COMBINATORIAL MORSE HOMOLOGY CALCULATIONS ON $SO(n)$

by

Turan Karakurt

B.S., Mathematics, Ankara University, 2015

M.S., Mathematics, Boğaziçi University, 2018

Submitted to the Institute for Graduate Studies in

Science and Engineering in partial fulfillment of

the requirements for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Boğaziçi University

2025

ACKNOWLEDGEMENTS

I would like to thank my advisor Prof. Ferit Öztürk of the Department of Mathematics at Boğaziçi University. He has profoundly inspired me with his perspective on life and his unparalleled geometric intuition.

I would like to thank Assoc. Prof. Ümit Işlak of the Department of Mathematics at Boğaziçi University. He has always taken care of me without hesitation.

I would like to thank to Assoc. Prof. Craig van Coevering and Prof. Arzu Boysal for their excellent Differential Geometry and Lie Groups courses.

I would also like to thank to my thesis committee members Prof. Yıldıray Ozan, Prof. Mahir Bilen Can and Prof. Ali Ulaş Özgür Kişisel.

I would thank to my dear friends Hasan Hüseyin Top, Şeyma Karadereli, Recep Altar Çiçeksiz, Arda Başarıcı and Oğuz Yılmaz for their companionship throughout my graduate school.

I would thank to Erol Barut. Without his brotherhood, I would not finish this study.

Finally, I am grateful to my family, especially my elder sister Ceren Karakurt; for their unconditional love and support.

ABSTRACT

COMBINATORIAL MORSE HOMOLOGY CALCULATIONS ON $SO(n)$

In this thesis we study the gradient flows of a specific Morse function $f_{A,B}$ on the special orthogonal group $SO(n)$. We determine the exact number of flows. We also find the exact conditions of existence of gradient flows between the critical points with consecutive indices. Finally, we put an order with a combinatorial setup on the critical point set. Thanks to this order, boundary maps can be written easily and we implement an algorithm to calculate the $\mathbb{Z}_2$-homology.

ÖZET

$SO(n)$ ÜZERİNDE KOMBİNATORİK MORSE HOMOLOJİ HESAPLAMALARI

Bu tezde spesiyal ortogonal üzerinde tanımlanan bir Morse fonksiyonun düşüm eğrilerini araştırdık. Bu eğrilerin tam olarak kaç tane olduğunu ve hangi kritik noktalar arasında var olduklarını ispatladık. Son olarak da kritik noktalar kümesi üzerine bir sıralama koyarak sınır fonksiyonlarının matris karşılıklarını yazdık. Bu sıralama ile birlikte Z_2-homoloji hesabı için bir algoritma oluşturduk.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

$\hat{\mathcal{M}}(x, y)$	Moduli space between critical points x and y
$N_G(H)$	Normalizer of H in G
$O(n)$	Real orthogonal group
$SO(n)$	Real special orthogonal group
$\mathfrak{so}(n)$	Real skew symmetric matrices

1. INTRODUCTION

Morse functions on homogeneous spaces and symmetric spaces have been analyzed for a long time of period, dating back to the paper "Nondegenerate Critical Manifolds" by R.Bott in 1954 [22]and later "Applications of the Theory of Morse to Symmetric Spaces" by R.Bott and H.Samelson in 1958 [3]. These works show how the "distance functions" or analogously, energy functionals and the height functions on homogeneous/symmetric spaces serve as Morse-Bott functions and they describe the topology of these spaces by constructing the corresponding cell decompositions. M.Atiyah's "Convexity and Commuting Hamiltonians" (1982) [10] and J.J. Duistermaat, J. Kolk, V. Varadarajan's "Functions, Flows and Oscillatory Integrals on Flag Manifolds and Conjugacy Classes in Real Semisimple Lie Groups" (1983) [12] papers also include the function defined on the adjoint orbit of a compact Lie group and given by $K(X,-)$ for fixed X where K is the Killing form.

In the study of H.I. Bozma, W.D Gillam and F.Öztürk [15], it is shown that the quadratic trace function on the orthogonal group $O(n)$ given by $f_{A,B}(X) = \mathrm{tr}(AXBX^T)$ is a Morse-Bott function for fixed symmetric matrices A and B. Parallel to the results of T.Frankel [16], they show that critical submanifolds of $O(n)$ diffeomorphic to disjoint union of Grassmannians. On the other hand, when both A and B have distinct eigenvalues then the function $f_{A,B}$ is a Morse function with a critical point set which is actually a subgroup of $O(n)$. This subgroup consists of signed permutation matrices, in other words, it is the hyperoctahedral group B_n.

Of course the restriction of $f_{A,B}$ to the special orthogonal group $SO(n)$ is also a Morse/Morse-Bott function and the critical point set (subgroup) $Cr(f_{A,B})$ is the signed permutation matrices with determinant 1. This group is an index 2 subgroup of B_n but its description differs by the parity. Independent from the parity, or the dimension of $SO(n)$ this subgroup has both algebraic and combinatorial properties: any critical point can be described by a product of a diagonal sign matrix and a unsigned permutation matrix in a unique way. Moreover, the Morse index of a critical point is given by the inversion number of the corresponding unsigned permutation matrix. With these properties, our aim is to calculate the Morse homology of $SO(n)$ in $\mathbb{Z}_2$ coefficients with respect to the function $f_{A,B}$ and the Riemannian metric given by the minus of trace form.

Since the Morse homology is based on the idea of "counting flows" we need to determine the gradient flows of $f_{A,B}$ on $SO(n)$, and how they appear between the critical points with consecutive indices. However, the number of critical points $|Cr(f_{A,B})| = 2^{n-1}n!$ hence, predicting the gradient flows or even determining the number of gradient flows has a very high complexity.

The main result of the present work is the determination of the gradient flow structure. More precisely we prove:

Theorem 1.1 (See Theorem 5.1 and 5.2)**.** *(i) Assume that $x = S\sigma$ is a critical point of $f_{A,B}$. Call σ for the corresponding unsigned permutation matrix. The number of gradient flows starting from x is equal to twice of the number of edges between σ and the permutations with inversion number $inv(\sigma) - 1$ in the Bruhat graph of the symmetric group.*

(ii) Assume that x and y are critical points with consecutive indices. Also assume that their corresponding permutation matrices are connected with an edge in the Bruhat graph. Then there is a transposition (i, j) which increases inversion number by 1 and send the unsigned part of x to the unsigned part of y. Moreover there exists a gradient flow between x and y if and only if y is obtained by rotating $\pm\pi/2$ degrees the plane spanned by i'th and j'th rows of x.

We prove this theorem in Chapter 5, Theorem 5.1 and Theorem 5.2.

Before obtaining above results we analyze the Morse functions over flag manifolds. Since our aim is studying on $SO(n)$, we are interested in the case of complex and real complete flag manifolds rather than for an arbitrary parabolic subgroup P of a given semisimple Lie group G and the quotient G/P. More explicitly, we can identify complex flag manifolds as adjoint orbits of the action of $SU(n)$ on its Lie algebra $\mathfrak{su}(n)$.

In Chapter 3, first we analyze the distance functions in more general sense, as in the J.Milnor's "Morse Theory" book [1]and then we describe the (height) function [2] with respect to the trace form on the adjoint orbit $SU(n).B$ given by

$$f_A : SU(n).B \to \mathbb{R} \tag{1.1}$$

$$x \mapsto \langle A, X \rangle \tag{1.2}$$

where A and B are purely imaginary, diagonal and regular. With these restrictions f_A become a Morse function.

R.R. Kocherlakota's 1995 paper "Integral Homology of Real Flag Manifolds and Loop Spaces of Symmetrical Spaces" [4] describes the gradient flows for the function f_A restricted to the real flag manifold. To do that, he defines a Cartan involution τ on a simply connected, compact Lie group, then sees the real flag manifold as the adjoint orbit of the fixed point set of τ. If we apply these to $SU(n)$ with the involution given by complex conjugation, then the fixed point set is $SO(n)$ hence $SO(n).B \subseteq SU(n).B$. Kocherlakota explains how the Morse theory on complex flag manifold $SU(n).B$ can be transferred onto the real flag manifold $SO(n).B$. He describes the gradient flows with respect to the restriction of the Kähler metric which is defined on the complex flag manifold and gives the exact number of flows. He also introduce a way to compute integral homology of real flag manifolds with only using the root systems and some parameters defined with respect to the Weyl group W. However, it turns out to be there is an ambiguity about the signs of flows and the integral homology results of Kocherlakota. Much later L.Rabelo and L.A.B. San Martin calculate integral homology of real flag manifolds in their 2019 paper "Cellular Homology of Real Flag Manifolds" [21]. In Chapter 4, we cover the results of Kocherlakota on the gradient flows on real flag manifolds.

In Chapter 5, we try to combine the datum of gradient flows on flag manifolds with the $SO(n)$ case. We realize that the function $f_{A,B}$ on $SO(n)$ and the function f_A on real flag manifold are related by the quotient map between $SO(n)$ and the adjoint orbit $SO(n).B$, i.e, $f_A = f_{A,B} \circ \pi$ with particular choices of the matrices A and B. Moreover we show that the gradient flows on $SO(n)$ (not for the metric defined as the trace form) can be obtained from the lifts of the gradient flows on the real flag manifold. To do that we use the Riemannian metric on $SO(n)$ defined as the pullback of the restriction of the Kähler metric on real flag manifold and we show that the quotient map is actually smooth covering map and a Riemannian submersion with respect to these metrics. Thus we prove our main results.

In Chapter 6, we introduce an order on the critical point set of $f_{A,B}$ and we construct the boundary maps with respect to this order. We see that the matrices correspond to the boundary maps consists of 2^{n-1} by 2^{n-1} symmetric block matrices in this order. As an example, we describe the gradient flows on $SO(3)$ in figures and we write the boundary maps in this case. For higher dimensions $(SO(4), SO(5))$ we collect our Sage based calculations in the website [23].

2. PRELIMINARIES

In this chapter we briefly mention about the necessary concepts and tools that are going to be used in following chapters.

2.1. Semisimple Lie Theory

First, we are going to explain some basic facts about Lie theory. Assume that G is a Lie group and $\mathfrak{g}$ is its Lie algebra. Details can be found in [5] and [9].

2.1.1. Killing Form and Cartan Decomposition

Definition 2.1. *Killing form K on $\mathfrak{g}$ is a symmetric bilinear form defined as*

$$K(X,Y) \mapsto tr(ad(X) \circ ad(Y)) \tag{2.1}$$

.

Lie algebra $\mathfrak{g}$ is called semisimple if it has no solvable ideals. Fortunately, there is a relatively easy way to determine whether the Lie algebra $\mathfrak{g}$ is semisimple or not:

Theorem 2.1. *Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form B is non-degenerate on $\mathfrak{g}$.*

Killing form does not only provide necessary information about the semisimplicity of $\mathfrak{g}$. Besides, it gives us a Morse function with the proper choice as to be appeared in the next chapter.

Now assume that $\mathfrak{g}$ is semisimple. Hence $\mathfrak{g}$ can not consisted entirely nilpotent elements. Otherwise, by the Engel's theorem [9] $\mathfrak{g}$ would be nilpotent. Hence $\mathfrak{g}$ contains semisimple elements (Over a algebraically closed field being semisimple is equivalent to being diagonalizable).

If $\mathfrak{g}$ is a linear semisimple Lie algebra then for any $x \in \mathfrak{g}$ there exists x_s and x_n such that $x = x_s + x_n$ where x_s and x_n denotes the semisimple and nilpotent parts of x respectively (Jordan decomposition). Then, $ad(x_s + x_n) = ad(x_s) + ad(x_n) = (adx)_s + (adx)_n$, which means that adjoint representation preserves the Jordan decomposition. So if $x \in \mathfrak{g}$ is semisimple (diagonalizable) then $ad(x)$ is also semisimple (diagonalizable).

Now consider a subalgebra $\mathfrak{t}$ in semisimple linear Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ which consists of semisimple elements. Then take a maximal subalgebra among such algebras. Then this maximal subalgebra generated by semisimple elements is abelian [9]. This subalgebra $\mathfrak{t}$ is called a Cartan subalgebra.

Fix a Cartan subalgebra $\mathfrak{t}$. [1] Since $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism, for any $x, y \in \mathfrak{t}$, $[adx, ady] = ad([x, y]) = 0$ since $[x, y] = 0$. Consequently, $ad(\mathfrak{t})$ is an abelian subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Thus, all the elements in $ad(\mathfrak{t})$ are simultaneously diagonalizable. So we have the same eigenspace decomposition for any adx. However, eigenvalues correspond to these eigenspaces may differ. Define $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x\}$ where α is a functional on $\mathfrak{t}$ and encodes the eigenvalue datum. The set of all $\alpha \in \mathfrak{t}^*$ is called the set of roots and denoted by Φ and $\mathfrak{g}_\alpha$ is called a root space. Therefore, we have the following decomposition, which is called the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{2.2}$$

Furthermore, we can restrict the Killing form onto Cartan subalgebra $\mathfrak{t}$, and in the semisimple Lie algebras this restriction still remains non-degenerate [9]. Thus, the function $K : \mathfrak{t} \to \mathfrak{t}^*$ given by $h \mapsto K(h, -)$ is an isomorphism. Under this isomorphism, there is unique t_α corresponds to the root α. Hence, we have the correspondence:

$$\Phi \longleftrightarrow \{t_\alpha | \alpha \in \Phi\} \tag{2.3}$$

[1] In this thesis we always pick Cartan subalgebra which consists of diagonal elements

and $\alpha(h) = K(t_\alpha, h)$.

Now we can list the orthogonality and integrality properties of roots and root spaces:

(i) For any $\alpha, \beta \in \Phi$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

(ii) If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

(iii) In the case of $\beta = -\alpha$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{t}$. Moreover, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = span(t_\alpha)$.

(iv) If $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ then, $[x, y] = K(x, y)t_\alpha$.

(v) If $\alpha \in \Phi$ and $x_\alpha \in \mathfrak{g}_\alpha$ then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ so that $x_\alpha, y_\alpha, [x, y] := h_\alpha$ span a three dimensional simple subalgebra of $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and the isomorphism is given by $x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(vi) If $x_\alpha, y_\alpha, h_\alpha$ are as in (v) then they satisfy the relations $[x_\alpha, y_\alpha] = h_\alpha, [h_\alpha, x_\alpha] = 2x_\alpha, [h_\alpha, y_\alpha] = -2y_\alpha$. Since $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha$, $\alpha(h_\alpha) = K(t_\alpha, h_\alpha) = 2$.

(vii) Again if $x_\alpha, y_\alpha, h_\alpha$ are as in (v), then

$$2 = K(t_\alpha, h_\alpha) = K(t_\alpha, K(x_\alpha, y_\alpha)t_\alpha) = K(x_\alpha, y_\alpha)K(t_\alpha, t_\alpha) \tag{2.4}$$

hence, $K(x_\alpha, y_\alpha) = \frac{2}{K(t_\alpha, t_\alpha)}$.

(viii) By (iv) and (vii) $h_\alpha = \frac{2}{K(t_\alpha, t_\alpha)} t_\alpha$.

2.1.2. Root Systems and Weyl Group

Consider a finite dimensional vector space over E over $\mathbb{R}$ endowed with positive definite symmetric bilinear form (,). Let r_α denotes the reflection through the hyperplane $\{\beta \in E | (\alpha, \beta) = 0\}$. Then,

$$r_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \tag{2.5}$$

.

Definition 2.2. *A subset Φ of E is called a roots system if:*

- *Φ is finite and $0 \notin \Phi$.*
- *$span(\Phi) = E$.*
- *For any $\alpha, \beta \in \Phi, \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.*
- *$r_\alpha(\Phi) \subseteq \Phi$.*
- *If $\alpha \in \Phi$ then the $-\alpha$ is the only nonzero multiple of α that is contained in Φ.*

If the last axiom of the previous definition is not satisfied then Φ is called restricted root system.

2.2. Morse Homology

A smooth function $f : M \to \mathbb{R}$ is called Morse function if at every critical point of f, Hessian is non-degenerate. Number of negative eigenvalues of the Hessian at the critical point is called the Morse index of a critical point. Critical points of a Morse function are isolated and in case M is closed, there are finitely many critical point. Number of critical points of a Morse function is bounded below by the dimension of homology of underlying manifold (i.e Morse inequalities: $b_k \leq c_k$). Moreover, number of critical points of index k, c_k, satisfy following equation called strong Morse inequalities,

$$\sum_{k=0}^{m}(-1)^{m-k}b_k \leq \sum_{k=0}^{m}(-1)^{m-k}c_k \quad \text{for all } m \geq 0 \tag{2.6}$$

In this section we follow [24].

In his 1982 paper [20], Edward Witten introduced Morse-Witten complex whose chain groups are generated by the critical points of a Morse function f and the boundary map is given by counting the number of (anti)-gradient flow of the function f. He also proved that homology of Morse-Witten complex is isomorphic to singular homology. We will give a short account for Morse-Witten complex and its homology so called Morse homology.

For an auxillary choice of Riemannian metric g , the gradient of a function f , ∇f , with respect to metric g is defined by following equation

$$g(\nabla^g f, X) = df(X)\forall X \in \mathfrak{X}(M) \tag{2.7}$$

And anti-gradient flow of f is given by

$$\begin{aligned} \frac{d\Phi^t(x)}{dt} &= -\nabla^g f \circ \Phi^t(x) \\ \Phi^0(x) &= x \end{aligned} \tag{2.8}$$

One can easily shows f is decreasing along anti-gradient flow lines. This has two important implications. First, every level set of f intersect a flow line at most once. Second, $\lim_{t\to\pm\infty} \Phi^t(x)$ is a critical point for every $x \in M$. First one is trivial and later is implied by the first and compactness.

Definition 2.3. *Let x, y be a critical point of f, stable and unstable manifolds of f at x are defined respectively as*

$$W^s(x) := \{p \in M : \lim_{t\to+\infty} \Phi^t(p) = x\} \text{ and } W^u(x) := \{p \in M : \lim_{t\to-\infty} \Phi^t(p) = x\}$$

and the set

$$\mathcal{M}(x, y) := U(x) \cap S(y) = \{p \in M : \lim_{t\to+\infty} \Phi^t(p) = y \wedge \lim_{t\to-\infty} \Phi^t(p) = x\}$$

is called space of connecting orbits.

Remark 2.1. *Let $p \in \mathcal{M}(x,y)$, then there exist unique gradient flow line $\gamma_p(t) = \Phi^t(p)$. So $\mathcal{M}(x,y)$ is the model space for space of gradient flow line flowing from x to y.*

Assume without loss of generality $f(x) > f(y)$ so that there exist a number $c \in (f(y), f(x))$. We know that $f^{-1}(c)$ will intersect every gradient flow in $\mathcal{M}(x,y)$ once thus the set $\hat{\mathcal{M}}_c(x,y) := \mathcal{M}(x,y) \cap f^{-1}(c)$ is in one-to-one correspondence with number of distinct gradient flow lines between x and y. For different choices of c we can identify the corresponding spaces by flow. Thus we will omit subscript c.

Remark 2.2. *Here is what we mean by distinct gradient flow. Let $\gamma(t)$ be a anti-gradient flow of f, and $t_0 \in \mathbb{R}$, $t_0 \cdot \gamma(t) := \gamma(t + T_0)$ is also a anti-gradient flow of f. We identify these two gradient flows.*

We are now ready to define Morse-Witten complex.

Definition 2.4. *Let g be a metric on M and f be morse function such that all its stable and unstable manifolds are intersecting transversally. The Morse-Witten chain complex associated to the Morse function f and the metric g in coefficient ring $\mathbb{Z}_2$, $(\mathcal{C}_\bullet(f,g), \partial_\bullet)$ is given by*

$$\mathcal{C}_k(f,g) := \bigoplus_{x \in Crit_k(f)} \mathbb{Z}_2 x \tag{2.9}$$

$$\partial_k x = \sum_{y \in Crit_{k-1}(f)} |\hat{\mathcal{M}}(x,y)| y \mod 2 \tag{2.10}$$

For the purpose of this study, we are interested in $\mathbb{Z}_2$ coefficient. However it is possible to define Morse-Witten complex in more general coefficient ring.

The transverse intersection of stable and unstable manifold is called Morse-Smale condition. a pair of morse function and metric satisfying it called Morse-Smale pair. Morse-Smale condition is a generic condition since transversality is generic. For Morse-Witten complex to be indeed a chain complex we need show that $\hat{\mathcal{M}}(x,y)$ is finite discrete set whenever index difference is one and should show that $\partial_{k-1} \circ \partial_k = 0$. We will start by investigating dimension of stable and unstable manifolds.

Theorem 2.2. *Let E_x^u to be sum of the negative eigenvalue space of hessian and E_x^s to be sum of the positive eigenvalue spaces for a critical point x. Then, unstable manifold $U(x)$ and $S(x)$ are embedded submanifold of M. Moreover,*

$$T_x U(x) = E_x^u \text{ and } T_x S(x) = E_x^s \tag{2.11}$$

Corollary 2.1. *Let x be a critical point of f, then*

$$\dim U(x) = codim S(x) = ind(x). \tag{2.12}$$

This is where transversality condition come handy. Transversality is sufficient condition for intersection of two submanifolds to be a submanifold. We summarize some implication of Morse-Smale condition in next theorem.

Theorem 2.3. *Let (f, g) be a Morse-Smale pair and x, y be critical points of f, then followings are true,*

(i) $\mathcal{M}(x, y)$ *is a submanifold.*

(ii) *if* $ind(x) > ind(y)$ *then* $\dim \mathcal{M}(x, y) = ind(x) - ind(y)$.

(iii) *if* $ind(x) \leq ind(y)$ *then* $\mathcal{M}(x, y) = \emptyset$ *unless* $x = y$.

This ensure that $\dim \hat{\mathcal{M}}(x, y) = 0$ whenever $ind(x) - ind(y) = 1$. Therefore it is a discrete set and finite if it is compact.

Proposition 2.1. *Let x,y be critical point of f and $ind(x) - ind(y) = 1$ then $\hat{\mathcal{M}}(x, y)$ is compact, thus it is finite.*

This ensure that boundary operator is well defined. We need a more generalized result to show that $\partial_k \circ \partial_{k-1} = 0$.

Definition 2.5. *Let (f, g) be a Morse-Smale pair and x, y be critical points of f such that $ind(x) - ind(y) = 2$. We say that a sequence in $p_k \in \hat{\mathcal{M}}(x, y)$ converges to broken orbit $(\gamma_1, \gamma_2) \in \hat{\mathcal{M}}(x, z) \times \hat{\mathcal{M}}(z, y)$ if p_k's representative in suitable level set converges to γ_1 and γ_2 with respect to Riemannian distance on M.*

We can give similar definition for higher difference in obvious manner but we only need it for index difference two.

Theorem 2.4. *Let (f, g) be a Morse-Smale pair and x, y be critical points of f such that $ind(x) - ind(y) = 2$. Then, every sequence in $\hat{\mathcal{M}}(x, y)$ has a subsequence converge to a broken trajectory.*

So if $ind(x) = k+1$ and $ind(y) = k-1$, $\hat{\mathcal{M}}(x, y) \cup \Big(\bigcup_{z \in Crit_k(f)} \hat{\mathcal{M}}(x, z) \times \hat{\mathcal{M}}(z, y) \Big)$ is compact with this topology.

Theorem 2.5. *Let (f, g) be a Morse-Smale pair and $x, y, z \in Crit(f)$ with indices $k-1, k, k+1$ respectively. Then there exist map*

$$\# : \hat{\mathcal{M}}(x, y) \times \hat{\mathcal{M}}(y, z) \times [r_0, \infty) \to \hat{\mathcal{M}}(x, z), ; (u, r, v) \to u \#_r v$$

satisfying

(i) *$\#$ is an embedding*
(ii) *$u \#_r v$ converge to broken trajectory (u, v) for all $u \in \hat{\mathcal{M}}(x, y), v \in \hat{\mathcal{M}}(y, z)$*
(iii) *Any sequence in $\hat{\mathcal{M}}(x, z)$ converging to a broken trajectory (u, v) has to be pass through $u \#_{[r, \infty)} v$*

The map in above theorem is called glueing map. The theorem implies the set of broken trajectories connecting x, z is actually boundary of connecting trajectories.

One can easily calculate

$$\partial_{k-1} \circ \partial_k(x) = \sum_{z \in Crit_{k-2}} \sum_{y \in Crit_{k-1}} |\hat{\mathcal{M}}(x,y)||\hat{\mathcal{M}}(y,z)|z \tag{2.13}$$

The coefficient of a critical point z is given by

$$|\partial \hat{\mathcal{M}}(x,z)| = |\bigcup_{y \in Crit_{k-1}} \hat{\mathcal{M}}(x,y) \times \hat{\mathcal{M}}(y,y)| = \sum_{y \in Crit_{k-1}} |\hat{\mathcal{M}}(x,y)||\hat{\mathcal{M}}(y,z)| \tag{2.14}$$

Which is number of element in the boundry of compact one dimensional manifold. Therefore it is even and vanishes in $\mathbb{Z}_2$ thus,

Theorem 2.6. *The Morse-Witten chain complex in $\mathbb{Z}_2$ coefficient $(\mathcal{C}_\bullet(f), \partial_\bullet)$ is indeen a chain complex i.e*

(i) $\partial_\bullet$ *is well defined.*

(ii) $\partial_{k-1} \circ \partial_k = 0$.

We will end this chapter with following two theorems:

Theorem 2.7. *For all pair (f^α, g^α) and (f^β, g^β) morse-smale pair there exist a natural homomorphism that preserves grading*

$$\Psi^{\alpha\beta} : H(\mathcal{C}_\bullet(f^\alpha, g^\alpha) \to H(\mathcal{C}_\bullet(f^\beta, g^\beta)) \tag{2.15}$$

satisfying,

(i) $(\Psi_\bullet^{\alpha\beta})^{-1} = \Psi_\bullet^{\beta\alpha}$.

(ii) $\Psi_\bullet^{\alpha\beta} \circ \Psi_\bullet^{\beta\gamma} = \Psi_\bullet^{\alpha\gamma}$.

The second theorem says there is no difference between Morse homology and the singular or cellular homology:

Theorem 2.8. *The homology of Morse-Witten complex of a Morse-Smale pair* (f, g) *is isomorphic to cellular homology of underlying manifold , i,e*

$$H(\mathcal{C}_{\bullet}(f, g)) \cong H^{cell}(M, \mathbb{Z}_2) \tag{2.16}$$

Following remark encapsulates the spirit of this thesis: Morse-Witten complex and the calculation of corresponding boundary maps are difficult to handle since it consists of the flow information. Ordinary differential equations which give the flows, in most cases, are very complex. However, in our case, combinatorics and the algebraic properties of the Lie group/homogeneous space make them reachable.

Remark 2.3. *Although Morse homology is isomorphic to cellular homology, it is impractical to calculate in most cases.*

3. MORSE FUNCTIONS ON COMPLEX FLAG MANIFOLDS

In this section, we will begin with a family of quadratic Morse functions on an arbitrary smooth embedded submanifold M in $\mathbb{R}^n$. It can be shown that the distance function on M to a point p is a Morse function for almost all $p \in \mathbb{R}^n$. Distance functions are very useful not only as examples of quadratic Morse functions but also in the sense that their Hessian matrices at critical points can easily be described along with the indices. After collecting necessary information about these functions, we will see that another family of quadratic Morse functions can be defined over complex flag manifolds by using the trace form on the Lie algebra $\mathfrak{su}$(n). Moreover, it will turn out that these quadratic Morse functions on flag manifolds are equal to some multiple of distance functions with a translation term. Then we will explain the stable/unstable manifolds in terms of Schubert calculus and finally we will give the full description of the gradient flows of Morse functions stemming from the trace form.

3.1. Morse Functions on Embedded Submanifolds

Assume that a smooth manifold M with dimension m is smoothly embedded in $\mathbb{R}^n$. In local coordinates $(u_1, u_2, ..., u_m)$ around some point in M, define this embedding as $x(u_1, u_2, ..., u_m) = (x_1(u_1, u_2, ..., u_m), x_2(u_1, u_2, ..., u_m), ..., x_n(u_1, u_2, ..., u_m))$ with smooth coordinate functions $x_1, x_2, ..., x_n$. Let $\mathbf{N}$ denote the total space of normal bundle of M, i.e,

$$\mathbf{N} = \{(q, v)|q \in M, v \in N_q(M)\} \tag{3.1}$$

Now we define the "end-point" function $E : \mathbf{N} \to \mathbb{R}^n$ as $E(q, v) = q + v$. which eventually leads us to have a Morse function.

Definition 3.1. *$p \in \mathbb{R}^n$ is called a focal point of M at q with multiplicity μ if $E(q,v) = p$ for some $v \in N_q(M)$ and the Jacobian of E $J_{(q,v)}(E)$ at (q,v) has nullity $\mu > 0$.*

We need a couple of definitions from differential geometry to describe $J_{(q,v)}(E)$ more explicitly.

Definition 3.2. *The first fundamental form of M with respect to the chosen coordinate system $(u_1, u_2, ..., u_m)$ and the embedding x is the symmetric $m \times m$ matrix defined as*

$$(g_{ij}) = (\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j})_{1 \le i,j \le m}. \tag{3.2}$$

Furthermore, the second fundamental form of M is a symmetric matrix (ℓ_{ij}) of vector valued functions defined as follows:

$$(\ell_{ij}) = \text{normal component of } \frac{\partial^2 x}{\partial u_i \partial u_j} \; . \tag{3.3}$$

The latter definition makes sense since $\frac{\partial^2 x}{\partial u_i \partial u_j}$ at point $q \in M$ is in $T_q(M) \bigoplus N_q(M)$.

Theorem 3.1. *$q + tv$ is a focal point of M at q with multiplicity μ if and only if the matrix $(g_{ij} - tv.\ell_{ij})$ is singular with nullity μ.*

Proof. Determine $n - m$ orthonormal vector fields $w_1(u_1, ..., u_m), ..., w_{n-m}(u_1, ..., u_m)$. Choose local coordinates $(u_1, ..., u_m, t_1, ..., t_{n-m})$ on the normal bundle $\mathbf{N}$ corresponding to the point $(\vec{x}(u_1, u_2, ..., u_m), \sum_{\alpha=1}^{n-m} t_\alpha w_\alpha(u_1, ..., u_m))$. Within these local coordinates function E is evaluated as

$$E(u_1, ..., u_m, t_1, ..., t_{n-m}) = x(u_1, u_2, ..., u_m) + \sum_{\alpha=1}^{n-m} t_\alpha w_\alpha(u_1, ..., u_m). \tag{3.4}$$

Then the Jacobian of E is of the form

$$\begin{bmatrix} & \vdots & \\ - & \frac{\partial x}{\partial u_i} + \sum_\alpha t_\alpha \frac{\partial w_\alpha}{\partial u_i} & - \\ & \vdots & \\ --- & w_\beta & --- \\ & \vdots & \end{bmatrix}$$

Now consider a matrix C with column vectors $\frac{\partial x}{\partial u_1}, ..., \frac{\partial x}{\partial u_m}, w_1, ..., w_{n-m}$. Since these vectors are linearly independent, C is invertible; therefore, multiplying by C does not change rank and nullity. Taking this into account, consider the matrix

$$J(E)C = \begin{bmatrix} (\frac{\partial x}{\partial u_i}.\frac{\partial x}{\partial u_j} + \sum_\alpha t_\alpha \frac{\partial w_\alpha}{\partial u_i}.\frac{\partial x}{\partial u_j}) & (\sum_\alpha t_\alpha \frac{\partial w_\alpha}{\partial u_\alpha}.w_\beta) \\ 0 & (\mathrm{Id})_{n-m \times n-m} \end{bmatrix}$$

It is clear that the nullity of $J(E)$ is equal to the nullity of the upper left hand block of $J(E)C$. Since w_α and $\frac{\partial x}{\partial u_j}$ are orthogonal

$$\frac{\partial}{\partial u_i}(w_\alpha \cdot \frac{\partial x}{\partial u_j}) = \frac{\partial w_\alpha}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + w_\alpha \cdot \frac{\partial^2 x}{\partial u_i \partial u_j} = 0 \quad . \tag{3.5}$$

Thus $w_\alpha.\frac{\partial^2 x}{\partial u_i \partial u_j} = -\frac{\partial w_\alpha}{\partial u_i}.\frac{\partial \vec{x}}{\partial u_j}$. Since w_α is a normal vector, $w_\alpha.\frac{\partial^2 x}{\partial u_i \partial u_j} = w_\alpha.\ell_{ij}$. Consequently, upper left hand block of $J(E)C$ is equal to the matrix $(g_{ij} - \sum_\alpha t_\alpha w_\alpha.\ell_{ij})$. □

Now, assume that all the $\frac{\partial x}{\partial u_i}$'s are orthonormal. This implies that the matrix (g_{ij}) is the identity matrix. Moreover, assume that the line ℓ contains all the points $q + tv$ for fixed unit normal vector v. Then,

Corollary 3.1. *The focal points of M at q along ℓ are $q+(1/K_i)v$, $1 \leq i \leq m$, $K_i \neq 0$ where K_i's are the eigenvalues* [2] *of the matrix $(v.\ell_{ij})$.*

Proof. By the previous lemma, $q+tv$ is a focal point if and only if the matrix $(g_{ij}-tv.\ell_{ij})$ is singular. Since (g_{ij}) is the identity matrix, for an element y in the null space of $(g_{ij} - tv.\ell_{ij})$, $y = (g_{ij})(y) = t(v.\ell_{ij})(y)$. Thus $q + tv$ is a focal point if and only if $1/t$ is an eigenvalue of $(v.\ell_{ij})$. □

Next, consider the distance function $s_p : M \to \mathbb{R}$ for a fixed $p \in \mathbb{R}^n$, $s_p(u_1, ..., u_m) = ||x(u_1, ..., u_m) - p||^2 \quad = x \cdot x - 2x \cdot p + p \cdot p$. When we look at the partial derivatives we obtain that $\frac{\partial s_p}{\partial u_i} = 2\frac{\partial x}{\partial u_i} \cdot (x - p)$. This implies,

Lemma 3.1. *q is a critical point of the function s_p if and only if $p - q \in N_q(M)$.*

Furthermore, the (i, j) entry of the Hessian of s_p at a critical point q is (provided with $p = q + tv$; v is a unit normal vector)

$$\frac{\partial^2 d_p}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_i}.\frac{\partial x}{\partial u_j} + \frac{\partial^2 x}{\partial u_i \partial u_j}.(x-p)\right) \tag{3.6}$$

$$= 2(g_{ij})_{ij} - 2t(v.\ell_{ij})_{ij} \tag{3.7}$$

Above equation leads us to the following:

Lemma 3.2. *The following statements are equivalent:*

(i) q is a non-degenerate critical point of s_p.

(ii) $(g_{ij} - tv.\ell_{ij})$ is non-singular.

(iii) p is not a focal point of M at q.

[2] K_i's are called the principal curvatures of M.

Finally, if (g_{ij}) is the identity matrix and $2(g_{ij} - tv.\ell_{ij})$ is non-singular, then for a negative eigenvalue λ and a corresponding eigenvector w, we have; $(2g_{ij} - 2tv.\ell_{ij})(w) = \lambda w$. Consequently, $\frac{2-\lambda}{2t}$ is an eigenvalue for $(v.\ell_{ij})$ thus equals K_r for some r. Then $2 - 2tK_r < 0$ since $\lambda < 0$. Equivalently $1/t < K_r$ and as a result, we conclude with:

Corollary 3.2. *If q is a non-degenerate critical point of s_p, then the index of q is equal to the number of eigenvalues of $(v.\ell_{ij})$ which are greater than $1/t$.*

3.2. Morse Functions on Complex Flag Manifolds

The next concept that we will be interested in is the complete complex flag manifolds. Although, all the upcoming statements, with proper adjustments are valid for any type of complex flag manifolds [3], we will stick to the complete ones. Similar treatment for complex Grassmanians can be found in [2].

Fix a diagonal matrix A in $\mathfrak{su}(n)$ with distinct eigenvalues and consider the adjoint orbit $SU(n).B = \{xBx^{-1} | x \in SU(n)\}$. Also denote the subspace of all purely imaginary diagonal traceless matrices by $\mathfrak{t}$ which is a Cartan subalgebra. Moreover, fix the inner product $\langle \ , \ \rangle$ on $\mathfrak{su}(n)$ given by $\langle X, Y \rangle = -\mathrm{Retr}(XY)$ which is non-degenerate, associative and Ad-invariant. Denote $SU(n)$ by G and denote $\mathfrak{su}(n)$ by $\mathfrak{g}$ for the rest of this section.

Next theorem explains under which conditions the inner product on $\mathfrak{su}(n)$ gives us a Morse function. Note that this function resembles the height function or the distance function on the Euiclidean space.

Theorem 3.2. *Suppose that $A \in \mathfrak{t}$ has distinct eigenvalues and define a function f_A as*

$$f_A : G.B \to \mathbb{R} \tag{3.8}$$

$$X \mapsto < A, X > \quad . \tag{3.9}$$

Then f_A is a Morse function.

Remark 3.1. *It is also possible to choose the matrix A non-diagonal. Indeed, since A is unitarily diagonalizable, there exists $y \in G$ such that $yAy^{-1} = D$. Then $f_D(X) =< D, X >=< yAy^{-1}, X >=< A, y^{-1}Xy >= f_A(y^{-1}Xy)$ by the Ad-invariance of the trace form. This implies $f_A = f_D \circ Ad(y)$. Hence f_A equals f_D composed with a diffeomorphism.*

Before the proof of Theorem 3.2, we state some facts about the adjoint orbit $G.B$ and the function f_A.

Lemma 3.3. *The tangent space $T_x(G.A)$ at the point x is $[\mathfrak{g}, x]$ and the normal space $N_x(G.A)$ is equal to $\{Y \in \mathfrak{g} | [Y, x] = 0\}$.*

Proof. Since the compact Lie group G acts on $G.A$ smoothly and transitively, for any $X \in T_x(G.A)$ there exists $Y \in \mathfrak{g}$ such that $\frac{d}{dt}(\mathrm{Ad}(\exp(tY)))\,|_{t=0}(x) = X$ based on the fact that the quotient map between G and $G.A$ is actually a submersion [18]. Besides, $\frac{d}{dt}(\mathrm{Ad}(\exp(tY)))\,|_{t=0}(x) = ad(\frac{d}{dt}(exp(tY)\,|_{t=0})(x) = [Y, x]$ so $T_x(G.A) \subseteq [\mathfrak{g}, x]$. Last equation also proves the converse direction since, for any $Y \in \mathfrak{g}$; $[Y, x]$ is the derivative of a curve $Ad(exp(tY))(x)$ in $G.A$ at $t = 0$.

On the other hand, if $Y \in N_x(G.A)$, then $\langle Y, T_x(G.A)\rangle = \langle Y, [\mathfrak{g}, x]\rangle = \langle Y, [x, \mathfrak{g}]\rangle = \langle [Y, x], \mathfrak{g}\rangle = 0$. Thus $[Y, x] = 0$ since the trace form is non-degenerate. $\square$

In the light of Lemma 3.3, we can make further observations:

Corollary 3.3. *Assume that a matrix x in $G.A \cap \mathfrak{t}$ has distinct eigenvalues. Then $N_x(G.A) = \mathfrak{t}$*

Proof. If the diagonal matrix x has distinct eigenvalues, then it is easy to check that any matrix which commutes with x is diagonal. Conversely, since $\mathfrak{t}$ is abelian $t \subseteq N_x(G.A)$. □

It can easily be seen that the directional derivative $D_X f_A$ of f_A in the direction of $X \in T_x(G.B)$ is $\langle X, A \rangle$ [2].

Lemma 3.4. $Cr(f_A) = \{q \in G.B | [q, A] = 0\} = G.B \cap \mathfrak{t}$.

Proof. If q is a critical point of f_A then all the directional derivatives are zero, hence $< X, A >= 0$ for any $X \in T_q(G.B)$. By the Lemma 3.3, $T_q(G.B) = [\mathfrak{g}, q]$ so, $X = [Y, q]$ for some $Y \in \mathfrak{g}$. This leads us to have $\langle X, A \rangle = \langle [Y, q], A \rangle = \langle Y, [q, A] \rangle = 0$ and consequently $[q, A] = 0$ due to the non-degenerateness of the trace form. Also, the second equality follows from A has distinct eigenvalues hence critical point q must lie in $\mathfrak{t}$. □

In this setting of critical points, we can make a further observation in terms of the Weyl group.

Lemma 3.5. *$Cr(f_A) = G.B \cap \mathfrak{t} = W.B$ where W is the Weyl group of $G = SU(n)$ and $W.A$ denotes the Weyl group orbit of A.*

Proof. Let $q = gAg^{-1}$ be in $G.A \cap \mathfrak{t}$ and characterize W as $N_G(T)/T$ where $T = \exp(\mathfrak{t})$. For one direction, it is enough to show that $gXg^{-1} \in \mathfrak{t}$ for any $X \in \mathfrak{t}$. The exponential map $exp : \mathfrak{t} \to T$ is surjective (T is both connected and compact) and $\exp(gXg^{-1}) =$

$g\exp(X)g^{-1} \in T$ so $g \in N_G(T)$. However, $[A, gXg^{-1}] = g[g^{-1}Ag, X]g^{-1} = 0$ because both $g^{-1}Ag$ and X are in $\mathfrak{t}$. This implies $[A, gXg^{-1}] = 0$ and since the centralizer of A only contains diagonal elements, gXg^{-1} must be in $\mathfrak{t}$. For the reverse direction, assume $g \in N_G(T)$. Then g induces diffeomorphism over T by conjugation, thus, its derivative at identity $\mathrm{Ad}(g) : \mathfrak{t} \to \mathfrak{t}$ sends A to gAg^{-1} which shows that gAg^{-1} must lie in $\mathfrak{t}$. $\square$

Now consider the function $s_A : G.B \to \mathfrak{g}$ defined as in the previous section. (We replace M by $G.B$, $\mathbb{R}^n$ by $\mathfrak{g}$ and the Euclidean inner product by the trace form.) Since $s_A(x) = \langle x, x\rangle - 2\langle A, x\rangle + \langle A, A\rangle$ we deduce that $f_A(x) = \frac{1}{2}(s_A(x) - \langle x, x\rangle - \langle A, A\rangle)$. Furthermore, $x \in G.A$ thus there exists $g \in G$ such that $x = gAg^{-1}$ thus; $\langle x, x\rangle = \langle gAg^{-1}, gAg^{-1}\rangle = \langle A, A\rangle$. Hence f_B is a scalar multiple of s_B plus a constant. As a result, we conclude that f_A and s_A have the same critical points and f_A is a Morse function if and only if s_A is a Morse function.

Furthermore, we know that B a is critical point of f_A since $[A, B] = 0$. Now assume that $q = gBg^{-1}$ is a critical point of f_A. Moreover, similarly with Remark 3.1; $f_A(x) = \langle A, x\rangle = \langle hAh^{-1}, hxh^{-1}\rangle = f_{hAh^{-1}}(hxh^{-1})$ for any $x \in G.B$ and $h \in G$. Thus B is a non-degenerate critical point of f_A if and only if q is also a non-degenerate critical point of $f_{gAg^{-1}}$. This implies that it is enough to only look at the non-degenerateness of critical point B to prove Theorem 3.2.

Now we can restate Lemma 3.2:

Lemma 3.6. *The following statements are equivalent:*

(i) *B is non-degenerate critical point of f_A.*

(ii) *B is non-degenerate critical point of s_A.*

(iii) *$A - B$ is not a focal point of $G.B$ at B.*

4. GRADIENT FLOWS ON REAL FLAG MANIFOLDS

In this chapter we will describe the gradient flows of a specific Morse function over a complete real flag manifold. This function will be defined similarly to f_A as in the previous chapter. Kocherlakota [4] explains the nature of the gradient flows and describes the stable/unstable manifolds of the corresponding critical points. Moreover he describes the integer homology of (generalized) real flag manifolds in terms of the root system of corresponding semisimple Lie algebra with some ambiguity of the signs. We will follow Kocherlakota's paper [4] throughout this chapter. However, we will only be interested in complete flag manifolds.

4.1. Root Spaces

Assume that $\tau : SU(n) \to SU(n)$ is given by $\tau(X) = \overline{X}$. Then τ is an involution with fixed point set $K = SO(n)$. Moreover, the differential $\tau_* : \mathfrak{su}(n) \to \mathfrak{su}(n)$ is also involution with -1 and $+1$ eigenspaces. It can easily be shown that the -1 eigenspace corresponds the the subspace $i\{$real traceless symmetric matrices$\}$ and the $+1$ eigenspace corresponds to the Lie algebra $\mathfrak{so}(n)$ of $K = SO(n)$.

For the rest of this chapter, assume that $G = SU(n)$, $\mathfrak{g} = su(n)$, $K = SO(n)$, $\mathfrak{k} = \mathfrak{so}(n)$, $\mathfrak{p}$ denotes the -1 eigenspace defined above (hence $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$) and lastly $\mathfrak{t}$ denotes the Cartan subalgebra in $\mathfrak{g}$ consisting of purely imaginary traceless diagonal matrices.

Here K acts on $\mathfrak{p}$ via conjugation and the orbits of this action are real flag manifolds. Assume that $B \in \mathfrak{t}$ is fixed as in the previous chapter. Specifically, we deal with the orbit space $K.B \subseteq \mathfrak{p}$ which is the real complete flag manifold.

Now, consider the complexification $\mathfrak{g}_{\mathbb{C}}$=$\mathfrak{sl}(n,\mathbb{C})$ of $\mathfrak{g}$ and the root space decomposition with respect to the $\mathfrak{t}_{\mathbb{C}}$:

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha\in\Phi} E_\alpha \tag{4.1}$$

where Φ denotes the set of roots and E_α's denote the corresponding root spaces. More explicitly, let α_{ij} denote the root defined by $\alpha_{ij}\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right) = \lambda_i - \lambda_j$. Then, Φ entirely consists of this type of roots [5]. Consequently, simple roots are $\{\alpha_{12}, ..., \alpha_{n-1,n}\}$. Now fix a diagonal matrix A with entries $i\lambda_1, ..., i\lambda_n$ along the diagonal with $\lambda_1 > \lambda_2 > ... > \lambda_n$. Then the set of positive roots Φ_+ is $\{\alpha_{ij}|i<j\}$ and the set of negative roots Φ_- is $\{\alpha_{ij}|i>j\}$ with respect to A. Note that this matrix A is compatible with the setting in the previous chapter, i.e. f_A is a Morse function on the complex flag manifold.

We also have a similar decomposition for $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha \tag{4.2}$$

where

$$\mathfrak{g}_\alpha = (E_\alpha + E_{-\alpha}) \cap \mathfrak{g} \quad . \tag{4.3}$$

More precisely, $\mathfrak{g}_\alpha$ is a 2-dimensional real vector space spanned by matrices in $\mathfrak{p}\setminus\mathfrak{t}$ and matrices in $\mathfrak{k}$. Moreover, $\mathfrak{g}_\alpha^{\mathbb{C}} = E_\alpha + E_{-\alpha}$ and $\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$.

Corollary 4.1. *If $x \in \mathfrak{t}\cap G.B$ is regular then $T_x(G.B) = \sum_{\alpha\in\Phi}\mathfrak{g}_\alpha$*

Proof. By Corollary 3.3, $N_x(G.B) = \mathfrak{t}$ and the decomposition (4.2) is an orthogonal decomposition with respect to the trace form. □

Furthermore, suppose that $\mathfrak{b}_+$ is the subspace $\mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha\in\Phi_+} E_\alpha$ and similarly $\mathfrak{b}_-$ is the subspace $= \mathfrak{t}_C \oplus \sum_{\alpha\in\Phi_-} E_\alpha$. Corresponding subgroups of $G_C = SL(n,\mathbb{C})$ are the Borel subgroups denoted by B_+ and B_- and in our case, these subgroups correspond to the upper triangular matrices in $SL(n,\mathbb{C})$ and the lower triangular matrices in $SL(n,\mathbb{C})$ respectively.

Before dealing with the topology of real flag manifolds we will collect some structural information about the complex flag manifolds.

4.2. Structures

It is known that, there is a one to one correspondence between the G-invariant almost complex structures on a flag manifold and the choice of the positive roots [6]. For a regular $x \in \mathfrak{t} \cap G.B$, $(\mathfrak{g}/\mathfrak{t})^{\mathbb{C}} = (T_x(G.B))^{\mathbb{C}} = \sum_{\alpha\in\Phi} E_\alpha$ by Corallary 4.1. Then set $T^{(1,0)} = \sum_{\alpha\in\Phi_+} E_\alpha$ and $T^{(0,1)} = \sum_{\alpha\in\Phi_-} E_\alpha$. Define a basis of $T_x(G.A)$ as $X_{ij} = E_{ij} - E_{ij}^T$ and $Y_{ij} = i(E_{ij} + E_{ij}^T)$ where E_{ij} is a matrix with ij entry is 1 and the rest of is 0. Also note that for $i < j$, E_{ij}'s span $T^{(1,0)}$ and for $i > j$, E_{ij}'s span $T^{(0,1)}$. If we set J as $J_x(X_{ij}) = Y_{ij}$ and $J_x(Y_{ij}) = -X_{ij}$ then $J_x^2 = -I$ and for $i < j$ $J_x(E_{ij}) = J_x(\frac{X_{ij}-iY_{ij}}{2}) = (\frac{Y_{ij}+iX_{ij}}{2}) = iE_{ij}$. Hence $T^{(1,0)}$ is the $(+i)$-eigenspace. Similarly, $T^{(0,1)}$ is the $(-i)$-eigenspace. Since G acts on $G.B$ transitively, we can extend J to whole manifold.

Moreover, this almost complex structure is integrable: vanishing of the Nijenhuis tensor is equivalent to checking $[T^{(1,0)}, T^{(1,0)}] \subseteq T^{(1,0)}$ [7]. However, root spaces satisfy $[E_\alpha, E_\beta] \subseteq E_{\alpha+\beta}$ hence $G.B$ is a complex manifold.

Additionally, there is a symplectic form ω on $G.B$ defined at B as follows:

$$\omega_A : T_B(G.B) \times T_B(G.B) \to \mathbb{R} \tag{4.4}$$

$$(X,Y) \mapsto \langle B, [X,Y] \rangle \tag{4.5}$$

This form is actually the Kirillov-Kostant-Souriau symplectic form [8] that is defined over the coadjoint orbit but under the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ (induced by the non-degenerate form $< , >$) it can be transferred to the subspace $T_B(G.B)$ of $\mathfrak{g}$. Then this 2-form is extended globally by the transitive action of G.

We can also define a bilinear 2-form g on $G.B$ by $g(X,Y) := \omega(X, JY)$ and with this form $G.A$ becomes a Kähler manifold:

Lemma 4.1. *$\omega(Ju, Jv) = \omega(u, v)$ and $\omega(u, Ju) > 0$. First equality implies that g is symmetric and the latter implies that g is positive definite.*

Proof. It is enough to show this implications at the point B. We know that the tangent space at B (or any regular point) is $\mathfrak{g}/\mathfrak{t} = \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$. If $\alpha, \beta \in \Phi$ and $u \in \mathfrak{g}_\alpha$, $v \in \mathfrak{g}_\beta$ in which $\beta \neq \pm\alpha$ then $[u, v] \in \mathfrak{g}_{\alpha+\beta}$. Also the decomposition at (4.2) is orthogonal with respect to the trace form and J rotates the symmetric part and the skew-symmetric part in each $\mathfrak{g}_\alpha$ hence $J(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$. As a result

$$\omega_B(Ju, Jv) = \langle B, [Ju, Jv]\rangle = 0 = \langle B, [u, v]\rangle = \omega_B(u, v) \quad . \tag{4.6}$$

Now assume for $\alpha \in \Phi$, $\mathfrak{g}_\alpha = span(X, Y)$ where X is skew-symmetric and Y is purely imaginary symmetric so that $J(X) = Y$ and $J(Y) = -X$. Then

$$\omega_B(JX, JY) = \langle B, [JX, JY]\rangle = \langle B, [Y, -X]\rangle = \langle B, [X, Y]\rangle = \omega_B(X, Y) \tag{4.7}$$

The last step is the situation of both u, v are skew symmetric (or symmetric) and both lie in the same eigenspace $\mathfrak{g}_\alpha$. However, $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ and $\mathfrak{k}$ is perpendicular to $\mathfrak{t}$. Moreover, J sends skew ones to symmetrics and vice versa. Consequently, in this case; J is still orthogonal with respect to ω. Positive definiteness can be shown in similar fashion. $\square$

Now, we are going to describe the gradient flows of f_A in $G.B$ with respect to the Kähler metric g [4].

4.3. Gradient Flows on $G.B$ and Stable(Unstable) Manifolds

Gradient vector field of the Morse function f_A with respect to the trace form is given by $\nabla f_A(x) = [[A, x], x]$ [4]. But $Ad(exp(t[A, x]))(x)$ is not the gradient flow even though $\frac{d}{dt} Ad(exp(t[A, x]))(x)\,_{|t=0} = [[A, x], x]$.

As mentioned in the previous section, $G.B$ has rich geometric structures. Compatibility relations between these structures give us a chance to determine gradient flows with ease.

Maximal torus T (corresponding to $\mathfrak{t}$) acts symplectically on $G.B$ by the adjoint action. Indeed,

$$\begin{aligned} Ad_g^*\omega_B(X, Y) &= \omega_{gBg^{-1}}(gXg^{-1}, gYg^{-1}) && (4.8)\\ &= \langle gBg^{-1}, [gXg^{-1}, gYg^{-1}]\rangle && (4.9)\\ &= \langle gBg^{-1}, g[X, Y]g^{-1}\rangle && (4.10)\\ &= \langle B, [X, Y]\rangle && (4.11)\\ &= \omega_B(X, Y) && (4.12) \end{aligned}$$

This action is hamiltonian ([4], [11]) hence induces moment mapping $\mu : G.A \to \mathfrak{t}^*$ given by $\mu(X) = \langle proj_{\mathfrak{t}}(X), -\rangle$ where $proj_{\mathfrak{t}}(X)$ denotes the orthogonal projection of X onto subspace $\mathfrak{t}$. As a result, the component μ^A of μ in the direction A is our Morse function f_A (μ^B being the hamiltonian). Let $A^{\#}$ denote the vector field generated by $\{exp(tA)|t \in \mathbb{R}\}$. Furthermore, since the action is hamiltonian, $df_A = \iota_{A^{\#}}\omega$. Since $SL(n, \mathbb{C})$ acts holomorpically on $G.B$ this action can be restricted to T, thus T acts holomorpically too. By collecting all these information, we obtain:

$$
\begin{aligned}
g(\nabla f_A, X) &= df_A(X) && (4.13)\\
&= \iota_{A^\#}\omega(X) && (4.14)\\
&= \omega(A^\#, X) && (4.15)\\
&= g(JA, X) && (4.16)
\end{aligned}
$$

Corollary 4.2. *Gradient flow of the Morse function f_A at $x \in G.B$ is given by $exp(itA).x$*

Remark 4.1. *Note that the action in the previous corollary is actually the action of $SL(n, \mathbb{C})$ on the flag manifold. Indeed, $iA \in \mathfrak{t}_{\mathbb{C}}$ and the adjoint orbit is identified with $SL(n, \mathbb{C})/B_+$ hence, in this setting, action is the left multiplication with the corresponding flag.*

Before passing through the real case we describe the unstable and stable manifolds of $G.B$.

Note that the Bruhat cells are given by $B_+\omega B_+/B_+$ where ω is a representative in $N(T)$ for the Weyl group W. These cells are actually the orbits of the action of B on the flag manifold G/B_+ given by the left multiplication. Besides, the critical points are also described by the aforementioned representatives of the Weyl group W. Following lemma from [4] explains the unstable/stable manifolds in terms of the Bruhat cells:

Lemma 4.2. *The unstable manifold $W^u(x)$ and the stable manifold $W^s(x)$ at the critical point x (correspondingly at coset wB_+) are equal to*

$$
\begin{aligned}
W^u(x) &= B_+\omega B_+/B_+ = B_+.x && (4.17)\\
W^s(x) &= B_-\omega B_-/B_- = B_-.x && (4.18)
\end{aligned}
$$

4.4. The Real Case

Similarly to the complex case, we can look at the adjoint orbit $SO(n).B$ of $SO(n)$ in the Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$ via the involution τ which is defined in the beginning of this chapter. In this case, the stabilizer of B is a discrete subgroup of $SO(n)$ and from now on, denote this group by $\mathit{\Gamma}$. Hence $SO(n)/\mathit{\Gamma} = SO(n).B$. On the other hand consider the action of $SL(n,\mathbb{R})$ on real flags via left multiplication. So we can make the following identifications:

$$SO(n).B = SO(n)/\mathit{\Gamma} = SL(n,\mathbb{R})/B^+ \tag{4.19}$$

where B^+ is the real upper triangular matrices with determinant 1. Note that $B^+ = B_+ \cap SL(n,\mathbb{R})$. Denote $SO(n)$ by K as before.

Above identifications have correspondents in the Lie algebra level: We can decompose split real form $\mathfrak{sl}(n,\mathbb{R})$ into root spaces with respect to the Cartan subalgebra $i\mathfrak{t}$:

$$\mathfrak{sl}(n,\mathbb{R}) = i\mathfrak{t} \oplus \sum_{\alpha\in\mathit{\Phi}} E_\alpha \tag{4.20}$$

Remark 4.2. *In the above decomposition -with abuse of notation- root spaces are actually the real part of the E_α's that are defined for the root space decomposition of $\mathfrak{sl}(n,\mathbb{C})$ at (4.1). Hence instead of writing $E_\alpha \cap \mathfrak{sl}(n,\mathbb{R})$ we keep the same notation for the root spaces.*

Now assume that ω and σ are in the Weyl orbit $W.B$ hence they are critical points. Assume that $ind(\sigma) = 1 + ind(\omega)$. Here the indices are the inversion numbers of corresponding permutations($W.B$ is identified with the symmetric group S_n). Then there is a way [4] to determine whether the moduli space $\tilde{\mathcal{M}}(\sigma,\omega)$ is empty or not. Similar arguments in the language of Toda flows can be found in [14].

Theorem 4.1. *Assume* $ind(\sigma) = 1 + ind(\omega)$ *for the critical points* σ *and* ω *in K.B. Then there exists a gradient flow between these two critical points if and only if there exists a reflection in* W *carrying* σ *to* ω*. Equivalently,* $\sigma \geq \omega$ *in the Bruhat order of* S_n

Kocherlakota gives the exact number of gradient flows whenever there exists a reflection that carrying critical points with consecutive indices.

Theorem 4.2 (Kocherlakota)**.** *If there exists a reflection between critical points* σ *and* ω *then there are exactly two gradient flows between them.*

In this setting, we can express the flows more combinatorically. Hasse diagram of the symmetric group with respect to the strong Bruhat order contains all the necessary information about flows:

Remark 4.3. *Here the reflection is a transposition in* S_n*. Since the indices are given by the inversion number, existing of a such a reflection is equivalent to existing of a transposition that increases inversion number by one. This means that the Bruhat diagram on* S_n *represents the flow diagram by adding extra edge beside the existing edges.*

Moreover, we are not only allowed to deal with the complete real flag manifolds, all the construction can be generalized to arbitrary reductive homogeneous space.

Remark 4.4. *All the arguments are valid for any reductive homogenous space [8].*

5. GRADIENT FLOWS ON $SO(n)$

After all the details about the flag manifolds and the Morse functions over these spaces, we are now ready to begin the case $SO(n)$. In the study of H.I. Bozma, W.D Gillam and F.Öztürk [15], it is shown that the quadratic trace function on the orthogonal group $O(n)$ given by $f_{A,B}(X) = tr(AXBX^T)$ is a Morse-Bott function for the fixed symmetric matrices A and B. Moreover, if A and B have different eigenvalues, then the function is Morse. In their setup, they compute the indices by the method so-called "perfect fillings" and they compare their results with the *linear case* studied in the paper of T.Frankel [16]. However, in this study, as in the case of flag manifolds; we are only interested in the Morse situation.

In this chapter, we will first describe the critical point set of $f_{A,B}$ and then we will give algebraic properties of this set. Then, we will combine the real flag manifold case and the $SO(n)$ case by showing, with particular choices, the function $f_{A,B}$ in $SO(n)$ and the function f_A in the real flag manifold behave similarly. This will lead us to determine the exact number of gradient flows on $SO(n)$.

5.1. Quadratic Morse Function on SO(n)

Assume A and B are real diagonal matrices and both have different eigenvalues. In the study [15], it is shown that the function given by

$$f_{A,B} : O(n) \to \mathbb{R} \tag{5.1}$$

$$X \mapsto tr(AXBX^T) \tag{5.2}$$

is a Morse function with the critical point set consists of signed permutation matrices. Lie theoretically, It is the Coxeter group B_n; Weyl group of $SO(2n+1)$. On the other hand this group splits as

$$B_n \cong \mathbb{Z}_2^n \rtimes S_n \quad . \tag{5.3}$$

Here, the factor $\mathbb{Z}_2^n$ corresponds to the normal subgroup of B_n consists of diagonal matrices with entries ± 1 and S_n corresponds to the (unsigned) permutation matrices. Hence we can decompose any critical point as a product of a sign matrix and an unsigned permutation matrix.

Remark 5.1. *Here the set of critical points B_n of $f_{A,B}$ is obviously a subgroup of $O(n)$.*

Now assume that a critical point is given by $D.\sigma$ where D is a diagonal sign matrix and σ is a permutation matrix. Then the index of $D.\sigma$ is given by [15]

$$ind(D.\sigma) = inv(\sigma) = |\{(i,j) : \sigma(i) > \sigma(j), i < j\}|. \tag{5.4}$$

In other words, the index of a critical point is the inversion number of corresponding unsigned permutation.

Next, consider the restriction of $f_{A,B}$ onto $SO(n)$. We have denoted this restriction again with $f_{A,B}$ since there will be no further information about the $O(n)$ case. From now on, $Cr(f_{A,B})$ denotes the set of critical points of $f_{A,B}$. Clearly, $Cr(f_{A,B})$ is the set of signed permutation matrices with determinant 1. Now we describe the algebraic properties of $Cr(f_{A,B})$:

Lemma 5.1. *$Cr(f_{A,B}) \trianglelefteq B_n$. When n is odd $Cr(f_{A,B}) \cong D_n$ where D_n consists of signed permutations with an even number of minus sign(equivalently, denotes the Weyl group of $SO(2n)$).*

Proof. Since $|Cr(f_{A,B})| = 2^{n-1}n!$, its index in B_n is 2 hence $Cr(f_{A,B}) \trianglelefteq B_n$. If n is odd then consider the map

$$\phi : D_n \to Cr(f_{A,B}) \tag{5.5}$$

$$\sigma \mapsto \begin{cases} \sigma & det(\sigma) = 1 \\ -\sigma & det(\sigma) = -1 \end{cases} \tag{5.6}$$

Since the determinant is a homomorphism, ϕ is also a homomorphism. Surjectivity is obvious. In addition, the sizes of these two groups are equal. Hence ϕ is an isomorphism. □

Unfortunately, when n is even $Cr(f_{A,B})$ is not isomorphic to D_n. For example, when $n = 2$, $Cr(f_{A,B}) \cong \mathbb{Z}_4$ but $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This situation of parity prevents us working only on Coxeter groups.

5.2. Gradient Flows on $SO(n)$

Remind that in chapter 3 and 4 matrices A and B were purely imaginary, diagonal and regular. Now assume that we are fixing such A and B. Then the function $f_{-iA,iB}$ is a Morse function on $SO(n)$. Moreover $f_{-iA,iB}(X) = tr(-iAXiBX^T) = tr(AXBX^{-1}) = f_A(XBX^{-1})$ for all $X \in SO(n)$.

The subset $\mathit{\Gamma}$ of $Cr(f_{A,B})$ which consists of diagonal sign matrices acts freely on $SO(n)$ by the right multiplication: $g.x = g$ implies $x = I$. Moreover this action is properly discontinuos. To prove this, we need to check that for any $g \in SO(n)$ whether there is a neighborhood U of x such that $\{x \in \mathit{\Gamma} | U \cap Ux \neq \emptyset\}$ is finite. However, for any neighborhood of x this set is finite since $\mathit{\Gamma}$ is finite. Moreover, free and properly discontinous action of a group on a Hausdorff space is a covering space action [17]. Hence,

Corollary 5.1. *The quotient map $\pi : SO(n) \to SO(n)/\mathit{\Gamma}$ is a covering map.*

In addition, we also know that if a compact Lie group acts on some smooth manifold transitively, then the quotient map with respect to any stabilizer is a submersion [18]. Hence π is a submersion. Since $\mathit{\Gamma}$ is finite, $dim(SO(n)) = dim(SO(n)/\mathit{\Gamma})$. This implies π is also a local diffeomorphism. Topological covering map and local diffeomorphism implies smooth covering [19] Hence,

Corollary 5.2. *The quotient map $\pi : SO(n) \to SO(n)/\mathit{\Gamma}$ is a smooth covering map.*

Now let $x \in SO(n)/\mathit{\Gamma}$ and let $\gamma_x(t)$ be the gradient flow line passing through x. Since $SO(n)/\mathit{\Gamma}$ is compact, gradient vector field is complete. Moreover, the map $\gamma_x : (\mathbb{R}, 0) \to (SO(n)/\mathit{\Gamma}, x)$ can be lifted into the covering space uniquely since $\mathbb{R}$ is simply connected: $(\gamma_x)_*(\pi_1(\mathbb{R}, 0)) \subseteq \pi_*(\pi_1(SO(n), g))$ where $\pi(g) = x$ and $\gamma_x(0) = x$.

It remains to show that these lifts are gradient flows: We have a unique bi-invariant Riemannian metric (up to scalar) on $SO(n)$ given by the the trace form as in the case of $SU(n)$. However, in order to describe gradient flows on $SO(n)$ as lifts of gradient flows on $SO(n)/\mathit{\Gamma}$ we need to introduce another Riemannian metric on $SO(n)$. The reason is that the trace form on $SO(n)$ and the restriction of the Kähler metric g on real flag manifold is not compatible. One obvious choice of a metric to solve this problem is the pullback $\pi^* g$.

Assume that $\tilde{\gamma}_x$ is the lift of the gradient flow line γ_x. We need to show that $\frac{d\tilde{\gamma}_x(t)}{dt} = \nabla f_{A,B}$. Now define $(\pi^* g)(X, Y) := g(d\pi(X), d\pi(Y))$. Then,

$$df_{A,B}(X) = (\pi^* g)(\nabla f_{A,B}, X) \tag{5.7}$$

$$= g(d\pi(\nabla f_{A,B}), d\pi(X)) \tag{5.8}$$

and

$$df_{A,B}(X) = df_A \circ d\pi(X) \tag{5.9}$$
$$= g(\nabla f_A, d\pi(X)) \tag{5.10}$$

Hence we obtain that, for all X,

$$(\pi^* g)(\nabla f_{A,B}, X) = g(d\pi(\nabla f_{A,B}), d\pi(X)) \tag{5.11}$$
$$= g(\nabla f_A, d\pi(X)) \tag{5.12}$$
$$= (\pi^* g)(d\pi^{-1}\nabla f_A, X) \tag{5.13}$$

As a result, $\nabla f_{A,B} = d\pi^{-1}\nabla f_A$. Since $\frac{d\pi \circ \tilde{\gamma}_x(t)}{dt} = \frac{d\gamma_x(t)}{dt} = \nabla f_A$, by the chain rule, we get the desired result. Thus we have proved that

Theorem 5.1. *Gradient flows on $SO(n)$ are the lifts of the gradient flows on the real flag manifold.*

Now assume that θ_1 and θ_2 are critical points of the function f_A over the real flag manifold with consecutive indices. Since we can identify $Cr(f_A)$ with the symmetric group S_n, Theorem 4.2 says that if there is a reflection r on the Weyl group such that $r(\theta_1) = \theta_2$, then there exists exactly two gradient flows between these critical points. By Theorem 5.1, we can lift these flows up to $SO(n)$ hence for any critical point $X \in \pi^{-1}(\theta_1)$ there are exactly two gradient flows starting from X and flow through $\pi^{-1}(\theta_2)$. But these arguments do not answer this question: between which critical points with consecutive indices are there gradient flows?

Assume Z_α is a root vector living in E_α (4.8) and $\alpha \in \varPhi_+$. Also assume that r_α is the the reflection corresponding to the root plane $\ker(\alpha)$ and $r_\alpha(\theta_1) = \theta_2$ where $\theta_1, \theta_2 \in Cr(f_A)$. Gradient flows between this points are given by the equation $\lim_{t\to\pm\infty} e^{tZ_\alpha}\theta_1 = \theta_2$. Now consider $X_\alpha = Z_\alpha - Z_{-\alpha}$ which is skew-symmetric and $\lim_{t\to\pm\frac{\pi}{2}} \mathrm{Ad}(\exp)(tX_\alpha)(\theta_1) = \lim_{t\to\pm\infty} e^{tZ_\alpha}\theta_1$. Then

Theorem 5.2. *Suppose $x \in \pi^{-1}(\theta_1)$ where $\theta_1 \in Cr(f_A)$. Then there are two gradient flows starting from x correspond to the reflection r_α and they are given by $\lim_{t\to\pm\frac{\pi}{2}} exp(tX_\alpha).x$. Hence flows starting from x land to $exp(\frac{\pi}{2}X_\alpha).x$ and $exp(-\frac{\pi}{2}X_\alpha).x$. Total number of gradient flows starting from x are equal to two times the number of transpositions that increases the inversion number of $\pi(x)$ by one.*

Proof. It is enough to show that the curve $\exp(tX_\alpha).x$ is a lift of a gradient flow on the real flag manifold. Indeed,

$$\begin{aligned}\pi(exp(tX_\alpha).x) &= exp(tX_\alpha).x.B.(exp(tX_\alpha).x)^{-1}\\ &= Ad(exp(tX_\alpha)).\pi(x)\\ &= Ad(exp(tX_\alpha)).\theta_1\end{aligned}$$

□

6. COMPUTATIONS

In this chapter we introduce a way to calculate Morse homology of $SO(n)$ in $\mathbb{Z}_2$ coefficients.

6.1. An Ordering on $Cr(f_{A,B})$

Assume that a reflection r_α sends critical point θ_1 to θ_2 in the real flag manifold. As stated in the previous chapter, this reflection also induces the gradient flows between the "layers" $\pi^{-1}(\theta_1)$ and $\pi^{-1}(\theta_2)$. Also note that $|\pi^{-1}(\theta_1)| = |\pi^{-1}(\theta_2)|$ since π is a covering map. Moreover,

Lemma 6.1. *If $x \in \pi^{-1}(\theta_1)$ then there exists $y \in \pi^{-1}(\theta_1)$ so that the gradient flows satisfy $exp(\frac{\pi}{2}X_\alpha).x{=}exp(-\frac{\pi}{2}X_\alpha).y$ and $exp(-\frac{\pi}{2}X_\alpha).x$ $=exp(\frac{\pi}{2}X_\alpha).y$. Hence for each reflection that carries layers, critical points appear as pairs.*

Proof. Assume that the reflection r_α with respect to the hyperplane $ker(\alpha) \subseteq \mathfrak{t}$ carries θ_1 to θ_2. Assume that the corresponding $\mathfrak{sl}(2,\mathbb{R})$ copy generated by the set $\{Z_\alpha, Z_{-\alpha}, [Z_\alpha, Z_{-\alpha}]\}$ and $\alpha \in \Phi_+$. Hence $X_\alpha = Z_\alpha - Z_{-\alpha} \in \mathfrak{so}(n)$. Also suppose that $\exp(\frac{\pi}{2}X_\alpha)$ and $\exp(-\frac{\pi}{2}X_\alpha)$ act as $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ rotations on the plane $\text{span}(e_i, e_j) \subseteq \mathbb{R}^n$. Hence if the signs of a critical point $x \in \pi^{-1}(\theta_1)$ are $\{\epsilon_1, \epsilon_2\}$ at i'th row and j'th row respectively, then the critical point with signs $\{-\epsilon_1, -\epsilon_2\}$ at i'th row and j'th row is sent to same critical points under the action of $\exp(\frac{\pi}{2}X_\alpha)$ and $\exp(-\frac{\pi}{2}X_\alpha)$. □

Any critical point in $SO(n)$ can be written as a product of a diagonal sign matrix and a unsigned permutation matrix uniquely. However, in this decomposition, corresponding sign matrix does not have to be in $SO(n)$. Hence this decomposition is actually occurs in $O(n)$ rather then $SO(n)$.

In the Morse-Witten chain complex, denote the chain group generated by the signed permutation matrices correspond to inversion number k by C_k. If the inversion number is odd then the determinant of related unsigned permutations are -1. Conversely, when the inversion number is even then the corresponding unsigned permutation matrices have determinant 1.

Now our aim is to give an order to the fixed basis (critical points) of C_k for all inversion numbers k regarding sign changes.

First, represent the group of k by k sign matrices with determinant 1 by Γ_k^+. Now denote a sign matrix in Γ_k^+ in the vector form $a = (a_1, a_2, ..., a_n)$. Then define an *embedding*

$$\varphi^+ : \Gamma_k^+ \to \Gamma_{k+1}^+ \tag{6.1}$$

$$(a_1, ...a_k) \mapsto (a_1, ..., a_k, 1) \tag{6.2}$$

Also note that Γ_k^+ is generated by the sign matrices with two consecutive -1 entries and the remaining entries are 1. Denote such matrix as $(l, l+1)$. Then it is clear that

$$\Gamma_{k+1}^+ = \varphi^+(\Gamma_k^+) \bigsqcup (\varphi^+(\Gamma_k^+)).(k, k+1) \tag{6.3}$$

In other words, we can inductively obtain all the sign matrices. Moreover, since the Γ_1^+ is trivially ordered, we can expand this order inductively. Thus, if Γ_k^+ is ordered as $\{A_1, ..., A_{2^{k-1}}\}$ then Γ_{k+1}^+ is ordered as $\{\varphi^+(A_1), ..., \varphi^+(A_{2^{k-1}}), (\varphi^+(A_1))(k, k+1), ..., (\varphi^+(A_{2^{k-1}}))(k, k+1)\}$.

Similarly, we define an order on sign matrices with determinant -1. Since this set is not a group we actually put an order on the quotient Γ_k/Γ_k^+ where $\Gamma_k \subseteq O(k)$ consisting of all the k by k sign matrices. We can define φ^- in the same way, and again multiplying with sign matrix with only minus entries occurring in the last two spot, we can obtain all the negative sign matrices.

Remark 6.1. *Above process is very similar to constructing $k+1$-dimensional cube from k- dimensional cube. When the vertices of k-dimensional cube represented by a binary sequences, $k+1$-dimensional cube can be obtained by adding 0 and 1 to the each vertex of k-dimensional cube.*

Corollary 6.1. *If the set of permutations in S_n with inversion number i ordered as $\{\sigma_1, ..., \sigma_L\}$ then the set of basis of C_i, $\bigcup_k \pi^{-1}(\sigma_k)$ can be ordered as*

$$\{A_1\sigma_1, ..., A_{2^{n-1}}\sigma_1, A_1\sigma_2, ..., A_{2^{n-1}}\sigma_2, ..., A_1\sigma_L, ..., A_{2^{n-1}}\sigma_L\}$$

where A_i's are ordered as above.

Matrices correspond to the boundary maps are formed by 2^{n-1} by 2^{n-1} blocks and each block corresponds to a reflection that carries critical points in the real flag manifold or edges of the Bruhat diagram of the symmetric group. Moreover,

Corollary 6.2. *With the above ordering, matrices correspond to the boundary maps are formed by 2^{n-1} by 2^{n-1} symmetric blocks where each block correspond to a transposition that increases inversion number by 1.*

6.2. An Example: $SO(3)$

Now we apply the arguments about critical points and the gradient flows to $SO(3)$ with following chain complex and we obtain the critical points in Figure 6.1.

$$0 \to \mathbb{Z}_2^4 \xrightarrow{\partial_1} \mathbb{Z}_2^8 \xrightarrow{\partial_2} \mathbb{Z}_2^8 \xrightarrow{\partial_3} \mathbb{Z}_2^4 \to 0 \tag{6.4}$$

$$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix} \quad \begin{pmatrix} -1&0&0\\0&1&0\\0&0&-1 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} 0&-1&0\\1&0&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} 0&1&0\\1&0&0\\0&0&-1 \end{pmatrix} \quad \begin{pmatrix} 0&-1&0\\-1&0&0\\0&0&-1 \end{pmatrix} \quad \begin{pmatrix} 1&0&0\\0&0&-1\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} -1&0&0\\0&0&1\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} 1&0&0\\0&0&1\\0&-1&0 \end{pmatrix} \quad \begin{pmatrix} 0&-1&0\\-1&0&0\\0&0&-1 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} \quad \begin{pmatrix} 0&-1&0\\0&0&-1\\1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0\\0&0&-1\\-1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&-1&0\\0&0&1\\-1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&-1\\-1&0&0\\0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1\\-1&0&0\\0&-1&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&-1\\1&0&0\\0&-1&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1\\0&-1&0\\1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&-1\\0&1&0\\1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1\\0&1&0\\-1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&-1\\0&-1&0\\-1&0&0 \end{pmatrix}$$

Figure 6.1. Critical points on $SO(3)$.

Above figure shows all the 24 critical points of the Morse function $f_{A,B}$ on $SO(3)$. They are given in the hierarchy of Morse indices, so that the top 4 matrices are corresponds permutations with inversion number 0, second line corresponds to 1, third line corresponds to 2 and the bottom corresponds to 3 (since we are interested in the negative gradient flows). We also respect to the order defined in chapter 6.1. With this ordering, we can easily predict the any other layers with knowing the only top layer.

Figure 6.2. in the next page shows the gradient flows on $SO(3)$. Note that the red flows between the top and the second layer and again the red flows between the third layer and the bottom represent the same action; induced by the corresponding reflection in the flag manifold. Similarly black and purple edges are written. This says us there is a sort of symmetry dictated by the strong Bruhat order combined with the order which we defined in section 6.1.

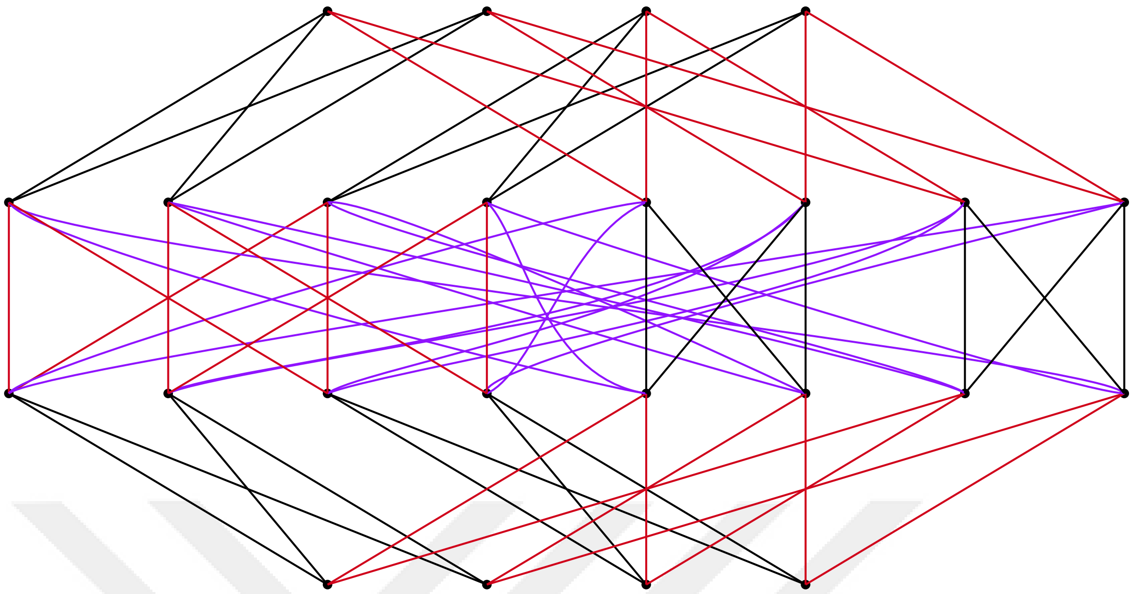

Figure 6.2. Gradient flows on $SO(3)$

Next we describe the boundary maps ∂_1, ∂_2 and ∂_3 correspond to the related inversion numbers. As we mentioned in the previous page, to describe these boundary maps, we use the order defined in section 6.1.

In this ordering, independent from the dimension; all the matrices are formed by symmetric blocks and each block correspond to the reflection in the complete real flag manifold.

Moreover, when we consider the blocks and the corresponding edges they can be seen as a graph with some *nice* properties:

Corollary 6.3. *Critical points and the flows between them correspond to the blocks between consecutive layers can be considered as a 2-balanced, regular, bi-partite graph. Hence the boundary maps are formed by the* 2^{n-1} *by* 2^{n-1} *blocks and each block is actually a bi-adjacency matrix of the corresponding bi-partite graph.*

Finally, Thanks to Sage homology package, we obtain the $\mathbb{Z}_2$-homology of $SO(3)$, $SO(4)$, $SO(5)$ and we collect all these datum in the website [23]. In the next page we give the aforementioned boundary maps for $SO(3)$.

$$\partial_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\partial_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\partial_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

7. CONCLUSION

In this thesis, we study the particular Morse functions both on flag manifolds and the special orthogonal group $SO(n)$. We describe the gradient flows on these spaces with proper metric. Moreover, we connect these gradient flows via the smooth covering map between $SO(n)$ and the quotient space $SO(n)/\Gamma$ where Γ consists of diagonal sign matrices in $SO(n)$. Then we determine how the gradient flows between the critical points with consecutive indices appear. With this information we set an order on the critical point set which helps to write boundary maps easily.

REFERENCES

1. Milnor, J., *Morse Theory*, Princeton University Press, Princeton-New Jersey, 1963.

2. Banyaga, A., Hurtubise, D., *Lectures on Morse Homology*, Springer-Dordrecht, 2004.

3. Bott, R., Samelson, H., "Applications of the Theory of Morse to Symmetric Spaces", *American Journal of Mathematics*, Vol. 80., No. 4., pp. 964-1029, 1958.

4. Kocherlakota, R.R., "Integral Homology of Real Flag Manifolds and Loop Spaces of Symmetrical Spaces", *Advances in Mathematics*, Vol. 110., No.1., pp. 1-46, 1995.

5. Fulton, W., Harris, J., *Representation theory, A First Course*, Graduate Texts in Mathematics, Readings in Mathematics, Vol. 129., Springer-Verlag, New York, 1991.

6. Borel, A., Hirzebruch, F., "Characteristic Classes and Homogeneous Spaces", *American Journal of Mathematics*, Vol. 80., No.2., pp. 964-1029, 1958.

7. Cannas da Silva, A., *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics, Springer Berlin, Heidelberg, 2008.

8. Alekseevsky, D.V., "Flag Manifolds", *Zbornik Radova*, Vol. 6. No. 14., pp. 3-35, 1997.

9. Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory*, Springer New York, New York, 1972.

10. Atiyah, M.F., "Convexity and Commuting Hamiltonians", *Bull. London Mathematical Society*, Vol. 14., No. 1, pp. 1-15, 1982.

11. Guillemin, V., Sternberg, S., "Convexity Properties of the Moment Mapping", *Invent Math*, Vol. 67., pp. 491-513, 1982

12. Duistermaat, J.J., Kolk, J.A.C., Varadarajan, V.S., "Functions, Flows and Oscillatory Integrals on Flag Manifolds and Conjugacy Classes in Real Semisimple Lie Groups", *Compositio Mathematica*, Vol. 49., No. 3., pp. 309-398, 1983.

13. Kirwan, F.C., "Cohomology of Quotients in Symplectic and Algebraic Geometry", *Princeton University Press*, Vol. 31, No. 104, 1984.

14. Chernyakov, Yu.B., Sharygin, G.I., Sorin, A.S., "Bruhat Order in Full Symmetric Toda System", *Comm. Math. Phys.*, Vol. 330., pp. 367-399, 2014.

15. Bozma, H.I., Gillam, W.D., Öztürk, F., "Morse-Bott Functions on Orthogonal Groups", *Topology and its Applications*, Vol. 265., No. 106807, 2019.

16. Frankel, T., "Critical Submanifolds of the Classical Groups and Stiefel Manifolds". *Princeton University Press*, Differential and Combinatorial Topology: A Symposium in Honor of Marston Morse, pp. 37-54, 1965.

17. Hatcher, A., *Algebraic Topology* , Cambridge University Press , Cambridge, 2002.

18. Warner, F.W., *Foundations of Differential Geometry and Lie Groups*, Graduate Texts in Mathematics, Springer-Verlag, New York, Vol. 94, 1983.

19. Lee, J.M., *Riemannian Manifolds: An Introduction to Curvature*, Springer-Verlag, New York, 1997.

20. Witten, E., "Supersymmetry and Morse Theory", *Journal of Differential Geometry*, Vol. 17, No. 4., pp. 661-692, 1982.

21. Rabelo, L., San Martin, L.A.B., "Cellular Homology of Real Flag Manifolds", *Indagationes Mathematicae*, Vol. 30., No. 4., pp. 745-772, 2019.

22. Bott, R., "Nondegenerate Critical Manifolds", *Annals of Mathematics*, Vol. 60, No. 4., pp. 248-261, 1954.

23. Barut, E., Karakurt, T., 2025, "Examples:$SO(3), SO(4), SO(5)$", https://github.com/oliverol01/Combinatorial-Morse-homology-calculatitions-on-SO-n-, accessed on May 11, 2025.

24. Weber, J., "The Morse–Witten Complex via Dynamical Systems", *Expositiones Mathematicae*, Vol. 24, No. 2., pp. 127-159, 2006.