}{\partial r} + \frac{u_\theta}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + u_z \frac{\partial \tau_{\theta\theta}}{\partial z} + 2 \frac{u_\theta}{r} \tau_{\theta r} \right) \quad (2.26h)$$

$$- De \left(2\tau_{\theta r} \frac{\partial u_\theta}{\partial r} + 2\tau_{\theta z} \frac{\partial u_\theta}{\partial z} + 2 \frac{u_r \tau_{\theta\theta}}{r} + 2 \frac{\tau_{\theta\theta}}{r} \frac{\partial u_\theta}{\partial \theta} \right) = 0,$$

$$\tau_{z\theta} + De \left(\frac{\partial \tau_{z\theta}}{\partial t} + u_r \frac{\partial \tau_{z\theta}}{\partial r} + \frac{u_\theta}{r} \frac{\partial \tau_{z\theta}}{\partial \theta} + u_z \frac{\partial \tau_{z\theta}}{\partial z} + \frac{u_\theta}{r} \tau_{zr} \right) \quad (2.26i)$$

$$- De \left(\tau_{zr} \frac{\partial u_\theta}{\partial r} + \tau_{r\theta} \frac{\partial u_z}{\partial r} + \tau_{zz} \frac{\partial u_\theta}{\partial z} + \tau_{\theta z} \frac{\partial u_z}{\partial z} + \frac{u_r \tau_{\theta z}}{r} + \frac{\tau_{\theta z}}{r} \frac{\partial u_\theta}{\partial \theta} \right. \\ \left. + \frac{\tau_{\theta\theta}}{r} \frac{\partial u_z}{\partial \theta} \right) = (1 - \beta) \frac{\partial u_\theta}{\partial z},$$

$$\tau_{zz} + De \left(\frac{\partial \tau_{zz}}{\partial t} + u_r \frac{\partial \tau_{zz}}{\partial r} + \frac{u_\theta}{r} \frac{\partial \tau_{zz}}{\partial \theta} + u_z \frac{\partial \tau_{zz}}{\partial z} \right) \quad (2.26j)$$

$$- De \left(2\tau_{rz} \frac{\partial u_z}{\partial r} + 2\tau_{zz} \frac{\partial u_z}{\partial z} + 2 \frac{\tau_{\theta z}}{r} \frac{\partial u_z}{\partial \theta} \right) = 0,$$

where, $De = \lambda \frac{U_0}{R_0}$ and $Wi = \epsilon^{-1} De$ for the scaling we have adopted (Eqs. 2.25b-2.25b).

The arguments leading to Eqs. 2.25a-2.25b are identical to those employed for the cartesian coordinate system. However, in contrast to it, the appearance of the circumferential stress component $\tau_{\theta\theta}$ in the Cauchy's equation of motion along the radial direction complicates the problem. In order to reduce the three-dimensional system to a two-dimensional one along the r -axis, $\tau_{\theta\theta}$ must also vanish. From Eq. 2.26h, if we assume $u_\theta = 0$, then we will obtain $\frac{\partial}{\partial \theta} = 0$.

2.4 Linear model

For the cartesian case, the non-linear system of reduced equations Eqs. 2.18a-2.18j can be linearized by considering a first-order perturbation in the Deborah number (or alternatively the ϵWi for PTT or FENE). We expand the dependent variables in De ,

$$u_k = u_k^{(0)} + De u_k^{(1)}, \quad (2.27a)$$

$$p = p^{(0)} + De p^{(1)}, \quad (2.27b)$$

$$\tau_{ij} = \tau_{ij}^{(0)} + De \tau_{ij}^{(1)}, \quad (2.27c)$$

where, $u_k = \{u, v, w\}$, $\tau_{ij} = \{\tau_{xx}, \tau_{yx}, \tau_{zx}, \tau_{yy}, \tau_{zy}, \tau_{zz}\}$, the superscripts $(\cdot)^{(0)}$ and $(\cdot)^{(1)}$ denote the leading-order and Deborah-order, respectively. Then, substituting Eqs. 2.27a-2.27c into Eqs. 2.18a-2.18j and collecting the like-order terms, we

obtain the leading-order system

$$\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} + a \frac{\partial w^{(0)}}{\partial z} = 0, \quad (2.28a)$$

$$\frac{\partial p^{(0)}}{\partial x} = \frac{\partial^2 u^{(0)}}{\partial y^2}, \quad (2.28b)$$

$$a \frac{\partial p^{(0)}}{\partial z} = \frac{\partial^2 w^{(0)}}{\partial y^2}, \quad (2.28c)$$

$$\tau_{xx}^{(0)} = 0, \quad (2.28d)$$

$$\tau_{xy}^{(0)} = (1 - \beta) \frac{\partial u^{(0)}}{\partial y}, \quad (2.28e)$$

$$\tau_{xz}^{(0)} = 0, \quad (2.28f)$$

$$\tau_{yy}^{(0)} = 2(1 - \beta) \frac{\partial v^{(0)}}{\partial y}, \quad (2.28g)$$

$$\tau_{yz}^{(0)} = (1 - \beta) \frac{\partial w^{(0)}}{\partial y}, \quad (2.28h)$$

$$\tau_{zz}^{(0)} = 0, \quad (2.28i)$$

and, the Deborah-order system

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + a \frac{\partial w^{(1)}}{\partial z} = 0, \quad (2.29a)$$

$$\frac{\partial p^{(1)}}{\partial x} = (1 - \beta) \frac{\partial^2 u^{(1)}}{\partial y^2} + \frac{\partial \tau_{xx}^{(1)}}{\partial x} + \frac{\partial \tau_{yx}^{(1)}}{\partial y} + a \frac{\partial \tau_{zx}^{(1)}}{\partial z}, \quad (2.29b)$$

$$\frac{\partial p^{(1)}}{\partial y} = 0, \quad (2.29c)$$

$$a \frac{\partial p^{(1)}}{\partial z} = \beta_s \frac{\partial^2 w^{(1)}}{\partial y^2} + \frac{\partial \tau_{xz}^{(1)}}{\partial x} + \frac{\partial \tau_{yz}^{(1)}}{\partial y} + a \frac{\partial \tau_{zz}^{(1)}}{\partial z}, \quad (2.29d)$$

$$\tau_{xx}^{(1)} = -De \left(\frac{D\tau_{xx}^{(0)}}{Dt} - 2\tau_{xy}^{(0)} \frac{\partial u^{(0)}}{\partial y} - 2\tau_{xx}^{(0)} \frac{\partial u^{(0)}}{\partial x} - 2a \frac{\partial u^{(0)}}{\partial z} \tau_{xz}^{(0)} \right), \quad (2.29e)$$

$$\tau_{xy}^{(1)} = -De \left(\frac{D\tau_{xy}^{(0)}}{Dt} - \tau_{yy}^{(0)} \frac{\partial u^{(0)}}{\partial y} - \tau_{xx}^{(0)} \frac{\partial v^{(0)}}{\partial x} - \frac{\partial u^{(0)}}{\partial x} \tau_{xy}^{(0)} - \frac{\partial v^{(0)}}{\partial y} \tau_{xy}^{(0)} - a \frac{\partial u}{\partial z} \tau_{yz}^{(0)} \right. \\ \left. - a \frac{\partial v}{\partial z} \tau_{xz}^{(0)} \right) + (1 - \beta) \frac{\partial u^{(1)}}{\partial y}, \quad (2.29f)$$

$$\tau_{xz}^{(1)} = -De \left(\frac{D\tau_{yz}^{(0)}}{Dt} - \tau_{xz}^{(0)} \frac{\partial u}{\partial x} - \tau_{xx}^{(0)} \frac{\partial w^{(0)}}{\partial x} - \tau_{yz}^{(0)} \frac{\partial u^{(0)}}{\partial y} - \tau_{xy}^{(0)} \frac{\partial w^{(0)}}{\partial y} \right. \\ \left. - a \tau_{zz}^{(0)} \frac{\partial u^{(0)}}{\partial z} - a \tau_{xz}^{(0)} \frac{\partial w^{(0)}}{\partial z} \right), \quad (2.29g)$$

$$\tau_{yy}^{(1)} = -De \left(\frac{D\tau_{yy}^{(0)}}{Dt} - 2\tau_{xy}^{(0)} \frac{\partial v^{(0)}}{\partial x} - 2\tau_{yy}^{(0)} \frac{\partial v^{(0)}}{\partial y} - 2a \frac{\partial v^{(0)}}{\partial z} \tau_{yz}^{(0)} \right) + 2(1 - \beta) \frac{\partial v^{(1)}}{\partial y}, \quad (2.29h)$$

$$\tau_{yz}^{(1)} = -De \left(\frac{D\tau_{yy}^{(0)}}{Dt} - \tau_{xz}^{(0)} \frac{\partial v^{(0)}}{\partial x} - \tau_{xy}^{(0)} \frac{\partial w^{(0)}}{\partial x} - \tau_{yz}^{(0)} \frac{\partial v^{(0)}}{\partial y} - \tau_{yy}^{(0)} \frac{\partial w^{(0)}}{\partial y} - \right. \\ \left. a \tau_{zz}^{(0)} \frac{\partial v^{(0)}}{\partial z} - a \tau_{yz}^{(0)} \frac{\partial w^{(0)}}{\partial z} \right) + (1 - \beta) \frac{\partial w^{(1)}}{\partial y}, \quad (2.29i)$$

$$\tau_{zz}^{(1)} = -De \left(\frac{D\tau_{yy}^{(0)}}{Dt} - 2\tau_{xz}^{(0)} \frac{\partial w^{(0)}}{\partial x} - 2\tau_{xy}^{(0)} \frac{\partial w^{(0)}}{\partial y} - 2a \frac{\partial w^{(0)}}{\partial z} \tau_{zz}^{(0)} \right) = 0, \quad (2.29j)$$

$$\frac{D\tau_{ij}^{(0)}}{Dt} = u^{(0)} \frac{\partial \tau_{ij}^{(0)}}{\partial x} + v^{(0)} \frac{\partial \tau_{ij}^{(0)}}{\partial y} + a w^{(0)} \frac{\partial \tau_{ij}^{(0)}}{\partial z}. \quad (2.29k)$$

$$\frac{D\tau_{ij}^{(0)}}{Dt} = u^{(0)} \frac{\partial \tau_{ij}^{(0)}}{\partial x} + v^{(0)} \frac{\partial \tau_{ij}^{(0)}}{\partial y} + a w^{(0)} \frac{\partial \tau_{ij}^{(0)}}{\partial z}. \quad (2.29l)$$

The solution to the leading order system of Eqs. 2.28a-2.28i is obtained by supplying appropriate boundary (and if relevant initial) conditions. In this work, we use the boundary conditions (and follow the procedure for deriving the Reynolds

equation, see [16])

$$u^{(0)}(x, y = 0, z) = 1, \text{ and } u^{(0)}(x, y = h, z) = 0, \quad (2.30a)$$

$$v^{(0)}(x, y = 0, z) = 0, \text{ and } v^{(0)}(x, y = h, z) = 0, \quad (2.30b)$$

$$w^{(0)}(x, y = 0, z) = 0, \text{ and } w^{(0)}(x, y = h, z) = 0, \quad (2.30c)$$

providing us the classical Newtonian Reynolds equation, describing the leading-order pressure for a two-dimensional contact, along with the leading-order velocity field,

$$\frac{a^2 h^3 \frac{\partial^2}{\partial z^2} p^{(0)}}{12} - \frac{a^2 h^2 \frac{\partial}{\partial z} h \frac{\partial}{\partial z} p^{(0)}}{4} - \frac{h^3 \frac{\partial^2}{\partial x^2} p^{(0)}}{12} - \frac{h^2 \frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(0)}}{4} + \frac{\frac{\partial}{\partial x} h}{2} \quad (2.31a)$$

$$u^{(0)} = \frac{y^2 \frac{\partial}{\partial x} p^{(0)}}{2} - \frac{y h \frac{\partial}{\partial x} p^{(0)}}{2} - \frac{y}{h} + 1, \quad (2.31b)$$

$$v^{(0)} = y^3 \left(-\frac{a^2 \frac{\partial^2}{\partial z^2} p^{(0)}}{6} - \frac{\frac{\partial^2}{\partial x^2} p^{(0)}}{6} \right) + y^2 \left(\frac{a^2 h \frac{\partial^2}{\partial z^2} p^{(0)}}{4} + \frac{a^2 \frac{\partial}{\partial z} h \frac{\partial}{\partial z} p^{(0)}}{4} \right. \\ \left. + \frac{h \frac{\partial^2}{\partial x^2} p^{(0)}}{4} + \frac{\frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(0)}}{4} - \frac{\frac{\partial}{\partial x} h}{2h^2} \right), \quad (2.31c)$$

$$w^{(0)} = \frac{a y^2 \frac{\partial}{\partial z} p^{(0)}}{2} - \frac{a y h \frac{\partial}{\partial z} p^{(0)}}{2}. \quad (2.31d)$$

In a similar manner, but with some additional manipulation, we can obtain the Reynolds equation for the Deborah-order system (expressions for the velocity

field are too large to print here, in a reasonable format)

$$\begin{aligned}
& - \frac{\beta_s a^4 h^5 \frac{\partial}{\partial z} p^{(0)} \frac{\partial^3}{\partial z^3} p^{(0)}}{48} - \frac{\beta_s a^4 h^5 (\frac{\partial^2}{\partial z^2} p^{(0)})^2}{48} - \frac{7\beta_s a^4 h^4 \frac{\partial}{\partial z} h \frac{\partial}{\partial z} p^{(0)} \frac{\partial^2}{\partial z^2} p^{(0)}}{48} \\
& - \frac{\beta_s a^4 h^4 \frac{\partial^2}{\partial z^2} h (\frac{\partial}{\partial z} p^{(0)})^2}{48} - \frac{\beta_s a^4 h^3 (\frac{\partial}{\partial z} h)^2 (\frac{\partial}{\partial z} p^{(0)})^2}{12} \\
& - \frac{\beta_s a^2 h^5 \frac{\partial}{\partial x} p^{(0)} \frac{\partial^3}{\partial z^2 \partial x} p^{(0)}}{48} - \frac{\beta_s a^2 h^5 \frac{\partial^2}{\partial x^2} p^{(0)} \frac{\partial^2}{\partial z^2} p^{(0)}}{40} \\
& - \frac{\beta_s a^2 h^5 \frac{\partial}{\partial z} p^{(0)} \frac{\partial^3}{\partial z \partial x^2} p^{(0)}}{48} - \frac{\beta_s a^2 h^5 (\frac{\partial^2}{\partial z \partial x} p^{(0)})^2}{60} \\
& - \frac{\beta_s a^2 h^4 \frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial^2}{\partial z^2} p^{(0)}}{12} - \frac{\beta_s a^2 h^4 \frac{\partial}{\partial x} h \frac{\partial}{\partial z} p^{(0)} \frac{\partial^2}{\partial z \partial x} p^{(0)}}{16} \\
& - \frac{\beta_s a^2 h^4 \frac{\partial}{\partial z} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial^2}{\partial z \partial x} p^{(0)}}{16} - \frac{\beta_s a^2 h^4 \frac{\partial}{\partial z} h \frac{\partial^2}{\partial x^2} p^{(0)} \frac{\partial}{\partial z} p^{(0)}}{12} - \frac{\beta_s a^2 h^4 \frac{\partial}{\partial x} p^{(0)} \frac{\partial}{\partial z} p^{(0)} \frac{\partial^2}{\partial z \partial x} h}{24} \\
& - \frac{\beta_s a^2 h^3 \frac{\partial}{\partial x} h \frac{\partial}{\partial z} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial}{\partial z} p^{(0)}}{6} + \frac{\beta_s a^2 h^3 \frac{\partial^3}{\partial z^2 \partial x} p^{(0)}}{24} + \frac{\beta_s a^2 h^2 \frac{\partial}{\partial z} h \frac{\partial^2}{\partial z \partial x} p^{(0)}}{8} - \frac{\beta_s h^5 \frac{\partial}{\partial x} p^{(0)} \frac{\partial^3}{\partial x^3} p^{(0)}}{48} \\
& - \frac{\beta_s h^5 (\frac{\partial^2}{\partial x^2} p^{(0)})^2}{48} - \frac{7\beta_s h^4 \frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial^2}{\partial x^2} p^{(0)}}{48} - \frac{\beta_s h^4 \frac{\partial^2}{\partial x^2} h (\frac{\partial}{\partial x} p^{(0)})^2}{48} \\
& - \frac{\beta_s h^3 (\frac{\partial}{\partial x} h)^2 (\frac{\partial}{\partial x} p^{(0)})^2}{12} + \frac{\beta_s h^3 \frac{\partial^3}{\partial x^3} p^{(0)}}{24} \\
& + \frac{\beta_s h^2 \frac{\partial}{\partial x} h \frac{\partial^2}{\partial x^2} p^{(0)}}{8} + \frac{\beta_s \frac{\partial^2}{\partial x^2} h}{12} + \frac{a^4 h^5 \frac{\partial}{\partial z} p^{(0)} \frac{\partial^3}{\partial z^3} p^{(0)}}{48} + \frac{a^4 h^5 (\frac{\partial^2}{\partial z^2} p^{(0)})^2}{48} \\
& + \frac{7a^4 h^4 \frac{\partial}{\partial z} h \frac{\partial}{\partial z} p^{(0)} \frac{\partial^2}{\partial z^2} p^{(0)}}{48} + \frac{a^4 h^4 \frac{\partial^2}{\partial z^2} h (\frac{\partial}{\partial z} p^{(0)})^2}{48} \\
& + \frac{a^4 h^3 (\frac{\partial}{\partial z} h)^2 (\frac{\partial}{\partial z} p^{(0)})^2}{12} + \frac{a^2 h^5 \frac{\partial}{\partial x} p^{(0)} \frac{\partial^3}{\partial z^2 \partial x} p^{(0)}}{48} + \frac{a^2 h^5 \frac{\partial^2}{\partial x^2} p^{(0)} \frac{\partial^2}{\partial z^2} p^{(0)}}{40} + \frac{a^2 h^5 \frac{\partial}{\partial z} p^{(0)} \frac{\partial^3}{\partial z \partial x^2} p^{(0)}}{48} \\
& + \frac{a^2 h^5 (\frac{\partial^2}{\partial z \partial x} p^{(0)})^2}{60} + \frac{a^2 h^4 \frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial^2}{\partial z^2} p^{(0)}}{12} + \frac{a^2 h^4 \frac{\partial}{\partial x} h \frac{\partial}{\partial z} p^{(0)} \frac{\partial^2}{\partial z \partial x} p^{(0)}}{16} \\
& + \frac{a^2 h^4 \frac{\partial}{\partial z} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial^2}{\partial z \partial x} p^{(0)}}{16} + \frac{a^2 h^4 \frac{\partial}{\partial z} h \frac{\partial^2}{\partial x^2} p^{(0)} \frac{\partial}{\partial z} p^{(0)}}{12} + \frac{a^2 h^4 \frac{\partial}{\partial x} p^{(0)} \frac{\partial}{\partial z} p^{(0)} \frac{\partial^2}{\partial z \partial x} h}{24} \\
& + \frac{a^2 h^3 \frac{\partial}{\partial x} h \frac{\partial}{\partial z} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial}{\partial z} p^{(0)}}{6} - \frac{a^2 h^3 \frac{\partial^2}{\partial z^2} p^{(1)}}{12} - \frac{a^2 h^3 \frac{\partial^3}{\partial z^2 \partial x} p^{(0)}}{24} - \frac{a^2 h^2 \frac{\partial}{\partial z} h \frac{\partial}{\partial z} p^{(1)}}{4} \\
& - \frac{a^2 h^2 \frac{\partial}{\partial z} h \frac{\partial^2}{\partial z \partial x} p^{(0)}}{8} + \frac{h^5 \frac{\partial}{\partial x} p^{(0)} \frac{\partial^3}{\partial x^3} p^{(0)}}{48} + \frac{h^5 (\frac{\partial^2}{\partial x^2} p^{(0)})^2}{48} \\
& + \frac{7h^4 \frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(0)} \frac{\partial^2}{\partial x^2} p^{(0)}}{48} + \frac{h^4 \frac{\partial^2}{\partial x^2} h (\frac{\partial}{\partial x} p^{(0)})^2}{48} + \frac{h^3 (\frac{\partial}{\partial x} h)^2 (\frac{\partial}{\partial x} p^{(0)})^2}{12} - \frac{h^3 \frac{\partial^3}{\partial x^3} p^{(0)}}{24} \\
& - \frac{h^3 \frac{\partial^2}{\partial x^2} p^{(1)}}{12} - \frac{h^2 \frac{\partial}{\partial x} h \frac{\partial^2}{\partial x^2} p^{(0)}}{8} - \frac{h^2 \frac{\partial}{\partial x} h \frac{\partial}{\partial x} p^{(1)}}{4} - \frac{\frac{\partial^2}{\partial x^2} h}{12}
\end{aligned} \tag{2.32}$$

The expressions in both the systems, i.e., the leading-order and the De -order, involve higher powers of a . Furthermore, third-order, and mixed partial derivatives of the leading order pressure emerge. If we set $a = 0$, then the linearized system of equations for a two-dimensional problem are easily retrieved.

The exact analytical solution of the leading-order system of Eqs. 2.31a-2.31d and the Deborah-order system Eq. 2.32 for arbitrarily shaped channels, $h = h(x, z)$, requires solving a second-order, non-homogeneous partial differential equation, and is difficult to retrieve. In this work, we will resort to a numerical treatment discussed in chapter 3.

2.5 Viscoelastic Reynolds equation

The viscoelastic Reynolds equation is derived from the parent thin-film reduced system of equations (Eqs. 2.18a-2.18j) by following the steps outlined in Ahmed and Biancofiore [23]. It is inspired, in large part, from the approach employed for generalized Reynolds equation [18, 44]. The essence of the procedure is to (i) retain all the integrals across the film for the unknown polymeric stress components and (ii) take advantage of the weak perturbation to the velocity due to viscoelastic effects. The latter allows assuming a velocity field that is known *a priori*, such as $\mathbf{u} = \mathbf{u}^{(0)} + De\mathbf{u}^{(1)}$. Therefore, the only unknowns in the constitutive relations are the scalar stress components (Eqs. 2.29e-2.29l). Note that this velocity-stress decoupling only holds within lubrication theory. From the linearized model, it can be shown, analytically for the two-dimensional channel and numerically for the three-dimensional channel, that the De -order perturbation to the velocity field vanishes when the channel height, $h(x, y, z)$ satisfies

$$h(x = 0, y, z) = h(x = 1, y, z), \quad (2.33a)$$

$$h(x, y, z = 0) = h(x, y, z = 1). \quad (2.33b)$$

Thus, we expect the flow along the thin channel for lubricants with weak elasticity to remain Newtonian. However, for strongly elastic flows, as De approaches unity, it is possible that the second-order perturbation in De , $\mathbf{u}^{(2)}$, has a strong

influence, i.e. $De^2\mathbf{u}^{(2)} \neq 0$. However, results for pressure driven flow along a converging channel show that this perturbation can vanish up to the third-order [51].

The three dimensional viscoelastic Reynolds equation takes the form

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\frac{h^3}{12} \frac{\partial p}{\partial z} \right) = \frac{\partial \mathcal{F}}{\partial x} + \frac{\partial \mathcal{G}}{\partial z}, \quad (2.34a)$$

$$\mathcal{F} = \frac{\beta}{2} h + \mathcal{A} \left(\frac{\partial \tau_{xx}}{\partial x} \right) + \mathcal{B}(\tau_{xy}) + a \mathcal{A} \left(\frac{\partial \tau_{xz}}{\partial z} \right), \quad (2.34b)$$

$$\mathcal{G} = a^2 \mathcal{A} \left(\frac{\partial \tau_{zz}}{\partial z} \right) + a \mathcal{A} \left(\frac{\partial \tau_{xz}}{\partial x} \right) + a \mathcal{B}(\tau_{yz}), \quad (2.34c)$$

where,

$$\mathcal{A}(\cdot) = \int_0^h \frac{y}{h} \int_0^h \int_0^{y'} (\cdot) dy' dy dy - \int_0^h \int_0^{y''} \int_0^{y'} (\cdot) dy' dy'' dy, \quad (2.35a)$$

$$\mathcal{B}(\cdot) = \int_0^h \frac{y}{h} \int_0^h (\cdot) dy' dy - \int_0^h \int_0^{y'} (\cdot) dy' dy, \quad (2.35b)$$

are integral operators.

Note the appearance of the aspect ratio in front of the terms with some influence of the spanwise direction. If we set $a = 0$, the VR reduces to the two-dimensional form given in Ahmed and Biancofiore [23] demonstrating the absence of a singularity for very small values of a . Furthermore, the normal stress τ_{zz} vanishes quadratically as opposed to the shear component τ_{xz} .

In a similar manner, we can obtain the viscoelastic Reynolds equation for the cylindrical configuration (r, θ, z) ,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r h^3}{12} \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{h^3}{12 r} \frac{\partial p}{\partial \theta} \right) = \frac{1}{r} \frac{\partial \mathcal{G}}{\partial \theta} + \frac{1}{r} \frac{\partial (r \mathcal{F})}{\partial r} \quad (2.36a)$$

$$\mathcal{F} = \mathcal{C} \left(\frac{1}{r} \frac{\partial (r \tau_{rr})}{\partial r} \right) + \mathcal{C} \left(\frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} \right) + \mathcal{C} \left(\frac{-\tau_{\theta \theta}}{r} \right) + \mathcal{D}(\tau_{zr}), \quad (2.36b)$$

$$\mathcal{G} = \frac{\beta}{2} h + \mathcal{C} \left(\frac{1}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} \right) + \mathcal{C} \left(\frac{1}{r} \frac{\partial \tau_{\theta \theta}}{\partial \theta} \right) + \mathcal{D}(\tau_{z\theta}), \quad (2.36c)$$

where,

$$\mathcal{C}(\cdot) = \int_0^h \frac{z}{h} \int_0^h \int_0^{z'} (\cdot) dz' dz dz - \int_0^h \int_0^{z''} \int_0^{z'} (\cdot) dz' dz'' dz, \quad (2.37a)$$

$$\mathcal{D}(\cdot) = \int_0^h \frac{z}{h} \int_0^h (\cdot) dz' dz - \int_0^h \int_0^{z'} (\cdot) dz' dz, \quad (2.37b)$$

are the integral operators. Note that by setting $\frac{\partial}{\partial \theta} = 0$, and $u_\theta = 0$, we obtain a problem confined along the radial direction.

2.6 Squeeze film lubrication

Eqs. 2.26a-2.26j comprise the full set of reduced equations, for an incompressible, isoviscous and isothermal viscoelastic lubricant film occupying the region between two surfaces. The film is said to be squeezed when the bounding surfaces, top $h_1(r, \theta, t)$, and bottom $h_2(r, \theta, t)$, approach each other at some velocity $V_1 = \frac{\partial h_1}{\partial t}$ and $V_2 = \frac{\partial h_2}{\partial t}$, respectively. The film thickness, $h = h_1 - h_2$, can then be used to determine the pressure from the Reynolds equation. The analytical solution of Eqs. 2.26a-2.26j (a set of coupled non-linear partial differential equations), for an arbitrary film thickness $h = h(r, \theta, t)$ is cumbersome even for a purely Newtonian

lubricant. In this work, we confine ourselves to the two-dimensional problem, i.e.,

$$\frac{1}{r} \frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0, \quad (2.38a)$$

$$\frac{\partial p}{\partial r} = \beta \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{zr}}{\partial z}, \quad (2.38b)$$

$$\frac{\partial p}{\partial \theta} = 0, \quad (2.38c)$$

$$\frac{\partial p}{\partial z} = 0, \quad (2.38d)$$

$$\tau_{rr} + De \left(\frac{\partial \tau_{rr}}{\partial t} + u_r \frac{\partial \tau_{rr}}{\partial r} + u_z \frac{\partial \tau_{rr}}{\partial z} \right) - De \left(2\tau_{rr} \frac{\partial u_r}{\partial r} + 2\tau_{zr} \frac{\partial u_r}{\partial z} \right) = 0, \quad (2.38e)$$

$$\tau_{zr} + De \left(\frac{\partial \tau_{zr}}{\partial t} + u_r \frac{\partial \tau_{zr}}{\partial r} + u_z \frac{\partial \tau_{zr}}{\partial z} \right) \quad (2.38f)$$

$$- De \left(\tau_{rr} \frac{\partial u_z}{\partial r} + \tau_{zr} \frac{\partial u_r}{\partial r} + \tau_{zr} \frac{\partial u_z}{\partial z} + \tau_{zz} \frac{\partial u_r}{\partial z} \right) = (1 - \beta) \frac{\partial u_r}{\partial z},$$

$$\tau_{zz} + De \left(\frac{\partial \tau_{zz}}{\partial t} + u_r \frac{\partial \tau_{zz}}{\partial r} + u_z \frac{\partial \tau_{zz}}{\partial z} \right) - De \left(2\tau_{rz} \frac{\partial u_z}{\partial r} + 2\tau_{zz} \frac{\partial u_z}{\partial z} \right) = 0, \quad (2.38g)$$

where, we assume

$$h_1 = H(t), \text{ and } h_2 = 0, \quad (2.39a)$$

$$u_\theta = \tau_{\theta\theta} = \tau_{\theta r} = 0, \text{ and } \frac{\partial(\cdot)}{\partial \theta} = 0. \quad (2.39b)$$

Eq. 2.39b enforces symmetry about the z -axis for the pressure, polymer stress and velocity field. In addition, we assume that the normal stress $\tau_{\theta\theta} = 0$, and that the surface profile, which is only a function of time, satisfies $h = 1 - t$. We seek to understand the variation in the vertical force $F_N = 2\pi \int_0^1 p r dr$ and the friction $F_{(r)} = 2\pi \int_0^1 \tau_{zr}|_{z=0} r dr$ versus the Deborah number and time.

2.6.1 Perturbation models

The two-dimensional non-linear system of Eqs. 2.38a-2.38g, is simplified by considering a perturbation in the dependent variables about De ,

$$u_k = u_k^{(0)} + Deu_k^{(1)} + \dots, \quad (2.40a)$$

$$p = p^{(0)} + Dep^{(1)} + \dots, \quad (2.40b)$$

$$\tau_{ij} = \tau_{ij}^{(0)} + De\tau_{ij}^{(1)} + \dots, \quad (2.40c)$$

where, $\tau_{ij} = \{\tau_{rr}, \tau_{zr}, \tau_{zz}\}$ and $u_k = \{u_r, u_z\}$. Substituting the perturbed variables into Eqs. 2.38a-2.38g and collecting the terms (on the same perturbation order) gives the leading-order system,

$$\frac{1}{r} \frac{\partial(ru_z^{(0)})}{\partial r} + \frac{\partial u_z^{(0)}}{\partial z} = 0, \quad (2.41a)$$

$$-\frac{\partial p^{(0)}}{\partial r} + \beta \frac{\partial^2 u_r^{(0)}}{\partial z^2} = 0 \quad (2.41b)$$

$$\tau_{rr}^{(0)} = 0, \quad (2.41c)$$

$$\tau_{zr}^{(0)} = (1 - \beta) \frac{\partial u_r^{(0)}}{\partial z}, \quad (2.41d)$$

$$\tau_{zz}^{(0)} = 2(1 - \beta) \frac{\partial u_z^{(0)}}{\partial z}, \quad (2.41e)$$

and the first-order system,

$$\frac{1}{r} \frac{\partial(ru_z^{(1)})}{\partial r} + \frac{\partial u_z^{(1)}}{\partial z} = 0, \quad (2.42a)$$

$$-\frac{\partial p^{(1)}}{\partial r} + \beta \frac{\partial^2 u_r^{(1)}}{\partial z^2} + \frac{1}{r} \frac{\partial(r\tau_{rr}^{(1)})}{\partial r} + \frac{\partial \tau_{zr}^{(1)}}{\partial z} = 0 \quad (2.42b)$$

$$\tau_{rr}^{(1)} = 2(1 - \beta) \left(\frac{\partial u_r^{(0)}}{\partial z} \right)^2, \quad (2.42c)$$

$$\tau_{zr}^{(1)} = (1 - \beta) \frac{\partial u_r^{(1)}}{\partial z}, \quad (2.42d)$$

$$\tau_{zz}^{(1)} = 2(1 - \beta) \frac{\partial u_z^{(1)}}{\partial z}. \quad (2.42e)$$

Similarly, one can obtain the systems for the higher order perturbations. However, for the sake of brevity, we do not explicitly write the system owing to its length and the large number of terms involved. Instead, we provide the final expressions

for the quantities of interest, i.e. the pressure distribution, vertical force and friction.

The solution to the leading-order system is

$$p^{(0)} = \frac{3(r^2 - 1)}{H^3} \frac{dH}{dt}, \quad (2.43a)$$

$$F_N^{(0)} = -\frac{3\pi}{2H^3} \frac{dH}{dt}, \quad (2.43b)$$

$$F_r^{(0)} = -\frac{2\pi}{H^3} \frac{dH}{dt}, \quad (2.43c)$$

for the first-order system,

$$p^{(1)} = -\frac{3(r^2 - 1)}{10H^4} \left(10H \left(\frac{d^2 H}{dt^2} \right) - 57 \left(\frac{dH}{dt} \right)^2 \right), \quad (2.44a)$$

$$F_N^{(1)} = \frac{3\pi}{2H^3} \frac{d^2 H}{dt^2} - \frac{171\pi}{20H^4} \left(\frac{dH}{dt} \right)^2, \quad (2.44b)$$

$$F_r^{(1)} = \frac{2\pi}{H^3} \frac{d^2 H}{dt^2} - \frac{17\pi}{5H^3} \left(\frac{dH}{dt} \right)^2, \quad (2.44c)$$

and the second-order system,

$$p^{(2)} = \frac{3(r^2 - 1)}{700H^5} \left(700H^2 \frac{d^3 H}{dt^3} - 11970H \frac{dH}{dt} \frac{d^2 H}{dt^2} + 19491 \left(\frac{dH}{dt} \right)^3 \right), \quad (2.45a)$$

$$F_N^{(2)} = -\frac{3\pi}{2H^3} \frac{d^3 H}{dt^3} - \frac{513\pi}{20H^4} \frac{dH}{dt} \frac{d^2 H}{dt^2} - \frac{58473\pi}{1400H^5} \left(\frac{dH}{dt} \right)^3, \quad (2.45b)$$

$$F_r^{(2)} = -\frac{2\pi}{H^3} \frac{d^3 H}{dt^3} + \frac{51\pi}{5H^3} \left(\frac{dH}{dt} \right) \left(\frac{d^2 H}{dt^2} \right) - \frac{4021\pi}{350H^4} \left(\frac{dH}{dt} \right)^3, \quad (2.45c)$$

We immediately note, that the pressure distribution remains quadratic regardless of the order of perturbation. In fact, it retains precisely the general form for the pressure

$$p^{(n)} = \frac{3(r^2 - 1)}{H^3} \mathcal{K}^{(n)} \left(H, \frac{dH}{dt}, \frac{d^2 H}{dt^2}, \dots \right), \quad (2.46)$$

where, $\mathcal{K}^{(n)}$ is some function of the film thickness (varying in time) and its derivatives. The highest order derivative and the highest polynomial degree, in surface height, is always $n + 1$. Similarly, for the net normal force and the friction, a general form is apparent. The expressions appear compact and also convenient, however, the explicit expressions for the general surface function $h = h(r, t)$ are too large to even print. In fact, utilizing the symbolic toolbox available in

Python, i.e., Sympy [55], the maximum perturbation order that can be attained, in reasonable amount of time, is the third-order.

The VR equation for the case of two-dimensional (r, z) squeeze flow is obtained by setting $\frac{\partial}{\partial \theta} = 0$ in Eq. 2.36a and assuming the circumferential stress $\tau_{\theta\theta} = 0$. The resulting simplified form is used to solve for the pressure subject to the boundary conditions

$$\left. \frac{\partial p}{\partial r} \right|_{(r=0)} = 0, \quad (2.47a)$$

$$p(r = 1) = 0, \quad (2.47b)$$

once the polymer stress is obtained from the solution of the reduced constitutive equations. When solving for the stress, however, the velocity field is considered constant and is assumed to be simply the leading-order, or Newtonian from Tanner and Pipkin [56]. Higher-order perturbations can be added to improve the velocity estimate, but for the simple surface motion considered here, the first-order is identically zero. The stress boundary condition, in this work, is simply

$$\left. \frac{\partial \tau_{ij}}{\partial r} \right|_{r=0} = 0, \quad (2.48)$$

and the initial condition is a stress-free state, i.e., $\tau_{ij} = 0$.

It is worthwhile to note the differences already apparent between the formulations of the simplified perturbed system and the VR model (2.36a-2.36a). Firstly, the transient nature of the problem implies that all quantities will vary with time and therefore, must accept some initial condition. However, the perturbed system does not yield a system that accepts an initial condition. This is understandable, since it is an expansion about the Newtonian solution which has a finite value of stress, as soon as the surface is moved. On the other hand, the VR model will accept an initial condition, reflecting the true physical nature of the stress which will be initially zero, since the polymers are assumed to be relaxed (at least initially). Then, upon applying a surface velocity $\frac{dh}{dt} = -1$ at $t = 0$, the response for the viscoelastic lubricant should be delayed and not instantaneous (but for a vanishingly small De).

2.7 Cavitation

Lubricant cavitation is characterized by a change of phase, from liquid to a vapor state, due to a drop in pressure [57]. When the surfaces are moving away from each other at high enough speeds, the fluid pressure drops, owing to the increase in fluid velocity. If at any point within the film, the pressure drops below the cavitation pressure, the liquid phase transitions to the vapor phase and/or dissolved gases are set free.

For a Newtonian solvent, the pressure distribution is corrected for the presence of ϕ fraction of vapor. The Reynolds equation, for a isoviscous, incompressible liquid phase takes the form (in a cartesian coordinate system)

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\frac{h^3}{12} \frac{\partial p}{\partial z} \right) = \frac{\partial(\phi h)}{\partial x}. \quad (2.49)$$

Under the linear-complementarity approach [2, 3] for modeling cavitating lubricated contacts, we impose the condition $p \cdot r = 0$, where, $r = 1 - \phi$ is the complementarity variable. When the Poiseuille contribution vanishes in the cavitating region, i.e. $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0$, then one solves for the film fraction instead of the pressure, where the quantity of interest is the fraction close to the interface.

For a non-Newtonian lubricant, with significant elastic contribution, a modification is required to the constitutive relations based on the assumption that the elasticity exists only in the liquid phase and the vapor phase does not contribute to the viscoelasticity of the mixture. Therefore, we have for the Oldroyd-B,

$$\boldsymbol{\tau} + \phi \lambda \overset{\vee}{\boldsymbol{\tau}} = 2\phi(1 - \beta)\hat{\mathbf{D}}, \quad (2.50)$$

a linear correction to the upper-convected rate operator (elastic contribution) and the viscous part in function of ϕ , which depends on the pressure. One immediately observes, for $\phi = 1$, i.e., a fully liquid region, we have the original expression for the Oldroyd-B and for a fully vapor region the polymer stress is zero. In practice, the growth of the vapor fraction follows $\frac{\partial \phi}{\partial x} = 0$.

Note, in the absence of this correction, the normal and shear stresses will predict a finite value arising from the polymer elasticity, which would act as if

the lubricant is in a liquid phase. Furthermore, the velocity field will be corrected apriori, since the Poiseuille component will vanish in the vapor region. However, the Couette component will be retained and give a non-zero elastic stress when simply input to the constitutive relation. The correction of Eq. 2.50, ensures the Couette velocity field (and its gradients) does not induce any extra liquid-phase stress. The numerical algorithm used in this work for capturing cavitation for viscoelastic lubricants is detailed in chapter 3.



Chapter 3

Numerical methods

The equations introduced in chapter 2 are difficult to treat analytically, especially for the three-dimensional channels. Therefore, an exact solution is often not attainable even for simple surface functions. A numerical approach, despite the added effort and demand for computational resources offers a viable alternative. However, the numerical method, involving appropriate treatment of the different differential (integral) operators, discretization strategy for the geometry (which may not be smooth or continuous) has a strong impact on the success of the solution procedure. In this chapter, we formulate the numerical technique used to solve the governing system of equations using the classical finite difference method (FDM), which is arguably the simplest discretization technique. Other approaches available in literature involve the finite volume method (FVM) and the finite element method (FEM). The latter two, although well suited for curvilinear meshes, require additional care when dealing with the differential (and integral) operators. For example, the FVM demands an additional interpolation scheme to evaluate, accurately, the relevant fluxes on the cell interface. Similarly, in the case of FEM, the construction of specialized elements for the treatment of the advective terms is required which merely adds to the complexity of the problem. Nevertheless, they are valid, and arguably, more robust techniques that can be employed for the numerical simulation of the viscoelastic problem.

This chapter is organized as follows. In section 3.1, the method used to treat the curved nature of the domain is introduced. In section 3.2, the strategy for an implicit discretization of the governing equations (including the general form of the Reynolds equation, and the scalar constitutive relations) is provided, and a simple stability analysis of the numerical system is performed in section 3.2.1. Finally, we offer a modification of the robust FNBS cavitation algorithm [58] in order to accommodate viscoelastic lubricants in section 3.3.

3.1 Grid transformation

The classical finite difference method, for an equidistant mesh in three-dimensions, satisfies the necessary requirements of consistency, convergence and, depending on the differential operator, stability as well. They can be easily constructed to any order of accuracy from the Taylor series expansion about some point in space. However, for a curved mesh, conforming to the curvature of the channel surface, the direct use of these classical schemes becomes problematic. To see this, consider a grid with the spacing between the nodes

$$x = i \cdot \frac{\ell_x}{N_x}, \quad (3.1a)$$

$$y = j \cdot \frac{h(x, z)}{N_y}, \quad (3.1b)$$

$$z = k \cdot \frac{\ell_z}{N_z}, \quad (3.1c)$$

where, N_x , N_y is the number of nodes and N_z , (i, j, k) are the nodal indices, along the x , y and z , respectively. Now, we consider two neighboring points $p_1 = (x_{(i,j,k)}, y_{(i,j,k)}, z_{(i,j,k)})$ and $p_2 = (x_{(i+1,j,k)}, y_{(i+1,j,k)}, z_{(i+1,j,k)})$, where i, j, k are nodal indices along the x , y and z directions, respectively. Using Eqs. 3.1a-3.1c, and assuming $p_1 = (x_0, y_0, z_0)$, we have $p_2 = (x_0 + \Delta x, y_0 \cdot \frac{\Delta y(x_0 + \Delta x, z_0)}{\Delta y(x_0, z_0)}, z_0)$. The point p_2 does not lie on the same y_0 -line and the difference between p_1 and p_2 will scale with the curvature of the grid. This directly affects the derivatives, for examples consider the first-order forward discretization, of some dependent

quantity $q(x, y, z)$,

$$\left. \frac{\partial q}{\partial x} \right|_{(x_0, y_0, z_0)} = \frac{q(x_0 + \Delta x, y_0, z_0) - q(x_0, y_0, z_0)}{\Delta x}. \quad (3.2)$$

In Eq. 3.2, we assume, in accordance with the definition of a partial derivative, that the location does not change along y and z . Expressed in terms of the nodal indices

$$\left. \frac{\partial q}{\partial x} \right|_{(i, j, k)} = \frac{q_{(i+1, j, k)} - q_{(i, j, k)}}{\Delta x}. \quad (3.3)$$

However, for a curved grid, the point $(x_0 + \Delta x, y_0, z_0)$ does not coincide with any point in the discretized set of points (or nodes) arising from Eqs. 3.1a-3.1c.

There are several remedies to this problem, e.g. an immersed boundary method will allow discretizing the geometry such that we can move in constant increments of Δx , Δy and Δz but demands a more involved treatment of the boundary nodes with special interpolation of the boundary values. One may also consider non-classical finite difference schemes with non-uniform spacing but those are not trivial to construct and cumbersome to obtain for higher accuracy to say the least.

We adopt a simpler approach, in which the physical space $\mathbf{x}(x, y, z)$ is transformed to a computational space $\mathbf{X}(X, Y, Z)$. The transformed space will be a cube (square) in three dimensions (two dimensions) and allow the use of the classical FDM schemes. The limits of this computational domain can be set based on the problem, but typically a unit cube is preferable in most situations, for simplicity.

The mapping between the physical and computational domain, for all problems considered in this work,

$$x = X; 0 \leq X \leq 1, \quad (3.4a)$$

$$y = Yh(x, z); 0 \leq Y \leq 1, \quad (3.4b)$$

$$z = Z; 0 \leq Z \leq 1, \quad (3.4c)$$

where, $h(x, z)$ is the surface function, allows the transformation of any point $\mathbf{x} = (x, y, z)$ in the channel to the computational space. Note that Eqs. 3.4a-3.4c

represent a unit box in which the grid points are equally spaced. However, this is not a constraint and, in general, the computational coordinates could occupy any range depending on the problem. In addition, we assume that the points along the x and z axes are equidistant to begin with and require no transformation. However, for certain problems it may be advantageous to use a non-uniform stencil to capture large gradients where necessary. In addition, for a cylindrical system, we simply change the definition of the physical space to $\mathbf{x} = (r, \theta, z)$ and the computational space $\mathbf{X} = (R, \Theta, Z)$. Then, Eqs. 3.4a-3.4c can be used in the same manner for a surface function $h(r, \theta)$.

Now we wish to transform the governing equations such that the dependent variables are functions of the computational coordinates by applying the general definition of the total derivative of a function $F = F(\eta_1, \eta_2, \dots)$ in terms of some other choice of independent variables $\zeta_1, \zeta_2, \dots$,

$$dF = \sum_i \frac{\partial F}{\partial \eta_i} d\zeta_i, \quad (3.5)$$

to evaluate the first, second and mixed derivatives of any dependent quantity for instance the velocity field, with respect to x, y and z . Note that our objective is to obtain differential forms in $\mathbf{x}$, but we will first express the total derivative in terms of $\mathbf{X}$ and later collect the equations to obtain the required forms. The first derivatives (gradients) are readily obtained from Eq. 3.5

$$\frac{\partial}{\partial X} = \frac{\partial x}{\partial X} \frac{\partial}{\partial x} + \frac{\partial y}{\partial X} \frac{\partial}{\partial y} + \frac{\partial z}{\partial X} \frac{\partial}{\partial z}, \quad (3.6a)$$

$$\frac{\partial}{\partial Y} = \frac{\partial x}{\partial Y} \frac{\partial}{\partial x} + \frac{\partial y}{\partial Y} \frac{\partial}{\partial y} + \frac{\partial z}{\partial Y} \frac{\partial}{\partial z}, \quad (3.6b)$$

$$\frac{\partial}{\partial Z} = \frac{\partial x}{\partial Z} \frac{\partial}{\partial x} + \frac{\partial y}{\partial Z} \frac{\partial}{\partial y} + \frac{\partial z}{\partial Z} \frac{\partial}{\partial z}. \quad (3.6c)$$

The second and mixed derivatives are a bit more involved and require the useful

of the transformation and comprises the transformation metrics, i.e. derivatives of the physical coordinates with respect to the computational coordinates. It is evident from Eq. 3.8 that $\mathbf{A}^{-1}$ will give us the required derivatives of the pressure, velocity and stress with respect to the physical domain in some linear combination of the derivatives existing purely in the computational domain. The detailed contents of $\mathbf{A}$ are not provided for the sake of brevity but can be obtained easily from Eqs. 3.6a-3.6c. Notice that all the metrics can be readily obtained via analytical treatment, for simple geometries, from Eqs. 3.4a-3.4c. However, for more complex surfaces such as, textured surfaces, rough surfaces or complex analytical expressions, the metrics can be obtained via an application of the classical finite difference schemes. Furthermore, had we initially expressed the required derivatives with respect to $\mathbf{x}$, e.g.

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x} \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x} \frac{\partial}{\partial Y} + \frac{\partial Z}{\partial x} \frac{\partial}{\partial Z}, \quad (3.9)$$

at the beginning of the derivation (Eqs. 3.6a-3.7f), the metrics would then depend on the physical domain, e.g. $\frac{\partial Y}{\partial x}$, and the differential operator $\frac{\partial}{\partial x}$, not being known a priori, would have become difficult to obtain. Whereas, the operator $\frac{\partial}{\partial X}$, for example, is given simply via a second order accurate central difference scheme (or any other).

3.2 Implicit discretization

The transformation from the curved physical domain to the uniform computational domain has allowed us to express all the required derivatives up to second order in space in terms of operators that can be readily discretized via classical constant-stencil finite difference schemes available in any introductory text on numerical methods. Here, we describe the use of implicit discretization techniques for the three-dimensional case which can be reduced to the two-dimensional case without much effort.

An implicit approach inherently requires all dependent variables to be evaluated at the unknown time step resulting in a system of algebraic equations that is

compactly represented as $\mathbf{A}u = b$, where $\mathbf{A}$ is the coefficient matrix arising from the discretization of the differential equation, u is the desired solution vector and b is some forcing. In contrast, the explicit approach can be obtained by splitting $\mathbf{A}$ into its diagonal and off-diagonal components, $\mathbf{A}^{(D)}$ and $\mathbf{A}^{(LU)}$, respectively, giving $\mathbf{A}^{(D)}u^{t+1} = \mathbf{A}^{(LU)}u^t + b$. The solution of the explicit version then involves matrix-vector products, and, assuming a sufficiently small time step size can converge to the steady-state solution. Clearly, it is a more appealing approach from a programming perspective.

The problem arises when the required step size is too small, leading to prohibitively long simulation times. This is especially true for viscoelastic flows, where the polymer relaxation time is on the order of several milliseconds, and the steady-state (if it exists) is achieved much later. Furthermore, the criteria for numerical stability, for explicit schemes, enforces a restriction on the maximum allowable time step size (Δt) as a function of the grid spacing ($\Delta X, \Delta Y, \Delta Z$). For the classical, one-dimensional advection equation, the stability criteria is the Courant-Fredrich-Levy (CFL) condition, where $\Delta t < CFL \cdot \Delta X/U_0$, for some advection velocity U_0 and becomes more restrictive if diffusion is added to the system, scaling with ΔX^2 instead.

Implicit schemes are, for the most part, impervious to such restrictions and decouple the time step size from the grid spacing, allowing for generously large values (if the steady-state solution is the goal). The stability analysis of implicit schemes for ordinary (and partial) differential equations began with the pioneering work of Dahlquist, the Danish mathematician, and are still the subject of interest today. Recent efforts in modeling viscoelastic flows specifically, have shown an interest in exploring implicit schemes for a resource-efficient solution. In particular, for thin viscoelastic lubricant films, the treatment of the constitutive relations is done via an explicit projection in time, assuming some appropriate initial condition. This severely limits the maximum allowable time step, since the (i) height of the film is very small, and (ii) the added non-linearity and coupling between the velocity and polymer stress is strong.

In this work, we adopt an fully-implicit approach for the Oldroyd-B constitutive equation and a semi-implicit approach for the non-linear constitutive relations such as, FENE-CR, FENE-P, Giesekus and PTT models. We assume that all the differential and integral operators in the physical coordinate system have been properly transformed via Eqs. 3.6a-3.7f as necessary in the governing equations. Furthermore, the specific difference expressions for each derivative are given in Appendix B. Here, we assume that a particular derivative, using the finite difference stencil, can be expressed as a matrix-vector product, for example $\frac{\partial p}{\partial X} = \mathbf{K}p$, where, $\mathbf{K}$ is a sparse coefficient matrix and p is some vector.

The Reynolds type equations, (e.g., the VR Eq. 2.34a) considered in this work, can be written as a matrix-vector system

$$\mathbf{A}p = f(h, \tau_{ij}), \quad (3.10)$$

where, for cartesian configurations $\mathbf{A} = \left\{ \frac{\partial}{\partial X} \left(\frac{h^3}{12} \frac{\partial}{\partial X} \right) + a^2 \frac{\partial}{\partial Z} \left(\frac{h^3}{12} \frac{\partial}{\partial Z} \right) (\cdot) \right\}$, and for cylindrical configurations $\mathbf{A} = \left\{ \frac{\partial}{\partial R} \left(\frac{Rh^3}{12} \frac{\partial}{\partial R} \right) + \frac{1}{R} \frac{\partial}{\partial \theta} \left(\frac{h^3}{12} \frac{\partial}{\partial \theta} \right) (\cdot) \right\}$. The forcing vector, f , is a function of the surface height, normal and shear stress components (see Eqs. 2.34a-2.34c). The sparse-matrix operator $\mathbf{A}$ can be obtained via discretizing the derivatives according to the stencils given in Appendix B.

The constitutive relations considered in this work can be written in a compact form

$$F\boldsymbol{\tau} + De\boldsymbol{\tau} = 2G(1 - \beta)\mathbf{D}, \quad (3.11)$$

where, F and G are non-linear functions of the stress tensor, De and other parameters. The material derivative in the upper-convected rate operator (and in F for the FENE models) is discretized via a biased, and locally (globally) second-order (third-order) accurate scheme for each component of the stress tensor, and fourth-order accurate central difference schemes for each component of the velocity gradient tensor. To arrive at a globally implicit system for Eq. 3.11, we first write the simpler semi-implicit procedure (using Oldroyd-B and for τ_{xx} , for

brevity)

$$\mathbf{A}_{(xx)}\tau_{(xx)}^{(t+1)} = b_{(xx)}^{(t)}, \quad (3.12a)$$

$$\mathbf{A} = \Delta t(\mathbf{M} + \mathbf{I} + \mathbf{D}_1), \quad (3.12b)$$

$$\mathbf{D}_1 = -2De \frac{\partial u}{\partial x}, \quad (3.12c)$$

$$b_{(xx)}^{(t)} = 2De\Delta t \left(\tau_{(xy)} \frac{\partial u}{\partial x} + a\tau_{(xz)} \frac{\partial u}{\partial z} \right)^{(t)} + \tau_{(xx)}^{(t)}, \quad (3.12d)$$

where, $\mathbf{I}$ is the identity matrix, $\mathbf{M} = U \frac{\partial}{\partial X} + V \frac{\partial}{\partial Y} + aW \frac{\partial}{\partial Z}$ is the material derivative operator, U, V, W are the velocities in the computational domain (arising from the grid-transformation via Eqs. 3.4a-3.4c), $b_{(ij)}$ is the forcing vector containing the terms involving stress components $\tau_{(mn)}$ ($i \neq m$ and $j \neq n$) at the previous time level t . In addition, it contains the non-linear terms (from F and G) at the known time level t and the corrections due to the second-order accurate upwind/downwind schemes. We will have a system similar to Eqs. 3.12a-3.12d, for each component of the stress tensor (omitted here for brevity, see Appendix C for the two-dimensional problem and additional comments for the FENE models).

While this approach is notably better than its fully-explicit variant, the fully implicit approach

$$\sum_{(IJ)} \mathbf{F}_{(IJ)}^{(ij)} \tau_{IJ}^{(t+1)} = b_{(ij)}^{(t)}, \quad (3.13)$$

where, the summation is taken over all components of the stress tensor (six for three-dimensional problems and three for two-dimensional problems) each having a coefficient matrix, denoted by $\mathbf{F}_{IJ}^{(ij)}$ for $\tau_{(IJ)}$ in the scalar equation for component $\tau_{(ij)}$, (after discretizing the differential operators by a proper upwind/downwind scheme). Eq. 3.13 couples all the components of the stress tensor into single unknown vector and is markedly more numerically efficient owing to its greater

stability. As an example, we write the algebraic system, explicitly, for τ_{xx}

$$\mathbf{F}_{(xx)}^{(xx)}\tau_{(xx)}^{(t+1)} + \mathbf{F}_{(xy)}^{(xx)}\tau_{(xy)}^{(t+1)} + \mathbf{F}_{(xz)}^{(xx)}\tau_{(xz)}^{(t+1)} = b_{(xx)}^{(t)}, \quad (3.14a)$$

$$\mathbf{F}_{(xx)}^{(xx)} = \Delta t(\mathbf{M} + \mathbf{I} + \mathbf{D}_1), \quad (3.14b)$$

$$\mathbf{F}_{(xy)}^{(xx)} = -2De\Delta t\left(\frac{\partial u}{\partial y}\right), \quad (3.14c)$$

$$\mathbf{F}_{(xz)}^{(xx)} = -2De\Delta t\left(a\frac{\partial u}{\partial z}\right) \quad (3.14d)$$

$$\mathbf{D}_1 = -2De\frac{\partial u}{\partial x}, \quad (3.14e)$$

$$b_{(xx)}^{(t)} = \tau_{(xx)}^{(t)}. \quad (3.14f)$$

Similar to Eqs. 3.14a-3.14f, we can write a system for each stress component, assembling them to form matrix-vector system which can be solved via an iterative approach. Note that the global coefficient matrix (from Eq. 3.13), assembled from all the stress components, in Eq. 3.13 has the size $(6N_X N_Y N_Z \times 6N_X N_Y N_Z)$ and the vectors formed from the scalar stress components τ_{ij} and forcing terms $b_{(ij)}$ have the size $6N_X N_Y N_Z$, where N_X , N_Y and N_Z are the nodes along the streamwise, crossfilm and spanwise directions, respectively.

The PTT, FENE and Giesekus models contain terms that are non-linear in the stress. For the FENE models, the term $f(1 + De\frac{Df^{-1}}{Dt})\tau_{ij}$, is simply linearized by considering the non-linear portion, i.e., $f(1 + De\frac{Df^{-1}}{Dt})^{(t)}\tau_{ij}^{(t+1)}$, at the time level (t) and adding to the coefficient matrix. Similarly, for the PTT models, both linear and exponential, the same treatment can be adopted. Note however, this requires updating the sparse coefficient matrix at each time step, but the added stability, in the form of a larger time step size, is significant. Alternatively, one may simply consider the entire term at time level (t) and include it in the forcing vector $b_{(ij)}^{(t)}$. We utilize the latter only for gradually varying surfaces, i.e. a weak convergence ratio (small eccentricity ratio for the journal bearing) and the former for three-dimensional configurations and/or steeply varying surfaces.

For the Giesekus model, one may again adopt the aforementioned procedure. However, the non-linear term in question, $\frac{\alpha W_i}{\epsilon(1-\beta)}\tau_{ik}\tau_{kj}$, does not involve a scalar function but a simple matrix product. For this, we linearize these terms via a

Taylor series expansion, assuming $F_{(ik)} = \frac{\alpha W_i}{\epsilon(1-\beta)} \tau_{(ik)}$

$$F_{(ik)} \tau_{(jk)} = F_{(ik)} \tau_{(jk)}|_t + \frac{\partial(F_{(ik)} \tau_{(jk)})}{\partial \tau_{(mn)}} \bigg|_t (\tau_{(mn)}^{(t+1)} - \tau_{(mn)}^{(t)}), \quad (3.15)$$

of $F_{(ik)} \tau_{(jk)}$ about the known time step (t). Notice that in the case for FENE-P, FENE-CR, and PTT, F only involves the normal stresses, via the trace of the stress, of which only the streamwise and spanwise components are $O(1)$ under the thin film approximation. For Giesekus, the expression involves multiple components of the stress. In practice, Eq. 3.15 is only required for steep configurations, e.g. a high eccentricity ratio for the journal bearing, or for highly undulating configurations. For gradual variations in the channel surface, the semi-implicit treatment can be adopted and the resulting difference in computational time taken (or allowable time step size) is not significant.

3.2.1 Numerical stability

A numerical stability analysis for the constitutive relations identifies a critical issue, namely the high Weissenberg number problem [59]. Apart from the well known CFL condition and special treatment of the advective terms, i.e., a bi-ased operator based on the direction of the flow, it shows a restriction on the grid spacing as a function of the Deborah (or Weissenberg) number. This restriction prevents solutions from converging regardless of an implicit or explicit discretization.

In the pioneering work of Fattal and Kupferman [60], proposing the log-conformation approach as a remedy for the HWNP, shows the explicit dependence of the grid spacing on De . This was a result of asserting that the eigenvalues of the coefficient matrix, for the stress at some point in the mesh, must remain positive and less than unity. A simpler, but arguably less holistic, approach to numerical stability is performed here, for the implicit discretization, for the purpose of visualizing the region of stability based the classical Von-Neumann analysis on a simplified one-dimensional approximation of the scalar equations, specifically the normal streamwise stress, i.e. τ_{xx} . Inherent in this sort of analysis, is the

assumption that if a single equation in the governing system of equations becomes unstable, then the larger system comprising all of the stress components cannot be solved. Our choice for the normal stress is motivated by the fact that it is zero for a purely Newtonian fluid, becoming finite for a viscoelastic fluid, and, therefore, likely to be the cause for numerical instability. For simplicity, we consider only

$$\tau_{xx} + De \left(\frac{\partial \tau_{xx}}{\partial t} + u \frac{\partial \tau_{xx}}{\partial x} + v \frac{\partial \tau_{xx}}{\partial y} \right) - 2De \tau_{xx} \frac{\partial u}{\partial x} = 0, \quad (3.16)$$

and, using the Eqs. 3.6a-3.6c, transform Eq. 3.16 into the computational domain (X, Y)

$$\tau_{xx} + De \left(\frac{\partial \tau_{xx}}{\partial t} + U \frac{\partial \tau_{xx}}{\partial X} + V \frac{\partial \tau_{xx}}{\partial Y} \right) - 2De \tau_{xx} \left(a - \frac{Y}{h} \frac{\partial h}{\partial X} b \right) = 0, \quad (3.17a)$$

$$U = u, \quad (3.17b)$$

$$V = \frac{v}{h} - u \frac{Y}{h} \frac{\partial h}{\partial X}, \quad (3.17c)$$

$$a = \frac{\partial u}{\partial X}, \text{ and } b = \frac{\partial u}{\partial Y}. \quad (3.17d)$$

where, h represents the surface function.

Discretizing via a simple forward difference scheme (assuming $U > 0$ and $V > 0$) via an explicit approach

$$\begin{aligned} \tau_{xx,(i,j)}^{(t+1)} &= \frac{\Delta t}{De} \left(-1 + \frac{1}{\Delta t} - \frac{U}{\Delta X} - \frac{V}{\Delta Y} + 2 \left(a - \frac{Y}{h} \frac{\partial h}{\partial X} b \right) \right) \tau_{xx,(i,j)}^{(t)} \\ &\quad + \frac{U \Delta t}{De \Delta X} \tau_{xx,(i-1,j)}^{(t)} + \frac{V \Delta t}{De \Delta Y} \tau_{xx,(i,j-1)}^{(t)}, \end{aligned} \quad (3.18)$$

and via an implicit approach

$$\begin{aligned} De \left(1 + \frac{1}{\Delta t} + \frac{U}{\Delta X} + \frac{V}{\Delta Y} - 2 \left(a - \frac{Y}{h} \frac{\partial h}{\partial X} b \right) \right) \tau_{xx,(i,j)}^{(t+1)} \\ - \frac{U}{De \Delta X} \tau_{xx,(i-1,j)}^{(t+1)} - \frac{V}{De \Delta Y} \tau_{xx,(i,j-1)}^{(t+1)} = \frac{De}{\Delta t} \tau_{xx,(i,j)}^{(t)}. \end{aligned} \quad (3.19)$$

In addition, the surface function is $h = 1 + e \cos(2\pi X)$ (defining the two-dimensional journal bearing). In expressing the surface function in terms of a single parameter e which governs the convergence ratio of the channel, we can also investigate the influence of steep surface gradients.

The Von-Neumann stability analysis requires that we approximate the solution as a wave, i.e., $\tau_{xx,(i,j)}^{(t)} = A_{(t)} e^{J(k_X X + k_Y Y)}$ and $\tau_{xx,(i,j)}^{(t+1)} = A_{(t+1)} e^{J(k_X X + k_Y Y)}$, where, k_X (k_Y) is the wave number along X (Y), and $J = \sqrt{-1}$ denotes the complex part. Then solving for the amplification factor $\zeta = A_{(t+1)}/A_{(t)}$ (which is a complex number) and posing the equality for stability, i.e., $|\zeta| < 1$, we find, in both explicit and implicit cases, that the numerical stability criteria is a function of the Deborah number, grid spacing, time step size, local velocity field, and its gradients. The analytical expressions in either case, more so for the implicit approach, are large and omitted for the sake of brevity. More importantly, we obtain, via numerical treatment, the phase diagrams, demarcating the stable region ($\zeta < 1$) and the unstable region ($\zeta > 1$) for different choices of parameters. Primarily, we wish to see how the De , and the velocity gradients influence the contours, and ultimately the choice, for ΔX , ΔY and Δt , since these are the only modifiable parameters, whereas the physical quantities (velocity, De , eccentricity ratio (via the surface function), are determined from the problem definition.

In Fig. 3.1, we plot the stability diagram of $\Delta t, \Delta X$ for $(X, Y) = (0.5, 0.5)$, $De = 0.5$, $(k_X, k_Y) = (2\pi, 2\pi)$, depicting the stable regions (shaded) and unstable regions (unshaded) for the explicit and implicit methods, as we increase the eccentricity (or steepness) of the channel. The stable region for explicit methods rapidly shrinks as opposed to the implicit methods, which remains unconditionally stable, at least initially. The shrinkage implies that fine meshes and large time steps will lead to failure when solving the simplified equation for τ_{xx} (Eq. 3.16), and one must resort to ever smaller time step sizes and coarse meshes. Note that the small value of De is insufficient in ensuring numerical stability for rapidly converging channels and singularities can arise owing to the geometry. This is a well known result, but what is surprising is the effect of coarsening the mesh induces some numerical stability under the thin-film approximation.

Similarly, In Fig. 3.2, we plot a contour of the phase space $\Delta t, \Delta X$ for $(X, Y) = (0.5, 0.5)$, $De = 0.5$, $(k_X, k_Y) = (2\pi, 2\pi)$, depicting the stable regions (shaded) and unstable regions (unshaded) for the explicit and implicit methods, as we increase the Deborah number. The implicit approach demonstrates the

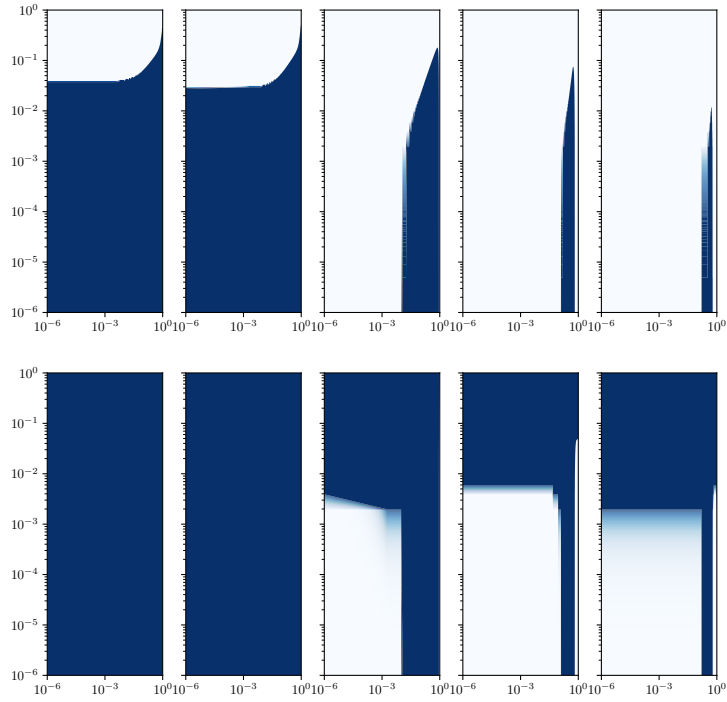


Figure 3.1: The contours of the time step size Δt against grid spacing $\Delta X = \Delta Y = \Delta$ depicting the stable region (shaded) and unstable region (un-shaded) for the explicit approach (top row) and the implicit approach (bottom row) for $e = 0.9, 0.91, 0.95, 0.99, 0.999$ (left to right). Assuming $(X, Y) = (0.5, 0.5)$, $De = 0.5$, $(k_X, k_Y) = (2\pi, 2\pi)$.

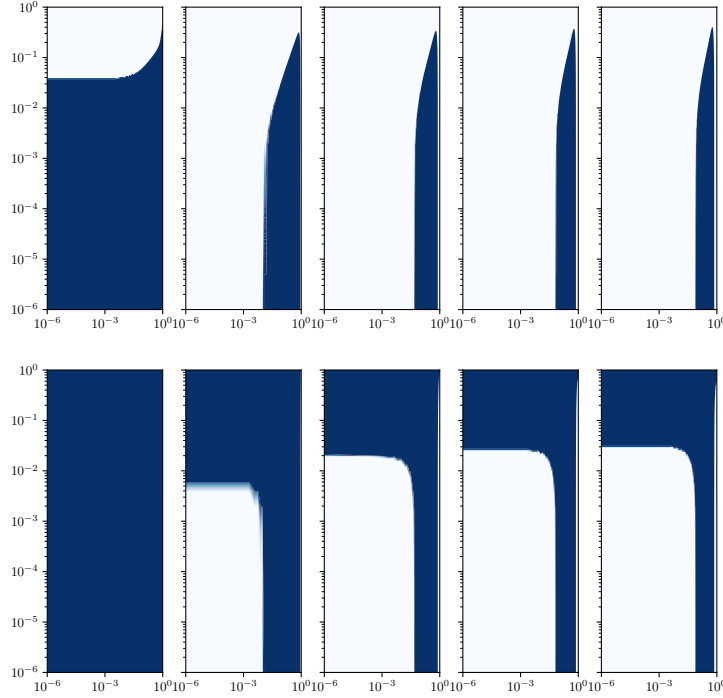


Figure 3.2: The contours of the time step size Δt against grid spacing $\Delta X = \Delta Y = \Delta$ depicting the stable region (shaded) and unstable region (un-shaded) for the explicit approach (top row) and the implicit approach (bottom row) for $De = 0.1, 0.2, 0.3, 0.4, 0.5$ (left to right). Assuming $(X, Y) = (0.5, 0.5)$, $De = 0.5$, $(k_X, k_Y) = (2\pi, 2\pi)$.

same trend as the explicit method, with no noticeable improvement in stability, contrary to what is often observed in the stability of classical linear equations. We deduce from this that the high Weissenberg number problem is not alleviated and numerical instability can be brought about, at large enough De , in a fully implicit approach as well, regardless of how small Δt and/or $\Delta X(Y)$.

We have demonstrated via the use of a simple equation and using the Von-Neumann stability analysis the decline in numerical stability versus the grid spacing and time step size (for a single point). It is possible to generate similar contours and investigate other parameters e.g., the influence of the wave number

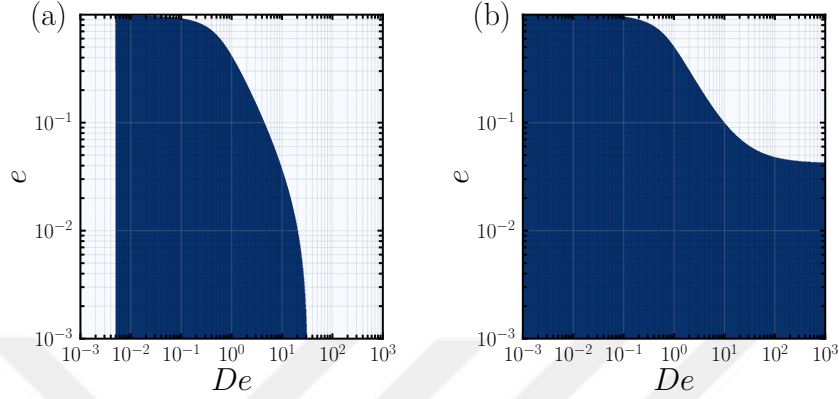


Figure 3.3: The stability phase diagram for e and De (shaded: stable, un-shaded: unstable) for the (a) fully-explicit method and (b) the implicit approach ($\Delta t = 0.01$, $\Delta X = \Delta Y = 0.1$, $k_X = k_Y = 2\pi$).

along X and Y (depicting a highly undulating stress profile). Interestingly, an observation that is not often discussed but which is present in the discussion of some works on viscoelastic simulations, is the decline (or at least stagnation) in numerical stability versus mesh refinement. In fact, coarsening seems to provide the desired stability but the accuracy of those results are naturally in question.

3.3 The FNBS algorithm for viscoelastic lubricants

It is well-known that lubricants undergo cavitation in when the local film pressure drops below the cavitation pressure, leading to the (i) release of dissolved gases, and (ii) phase change from liquid to a liquid-vapor mixture. We describe the implicit implementation of the robust FNBS algorithm for viscoelastic lubricants via the finite difference method.

We can compactly write the viscoelastic Reynolds Eq. 2.34a for a two-phase (liquid-vapor) lubricant mixture as

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\frac{h^3}{12} \frac{\partial p}{\partial z} \right) = \frac{\partial}{\partial x} \left((1 - \phi) \mathcal{F} \right) + \frac{\partial}{\partial z} \left((1 - \phi) \mathcal{G} \right), \quad (3.20a)$$

(see Eq. 2.34a-2.34c for the definition of $\mathcal{F}$ and $\mathcal{G}$), where, ϕ is the complementarity variable related to the pressure, for each node (m) in the mesh, via

$$p_{(m)} + \phi_{(m)} = -\sqrt{p_{(m)}^2 + \phi_{(m)}^2}. \quad (3.21)$$

We can discretize Eq. 3.20a and Eq. 3.21 using the FDM and then linearize in p and ϕ via the Newton-Raphson method (since Eq. 3.21 is non-linear), obtaining a coupled linear system

$$\mathbf{A}p^{(k+1)} + \mathbf{B}\phi^{(k+1)} = b_1^{(k)}, \quad (3.22a)$$

$$\mathbf{J}_p p^{(k+1)} + \mathbf{J}_\phi \phi^{(k+1)} = b_2^{(k)}, \quad (3.22b)$$

where, k is the iteration level, $\mathbf{J}_p = 1 - \frac{p_{(m)}^{(k)}}{\sqrt{p_{(m)}^{(k)2} + \phi_{(m)}^{(k)2}}}$ and $\mathbf{J}_\phi = 1 - \frac{\phi_{(m)}^{(k)}}{\sqrt{p_{(m)}^{(k)2} + \phi_{(m)}^{(k)2}}}$ are diagonal matrices, k is the iteration level. The coefficient matrix for ϕ is related to the differential operator $\mathbf{B} = \frac{\partial \mathcal{F}}{\partial x} + \frac{\partial \mathcal{G}}{\partial z} + \mathcal{F} \frac{\partial}{\partial x} + \mathcal{G} \frac{\partial}{\partial z}$, $b_1 = \mathbf{A}p^{(k)} - \frac{\partial}{\partial x}((1 - \phi^{(k)})\mathcal{F}) + \frac{\partial}{\partial z}((1 - \phi^{(k)})\mathcal{G}) - \frac{\partial \mathcal{F}}{\partial x} - \frac{\partial \mathcal{G}}{\partial z}$, and $b_2 = p^{(k)} + \phi^{(k)} + \sqrt{p^{(k)2} + \phi^{(k)2}}$.

The solution procedure involves inverting the system of Eqs. 3.22a-3.22b to obtain successive updates to p and ϕ , and proceed until convergence is achieved. It was observed that iterative solvers, such as the conjugate gradient method, cannot be utilized owing to the poor conditioning of the system. If we attempt to use a good preconditioner, then the numerical solutions tends to proceed for a few iterations after which the solution rapidly diverges. The use of a direct solver is, therefore, a requirement. Further optimization of the iterative process can be achieved via manipulating the rows (based on the particular value of pressure and ϕ for a given node (m)) of the global coefficient matrix formed via $\mathbf{A}$, $\mathbf{B}$, $\mathbf{J}_p$ and $\mathbf{J}_\phi$. This was not necessary in this particular case, since the added efficiency is not required and the classical Newton-Raphson procedure was sufficient.

However, care must be taken in treating the discretization of $\mathbf{B}$. Notice, that when the film is fully vapor region, then we have $p = 0$, and Eq. 3.22a changes type from parabolic to hyperbolic in ϕ . As such, a hybrid scheme (see appendix B) must be employed when discretizing $\mathbf{B}$ to avoid numerical instabilities. In addition, greater accuracy is achieved if a second-order accurate (or higher) difference scheme is used, but convergence might be delayed owing to its reduced stability.

Chapter 4

Shear thinning and viscoplastic flow

In this study we present a new approach, i.e. the modified viscosity (MV) technique, which (i) extends the range over which ASD methods operate, (ii) retain an appreciable degree of accuracy when computing the pressure and the load carrying capacity for different geometries and (iii) is valid for three-dimensional hydrodynamically lubricated contacts. The paper is organized as follows; in section 4.1 we describe the new mathematical approach, i.e. MV, to describe shear thinning lubricants with summarizing also the previous models found in literature. Afterwards, the numerical procedure is described in section 4.2. In section 4.3 we compare MV and GR showing the benefits of using MV and in section 4.4, a discussion on the optimal location for the approximate shear strain rate is presented. Finally, conclusions are drawn in section 9.1. The work presented here has been published in Ahmed, H., & Biancofiore, L. (2022). *A modified viscosity approach for shear thinning lubricants*. Physics of Fluids, 34(10).

4.1 Mathematical model

We present the governing equation for modeling complex rheology lubricants in section 4.1.1, followed in section 4.1.2 by a brief overview of the models and techniques available in literature. Afterwards, we introduce the modified viscosity technique in section 4.1.3. The rheology of the lubricant is covered in section 4.1.4, while the approach to describe cavitation is given in section 4.1.5. Finally, the contact geometries examined in our work are described in section 4.1.6.

4.1.1 Problem formulation

The pressure distribution in a fully lubricated contact (see Fig. 4.1)

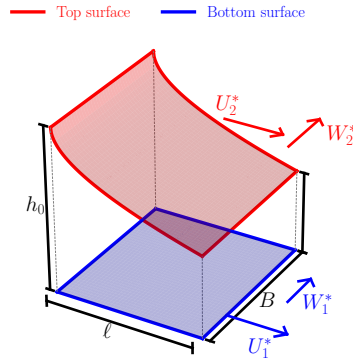


Figure 4.1: A general three-dimensional lubricated channel of length ℓ and width B . The surfaces are sliding with streamwise velocity U_1^* and U_2^* and spanwise velocity W_1^* and W_2^* .

for a Newtonian, compressible and piezoviscous lubricant is defined by the dimensional Reynolds equation (RE)

$$\frac{\partial}{\partial x^*} \left(\frac{h^{*3}}{12} \frac{\rho}{\eta^*} \frac{\partial p^*}{\partial x^*} \right) + \frac{\partial}{\partial z^*} \left(\frac{h^{*3}}{12} \frac{\rho}{\eta^*} \frac{\partial p^*}{\partial z^*} \right) = \frac{\partial(\rho h^*)}{\partial t^*} + \frac{1}{2} \frac{\partial(\rho h^*(U_1^* + U_2^*))}{\partial x^*} + \frac{1}{2} \frac{\partial(\rho h^*(W_1^* + W_2^*))}{\partial z^*}, \quad (4.1)$$

where x^* , y^* and z^* represent the streamwise, wall-normal and the spanwise directions, respectively, t^* the time, η^* is the viscosity, h^* is the surface height, p^*

is the pressure, ρ is the density, U_1^* (U_2^*) are the streamwise lower (upper) surface velocity and W_1^* (W_2^*) is the spanwise velocity of the lower (upper) surface. A solution satisfying Eq. 4.1 is obtained by providing the surface function h^* along with the boundaries motion and by knowing the rheology of the lubricant.

We non-dimensionalize Eq. 4.1 by using

$$\begin{aligned} x &= \frac{x^*}{\ell}, & y &= \frac{y^*}{h_0}, & z &= \frac{z^*}{B}, & h &= \frac{h^*}{h_0}, & \phi &= \frac{\rho}{\rho_0}, & \eta &= \frac{\eta^*}{\eta_0}, & U_1 &= \frac{U_1^*}{U_{ref}}, \\ U_2 &= \frac{U_2^*}{U_{ref}}, & W_1 &= \frac{W_1^*}{U_{ref}}, & W_2 &= \frac{W_2^*}{U_{ref}}, & p &= \frac{p^*}{p_0}, & \Lambda &= \frac{\eta_0 U_{ref} \ell}{h_0^2 p_0}, & a &= \frac{\ell}{B}, & t &= \frac{t^* \ell}{U_{ref}}, \end{aligned} \quad (4.2)$$

where h_0 is the entrance height, ℓ is the channel length, B is the channel width, $U_{ref} = U_1^* + U_2^* + \frac{\ell}{h_0} \max(|\frac{\partial h^*}{\partial t^*}|)$ is the reference velocity, η_0 , ρ_0 and p_0 are the viscosity, density and pressure, respectively, at the entry. In addition to this non-dimensionalization, we neglect the boundary motion along the spanwise direction, i.e. $W_1 = W_2 = 0$

$$\frac{\partial}{\partial x} \left(\frac{h^3 \phi}{12 \eta} \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\frac{h^3 \phi}{12 \eta} \frac{\partial p}{\partial z} \right) = \Lambda \frac{\partial(h\phi)}{\partial t} + \frac{\Lambda(U_2 + U_1)}{2} \frac{\partial(h\phi)}{\partial x}. \quad (4.3)$$

In this manner Eq. 4.3 is parameterized in terms of two dimensionless quantities, i.e. (i) the bearing number Λ and (ii) the aspect ratio a of the contact. Eq. 4.3 is subject to the boundary conditions

$$p(0, z) = p(1, z) = 0, \quad (4.4a)$$

$$p(x, 0) = p(x, 1) = 0. \quad (4.4b)$$

For the transient simulations performed in this study the initial condition for the pressure is $p(x, z) = 0$ and the density is $\phi(x, z) = 1$ at $t = 0$.

The use of Eq. 4.3 becomes limited for lubricants with a complex rheology, i.e. a non-Newtonian lubricant or with significant thermophysical variations in the lubricant properties. For the latter, one can assume (i) isothermal conditions or (ii) only streamwise temperature variation [17]. However, no similar simplifications exist for a non-Newtonian lubricant. Furthermore, shear thinning is exhibited by nearly all polymer enhanced lubricants pointing out the need for a correction to Eq. 4.3.

For a shear thinning lubricant the viscosity μ is a function of the instantaneous shear strain rate $\dot{\gamma}$, which varies across the film gap y , i.e. $\dot{\gamma} = f(x, y, z)$, whereas the pressure field remains invariant across the film. If the viscosity is calculated using the shear strain rate distribution across the film, it cannot be substituted into Eq. 4.3. Therefore, modifications of the Reynolds equation or the functional form of the viscosity-shear relationship are required. We discuss some of the general ways to approach this issue in section 4.1.2. It should be noted that in this work we highlight only the issues regarding the applicability of Eq. 4.1 for shear thinning lubricants. However, a similar argument exists for the direct use of Eq. 4.1, without appropriate modifications to the mathematical form, for a piezoviscous lubricant under certain conditions [16].

4.1.2 Available models

In this section we describe the methods and the models available in literature to account for shear thinning. We can separate the major attempts into three general categories; (i) the generalized Reynolds equation in section 4.1.2.1, (ii) an analytically derived RE using a given shear thinning constitutive relation [61] in section 4.1.2.2 and (iii) the use of an approximate shear distribution (ASD) coupled with the Reynolds equation (RE) in section 4.1.2.3.

4.1.2.1 The generalized Reynolds equation (GR)

The generalized Reynolds equation accounts for all cross-film variations in the fluid properties. GR was first derived by Dowson [18], but variations of the original form are found in many studies introducing effects such as fluid inertia, slip at the wall [62] and viscoelasticity [63]. The details regarding the derivation are given in the original work [18]. The mathematical model consists of an integral-differential equation for pressure,

$$\frac{\partial}{\partial x} \left(m_2 \phi \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(m_2 \phi \frac{\partial p}{\partial z} \right) = \Lambda \frac{\partial(h\phi)}{\partial t} + \Lambda \frac{\partial}{\partial x} \left((U_2 - U_1) \frac{m_1}{f_0} \phi + U_1 h \phi \right), \quad (4.5)$$

where m_1 , m_2 , f_0 , f_1 are integrals defined as,

$$\begin{aligned} f_0 &= \int_0^h \mu^{-1} dy, \\ f_1 &= \int_0^h \mu^{-1} y dy, \\ m_1 &= \int_0^h \left(\int_0^y \mu^{-1} dy' \right) dy, \\ m_2 &= \frac{f_1}{f_0} m_1 - \int_0^h \left(\int_0^y \mu^{-1} y' dy' \right) dy. \end{aligned} \quad (4.6)$$

The shear strain rates $\dot{\gamma}_{xy}$ and $\dot{\gamma}_{zy}$ are

$$\dot{\gamma}_{xy} = \frac{\mu^{-1}}{\Lambda} \frac{\partial p}{\partial x} \left(y - \frac{f_1}{f_0} \right) + \frac{\mu^{-1}}{f_0} (U_2 - U_1), \quad (4.7)$$

$$\dot{\gamma}_{zy} = \frac{a\mu^{-1}}{\Lambda} \frac{\partial p}{\partial z} \left(y - \frac{f_1}{f_0} \right). \quad (4.8)$$

GR can accurately capture the variation in the viscosity across the film due to pressure, shear and temperature. It expresses this variation in terms of the unknown integrals appearing in Eqs. 4.6-4.8, which must be computed separately. Despite the accuracy of the final solution, the integral operations can become computationally expensive with an increase in (i) the mesh size, (ii) non-linearity, i.e. at high bearing numbers, or (iii) if the problem is transient, e.g a periodic squeeze flow. Nonetheless, we must emphasize that it can account for any constitutive relation, thermophysical variation and density gradients across the channel.

4.1.2.2 Analytically modified Reynolds equation (AMRE)

The second approach involves analytically deriving a modified version of RE from the fundamental governing equations and a non-linear constitutive relation that defines shear thinning. For this reason we call it analytically modified Reynolds equation (AMRE). Examples of the constitutive relations used in AMRE are the power-law equations [21] and Ree-Eyring equation for the one-dimensional case [61]. There are several advantages in this approach: (i) firstly shear thinning

has been analytically incorporated into the resulting Reynolds equation and (ii) the computational method for the numerical solution will be fast and efficient. The latter is appealing for studies involving large meshes. However, there are several cases where this approach cannot be used, such as (i) two-dimensional problems that generally do not yield an analytical Reynolds equation, (ii) cases involving cavitation or transient effects and (iii) lubricants that obey complicated constitutive relations, such as the differential types commonly used to couple shear thinning and viscoelasticity [64].

4.1.2.3 Approximate shear distribution (ASD) approach

Finally we discuss the approximate shear distribution approach, which is the category of our new approach (i.e. the modified viscosity technique) that will be described in section 4.1.3. The basic premise of ASD is to approximate the shear strain rate by an expression independent of the cross-film coordinate, i.e. it must not vary across the channel height. This is an empirical approach since the shear strain rate is parabolic for Newtonian flows [16] and becomes more complex for shear thinning lubricants. The primary motivations for an ASD type approach is (i) its simplicity and numerical efficiency compared to the GR model and (ii) it is not restricted by the choice of shear thinning formulation. Furthermore, we can extend it trivially to three dimensional contacts without increasing the numerical complexity leading to faster computations. However, the approximation of the shear strain rate leads to a decline in the accuracy as the non-linearity of the problem increases.

For practical purposes a function not dependent on y must be chosen to represent the velocity gradient field. Several studies largely adhere to the use of the mid-plane shear strain rate [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75], i.e. assuming $\dot{\gamma} = (U_2 - U_1)/h$. In some cases a mathematically equivalent film averaged shear strain rate function is employed. Additionally, one may choose the function based on the constitutive relation. For example, it can be obtained from the film-averaged second invariant of the strain rate in the case of a power-law type model [76, 77]. For the most part, the ASD approach is motivated by observing

that the variation in the solution of Eq. 4.3 does not vary for other choices of the shear strain rate. This has been shown to be valid for many types of lubricated contacts, for e.g. journal bearings [69]. Similarly, one can assume that the shear strain rate is defined at the moving surface [17, 78, 3], i.e. $\dot{\gamma} = f(x, y = 0, z)$ or $\dot{\gamma} = f(x, y = h, z)$. This particular choice is intuitively appealing since integral quantities such as friction and load are calculated at the channel boundaries. The only difference is that at the top or bottom surface, we must account for the pressure gradient, whereas in the mid-plane or averaged shear the pressure gradient is not involved. Table 4.1 lists the three possible choices for the shear strain rate. Clearly, if one opts for an ASD type approach the viscosity obtained by applying the correction for shear thinning does not vary across the film gap.

Shear distribution	Shear strain rate
ASD ⁽¹⁾	$\dot{\gamma} = \frac{1}{2\mu} \frac{dp}{dx} + \frac{(U_2 - U_1)}{h}$
ASD ⁽²⁾	$\dot{\gamma} = \frac{(U_2 - U_1)}{h}$
ASD ⁽³⁾	$\dot{\gamma} = -\frac{1}{2\mu} \frac{dp}{dx} + \frac{(U_2 - U_1)}{h}$

Table 4.1: The shear strain rate distributions for the ASD type models.

4.1.3 Modified viscosity approach

In this section we present a new ASD approach for capturing the cross-film variation in viscosity due to shear thinning. It is the aim of our work to capitalize on the advantages of the ASD approach, whilst minimizing the inherent disadvantages. For this reason, we seek a simple but effective modification to the viscosity function that accurately computes the pressure along the contact when compared with GR but avoids the integration operations. Since this approach seeks to extend the available ASD methods, we refer to it as the modified viscosity-ASD or simply MV.

Consider the one-dimensional scalar x -momentum equation for a thin-film,

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) = 0, \quad (4.9)$$

where μ is a function that varies along both the steamwise and the wall normal direction. We expand Eq. 4.9 and replace $\frac{\partial u}{\partial y}$ by $\dot{\gamma}_{xy}$ to obtain

$$-\frac{\partial p}{\partial x} + \frac{\partial \mu}{\partial y} \dot{\gamma}_{xy} + \mu \frac{\partial \dot{\gamma}_{xy}}{\partial y} = 0. \quad (4.10)$$

Rearranging and simplifying Eq. 4.10,

$$-\frac{\partial p}{\partial x} + \left(\mu + \frac{\partial \mu}{\partial \dot{\gamma}_{xy}} \dot{\gamma}_{xy} \right) \frac{\partial^2 u}{\partial y^2} = 0, \quad (4.11)$$

where the term in the parenthesis is the modified viscosity due to the shear strain rate, i.e.

$$\mu_m = \mu + \frac{\partial \mu}{\partial \dot{\gamma}_{xy}} \dot{\gamma}_{xy}. \quad (4.12)$$

We extend the definition of the modified viscosity to two-dimensional contacts by replacing the shear strain rate in the xy -plane ($\dot{\gamma}_{xy}$) by the expression for the equivalent shear strain rate,

$$\dot{\gamma}_{eq} = \sqrt{\dot{\gamma}_{xy}^2 + \dot{\gamma}_{zy}^2}, \quad (4.13)$$

where $\dot{\gamma}_{zy}$ is the shear strain rate in the $(y-z)$ plane. Note that if the bounding surfaces do not move along neither the wall normal direction nor the spanwise direction, as in most of the cases studied in this paper but the squeeze flow (see section 4.1.6), the magnitude of $\dot{\gamma}_{zy} \ll \dot{\gamma}_{xy}$ [65]. We summarize the modified viscosity approach as a system of equations comprising of a single non-linear Reynolds equation and an expression for the modified viscosity,

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12 \mu_m} \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\frac{h^3}{12 \mu_m} \frac{\partial p}{\partial z} \right) = \Lambda \frac{\partial(h\phi)}{\partial t} + \frac{\Lambda(U_2 + U_1)}{2} \frac{\partial(h\phi)}{\partial x}, \quad (4.14a)$$

$$\mu_m = \mu + \frac{d\mu}{d\dot{\gamma}_{eq}} \dot{\gamma}_{eq}, \quad (4.14b)$$

$$\dot{\gamma}_{xy} = \frac{1}{2\Lambda\mu_m} \frac{\partial p}{\partial x} h^2 (2\chi - 1) + (U_2 - U_1) \frac{1}{h}, \quad (4.14c)$$

$$\dot{\gamma}_{zy} = \frac{a}{2\Lambda\mu_m} \frac{\partial p}{\partial z} h^2 (2\chi - 1), \quad (4.14d)$$

where $0 \leq \chi \leq 1$ is a fraction of the film height h and denotes the location $y = \chi h$ where the strain rate is chosen to approximate its entire field.

Examining the expression for the modified viscosity (Eq. 4.14b), we find that it contains two functions; (i) the viscosity-shear relationship μ defined by the empirical constitutive relation (see section 4.1.4) and (ii) an additional term depending on the strain rate which constitutes the modified viscosity. Since the modified viscosity accounts for the rate of change μ due to shear strain rate $\dot{\gamma}_{eq}$, it must be a smooth function of $\dot{\gamma}_{eq}$. There is no restriction on which y -location, i.e. the value of χ , to take the strain rate for MV. However, not all values of χ guarantee accurate results. We will comment more on this in the results section.

4.1.4 Lubricant properties

The rheology examined in our work is non-linear and requires additional dimensionless quantities to be defined. The values for the empirical constants appearing in this section are assumed for all the cases considered in section 4.3, if not stated otherwise. We consider a lubricant whose (i) viscosity depends on pressure (i.e. piezoviscosity) and strain rate and (ii) density depends on pressure (i.e. liquid compressibility). The constitutive relations are non-dimensionalized using Eqs. 4.2.

The effect of pressure on viscosity is modeled via the Roelands model

$$\eta = e^{\kappa \cdot (-1 + (1 + \frac{p}{p_r})^z)}. \quad (4.15)$$

where, $\kappa = 7.3674$, $z = 0.34$, $p_r = 0.1983 \times 10^5$ are the dimensionless empirical constants. Note that (i) we have assumed the reference pressure $p_0 = 1 \times 10^5 Pa$ and (ii) κ , although dimensionless, is dependent on the ambient viscosity. Therefore, the value of κ may be restated for certain cases later in the manuscript.

Shear thinning is modeled via either (i) the Ree-Eyring constitutive relation

$$\mu = \frac{\tau_0}{\dot{\gamma}} \sinh^{-1} \left(\frac{\eta \dot{\gamma}}{\tau_0} \right), \quad (4.16)$$

or (ii) the Bair-Winer constitutive relation

$$\mu = \frac{\tau_0}{\dot{\gamma}} \tanh \left(\frac{\eta \dot{\gamma}}{\tau_0} \right), \quad (4.17)$$

where τ_0 is the dimensionless limiting (yield) stress and $\dot{\gamma}$ is the shear strain rate. The dimensionless limiting stress is defined as $\tau_0 = \tau_0^* U_{ref} \eta_0 / h_0^*$. For all cases the value of the dimensional limiting stress τ_0^* is fixed at $\tau_0^* = 7 \times 10^6$ Pa. However, the choice of the reference velocity (U_{ref}) and the entrance height (h_0) can result in a different dimensionless value. Therefore, we must specify τ_0 for each case considered. Note that although the Ree-Eyring relation has been widely used coming out from molecular theory [79] there exists a debate [80] surrounding the general validity of this relation in certain cases [43]. We compare the two constitutive relations for $\tau_0 = 70$ and assuming $\eta = 1$ in Fig. 4.2. For the same set of parameters, the Bair-Winer constitutive relation exhibits a larger decrease in viscosity against the shear strain rate. Although, in this study Eqs. 4.16 and 4.17 only are used as the constitutive relations for shear thinning, our MV technique can be coupled with any shear thinning relation available in literature.

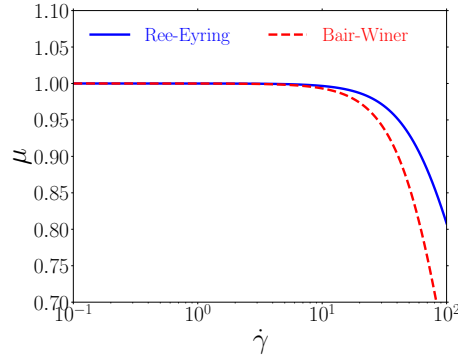


Figure 4.2: A comparison of the Ree-Eyring and Bair-Winer constitutive relation for $\tau_0 = 70$ and viscosity $\eta = 1$ versus the shear strain rate.

The effect of pressure on the density is modeled via the Dowson-Higginson model [81],

$$\phi = \frac{c_1 + c_2 p}{c_1 + p}, \quad (4.18)$$

where $c_1 = 5.9 \times 10^3$ and $c_2 = 1.34$ are the dimensionless empirical constants. We have used a reference pressure of $p_0 = 1 \times 10^5$ Pa to obtain c_1 .

4.1.5 Cavitation

The pressure developed in a lubricant film will be always above the pressure at the inlet for channels with a converging geometry along the streamwise direction. However, if there is a divergence in the channel the film pressure could decrease below the inlet pressure. Unless the bearing is pressurized [82] or the divergence is not significant, the film pressure can drop below the cavitation pressure and the lubricant film no longer occupies a purely liquid state. Vapor may be formed or dissolved gases within the lubricant film can emerge and occupy a finite region within the film. The exact value of the cavitation pressure depends on the chemistry of the lubricant [9]. In this work, we only consider the onset of cavitation due to a decrease in the film pressure.

Modeling cavitation requires accounting for the sudden change in the phase of the lubricant. If we consider Eq. 4.3, the dimensionless density acts as a measure of the film fraction, i.e. how much liquid or gas is present at a given point $\phi(x, z)$. If $\phi \geq 1$, the lubricant is said to be in the full-film region and if $\phi < 1$, we have cavitation and the pressure is simply equal to the cavitation pressure. In order to obtain the film pressure due to a cavitating contact we must also simultaneously obtain ϕ [83]. We have opted for the Elrod-Adams $p - \phi$ algorithm [83] to model cavitation in this study. Our choice is motivated by the fact that the algorithm has been (i) widely used in literature, (ii) works for a wide range of geometries and (iii) its implementation is fairly straightforward regardless of the numerical discretization method. Some remarks on the implementation of the cavitation algorithm are given in section 4.2.2.

The MV approach is not bound to the choice of the cavitation algorithm. One can also use stronger and more robust algorithms, such as the linear complementarity approach (LCP) [3, 84] or the Fischer-Burmeister-Newton-Schur (FNBS) [58]. However, they are arguably more complex and go beyond the scope of this work. The simplicity of the Elrod-Adams $p - \phi$ is suitable for this study.

4.1.6 Channel geometry

In this work we examine five different channels, see Fig. 4.3. They can be divided into three groups; (i) non-cavitating sliding/rolling contacts which includes the ridge and the plane slider, (ii) cavitating sliding/rolling contacts, such as the journal bearing and inclined pocketed pad, and (iii) a flat plate undergoing pure squeeze motion.

1. The ridge type geometry, which approximates a protrusion due to deformation of the channel surface or a single unit of the surface roughness, is depicted in Fig. 4.3(a). The ridge height is defined as

$$h = A(10^{-10(x-x_c)^2} \cos(2\pi(x-x_c))), \quad (4.19)$$

where A is the amplitude and x_c is the center of the ridge. In this work we assume $A = 0.5$ and $x_c = 0.5$.

2. The plane slider, which approximates a thrust pad in tribological applications, is shown in Fig. 4.3(b). Its height is defined as,

$$h = 1 + (\alpha - 1)x, \quad (4.20)$$

where $\alpha = 0.5$ is the minimum height of the channel.

3. In Fig. 4.3(c) we illustrate an unwrapped geometry of a journal bearing, which is commonly used in rotating mechanical elements, such as shafts for instance. The journal bearing surface function is

$$h = 1 + \epsilon \cos(2\pi x), \quad (4.21)$$

where ϵ is the eccentricity ratio. Note that the contact point is situated near $x = 0.5$.

4. The inclined pad with a diverging pocket is depicted in Fig. 4.3(d). This is used to approximate certain textures on a bearing surface. For our work we have chosen a pocketed pad used by Giacomini et. al. [2], whose height

is defined as

$$h = \begin{cases} \frac{15}{11} + (\alpha - 1)x, & \text{if } 0.2 \leq x \leq 0.5 \text{ and } 0.325 \leq z \leq 0.675 \\ 1 + (\alpha - 1)x, & \text{otherwise,} \end{cases} \quad (4.22)$$

where $\alpha = \frac{10}{11}$. Eq. 4.22 describes a pocket that is centered along the spanwise direction with a varying height defined by the inclination of the pad.

5. Finally, we have a flat plane with no inclination, shown in Fig. 4.3(e), which is used to study the MV approach for pure squeeze flows. Therefore, the surface function h depends only on time

$$h = 0.375 + 0.125 \cos(\omega t), \quad (4.23)$$

where ω is the frequency, which is set to $\omega = 4\pi$ in our work. Eq. 4.23 gives the maximum height $h_{max} = 0.5$ and the minimum height $h_{min} = 0.25$.

Although the surface functions show no dependency on the spanwise direction z , we have constructed the three-dimensional geometries by extruding along z . Furthermore, we note that Eq. 4.3 is expressed in terms of the aspect ratio a . In section 4.3 we will often show the pressure distribution along the channels. For the MV and the GR models, this pressure distribution is taken along the intersection of the surface and the mid-plane in the spanwise direction (i.e. the plane located at $z = 0.5$).

4.2 Numerical methods

We present first in section 4.2.1 the discretization using finite differences for the models given in section 4.1. The cavitation algorithm and its numerical method are described in section 4.2.2. Finally the different meshes used for our geometries are shown in section 4.2.3.

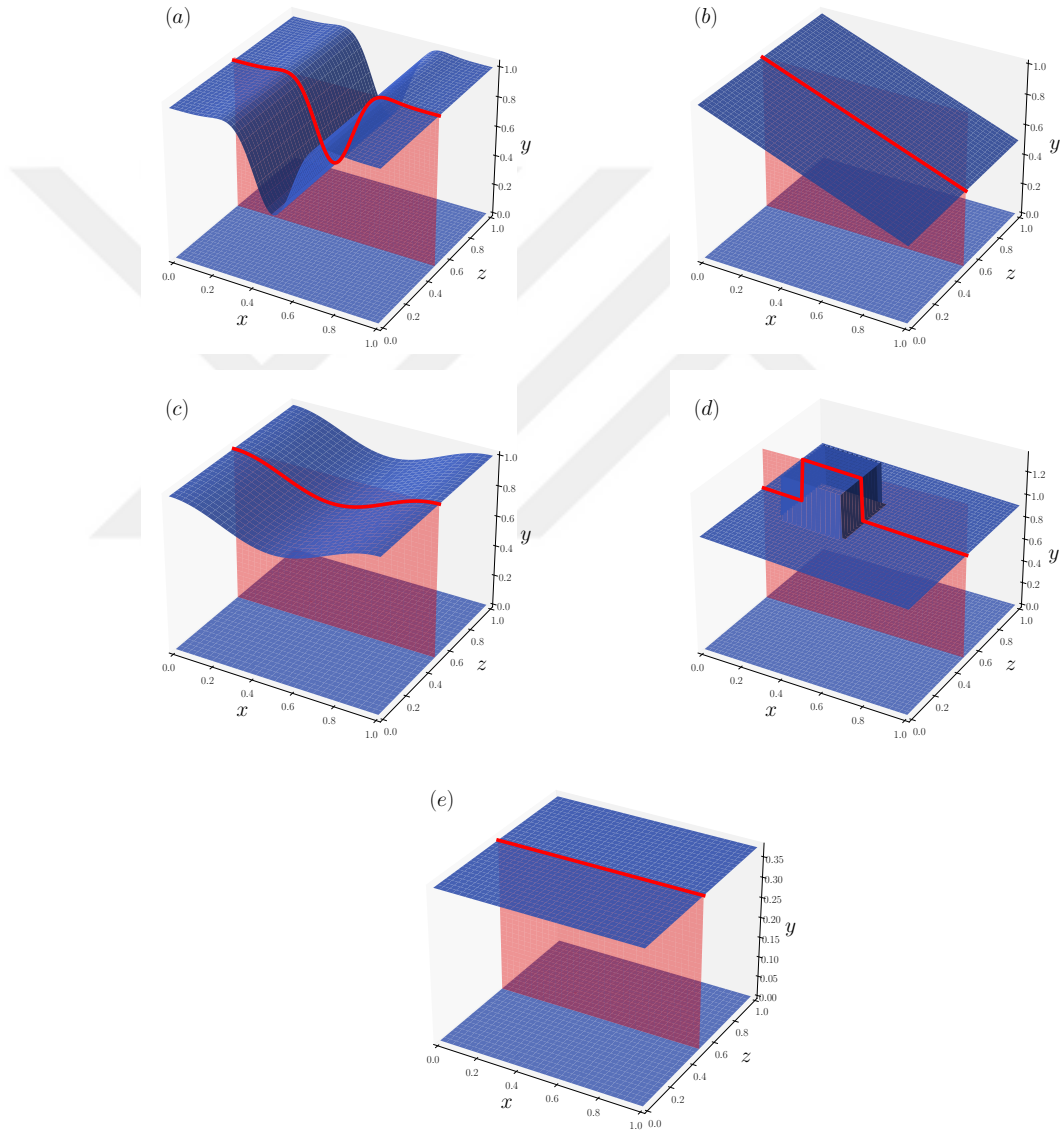


Figure 4.3: The (a) ridge, (b) plane slider, (c) journal bearing unwrapped geometry, (d) inclined pad pocket and (e) flat plate geometries used in this work. The mid-plane depicted in red contains the surface points located at $z = 0.5$.

4.2.1 Discretization

The governing partial differential equations presented in the preceeding section can be generalized into a single compact form

$$\frac{\partial}{\partial x} \left(\zeta \frac{\partial p}{\partial x} \right) + a^2 \frac{\partial}{\partial z} \left(\zeta \frac{\partial p}{\partial z} \right) = \Lambda \frac{\partial(h\phi)}{\partial t} + \Lambda \frac{\partial(\xi\phi)}{\partial x}, \quad (4.24)$$

where the definitions of ζ and ξ are provided for each model in Table 4.2. We use finite differences to convert the differential form into a discrete algebraic form in terms of ζ and ξ . The compact form of Eq. 4.24 allows us to generalize the numerical solution of the algebraic system of equations regardless of the type of model, since each model differs only in the definition of ζ and ξ . Note that we have used the trapezoidal rule for evaluating the integrals appearing in ζ and ξ for GR.

Model	ζ	ξ
ASD	$\frac{h^3}{12} \frac{\phi}{\mu}$	$\frac{(U_1+U_2)h}{2}$
MV	$\frac{h^3}{12} \frac{\phi}{\mu_m}$	$\frac{(U_1+U_2)h}{2}$
GR	$m_2\phi$	$U_1h + (U_2 - U_1)\frac{m_1}{f_0}$

Table 4.2: The expressions for ζ and ξ introduced in Eq. 4.24 for RE, MV and GR.

Using a cell centered finite difference scheme for the diffusion terms, i.e. the left hand side of Eq. 4.24, we obtain the expansion for the diffusion along x ,

$$\frac{\partial}{\partial x} \left(\zeta \frac{\partial p}{\partial x} \right) = \frac{1}{2\Delta x^2} ((\zeta_{i+1,j} + \zeta_{i,j})p_{i+1,j} + (\zeta_{i-1,j} + \zeta_{i,j})p_{i-1,j} - (\zeta_{i+1,j} + 2\zeta_{i,j} + \zeta_{i-1,j})p_{i,j}), \quad (4.25)$$

and the diffusion along z ,

$$a^2 \frac{\partial}{\partial z} \left(\zeta \frac{\partial p}{\partial z} \right) = \frac{a^2}{2\Delta z^2} ((\zeta_{i,j+1} + \zeta_{i,j})p_{i,j+1} + (\zeta_{i,j-1} + \zeta_{i,j})p_{i,j-1} - (\zeta_{i,j+1} + 2\zeta_{i,j} + \zeta_{i,j-1})p_{i,j}). \quad (4.26)$$

The convective term on the right hand side of Eq. 4.24 is discretized using a second order central difference scheme

$$\Lambda \frac{\partial(\xi\phi)}{\partial x} = \frac{\Lambda}{2\Delta x} (\phi_{i+1,j}\xi_{i+1,j} - \phi_{i-1,j}\xi_{i-1,j}). \quad (4.27)$$

In general, for all surfaces we have checked higher-order schemes for both the central and forward difference convective operators via a deferred-correction approach [85] and observed no change in the (i) converged solution or the (ii) rate of convergence. The central difference scheme works well for all the channels except the inclined pocketed pad, see Fig. 4.3(d). This is mainly due to the discontinuity in the surface function defining the pocket, which adversely affects the Elrod-Adams $p - \phi$ algorithm. A simple remedy is to use a first order upwind scheme

$$\Lambda \frac{\partial(\xi\phi)}{\partial x} = \frac{\Lambda}{\Delta x} (\phi_{i,j} \xi_{i,j} - \phi_{i-1,j} \xi_{i-1,j}). \quad (4.28)$$

The use of an upwind scheme reduces the global accuracy of the model to first order, i.e. $O(\Delta x, \Delta z)$. Higher-order upwind schemes seem to worsen the solution for the pocketed pad.

Finally, the transient term on the right hand side of Eq. 4.24 is discretized using a first-order forward difference scheme

$$\Lambda \frac{\partial(h\phi)}{\partial t} = \frac{\Lambda}{\Delta t} (h_{i,j}^{t+1} \phi_{i,j}^{t+1} - h_{i,j}^t \phi_{i,j}^t) + \Lambda \left(\frac{\partial h}{\partial t} \right)_{i,j} \phi_{i,j}^{t+1}, \quad (4.29)$$

where $\frac{\partial h}{\partial t}_{i,j}$ is the squeeze velocity at the node i, j . This term is only relevant for the pure squeeze flow problem examined in section 4.3.4.

We can collect the terms from Eqs. 4.25-4.28 to form a system of equations

$$W_{i,j} p_{i-1,j} + E_{i,j} p_{i+1,j} + C_{i,j} p_{i,j} + N_{i,j} p_{i,j+1} + S_{i,j} p_{i,j-1} = b_{i,j}, \quad (4.30)$$

where the left hand side of Eq. 4.30 forms a sparse matrix and the right hand side is a forcing function in ϕ . The non-linear system of Eqs. 4.30 is solved via an iterative algorithm that uses Gauss-Seidel with relaxation coefficient ω_{relax} for non-cavitating channels and is summarized in Fig. 4.4. The algorithm terminates when a tolerance of $M_p = 10^{-6}$ is achieved. Note that we have over relaxed the system, i.e. $\omega_{relax} = 1.95$, for most of the simulations conducted in this work for non-cavitating sliders. However, in some cases stated in the results section we had to under relax by choosing $\omega_{relax} < 1$. If cavitation occurs, ω_{relax} is set to $\omega_{relax} = 1$.

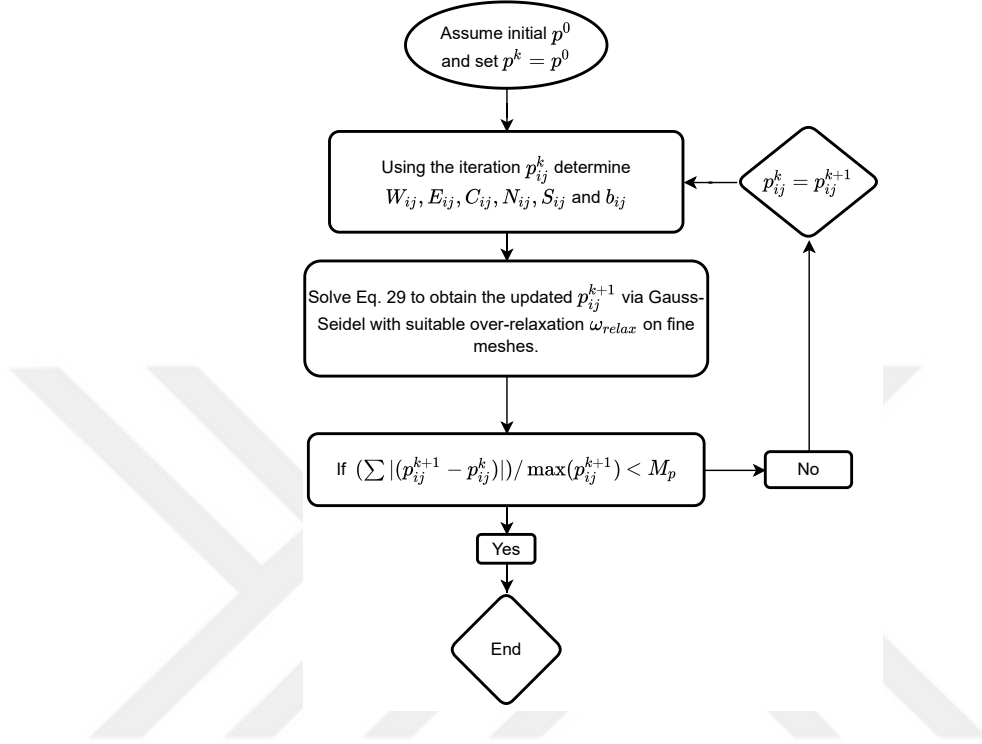


Figure 4.4: The fixed point iterative algorithm to obtain the film pressure p for sliding and non-cavitating contacts.

4.2.2 Cavitation algorithm

In this section we briefly introduce the Elrod-Adams $p - \phi$ algorithm to capture cavitation (see section 4.1.5) and make some remarks regarding the overall utility of the algorithm. The numerical procedure differs slightly from the fixed point iteration given in Fig. 4.4 since the Elrod-Adams $p - \phi$ algorithm requires an additional equation for the density ϕ . It is easily obtained by rearranging the terms in Eq. 4.29 and Eq. 4.30 for $\phi_{i,j}^{t+1}$

$$\phi_{i,j}^{t+1} = \frac{\Lambda}{\Delta t} h_{i,j}^t \phi_{i,j}^t + \frac{1}{c_{ij}} \left(W_{i,j}^t p_{i-1,j}^{t+1} + E_{i,j}^t p_{i+1,j}^{t+1} + C_{i,j}^t p_{i,j}^{t+1} + N_{i,j}^t p_{i,j+1}^t + S_{i,j}^t p_{i,j-1}^t \right), \quad (4.31)$$

where $c_{i,j} = \Lambda \frac{\partial h}{\partial t} + \frac{\Lambda}{\Delta t} h_{i,j}^{t+1}$. In Fig. 4.5 we outline the steps of the Elrod-Adams $p - \phi$ algorithm. Note that we must (i) provide an initial condition for ϕ and (ii) solve for ϕ when the pressure falls below cavitation pressure. An additional tolerance for ϕ is introduced as M_ϕ and is set to $M_\phi = 10^{-6}$.

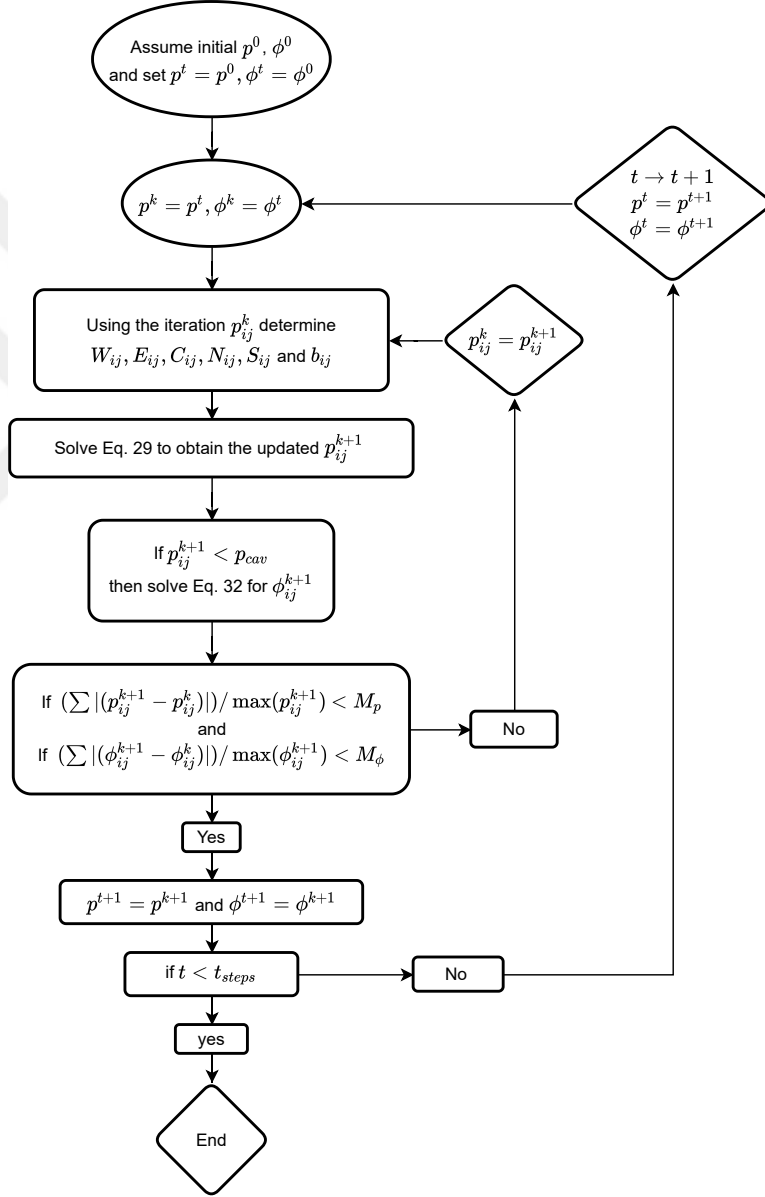


Figure 4.5: The fixed point iterative algorithm to obtain the film pressure p and the dimensionless density ϕ for cavitating contacts.

Furthermore, it is worth noting that the Elrod-Adams algorithm provides a single differential equation and avoids the need for two separate differential equations by means of a switch function [86]. However, it is limited to (i) compressible lubricants and (ii) renders the problem entirely in terms of the film fraction ϕ [87]. Furthermore, the switch function is inefficient for the pocketed pad and delivers an incorrect estimate for the pressure peak.

Note we do not over-relax the system. More efficient methods such as the multigrid [88] or the preconditioned bi-conjugate gradient descent method [89] can be used only for non-cavitating channels. A matrix representing this system cannot be constructed, since the Elrod-Adams $p - \phi$ algorithm involves correcting the pressure and density as it sweeps across the nodal points in the mesh. A Gauss-Seidel-like solver [83] can easily accommodate this requirement and then is adopted in this work.

4.2.3 Numerical domain

The degree of non-linearity of the problem changes for the different geometries and strongly affects the mesh quality. For all the geometries described in section 4.1.6 we have used thus different values of N_x , N_y and N_z , i.e. the number of nodes along x , y and z axes, respectively. In this work, we refine the mesh by doubling the number of nodes along each axis, starting from a coarse mesh, until we satisfy the requirement $(\max(p)_{fine} - \max(p)_{coarse}) / \max(p)_{fine} < 10^{-3}$. Note that the tolerance criteria was chosen by considering that at high bearing numbers the relative change in the solution, especially at the peak pressure, is large. The final mesh sizes for all the geometries are listed in Tab. 4.3.

4.3 Results

In this section we present a detailed comparison of the MV approach and the GR model for different channels presented in section 4.1.4. For each channel we (i)

Channel geometry	MV (N_x, N_z)	GR (N_x, N_y, N_z)
Ridge	128×128	$128 \times 64 \times 128$
Plane slider	128×128	$128 \times 64 \times 128$
Unwrapped journal bearing	64×64	$64 \times 64 \times 64$
Inclined pocketed pad	128×128	$128 \times 64 \times 128$
Squeeze flow	128×128	$128 \times 64 \times 128$

Table 4.3: Grid sizes for the different channel geometries.

establish the validity of the MV approach, (ii) perform a comparison with the GR model for the load,

$$L = \int_0^1 \int_0^1 p dx dz, \quad (4.32)$$

the maximum pressure and the computational time. The main focus is on (i) the computational efficiency of MV with respect to the GR model which is measured as the time necessary to reach convergence and (ii) the optimal choice of the strain rate location χ . The simulations are performed using an in-house code written in Python 3 and the processor is an AMD Ryzen 3600 3.4 GHz CPU with 16 Gb of ram.

4.3.1 Non-cavitating contacts

In this section we (i) validate the modified viscosity approach by comparing with results available in literature for the ridge profile, (ii) assess its computational efficiency for the ridge by varying Λ and (iii) extend the analysis to the plane slider.

4.3.1.1 Ridge: validation of MV

Validation of the MV approach is performed by comparing the pressure distribution along the ridge with the GR model and results available in literature [1]. The geometry of the ridge is useful for validating the proposed approach since it (i) produces large shear strain rates thereby increasing the effect of shear thinning

and (ii) is a physically relevant structure. While the GR model does not require validation in the limits of the lubrication theory, it serves as an alternative to AMRE by using the Ree-Eyring constitutive relation [1]. Thus, for all cases we simply resort to the GR model as a benchmark for the load, the pressure peak and the pressure distribution. In this section we will use the Ree-Eyring constitutive relations for modeling shear thinning and, for MV, χ is chosen to be $\chi = 0.5$ to be consistent with most of the literature (see section 4.1.2.3).

The pressure distribution along the ridge for a piezoviscous, compressible and shear thinning lubricant is given in Fig. 4.6(a) for $\chi = 0.5$, $\Lambda = 250$, $\kappa = 7.3674$, $\tau = 70$ and $a = 0.001$. It demonstrates that the MV approach can correctly predict the pressure distribution and is in full agreement with the (i) GR model and (ii) AMRE [61, 1]. All the features of the distribution, i.e. the maxima and minima are accurately determined. Furthermore, this precise prediction of the pressure indicates that the equivalent shear thinning function serves as a valid approximation. However, we must note that the resulting viscosity obtained after applying the shear thinning formulation may be entirely fictitious since we do not impose that the lubricant viscosity is invariant across the gap.

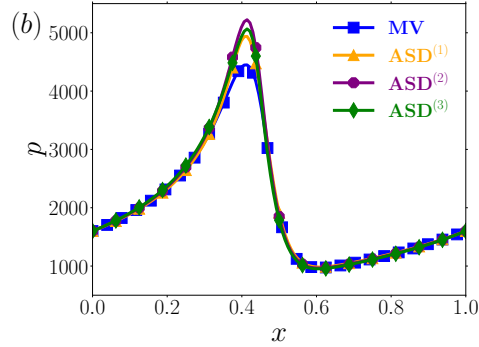


Figure 4.6: The pressure distribution of the modified viscosity approach (MV with $\chi = 0.5$) compared against (a) AMRE [1] and the GR model and (b) available ASD methods. The following parameters have been used $\Lambda = 250$, $\kappa = 7.3674$, $\tau = 70$ and $a = 0.001$.

Next we examine the relative improvement over the ASD methods (for the definition of $ASD^{(1)}$, $ASD^{(2)}$ and $ASD^{(3)}$, see table 4.1). Fig. 4.6(b) shows that

the pressure distribution obtained from the ASD models is not accurately predicted in the vicinity of the pressure peak. A noticeable overestimation of the maximum pressure occurs regardless of the choice of the ASD. In fact, all the ASD distributions lie close to each other. Therefore, the decrease in accuracy is inherent within the ASD models and not due to any particular choice of shear strain rate. We infer that this overestimation of the pressure peak manifests because the shear thinning effect is underestimated in the ASD method. This is clearly evident in the ridge because the shear thinning effects dominate. Thus, problems involving significant shear thinning may not be accurately modeled via conventional ASD methods.

The operational non-linear range of these ASD methods cannot be known a priori. We estimate the effect of the non-linearity on the accuracy of the ASD by varying the bearing number. In Figs. 4.7(a) and 4.7(b), we compare the load and the pressure peak for the MV and the ASD approach versus the bearing number. There is an excellent agreement between all models for low to moderate values of Λ . Until $\Lambda = 250$ the difference in the predicted load and pressure peak is small. It grows dramatically as we further increase the bearing number. Note that the deviation in the load is small for $\Lambda = 300$. This is because the discrepancy manifests initially close to the peak value and propagates towards the boundaries with increasing Λ . Since the load is an integral quantity, the variation close to the peak is not immediately reflected in its value.

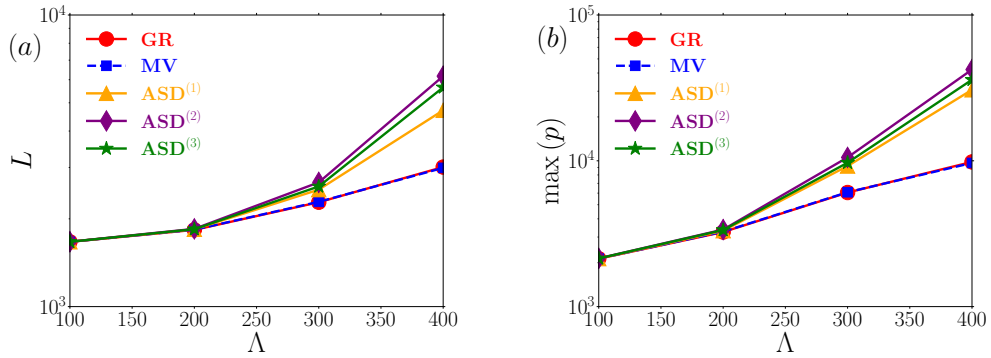


Figure 4.7: The (a) load and (b) maximum pressure for several models; (i) MV with $\chi = 0.5$, (ii) ASD⁽¹⁾, (ii) ASD⁽²⁾ and (iii) ASD⁽³⁾ for the ridge versus Λ for $\kappa = 7.3674$, $\tau = 70$ and $a = 0.001$.

The MV approach is clearly valid over the range where ASD accurately operate. We now investigate the limits of the MV approach for the ridge case. Figs. 4.8(a) and 4.8(b) show the comparison between the load carrying capacity and the pressure peak versus Λ for two values of the aspect ratio. For $a \ll 1$, the difference in the load and in the pressure peak remains small over the entire range of Λ . However, as the width increases, i.e. $a = 1$, the deviation gradually increases until reaching a percentage error of 12.69% and 6.5% for $\Lambda = 600$ for the load and the maximum pressure, respectively. Note that the ridge represents an asperity under the conditions of elastic deformation and, therefore, $\Lambda = 600$ is very high. In conclusion, for channels with aspect ratios around $a = 1$, we can expect a suitable performance of the MV within a physically valid range of Λ .

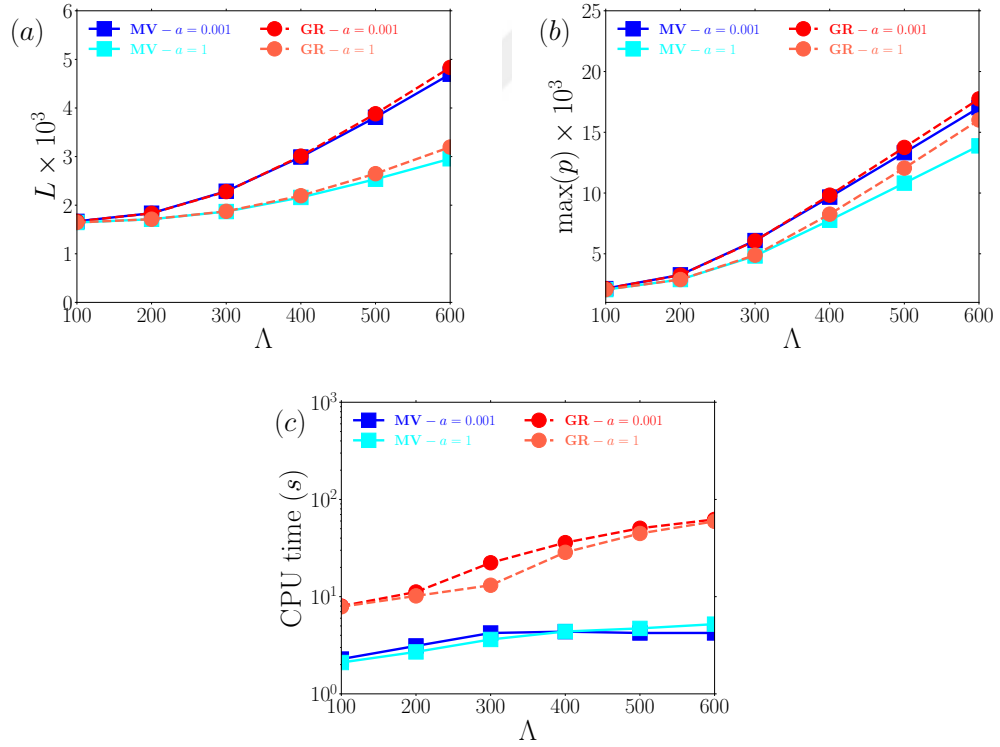


Figure 4.8: A comparison of the (a) load, (b) maximum pressure and (c) computational time versus Λ between the (i) MV with $\chi = 0.5$ (blue lines) and (ii) the GR model (red lines) for the ridge contact with $\kappa = 7.3674$ and $\tau = 70$ for two different values of the aspect ratio $a = 0.001$ and $a = 1$.

After illustrating the accuracy of the MV approach for a given range of Λ , we

investigate its second advantage, which is also common to all ASD approaches, i.e. its computational efficiency. Fig. 4.8(c) compares the computational (CPU) time of the GR model versus the MV approach. The difference is at least one order of magnitudes apart. The MV approach requires a few seconds to achieve the steady-state solution of Eq. 4.14a, whereas the GR model is almost 10-15 times slower.

4.3.1.2 Plane slider

In this section we apply our new approach to the plane slider but use the more sensitive Bair-Winer constitutive relation (Eq. 4.17). Similarly to the analyses conducted for the ridge, we examine the capacity of the MV approach in accurately predicting both the load and the pressure peak and comment on its computational efficiency.

Figs. 4.9(a)-(c) compare the load, the pressure peak and the CPU time for the proposed MV approach and the GR model in function of the bearing number for $\alpha = 0.5$, $\tau_0 = 820.44$ and $\kappa = 6.4511$. For the range of Λ examined, we find a perfect agreement between the load and maximum pressure for both (i) $a = 0.001$ and (ii) $a = 1$. This implies that the MV approach can accurately capture the pressure distribution along the bearing with exactly the same maximum value of the pressure. By continuing to increase Λ , a deviation of 2% in the pressure peak is observed at $\Lambda = 1500$. However, the pressure is outside the practical application of hydrodynamic lubrication at such a large value of Λ .

Fig. 4.9(c) shows that the computation time for the MV approach is much smaller than that for the GR model. Unlike the ridge, the computational time rises gradually with Λ with a sharp increase near $\Lambda = 1000$, since, for the plane slider the relative increase of the pressure above the boundary value ($p_0 = 0$) is very large resulting in extreme pressure gradients. In the standard fixed point iteration scheme adopted in this work (Fig. 4.4), the use of under-relaxation $\omega_{relax} = 0.1$ is required to prevent divergence due to the large growth of the pressure gradients, increasing the required simulation time. The computational

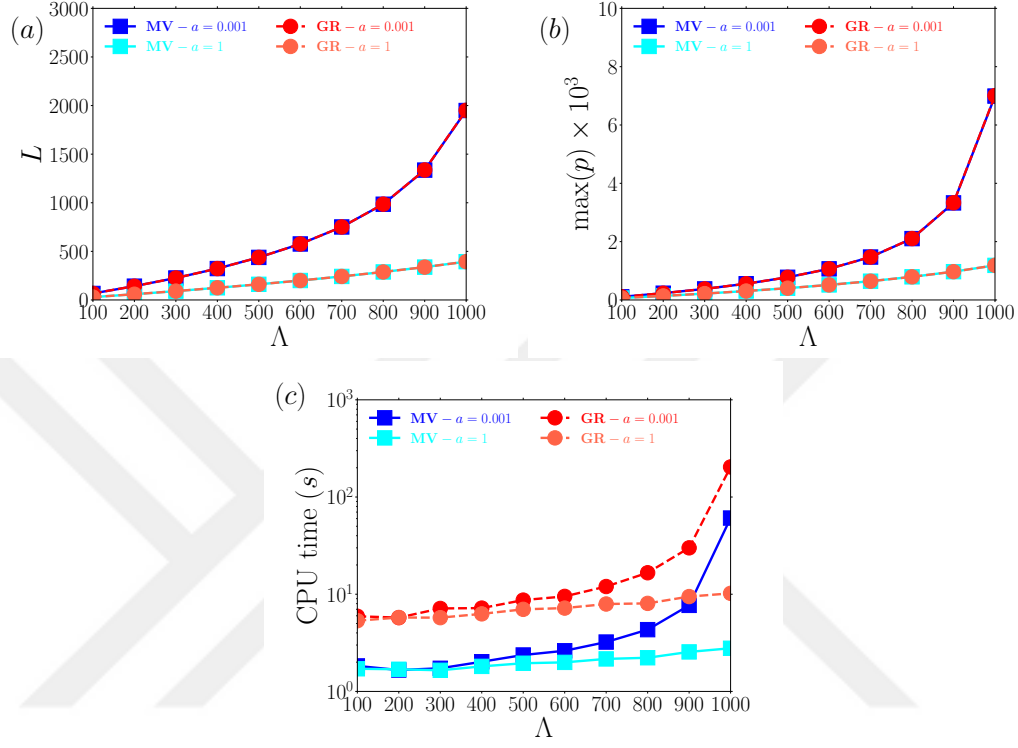


Figure 4.9: A comparison of the (a) load, (b) maximum pressure and (c) computational time between the MV approach with $\chi = 0.5$ (blue lines) and the GR model (red lines) for the plane slider with, $\alpha = 0.5$, $\tau_0 = 875$ and $\kappa = 6.4511$ for two different values of the aspect ratio $a = 0.001$ and $a = 1$.

time for the higher aspect ratio remains unchanged because the pressure is further diffused along the spanwise direction. This additional diffusion curtails the numerical buildup of large pressure gradients at each iteration. In fact, no under-relaxation is required for $a = 1$.

4.3.1.3 Optimal location for sliding contacts

The MV approach allows for a wide range of choices for the optimal strain rate location $y = \chi h$. In order to predict the best choice for χ , we exploit the functional parameters appearing in the general form of the governing equations Eq. 4.24, see section 4.2. The GR model and the MV (or in general any ASD) approach differ in the expressions for ζ and ξ , see Tab. 4.2. They are equivalent only for

the isothermal and Newtonian case, but differ completely for a non-Newtonian lubricant. Therefore, we introduce two error measures that can capture this difference

$$E_\zeta = \sqrt{\int_0^1 \frac{(\zeta_{GR} - \zeta_{MV})^2}{\zeta_{GR}^2}} \quad \text{and} \quad (4.33a)$$

$$E_\xi = \sqrt{\int_0^1 \frac{(\xi_{GR} - \xi_{MV})^2}{\xi_{GR}^2}}. \quad (4.33b)$$

The optimal value of χ is such that it minimises E_ζ and E_ξ , i.e. the solution of $\frac{\partial E_\zeta}{\partial \chi} = 0$ and $\frac{\partial E_\xi}{\partial \chi} = 0$. To achieve this minimization rigorously would require an analytical expression for E_ζ and E_ξ in terms of χ . For the latter, the matter is straight forward because ξ_{MV} is independent of χ . Therefore, $\frac{\partial E_\xi}{\partial \chi} = 0$ for all parameters regardless of χ . This simply implies that the model can reach to a sufficiently accurate answer, but cannot fully approximate the solution for the GR model. However, the former expression, i.e. $\frac{\partial E_\zeta}{\partial \chi} = 0$, deserves some attention. Note that it cannot be analytically obtained because of the (i) implicit integral expressions in ζ_{GR} and the (ii) implicit coupling between μ and $\dot{\gamma}_{eq}$.

Therefore, we proceed via a numerical treatment of Eq. 4.33a. Furthermore, we (i) simplify our approach by assuming a one-dimensional ridge contact because of its sensitivity to shear thinning effects and (ii) use the Ree-Eyring constitutive relation. In this manner, the optimal location resulting from this analysis will hold for simpler geometries, such as the plane slider, journal bearing etc.

Fig. 4.10(a) show the variation of E_ζ versus χ for several values of Λ . We make two key observations; (i) E_ζ increases with Λ , implying that the error can be minimized but not eliminated and, secondly, (ii) the distribution is approximately quadratic possessing a single minimum in χ . This minimum exists close to $\chi = 0.5$ and becomes increasingly prominent as Λ increases. Interestingly, as we move towards the top or the bottom surface where the contribution of the Poiseuille flow to the strain rate is strongest, i.e. $\chi = 0$ or $\chi = 1$, the error significantly increases.

Fig. 4.10(b) shows the variation in the percentage load. The minimum is

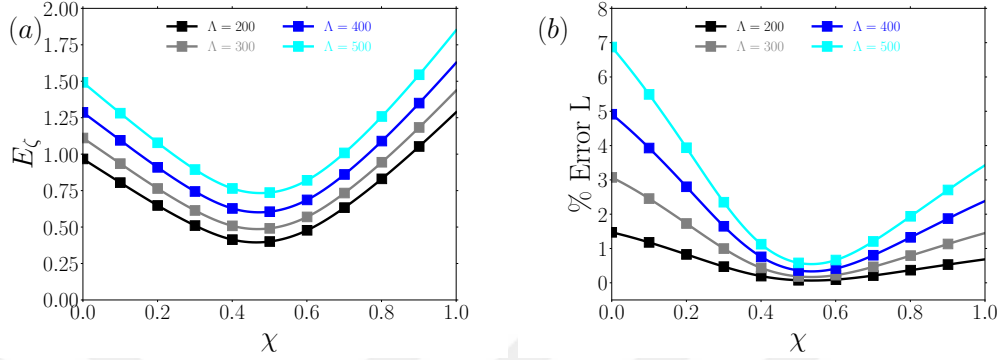


Figure 4.10: Distribution of the (a) E_ζ and the percentage error in the (b) load χ for different values of Λ for the ridge. The parameters used are $\kappa = 7.3674$ and $\tau_0 = 7$.

not exactly $\chi = 0.5$ but around this value. Despite the small discrepancy, this confirms our hypothesis that the optimal location can be qualitatively connected to minimum value of E_ζ . It should be noted that (i) the choice of $\chi = 0.5$ requires less computational effort, since the absence of the shear due to the pressure gradient decreases the non-linearity of the problem and (ii) the error is in general small for all the analyzed cases (maximum value is less than 7%), especially at low Λ and for $0.3 < \chi < 0.8$. This latter point shows that MV has an inherent accuracy independently on the value of χ .

4.3.2 Cavitating sliding contacts

In this section we examine the performance of the MV approach for cavitating channels. We focus on (i) a journal bearing unwrapped geometry in section 4.3.2.1 and (ii) an inclined pocketed pad in section 4.3.2.2.

4.3.2.1 Journal bearing

We validate the MV approach and GR model by comparing the load and the pressure peak against data available in literature [2] for the journal bearing. Note

that the load for the journal bearing is found by considering the force along x ,

$$W_x = 2\pi \int_0^1 \int_0^1 p \sin(2\pi x) dx dz, \quad (4.34)$$

and along y ,

$$W_y = 2\pi \int_0^1 \int_0^1 p \cos(2\pi x) dx dz, \quad (4.35)$$

which gives the load L ,

$$L = \sqrt{(W_x^2 + W_y^2)}. \quad (4.36)$$

Figs. 4.11(a) and 4.11(b) show the variation in the load and the pressure peak versus the eccentricity ratio for a Newtonian, isoviscous and incompressible lubricant with $\Lambda = 100$ and $a = \pi$. The MV and the GR model are able to replicate the published data using the Elrod-Adams $p - \phi$ algorithm.

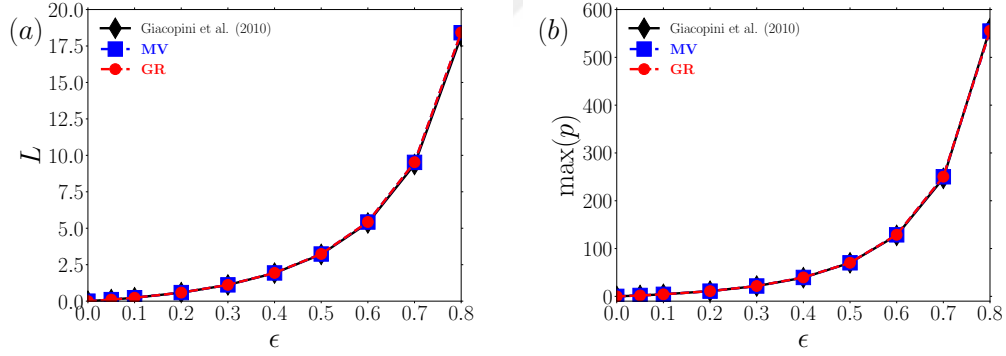


Figure 4.11: A comparison of the variation in the (a) load and (b) maximum pressure versus the eccentricity ratio between the MV approach (blue lines) against (i) the GR model (red lines) and (ii) LCP model of [2] for the journal bearing with $\Lambda = 100$ and $a = \pi$.

The test case used for validating the models is now employed for a piezoviscous, shear thinning and compressible lubricant. We keep $\chi = 0.5$ for the considerations drawn in section 4.1.3. Figs. 4.12(a)-(c) compare the load, the pressure peak and the computational time, respectively, for different aspect ratio versus Λ for GR for $\epsilon = 0.1$, $\tau_0 = 1925$ and $\kappa = 6.4511$. The results for the load and the maximum pressure are identical. The increased non-linearity due to cavitation is accurately captured by the MV approach in the range of examined bearing numbers. Therefore, in this case an accurate prediction of the load and the

maximum pressure also implies a correct prediction of the cavitation location as well.

The difference in CPU time between the models has increased up to two orders of magnitude due to the presence of cavitation. Note that the computational time decreases for large Λ because the effect of the shear thinning is dominant, leading to a numerical damping of the growth in viscosity at each iteration. The large difference is due to the construction of the numerically costly integrals at each iteration in the GR model. This difference could be reduced via the use of more advanced numerical integration methods which may lower the requirement of nodes along the film thickness. However, regardless of the sophistication of the numerical integration scheme, we will still require a finite time at each iteration to construct the integrals.

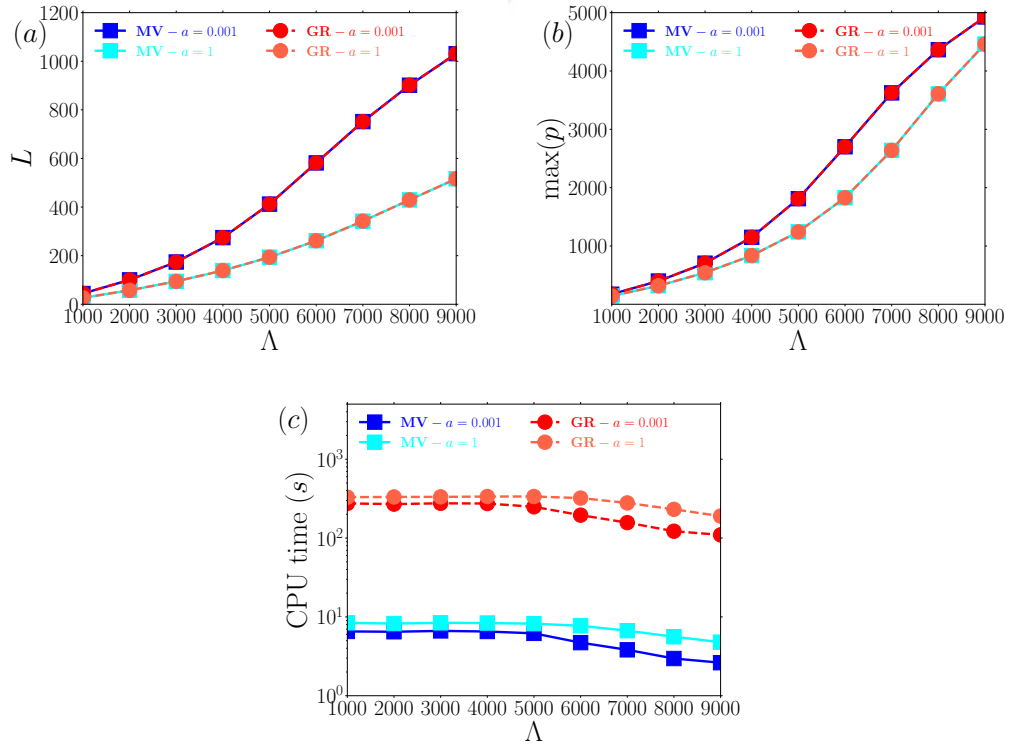


Figure 4.12: A comparison of the (a) load, (b) maximum pressure and (c) computational time between the MV approach with $\chi = 0.5$ (blue lines) and the GR model (red lines) for two different values of the aspect ratio $a = 0.001$ and $a = 1$. The eccentricity ratio is fixed at $\epsilon = 0.1$, $\tau_0 = 1925$ and $\kappa = 6.4511$.

4.3.2.2 Pocketed slider

In this section we compare the MV approach and the GR model for the inclined pocketed pad depicted in Fig. 4.3(d). Firstly, we validate against results available in literature and then present the comparison for the load, the maximum pressure and computational time.

Fig. 4.13 shows the pressure distribution along an inclined pocketed pad for an incompressible and Newtonian film. The MV approach (with $\chi = 0.5$) and GR model are in agreement with published data [3]. However, due to the presence of a sharp change in surface height, we observe a dramatic rise in the pressure. This implies that the Elrod-Adams $p - \phi$ algorithm requires several iterations to converge. Furthermore, the use of a central difference scheme for the convective term, even in the full-film region, leads to inaccurate results. Specifically, we found that the maximum pressure is not correctly predicted even though the cavitation region is captured. Therefore, we have used a first-order upwind scheme for this particular geometry, see section 4.2.1.

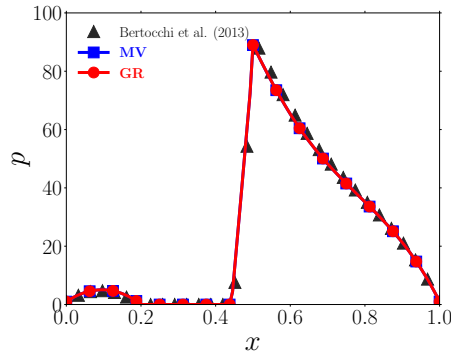


Figure 4.13: Pressure distribution for a pocketed inclined pad (see Fig. 4.3(d)) for the MV approach (blue squares), the GR model (red circles) and the results from the work of [3] (black triangles) with parameters $\Lambda = 1652$ and $a = 1$.

We modify now the lubricant properties to account for piezoviscosity, shear thinning and compressibility. Figs. 4.14(a)-(c) compare the load, the maximum pressure and the computational time for the MV approach and the GR model for the same geometry of Fig. 4.13. The models are in good agreement for the range

of analyzed bearing numbers. Fig. 4.14(c) indicates a rise in computation time for both the MV and the GR model with respect to the journal bearing case (see section 4.3.2.1). This is due to the large increase in the total number of iterations to reach convergence which can be attributed to (i) the use of a first-order upwind scheme and (ii) the choice of the cavitation algorithm. The computational time averages around 10000 – 12000s for GR and 100 – 200s for the MV approach. Clearly, GR is prohibitively slow for pocket type channels. Furthermore, we note that the computational time versus Λ does not vary indicating that the continuous construction of the numerically costly integrals is the primary reason for the slowness of GR.

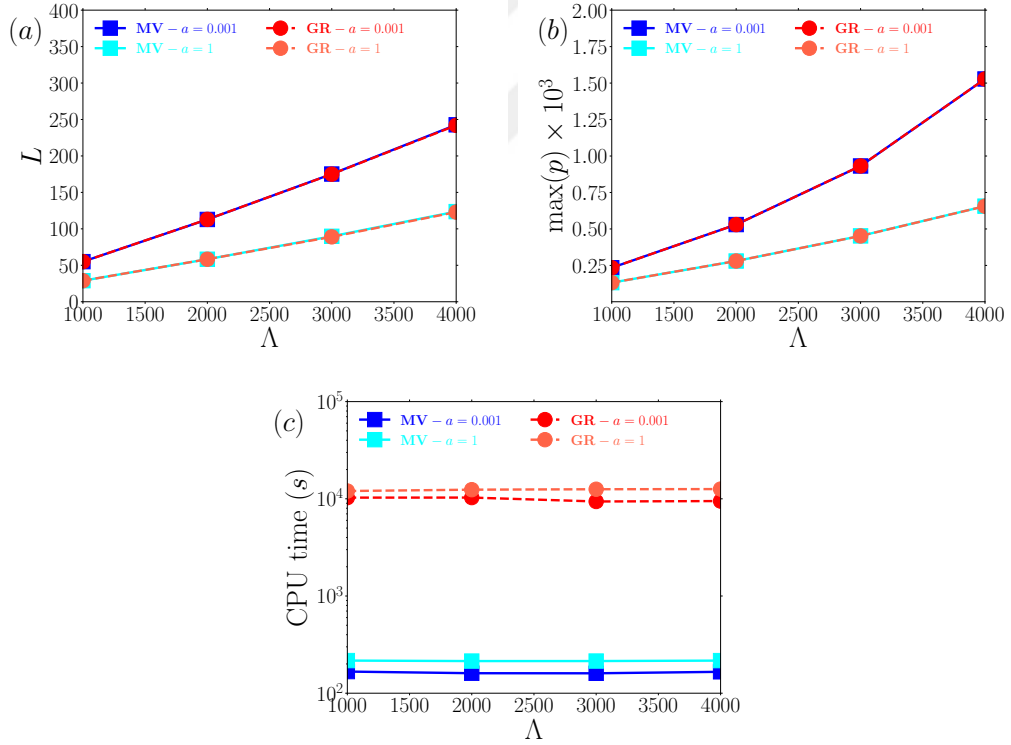


Figure 4.14: A comparison of the (a) load, (b) maximum pressure and (c) computational time between the MV approach with $\chi = 0.5$ (blue lines) and the GR model (red lines) for the inclined pocketed pad for two different values of the aspect ratio $a = 0.001$ and $a = 1$. The parameters used are $\kappa = 5.0648$ and $\tau_0 = 550$.

4.3.3 Rolling contacts

In this section we examine the performance of the MV approach for rolling and sliding contacts. The slide-to-roll ratio or $SRR = (U_2 - U_1)/(U_2 + U_1)$ measures the amount of sliding compared to the rolling motion of the boundaries. In the case of a purely rolling contact, i.e. $SRR = 0$, the strain rate induced by the boundary motion vanishes. Predicting the shear thinning behavior in this limit will lead to erroneous results when using simplified models that rely on the (i) mid-plane shear rate (see section 4.1.2) or the (ii) assumption of a Couette dominated flow [90, 15]. In order to avoid this failure, one must use the numerically demanding integral models such as the GR or an ASD approach with the strain rate defined far from the mid-plane. However, the latter, as seen in fig. 4.7, can lead to significant error when shear thinning is significant. For simplicity, we will consider the one-dimensional ridge contact only and the Ree-Eyring constitutive relation.

In order to avoid encountering the same issue we first find the optimal choice for the strain rate by considering E_ζ for the case of rolling and sliding. Fig. 4.15(a) shows the variation of E_ζ versus χ for different values of SRR . The shape of the distribution varies significantly from $SRR = 0$ to $SRR = 1/3$. For larger values of SRR the profile converges to the distribution of the pure sliding case. The rolling-only case ($SRR = 0$) has two minimum points ($\chi = 0.3$ and $\chi = 0.7$) in the distribution of E_ζ which converge into a single point, close to $\chi = 0.5$, as SRR increases until reaching the sliding-only case ($SRR = 1$). Moreover, for all values of SRR , E_ζ is maximum at the boundaries.

Fig. 4.15(b) show the percentage error in the load versus χ . The trend in E_ζ for pure rolling is reflected qualitatively, showing that the error is minimum around the same values of χ for which E_ζ is minimum. As we increase the contribution of sliding to the boundaries' motion, the error distributions tend to become nearly flat close to $\chi = 0.5$ which is the optimal value for pure sliding contacts. We again observe, similar to the sliding contacts, that the intuitive choice of $\chi = 0$ is not suitable as it tends to overestimate the effect of shear thinning.

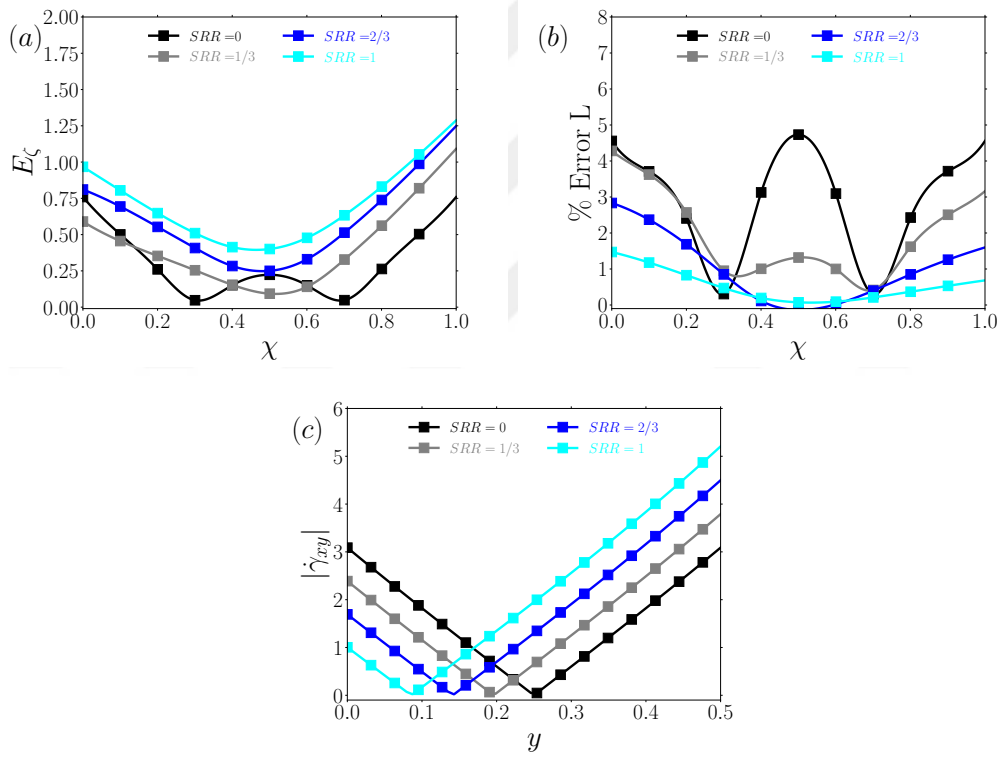


Figure 4.15: Distribution of the (a) error E_ζ , the (b) percentage error in the load versus χ and the (c) shear strain rate $\dot{\gamma}_{xy}$ across the contact at $x = 0.5$ and $z = 0.5$ for different values of SRR . The parameters used are $\Lambda = 200$, $\kappa = 7.3674$ and $\tau_0 = 7$.

The distribution of E_ζ versus χ is explained by considering the shear strain rate profile across the contact. Fig. 4.15(d) shows the shear strain rate profile across the contact at $x = 0.5$ and $z = 0.5$ for the ridge for different values of SRR . For $SRR = 0$ the strain rate distribution is anti-symmetric about $y = 0.25$ (corresponding to $\chi = 0.5$). In this limit, the strain rate becomes zero when averaged across the full film height h . It is not surprising then that the error is minimized for values of χ around $\chi = 0.25$ or $\chi = 0.75$, i.e. $y = h/4$ and $y = 3h/4$, since they correspond exactly to the average value of the absolute value of the strain rate, which varies linearly for $SRR = 0$. On the other hand, when $SRR = 1$ the distribution shifts such that the film average is not zero allowing the use of the mid-plane (or averaged) strain rate as a suitable choice.

To test this finding we illustrate in Figs. 4.16(a) and 4.16(b) the load and the maximum pressure, respectively, versus the SRR for $\chi = 0.25$, $\Lambda = 300$, $\kappa = 7.3674$ and $\tau_0 = 7$ by setting $\chi = 0.25$ for MV. Both quantities decrease gradually with increasing values of the SRR . The steady decline in both quantities versus increasing SRR is mainly due to increased shearing of the lubricant as the boundary motion dominates the flow. MV with $\chi = 0.25$ is in good agreement with the GR model and estimates both quantities with good accuracy. More importantly, it does not fail to account for shear thinning at $SRR = 0$ with $< 2\%$ error in the load and peak pressure. The deficiency observed in the most simplified models for shear thinning and some ASD approaches is not reproduced in our proposed MV approach.

4.3.4 Squeeze film flow

In this section we compare the performance of the MV approach against the GR model for a film squeezed by a flat surface. Since we focus only on the validity of the MV approach, the frequency, the minimum depth of squeeze and the squeeze velocity are kept constant, see section 4.1.6. The strain rate cannot be defined at the mid-plane similarly to the rolling-only case. Hence, we will first find the optimal χ for the squeeze flow in section 4.3.4.1 and afterwards we analyze the

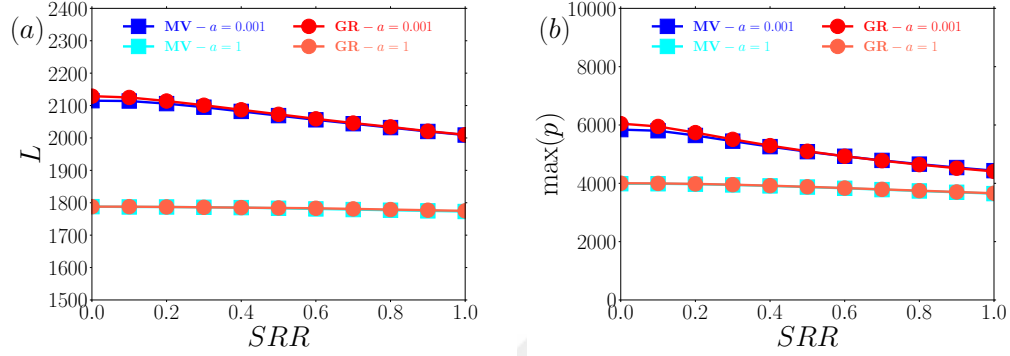


Figure 4.16: A comparison of the (a) load and the (b) maximum pressure in function of SRR between the MV approach with $\chi = 0.25$ (blue lines) and the GR model (red lines) for the ridge for two different values of the aspect ratio $a = 0.001$ and $a = 1$. The parameters used are $\chi = 0.25$, $\Lambda = 250$, $\kappa = 7.3674$ and $\tau_0 = 70$.

performance of MV using this choice in section 4.3.4.2.

4.3.4.1 The optimal location χ for squeeze film flow

We have seen that χ must be chosen carefully depending on the nature of the boundaries' motion and is found from the distribution of E_ζ , see sections 4.3.1.3 and 4.3.3. Unlike the previous cases, for squeeze film flow the error measure E_ζ will also vary versus time. To simplify our analysis, we consider only the (i) downward motion of a one-dimensional flat plate and use the (ii) Bair-Winer constitutive relation.

Fig. 4.17(a) compares the distribution of E_ζ for different values of χ . Note that we have simulated only the downward motion of the squeezing cycle, since in the upward motion the pressure (and its gradient) are zero due to cavitation. From Fig. 4.17(a) we notice that the error E_ζ is maximum for $\chi = 0$ and minimum and $\chi = 0.25$ throughout the cycle. This suggests that $\chi = 0.25$ is also a suitable choice for squeezing. In fact, the discussion about the anti-symmetric strain rate profile (see section 4.3.3) for the rolling-only case holds for the squeeze film. This is because in both cases we have only the Poiseuille term in the velocity profile.

Therefore, we are not surprised that $\chi = 0.25$ minimizes the error between GR and MV.

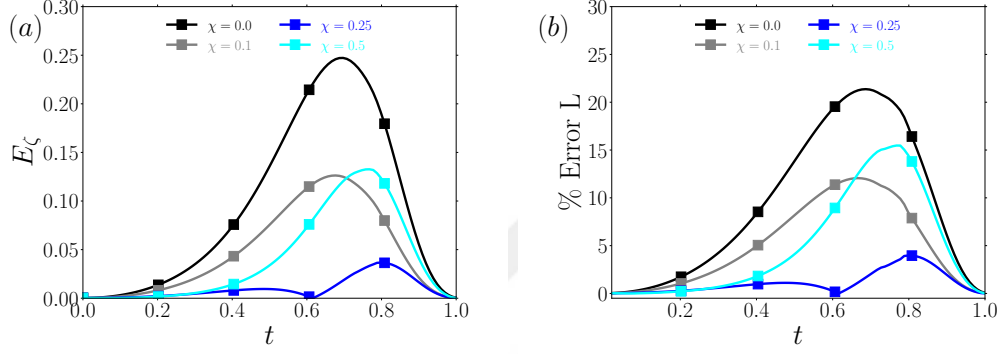


Figure 4.17: Distribution of the (a) E_ζ and the (b) percentage error in load for a purely downward squeeze motion for different values of χ . The parameters used are $\kappa = 4.777$, $\tau_0 = 7$ and $\Lambda = 10$.

Fig. 4.17(b) shows the percentage error in the load. Qualitatively the evolution of both the quantities mimics the distribution of E_ζ clearly pointing out that $\chi = 0.25$ is the most suitable choice. The maximum error for the load for $\chi = 0.25$ is less than 5% but can reach almost 25% for other choices. Note that we have used an artificial value for the yield stress τ_0 to enhance the effects of shear thinning. Therefore, this analysis and its outcomes will also hold for less stringent (but more realistic) conditions.

4.3.4.2 Validation and performance of MV for the squeeze flow

Fig. 4.18 compares the evolution of the location of the onset of cavitation X_{cav} along the streamwise direction for three squeezing cycles. The MV approach and the GR model compare favorably with data available in literature [4]. Note that the lubricant was assumed to be isoviscous, incompressible and Newtonian.

We now extend our analysis to a non-linear case involving a piezoviscous, shear thinning (with using the Bair-Winer constitutive relation) and compressible lubricant. We simulate only a single squeeze cycle since it is sufficient to

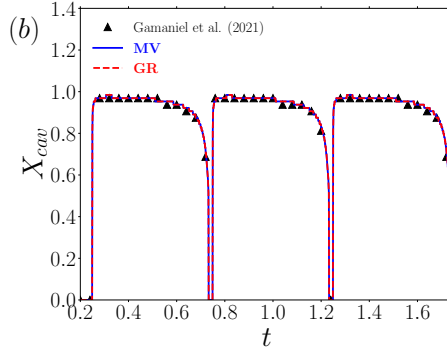


Figure 4.18: The evolution of the cavitation location along the center plane, i.e. $z = 0.5$, for the flat plate defined by Eq. 4.23 for the MV (blue lines) and the GR (red lines) compared with the work of [4] (black triangles) using $\Lambda = 100$ and $a = 1$.

compare MV (with $\chi = 0.25$) and GR. We define the cycle maximum load as $\langle L \rangle = \max(L(t))$ and the cycle maximum pressure as $\langle \max(p) \rangle = \max(p(t))$. Note that the average load or average pressure are significantly different and are insensitive to the bearing number. This occurs because the presence of cavitation dominates the analyzed cyclic motion and the pressure remains zero for most of the cycle. For this reason we have opted to show the maximum load and maximum pressure occurring during the cycle.

Figs. 4.19(a) and 4.19(b) compare the load and the pressure peak for the MV and the GR model for the same test case of Fig. 4.18. For each value of the bearing number we perform a single complete squeezing cycle. Firstly, it must be noted that the reference velocity of the purely squeeze problem is relatively small. This means that the dimensionless yield stress is small as well entailing that the shear thinning effect is not significant. We artificially enhance the effect of shear thinning by keeping the yield stress $\tau_0 = 7$, nearly a 1000 times stronger as for the ridge case. However, this may not be physical and represents polymer additives with a very small shear modulus. We find from Figs. 4.19(a) and 4.19(b) that MV and GR are in excellent agreement for the range of Λ herein examined. We should note that this agreement may stem from a weak dependence of the problem on shear thinning. We have verified that at very large values of Λ , shear thinning does play a key role in shaping the load and

the pressure evolution. However, those regimes are largely non-physical, since impractical values of the load and maximum pressure are obtained. For example, at $\Lambda = 1000$ the dimensional pressure peak exceeds several hundred GPa by using values for the other parameters (e.g. viscosity, wall velocity) of common lubricant oils under usual operating conditions.

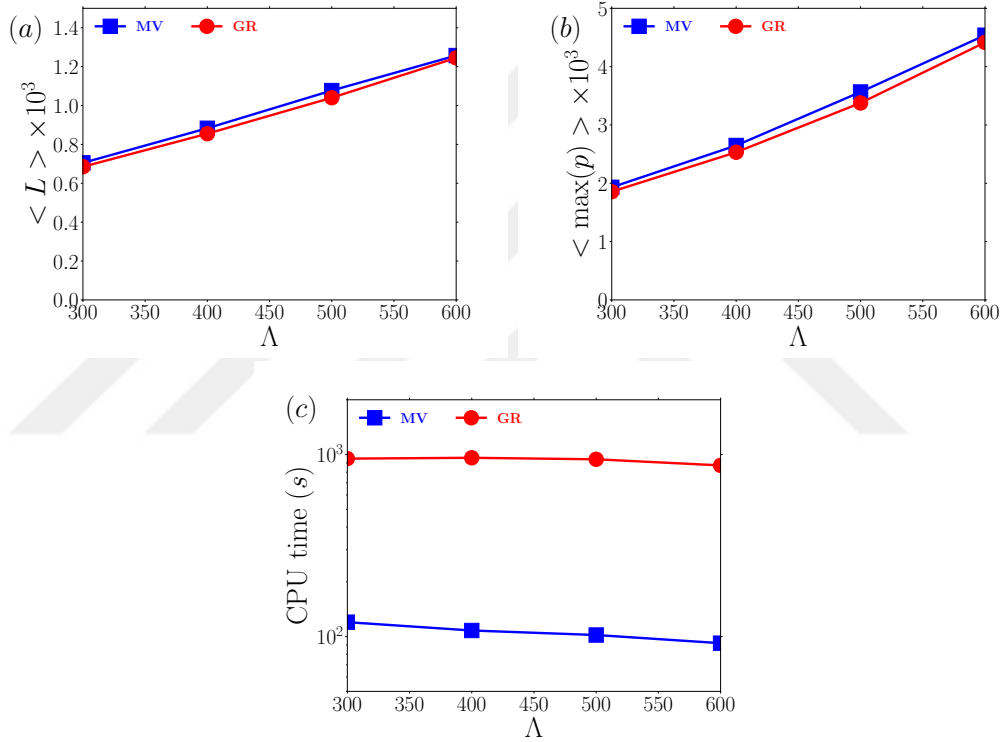


Figure 4.19: Comparison between the MV with $\chi = 0.25$ (blue line) and GR (red line) of the (a) cycle-maximum load, (b) cycle-maximum pressure and (c) computational time taken for a flat plane undergoing squeeze motion. The parameters used are $\kappa = 4.777$, $\tau_0 = 7$ and $a = 1$.

By observing Fig. 4.19(c), we find that the MV approach is several times faster than the GR model. Although the bearing number increases, the computational time is not significantly affected. This is because we adopted a purely explicit approach for the time marching. Note that the value of the yield stress used is 1000 times smaller than the ridge case. Below this value, no differences in the the maximum cycle load and the pressure exist between the GR model and MV approach.

4.4 Discussions: a priori choice of the optimal

χ

We have examined the error term E_ζ for the different configurations in earlier sections. For each regime of motion, the distribution of E_ζ dictates the optimal choice of χ . We found that the optimal χ is (i) around $\chi = 0.5$ for pure sliding, (ii) around $\chi = 0.25$ for contacts in which rolling is dominant and for squeezing contacts. However, in these analyses, we have relied upon the numerical solutions obtained from the GR and the MV model then an "a posteriori" method to check the optimal solution. In this section an alternative "a priori" approach is discussed to obtain the distribution of E_ζ .

We seek the value of χ where E_ζ is minimum, since that governs the accuracy of the MV approach (the same goes for all ASD methods, in general). Ideally, the expression for E_ζ , obtained analytically, should be minimized against χ to determine the optimal value, i.e. solving $\frac{\partial E_\zeta}{\partial \chi} = 0$. However, the shear strain rate from the GR (Eq. 4.7) is an implicit expression because the viscosity is a function of the strain rate. Therefore, decoupling the viscosity μ from $\dot{\gamma}_{xy}$ for the shear thinning case via an analytical approach is difficult. However, we can proceed with a semi-analytical approach by decoupling the viscosity-strain rate relationship.

One method of achieving this decoupling is by assuming a Newtonian (but piezoviscous) strain rate, i.e.

$$\dot{\gamma}_{xy} = \frac{1}{2\eta} \frac{dp}{dx} (2y - h) + (U_2 - U_1) \frac{1}{h}, \quad (4.37)$$

for the calculation of μ via a given constitutive relation. Note that Eq. 4.37 is a function of η and not of the non-Newtonian viscosity μ . The pressure gradient in Eq. 4.37 can be obtained by numerically solving the Reynolds equation (see Eq. 4.1) for a one-dimensional contact. In this manner, there is no need to solve the full governing equations (Eq. 4.5 for the GR and Eq. 4.14a for the MV). We still require the assumption of a piezoviscous lubricant otherwise the strain rate is too weak to induce any shear thinning, at least for the Ree-Eyring and Bair-Winer

constitutive relations, with the parameters considered in this study.

This approach is tested for the one-dimensional ridge contact with the Ree-Eyring constitutive relation. Fig. 4.20 shows the distribution of E_ζ for several values of SRR versus χ for $\Lambda = 200$ for a sliding and rolling considering only the Newtonian shear strain rate. The distribution is in qualitative agreement with the results presented in earlier sections, see, in particular, Fig. 4.10a and Fig. 4.15a. Moreover, the optimal value of χ is also consistent. For pure rolling, we observe the minimum of E_ζ close to $\chi = 0.25$ and upon increasing SRR to $SRR = 1$ representing pure sliding, the minima shifts towards $\chi = 0.5$, again in agreement to behavior of E_ζ found by the a posteriori analysis, see Fig. 4.15(a). For a purely squeeze flow the optimal choice for χ is no different from the pure rolling case. This is because in both cases we have a purely parabolic velocity profile due to the Poiseuille flow only. When the boundaries exert some sliding motion ($SRR > 1/3$) the optimal choice will shift towards $\chi = 0.5$, similarly to Fig. 4.15 where the full governing equations for both GR and MV were taken into account.

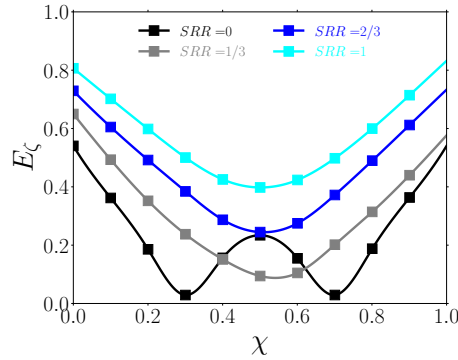


Figure 4.20: Distribution of the error E_ζ for sliding and rolling for different values of SRR at $\Lambda = 200$ versus χ for $\kappa = 7.3674$ and $\tau_0 = 7$.

There are several advantages of this a priori approach. Firstly, it is performed with minimal computational effort. Secondly, owing to the generality of the method, any combination of a surface function and a constitutive relation can be examined for a wide range of parameters. Thus, for very complex surfaces such as those mimicking surface texture or surface roughness, the optimum value of χ

can be straightforwardly found. In our analysis, the values for χ are sufficient to achieve an error $< 5\%$ in the load and the peak pressure for the most extreme cases as well. However, some lubrication problems may require further accuracy in the solution for the pressure. For such cases, the proposed a priori estimation method can provide an efficient tool for choosing the exact optimal value of χ based on the boundaries' motion to minimize the error. We should highlight that for large Λ the Newtonian piezoviscous solution could become numerically impossible to achieve since the viscosity become too high without the mitigating effect of shear thinning. In this scenario, however, the trend can be captured by studying smaller bearing numbers since, as it was shown in Fig. 4.10b, an increase in Λ enhances the error, but does not qualitatively change its behavior.

Chapter 5

Viscoelastic thin-film squeeze flow

Thin lubricating films confined between two surfaces exhibit a range of behavior depending on the type of surface motion. Here, we consider a case in which a film is squeezed when the two surfaces, initially separated, approach each other at some velocity, which may not be constant. If the film formed between these surfaces is quite thin in comparison to the length it spans, and the approach velocity of the surfaces is small, then one may apply the assumptions of lubrication theory and arrive at a simplified set of equations which could then be used for analysing the film's behavior. The mathematical model governing the behavior for a film comprising a Newtonian solvent and a elastic polymer additive were introduced in chapter 2. Two simplified modelling approaches were developed to overcome the mathematical complexity (specifically, the non-linear coupling between the dependent variables) of the governing set of equations, namely the perturbation method (see section 2.6) and the viscoelastic Reynolds approach (Eq. 2.36a-2.36c). In this chapter, we present the results obtained from using these two approaches, offering a detailed comparison of their advantages (disadvantages) for the case of two flat cylindrical plates; for constant surface speed in section 5.1 and variable surface speed in section 5.2.

We consider the Oldroyd-B constitutive relation which represents a purely elastic polymer with infinite stretch. However, we point out that experiments have demonstrated features, such as an overshoot in the stress (pressure), that the Oldroyd-B model cannot capture fully [54, 91]. However, the analysis performed is for the case of creeping flow which may not conform to the assumptions of lubrication theory, but only those of a low-Reynolds number (Stokes flow). Furthermore, several necessary assumptions on the convected rate operator have been made to simplify the problems.

5.1 Constant surface speed

The simplest case is that for a film squeezed between two flat plates at a constant approach velocity, i.e., $H = 1 - t$, where t is the non-dimensional time. In this setting, t can also be interpreted as the total fraction of the initial film height. The perturbation model up to the second-order in De can then be further simplified by noting $\frac{dH}{dt} = -1$, $\frac{d^2H}{dt^2} = \frac{d^3H}{dt^3} = 0$.

The pressure distribution, normal force F_N and the friction force on the bottom plate F_r for the leading-order is

$$p^{(0)} = \frac{3(r^2 - 1)}{H^3} \frac{dH}{dt}, \quad (5.1a)$$

$$F_N^{(0)} = -\frac{3\pi}{2H^3} \frac{dH}{dt}, \quad (5.1b)$$

$$F_r^{(0)} = -\frac{2\pi}{H^3} \frac{dH}{dt}, \quad (5.1c)$$

for the first-order,

$$p^{(1)} = -\frac{3(r^2 - 1)}{10H^4} \left(-57 \left(\frac{dH}{dt} \right)^2 \right), \quad (5.2a)$$

$$F_N^{(1)} = -\frac{171\pi}{20H^4} \left(\frac{dH}{dt} \right)^2, \quad (5.2b)$$

$$(5.2c)$$

$$F_r^{(1)} = -\frac{17\pi}{5H^3} \left(\frac{dh}{dt} \right)^2, \quad (5.2d)$$

and the second-order system,

$$p^{(2)} = \frac{3(r^2 - 1)}{700H^5} \left(19491 \left(\frac{dH}{dt} \right)^3 \right), \quad (5.3a)$$

$$F_N^{(2)} = -\frac{58473\pi}{1400H^5} \left(\frac{dH}{dt} \right)^3, \quad (5.3b)$$

$$F_r^{(2)} = -\frac{4021\pi}{350H^4} \left(\frac{dh}{dt} \right)^3. \quad (5.3c)$$

Note that when the approach velocity is constant, $\frac{\partial h^*}{\partial t^*} = -V_0$, where, V_0 is a constant, and no apparent frequency is identifiable in the system, then the Deborah number and Weissenberg are simply related by the thin-film ratio, similar to the pure sliding contact. For some constant plate radius R_0 and initial height $h^*(t^* = 0) = H_0$, we have $Wi = \lambda \frac{U_0}{H_0}$, and $De = \lambda \frac{V_0}{H_0}$, where, $V_0 = \epsilon U_0$, and, therefore, $De = \epsilon Wi$. The Weissenberg number is generally chosen to represent the relative time scale of deformation, i.e., $Wi = \lambda \dot{\gamma}^*$, which is, for the pure squeezing problem, simply $\dot{\gamma}^* = \frac{U_0}{H_0}$.

In Fig. 5.1, we plot a comparison of the second-order perturbed model and the VR approach for the net vertical force (F_N) and the friction (at the bottom plate) (F_r) using $\beta = 1, \epsilon = 0.001$ for different values of the Deborah number. Grid independent results (for each time step) were obtained using 128 nodes along the r -axis and 32 nodes along z -axis (for VR only). The time step size $dt = t_f e^{10(k/N-1)}/N$, where, $t_f = 0.9$ is the total time of motion, $N = 256$ are the number of time steps, for each step k was unequal in order to capture the initial transient phase. The numerical techniques employed are detailed in chapter ??.

As time progresses and the film height decreases, the load and friction exponentially increase for the Newtonian and the viscoelastic cases. The vertical force developed on the plate is due to the resistance offered by the viscous lubricant when being forced out of the channel. Similarly, the friction on the pad increases, since the fluid velocity must increase (along the radial direction) to account for the decreasing flow cross-section ($2\pi R_0 H(t)$). A sharper velocity profile (although it is still quadratic along z) leads to larger velocity gradients and, therefore, larger shear strain rates.

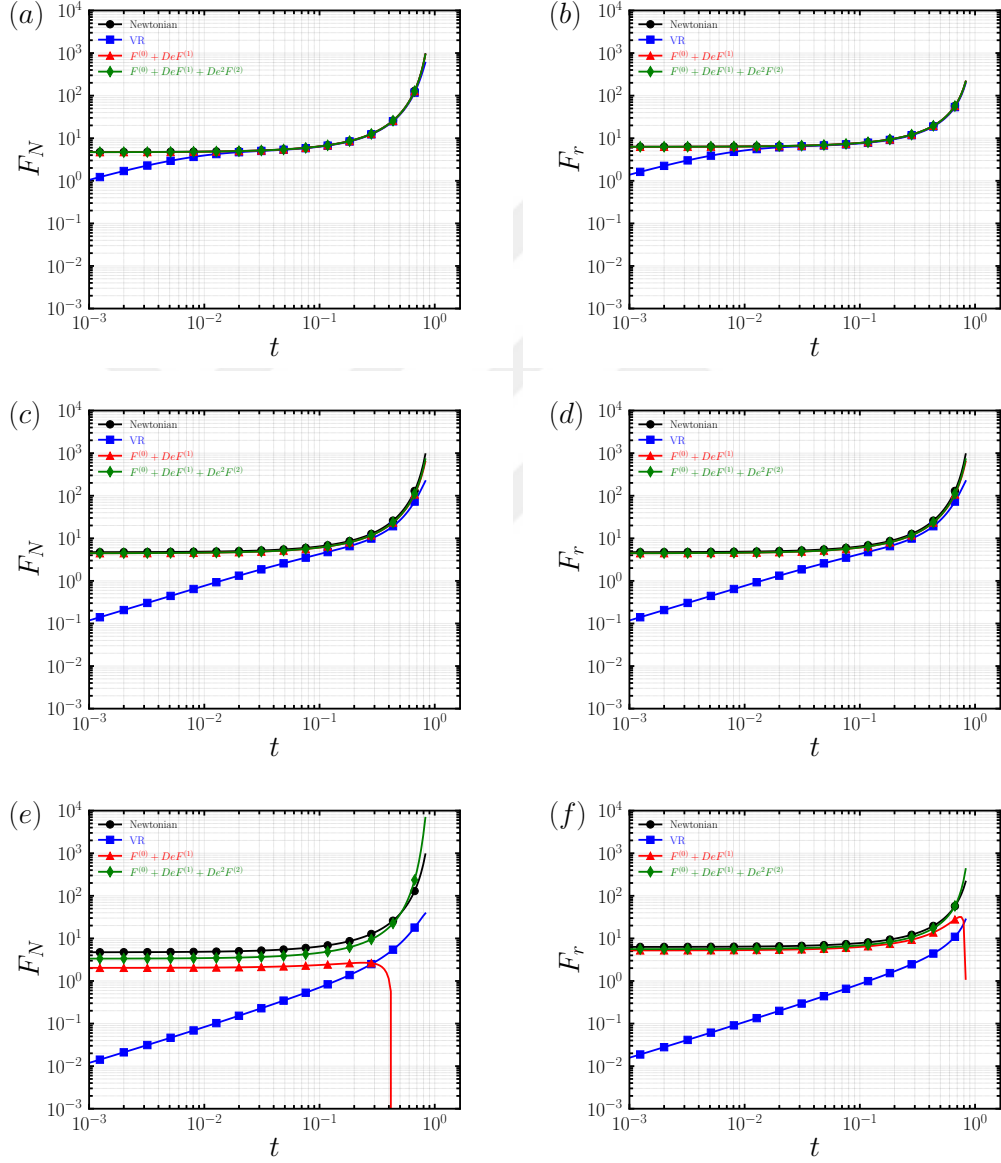


Figure 5.1: The variation in the net vertical force F_N and the friction F_r at the bottom plate versus time for (a) and (b) $De = 0.001$, (c) and (d) $De = 0.01$, and (e) and (f) $De = 0.1$, respectively, for the second-order perturbed model and the VR approach using $\beta = 0$, and $\epsilon = 0.001$.

A good agreement is observed for small values of De , consistent with perturbation theory, when the transient effects have sufficiently vanished ($t > 0.01$). In the instant when the motion starts ($t = 0$), we observe a large difference between the perturbed models and the VR, but diminishes as time progresses. Firstly, the VR model, not only accepts any initial condition, but caters to the realistic condition of zero initial polymer stress. Secondly, the perturbation models cannot capture this, since they are a Taylor-series expansion (in De) about the Newtonian case for which the stress develops instantly (owing to a zero relaxation time). We can easily expect that this behavior will be observed regardless of the order of perturbation. It is an odd outcome, since viscoelasticity is inherently a transient phenomena, arising entirely due to the dependency of the polymer stress on its time history.

Not surprisingly, as we increase the strength of viscoelastic effects, the transient phase stretches owing to the late response time of the polymers. For the highly elastic case, depicted in Figs. 5.1(e)-(f), the prediction of the load and friction is in stark contrast among the models. Firstly, the first and second-order perturbed models predict an opposing trend, the latter being positive whereas, the former is negative. This is seen easily from Eqs. ?? in which the influence of the surface speed, $\frac{dh}{dt}$, seems to change sign as the perturbation order increases. For a negative speed, this implies that all even (odd) order perturbations will always predict a positive (negative) load. The VR model predicts a monotonically increasing load which, notice, stays somewhat between the predictions of the first and second-order perturbation.

The prediction for the surface friction for different De follows the same trend as the load. Note that the velocity field is only weakly disturbed due to the elasticity of the polymer. In fact, one may assume it to be Newtonian (in this channel geometry) for small De . Then, the difference in friction, across Figs. 5.1(b)-(d)-(f), is not due to the shear strain rate but rather the latency in the polymer shear stress which is unable to catch up to its Newtonian equivalent.

We infer, based on the results, for moderate values of De , both the VR model and the perturbed system predict results with a very small variation provided

the initial transient phase has passed. The length of this phase depends on the Deborah number. However, the initial phase cannot be modeled via the perturbed system regardless of its order.

5.2 Irregular motion

Viscoelasticity is inherently a temporal phenomena, therefore, we wish to see the effect of surface acceleration, and deceleration, on the response of the film. Note, these are not inertial effects, since we have assumed the Reynolds number to be small. This requirement, in general, is not trivial to enforce, since the Re is affected by the reference height and applied frequency. Therefore, in addition to the thin-film approximation, we must also ensure that the frequency at which the surface oscillates, remains small.

We consider the simple motion

$$h^* = 2H_0 + d^* \cos(2\pi\omega^*t^*), \quad (5.4)$$

where, H_0 is the reference height, d^* and ω^* are the amplitude and frequency of the motion, respectively. We non-dimensionalize the motion using the scaling provided in chapter 2

$$h = 2 + d \cos(2\pi t), \quad (5.5a)$$

$$\frac{dh}{dt} = -2\pi d \sin(2\pi t), \quad (5.5b)$$

$$\frac{d^2h}{dt^2} = -4\pi^2 d \cos(2\pi t), \quad (5.5c)$$

$$\frac{d^3h}{dt^3} = 8\pi^3 d \sin(2\pi t), \quad (5.5d)$$

where, $t = \omega^*t^*$ and $d = d^*/H_0$ is the modified amplitude.

In order to determine the appropriate relation between the Deborah and Weissenberg numbers, the reference velocity (required for determining the residence and deformation time scales) along the radial, U_0 , and vertical, V_0 , must be settled. The approach velocity in the oscillating case is not constant, and varies

with time, depending strongly on the functional form. One must carefully choose the reference velocities which directly affects the choice of Wi . However, the choice of the Deborah number is straight forward, i.e., $De = \lambda\omega^*$, ω^* is the frequency. From Eq. 5.4, we choose the reference vertical velocity as $V_0 = \omega^*d^*$ (ignoring the 2π), which gives $Wi = \lambda \frac{U_0}{\epsilon H_0} = \lambda \frac{R_0 \omega^* d^*}{H_0^2}$. This yields the relationship $De = \epsilon Wi \frac{d}{H_0}$. For lubricated contacts, we have a finitely large shear strain rate (and shear stress), which strongly affects the polymeric additive in the form of shear thinning and stretching, which implies Wi is $O(1)$. If $d \approx H_0$, then De and Wi are related by ϵ as in the constant speed case. However, if $d^* < H_0$, then the upper-convected rate operator (assuming $\frac{\partial}{\partial \theta} = 0$)

$$\lambda \overset{\nabla}{\tau}^* = \lambda \frac{\partial \tau^*}{\partial t^*} + u_r^* \frac{\partial \tau^*}{\partial r^*} + u_z \frac{\partial \tau}{\partial z^*} - \mathbf{L}^{*T} \tau^* - \tau^* \mathbf{L}^*, \quad (5.6)$$

reduces to

$$De \overset{\nabla}{\tau} = De \left\{ \frac{\partial}{\partial t^*} + \left[\frac{d^*}{H_0} \right] \left(u_r^* \frac{\partial}{\partial r^*} + u_z \frac{\partial}{\partial z^*} - \mathbf{L}^T \tau - \tau \mathbf{L} \right) \right\}. \quad (5.7)$$

If we assume a small amplitude motion, i.e., $\frac{d^*}{H_0} \ll 1$, then the convective transport and the effect of the stretch terms are negligible. The Oldroyd-B is reduced to the simple linear Maxwell model and only the relative time history of the polymer stress is important. Interestingly, if $\frac{d^*}{H_0} \gg 1$, then the local temporal variation can be ignored.

We plot the time variation of the load and friction in Fig. 5.2 for different values of De , assuming $\epsilon = 0.001$, $\beta = 0$. For the sake of simplicity, we have assumed that $\frac{d^*}{H_0} = 1$ and, therefore, $De = \epsilon Wi$ allowing for a comparison with the constant speed case.

We observe a good agreement with between the models and a very small deviation from the Newtonina case for small values of De . For a strongly elastic case, $De = 0.1$ as shown in Figs. 5.2(e)-(f), a stark contrast is observed between the perturbed models and the VR. This difference is more pronounced in the load (Fig. 5.2(e)) and less so in the friction force ((Fig. 5.2(f)). Although, elastic effects are stronger, we have observed, from the constant speed case, the effect of viscoelasticity is to diminish the stress owing to a time and spatial delay

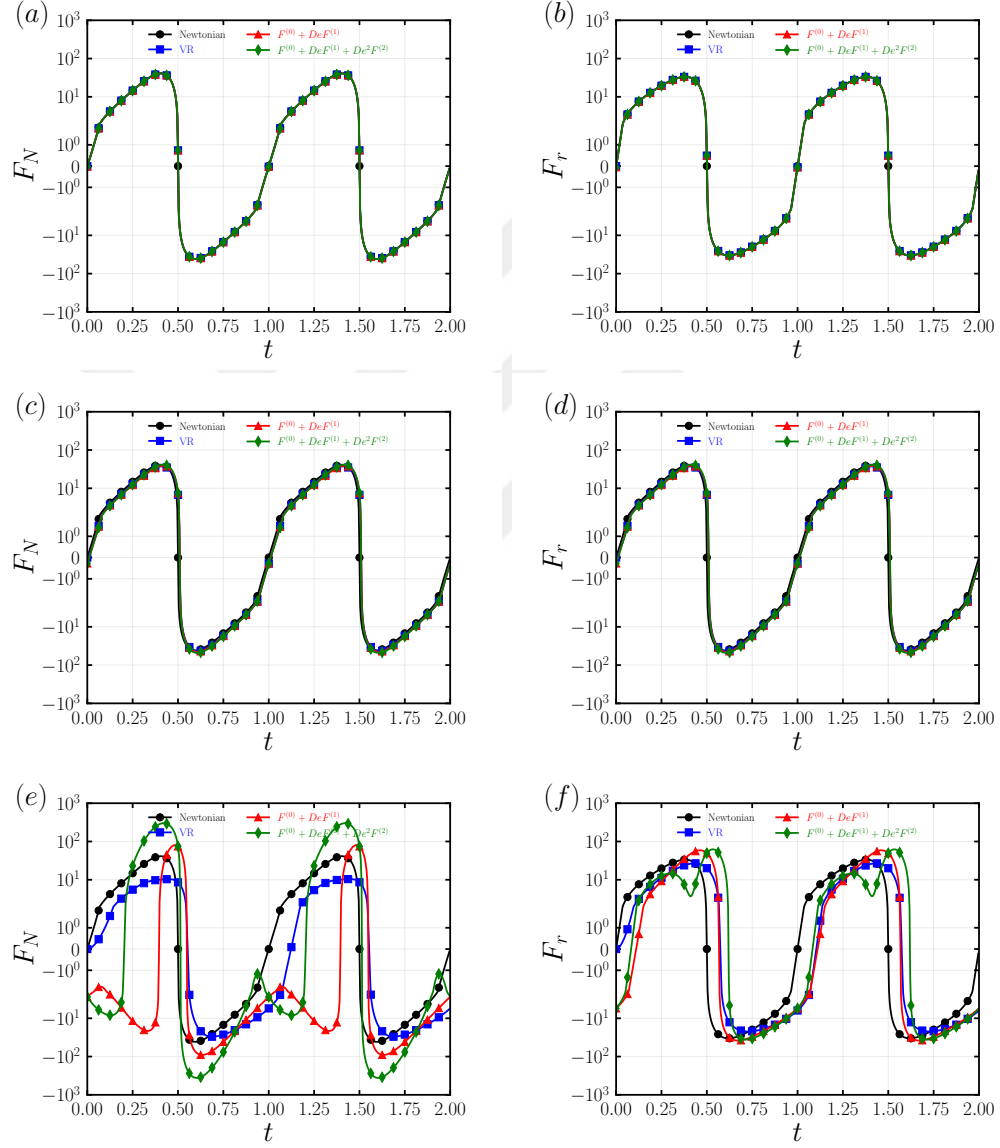


Figure 5.2: The variation in the net vertical force F_N and the friction F_r at the bottom plate versus time for (a) and (b) $De = 0.001$, (c) and (d) $De = 0.01$, and (e) and (f) $De = 0.1$, respectively, for the second-order perturbed model and the VR approach using $\beta = 0$, and $\epsilon = 0.001$, for a thin lubricant film subjected to oscillating motion.

in developing the stress. Therefore, the VR approach qualitatively reproduces this effect by exhibiting a load response that (i) follows the trend of the Newtonian case ($De = 0$) and (ii) also portrays the reduced response. However, in stark contrast is the result from the perturbation approach. Both the first and the second-order models produce dramatically different trends, leading to load overshoots during film compression and undershoots during film expansion. In particular, it exceeds the load predicted by the Newtonian case, which is at odds results currently available in literature.

One possible explanation for the discrepancy of the perturbation approach (at least up to the second-order) is the inclusion of only the lower derivatives of the surface motion. The general form for the normal force and surface friction, given by Eq. 2.43a, shows the influence of only the $\frac{d^{n+1}}{dt^{n+1}}$ derivative, where, n is the perturbation-order. As such, the motion we have chosen (Eqs. 5.5a-5.5d) provides an infinitely differentiable motion which the perturbation models do not capture. One can argue that their influence does not show up for small De , but this does not necessarily hold for each time step. As the time progresses and the film height decreases, the effective Weissenberg number, i.e., $Wi_{eff} = \lambda \frac{U_0}{H(t > t_0)}$ is increasing and the viscoelastic effects are stronger.

Chapter 6

Viscoelastic thin film lubrication

In this chapter, try to understand the significance of lubricant elasticity on the important attributes of a thin lubricating film, such as the pressure and surface friction, formed between two impermeable surfaces in relative motion. These load carrying films are commonly found in bearings and gears, and can strongly influence the performance of these mechanical elements. Likewise, human tribological joints, such as synovial joints, operate on a similar lubrication mechanism.

It is organized as follows: Firstly, the mathematical problem is formulated using the fundamental conservation laws, and a discussion pertaining to the application of lubrication theory is presented in section 6.2. Then, the numerical techniques employed for the solution of the governing equations are provided in section 3. Finally, we analyse and discuss the results for two-dimensional contacts in section 6.3. The work presented has been published, see Ahmed, H., & Biancofiore, L. (2023). *Modeling polymeric lubricants with non-linear stress constitutive relations*. Journal of Non-Newtonian Fluid Mechanics, 321, 105123.

6.1 Geometry

The pressure distribution within a thin lubricating film is strongly influenced by the surface geometry. In this study, we consider two-dimensional contacts, ignoring the spanwise variation of the channel. The first proposed geometry is the parabolic slider given in Fig. 6.1, used only to compare the proposed approach for the FENE-CR and FENE-P models against results available in literature and DNS.

$$h = 2x^2 + 2x + 1. \quad (6.1)$$

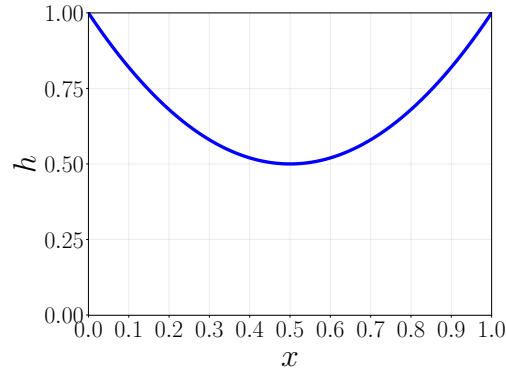


Figure 6.1: The film thickness along the stream wise direction for a parabolic slider bearing.

In a journal bearing the film thickness varies tangentially along the surface of the journal, as shown in Fig. 6.2a, favoring a cylindrical coordinate system for mathematical modeling. Alternatively, the same channel can be formed in a Cartesian coordinate system by unwrapping the film thickness along the tangential direction yielding an expression for the film height

$$h = 1 + e \cos(2\pi x), \quad (6.2)$$

in terms of the streamwise coordinate axis x and the eccentricity ratio $0 < e < 1$. The latter influencing the maximum and the minimum height of the channel, as shown in Fig. 6.2(b) for $e = 0.5$ and $e = 0.9$.

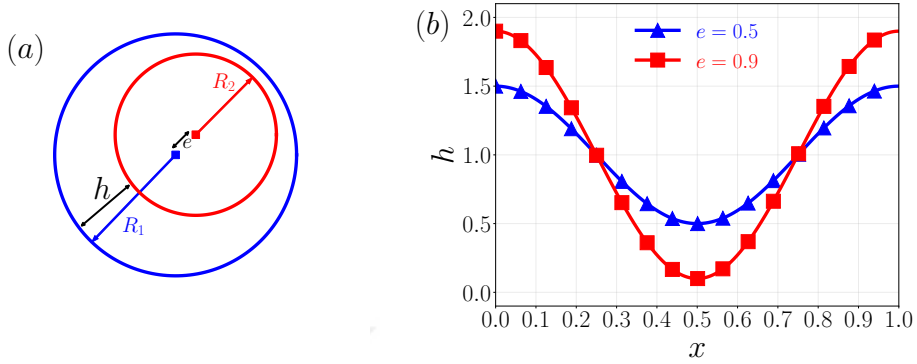


Figure 6.2: (a) A two dimensional depiction of a journal bearing with bearing (journal) radius R_1 (R_2) and the (b) film thickness along the stream wise direction of the equivalent unwrapped channel for two different eccentricity ratios.

As opposed to the parabolic slider, Eq. 6.2 models the full bearing, including regions far from the contact point. Using Eq. 6.2, we can analyze the viscoelastic effects throughout the bearing, not just in the vicinity of the contact point that generates the majority of the load carrying capacity. This also reduces the problem in choosing suitable values for the boundary pressure to some extent, as explained in section 6.2.1.

6.2 Mathematical modeling

The equations governing the flow of an incompressible, viscoelastic lubricant through a thin, two-dimensional, confined channel, depicted in Fig. 6.3,

consist of the conservation of mass

$$\nabla^* \cdot \mathbf{u}^* = 0, \quad (6.3)$$

and the conservation of momentum,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* = -\nabla^* p^* + \nabla^* \cdot \boldsymbol{\sigma}^*, \quad (6.4)$$

where $\mathbf{u}^* = (u^*, v^*)$ is the velocity field, p^* is the pressure and $\boldsymbol{\sigma}^*$ is the Cauchy stress tensor. For a lubricant mixture comprising a Newtonian solvent and a

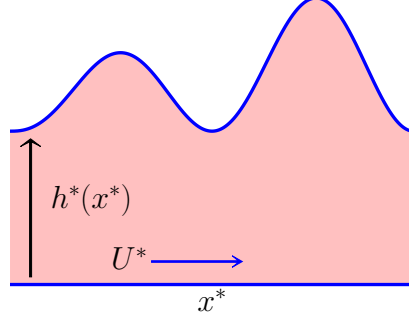


Figure 6.3: A fully lubricated, arbitrarily shaped thin channel of height $h^*(x^*)$ with the bottom boundary moving along the streamwise direction x^* with speed U^* .

non-Newtonian polymer additive, the stress tensor is split into two components

$$\boldsymbol{\sigma}^* = \boldsymbol{\tau}_s^* + \boldsymbol{\tau}^*, \quad (6.5)$$

namely, the (i) Newtonian solvent stress $\boldsymbol{\tau}_s^* = 2\eta_s \mathbf{D}^*$ and the (ii) viscoelastic polymer stress $\boldsymbol{\tau}^*$, where η_s is the solvent viscosity and $\mathbf{D}^* = 1/2(\nabla^* \mathbf{u}^* + \nabla^{*T} \mathbf{u}^*)$ is the deformation rate tensor. To close the system Eqs. 6.3-6.5, a constitutive relation is required for $\boldsymbol{\tau}^*$. The choice depends on several factors such as, the polymer rheology and flow characteristics [92]. For example, the Oldroyd-B constitutive relation

$$\boldsymbol{\tau}^* + \lambda \overset{\nabla}{\boldsymbol{\tau}}^* = 2\eta_p \mathbf{D}^*, \quad (6.6)$$

where, λ is the polymer relaxation time and η_p is the polymer viscosity, and $\overset{\nabla}{\boldsymbol{\tau}}^* = \frac{\partial \boldsymbol{\tau}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \boldsymbol{\tau}^* - \boldsymbol{\tau}^* \nabla^* \mathbf{u}^* - \nabla^{*T} \mathbf{u}^* \boldsymbol{\tau}^*$ is the upper-convected rate operator, describes a dilute, isoviscous, and incompressible polymer solution exhibiting viscoelastic effects. However, it does not account for non-linear effects such as, (i) the finite extensibility of the polymer chains and (ii) the effect of shear thinning.

In this study, we aim to compare two distinct non-linear effects, namely the limited extensibility of the polymers and the shear thinning of the lubricant, without simplifying the polymeric stress tensor. For this, we utilize the viscoelastic Reynolds approach; a numerical technique that takes advantage of the weak perturbation to the velocity field due to viscoelastic effects [23]. It incorporates the

Table 6.1: Functions F , G and H for the different constitutive relations appearing in Eqs. 6.8.

Model	F	G	H
Oldroyd-B	1	1	0
FENE-CR	$f(1 + \epsilon Wi \frac{D}{Dt}(1/f))$	f	0
FENE-P	$f(1 + \epsilon Wi \frac{D}{Dt}(1/f))$	$a - f(a(1 - \beta) \frac{D}{Dt}(1/f))$	

polymer constitutive relation without any restriction on the stress and compares favorably with direct numerical simulations for Oldroyd-B for a boundary driven flow in a thin lubricated channel [23].

It consists of a Reynolds type pressure equation

$$\begin{aligned} \frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) &= \frac{\beta}{2} \frac{dh}{dx} + \frac{d}{dx} \left(\int_0^h \frac{y}{h} \int_0^h \tau_{xy} dy' dy - \int_0^h \int_0^{y'} \tau_{xy} dy' dy \right) \\ &+ \frac{d}{dx} \left(\int_0^h \frac{y}{h} \int_0^h \int_0^{y'} \left(\frac{\partial \tau_{xx}}{\partial x} \right) dy' dy dy - \int_0^h \int_0^{y'} \int_0^{y''} \left(\frac{\partial \tau_{xx}}{\partial x} \right) dy'' dy' dy \right), \end{aligned} \quad (6.7)$$

and the scalar equations for the components of the polymer stress

$$F\tau_{xx} + \epsilon Wi \left(\frac{D\tau_{xx}}{Dt} - 2\tau_{xy} \frac{\partial u}{\partial y} - 2\tau_{xx} \frac{\partial u}{\partial x} \right) = 0, \quad (6.8a)$$

$$F\tau_{xy} + \epsilon Wi \left(\frac{D\tau_{xy}}{Dt} - \tau_{yy} \frac{\partial u}{\partial y} - \tau_{xx} \frac{\partial v}{\partial x} \right) = G(1 - \beta) \frac{\partial u}{\partial y}, \quad (6.8b)$$

$$F\tau_{yy} + \epsilon Wi \left(\frac{D\tau_{yy}}{Dt} - 2\tau_{xy} \frac{\partial v}{\partial x} - 2\tau_{yy} \frac{\partial v}{\partial y} \right) = 2G(1 - \beta) \frac{\partial v}{\partial y} + H, \quad (6.8c)$$

where the functions F , G and H (neglecting the ϵ^2 -order terms) for Oldroyd-B, FENE-CR and FENE-P are given in Tab. 6.1.

Finally, the components of the velocity field assumed for Eqs. 6.8a-6.8c,

$$u = u^{(0)} + \epsilon Wi u^{(1)}, \quad (6.9a)$$

$$v = v^{(0)} + \epsilon Wi v^{(1)}, \quad (6.9b)$$

are obtained from the sum of the leading order (Newtonian) components ($u^{(0)}$, $v^{(0)}$) and the first order perturbation due to ϵWi , i.e. ($u^{(1)}$, $v^{(1)}$). The Newtonian

velocity field is readily obtained by solving the system

$$\frac{d}{dx} \left(\frac{h^3}{12} \frac{dp}{dx} \right) = \frac{1}{2} \frac{dh}{dx}, \quad (6.10a)$$

$$u^{(0)} = \frac{1}{2} \frac{dp}{dx} (y^2 - yh) + \left(1 - \frac{y}{h} \right), \quad (6.10b)$$

$$v^{(0)} = -\frac{1}{2} \frac{d^2 p}{dx^2} (y^2 - yh) + \frac{1}{2} \frac{dp}{dx} \left(y \frac{dh}{dx} \right) - \frac{dh}{dx} \frac{y}{h^2}, \quad (6.10c)$$

where Eq. 6.10 is the classical Reynolds equation. The first order perturbation in ϵWi forms a similar system

$$\frac{d}{dx} \left(\frac{1}{12(1-\beta)} \frac{dp^{(1)}}{dx} \right) = \frac{1}{h} \left(\frac{dh}{dx} \right)^2 \left(3 \frac{h_m^2}{h^2} - \frac{3}{2} \frac{h_m}{h} \right) + \frac{d^2 h}{dx^2} \left(-\frac{1}{3} - \frac{3}{2} \frac{h_m^2}{h^2} + \frac{3}{2} \frac{h_m}{h} \right), \quad (6.11a)$$

$$u^{(1)} = \frac{1}{2} \frac{dp^{(1)}}{dx} (y^2 - yh) + \frac{1}{h} \frac{dh}{dx} \left(1 - 3 \frac{h_m}{h} \right) \left(2 - 3 \frac{h_m}{h} \right) (y^2 - yh), \quad (6.11b)$$

$$v^{(1)} = \frac{1}{2} \frac{d^2 p^{(1)}}{dx^2} \left(\frac{y^3}{3} - \frac{y^2 h}{2} \right) + \frac{1}{2} \frac{dp^{(1)}}{dx} \left(\frac{y^3}{3} - \frac{y^2}{2} \frac{dh}{dx} \right) \\ \left\{ \frac{1}{h} \frac{d^2 h}{dx^2} \left(1 - 3 \frac{h_m}{h} \right) \left(2 - 3 \frac{h_m}{h} \right) - \frac{1}{h^2} \left(\frac{dh}{dx} \right)^2 \left(1 - 3 \frac{h_m}{h} \right) \left(2 - 3 \frac{h_m}{h} \right) \right\} \left(\frac{y^3}{3} - \frac{y^2 h}{2} \right) \quad (6.11c)$$

$$+ \frac{9}{h} \frac{dh}{dx} \left\{ \left(\frac{h_m}{h^2} \frac{dh}{dx} \right) \left(1 - 2 \frac{h_m}{h} \right) \right\} \left(\frac{y^3}{3} - \frac{y^2 h}{2} \right) \\ + \frac{1}{h} \frac{dh}{dx} \left(1 - 3 \frac{h_m}{h} \right) \left(2 - 3 \frac{h_m}{h} \right) \left(\frac{y^3}{3} - \frac{y^2}{2} \frac{dh}{dx} \right),$$

where $p^{(1)}$ is the ϵWi -order pressure and $h_m = \frac{\int_0^1 h^{-2} dx'}{\int_0^1 h^{-3} dx'}$.

It is worthwhile to note that the first order perturbation to the pressure and velocity, Eqs. 6.11a-6.11c, for geometries in which the height at the entry and the exit (e.g. the journal bearing and the parabolic slider introduced in section 6.1) is zero [23]. Furthermore, the linearized components of the velocity for the FENE-CR and FENE-P reduce to the Oldroyd-B approximation because the effect of finite polymer extensibility, captured within the function $1/f$, is at least second-order in ϵWi . Thus, in a typical linearization approach where only the first order terms in ϵWi are kept, it is not accounted for. Note, for some cases the first order perturbation may not vanish, e.g. the plane slider bearing [29], and it may be

necessary to include their effect for large values of Wi . In this manner, for the journal bearing and the parabolic slider, using VR, a potential inaccuracy (order $\epsilon^2 Wi^2$) could incur in the velocity field and propagate to the polymer stress via the constitutive relation. However, for the range of considered Wi , this was not the case as evidenced by the results given in section 6.3.2.

6.2.1 Boundary conditions

The closure of the governing system of equations, Eqs. 6.3-6.5, requires appropriate boundary and initial conditions. For the steady state case, the pressure distribution within the thin channel depends largely on the imposed boundary conditions. When dealing with a Newtonian lubricant, we can simply impose a fixed value at the open boundary, such as an ambient pressure condition at the channel inlet and outlet. This approach is plausible because the total stress tensor

$$\mathbf{T}^* = \boldsymbol{\tau}_s^* - \mathbf{I}p^*, \quad (6.12)$$

can be split into the extra stress $\boldsymbol{\tau}_s^*$, which is known from the velocity field, and the pressure p . Furthermore, the streamwise normal stress is much smaller than the pressure, i.e., on the order $O(\epsilon^2)$ and can be ignored. As a result, it vanishes at the channel inlet and outlet, having negligible impact on the pressure. On the other hand, for a viscoelastic lubricant the stress is taken to be the sum of the Newtonian solvent stress and the polymeric stress. The latter, being non-Newtonian, does not have a clear relationship with the pressure [93].

For thin lubricating films, the lubricant is strongly non-Newtonian and the streamwise normal stress is large, i.e., on the same order of magnitude of the pressure. In this case, it can be argued that simply imposing a fixed pressure at the boundaries, specifically the channel entrance and exit, is insufficient. Instead, the total normal stress must vanish. However, this requirement cannot be satisfied simultaneously with the established outcome of the lubrication theory, which states $\frac{\partial p}{\partial y} = 0$, even when the normal stresses vary significantly across the film. In such a case, an average of the normal stress can be considered at the boundary.

In this work, we have not dealt with this issue since the choice of a journal bearing as our physical domain (section 6.1) allows us to impose periodic boundary conditions along the streamwise direction as follows:

$$p(x = 0) = p(x = 1), \quad (6.13a)$$

$$\tau_{ij}(x = 0) = \tau_{ij}(x = 1), \quad (6.13b)$$

where τ_{ij} are the scalar components of the polymer stress. Eqs. 6.13 are both physically accurate and intuitively appealing for the journal bearing and similar geometries where the flow inlet wraps around to the flow outlet. However, it is important to note that the larger issue of relating the mechanical pressure to the normal stress at the boundary remains unresolved. Regardless of whether the pressure is considered independently of the normal stresses or is made to be some dependent quantity at the boundaries this argument remains open for potential future work (see [29, 23] for more details).

6.3 Results and discussion

6.3.1 Two dimensional contacts

In this section, we present and discuss the influence of viscoelasticity in lubricated contacts as described by the VR approach applied to the FENE models. Firstly, an extensive validation of the VR is performed for the FENE-P and FENE-CR constitutive relations for the parabolic slider against the DNS in section 6.3.1.1. We then apply the VR model to the profile mimicking a journal bearing and examine the load and gauge pressure for different combinations of the different parameters, i.e., Wi , L^2 and e , in section 6.3.1.2. We also state the corresponding De number, although for the class of problems studied in this work, it remains small, indicating a weakly viscoelastic flow (note however, that the lubricant can be strongly elastic and sensitive to high shear, i.e. large Wi).

6.3.1.1 Validation

The VR approach has been validated for an Oldroyd-B fluid passing through a parabolic channel subjected to sliding motion [23]. Here, we present a validation of the FENE-CR and FENE-P models using the same parabolic channel described also in section 6.1. A comparison is presented for the Oldroyd-B, FENE-CR and FENE-P over a wide range of Wi . We also validate the FENE models over a wide range of the finite extensibility parameter. In addition to the FENE models, a validation against Wi is given in Appendix A using the same parabolic slider but for the PTT and Giesekus constitutive relations to prove the flexibility of VR. However, they are not discussed further in this study.

The pressure distribution along the channel, obtained using the different constitutive relations, is shown in Fig. 6.4(a) for $Wi = 20$ ($De = 0.08$), $L^2 = 10^3$, $\epsilon = 0.004$ and $\beta = 0.8$. The pressure predicted using the VR approach is in excellent agreement with the DNS, solved via the log-conformation representation. Despite the additional non-linearity present in the FENE-CR and FENE-P constitutive relations, the VR approach accurately predicts the pressure distribution implying the robustness of the approach.

The load carrying capacity

$$F_\ell = \int_0^1 p \, dx, \quad (6.14)$$

versus Wi is given in Fig. 6.4(b) for the same set of constitutive relations using $L^2 = 10^3$, $\epsilon = 0.004$ and $\beta = 0.8$. We find an excellent agreement between the VR approach and DNS over the range of Wi examined. Additionally, the load is accurately predicted against the the polymer extensibility parameter, as shown in Fig. 6.4(c). This accuracy is reflected in the small percentage errors in the load, as shown in Tab. 6.2.

Furthermore, both the FENE-CR and FENE-P predict an increase in the load for small values of Wi . The spatial distribution of pressure is visibly different between the model examined, with subtle variations close to the peak and center. Finally, the load not only varies strongly with Wi but exhibits a non-linear

Table 6.2: The percentage error in the load carrying capacity $E = |F_\ell^{VR} - F_\ell^{DNS}|/|F_\ell^{DNS}| \times 100$ depicted in figs. 3(b) and 3(c) for the FENE-CR and FENE-P models.

Model	$Wi = 5, L^2 = 10^3$	$Wi = 30, L^2 = 10^3$	$Wi = 10, L^2 = 10^2$	$Wi = 10, L^2 = 10^4$
FENE-CR	0.5	0.8	0.4	0.55
FENE-P	2.04	1.73	2.29	1.22

behavior versus L^2 , becoming also negative for FENE-P at small values of the extensibility parameter.

It is worthwhile to point out the computational efficiency of VR in relation to the DNS. For $Wi = 30$, the simulation time required for conducting DNS was approximately one week using eight 2.60 GHz Intel Xeon processors on a 512×32 cell grid. Although no issues with regards to the HWNP were observed for the FENE models (in accordance with literature [60]), the severe limitation in time step due to the increased (i) non-linearity of the constitutive relations and (ii) a very small stable time step led to an impractical simulation time. These issues could, perhaps, be alleviated via an implicit treatment of the discretized equations [94] but goes well beyond the scope of this work. On the other hand, the VR requires only a few minutes (for the same mesh resolution used in the DNS). It is also more amenable to advanced numerical techniques, since the pressure equation is decoupled from the stress.

A detailed analysis of the influence of these additional parameters on the aforementioned quantities is performed in the proceeding sections. However, we focus on the journal bearing due to (i) practical considerations and (ii) an argument pertaining to the boundary conditions, see sections 6.1 and 6.2.1, respectively, for details.

6.3.1.2 Journal bearing

We examine the spatial evolution of the pressure and the load carrying capacity for a periodic channel formed by unwrapping the journal bearing (see section

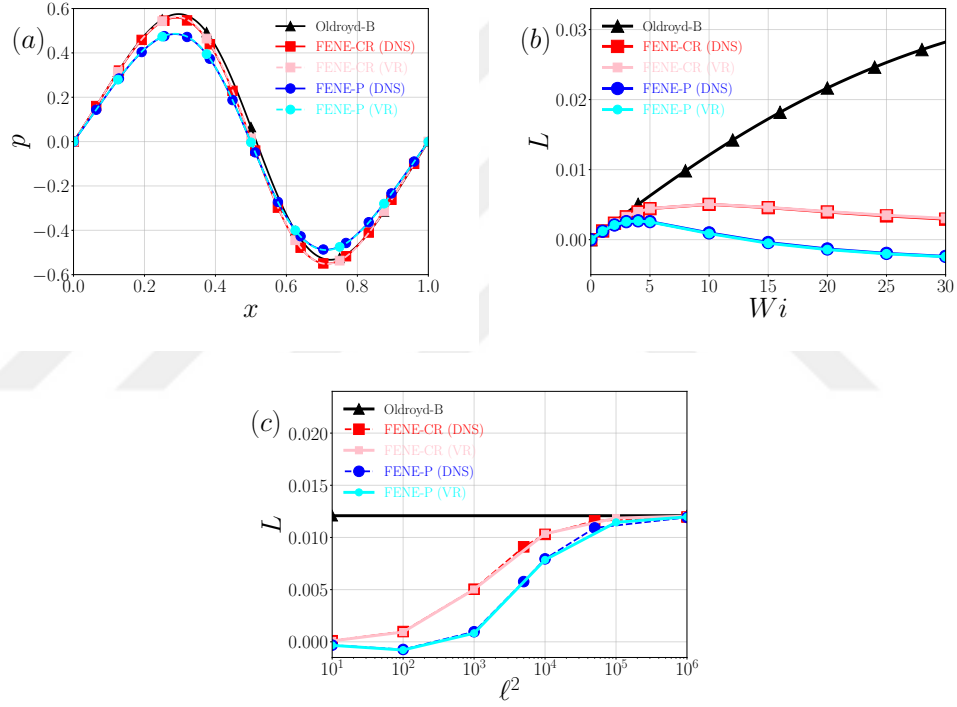


Figure 6.4: A comparison between DNS and VR using different constitutive relations for the parabolic slider bearing of (a) the pressure profile along the channel ($Wi = 20$ ($De = 0.08$), $L^2 = 10^3$, $\beta = 0.8$ and $\epsilon = 0.004$), the load carrying capacity (b) versus Wi ($L^2 = 10^3$, $\beta = 0.8$ and $\epsilon = 0.004$), and (c) versus L^2 ($Wi = 10$ ($De = 0.04$), $\beta = 0.8$ and $\epsilon = 0.004$). Note that the load for a Newtonian fluid is zero and $Wi = 30$ corresponds to $De = 0.12$.

6.1). Having validated the VR approach for the non-linear constitutive relations, i.e. FENE-CR and FENE-P, we systematically examine the influence of the Weissenberg number, the extensibility parameter, and the journal eccentricity ratio on the pressure and load using the VR approach only.

6.3.1.3 Influence of the Weissenberg number

The pressure distribution along the channel governs the load carrying capacity of the film. For $Wi = 35$, $L^2 = 10000$, $\beta = 0.8$, $e = 0.5$, and $\epsilon = 0.001$ the streamwise variation of the pressure is shown in Fig. 6.5(a). Viscoelasticity has a strong impact on the pressure and load carrying capacity; a net increase is visible across the models, confirming the findings available in literature [4, 23]. Stark differences are observed in the (i) peak pressure regions and the (ii) position of the transition from positive to negative pressure. Compared to Oldroyd-B, which predicts the highest peak pressure, both FENE-CR and FENE-P lag behind. These small shifts in the pressure lead to a net positive or negative load carrying capacity.

For the same set of parameters of Fig. 6.5(a), we compare the load versus Wi in Fig. 6.5(b). All models predict an increase over the range of Wi examined. The range of Wi that we have considered covers a linear region, i.e., a weakly viscoelastic lubricant, and a non-linear region at large values of Wi , i.e., a highly viscoelastic lubricant. In the linear region, which extends until $Wi = 5$, the same load is predicted by each model. After this initial linear phase, the onset of non-linear behavior is immediately visible; spanning moderate to large values of Wi . In this range, both FENE-CR and FENE-P depart from the load predicted by Oldroyd-B, which is monotonically increasing, and display a gradually saturating curve. Additionally, the FENE-P model predicts a sharp decline in the load for large values of Wi by virtue of the shear thinning effect.

Now, we increase the journal eccentricity ratio from $e = 0.5$ to $e = 0.9$, and examine in figs. 6.5(c)-(d) the pressure distribution and load for the same set of parameters. The pressure distribution is now sharper, reflecting the increase in

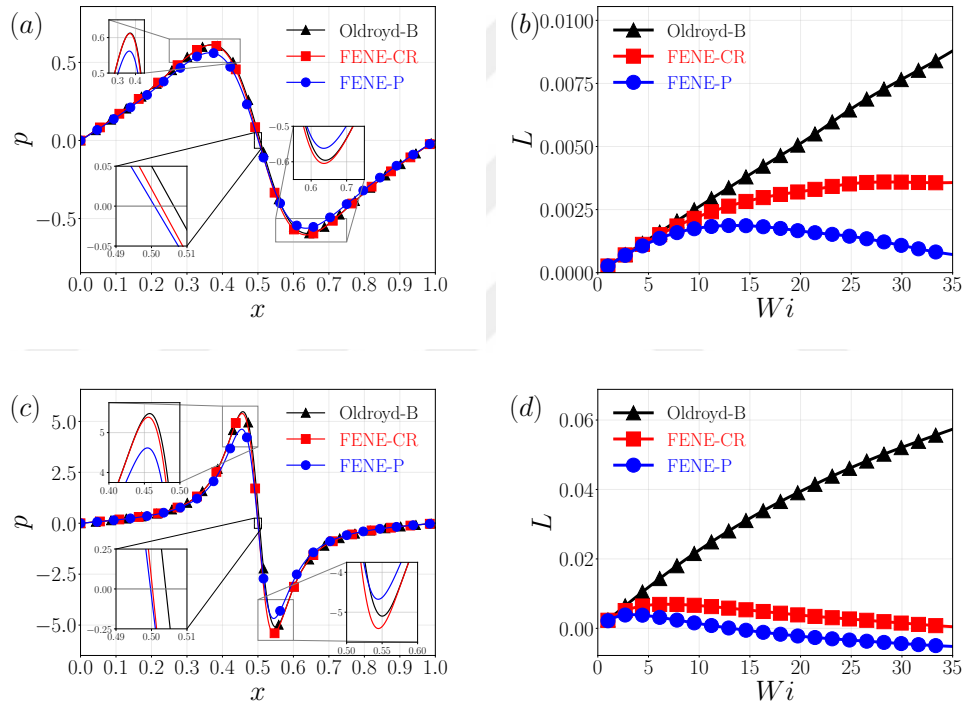


Figure 6.5: The (a) pressure distribution along the channel ($e = 0.5$, $Wi = 35$ ($De = 0.035$), $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$), the (b) load versus Wi ($e = 0.5$, $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$), the (c) pressure distribution along the channel ($e = 0.9$, $Wi = 35$ ($De = 0.035$), $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$) and the (d) load versus Wi ($e = 0.9$, $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$) for the journal bearing.

surface gradients and the differences in the (i) pressure peaks and (ii) transition region are more pronounced. If we examine the load carrying capacity in Fig. 6.5(d), the linear region shrinks and a non-linear response is exhibited even for small values of Wi . In fact, the saturation in the load occurs much earlier for both FENE-CR and FENE-P followed by a gradual decline in the load versus Wi . Note that FENE-P eventually predicts a net negative load, whereas the starkly contrasting Oldroyd-B continues to output a net positive increase.

It is clear that the additional parameters, i.e., the (i) finite polymer extensibility and (ii) journal eccentricity ratio, play a significant role in determining the total load carrying capacity of the film along with the Weissenberg number. Furthermore, observing figs. 6.5(b) and 6.5(d), the non-linear region tends to appear earlier for large e . This implies that a linearized model for FENE-CR and FENE-P, or any non-linear constitutive relation, must consider a second order (or higher [95]) perturbation in ϵWi .

6.3.1.4 Influence of the polymer finite extensibility

The extension of the polymers along the streamwise and the cross film direction is governed by components of the polymer stress tensor along those directions, i.e., τ_{xx} and τ_{yy} , respectively. The Oldroyd-B constitutive relation allows an infinite extension which can lead to large, and possibly nonphysical, values of the polymer normal stress components. The FENE-CR and FENE-P models enforce an upper limit on the maximum extension of the polymer chain via the finite extensibility parameter L . We examine its effect on the load carrying capacity in this section.

The variation of the load versus the polymer finite extensibility for different constitutive relations is depicted in Fig. 6.6. The trend in the load versus L^2 , although sensitive to e and Wi , mimics an s-shaped curve that connects the two limiting cases of small to nearly infinite polymer extensibility. The lower (upper) limit, i.e., small (large) values of L , mimics a rigid (highly extensible) polymer. We denote these limiting regions, for simplicity as the tail and head of the curve. As we increase L^2 , moving from the tail-end to the head, the load initially declines

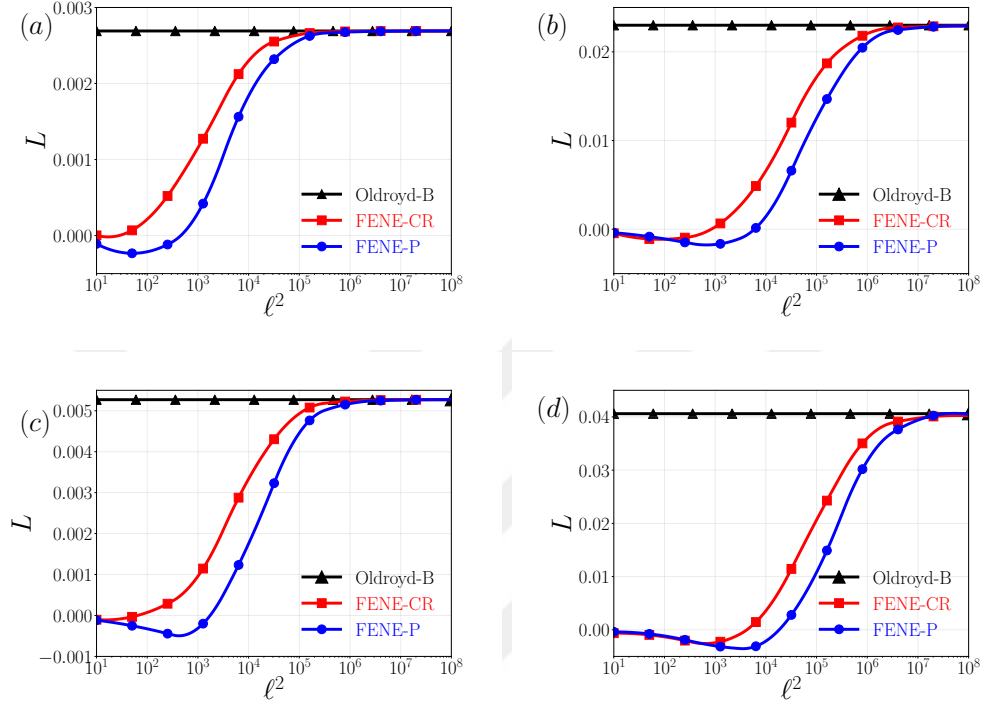


Figure 6.6: The load carrying capacity versus the polymer extensibility parameter for (a) $Wi = 10$ ($De = 0.01$), $e = 0.5$, (b) $Wi = 10$ ($De = 0.01$), $e = 0.9$, (c) $Wi = 20$ ($De = 0.02$), $e = 0.5$ and (d) $Wi = 20$ ($De = 0.02$), $e = 0.9$. In the all cases $\beta = 0.8$, and $\epsilon = 0.001$.

and then recovers, following a sharp increase, to the maximum value predicted by the Oldroyd-B.

As we increase the eccentricity ratio, moving from Fig. 6.6(a) to Fig. 6.6(b), we observe some key differences in the shape of the curve. Not only does the load increase in magnitude over the same range of L^2 , the initial negative load region stretches over a larger range and a greater value of L^2 is needed to achieve the maximum possible load. Furthermore, across the lower limit of the curve the FENE-CR model also predicts a negative load even though it does not account for shear thinning.

Now we increase the strength of viscoelasticity for the same values of e from $Wi = 10$ ($De = 0.01$) to $Wi = 20$ ($De = 0.02$), as shown in figs. 6.6(c) and

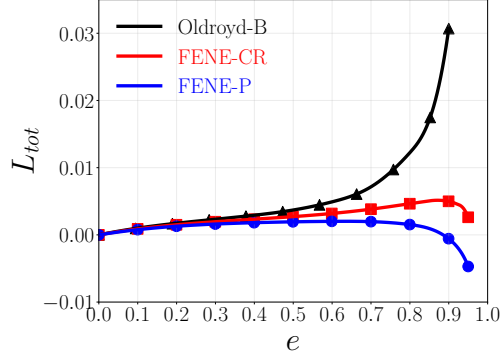


Figure 6.7: A comparison of the load carrying capacity versus the journal eccentricity ratio using $Wi = 15$ ($De = 0.015$), $L^2 = 10^4$, $\epsilon = 0.001$, and $\beta = 0.8$.

6.6(d). In this instance, the range of L^2 spanned by the tail and the head are not affected. A sharpening of the tail region can be observed, especially for FENE-P, for both values of the eccentricity ratio. Again, the FENE-CR as well exhibits a negative load carrying capacity for high eccentricity ratios.

6.3.1.5 Influence of the journal eccentricity ratio

The journal eccentricity ratio is a measure of the distance between the centers of the shaft and the journal bearing and unlike the Wi and L , it is purely a geometric parameter. A high ratio leads to a larger separation of the surfaces and yields higher (lower) shear strain rates close to the convergent (divergent) portion of the equivalent channel. Thus, it impacts not only the channel height but also the surface gradients. The variation of this parameter and its impact on the load carrying capacity is examined in this section.

A comparison between FENE models and Oldroyd-B of the load versus eccentricity ratio for $Wi = 15$ ($De = 0.015$) is shown in Fig. 6.7. All models predict an increase in the load versus e with an initial growth that is linear for small values of the eccentricity ratio; followed by (i) an exponential increase predicted by Oldroyd-B, (ii) a gradual increase predicted by FENE-CR and (iii) a gradual decline for FENE-P for large values of e . The high strain rate over the

range of large e increases the normal stress (which depends quadratically on the shear rate) and causes pressure growth [23]. However, the FENE models place a restriction on this growth, by limiting the maximum possible extension of the polymer, thereby inhibiting this beneficial effect, resulting in a decline in load for high eccentricity ratios.

6.3.2 Discussions

We have seen the strong influence of the Weissenberg number, polymer extensibility parameter and the journal eccentricity ratio on the load carrying capacity for the thin lubricating film in a journal bearing. In addition, the combination of these parameters have a strong influence on the variation of the load versus any single parameter. An explanation of the trends observed in the previous sections, for the different models is presented in section 6.3.2.1 and a focused discussion on the shear thinning effect in FENE-P is provided in section 6.3.2.2.

6.3.2.1 Load due to viscoelasticity

Lubricant viscoelasticity raises the load carrying capacity primarily through a growth in the streamwise polymer normal stress (τ_{xx}) which increases with Wi . This growth is countered by a decay due to the polymer shear stress (τ_{xy}). The exact influence of these two stress components on the pressure is shown, mathematically, in Eq. 6.7. However, despite the explicit appearance of the normal and shear stress, the terms involve several integral expressions that make an analytical treatment quite cumbersome. On the other hand, a numerical treatment gives the pressure due to each component and also, after integration along the channel, the contribution to the load carrying capacity of the film.

To visualize the relative contribution of the normal (shear) stress to the load carrying capacity, denoted as $F_{l,xx}$ ($F_{l,xy}$) respectively, we assume that the normal (shear) stress is decomposed into a purely (i) elastic (EL) and (ii) a viscous (VI)

Model	$\tau_{xx}^{(EL)}$	$\tau_{xy}^{(EL)}$	$\tau_{xy}^{(VI)}$
Oldroyd-B	$-\epsilon Wi \bar{\tau}_{xx}$	$-\epsilon Wi \bar{\tau}_{xy}$	$(1 - \beta) \frac{\partial u}{\partial y}$
FENE-CR	$-\tau_{xx} \epsilon Wi \frac{D(f^{-1})}{Dt} - \frac{\epsilon Wi}{f} \bar{\tau}_{xx}$	$-\tau_{xy} \epsilon Wi \frac{D(f^{-1})}{Dt} - \frac{\epsilon Wi}{f} \bar{\tau}_{xy}$	$(1 - \beta) \frac{\partial u}{\partial y}$
FENE-P	$-\tau_{xx} \epsilon Wi \frac{D(f^{-1})}{Dt} - \frac{\epsilon Wi}{f} \bar{\tau}_{xx}$	$-\tau_{xy} \epsilon Wi \frac{D(f^{-1})}{Dt} - \frac{\epsilon Wi}{f} \bar{\tau}_{xy}$	$a \frac{(1-\beta)}{f} \frac{\partial u}{\partial y}$

Table 6.3: Elastic (EL) and viscous (VI) components of the normal and shear stress. The function f is defined separately for FENE-CR and FENE-P, see table. 2.1.

component

$$\tau_{ij} = \tau_{ij}^{(EL)} + \tau_{ij}^{(VI)}, \quad (6.15)$$

where the stress due to purely elastic effects is $\tau_{ij}^{(EL)}$ and due to viscous effects is $\tau_{ij}^{(VI)}$. The latter will be relevant only for FENE-P which exhibits shear thinning. This split between elastic and viscous stress is commonly employed in literature for the simulation of viscoelastic materials [96, 97, 98]. However, we make this decomposition purely for the sake of analyzing the contribution to the overall load carrying capacity. The mathematical forms for the normal and shear stress components are given for the different models in Tab. 6.3.

In Fig. 6.8, we present the separate components of the load due to the normal stress ($F_{l,xx}$) and the shear stress ($F_{l,xy}$) as a function of the journal eccentricity ratio. The latter is further broken down to show the influence of the polymer elasticity ($F_{l,xy}^{EL}$) and shear thinning ($F_{l,xy}^{VI}$) for $Wi = 15$, $L^2 = 10000$, $\epsilon = 0.001$, and $\beta = 0.8$. We had observed the total load versus eccentricity ratio for all models rises in Fig. 6.7; increasing (declining) for weakly (highly) eccentric bearings the FENE-CR and FENE-P models. This is primarily due to a balance between the exponential growths of the load due to the normal stress [23], shown in Fig. 6.8(a), and the decline due to shear stress, shown in Fig. 6.8(b). For small to moderate values of e , a linear growth is observed in the normal load, whereas the shear load is practically zero. However, as e increases further, the latter eventually supersedes. This leads to an overall decline in the total load.

Note that the exponential increase (decrease) of $F_{l,xx}^{EL}$ ($F_{l,xy}^{EL}$) of the load exists for Oldroyd-B. However, it does not display any, nor suggests any forthcoming,

decline in the total load within the strongly eccentric region. This implies a dominant (possibly non-physical) increase in the normal stress, and a more realistic prediction in FENE-CR and FENE-P owing to the finite extension. Interestingly, the decline due to the shear stress is almost the same for all models. Therefore, the net effect is an unimpeded growth in Oldroyd-B, as opposed to a slow increase for FENE-CR and FENE-P.

The load due to shear thinning ($F_{l,xy}^{(VI)}$) versus e , depicted in Fig. 6.8(d), is predictably zero for Oldroyd-B and FENE-CR. However, for FENE-P, we observe an initial decline for small e . This is followed by an unexpected rise in the load, eventually becoming positive for $e > 0.92$. As we increase the eccentricity ratio, the average shear rate across the channel does increase, implying a greater average polymer shear stress and lower average shear viscosity due to shear thinning. In lubrication, a decline in the shear viscosity invariably leads to a net decrease in the pressure distribution. This apparent gain in load, although small, may allude to a thickening of the lubricant, but it turns out not to be the case.

This sharp increase in load due to shear thinning is explained by re-examining the pressure distribution in Fig. 6.5(c). We find that the pressure, across the channel, does indeed decline in comparison to Oldroyd-B and FENE-CR, reflecting the impact of shear thinning on the viscosity. However, the change in the positive and negative regions of the curve is not equal. In fact, a slightly larger change is observed in the negative pressure region. Upon integrating the pressure along the channel, the net effect is a rise in the load due to shear thinning, although the magnitude is quite small.

6.3.2.2 The load due to shear thinning

The FENE-P constitutive relation presents the highest degree of non-linearity by combining the effects of (i) polymer elasticity, the (ii) finite extension of the polymer and (iii) shear thinning. These effects create interesting trends that are absent for Oldroyd-B and FENE-CR. We have seen, in section 6.3.2.1, that the load due to shear thinning predicted by FENE-P can increase at high values of the

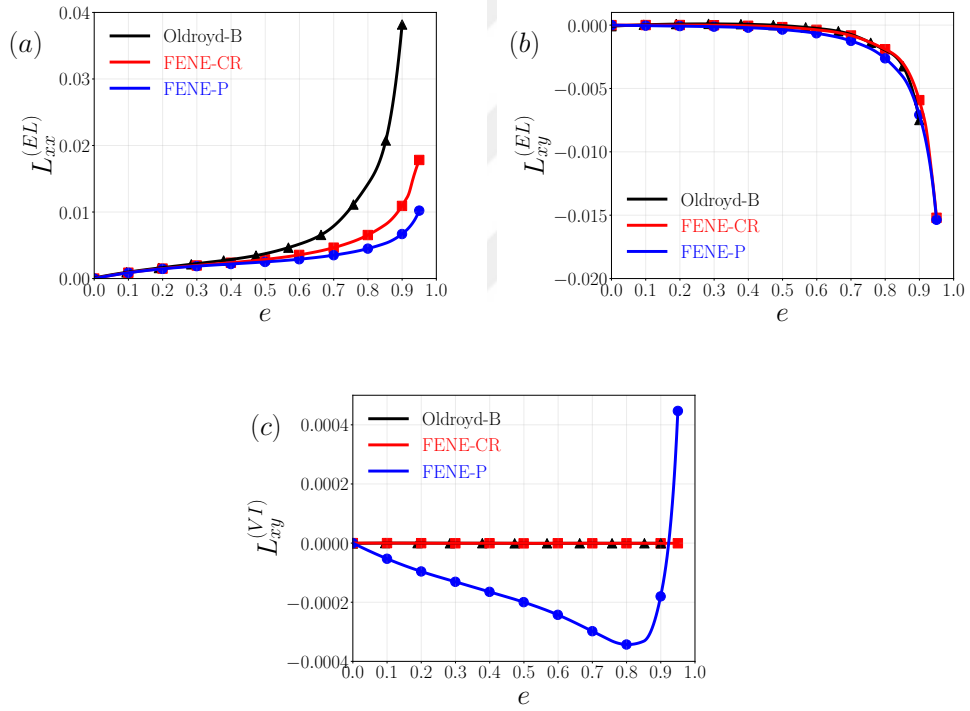


Figure 6.8: The variation of the component of load due to (a) normal stress, the (b) elastic and (c) viscous part of the shear stress versus the journal eccentricity ratio using $Wi = 15$ ($De = 0.015$), $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$.

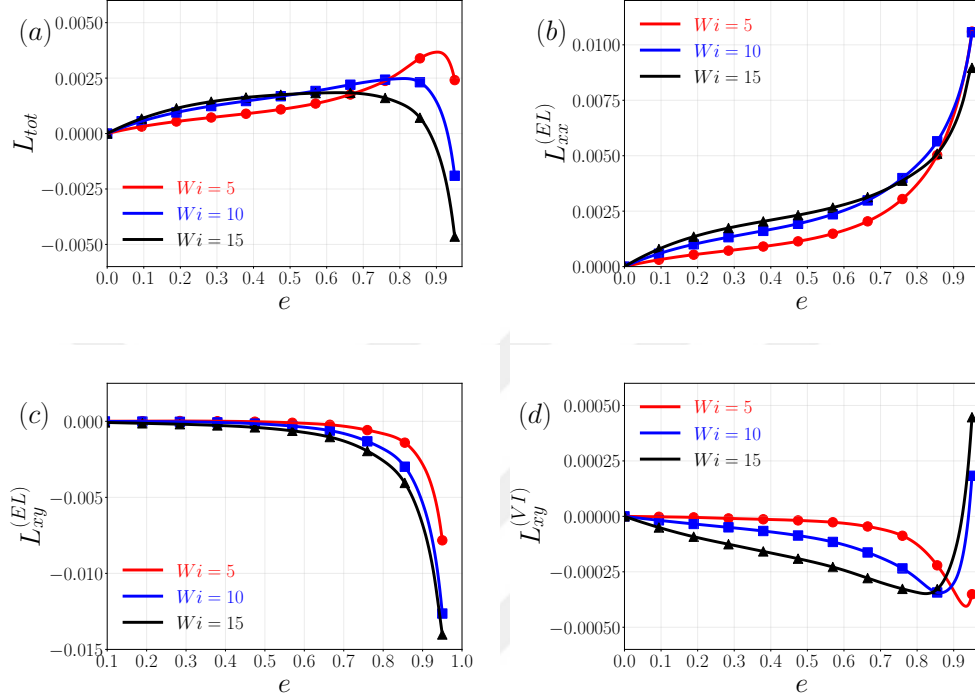


Figure 6.9: The variation of the (a) total load carrying capacity, the component of load due to (b) normal stress, the (c) elastic and (d) viscous part of the shear stress versus the journal eccentricity ratio for the FENE-P (table 2.1) using $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$. Note that $Wi = 5, 10$ and 15 correspond to $De = 0.005, 0.01$ and 0.015 , respectively.

eccentricity ratio. We now examine, in detail, the effect of raising the Weissenberg number on the load due to shear thinning.

In a similar manner to Fig. 6.8, the components of the load for FENE-P are presented in Fig. 6.9 for $Wi = 5, 10$ and 15 ($De = 0.005, 0.01$ and 0.015), while the remaining parameters are kept the same. The total load versus e , shown in Fig. 6.9(a), increases linearly for small values of e and then declines rapidly for large e . The increase (decrease) is due to the elastic normal (shear) stress contributions, shown in Fig. 6.9(b) (Fig. 6.9(c)). The change in the trend, from an initial ascending phase to a rapid decline, is inversely related with Wi ; with the largest increase observed for the smallest Wi .

This is mainly due to the competing effects of the stress advection and generation encapsulated within the upper-convected rate operator

$$\overset{\nabla}{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} - \boldsymbol{\tau} \nabla \mathbf{u} - \nabla \mathbf{u}^T \boldsymbol{\tau}, \quad (6.16)$$

where, the material derivative propels the stress along the flow field, predominantly the streamwise direction, and the stretch terms serve as a source for generating stress [23]. The effect of the eccentricity is embedded within the velocity field. At large values of the eccentricity ratio (Weissenberg number), the channel surface gradients (upper-convected rate) increase(s). The changes in the channel shape are reflected in the velocity field which modifies the stress distribution across and along the channel. As a result, the film's resistance (related to the shear stress) is modified, moved along the direction of the flow towards the negative pressure region [23], presenting a larger negative shear load versus e , see Fig. 6.9(c).

However, from Fig. 6.9(d), the eccentricity ratio has a reciprocal effect on the load due to shear thinning, i.e. increasing it for extreme values of e . This is because the pressure profile favors a slightly larger positive region as opposed to the negative region, since the effect of shear thinning (given by the function $a\eta_p/f$ in table 2.1) is not symmetric along the channel. To demonstrate this, we consider a modified version of the FENE-P model

$$\boldsymbol{\tau}^* + \lambda \frac{1}{f^*} \overset{\nabla}{\boldsymbol{\tau}}^* = 2\eta_p \frac{a}{f^*} \mathbf{D}^*, \quad (6.17)$$

that disregards the additional advection of $1/f$ for numerical simplicity and analytical treatment [99, 100]. Furthermore, for a weakly viscoelastic fluid, it may only slightly compromise on accuracy but provide computational efficiency.

A comparison of the load due to shear thinning for the FENE-P model given by Eq. ?? and its modified version (FENE-P Mod.) given by Eq. 6.17 is presented in Fig. 6.10. The load predicted in both cases declines due to the advection of the stress. However, in the case of the modified version that lacks the additional advection of the function $1/f$ which only serves to diminish the stress, a larger net resistance gathers in the negative portion which promotes a greater (lesser) negative (positive) pressure region, thus prolonging the decline in load versus e .

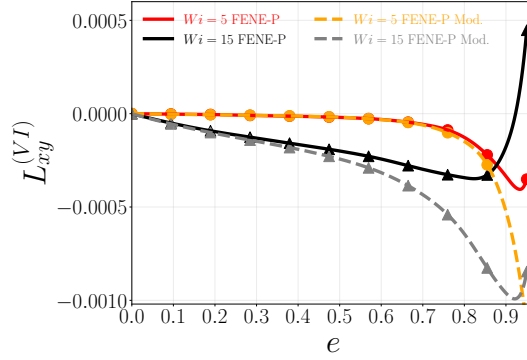


Figure 6.10: The variation of the the component of the load due to viscous part of the shear stress versus the journal eccentricity ratio for the FENE-P (table 2.1, solid lines) and the modified FENE-P (Eq. 6.17, dashed lines) using $L^2 = 10^4$, $\epsilon = 0.001$ and $\beta = 0.8$. Note that $Wi = 5$ and 15 correspond to $De = 0.005$ and 0.015 , respectively.

The slightly larger resistance of the negative region disturbs the symmetry of the pressure distribution, favoring, then, a net total increase in the load. This does not imply a net increase in the shear viscosity (shear thickening), but is merely an indicator of an asymmetric pressure profile. For non-pressurized bearings, the negative pressure region produces a change in the lubricant phase due to cavitation. In that case, we expect to have a more pronounced increase in the total load [4] and an expected decrease in the load due to shear thinning.

6.4 Cavitation

In this section, we investigate the influence strongly elastic polymers on the load carrying capacity for a lubricant which can experience cavitation at low enough pressure. We have used the methodology described in chapter 2 to capture the lubricant phase change such that the polymer elasticity only acts according to the fraction of liquid present. In gaseous phase, the influence of the polymer viscosity and polymer relaxation time is switched off. Finally, we have used the FNBS algorithm for modeling cavitation as opposed to the Elrod-Adams algorithm for

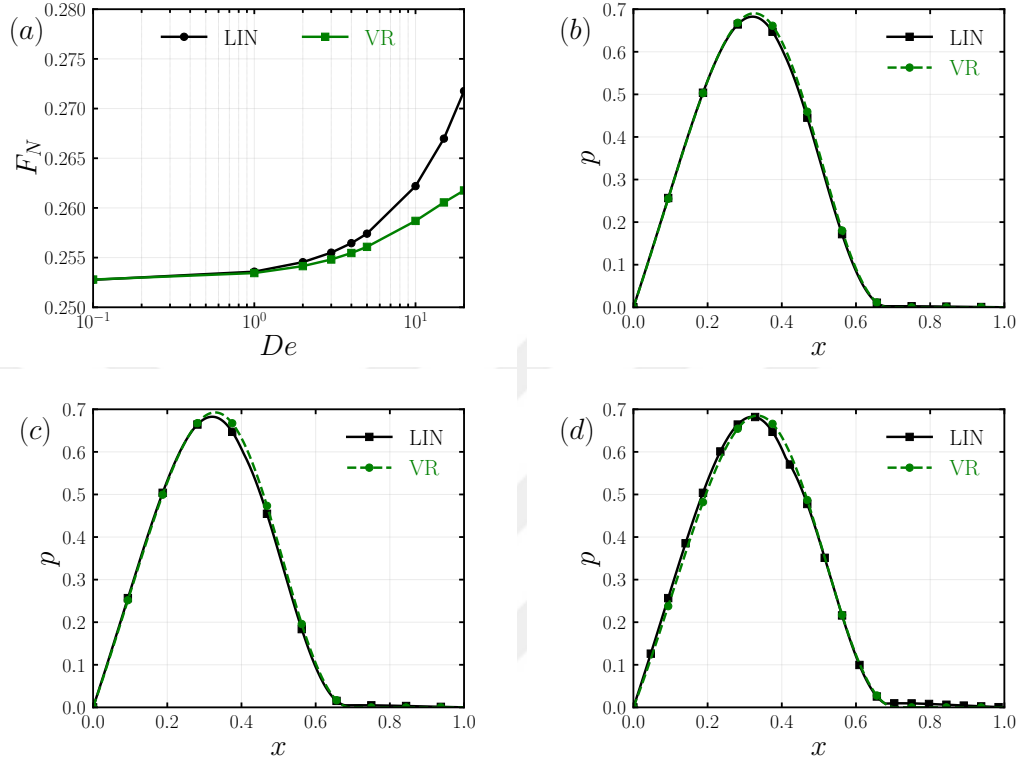


Figure 6.11: The (a) variation in the maximum load carrying capacity versus the Weissenberg number, pressure distribution along the channel for (b) $De = 0.01$, (c) $De = 0.02$, and (d) $De = 0.1$ for the linearized model and the VR approach for the journal bearing $e = 0.5$, using $\epsilon = 0.001$, $\beta = 0.2$.

viscoplastic lubricants, since it is more robust and far more efficient, see chapter 3.

In Fig. 6.11 we examine the influence of the polymer viscoelasticity on the load carrying capacity (Fig. 6.11(a), and the pressure distribution along the channel (6.11(b)-(d)) as we increase the polymer's elasticity, for the journal bearing $e = 0.5$, using $\epsilon = 0.001$, $\beta = 0.2$.

A comparison of the load predicted via the linearized model (LIN) and the VR approach shows good agreement for small values of the De . As viscoelastic effects strengthen, an increase in load is observed for both models. The deviation between the estimations between the two models also increases. The linearized model predicts a larger force than the VR for moderate to larger values of De .

The pressure distribution along the channel for both models shows a net positive region, in which the lubricant exists purely in the liquid phase, followed by a vapor region, in which a mixture exists. In the latter portion, the pressure is zero. However, the LIN model predicts, for large De , a slightly positive pressure indicating reformation of the film. However, the VR, based on the current understanding of cavitating sliding contacts, predicts a vanishing pressure field (where the lubricant pressure is equal to the cavitation pressure). The difference in behavior between the two models is likely due to an excess contribution from the elastic portion that has been properly ruled out from the VR approach but not from the linearized model.

Specifically, we find that the polymer normal stress which depends quadratically on the shear strain rate for LIN, $\tau_{xx}^{(1)} = 2(1 - \beta)\dot{\gamma}_{(yx)}^{(0),2}$, can create a significant increase in pressure. Even if the pressure gradient vanishes correctly in the cavitation zone, thereby eliminating the Poiseuille flow contribution to the shear strain rate, the Couette contribution exists and predicts a larger normal stress. It should be corrected, at least to the first-order, by the liquid-vapor film fraction, ϕ , in a manner that removes the extra contribution from a purely liquid polymer state, i.e., $\tau_{xx}^{(1)} = 2\phi^{(0)}(1 - \beta)\dot{\gamma}_{(yx)}^{(0),2}$. In doing so, we note for a fully liquid phase, we retrieve the first-order linear estimate and in the mixed phase, a linear variation versus ϕ will be obtained. We have assumed, however, that this interaction varies linearly and only the leading-order film fraction can be used, since the first-order approximate will render the expression second-order in De .

Finally, we address the advective response of the pressure field versus De observed from the VR approach in Figs. 6.11(b)-(d). The net pressure is some function of the polymer stress that advects along the flow direction (varying locally, in magnitude, as well). This causes a shift in the liquid-vapor interface, and a reduction in the pressure close to the inlet which inevitably dominates leading to a saturation and net decline in total load carrying capacity for large values of De . We report a load carrying capacity of $F_N = 0.2488$ for $De = 0.2$, a decrease from the Newtonian value of $F_N = 2.52$. While the normal stress generates additional pressure, it moves towards the cavitated zone and is nullified owing to the increasing vapor fraction.

Chapter 7

Three dimensional channel configurations

In this chapter, we examine the influence of finite polymer elasticity on the meaningful parameters in three-dimensional lubricated contacts, namely the load carrying capacity and the friction experienced by the channel walls. The results are presented, then discussed and critiqued, for two particular coordinate configurations. We examine the aforementioned quantities (the definitions introduced in the subsequent sections), for channels readily conforming to the relatively simple (i) cartesian coordinate system $\mathbf{x} = (x^*, y^*, z^*)$ in section 7.1, and the arguably more involved (ii) cylindrical coordinate system ($\mathbf{x} = (r^*, \theta, z^*)$) in section 7.2.

Numerically, there is little need for two different coordinate systems, since there are methods (such as the finite volume method or the finite element method) that readily adapt to a curved control volume (element). However, the effort required to adapt these methods can be quite involved for three-dimensional problems and the use of high-order accurate discretizations are not trivial. Furthermore, certain problems such as, the rotation of two parallel (but not necessarily flat) surfaces, i.e., in a rheometer, readily conform to a cylindrical configuration. Biological contacts such as, the human eye [101] (or the synovial joints) are well suited to spherical configuration for that matter. Therefore, we apply the models developed

in chapter 2 to these two different coordinate systems.

7.1 Cartesian configurations

The governing equations for the full viscoelastic flow problem, for sliding contacts, were introduced in chapter 2. Here, we apply the linezrized model Eqs. ?? and the viscoelastic Reynolds approach Eqs. 2.34a-2.34c. When applied to a thin lubricant film separating two surfaces in relative motion, they describe, subject to the continuum approximation, the pressure, velocity and stress distribution. We have assumed a pure sliding configuration, in which the surface at $y = 0$ moves with constant velocity along the streamwise direction.

7.1.1 Results and discussion

We now examine the load

$$F_N = \int_0^1 \int_0^1 p \, dx dz, \quad (7.1)$$

the bottom surface friction along the streamwise direction

$$F_{(YX)} = \int_0^1 \int_0^1 |\tau_{yx}|_{Y=0} \, dx dz, \quad (7.2)$$

and the average point-wise bottom force = along the spanwise direction

$$F_{(YZ)} = \int_0^1 \int_0^1 |\tau_{yz}|_{Y=0} \, dX dZ, \quad (7.3)$$

for different values of the channel aspect ratio and Deborah number. The latter expression requires taking the absolute value of the spanwise shear stress, since it is, for most cases, entirely anti-symmetric about the z -axis leading to zero net force along z . However, from a practical perspective, the wear and tear mechanism, at some point $(x, y = (0, h), z)$ along the surfaces depends on the force at that point in the channel.

In Fig. 7.1(a) we examine the parabolic channel (mimicing a parabolic slider bearing with finite width), and in Figs. In Fig. 7.1(c)-(d), the variation in the lift force, streamwise and spanwise average force, respectively, for $\epsilon = 0.001$, and $\beta = 0.2$.

An immediate observation, from Fig. 7.1(b), is that the effect of increasing polymer elasticity on the net vertical force (or simply the lift) was positive over a wide range of De , similar to the two-dimensional parabolic slider. We infer that the mechanisms involved in this three-dimensional configuration for additional load generation are not dissimilar to the two-dimensional case, namely a growth in the normal stresses. It is not clear at this stage the exact relation between the pressure and the normal stress that leads to this gain in load, since the latter quantity (although acts along the same direction) varies across the channel, whereas the pressure does not. It is more meaningful to comment on the additional parameter a , and the differences in the two models employed.

We increase the aspect ratio, thereby assuming that the width of the channel is comparable to the length, to examine the change in the force measures (Eqs. 7.1-7.3). Firstly, we make a comparison of the linearized model versus the VR approach. For small values of De , over the examined range of a , the models predict the same forces. As we increase De , enhancing the elasticity of the polymer, we observe the expected differences in the magnitude of the load but an almost identical prediction for the surface forces, Figs. 7.1(b) and 7.1(c). Furthermore, increasing the aspect ratio tends to diminish the differences in the two models.

The decline in the total load carrying capacity of the film versus a , observed in Fig. 7.1(b), is attributed to the additional diffusion of pressure along the spanwise direction. The boundaries along the spanwise direction are at ambient pressure, and for an wide channel ($a \ll 1$), are sufficiently far apart. In this case, the pressure variation close to boundary is not felt by the bulk fluid. Now, if we bring these boundaries closer, mimicked by increasing a (for narrow channels), the bulk fluid is perturbed over this small distance and, because the pressure must decay to ambient at the boundaries, leads to a net decline in load. In addition, when the aspect ratio increases, the streamwise normal and shear stresses decline, owing

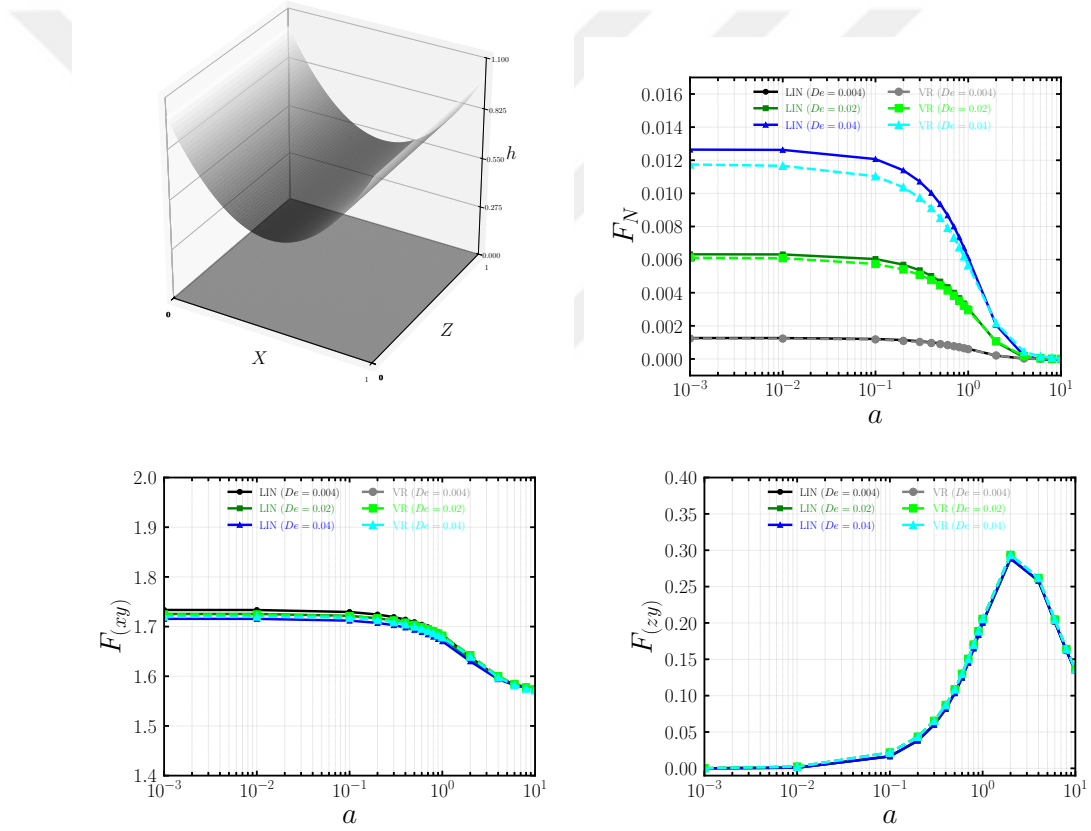


Figure 7.1: The variation in the (a) net vertical force, (b) average streamwise force and (c) the average spanwise force versus the channel aspect ratio for the parabola ($\alpha = 0.5, \epsilon = 0.004, \beta = 0.8$) using the Oldroyd-B constitutive relation for different values of the Deborah number.

to the weakening of the pressure gradient along x , leading to a further decline in force. Viscoelasticity, enhances the load carrying capacity up to $a = O(1)$, since the normal spanwise stress τ_{zz} , similar to the normal streamwise stress τ_{xx} which contributes positively to the load, increases versus a .

Next, we consider a high eccentricity, $e = 0.9$, journal bearing and examine the same force measures (Eqs. 7.1-7.3) in Fig. 7.2. In this case, the high eccentricity leads to a sharp convergence of the channel height at $x = 0$. Similar to the parabolic slider, we find good agreement between the LIN and VR models. However, the difference at larger De is more pronounced. In this case, the relation between the Deborah number and the Weissenberg is not linear and the effective Weissenberg, Wi_{eff} in the channel is quite large. The Deborah number being defined as $\lambda U_0/R_0$, whereas, the Weissenberg (factoring in the high eccentricity) is $\lambda U_0/H_0(1-e)$, where, the reference height, H_0 , is typically taken as the journal clearance which is constant for a bearing. We will note that for moderate values of the eccentricity ratio $Wi_{eff} \approx Wi$.

When comparing the parabolic slider and the high eccentricity un-wrapped journal bearing, there is little difference in the force versus a trends and, for this reason, we infer that the competing mechanisms of stress advection and generation (loss) governing the increase/decrease in load remain the same (details are found in [102]). We note that the friction force does not vary strongly versus the De , but is erroneously over-predicted by the first-order linear model, most likely due to an incorrect estimation of the wall-normal stress component (τ_{yy}).

Additionally, a high eccentricity ratio demarcates the difference in the LIN and VR approach at large values of a , with LIN over-estimating the load carrying capacity. In general, the linearized model overestimates both the load enhancement due to normal stresses, but also the diminishing effect due to large a . However, in terms of computational efficiency, LIN possesses a significant advantage over the VR (and naturally, DNS). In fact, a second-order perturbation might even prove sufficient for moderate values. However, obtaining the expressions (owing to their length) can be quite challenging, and increasing the eccentricity further will again require an even higher perturbation-order.

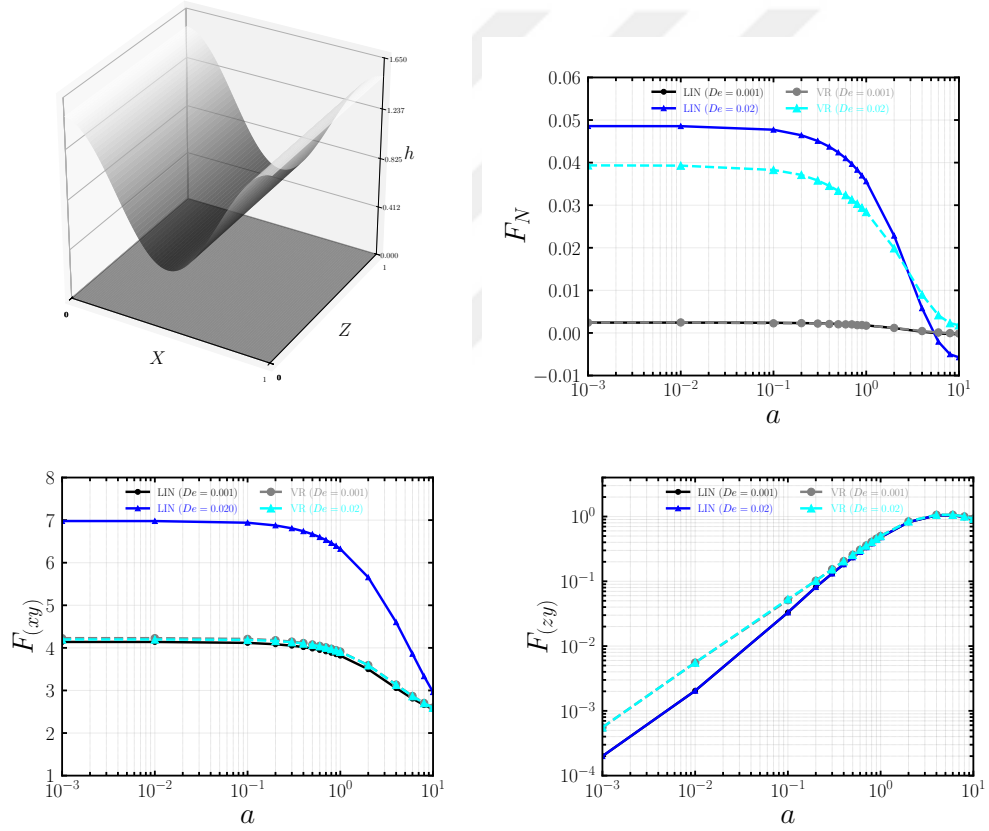


Figure 7.2: The variation in the (a) net vertical force, (b) average streamwise force and (c) the average spanwise force versus the channel aspect ratio for the journal bearing ($e = 0.9, \epsilon = 0.001, \beta = 0.8$) using the Oldroyd-B constitutive relation for different values of the Deborah number.

In Fig. 7.3, we provide a comparison of the prediction for the load, average force along the streamwise and spanwise directions for different non-linear models versus the channel aspect ratio using the high eccentricity journal bearing geometry for two different values of De . For the Phan-Than-Thien models we have used $\kappa = 10^{-4}$, and for the FENE models $L^2 = 10000$. For the simulations, the mesh size and time step used were $(N_X, N_Y, N_Z = 257 \times 33 \times 65)$ and $\Delta t = 0.01$, respectively, achieving convergence for each a .

The trends in load versus the aspect ratio are similar for all the models and the difference exists in the predicted magnitude, arising due to the finite extension of the polymer chains and presence of shear thinning. The latter is more evident in the prediction for the average streamwise force (or friction along x). Owing to the decline in the effective shear viscosity, $\eta_{shear} = \frac{\tau_{yx}}{\dot{\gamma}_{yx}}$, (even for $De = 0.005$) captured by the FENE-P, and PTT models, the average streamwise force decreases when Wi is increased (from Fig. 7.3(c) to Fig. 7.3(d)). Note, we must refer to the Wi in this case, for appropriateness, since it incorporates the shear time scale. Interestingly, the average spanwise force undergoes negligible change. This could be indicative of a small De which is not large enough to cause a change in that direction. Moreover, the spanwise shear rate $\dot{\gamma}_{zy}$ is small in comparison to the streamwise shear strain rate.

While physically we obtain more or less similar behavior to the Oldroyd-B constitutive relation, the numerical treatment for the models strongly influenced the success of the simulations. In particular, for the FENE models, the semi-implicit treatment of the non-linear terms (see section 3.2 for details) was required for convergence at large values of the time step. An explicit treatment led to very small time step sizes and consequently prolonged simulation times (for example, $\Delta t = 0.0001$ was required for the FENE models) despite implicitly retaining the upper-convected rate operator.

Moreover, the PTT-linear and PTT-exponential models provide nearly the same results (for the load and friction) as the FENE-P without significant numerical burden. Not only is it easier to achieve convergence, but the restriction

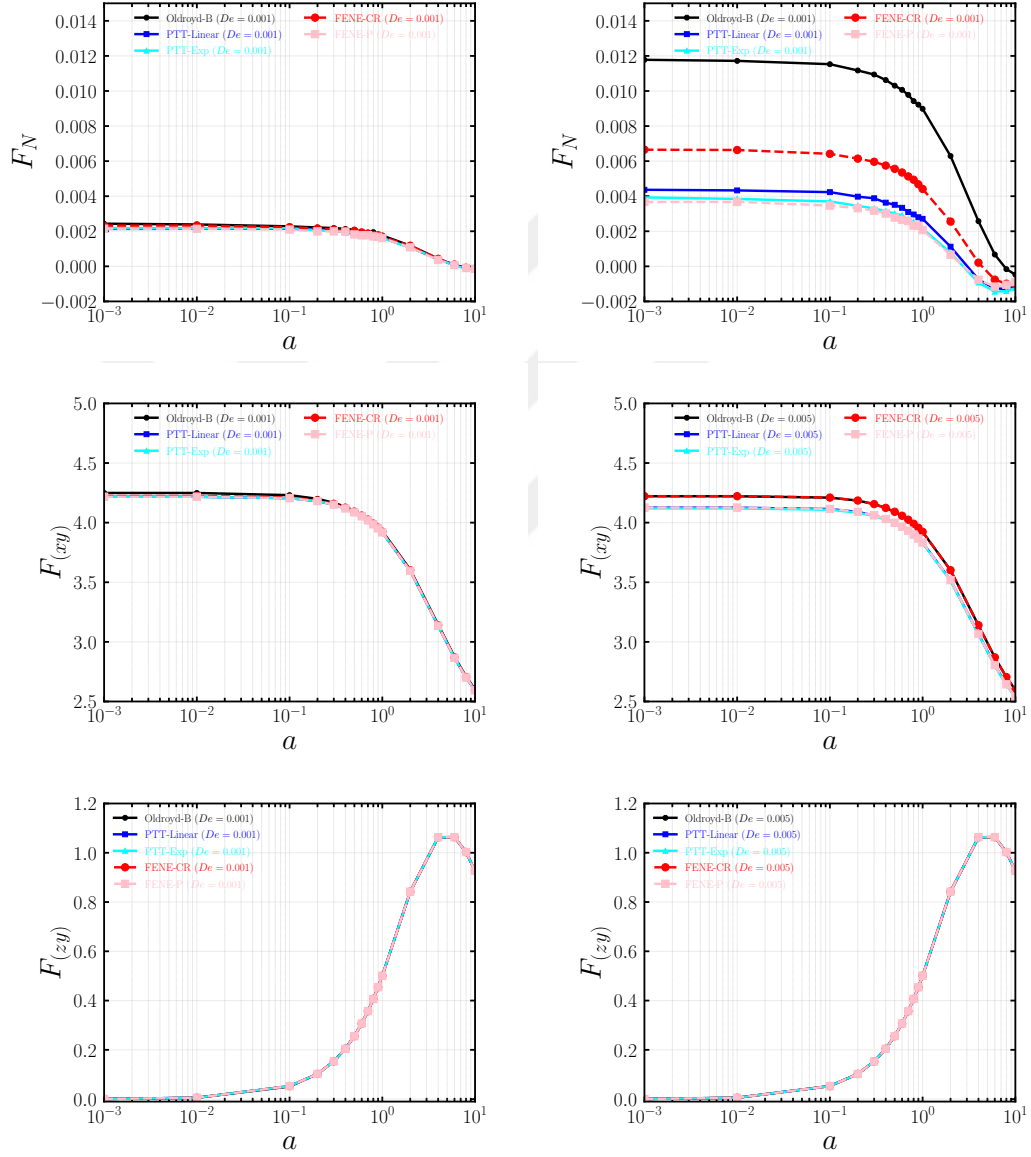


Figure 7.3: The variation in the net vertical force, (a) $De = 0.005, Wi = 5$ (b), average streamwise force (c) $De = 0.001, Wi = 1$ and (d) $De = 0.005, Wi = 5$, and the average spanwise force (e) $De = 0.001, Wi = 1$ and (f) $De = 0.005, Wi = 5$, versus the channel aspect ratio for the journal bearing ($e = 0.9, \epsilon = 0.001, \beta = 0.8$) using different non-linear constitutive relations.

on the maximum time step size is less severe. The issues pertaining to the numerics can be alleviated via the construction of linear models (in De) as with the Oldroyd-B. However, the choice of scaling can have a strong influence on the results, and an appropriate scaling for non-linear models is not available in literature. If we naively adopt the scaling introduced in chapter 2, then the non-linear response is lost and the model degrades to the Oldroyd-B in the first-order, necessitating at least a second-order (or higher) model. As discussed for the LIN model, higher-order models may be difficult to obtain for three-dimensional channels and could involve high-order mixed partial derivatives of the pressure that could complicate the numerical approach. Furthermore, no analytical solution for the leading-order (or the first-order) is available, since the problem would involve a system of non-homogenous partial differential equations.

Given their equivalent response, at least for the range of De considered here, linearized models (in κ) for PTT models can be constructed, since the non-linear terms are simpler. However, there is no mechanism for preventing shear thinning in the PTT models. Therefore, it would prove difficult, in a linearized setting to separate the two distinct effects of finite chain extension and shear thinning (reversible).

7.2 Cylindrical configurations

In the previous section, we examined the effects of finite polymer elasticity on the load carrying capacity of the lubricant film and the friction on the surface of the channel wall ($y = 0$). Having compared the linearized model with the viscoelastic Reynolds approach, we find the VR to provide suitable results for both small and moderate values of the Deborah number. Here, we examine the load carrying capacity for lubricated channels ideally conforming to a cylindrical configuration. In particular, we focus on two specific cases, namely (i) the pocketed (or textured) surface and (ii) the cone-and-plate rheometer adapted from literature [103, 104, 105, 106, 5].

The VR model is given by Eqs. 2.36a-2.36c, and is subject to the boundary conditions

$$u_r(R, \theta, Z = 0) = 0, u_r(R, \theta, Z = 1) = 0, \quad (7.4a)$$

$$u_\theta(R, \theta, Z = 0) = 1, u_\theta(R, \theta, Z = 1) = 0, \quad (7.4b)$$

$$u_z(R, \theta, Z = 0) = 0, u_z(R, \theta, Z = 1) = 0, \quad (7.4c)$$

$$p(R, \theta = 0) = p(R, \theta = 2\pi), \quad (7.4d)$$

$$\frac{\partial p}{\partial R}(R = 0, \theta) = 0, p(R = 1, \theta) = 0, \quad (7.4e)$$

$$\frac{\partial \tau_{ij}}{\partial r}(R = 0, \theta, Z) = 0, \quad (7.4f)$$

$$\tau_{ij}(R, \theta = 0, Z) = \tau_{ij}(R, \theta = 2\pi, Z), \quad (7.4g)$$

$$\frac{\partial \tau_{ij}}{\partial r}(R, \theta, Z = 0) = 0, \quad (7.4h)$$

and initial condition $\tau_{ij}(R, \theta, Z) = 0$. Note, in the boundary condition, along the radial direction, for the stress components is stated with respect to the physical coordinate system and requires transformation to the computational domain via Eqs. 3.6a-3.6c, giving

$$\frac{\partial \tau_{ij}}{\partial r} = \frac{\partial \tau_{ij}}{\partial R} - \frac{Z}{h} \frac{\partial h}{\partial R} \frac{\partial \tau_{ij}}{\partial Z}. \quad (7.5)$$

From Eq. 7.5, if $\frac{\partial h}{\partial R}|_{R=0, \theta} = 0$, then $\frac{\partial \tau_{ij}}{\partial r}(R = 0, \theta, Z) = \frac{\partial \tau_{ij}}{\partial R}(R = 0, \theta, Z) = 0$. This latter condition is satisfied for the channel surface geometries considered in this work.

Firstly, owing to the added complexity of the governing equations in cylindrical coordinates, we present a validation of the numerical methods adopted in this study for the case of the textured pad depicted in Fig. 7.4(a) and a comparison of the net vertical force (in Newtons) exerted on the pad, due to the asymmetrical textures, versus the inclination angle of the textures ($\beta_{texture}$) for a Newtonian fluid.

We find a good agreement between the predicted results using the classical FDM (see Appendix B), for $(N_R, N_\theta) = (256 \times 256)$, where N_R, N_θ are the nodes along the radial and circumferential direction, respectively, and the pseudo-spectral method used in Schuh et al. [5] over the entire range of the texture angle,

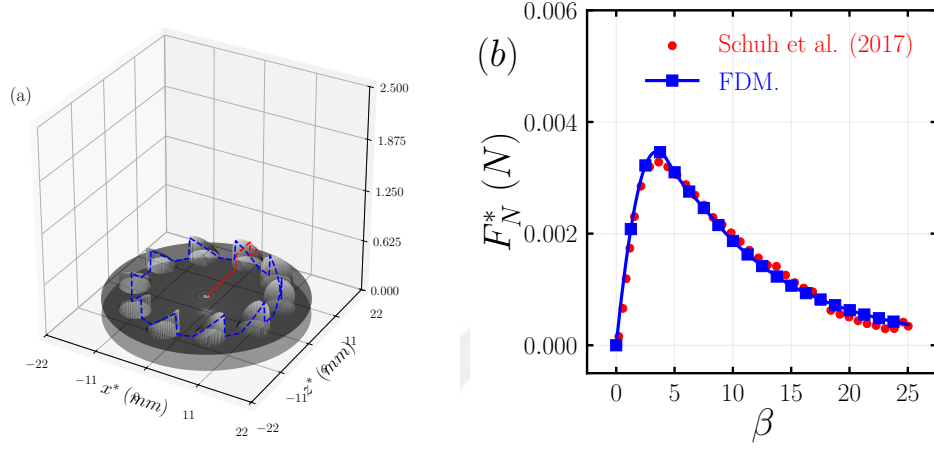


Figure 7.4: (a) The cylindrical pad of radius $R^* = 20\text{mm}$, with asymmetrical textures inclined at $\beta_{\text{texture}} = 2.5^\circ$. (b) the dimensional net vertical force exerted on the pad versus the inclination of the textures using the finite difference method, compared with the results from Schuh et al. (2017).

β_{texture} . Moreover, the variation in the normal force is smooth and free of any discrepancies for steeper textures. A small difference is observed between the results which is attributed to (i) the different numerical schemes adopted in this work and (ii) the use of the full circular pad as opposed to the single sector approach. In the latter, we simulate only a single texture (using a highly resolved mesh) and assume the pressure field is identical over N_{texture} number of textured sectors. This holds true only if the disturbance in the pressure (which is zero in the absence of the texture) does not overlap between neighboring sectors. This constraint is satisfied for a small enough texture radius.

We now introduce a viscoelastic lubricant, for the same textured pad, and examine the variation in the force versus the Deborah number, Fig. 7.5, assuming $\epsilon = 0.01$, $\beta = 0.2$ and $\beta_{\text{texture}} = 2.5^\circ$.

We find a disparity between the predictions in load. For very small De , both models are in good agreement, however, an overshoot for the Giesekus is observed, exceeding the load predicted by the Oldroyd-B model. Following this overshoot, the load declines owing to the influence finite extension of the polymer and shear

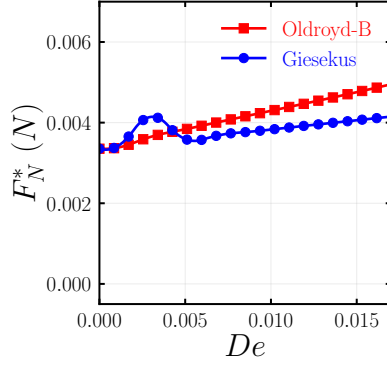


Figure 7.5: A comparison of the dimensional net vertical force predicted by the Oldroyd-B and Geisecus model for a single texture with an inclination of $\beta_{texture} = 2.5^\circ$.

thinning at large De (or Wi). We remark, that this initial overshoot matches the qualitatively the results for the cone-and-plane rheometer used in the work by Schuh et al. [?], if the rotation rates are extracted from the De . However, it could be a spurious effect of the numerical method and a more robust numerical approach such as the finite element method or the finite volume method may be required for accurate resolution of the spatial derivatives close to the pocket.

In general, for both models, we observe a monotonically increasing trend as the polymer elasticity is increased. Moreover, the load increment is not negligible over the range of De examined. Clearly, the addition of elastic polymers further enhance the load generation due to the asymmetric textures and the forces involved are in addition to the purely viscous forces in the Newtonian case. The additional gain in load can be attributed to the finite normal stress along the radial and circumferential directions.

However, the values of De are quite small and the gain in load is significant. In fact, one might assume, based on the small values, that viscoelasticity is negligible. However, this is purely do the scaling we have chosen which may not be reflective of the presence of the asymmetrical textures, spanning a small section of the total pad, and, the constant channel height elsewhere (majority of the pad). Furthermore, polymer molecules are subject to varying length and time

scales within the pocket along the radial direction (since the depth of the pocket is not constant). In addition, the rotation rate for the pad is constant, but the local surface velocity varies radially. Therefore, we revisit the definition of the De ; which is the ratio of the polymer relaxation time to the residence time scale to better reflect this topology.

If we choose a residence time scale in which the pocket makes a dominating contribution, i.e. the time taken for a polymer to traverse the pocket along the circumferential direction, $t_{res}^* = R_p \Omega / 2R_t$, where R_t is the radius of the texture, Ω is the rotation rate and R_p is the distance to the pocket, then we find that $De_{texture} = \lambda / t_{res}^* = \lambda R_0 \Omega / R_t$ and, $Wi = \lambda \dot{\gamma}^* = \lambda / R_0 \Omega / H_0$, where H_0 is the reference height. Using the values given in Schuh et al. [5], we have $De_{texture} \approx 4De$. This is when we consider a polymer chain elongating along the center line of the pocket. It will vary based significantly for other locations. The key point, is the Deborah number for the texture is much larger than the De_{pad} . This is also indirectly reflected in the difficulty in obtaining converged results using VR. The simulations diverge beyond $Wi = 2$ and require a much smaller Δt compared to the cartesian case (for successful cases). Generally, the VR approach fails (or the iterative procedure stagnates/slows down) when the value of De is large. These indirectly point to a far larger Deborah number than is experienced by the polymer. It may be worthwhile to obtain the linearized model for the cylindrical sliding contact, when only opting to examine the forces due to these textures, since the magnitude of such forces is quite small when considering the large applied loads in bearings and gears. A small error will be incurred in predicting the load via the linearized model (as seen in the cartesian case).

Next, we examine the net vertical force for a cone-and-plate rheometer configuration,

$$h = 1 + \tan(\delta)R, \quad (7.6)$$

varying the angle of the cone along the radial direction, denoted as δ . In this configuration, we have $\frac{\partial h}{\partial \theta} = 0$, and $\frac{\partial h}{\partial r} = \tan(\delta)$. Owing to a constant channel height along θ , the contribution to the load due to rotation will vanish, since there is no sliding effect.

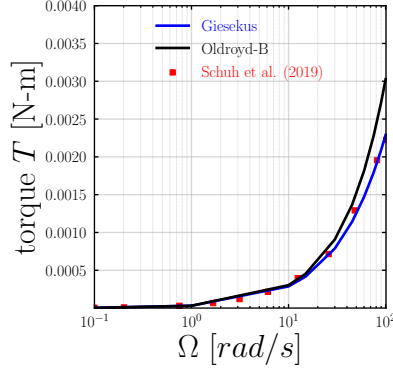


Figure 7.6: The net torque versus the angular rate of rotation for the Oldroyd-B, Giesekus model (this work, using the FDM) compared against the results from Schuh et al. [5].

Before proceeding, we validate our model against the results presented by Schuh et al. [5] for the torque on the lower plate $T = \int_0^1 \int_0^{2\pi} r^2 \tau_{\theta z}|_{z=0} dr d\theta$ using the numerical methods employed for the pocketed pad. We find a good agreement for the net torque produced (after considering the dimensional parameters contained in the original work) versus the rotation speed. However, a comparison of the net vertical force could not be made because the original work (i) considered inertial effects, i.e., for a non-negligible Re , and (ii) required corrections for different sources of error in the rheometer.

Fig. 7.7 shows the variation in the net vertical force versus the cone inclination angle for different values of De ; ($\epsilon = 0.01$, $\beta = 0.2$). For positive values, the inclination of the cone represents a diverging channel along the radial direction, and for negative values we have a converging channel. The cross-section, as seen from the Z -direction is the same for both the positive and negative inclination, if the absolute value of δ is the same. Spanning both the positive and negative range, we examine whether the polymer's elasticity, which has a strong tendency to disturb the symmetric distribution of pressure (for symmetric channels), can feel the inverted configuration. Note, that for a Newtonian fluid, the pressure field is zero across the entire channel and the vertical force is zero. Hence, any load gain (or loss) is due to viscoelastic effects.

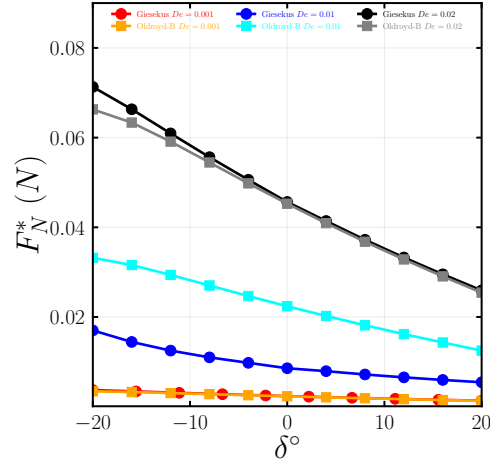


Figure 7.7: The variation in net vertical force versus the cone inclination angle for the Oldroyd-B and the Giesekus ($\alpha = \epsilon^2$) constitutive relations for (a) $De = 0.01$, $\epsilon = 0.01$, $\beta = 0.8$.

We find a that the load decreases as we sweep the cone inclination angle from negative to positive. For the $\delta = +\delta_0^\circ$, the forces developed are smaller compared to $\delta = -\delta_0^\circ$. We find a maximum occurs when the channel is a convergent profile ($\delta < 0$) and approaches a minimum for a divergent profile. Despite the same surface gradient, represented by $\tan(\delta^\circ)$, we obtain different predictions for the load carrying capacity, which would imply that the surface height itself is also important. This is interesting, since the net load (dimensionless) for a plane slider bearing (in which the surface has constant slope) depends on the surface gradient only.

Next, we observe, by increasing De , the effect of polymer elasticity is to enhance the load for a given angle. This would suggest the mechanisms involved in this cylindrical configuration differ from their cartesian counterpart because for the constant slope cartesian surface, we find a decreasing load carrying capacity versus De . A point to note, it that in the latter case, the lubricant is forced by the moving boundaries, along the streamwise direction (i.e, along the slope). A finite generation of streamwise normal stresses, would then, decrease the required pressure needed to push the fluid out. Whereas, for the rotating plates, the fluid

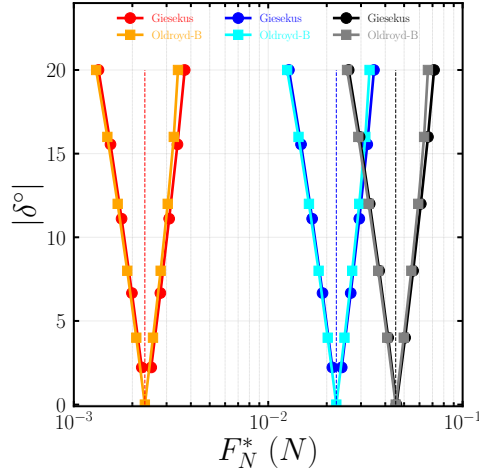


Figure 7.8: The change in the cone inclination angle against the net vertical force for the Oldroyd-B and the Giesekus ($\alpha = \epsilon^2$) constitutive relations for (a) $De = 0.01$, $De =$

is forced along the circumferential direction (θ) and pushed out radially. This, at least intuitively, carries similarities to the Weissenberg rod-climbing effect, in which the fluid tends to rise up the rod due to finite circumferential stress. Perhaps, a similar mechanism is at play for the rotating circular plates.

An interesting v-shaped diagram emerges, centered about the load for flat plates, if we swap the axis of Fig. 7.7. To demonstrate the asymmetric nature of load versus the inclination angle, we take the absolute value of δ and plot it versus the net vertical force, as shown in Fig. 7.8, for different values of the De and for the Oldroyd-B and Giesekus model, using $\beta = 0.2$, $\alpha = 0.01$, $\epsilon = 0.01$. The right branch, for each model and De , indicates the load for negative δ and the left branch for positive values.

The left and right hand side branches of the v-shaped curve are not symmetric about the center line $F_N(\delta = 0^\circ)$. This asymmetry is not affected by the Wi , but only slightly by the additional parameter α .

A spurious effect is the larger load predicted by the Giesekus model for $\delta =$

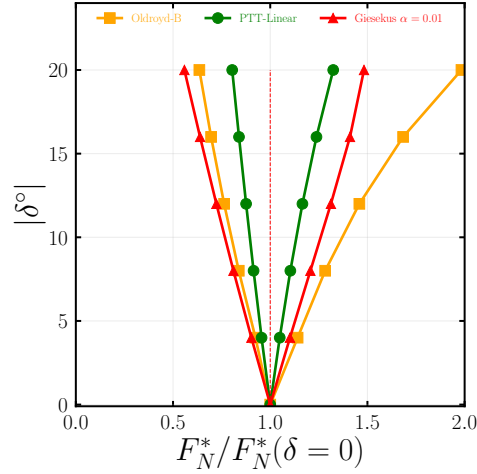


Figure 7.9: The change in the cone inclination angle against the net vertical force for the Oldroyd-B and the Giesekus ($\alpha = \epsilon$), PTT-linear ($\kappa = 0.1$), constitutive relations for $De = 0.1$, $\beta = 0.8$ and $\epsilon = 0.01$.

-20° . In general, the largest contribution to the load due to elasticity results from the Oldroyd-B model which is perfectly linear. Subsequent models, built on Oldroyd's methodology for dilute polymer solutions, take into account non-linear effects of finite extension of the polymer chains and shear thinning (possibly extending up to chain failure, i.e, molecular scission). We plot the phase diagram, similar to Fig. 7.8, but include the PTT-linear, FENE-CR and FENE-P models in Fig. 7.9.

The additional non-linearities are consistently described via the PTT and FENE models. By reducing the maximum possible extension of the polymer, the span of the expected force for the given inclination angles reduces. Interestingly, the right branch, for each model, exhibits a non-linear trend versus δ . For this cone configuration, the curves for Oldroyd-B collapse onto a single curve for all values of De , indicating a linear trend versus De . Non-linear behavior for the Oldroyd-B, not shown here, is observed at very large δ and De .

Finally, there are numerical considerations to be made for the cylindrical configuration. Unlike the cartesian coordinate system, the stability requirements

seem to be more restrictive, more so for the non-linear models. For example, the success of the numerical discretization, even in the fully-implicit schemes utilized here, still depend on the grid spacing and time step size (see chapter 3 for more details). However, a much more refined grid was achievable in the cartesian case than for the cylindrical problem. Furthermore, the maximum De possible is also must lower and, surprisingly, the FENE models, despite the additional physical stability they offer, seemed to suffer more from numerical divergence for the same grid size. The most stable non-linear model was the PTT-linear which seemed to provide results for very large values of De and for much greater resolutions.

Chapter 8

Non-isothermal lubrication

Thin lubricant films in mechanical components operating at very high speeds are subject to high shear strain rates. For a non-Newtonian lubricant this can induce shear thinning, and a strong elastic response. However, the heating of the film may also be important owing to viscous dissipation. In this chapter, we look at the effects of viscoelasticity on the thermal characteristics of the film.

The governing equations for a thermal film consist of the continuity equation, Cauchy's equations of motion (introduced in chapter 2) and the energy equation for a lubricant mixture with a Newtonian base solvent and polymer additive (without any heat source)

$$\frac{D(\rho^* c_p^* T^*)}{Dt^*} = \nabla^* \cdot \nabla^* (\kappa^* T^*) + \Phi, \quad (8.1)$$

where, T^* is the film temperature, ρ^* is the density of the lubricant, c_p^* is the specific heat capacity at constant pressure, κ^* is the thermal conductivity, Φ is the dissipation function, and $\frac{D}{Dt^*}$ is the material time derivative. If no assumptions are made, then Eq. 8.1 is non-linear, since the thermodynamic properties can vary with the temperature.

The classical form of the energy equation used for modeling the temperature distribution, typically only contains the advection of energy, i.e., the left hand side of Eq. 8.1, and the diffusion of energy via Fourier's law of heat conduction,

i.e., the right hand side of Eq. 8.1 with $\Phi = 0$. However, for a fluid (liquid or gas), the dissipation function is described as

$$\Phi = \boldsymbol{\sigma}^* : \mathbf{D}^*, \quad (8.2)$$

and using Eq. 2.6 for the total stress tensor $\boldsymbol{\sigma}^*$,

$$\Phi = -\mathbf{I}p : \mathbf{D}^* + 2\eta_s \mathbf{D}^* : \mathbf{D}^* + \boldsymbol{\tau}_p^* : \mathbf{D}^*, \quad (8.3)$$

where, η_s is the solvent viscosity (note that the polymer viscosity does not appear). Eq. 8.3 is further simplified by using the property, $\mathbf{I} : \mathbf{D}^* = \boldsymbol{\nabla} \cdot \mathbf{u} = 0$, where $\mathbf{u}^*$ is the velocity field. Thus, we have for the viscous dissipation of the fluid,

$$\Phi = 2\eta_s \mathbf{D}^* : \mathbf{D}^* + \boldsymbol{\tau}_p^* : \mathbf{D}^*, \quad (8.4)$$

the contribution of the Newtonian solvent stress, in which the viscosity can be a function of the temperature, and the non-Newtonian polymer stress. Substituting Eq. 8.4 into Eq. 8.1, gives the energy equation for a non-Newtonian lubricant mixture in the absence of source terms

$$\frac{D(\rho^* c_p^* T^*)}{Dt^*} = \boldsymbol{\nabla}^* \cdot \boldsymbol{\nabla}^* (\kappa^* T^*) + 2\eta_s \mathbf{D}^* : \mathbf{D}^* + \boldsymbol{\tau}_p^* : \mathbf{D}^*. \quad (8.5)$$

For gases, the viscosity is typically small enough to ignore the effects of dissipation. However, a liquid lubricant is highly viscous and can respond strongly to high enough temperatures found in, for example, piston-rings in combustion engines, journal bearings in engine shafts etc.

For the proceeding work, we make several simplifications. Firstly, our interest lies only in examining the temperature distribution, due to the polymer stress arising due to a finite elastic response of the polymer ($\lambda \neq 0$). Secondly, the variation of the temperature field due to the dependence of the lubricant properties on the temperature is not considered, since this has been studied extensively in literature (although under very specific conditions). Therefore, we assume that all properties remain invariant with respect to the temperature and are assumed constant, i.e., $\rho^* = \rho_0$, $c_p^* = c_{p,0}$, $\kappa^* = \kappa_0$, $\eta_s^* = \eta_s$ [107, 108]. Lastly, in contrast to the approaches made within the tribology which rely on the dimensional form of

the energy equation, we apply lubrication theory to simplify Eq. 8.5 and satisfy ourselves with its outcome.

We substitute the scaling factors for the length (Eq. 2.9), velocity (Eq. 2.10), pressure and stress (Eqs. 2.17a-2.17b), introduced in chapter 2, and in addition, the temperature scaling $T = \frac{T^* - T_w}{T_w - T_0}$ into Eq. 8.5 to obtain the scaled energy equation

$$\epsilon Re Pr \frac{DT}{Dt} = \epsilon^2 \frac{\partial T^2}{\partial x^2} + \frac{\partial T^2}{\partial y^2} + Br \beta \left(\frac{\partial u}{\partial y} \right)^2 + Br \left(\tau_{xx} \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial u}{\partial y} \right), \quad (8.6)$$

where, $\frac{D}{Dt} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$, $Re = \frac{\rho_0 U_0 H_0}{\eta_0}$ is the Reynolds number, $\epsilon = \frac{H_0}{\ell_x}$ is the thin-film ratio, $Pr = \frac{\eta_0 c_{p,0}}{\kappa_0}$ is the Prandtl number, $Br = \frac{\eta_0 U_0^2}{\kappa(T_w - T_0)}$ is the Brinkman number, T_w is the wall temperature, T_0 is the fluid reference temperature, $\beta = \frac{\eta_s}{\eta_0}$ is the solvent concentration. Note also, $Pe = Re Pr$ and $Br = Ec Pr$, where, Pe is the Peclet number, and Ec is the Eckert number. The Brinkman number is the ratio of the heat dissipated by the fluid to that conducted via the walls, and a small value indicates that the viscous dissipation is not large enough to raise the fluid temperature significantly.

Applying the thin-film limit, i.e., $\lim_{\epsilon^2 \rightarrow 0}$, on Eq. 8.7,

$$\epsilon Re Pr \frac{DT}{Dt} = \frac{\partial T^2}{\partial y^2} + Br \beta \left(\frac{\partial u}{\partial y} \right)^2 + Br \left(\tau_{xx} \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial u}{\partial y} \right), \quad (8.7)$$

we find the temperature diffusion along the streamwise x direction vanishes. At this point, it is important to mention, that there are several variations of Eq. 8.7 available in literature owing to a dimensional approach, as opposed to the classical lubrication theory. For example, in some cases, the diffusion of the film is entirely neglected, even across the film while in others, the material derivative is truncated to keep only the term along x . The former seems erroneous under the scaling we have chosen, however, it is difficult within microfluidic contexts to consider retaining the transport of a scalar along the streamwise direction and neglecting it along the cross-film direction. As stated previously, in this work, we consider only the application of Eq. 8.7.

Now, the left hand side of Eq. 8.7, contains ϵRe , both of which are small for a thin film. In fact, it is precisely this assumption that is made for the thin-film

reduced Cauchy's equations of motion. However, the Prandtl number is large and cannot be neglected. Therefore, it cannot be ignored a priori and must be retained. In the special case of a vanishingly small Re , for example, in ultra-thin films encountered in gas film lubrication, we can assume the advection to be negligible.

8.1 Geometry and boundary conditions

In this work, we have restricted ourselves to analyzing the un-wrapped journal bearing which defines the surface height $h = 1 + e \cos(2\pi x)$ in terms of a single parameter e , i.e., the eccentricity ratio, (see Fig. 6.2 for the schematic representation). For our purposes, the physical definition of e is meaningful, but we treat it as a parameter that allows adjusting the steepness of the channel walls; a large e gives a highly converging geometry.

The boundary conditions for the temperature, for a periodic channel, are of the two classical types, Dirichlet (fixed wall temperature) and Neumann (insulated wall). For a fixed temperature at the walls,

$$T(X = 0, Y) = T(X = 1, Y), \quad (8.8a)$$

$$T(X, Y = 0) = 0, \text{ and } T(X, Y = 1) = 0, \quad (8.8b)$$

and for an insulated shaft (upper wall), i.e., a Neumann condition,

$$T(X = 0, Y) = T(X = 1, Y), \quad (8.9a)$$

$$T(X, Y = 0) = 0, \text{ and } \left. \frac{\partial T}{\partial y} \right|_{(X, Y=1)} = 0. \quad (8.9b)$$

Note, for a curved boundary, we use the transformation of the physical to the computational domain via Eqs. 3.4a-3.4c, which gives $\frac{\partial T}{\partial y} = \frac{1}{h} \frac{\partial T}{\partial Y}$, and since $\frac{1}{h} \neq 0$, $\left. \frac{\partial T}{\partial y} \right|_{(x, y=h)} = \left. \frac{\partial T}{\partial Y} \right|_{(X, Y=1)} = 0$.

The journal bearing represents a periodic channel along the streamwise direction X and, it would be natural to enforce periodicity along that axis. However,

in Eq. 8.7, the absence of a diffusion operator along X means that periodicity can only be applied in the advective component. The advection operator on the other hand, acts only one-way, meaning it carries the physical quantity along the flow direction without any constraint on maintaining the same value at the boundaries. In practice, for mechanical bearings, the oil is continuously supplied through an inlet and is mixed with the oil that has been heated. Thus, at the inlet a fixed temperature is maintained via the mixing action implying a Dirichlet condition. Nonetheless, we enforce periodicity at the channel inlet and outlet for simplicity (since our goal is to observe the influence of polymer elasticity on the temperature), and shall see later, that it is indirectly enforced due to the symmetric distribution of the shear strain rate.

8.2 Influence of polymer elasticity

The temperature distribution from Eq. 8.7 varies along and across the channel. It is a function of the dissipation due to the Newtonian solvent and polymer additive shear stress, and the polymer elastic energy due to stretching. To examine how finite polymer elasticity can affect this temperature distribution, we use the (i) linearized and the (ii) viscoelastic Reynolds approach (Eqs. 2.34a-2.34c).

When we include the polymer elasticity via the linearized model which assumes small values of De , we can obtain analytical solutions for the average temperature distribution of the film for the cases where $Pe = 0$, and assuming the lubricant properties are not a function of the temperature. In this case, the Eq. 8.7 reduces to,

$$\frac{\partial T^2}{\partial y^2} = -Br\beta \left(\frac{\partial u}{\partial y} \right)^2 - Br \left(\tau_{xx} \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial u}{\partial y} \right), \quad (8.10)$$

and the film-height averaged temperature $T_{avg} = \int_0^1 T dY$ for the Dirichlet (fixed

wall temperature) case

$$\begin{aligned}
T_{avg} = & Br \left(\frac{1}{40} h^4 \left(\frac{dp}{dx} \right)^2 - \frac{1}{12} h^2 \frac{dp}{dx} + \frac{1}{3} \right) \\
& + DeBr(1 - \beta) \left(-\frac{1}{343} h^6 \left(\frac{dp}{dx} \right)^2 \frac{d^2p}{dx^2} - \frac{1}{87} h^5 \frac{dh}{dx} \left(\frac{dp}{dx} \right)^3 + \frac{1}{103} h^4 \frac{dp}{dx} \frac{d^2p}{dx^2} \right. \\
& \left. + \frac{1}{14} h^3 \frac{dh}{dx} \left(\frac{dp}{dx} \right)^2 - \frac{1}{240} h^2 \frac{d^2p}{dx^2} - 0.1625h \frac{dh}{dx} \frac{dp}{dx} + \frac{1}{8h} \frac{dh}{dx} \right),
\end{aligned} \tag{8.11a}$$

and the Neumann (insulated upper wall) is

$$\begin{aligned}
T_{avg} = & Br \left(\frac{1}{40} h^4 \left(\frac{dp}{dx} \right)^2 - \frac{1}{12} h^2 \frac{dp}{dx} + \frac{1}{3} \right) \\
& + Br(1 - \beta)De \left(-0.0029h^6 \left(\frac{dp}{dx} \right)^2 \frac{d^2p}{dx^2} - 0.011458h^5 \frac{dh}{dx} \left(\frac{dp}{dx} \right)^3 + 0.00972h^4 \frac{dp}{dx} \frac{d^2p}{dx^2} \right. \\
& \left. + 0.07291h^3 \frac{dh}{dx} \left(\frac{dp}{dx} \right)^2 - \frac{1}{240} h^2 \frac{d^2p}{dx^2} - 0.1625h \frac{dh}{dx} \frac{dp}{dx} + \frac{dh}{8h} \right).
\end{aligned} \tag{8.12a}$$

In Fig. 8.1(a), we plot the mean temperature distribution along the channel for the Newtonian case, and viscoelastic case via the LIN and VR approach for $De = 0.01$, $e = 0.5$, $\epsilon = 0.001$ and $Pe = 0$. We find the predicted temperature profile is the same between the three cases which would imply that polymer elasticity does not have a strong influence. If we plot the minimum film temperature versus the De , depicted in Fig. 8.1(b), a non-linear trend will emerge for large values of De , and the trend for the LIN and VR model are not similar; decreasing as opposed to increasing. The difference manifests due an underestimation of the advective transport in the LIN (see [23]).

Using the VR, we examine the temperature distribution in the channel for different values of the De for a fixed wall temperature, as shown in Fig. 8.2(a)-(c), and the mean temperature profile along x in Fig. 8.2(d) for $\epsilon = 0.001$, $Br = 1$, $Pe = 0$. A weak movement of the temperature is observed in the temperature contours close to the center of the channel, where dissipation is expected to be highest, and the symmetry of the temperature profile is not preserved. It is

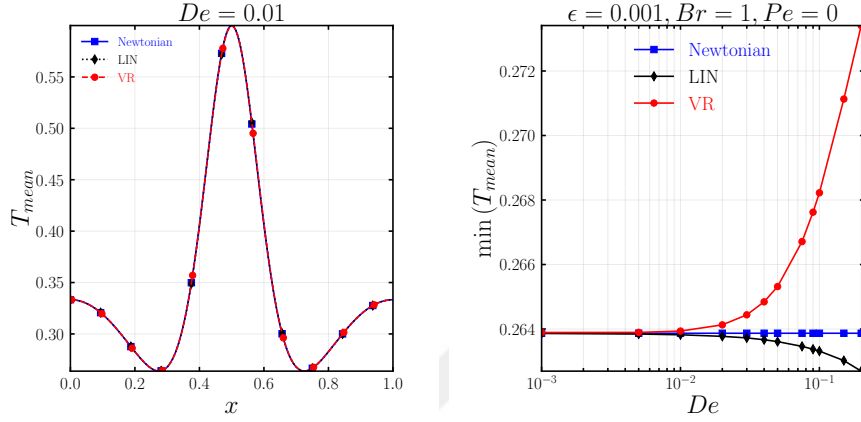


Figure 8.1: The variation of the mean film temperature along the channel for (a) $De = 0.01$, and (b) the minimum film temperature versus De . In each case, $\epsilon = 0.001$, $Br = 1$, $Pe = 0$.

evident that this is due to the movement of the polymer stress components and not the advection of energy, since $Pe = 0$. Most notably, the magnitude of the temperature is not strongly affected, seen in Fig. 8.2(d), but its profile shows a translation along the sliding direction. This demonstrates, under the assumptions we have made, the weak influence of polymer elasticity on the magnitude of the temperature (note that the polymer concentration is not large), but a strong influence on transporting energy along the flow direction.

Next, we examine the temperature distribution in the channel for different values of the De for a fixed wall temperature but include the effect of energy transport, as shown in Fig. 8.3(a)-(c), and the mean temperature profile along x in Fig. 8.3(d) for $\epsilon = 0.001$, $Br = 1$, $Pe = 1000$. No noticeable change occurs, since the advective component scales with ϵ which is small.

In a similar manner, we examine the temperature distribution in the channel for different values of the De for an insulated upper-wall, and neglecting energy advection, as shown in Fig. 8.4(a)-(c), and the mean temperature profile along x in Fig. 8.4(d) for $\epsilon = 0.001$, $Br = 1$, $Pe = 0$. In this case, the stress transport does not strongly influence the temperature profile. The mean temperature distribution shows a decline in maximum temperature, occurring at the channel

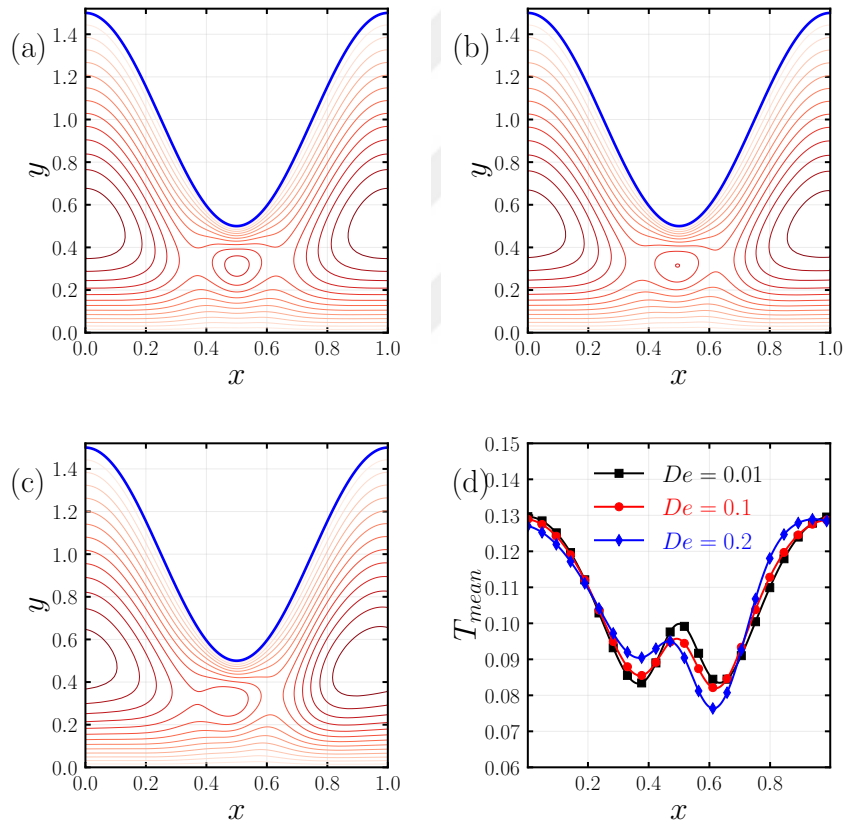


Figure 8.2: The film temperature distribution within the channel (fixed wall temperature) for (a) $De = 0.01$, (b) $De = 0.1$, (c) $De = 0.2$, and the average film temperature for different values of De . In each case, $\epsilon = 0.001$, $Br = 1$, $Pe = 0$.

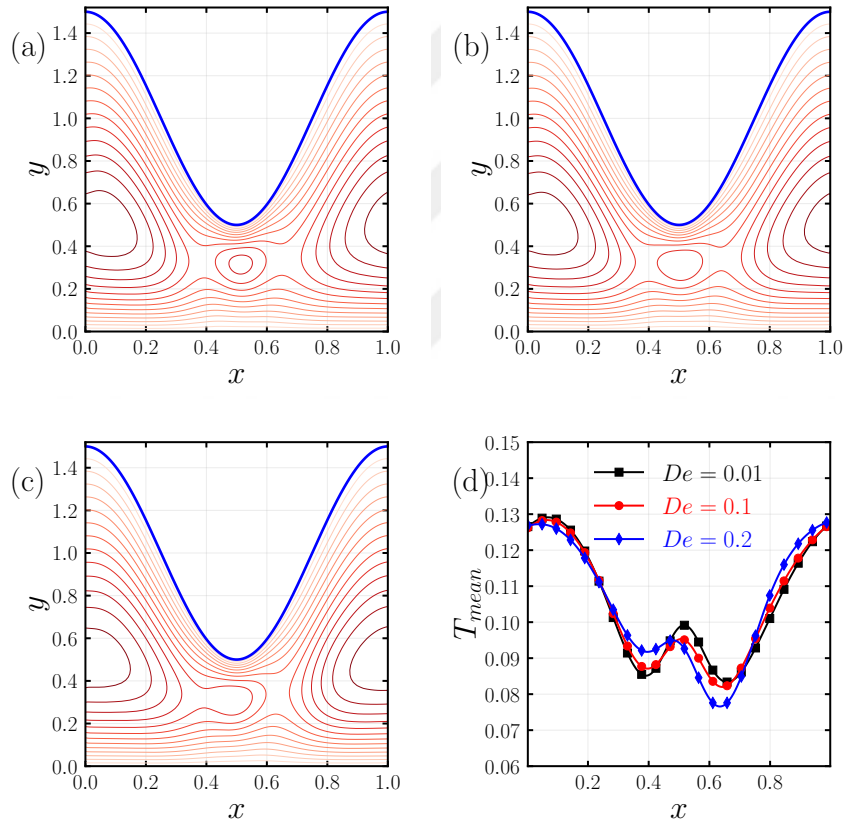


Figure 8.3: The film temperature distribution within the channel (fixed wall temperature) for (a) $De = 0.01$, (b) $De = 0.1$, (c) $De = 0.2$, and the average film temperature for different values of De . In each case, $\epsilon = 0.001$, $Br = 1$, $Pe = 1000$.

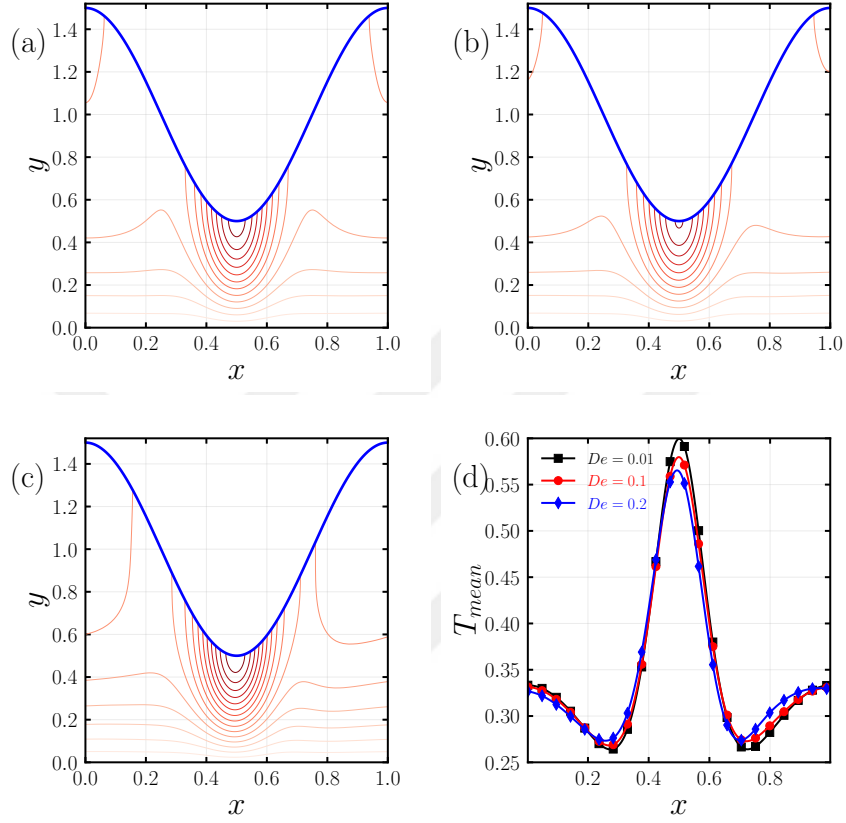


Figure 8.4: The film temperature distribution within the channel (insulated top boundary) for (a) $De = 0.01$, (b) $De = 0.1$, (c) $De = 0.2$, and the average film temperature for different values of De . In each case, $\epsilon = 0.001$, $Br = 1$, $Pe = 0$.

center.

In a similar manner, we examine the temperature distribution in the channel for different values of the De for an insulated upper-wall, and include the convection of energy, as shown in Fig. 8.5(a)-(c), and the mean temperature profile along x in Fig. 8.5(d) for $\epsilon = 0.001$, $Br = 1$, $Pe = 0$. In this case, a strong movement of the temperature along the flow direction is observed in the contours, reflected in the mean temperature distribution.

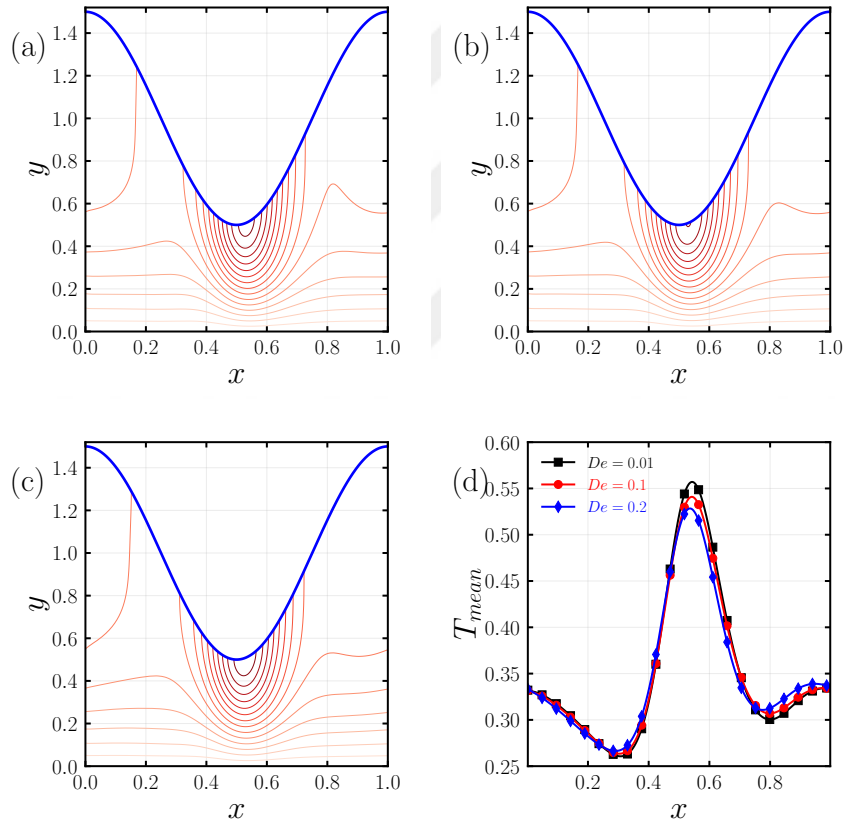


Figure 8.5: The film temperature distribution within the channel (insulated top boundary) for (a) $De = 0.01$, (b) $De = 0.1$, (c) $De = 0.2$, and the average film temperature for different values of De . In each case, $\epsilon = 0.001$, $Br = 1$, $Pe = 1000$.

8.3 Discussion

The influence of viscoelasticity on the temperature, assuming constant lubricant properties, is not strong. In fact, it manifests (for large values of De), via an additional movement of the energy owing to a transport of the stress along the channel. This transport changes the energy due to dissipation. However, the magnitude, which is of greater importance, remains largely unaffected by the De . It is affected, equally, in the Newtonian and non-Newtonian cases by the Brinkman number. In the thin-film reduced energy equation (Eq. 8.7), these two parameters appear as a product. One could also re-scale the temperature with Br and thus retain a single parameter for the case where $Pe = 0$.

It is also important to recognize that the variation in fluid properties can be significant at very high temperatures which may be a result of the operating environment of the bearing as opposed to fluid friction. In this case, the thermal conductivity, density and viscosity (both solvent and polymer) become functions of temperature. A detailed treatment would require coupling the momentum equations, with the viscoelastic constitutive relations and the energy equation for the numerical solution of the film temperature. In this setting, the use of the Generalized Reynolds equation (Eq. 4.5), which is an integral-differential equation, would become necessary. Clearly, this holistic approach is more robust but numerically demanding and perhaps not required for most applications. Note that the effect of the temperature on the viscosity, the key property governing the film pressure, is to diminish it. This would suggest, that if the viscosity declines versus temperature, then the viscoelastic effect would also in effect reduce, since $\eta_p(T^*) < \eta_p(T^* = T_0)$, leading to a reduction in shear and normal stress. Nonetheless, it could be an improvement over the simple feed-forward approach considered in this thesis.

Chapter 9

Conclusions and future work

In this chapter, we summarize and conclude the key points presented in the different chapters, finishing with a section pointing to possible extensions and areas for improving upon this study.

9.1 Conclusions

A new approach is proposed for the numerical simulation of a three-dimensional contact lubricated by a shear thinning liquid film. This approach is based on a modification of the viscous shear thinning function. For this reason we call it: modified viscosity (MV) approach. Both the accuracy and the computational time were compared between the proposed approach and the generalized Reynolds equation, which is taken as the benchmark for complex rheology lubricants.

The available ASD methods and the MV approach for a lubricant are compared when exposed to pressure levels typical of elastohydrodynamic contacts. We find that the available ASD methods are accurate for an appreciable range of the bearing number but their accuracy diminishes as the effect of shear thinning becomes significant. Furthermore, the proposed approach retains the numerical efficiency of the ASD based methods and greatly extends the valid operating

range of these methods. This increased accuracy of MV holds well for small aspect ratios. As the aspect ratio increases, the valid range shrinks but remains within the limits of physically relevant bearing numbers.

The computational time was compared for different lubricating modes, i.e. (i) a pure sliding case for converging and diverging channels, (ii) a sliding and rolling contact, and (iii) a pure squeezing case. We find that the improvement in numerical efficiency of MV with respect to GR is appreciable for non-cavitating channels, such as the plane slider, and externally pressurized channels, such as the ridge. However, the MV approach appears to be at least one order of magnitudes faster than GR, in profiles where cavitation occurs. In particular, the average computational time differed by a factor of (i) 10 for the unwrapped geometry mimicking a journal bearing and (ii) 50 for the pocketed pad bearing.

An analysis pertaining to the optimal choice for the location of the approximate shear strain rate is provided. For purely sliding contacts the optimal location is around $y = h/2$, while it is close to $y = h/4$ for both rolling contacts and squeeze film flows. The latter arises due to a strain rate profile that is perfectly anti-symmetric about $y = h/2$ when $SRR = 0$ preventing the use of an average shear rate. Furthermore, we have presented an a priori approach that can estimate this optimal location for any combination of the surface profile and shear thinning constitutive relation. In this a priori method, we calculate the shear rate for both GR and MV by using the Newtonian (but piezoviscous) shear rate instead of the non-Newtonian one. The behavior of E_ζ is qualitatively the same if we use either the Newtonian or the non-Newtonian shear rate. Therefore, both quantities are minimized for the same value of χ .

Similar to the sliding cavitating cases, we observe an increased numerical efficiency of MV at, in practice, no cost with regard to the accuracy for the pure squeeze problem. The load and pressure peak matched closely with the GR model within a reasonable range of bearing numbers. The computational time differs by one order of magnitude. If an additional constraint of higher mesh quality is imposed (due to a complex squeezing geometry for instance), our results indicate that the use of GR will become prohibitive due to the dramatic increase in (i)

computational time and (ii) needs of storage.

It should be observed that at large values of Λ most of the cases tested in this work reach pressures characteristic of elastohydrodynamic lubrication (EHL). For this reason, not explicitly examining the effect of EHL on the valid range of bearing numbers for MV raises additional questions. The goal of this work is to validate the MV approach and show its qualities. The next logical step is to consider the deformation of the solid surfaces in combination with MV, to be able to fully describe the physics of lubricated contacts at high pressures. We plan to incorporate this in a future work involving both EHL line and point contact problems with shear thinning lubricants [19, 17]. We believe that MV combined to the solid deformation will show more clearly its advantage in computational efficiency with respect to GR due to the increase in non-linear coupling present in EHL problems.

In conclusion, the proposed approach offers a numerically appealing alternative to the generalized Reynolds equation. It is valid and numerically appealing for a wide range of physical problems. It can be used to study lubricant behavior under high shear for a wide range of Λ , while the use of ASD methods without the MV correction may lead to erroneous results and must be used cautiously. Furthermore, MV will prove beneficial for simulations of textured [109] and/or rough surfaces [110] where fine meshes are required. In addition, studies that aim to couple dynamic motion of bearings, rotating motion of microchannels [111], capillary action [112] and other transient effects such as viscoelasticity [113, 23] can greatly benefit from MV. Interesting applications involving hydromagnetic and shear thinning fluids can also be extended via the MV approach [114], since this particular case involves expanding the model to include the effects of the electric and magnetic fields increasing the computational needs. The MV approach can offer a computationally appealing alternative to the GR model in these cases.

In this work, we have modeled the effects of viscoelasticity using the FENE-CR and FENE-P constitutive relations on the pressure distribution and the load carrying capacity of a journal bearing by using the viscoelastic Reynolds (VR) approach. The FENE-CR and FENE-P models introduce additional non-linear

behavior such as, finite extension of the polymer chains and shear thinning. The VR approach is validated for the parabolic slider (Fig. 6.1) using the FENE-CR and FENE-P models over a wide range of Wi , and L against DNS. The pressure distribution and load predicted by the VR approach is in excellent agreement with the DNS over the range of parameters examined. Furthermore, VR is computationally efficient in comparison to the DNS.

After validating the VR approach also for the FENE models, it was applied to an unwrapped journal bearing profile. Here, the variation in the pressure distribution and the load were examined against three parameters namely, (i) the Weissenberg number Wi , (ii) the finite extensibility parameter L , and (iii) the journal eccentricity ratio e . It was observed that all these three parameters have a strong influence on the load carrying capacity. In particular, the load increases versus strengthening viscoelastic effects (as Wi increases). For small values of Wi , the FENE-CR and FENE-P exhibit a strong non-linear trend, unlike a monotone increase predicted by the Oldroyd-B. For large Wi , a saturation and eventual decline in the load is observed for the FENE models.

In a similar manner, the dependency of the load on L^2 is examined. By increasing (decreasing) this parameter, the maximum stretch of the polymer chains is promoted (curtailed). For small values, the predicted load is almost Newtonian and increases to a maximum (the value obtained from Oldroyd-B) when an infinite stretch is allowed. In between these two limiting scenarios, a strong non-linear region exists influenced weakly (strongly) by Wi (e). Furthermore, for a rigid polymer (weakly extensible chain), a negative load is observed for both FENE-CR and FENE-P. This trend spans a longer (shorter) region as the eccentricity ratio is increased (decreased) and it takes a larger value of L^2 to reach the maximum possible load.

Finally we consider the effect of the journal eccentricity on the load carrying capacity. As e is increased, mimicking a sharper channel, the load initially increases. This initial positive effect subsides for larger values for the FENE-CR and FENE-P, whereas the Oldroyd-B simply predicts an exponential increase. For very large values of e , a rapid decline in load is observed with the Weissenberg

number seeming to exaggerate this effect.

Interestingly, the Wi and e influence the load by altering the advective transport of the stress from the positive, high pressure region into the negative, low pressure region. This movement of stress manifests as a loss (gain) in resistance in the positive (negative) part. Owing to this, the net load, obtained by integrating and summing the positive and negative regions, declines. However, for the shear thinning FENE-P fluid, this movement of stress results in a slightly uneven shear viscosity across the channel which contributes positively to the total load for cases with very high eccentricity ratio.

The proposed model provides a numerically efficient, accurate and easily extendable approach for modeling viscoelastic effects in thin lubricated contacts undergoing sliding motion. It can readily account for additional non-linear behavior due to finite extension of the polymer chains and shear thinning for arbitrarily shaped channels boundaries. Its main advantage is enabling a scalable modeling approach that combines several non-linear effects such as, cavitation [58], transient/squeezing motion of the boundaries [115], lubricant slippage at the boundaries [116], channel surface textures/roughness [109], pressure [117, 118] and temperature dependence [119] of the viscosity, and solid deformation [120], to name a few. In the presence of these effects, standard techniques like robust DNS can become computationally inefficient and simplified linearized models may be too difficult to obtain. As such, the VR can be coupled with any constitutive relation for the polymer stress, lubricant viscosity and/or density to a high degree of accuracy.

The VR approach was extended towards three-dimensional channels, for both cartesian and cylindrical coordinate systems. A linearized model for three dimensional sliding contacts was also presented. We examined using the two models the load carrying capacity, and the average force along the streamwise and spanwise directions, for the extruded parabolic slider and journal bearing for different values of the De and the channel aspect ratio.

Firstly, we note that as the aspect ratio was varied, a non-zero load is predicted

and decreases as the channel width becomes comparable to its length. The predictions of the linearized model and the VR approach are in good agreement for small values of De (similar to the two-dimensional problems) but a discrepancy is observed in the case of the highly eccentric extruded journal bearing. There the predictions of the first-order linearized approach fail for small values. Using the VR, we also compared under the same conditions and channel geometries, the predictions of the various non-linear constitutive relations. The variation of the quantities versus the channel aspect ratio was qualitatively the same but quantitatively damped, in comparison to the Oldroyd-B, owing to the effects of constrained polymer chain extension and shear thinning (in FENE-P). Moreover, we suspect the mechanisms generating/diminishing load, involving the polymer normal stresses along the streamwise and spanwise direction, are the same as for the two-dimensional channels.

Similarly, we examined the variation in the net vertical force for the textured pad and the cone-and-plate rheometer, both of which conform to a cylindrical coordinate system. We observe, in both cases, an increase in the load versus the Deborah number owing to the finite increase in polymer radial and circumferential stress components. Again, the competing effects of stress advection and generation lead to an overall gain in load. From a numerical perspective, most notably, convergence for the pocketed pad bearing was difficult to achieve as De increased.

Finally, we also examined the influence of polymer viscoelasticity on the load and the friction in the case of vertical channel boundary motion or simply squeeze film motion. This was done via the VR and using a second-order perturbation solution in De . A comparison between the two showed striking differences, even at small values of De , versus time. Most notably, the absence of an initial condition in the first and second-order perturbed solutions, predicts a finite initial load for an initially unstressed polymer. Whereas, the VR approach can accommodate an initial condition and predict a load reflecting an initially unstressed polymer.

9.2 Future work

The impact of strongly non-Newtonian lubricants, and fluids, in general, has attracted attention in several areas of study. The primary hurdles from a computational perspective are the inefficiency of robust modeling techniques such as the DNS for simulating these flows, especially for three-dimensional lubricated contacts. We have focused on two specific non-Newtonian effects, namely shear thinning and viscoelasticity, which require non-linear models and special algorithms for computational modeling. The approach/s introduced in this work, the perturbation models, and the viscoelastic Reynolds equation compared favorably with DNS, offering a computationally efficient alternative. This study offers several potential research directions, summarized below.

For viscoplastic or shear thinning lubricants, the modified viscosity approach works over a wide range of the bearing number, demonstrating its feasibility in simulating the pressure distribution in heavily loaded lubricated (with strongly non-linear properties) contacts. More generally, it represents a technique by which the three-dimensional viscosity of the channel can be reasonably approximated by a two-dimensional distribution. In a similar manner, one can construct high-order approximations leading to further improvements in load estimation. In addition, perturbation approaches have not yet been employed for shear thinning or viscoplastic lubricants (more generally) can be considered as an alternative methods for constructing reduced-order models. On the numerical front, we employed a simplified Gauss-Seidel method for the solution of the algebraic system of equations. It is also possible to create specialized non-linear multigrid methods (Full-Approximation Schemes) for the efficient solution of these reduced-order models, thereby further improving the computational efficiency. Although, this may not be necessary for smooth channels, but rough or textured surfaces are demanding even without the non-linear variation in lubricant properties.

Viscoelastic lubricants with strongly elastic polymer additives are difficult to model in a general setting. For thin-film lubricants, they present additional numerical complexity. In this work the VR approach of Ahmed and Biancofiore [23]

was, extended to three-dimensional cartesian and cylindrical configuration, and compared with DNS and first-order linearized models, showing a good agreement, for linear and non-linear polymer stress constitutive relations. However, it can be further improved by considering the log-conformation-representation to overcome the issues at larger values of De . In the original work, a polynomial-reduction of the stress was performed (for two-dimensional channels) to increase the efficiency of the approach which can also be extended to three-dimensional channels.

An extension to spherical coordinates could also be an interesting avenue for further research. Biological systems involving lubrication, such as human biological joints and the surface of the human eye conform to more complex topologies, which may be better adapted using a curvilinear coordinate system. Lineaized models for these systems may also be of interest, since the numerical solution of the governing equations, in general, are difficult to obtain for curvilinear configurations, even in a purely Newtonian setting.

The numerical technique used in this work relied heavily upon an implicit discretization of the various differential equations. These methods are gaining more traction as standard explicit approaches and algorithms often lead to prohibitively long simulation times owing to (i) increased non-linearity and the requirement for (ii) fine meshes. A semi-implicit approach must be used for the non-linear constitutive relations. We demonstrated one possible method for treating the non-linear terms, which resulted in improved convergence, but it can be further improved for optimizing convergence. It can be of importance for general applications and not just lubricated contacts.

An attempt at understanding the numerical stability via the Von-Neumann analysis was made for the polymer normal stress in a two-dimensional setting. This can be greatly improved by considering the influence of (either separately, or in a weakly-coupled fashion) the remaining stress components or via the global stability analysis using the discretized coefficient matrix representing the full system of equations. This would help shed some light on the high Weissenberg number problem to which no permanent solution exists.

The response of the load to polymer elasticity depended strongly on the variation of the channel surface, which for most practical problems is not smooth and, in special cases, textured by design for improving performance. Optimization of the surface for maximizing the additional gain in load carrying capacity by exploitation of the finite growth in the polymer normal stress could be an interesting approach. However, it is evident from the results presented in this work, that first-order linear models may not be useful when elastic effects dominate.



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Appendix A

Other polymer constitutive relations

The VR approach can be applied to any constitutive relation. In section 6.3.1.1, the approach is validated for the FENE-CR (Eq. ??) and FENE-P (Eq. ??). In this section, we also validate it for the thin film reduced (i) PTT linear model

$$\left[1 + \kappa \frac{Wi}{\epsilon(1-\beta)} tr(\boldsymbol{\tau})\right] \boldsymbol{\tau} + \epsilon Wi \overset{\nabla}{\boldsymbol{\tau}} = 2(1-\beta) \mathbf{D}^r, \quad (\text{A.1})$$

(ii) PTT exponential model

$$\left[1 + e^{\kappa \frac{Wi}{\epsilon(1-\beta)} tr(\boldsymbol{\tau})}\right] \boldsymbol{\tau} + \epsilon Wi \overset{\nabla}{\boldsymbol{\tau}} = 2(1-\beta) \mathbf{D}^r, \quad (\text{A.2})$$

and (iii) Giesekus model

$$\boldsymbol{\tau} + \epsilon Wi \overset{\nabla}{\boldsymbol{\tau}} + \alpha \frac{\epsilon Wi}{(1-\beta)} (\boldsymbol{\tau} \cdot \boldsymbol{\tau}) = 2(1-\beta) \mathbf{D}^r, \quad (\text{A.3})$$

where κ and α are model specific parameters and $\mathbf{D}^r = \frac{1}{2} \begin{Bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 2 \frac{\partial v}{\partial y} \end{Bmatrix}$ is the thin film reduced deformation rate tensor. Note that for each model, terms on the order of $O(\epsilon^2)$ have been ignored. For example, in $tr(\boldsymbol{\tau})$ the normal cross-film stress component τ_{yy} is $O(\epsilon^2)$. However, there are a few such terms in the constitutive relations and retaining them in the dimensionless form does not affect

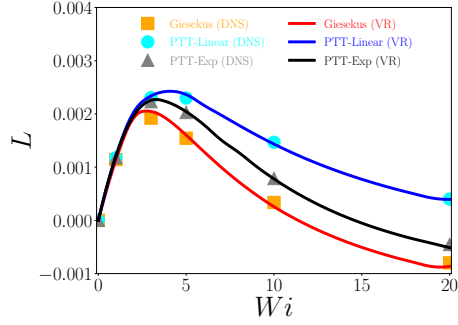


Figure A.1: A comparison of the variation in load for different constitutive relations versus Wi using $\alpha = 0.002$, $\kappa = 0.002$, $\epsilon = 0.004$, and $\beta = 0.8$ for the parabolic slider. Note that $Wi = 20$ corresponds to $De = 0.08$.

the final solution. Furthermore, for the PTT models, we assume only the upper convected rate instead of the Gordon-Schowalter derivative.

In Fig. A.1, we present a comparison of the load carrying capacity between the DNS (using OpenFOAM) and the VR approach over a wide range of Wi for the different constitutive relations given in Eqs. A.1-A.3. The VR compares favorably with the results from the DNS and is able to capture the strong non-linear effects that emerge at $Wi = 3$. This further demonstrates the robustness of the VR approach. Finally from a computational perspective, the DNS at $Wi = 20$ can take several days to achieve steady-state in contrast to a few minutes required by VR.

Appendix B

Finite difference stencils

The discretization of the differential operators is performed via the finite difference method. In this work, we cover three-dimensional channels for both cartesian and cylindrical coordinate systems. Therefore, the notation for the difference stencils about some node (i, j, k) provided here for some quantity M (representing for example, the film pressure p , velocity components u and v , the film height h e.t.c) along a coordinate axis S . For a cartesian configuration, we replace S with (X, Y, Z) , while for a cylindrical configuration by (R, θ, Z) .

The first order derivatives are

$$\frac{\partial}{\partial S} = \frac{1}{2}(U_S + |U_S|)\frac{\partial}{\partial S}\Big|_{UW} + \frac{1}{2}(|U_S| - U_S)\frac{\partial}{\partial S}\Big|_{DW}, \quad (\text{B.1a})$$

$$\frac{\partial M}{\partial S}\Big|_{UW} = \frac{M_{(i,j)} - M_{(i-1,j)}}{\Delta \hat{x}} \left(1 + \frac{1}{2}\psi(r_{i,j}) - \frac{1}{2}\frac{\psi(r_{i-1,j})}{r_{i-1,j}}\right), \quad (\text{B.1b})$$

$$\frac{\partial M}{\partial S}\Big|_{DW} = \frac{M_{i+1,j} - M_{i,j}}{\Delta \hat{x}} \left(1 + \frac{1}{2}\psi(r_{i,j}^{-1}) - \frac{1}{2}\frac{\psi(r_{i+1,j})}{r_{i+1,j}}\right), \quad (\text{B.1c})$$

$$\frac{dM}{dS}\Big|_{(i,j)} = \frac{1}{\Delta S} \left(\frac{1}{12}M_{(i-2,j)} - \frac{2}{3}M_{(i-1,j)} + \frac{2}{3}M_{(i+1,j)} - \frac{1}{12}M_{(i+2,j)}\right), \quad (\text{B.1d})$$

$$\frac{dM}{dS}\Big|_{(i,j)} = \frac{1}{\Delta S} \left(\frac{1}{12}M_{(i-2,j)} - \frac{2}{3}M_{(i-1,j)} + \frac{2}{3}M_{(i+1,j)} - \frac{1}{12}M_{(i+2,j)}\right), \quad (\text{B.1e})$$

where, ΔS is the spacing along the coordinate axis, $r_{i,j} = \frac{M_{i+1} - M_i}{M_i - M_{i-1}}$, $\psi(r_{(i,j)}) = \frac{r_{(i,j)}^2 + r_{(i,j)}}{1 + r_{(i,j)}^2}$, U_S is the velocity along S at a node (i, j, k) in the mesh.

For the second derivative, the difference schemes are

$$\left. \frac{d^2 M}{dS^2} \right|_{(i,j,k)} = \frac{1}{\Delta S^2} \left(M_{(i-1,j,k)} - 2M_{(i,j,k)} + M_{(i+1,j,k)} \right), \quad (\text{B.2a})$$

$$\begin{aligned} \left. \frac{d^2 M}{dS^2} \right|_{(i,j,k)} &= \frac{1}{\Delta S^2} \left(-\frac{1}{12}M_{(i-2,j,k)} + \frac{4}{3}M_{(i-1,j,k)} - \frac{5}{2}M_{(i,j,k)} \right. \\ &\quad \left. + \frac{4}{3}M_{(i+1,j,k)} - \frac{1}{12}M_{(i+2,j,k)} \right). \end{aligned} \quad (\text{B.2b})$$

In Eqs. B.1a-B.2b, we have considered, for brevity, a stencil covering nodes along the first axis, $i + 1, i + 2$ etc. The same expressions will hold when considering neighboring nodes along j and k . A slight modification to Eqs. B.2a-B.2b is required at the channel boundaries when using periodic boundary conditions. Assuming the periodic boundary exists along the axis spanned by varying the index i , then for the channel inlet nodes ($i = 1, j, k$)

$$\begin{aligned} \left. \frac{d^2 M}{dS^2} \right|_{(1,j,k)} &= \frac{1}{\Delta S^2} \left(-\frac{1}{12}M_{(N-2,j,k)} + \frac{4}{3}M_{(N-1,j,k)} \right. \\ &\quad \left. - \frac{5}{2}M_{(1,j,k)} + \frac{4}{3}M_{(2,j,k)} - \frac{1}{12}M_{(3,j,k)} \right), \end{aligned} \quad (\text{B.3a})$$

$$\left. \frac{dM}{dS} \right|_{(1,j,k)} = \frac{1}{\Delta \hat{x}} \left(\frac{1}{12}M_{(N-2,j)} - \frac{2}{3}M_{(N-1,j)} + \frac{2}{3}M_{(2,j)} - \frac{1}{12}M_{(3,j)} \right), \quad (\text{B.3b})$$

and for the channel outlet we simply set $M_{(N,j,k)} = M_{(1,j,k)}$ which ensures periodicity (note that finite difference stencils are symmetric). The boundary conditions considering nodes along the channel walls, i.e. the boundaries located at $Y = 0$ and $Y = 1$ (for a cartesian grid), are relevant only for the polymer stress components. Furthermore, the constitutive relations are hyperbolic and require only a single boundary condition which, ideally, should be aligned with the flow direction. However, the vertical velocity field vanishes at the boundaries and selecting the proper direction is not strictly required. Thus, we apply the zero-gradient boundary condition for the nodes along the bottom channel wall, i.e., $(i, j = 1, k)$, and using a third-order forward difference stencil

$$\left. \frac{\partial \tau}{\partial \hat{y}} \right|_{(i,1)} = \frac{1}{\Delta Y} \left(-\frac{11}{6}\tau_{(i,1,k)} + 3\tau_{(i,2,k)} - \frac{3}{2}\tau_{(i,3,k)} + \frac{1}{3}\tau_{(i,4,k)} \right) = 0, \quad (\text{B.4})$$

for all components of τ . The same difference stencil will be applied along the streamwise (radial) and spanwise (circumferential) directions, after replacing the indices as necessary.

For the components of the velocity gradient tensor appearing in the constitutive relations, the use of the fourth-order (or higher) stencils becomes imperative compared to the second-order accurate version to avoid errors stemming from a coarse discretization of the problem. In addition to this, the velocity components exist on a two dimensional grid which requires the pressure to be known at each point $(\hat{x}, \hat{y})$. Thus, the pressure obtained from VR must be projected onto the two dimensional grid via an interpolation scheme. Therefore, two levels of numerical error exist, the first in the derivative itself and the second in the projection phase. The latter can be avoided if the nodes along $\hat{x}$ coincide for the pressure, velocity and polymer stress components.

The convective operator appearing in the constitutive relations (including the function $1/f$ appearing in the FENE-CR and FENE-P constitutive relations), which are hyperbolic in nature, is treated via Eqs. B.1a-B.1e that comprise a hybrid total value diminishing (TVD) scheme, where, $\psi(r_{i,j,k})$ is the Van-Albada flux limiter [121]. For sliding surfaces, the results did not vary significantly between the choice of the flux limiter, but for smoothly varying functions it is recommended to use a mathematically smooth function as the limiter. For surfaces subjected to vertical motion (squeezing) the choice of the limiter had a strong influence. The use of a TVD scheme for the convective terms guarantees monotonicity which is a critical requirement for viscoelastic flows that exhibit strong gradients in the stress, especially within thin film flows.

Appendix C

Semi-implicit numerical approach for two-dimensional channels

In addition to a high-order accurate and monotone discretization of the differential equations, we also satisfy the requirement for numerical stability by enforcing a semi-implicit treatment of the discretized constitutive relations. For the compact representation Eq. 6.8a-c,

$$\epsilon W i \left(\frac{\tau_{xx}^{t+1} - \tau_{xx}^t}{\Delta t} + U \frac{\partial \tau_{xx}^{t+1}}{\partial \hat{x}} + V \frac{\partial \tau_{xx}^{t+1}}{\partial \hat{y}} - 2\tau_{xx}^{t+1} \frac{\partial u}{\partial x} \right) = \epsilon W i 2\tau_{xy}^t \frac{\partial u}{\partial y} - F^t \tau_{xx}^t, \quad (\text{C.1a})$$

$$\epsilon W i \left(\frac{\tau_{xy}^{t+1} - \tau_{xy}^t}{\Delta t} + U \frac{\partial \tau_{xy}^{t+1}}{\partial \hat{x}} + V \frac{\partial \tau_{xy}^{t+1}}{\partial \hat{y}} \right) \quad (\text{C.1b})$$

$$= f^t G^t (1 - \beta) \frac{\partial u}{\partial y} + \epsilon W i \left(\tau_{yy}^t \frac{\partial u}{\partial y} + \tau_{xx}^t \frac{\partial v}{\partial x} \right) - F^t \tau_{xy}^t,$$

$$\begin{aligned} & \epsilon W i \left(\frac{\tau_{yy}^{t+1} - \tau_{yy}^t}{\Delta t} + U \frac{\partial \tau_{yy}^{t+1}}{\partial \hat{x}} + V \frac{\partial \tau_{yy}^{t+1}}{\partial \hat{y}} - 2\tau_{yy}^{t+1} \frac{\partial u}{\partial y} \right) \quad (\text{C.1c}) \\ & = 2f^t G^t (1 - (1 - \beta)) \frac{\partial v}{\partial y} + 2\epsilon W i \tau_{xy}^t \frac{\partial v}{\partial x} - F^t \tau_{yy}^t + H^t, \end{aligned}$$

where t denotes the time step and the functions F , G and H are provided in Tab. 6.1. The terms in Eqs. C.1 at the time level t and $t + 1$ are treated explicitly and implicitly, respectively. Therefore, we have for each component of the stress

a system of algebraic equations

$$\mathbf{A}\tau = \mathbf{b}^t, \quad (\text{C.2})$$

where, $\mathbf{A}$ is the coefficient matrix of size $(N_x N_y \times N_x N_y)$, N_x and N_y are the number of nodes along the streamwise and cross-film direction, respectively and $\mathbf{b}^t$ is the forcing vector at time level t . The coefficient matrix is sparse and relatively large that causes the inefficiency of direct methods. For this reason, we utilize the gradient descent solver (BiPCGStab [89]), which is an efficient iterative approach, for the numerical solution of Eq. C.2.

A similar system of equations will be obtained for the pressure, Eq. 6.7, of size $N_x \times N_x$ which is not tri-diagonal because of the (i) fourth-order discretization and (ii) periodic boundary conditions. This system is also sparse in nature, although the size is comparatively smaller and, therefore, is solved iteratively as well. The iterations within the solvers are terminated upon satisfying the second norm

$$\|E^{t+1,k+1} - E^{t+1,k}\| \leq 10^{-12}, \quad (\text{C.3})$$

and steady state for the system stress components is detected by satisfying the second norm of neighboring time levels

$$\|\tau_{ij}^{t+1} - \tau_{ij}^t\| \leq 10^{-12}. \quad (\text{C.4})$$

We make a few remarks on the semi-implicit treatment of the constitutive relations. The equations can be readily treated via a purely explicit approach. However, for high eccentricity ratios ($e > 0.9$) the explicit approach rapidly becomes inefficient because the stable time step size reduces significantly. The same can be said for large and extremely small values of Wi saramito2014modified. Furthermore, the non-linear constitutive relations apply additional restrictions on the maximum time step size which renders the numerical solution slow for the purely explicit case.

However, in the semi-implicit explicit (IMEX) case the time step size is large enough to achieve solutions within a few minutes even for very large degree of freedoms. For example, the numerical solution of the semi-implicit Oldroyd-B is

stable for $\Delta t = 0.1$ up to $e = 0.99$ and $Wi = 50$ as opposed to the explicit step size $\Delta t = 10^{-4}$ for $e < 0.5$ and $Wi = 50$ for a mesh size 512×32 . The IMEX can be superseded by a fully implicit approach which removes the dependency on the time step entirely. This can enable even faster computation but at the cost of numerical complexity, requiring advanced programming techniques. Nonetheless, the latter appears to be getting traction due to its overall efficiency [94].

