STATISTICAL INFERENCE OF STRESS-STRENGTH RELIABILITY OF SOME COHERENT SYSTEMS WITH COLD STANDBY REDUNDANCY

by

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To my family...

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ABSTRACT

STATISTICAL INFERENCE OF STRESS-STRENGTH RELIABILITY OF SOME COHERENT SYSTEMS WITH COLD STANDBY REDUNDANCY

In this dissertation, statistical inference methods are considered for the series and parallel system structures that comprise original components and standby redundancy components. Estimation of stress-strength reliability and mean remaining strength for the series and parallel systems are obtained when the underlying distributions are exponential. For this purpose, the classical and Bayesian techniques are implemented. Next, corresponding asymptotic confidence intervals via Fisher information matrix or observed information matrix and highest probability density credible intervals are derived. A simulation study is implemented for statistical inference for each system structure. The performances of maximum likelihood and Bayesian estimates are compared via Monte Carlo simulations. As a final step, real-life data are analyzed for each case.

ÖZET

SOĞUK BEKLEME YEDEKLİLİĞİNE SAHİP BAZI TUTARLI SİSTEMLERİN GÜVENİLİRLİĞİNİN İSTATİSTİKSEL ÇIKARIMI

Bu tezde, orijinal bileşenler ve yedek bileşenler içeren seri ve paralel sistem yapıları için istatistiksel çıkarım yöntemleri ele alınmıştır. Seri ve paralel sistemler için güvenilirlik ve ortalama kalan mukavemet tahminleri, temeldeki dağılımlar üstel seçilerek elde edilmiştir. Bu amaçla klasik ve Bayes teknikleri göz önünde bulundurulmuştur. Daha sonra, Fisher bilgi matrisi veya gözlemlenen bilgi matrisi aracılığıyla asimptotik güven aralıkları ve Bayes güven aralıkları elde edilmiştir. Her sistem yapısında istatistiksel çıkarım için bir simülasyon çalışması gerçekleştirilmiştir. Maksimum olabilirlik ve Bayes tahminlerinin performansı Monte Carlo simülasyonları ile karşılaştırılmıştır. Son adım olarak, her durum için gerçek hayat verileri analiz edilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

C_X	Component X
$exp(\alpha)$	Exponential distribution with parameter α
$Gamma(\alpha, \beta)$	Gamma distribution with parameters α and β
$GE(\alpha, \beta)$	Generalized exponential distribution with parameters α and β
L	Likelihood function
ℓ	Log-likelihood function
R_{PC}	Stress strength reliability for parallel system at component level
R_{PS}	Stress–strength reliability for parallel system at system level
R_{SC}	Stress–strength reliability for series system at component level
R_{SS}	Stress–strength reliability for series system at system level
$\Gamma(.)$	Gamma function
σ^2	Variance
Φ	Mean remaining strength
ACI	Asymptotic confidence interval
AL	Average length
A-D	Anderson Darling
cdf	Cumulative density function
cp	Coverage probability
cps	Coverage probabilities
CI	Confidence interval
C-VM	Cramer–von Mises
ER	Estimated risk
HPD	Highest probability density
i.i.d.	Independently and identically distributed
K-S	Kolmogorov–Smirnov
MCMC	Markov chain Monte Carlo
ML	Maximum likelihood

MLE	Maximum likelihood estimation
MRS	Mean remaining strength
MSE	Mean squared error
pdf	Probability density function
SE	Squared error

1. INTRODUCTION

1.1. OVERVIEW AND MOTIVATION

A series system operates in the case that each component functions so that a series system will fail immediately upon at least one component fails. For instance, a lightning chain represents a series system since all bulbs aligned as series on a line. If one bulb burned out, the electric path can break, and this will affect the other bulbs in the loop thus; they will not light up. Solar panels also work in the same way when they are connected in series. Let us consider electronic devices. It is essential to lengthen the battery's life. Battery packs provide high voltage when connected in series [1]. A parallel system fails if every component fails and therefore at least one working component is sufficient to run such a system properly. Components of a parallel system can also be referred to as redundancy components. Assume that five bulbs are connected to the same battery separately, in parallel. If some of the bulbs complete their lifetimes, the others will not be affected and continue burning. In real life, batteries can be connected in parallel for high capacity in battery systems. Also, parallel connection of pipes to plumbing is widely used in pumping systems. In such systems, parallel connected pipelines help to supply different flow rate requirements. When a solar panel system is taken into account, the solar installers mostly prefer the hybrid of series and parallel system to lessen the defect. There exist many applications of parallel and series configurations in engineering systems.

In reliability analysis, the stress-strength probability is a major concern for some experiments in the fields including industrial engineering, bioengineering, hydrology, economics, and survival analysis. In an elementary form, the stress-strength model expresses the reliability of a component or system with regard to random variables. The reliability is described as $P(X < Y)$ where X denotes the random stress experienced by the system, and Y represents the random strength to get over the stress. If stress overtakes the strength, the system can not operate. Birnbaum [2] presented this core idea. Then, Birnbaum and McCarty [3] and Church and Harris [4] improved this subject. Estimation problems for the reliability of a coherent system such as series, parallel, and multicomponent systems are prominent issues in the literature.

A k-out-of-n:G system consists of n i.i.d. strength components, common stress, and such a system functions at least k out of the n components run. When $k = 1$ and $k = n$, the k-out-of-n:G system turns out to be a parallel and series system, respectively. Some recent research contributions to the topic about multicomponent stress-strength reliability are presented in the following papers: Eryilmaz [5], Pakdaman and Ahmadi [6], Basirat et al. [7], Dey et al. [8] Kizilaslan [9], Rasethuntsa and Nadar [10], Kızılaslan and Nadar [11], Ahmadi and Ghafouri [12], Akgül [13], Çetinkaya and Genç [14] , Kayal et al. [15], Jovanovic et al. [16], Kohansal and Shoaee [17], Heidari et al. [18] and the references therein.

In the stress-strength model, it is possible to know how long the component or system can stay safe under the influence of stress on average. The remaining strength Y of the system under stress X is a conditional random variable $(Y - X \mid Y > X)$. MRS of the component or system is expressed as the expected remaining strength exposed to the stress X, i.e. $\Phi = E(Y - X \mid Y > X)$. Guess et al. [19] gives the illustrations of mean residual life function to examine the strength of medium density fiberboard data. The MRS can be used to determine the safety thresholds for reliability under the stress component. The models of MRS of the k-out-of-n:F system, series, and parallel systems under the common stress have been presented by Gurler [20]. Later, Gurler et al. [21] studied MRS of the two-component parallel and series systems when dependent components exposed to random stress. In that paper, three different bivariate exponential models are chosen as strengths of the components, and simulation study for the MLEs of MRS for the two-component parallel system are carried out. Bairamov et al. [22] considered the MRS of a k-out-of-n:F, parallel, and series systems considering exchangeable strength under common stress. In 2019, Kızılaslan [23] presented the maximum likelihood and Bayes estimates of the MRS of the parallel systems in the stress-strength model.

A coherent system is a significant concept in reliability theory and survival analysis. For more details, the monograph by Barlow and Proschan [24] can be recommended. To optimize the lifetime of a coherent system, it is preferable to include redundant components (or spares) in the system. Standby redundancy is a commonly used technic to rise the system's reliability and frequently utilized in areas such as airplanes, data storage centers, or communications systems. In the literature, the reliability analysis of a coherent system that does not include standby components has been thoroughly considered, but not much examination is done for the estimation of the reliability of systems utilizing standby redundancy. Standby redundancy (cold standby), active redundancy (hot standby), and warm standby are the redundancy types that are used frequently. In the cold standby, redundancy components are put on standby and begin functioning when the original components fail. In such a system, standby components do not fail. In textile manufacturing systems, to analyze the reliability of a powerloom plant [25] and carbon recovery process applied in fertilizer plants [26] cold standby redundancy is preferred to increase the system reliability. Coit [27] handled optimization problem of cold standby for non-repairable series-parallel systems to increase the reliability and conclude that the cold standby promotes maximizing the system reliability. Recently, Dembińska et al. [28] examined the reliability problem of k-out-of-n systems with a cold standby unit where components have discrete lifetimes with stochastic ordering. In the hot standby, redundancy components are placed parallel to the original components and work simultaneously with them. These components are also called active redundant components. Therefore, the lifetimes of hot standby components can decrease even if they do not have to take over the operation. Reliability analysis of a system with two hot standby components is handled by [29] with a real case example using industrial data. Levitin et al. [30] evaluated the reliability function of 1-out-of-N:G systems that contain both hot and cold standby and examined cost minimization. The warm standby is said to be among the hot and the cold standby. It is called general standby because it contains both the hot standby and the cold standby. In such a system, the standby component works under lighter conditions. When the original component fails, the standby unit will be operating under regular stress in place of milder stress. The warm standby case for the units of general distributions was examined by Cha et al. [31]. Reliability analysis of a k-out-of-n:G system including one warm standby unit and also mean residual life of the systems are obtained by Eryilmaz [32]. The optimal redundancy allocation considering warm standby series and parallel systems is handled by Levitin et al. [33].

The performance of a coherent system arranged of n independent components may be improved by adding n standby redundancy components to the original components one by one or constructing a duplicate system using standby components similar to the original system. For instance, let a series system including n original components and n standby redundancy components. In this case, these standby components can be used either at component level or system level. Connecting a standby component(s) to a system increases the system reliability. That is why system engineers want to answer which type of standby redundancy can extend the lifetime at a n-component system.

Implementing standby redundancy to a series / parallel system prolongs its lifetime. Since in standby redundancy, redundant component is put into operation when the main component corrupts. Therefore, lifetime of the system is the sum of the lives of the original component and the standby component. Let $\phi\,(X_1, \dots, X_n)$ denote the lifetime of coherent system with component lifetimes $X_1, \dots, X_n$. The lifetime of the system after standby redundancy at component level is

$$T_C = \phi\,(X_1 + Y_1, \dots, X_n + Y_n)\,, \tag{1.1}$$

where X_i and Y_i are the lifetimes of component C_{X_i} and standby component C_{Y_i}, $i = 1, \dots, n$, respectively. The lifetime of the system with standby redundancy at system level is given as

$$T_S = \phi\,(X_1, \dots, X_n) + \phi\,(Y_1, \dots, Y_n)\,. \tag{1.2}$$

The following figures explain the system structures that are used in this thesis. Assume that we have a series system with components C_{X_i}, $i = 1, \dots, n$, as shown in the Figure 1.1. For this construction, standby redundancy at component level may be applied to such a system by inserting the standby components $C_{Y_1}, \dots, C_{Y_n}$, as in Figure 1.1 with the original components $C_{X_1}, \dots, C_{X_n}$ respectively. When we consider a set of original component and cold standby component connected at component level, the lifetime of each (C_{X_i}, C_{Y_i}) is $X_i + Y_i$. Since the whole system consists of the connection of these elements as serial, the smallest total lifetime of these pairs determine the system lifetime. Thus, for a series system at component level, the lifetime of the system is

$$T_C = \min\,(X_1 + Y_1, \dots, X_n + Y_n)\,. \tag{1.3}$$

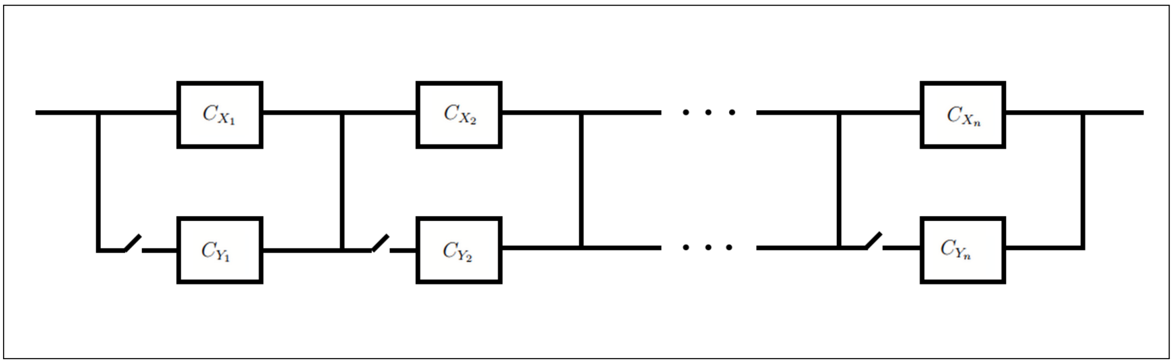

Figure 1.1. Standby at component level of a series system

A series system at system level consists of original components C_{X_i} and the cold standby components C_{Y_i} $i = 1, \dots, n$ as in the Figure 1.2. Adding a series system of standby components directly to the whole system represents the system level concerning redundancy. For our structure, the sum of the shortest lifetime of original components and the shortest lifetime of standby components determines the system lifetime. For a series system at system level, the lifetime of the system is given as

$$T_S = \min(X_1, \dots, X_n) + \min(Y_1, \dots, Y_n)\,. \tag{1.4}$$

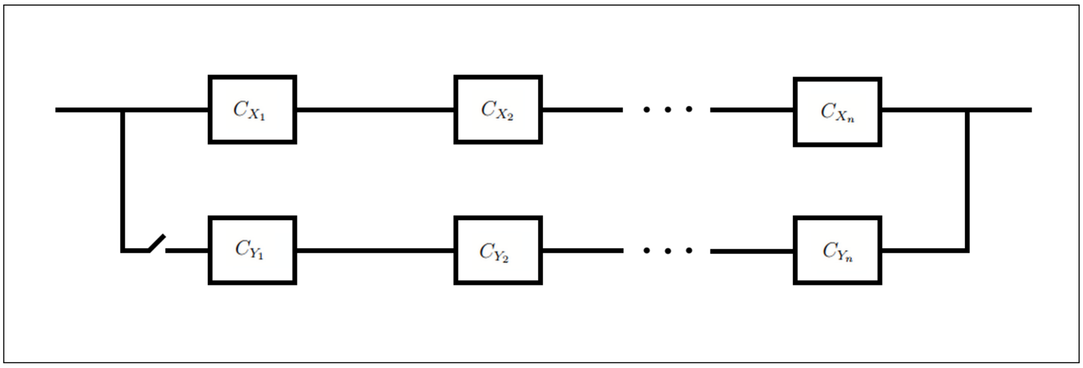

Figure 1.2. Standby at system level of a series system

Standby redundancy at component level for a parallel system is implemented by inserting standby components $C_{Y_1}, \dots, C_{Y_n}$ to each of the original components $C_{X_1}, \dots, C_{X_n}$ as in Figure 1.3, respectively. In this case, the lifetime of each (C_{X_i}, C_{Y_i}) is $X_i + Y_i$. For the whole parallel system to fail, all components must fail. Therefore, the one with the longest lifetime of these pairs determines the lifetime of the system. For a parallel system at component level, the lifetime of the system is as follows:

$$T_C = \max(X_1 + Y_1, \dots, X_n + Y_n) \tag{1.5}$$

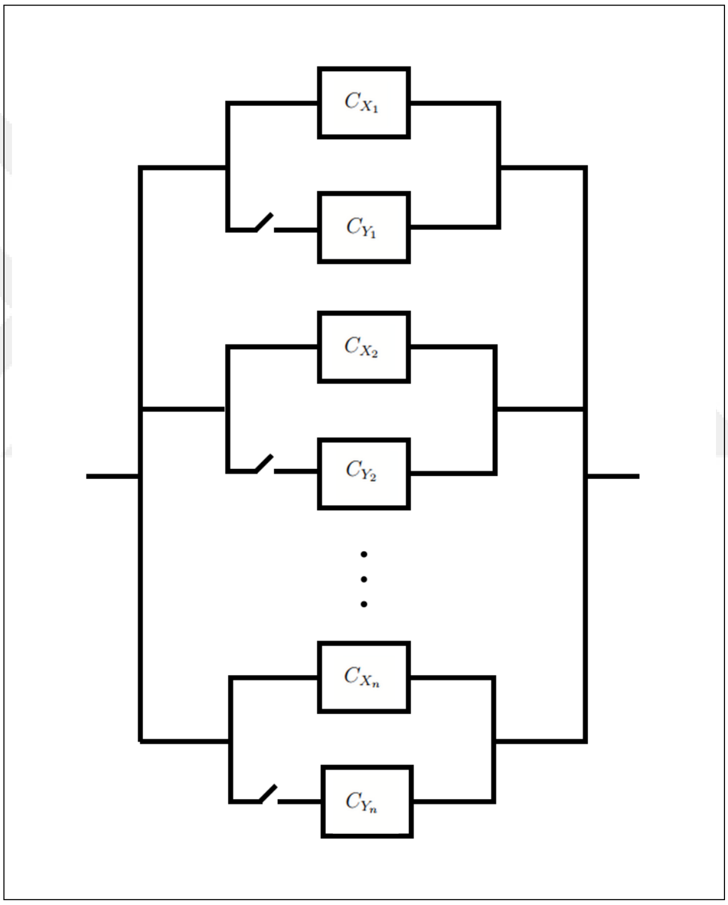

Figure 1.3. Standby at component level of a parallel system

For a parallel system that consists of original components C_{X_i} as in Figure 1.4, the standby components, C_{Y_i} $i = 1, \dots, n$, at system level can be settled to the original system as an alternative system. Since we settle two parallel systems as in Figure 1.4, when the original system fails, the cold standby system will operate. Thus, to compute the lifetime of the whole system, the longest lifetime of original components will be added to the longest lifetime of standby components. In this manner, at system level the lifetime is given by

$$T_S = \max(X_1, \dots, X_n) + \max(Y_1, \dots, Y_n)\,. \tag{1.6}$$

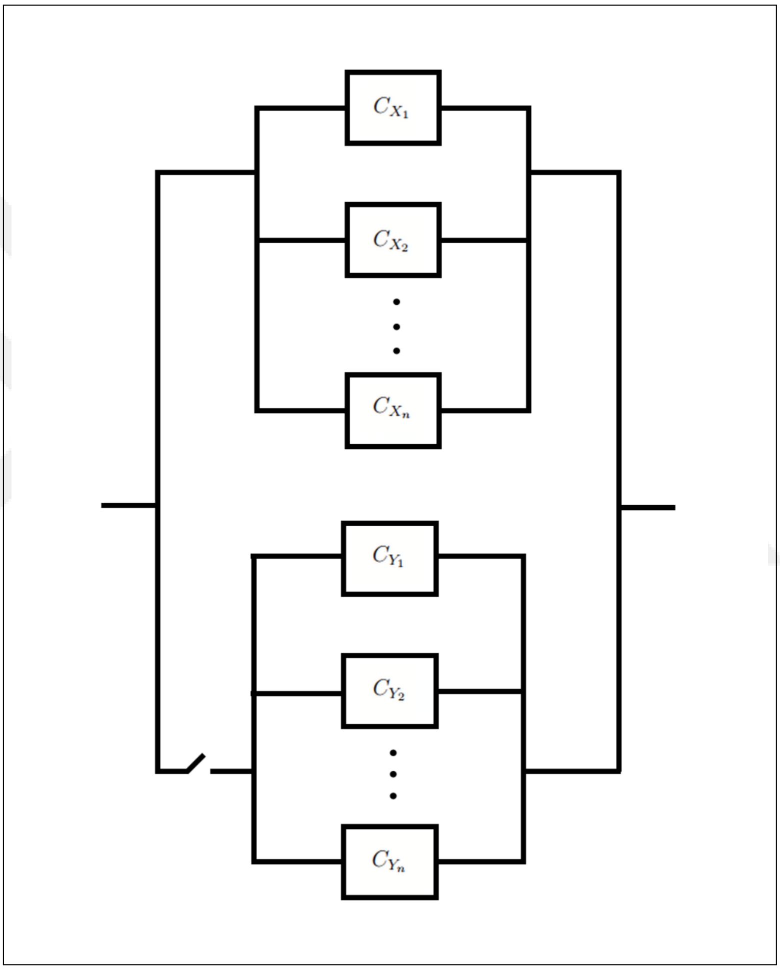

Figure 1.4. Standby at system level of a parallel system

In the literature, the stochastic comparison of coherent systems when the cold standby redundancy applied at component level and system level has been considered by many researchers. One can see the paper Shen and Xie [34] and Kumar [35]. Shen and Xie [34] studied the impact of standby redundancy at system and component levels in series and parallel systems, and compared the superiority of the levels. It is proved that standby redundancy is more efficient at the component (system) level for a series (parallel) system. The difference between component redundancy and system redundancy was investigated by Boland and El-Neweihi [36] concerning the hazard rate ordering. Boland and El-Neweihi [36] extended the stochastic comparison results of the cold standby levels for the series and parallel systems using the usual stochastic ordering and additionally the hazard rate ordering. They also obtained some results about the hazard rate ordering of the k-out-of-n:G system with cold standby redundancy. In view of different stochastic orders, Hazra and Nanda [37] indicate that sparing at the component level is surpassing to that at the system level. For the series system, Zhao et al. [38] presented the likelihood ratio ordering result for n exponential components when the active components and standby components are identical. Similar results for the parallel system were proved for n = 2 case. Eryilmaz and Tank [39] considered a series system with two dependent components and only one cold standby component. Eryilmaz [40] studied the effect of cold standby allocation to a coherent structure at system and component levels using mean time to failure ordering concluding that the effect of redundancy type mostly hinges on the structure of the system. Tuncel [41] studied the residual lifetime of a single component system with a cold standby component when the lifetimes of these components were dependent. Yan et al. [42] studied series and parallel structures of two constituent components with one standby redundancy component when all components follow exponential distribution. Roy and Gupta [43], [44] considered the reliability of a k-out-of-n and coherent systems arranged of two cold standby components, respectively. How or where to allocate redundancy components in a coherent system is another interesting problem. Chen et al. [45] discussed the problem of ideal allocation methods for two standby redundancies in a two-component series/parallel structure.

1.2. DEFINITIONS

Definition 1.2.1 (Structure function). The state of the system or each component is a discrete random variable which takes only two possible values that indicate a working state or failure state. If x_i represent the state of the component i for $1 \leq i \leq n$ and it is given by

$$x_i = \begin{cases} 1 & \text{if component } i \text{ works,} \\ 0 & \text{if component } i \text{ fails.} \end{cases} \tag{1.7}$$

The states of all components are represented by the component state vector $\mathbf{x} = (x_1, \ldots, x_n)$. The states of all components identify the state of the system which has Bernoulli distribution. If Φ denotes the state of the system, then

$$\Phi = \begin{cases} 1 & \text{if the system works,} \\ 0 & \text{if the system fails.} \end{cases} \tag{1.8}$$

and $\Phi = \Phi(\mathbf{x}) = \Phi(x_1, \ldots, x_n)$ is called the structure function of the system, see Kuo and Zuo [46].

The structure function of the series and parallel systems are written respectively by

$$\Phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \min\{x_1, \ldots, x_n\}, \tag{1.9}$$

$$\Phi(\mathbf{x}) = 1 - \prod_{i=1}^{n} (1 - x_i) = \max\{x_1, \ldots, x_n\}. \tag{1.10}$$

Definition 1.2.2. A component in a system is irrelevant if the state of this component does not change the state of the system. If a component is relevant, then the state of the system is dictated by at least one component in the system. That is, the system works whenever component i works and, the system corrupts in the event of failure of component i.

Definition 1.2.3. A system is called coherent if its structure function $\Phi(\mathbf{x})$ is non-decreasing in each x_i for $1 \leq i \leq n$ and each component is relevant.

Definition 1.2.4 (Order Statistics)**.** Let $X_1, \ldots, X_n$ be a set of random sample of size n from a population with the distribution function $F_X(x)$. The r^{th} order statistic is the r^{th} smallest value in the sample denoted by $X_{(r)}$ and $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r-1)} \leq X_{(r)} \leq \ldots \leq X_{(n)}$ are called the order statistics [47].

Definition 1.2.5 (The Distributions of the Smallest Order Statistics)**.** Let $X_1, \ldots, X_n$ be a random sample of size n with cdf $F(x)$. The distribution function of $X_{(1)} = \min(X_1, \ldots, X_n)$ is given as

$$
\begin{aligned}
F_{X_{(1)}}(t) &= P(\min(X_1, \ldots, X_n) \leq t) && (1.11)\\
&= 1 - P(X_1 > t, \ldots, X_n > t) && (1.12)\\
&= 1 - \prod_{i=1}^{n} P(X_i > t) && (1.13)\\
&= 1 - \prod_{i=1}^{n} (1 - P(X_i \leq t)) && (1.14)\\
&= 1 - \prod_{i=1}^{n} (1 - F(t)) && (1.15)\\
&= 1 - (1 - F(t))^n. && (1.16)
\end{aligned}
$$

Then, the pdf of $X_{(1)}$ is

$$
f_{X_{(1)}}(t) = \frac{d}{dt} F_{X_{(1)}}(t) = n(1 - F(t))^{n-1} f(t). \tag{1.17}
$$

for the continuous random variables.

Definition 1.2.6 (The Distributions of the Largest Order Statistics)**.** Let $X_1, \ldots, X_n$ be a random sample of size n with cdf $F(x)$. The distribution function of $X_{(n)} = \max(X_1, \ldots, X_n)$ is as follows:

$$F_{X_{(n)}}(t) = P(\max(X_1, \dots, X_n) \leq t) \tag{1.18}$$

$$= P(X_1 \leq t, \dots, X_n \leq t) \tag{1.19}$$

$$= \prod_{i=1}^{n} P(X_i \leq t) \tag{1.20}$$

$$= \prod_{i=1}^{n} F_{X_i}(t) = [F(t)]^n. \tag{1.21}$$

Hence, the pdf of $X_{(n)}$ is

$$f_{X_{(n)}}(t) = n[F(t)]^{n-1} f(t). \tag{1.22}$$

for the continuous random variables.

Definition 1.2.7 (The Stress-Strength Reliability)**.** Let X represents the random stress experienced by the system, and Y is the random strength to come through the stress. For the independent random variables X and Y, the reliability or stress strength parameter is defined as the probability of not failing and is equal to $R := P(X < Y)$. The system fails provided that the stress exceeds the strength. Let a random vector (X, Y) with the pdf $f(x, y \mid \theta)$. θ is an unknown scalar or vector-valued parameter in parameter space Θ. Based on the observations $(X_1, Y_1), \dots, (X_n, Y_n)$, the purpose is to estimate the reliability. The joint pdf of independent X and Y is expressed as

$$f(x, y \mid \theta) = f_X(x \mid \theta) f_Y(y \mid \theta), \tag{1.23}$$

where the number of observations for X and Y does not have to be equal. By definition, reliability can be obtained by

$$R(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y \mid \theta) I(x < y) dx dy \tag{1.24}$$

where $I(.)$ is the indicator function. If X and Y are independent having pdfs $f_X(x \mid \theta)$ and $f_Y(y \mid \theta)$ and cdfs $F_X(x \mid \theta)$ and $F_Y(y \mid \theta)$, respectively, $R(\theta)$ can be expressed as

$$R(\theta) = \int_{-\infty}^{\infty} F_X(z \mid \theta) f_Y(z \mid \theta) dz = \int_{-\infty}^{\infty} (1 - F_Y(z \mid \theta)) f_X(z \mid \theta) dz. \tag{1.25}$$

A comprehensive review of the topic is given by Kotz et al. [48] and the references therein. There exists a plentiful amount of work on the estimation problem of stress-strength reliability for several distributional assumptions on X and Y.

- **Inferential Methods for Bayesian Analysis**

  Over the last decades, the interest in Bayesian methods increases rapidly. Berger [49] gives an outline of the interdisciplinary application areas of Bayesian analysis and some approaches to Bayesian analysis. Especially for small samples, choosing a prior plays a leading role in Bayesian analysis. There exist various approaches, but the most common ones are objective, subjective and empirical approaches. For the empirical approach, it is possible to estimate the prior from the given data. The subjective approach depends on the researcher's belief and the previous information before the analysis. Berger [50] presents pros of the objective method and cons of the subjective method. According to that paper, some frame of references can help to define the objective approach. The posterior distribution is obtained depending on this choice. In Bayesian estimation, posterior density, as the combination of the prior distributions and likelihood, obtained in a closed-form requires the usage of analytical or numerical approximation methods. For this purpose, the following Lindley's approximation, Lindley [51] and Markov Chain Monte Carlo (MCMC) methods are implemented in this thesis.

- **The Lindley Approximation**

Lindley [51] presented an approximate method for the purpose of computing the ratio of two integrals. This operation, applied to the posterior expectation of the function $u(\theta)$ for a given **x**, is as follows:

$$E(u(\theta)\,|\mathbf{x}) = \frac{\int u(\theta)e^{Q(\theta)}d\theta}{\int e^{Q(\theta)}d\theta}, \tag{1.26}$$

where $u(\theta)$ is a smooth positive function of the parameter space, $Q(\theta) = \ell(\theta) + \rho(\theta)$, $\ell(\theta)$ is the logarithm of the likelihood function and $\rho(\theta)$ is the logarithm of the prior density of θ. Using Lindley's approximation, $E(u(\theta)\,|\mathbf{x})$ is approximately estimated by

$$E(u(\theta)\,|\mathbf{x}) = \left[u + \frac{1}{2}\sum_i\sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l\right]_{\hat{\theta}} \tag{1.27}$$

$$+ \text{ terms of order } n^{-2} \text{ or smaller,}$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_m)$, $i, j, k, l = 1, \dots, m$, $\hat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $L_{ijk} = \partial^3\ell/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$, and $\sigma_{ij} = (i,j)$th element in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters.

Remark. For the three parameter case $\theta = (\theta_1, \theta_2, \theta_3)$, Lindley's approximation gives the approximate Bayes estimate as

$$\hat{u}_B = E(u(\theta)\,|\mathbf{x}) = u + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) + 0.5\,[A\,(u_1\sigma_{11} + u_2\sigma_{12} \tag{1.28}$$

$$+u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})]\,,$$

evaluated at $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3)$, where

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \;\; i = 1, 2, 3, \tag{1.29}$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}, \; a_5 = 0.5(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}), \tag{1.30}$$

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331}, \tag{1.31}$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332}, \tag{1.32}$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333}. \tag{1.33}$$

An application of Lindley's approximation method for three-parameter case in reliability estimation can be found in the paper of Ahmadi and Ghafouri [12]. Akgül [13], Kohansal and Shoaee [17] implemented this approximation for the reliability estimation in multicomponent stress-strength models.

- **MCMC Method**

MCMC is a simulation technique based on an iterative scheme to sample from the posterior distributions using Markov chains. In Bayesian statistics, MCMC methods are preferred due to the adversity of sampling posterior distributions. For simple models, it is feasible to obtain posterior distributions analytically. Nevertheless, for most posterior densities, direct methods do not work. MCMC method provides useful insights into generate samples from the posterior distribution with the help of a Markov chain that converges to the posterior distribution and these samples are used to estimate the expectations of quantities of interest. Also, the simulation algorithm is applicable to high dimensional models. In recent years, MCMC methods, containing the Metropolis-Hastings algorithm and Gibbs sampler are preferred by many authors for the purpose of generating sample from models with high complexity. The detailed information about MCMC method can be attained in [52].

- **The Gibbs Sampling Algorithm**

The Gibbs sampler is a Monte Carlo Method that is proposed in Geman and Geman [53]. This method is suitable to obtain samples from the full conditional probability distributions of random variables (each parameter) and follows an iterative scheme. Namely, posterior samples of the parameters can be generated by using the posterior conditional density functions. Assume that one aims to generate N samples of $\theta = (\theta_1, \ldots, \theta_k)$ with pdf $p(\theta)$. Let $\theta_1, \ldots, \theta_k$ represent some grouping of θ and the full conditional distributions are given as below:

$$\pi(\theta_1 \mid \theta_2, \ldots, \theta_k; \mathbf{z}), \pi(\theta_2 \mid \theta_1, \ldots, \theta_k; \mathbf{z}), \ldots, \pi(\theta_k \mid \theta_1, \ldots, \theta_{k-1}; \mathbf{z}). \tag{1.34}$$

Gibbs sampling is useful when these full conditional distributions can be sampled effortlessly than the joint distribution. Then a draw $(\theta_1, \ldots, \theta_k)$ can be obtained according to the following steps:

(i) Determine the initial value $\theta^{(0)} = (\theta_1^{(0)}, \ldots, \theta_k^{(0)})$.

(ii) Start with $i = 1$.

(iii) Generate $\theta_1^{(i)}$ from $\pi_1^*(\theta_1 \mid \theta_2, \ldots, \theta_k; \mathbf{z})$.

(iv) Generate $\theta_2^{(i)}$ from $\pi_2^*(\theta_2 \mid \theta_1, \ldots, \theta_k; \mathbf{z})$.

(v) Generate $\theta_k^{(i)}$ from $\pi_k^*(\theta_k \mid \theta_1, \ldots, \theta_{k-1}; \mathbf{z})$.

(vi) Set $i = i + 1$ and repeat steps (iii)-(v) for $j = 1, \ldots, N$.

New sampled value of a parameter is called as an updated value.

- **The Metropolis-Hastings Algorithm**

Metropolis-Hastings sampler is introduced by Metropolis et al. [54] and then generalized by Hastings [55]. Actually, Gibbs sampler is a certain case of Metropolis Hastings algorithm. This method focuses on generating random samples from any distribution of complex form up to a normalizing constant. Let the full conditional density $\pi_1^*(\theta_1 \mid \theta_2, \ldots, \theta_k; \mathbf{z})$ be intractable and $q(\theta_1^* \mid \theta_2, \ldots, \theta_k; \mathbf{z})$ represents the proposal density which can generate θ_1^*. The M-H algorithm for $\pi_1^*(\theta_1 \mid \theta_2, \ldots, \theta_k; \mathbf{z})$ is given as below:

(i) Determine an initial value $\theta^{(0)} = (\theta_1^{(0)}, \dots, \theta_k^{(0)})$.

(ii) Propose a value for θ_1 by drawing $\theta^* \sim q(\theta_1^* \mid \theta_2^{(j-1)}, \dots, \theta_k^{(j-1)}; \mathbf{z})$.

(iii) Calculate the acceptance probability

$$\rho(\theta_1^{(j-1)},\theta^*)=\min\left[1,\frac{\pi_1^*(\theta_1^* \mid \theta_2^{(j-1)},\dots,\theta_k^{(j-1)};\mathbf{z})q(\theta_1^{(j-1)}|\theta_2^{(j-1)},\dots,\theta_k^{(j-1)};\mathbf{z})}{\pi_1^*(\theta_1^{(j-1)}|\theta_2^{(j-1)},\dots,\theta_k^{(j-1)};\mathbf{z})q(\theta_1^*|\theta_2^{(j-1)},\dots,\theta_k^{(j-1)};\mathbf{z})}\right]. \tag{1.35}$$

(iv) Generate $U \sim Uniform(0, 1)$.

(v) If $U \leq \rho(\theta_1^{(j-1)}, \theta^*)$, accept the proposal and set $\theta_1^{(j)} = \theta^*$. Otherwise, reject the proposal and let $\theta_1^{(j)} = \theta^{(j-1)}$.

(vi) Set $j = j + 1$ and repeat the procedure of the steps (i)-(iv) for $j = 1, \dots, N$.

1.3. THE AIM OF THE THESIS

The estimation problem of stress-strength reliability is reviewed in a widespread amount of studies. In recent years, although the standby redundancy structures have been considered in terms of stochastic ordering but to the best of our knowledge, the comparison of the statistical inference with standby redundancy in series and parallel systems has not been paid much attention. When stress and strength variables are independent and non-identical, UMVU, ML, and empirical Bayes estimators for R are given in [56]. MLE of the stress-strength reliability for a parallel system under the hybrid of active, warm, and cold standby components was studied by Siju and Kumar [57]. The reliability estimation when the standby redundancy system has a certain number of same subsystems with series structure was considered by Liu et al. [58] for the generalized half-logistic distribution hinge on progressive Type-II censoring sample.

In this thesis, we handle the estimation problem of stress-strength reliability and MRS for series and parallel systems when underlying distributions are exponential. For both systems, two types of cold standby redundancy allocation have been considered, namely, component level and system level. In the estimation part, classical and Bayesian approaches are handled depending on the system structure. ML, MCMC, and Lindley's approximation methods are implemented. Accordingly, due to the simulation results, comparisons for these methods are done with respect to MSE and ER values. Lastly, real-life studies are given to illustrate the presented estimates.

2. STATISTICAL ANALYSIS FOR THE SERIES SYSTEM WITH STANDBY COMPONENTS

In this chapter, we aim to introduce the stress-strength reliability and MRS of the series system when the standby components are applied at system and component levels. The major contribution of this chapter is to investigate the performance of the stress-strength reliability and MRS estimators of the series system in conjunction with cold standby cases. System and standby components are supposed to be independent but not identically distributed random variables that follow the exponential distribution. Moreover, a comparison of the stress-strength reliability of the interested series systems and the series system without standby components was made with graphics.

2.1. MODEL DEFINITION

Consider a series system with n independent and identically distributed components, also known as n-out-of-n system. Suppose that $X_1, \dots, X_n$ denote the lifetimes of the strength components and $Y_1, \dots, Y_n$ are the lifetimes of independent standby strength components following exponential distribution with parameters α and β, respectively. These standby redundancy components can be added to the system at component level or system level. Suppose that T is the common stress variable that is experienced by the series system and follows the exponential distribution with parameter θ.

• For the series system at component level, the standby redundancy components are added to each particular component of the system. The related figure belongs to such a system is given in Figure 1.1. In this situation, the total lifetime of each strength component consists of $Z_i = X_i + Y_i$, $i = 1, \dots, n$. Then, the cdf and pdf of Z_i, $i = 1, \dots, n$ can be written as

$$F_{Z_i}(z) = P(X + Y \leq Z) = \int_{y=0}^{z} \int_{x=0}^{z-y} f_x(x) f_y(y) dx dy \tag{2.1}$$

$$= \int_0^z F_x(z-y) f_y(y) dy \tag{2.2}$$

$$= \begin{cases} 1+\frac{1}{\beta-\alpha}(\alpha e^{-\beta z}-\beta e^{-\alpha z}) & ,\alpha \neq \beta \\ 1-e^{-\alpha z}(1+\alpha z) & ,\alpha = \beta \end{cases} , \tag{2.3}$$

and

$$f_{Z_i}(z) = \begin{cases} \frac{\alpha\beta}{\beta-\alpha}(e^{-\alpha z}-e^{-\beta z}) & ,\alpha \neq \beta \\ \alpha^2 z e^{-\alpha z} & ,\alpha = \beta \end{cases} . \tag{2.4}$$

We have seen that Z_i, $i = 1, \dots, n$ follow the Gamma distribution with parameters α and 2 when the active strength and standby redundancy components are identical, i.e. $\alpha = \beta$. Since T is the common stress variable in the series system made up of Z_i, $i = 1, \dots, n$, the stress-strength reliability of the system at component level is

$$R_{SC} = P(\min(Z_1, \dots, Z_n) > T) = P\left(Z_{(1)} > T\right) \tag{2.5}$$

$$= \int_0^\infty P(Z_{(1)} > T \,|\, T = t) f_T(t)dt \tag{2.6}$$

$$= \int_0^\infty [P(Z_{(1)} > t)]^n f_T(t)dt \tag{2.7}$$

$$= \int_0^\infty \left(\frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\beta - \alpha}\right)^n \theta e^{-\theta t}dt \tag{2.8}$$

$$= \frac{1}{(\beta-\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \frac{\alpha^k \beta^{n-k}\theta}{\alpha(n-k)+\beta k+\theta}. \tag{2.9}$$

Therefore we obtain the reliability function as given below:

$$R_{SC} = \begin{cases} \frac{1}{(\beta-\alpha)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\alpha^k \beta^{n-k} \theta}{\alpha(n-k)+\beta k+\theta}, & \alpha \neq \beta \\ \sum_{k=0}^{n} \binom{n}{k} \Gamma(k+1) \frac{\alpha^k \theta}{(\alpha n+\theta)^{k+1}}, & \alpha = \beta \end{cases} \tag{2.10}$$

where $Z_{(1)} = \min(Z_1, ..., Z_n)$ and $\Gamma(.)$ is the Gamma function.

• At the system level, the standby redundancy components form a duplicate series system for the original series system. Hence, the total lifetime of these series systems becomes $Z_{(1)} = X_{(1)} + Y_{(1)}$ where $X_{(1)}$ and $Y_{(1)}$ follow the exponential distribution with parameters $n\alpha$ and $n\beta$, respectively. Then, the pdf and cdf of $Z_{(1)}$ can be expressed as

$$f_{Z_{(1)}}(z) = \int_0^z f_{X_{(1)}}(z-y) f_{Y_{(1)}}(y) dy \tag{2.11}$$

$$= \frac{n\alpha\beta}{\beta-\alpha}(e^{-\alpha nz} - e^{-\beta nz}) \tag{2.12}$$

and

$$F_{Z_{(1)}}(z) = \int_0^z \frac{n\alpha\beta}{\beta-\alpha}(e^{-\beta tn} - e^{-\alpha tn}) dt \tag{2.13}$$

$$= 1 + \frac{\alpha e^{-\beta nz} - \beta e^{-\alpha nz}}{\beta - \alpha} \tag{2.14}$$

for $\alpha \neq \beta$. Also, Z_i, $i = 1, \dots, n$ follow the Gamma distribution with parameters $n\alpha$ and 2 for $\alpha = \beta$. Under the common stress variable T, the stress-strength reliability of this system is

$$R_{SS} = P(Z_{(1)} > T) = \int_0^\infty P(Z_{(1)} > T | T = t) f_T(t) dt \tag{2.15}$$

$$= \int_0^\infty -\frac{\alpha e^{-\beta nz} - \beta e^{-\alpha nz}}{\beta - \alpha} \theta e^{-\theta t} dt \tag{2.16}$$

$$= \frac{\theta \left[n(\alpha + \beta) + \theta\right]}{(\alpha n + \theta)(\beta n + \theta)}, \tag{2.17}$$

for any α and β. In the following sections, the equal strength parameters case ($\alpha = \beta$) has not been considered. In that case, the reliability problem is similar to the simple stress strength reliability of the Gamma components. For more details about this case see the following papers: Ali et al. [59] and Nojosa and Rathie [60].

Moreover, $2n$-component series system without standby components is considered in the stress-strength model. Let $X_1, \ldots, X_{2n}$ be the lifetimes of strength components and T is the common stress variable which follow the exponential distribution with parameter α and θ, respectively. In that situation, $2n$ strength components constitute a series system, $X_{(1)}$ is the lifetime of series system and $X_{(1)} \sim exp(2n\alpha)$. Accordingly, the stress-strength reliability of this series system is

$$R = P(X_{(1)} > T) = \frac{\theta}{2\alpha n + \theta}. \tag{2.18}$$

It is known that adding standby components increases system reliability. In this regard, we want to show how these standby components affect the stress-strength reliability of the system in our case with graphs. In Figure (2.1), the stress-strength reliability values of the aforementioned three different series systems are plotted by using Equations (2.10), (2.17) and (2.18) based on different parameter sets. It is seen that standby redundancy cause an escalating effect on system reliability. Therefore, the standby systems can be preferable under suitable circumstances.

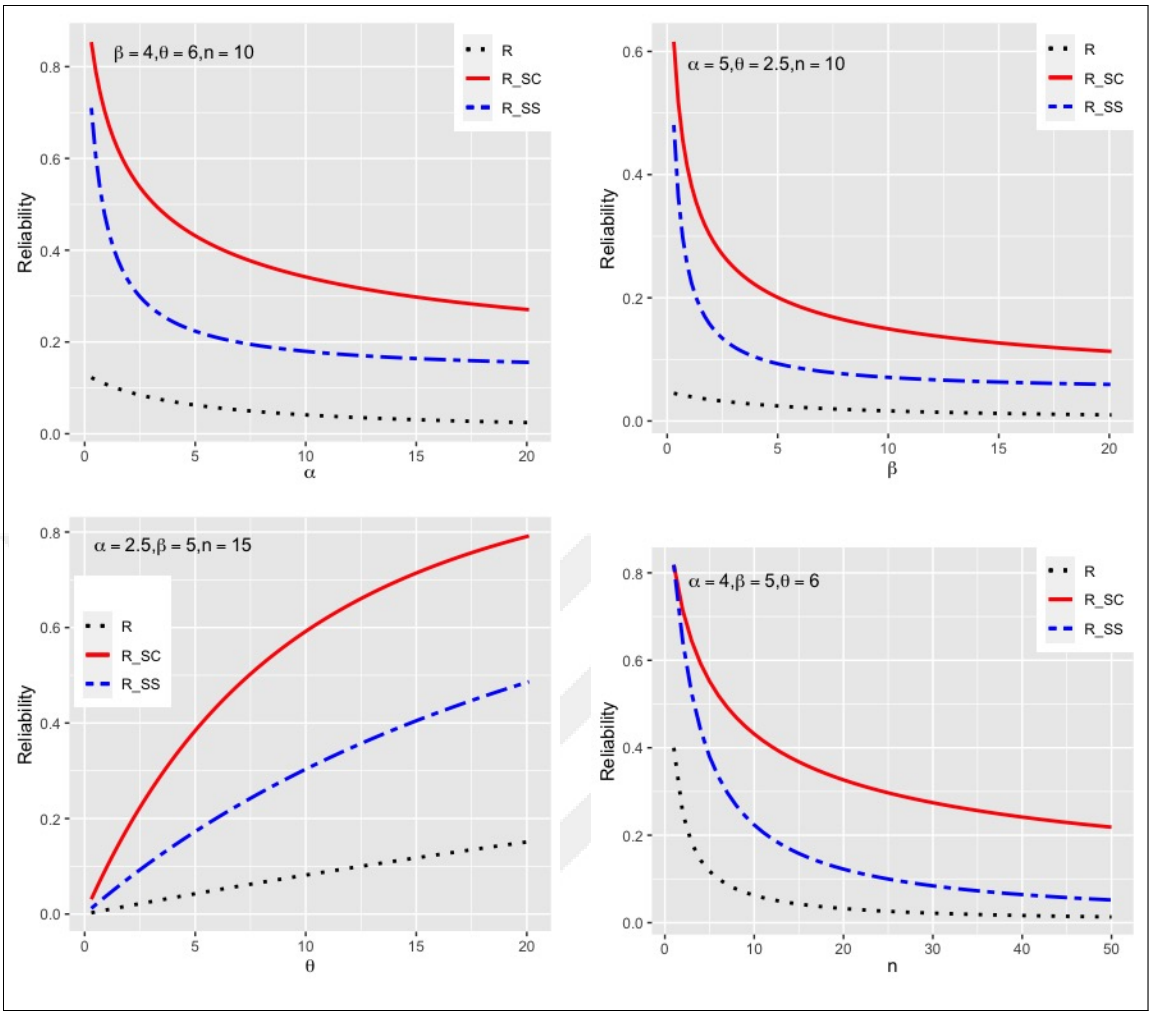

Figure 2.1. Plots for R, R_{SC} and R_{SS}

The rest of this chapter is settled as follows. In Section 2.2, we obtain ML estimator and Bayesian estimator of the stress-strength reliability; R_{SC} and MRS; Φ_{SC} for the series system with cold standby redundancy at component level. The ACIs and the HPD credible intervals for R_{SC} and Φ_{SC} are also given in this section. Then, Lindley's approximation and MCMC methods using hybrid M-H and Gibbs sampling algorithms are implemented under Bayesian estimation. In Section 2.3, a simulation study is performed to analyze the performance of the aforementioned estimates for the stress-strength reliability and MRS, and also real data analysis is given. In Section 2.4, we derive ML and MCMC estimators of the stress-strength reliability; R_{SS} and MRS; Φ_{SS} for the series system with cold standby redundancy at system level. The ACIs and the HPD credible intervals for R_{SS} and Φ_{SS} are also constructed. In Section 2.5, we carry out a simulation study to compare the performance of the aforementioned estimates and then an analysis of a real data set is given for the purpose of illustration.

Finally, we conclude the work on the series system with some remarks in Section 2.6.

2.2. ESTIMATION OF R_{SC} AND Φ_{SC}

In this section, estimation of the stress-strength reliability and MRS of the series system with standby redundancy at component level are investigated.

2.2.1 MLE of R_{SC}

Assume that m systems are put on a test each with n original components and n cold standby components in the series system. Then, the strength data are represented as $Z_{i1}, \ldots Z_{in}$, $i = 1, \ldots, m$ and stress is $T_i, i = 1, \ldots, m$. The likelihood function of α, β and θ for the observed sample is

$$L(\alpha,\beta,\theta;\mathbf{z},\mathbf{t}) = \prod_{i=1}^{m}\left(\prod_{j=1}^{n} f_Z(z_{ij})\right) f_T(t_i) \tag{2.19}$$

$$= \left(\frac{\alpha\beta}{\beta-\alpha}\right)^{nm} \exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right)\theta^m e^{-\theta\sum\limits_{i=1}^{m} t_i}. \tag{2.20}$$

The log-likelihood function is

$$\ell(\alpha,\beta,\theta;\mathbf{z},\mathbf{t}) = nm\left(\ln(\alpha\beta) - \ln(\beta-\alpha)\right) + m\ln\theta + \sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right) - \theta\sum_{i=1}^{m} t_i. \tag{2.21}$$

By partially differentiating (2.21) with respect to α and β, we have the following likelihood equations as

$$\frac{\partial \ell}{\partial \alpha} = nm\left(\frac{1}{\alpha} + \frac{1}{\beta - \alpha}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}e^{-\alpha z_{ij}}}{e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}} = 0, \tag{2.22}$$

$$\frac{\partial \ell}{\partial \beta} = nm\left(\frac{1}{\beta} - \frac{1}{\beta - \alpha}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}e^{-\beta z_{ij}}}{e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}} = 0. \tag{2.23}$$

The MLEs of the parameters α and β, i.e. $\widehat{\alpha}$ and $\widehat{\beta}$, can be acquired by solving the non-linear equations in (2.22) and (2.23) simultaneously. This non-linear equation system can be solved by using numerical methods such as Newton–Raphson and Broyden's methods. The MLE of θ is derived as $\widehat{\theta} = m/\sum_{i=1}^{m} t_i$. After obtaining $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\theta}$, the MLE of R_{SC}, i.e. $\widehat{R}_{SC}^{MLE}$, is computed from Equation (2.10) by applying the invariance property of MLE.

2.2.2 Asymptotic Distribution and Confidence Interval for R_{SC}

The observed information matrix of $\boldsymbol{\tau} = (\alpha, \beta, \theta)$ is defined as

$$J(\boldsymbol{\tau}) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \theta} \\ \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \beta} & \frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}. \tag{2.24}$$

In our case, the elements of this matrix are derived as $J_{13} = J_{31} = J_{23} = J_{32} = 0$, $J_{33} = m/\theta^2$,

$$J_{11} = nm\left(\frac{1}{\alpha^2} - \frac{1}{(\beta - \alpha)^2}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1 - e^{-z_{ij}(\beta-\alpha)})^2}, \tag{2.25}$$

$$J_{12} = J_{21} = \frac{nm}{(\beta - \alpha)^2} - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1 - e^{-z_{ij}(\beta-\alpha)})^2}, \tag{2.26}$$

$$J_{22} = nm\left(\frac{1}{\beta^2} - \frac{1}{(\beta-\alpha)^2}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1-e^{-z_{ij}(\beta-\alpha)})^2}. \tag{2.27}$$

The expectation of the observed information matrix $I(\boldsymbol{\tau}) = E\left(J(\boldsymbol{\tau})\right)$ is not available in closed form. It can be evaluated by using numerical integration methods. Then, $\hat{R}_{SC}^{MLE}$ is asymptotically normal with mean R_{SC} and asymptotic variance

$$\sigma_{R_{SC}}^2 = \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial R_{SC}}{\partial \boldsymbol{\tau}_i}\frac{\partial R_{SC}}{\partial \boldsymbol{\tau}_j} I_{ij}^{-1}, \tag{2.28}$$

where I_{ij}^{-1} is the $(i,j)^{th}$ element of the inverse of $I(\boldsymbol{\tau})$, see Rao [61].

$$\sigma_{R_{SC}}^2 = \left(\frac{\partial R_{SC}}{\partial \alpha}\right)^2 I_{11}^{-1} + 2\frac{\partial R_{SC}}{\partial \alpha}\frac{\partial R_{SC}}{\partial \beta} I_{12}^{-1} + \left(\frac{\partial R_{SC}}{\partial \beta}\right)^2 I_{22}^{-1} + \left(\frac{\partial R_{SC}}{\partial \theta}\right)^2 I_{33}^{-1} \tag{2.29}$$

Note that $I(\boldsymbol{\tau})$ can be replaced by $J(\boldsymbol{\tau})$ when $I(\boldsymbol{\tau})$ is not obtained. Therefore, an asymptotic $100(1-\gamma)$ percent confidence interval for R_{SC} is obtained as $(\hat{R}_{SC}^{MLE} \pm z_{\gamma/2}\hat{\sigma}_{R_{SC}})$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_{SC}}$ is the value of $\sigma_{R_{SC}}$ at the MLE of the unknown parameters.

2.2.3 Bayesian Estimation of R_{SC}

In this subsection, we consider Bayesian estimation of R_{SC}. In the Bayes approach, it is assumed that we have prior information about the unknown parameters. Suppose that α, β and θ have independent gamma priors with parameters (a_i, b_i), $i = 1, 2, 3$, respectively. The pdf of a gamma random variable X with parameters (a,b) is given by

$$g(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb},\ x > 0,\ a,b > 0, \tag{2.30}$$

and expressed by Gamma(a,b). The joint posterior density function of α, β and θ is

$$\pi(\alpha,\beta,\theta \mid \mathbf{z},\mathbf{t}) = \frac{L(\alpha,\beta,\theta;\mathbf{z},\mathbf{t})g(\alpha)g(\beta)g(\theta)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha,\beta,\theta;\mathbf{z},\mathbf{t})g(\alpha)g(\beta)g(\theta)\, d\alpha\, d\beta d\theta} \tag{2.31}$$

and after some computations we have

$$\pi(\alpha,\beta,\theta|\mathbf{z},\mathbf{t}) = I(\mathbf{z},\mathbf{t})\alpha^{nm+a_1-1}\beta^{nm+a_2-1}(\beta-\alpha)^{-nm}\theta^{a_3+m-1}e^{-\alpha b_1-\beta b_2-\theta\left(b_3+\sum_{i=1}^{m} t_i\right)} z_{\alpha,\beta} \tag{2.32}$$

where $I(\mathbf{z},\mathbf{t})$ is the normalizing constant

$$\frac{I(\mathbf{z},\mathbf{t})^{-1}}{\Gamma(a_3+m)}\left(b_3+\sum_{i=1}^{m} t_i\right)^{a_3+m} = \int_0^\infty \int_0^\infty \left(\frac{\alpha\beta}{\beta-\alpha}\right)^{nm} \alpha^{a_1-1}\beta^{a_2-1}e^{-\alpha b_1-\beta b_2} z_{\alpha,\beta} d\alpha d\beta \tag{2.33}$$

where

$$z_{\alpha,\beta} = \exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right). \tag{2.34}$$

Bayes estimator of R_{SC}, i.e. $\widehat{R}_{SC}^{Bayes}$, under the SE loss function is

$$\widehat{R}_{SC}^{Bayes} = \int_0^\infty \int_0^\infty \int_0^\infty R_{SC}\pi(\alpha, \beta, \theta; \mathbf{z}, \mathbf{t}) d\alpha d\beta d\theta. \tag{2.35}$$

Since the above integral is not computed analytically, some approximation methods are required to obtain approximate Bayes estimate. In the next part, we consider the Lindley's approximation and MCMC methods.

• **Lindley's approximation**

For the three parameter case, let $(\theta_1, \theta_2, \theta_3) \equiv (\alpha, \beta, \theta)$, and $u \equiv u(\alpha, \beta, \theta) = R_{SC}$ from Equation (2.10). First, $\sigma_{ij},\ i, j = 1, 2, 3$ are computed with the partial derivatives $L_{ij} = -J_{ij},\ i, j = 1, 2, 3$. Second, we evaluate the constants as $\rho_1 = ((a_1 - 1)/\alpha) - 1$, $\rho_2 = ((a_2 - 1)/\beta) - b_2$ and $\rho_3 = ((a_3 - 1)/\theta) - b_3$ using the logarithm of the prior density.

$$L_{111} = 2nm\left(\frac{1}{\alpha^3} + \frac{1}{(\beta-\alpha)^3}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1 + e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1 - e^{-z_{ij}(\beta-\alpha)}\right)^3}, \tag{2.36}$$

$$L_{121} = L_{112} = -2nm\frac{1}{(\beta-\alpha)^3} + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1 + e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1 - e^{-z_{ij}(\beta-\alpha)}\right)^3}, \tag{2.37}$$

$$L_{122} = L_{221} = 2nm\frac{1}{(\beta-\alpha)^3} - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1 + e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1 - e^{-z_{ij}(\beta-\alpha)}\right)^3}, \tag{2.38}$$

$$L_{222} = 2nm\left(\frac{1}{\beta^3} - \frac{1}{(\beta-\alpha)^3}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1 + e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1 - e^{-z_{ij}(\beta-\alpha)}\right)^3} \tag{2.39}$$

and $L_{333} = 2m/\theta^3$. Also, $u_{ij},\ i, j = 1, 2, 3$ are computed by taking the partial derivative of R_{SC} from Equation (2.10). Lastly, we obtain $A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + \sigma_{22}L_{221}$, $B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + \sigma_{22}L_{222}$, and $C = \sigma_{33}L_{333}$. After that, the approximate Bayes estimator of R_{SC} is given as

$$\hat{R}_{SC}^{Lindley} = R_{SC} + [u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5] + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})], \quad (2.40)$$

where all the parameters are evaluated at MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$.

• **MCMC method**

The joint posterior density function of α, β and θ given data is stated in (2.32). Then, the marginal posterior density functions of the parameters are given respectively as

$$\theta\,|\mathbf{t} \sim \text{Gamma}\left(m + a_3, b_3 + \sum_{i=1}^{m} t_i\right), \quad (2.41)$$

$$\pi(\alpha\,|\beta, \mathbf{z}) \;\propto\; \alpha^{nm+a_1-1}(\beta-\alpha)^{-nm}e^{-\alpha b_1}\exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right), \quad (2.42)$$

$$\pi(\beta\,|\alpha, \mathbf{z}) \;\propto\; \beta^{nm+a_2-1}(\beta-\alpha)^{-nm}e^{-\beta b_2}\exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right). \quad (2.43)$$

Hence, samples of θ can be readily generated by using a gamma distribution. Since the marginal posterior distributions of α and β are not well-known, it is not feasible to generate sample directly by applying standard methods. Hence, we use hybrid M-H and Gibbs sampling algorithm to generate samples from $\pi(\alpha\,|\beta, \mathbf{z})$ and $\pi(\beta\,|\alpha, \mathbf{z})$, (for more details see Gelman et al. [52] and Tierney [62]).

Step 1: Start with initial guess $\alpha^{(0)}, \beta^{(0)}$.

Step 2: Set $i = 1$.

Step 3: Generate $\theta^{(i)}$ from Gamma$(m + a_3, b_3 + \sum_{J=1}^{m} t_J)$.

Step 4: Generate $\alpha^{(i)}$ from $\pi(\alpha\,|\beta,\mathbf{z})$ using the Metropolis-Hastings algorithm with the proposal distribution $q_1(\alpha) \equiv N(\alpha^{(i-1)},1)$ as follows:

(i) Let $v = \alpha^{(i-1)}$.

(ii) Generate w from the proposal distribution q_1.

(iii) Let $p(v,w) = \min\left\{1, \dfrac{\pi(w\,|\beta^{(i-1)},\mathbf{z})\; q_1(v)}{\pi(v\,|\beta^{(i-1)},\mathbf{z})\; q_1(w)}\right\}$.

(iv) Generate u from Uniform$(0,1)$. If $u \le p(v,w)$, then accept the proposal and set

$\alpha^{(i)} = w$; otherwise, set $\alpha^{(i)} = v$.

Step 5: Similarly, $\beta^{(i)}$ is generated from $\pi(\beta\,|\alpha,\mathbf{z})$ using the Metropolis-Hastings algorithm with the proposal distribution $q_2(\beta) \equiv N(\beta^{(i-1)},1)$.

Step 6: Compute the $R_{SC}^{(i)}$ at $(\alpha^{(i)},\beta^{(i)},\theta^{(i)})$.

Step 7: Set $i = i+1$.

Step 8: Repeat the procedure of the Steps 2-7 and get the posterior sample $R_{SC}^{(i)}$ for $i = 1,\dots,N$.

This sample is wielded to compute Bayes estimate and to set the HPD credible interval for R_{SC}. Bayes estimate of R_{SC} under a SE loss function is expressed as

$$\hat{R}_{SC}^{MCMC} = \frac{1}{N-M}\sum_{i=M+1}^{N} R_{SC}^{(i)}, \tag{2.44}$$

where M is the burn-in period. The HPD $100(1-\gamma)$ percent credible interval of R_{SC} is obtained by the approach of Chen and Shao [63].

2.2.4 Inference on Φ_{SC}

The MRS of the series system under the stress T, is the expected remaining strength, and defined by the following conditional expectation $\Phi_{SC} = E(Z_{(1)} - T \,|\, Z_{(1)} > T)$. The cdf of the conditional random variable $\psi \equiv (Z_{(1)} - T \,|\, Z_{(1)} > T)$ is

$$F_{\psi}(x) = P(Z_{(1)} - T \le x \,|\, Z_{(1)} > T) = \frac{P(Z_{(1)} \le T + x, Z_{(1)} > T)}{R_{SC}}. \tag{2.45}$$

Then, conditioning on $T = t$, we have

$$P(Z_{(1)} \le T + x, Z_{(1)} > T) = \int_0^{\infty} P(t < Z_{(1)} \le t + x)\, dF_T(t) \tag{2.46}$$

$$= \int_0^{\infty} [F_{Z_{(1)}}(t + x) - F_{Z_{(1)}}(t)]\, dF_T(t) \tag{2.47}$$

$$= \int_0^{\infty} \left[\frac{(\beta e^{-\alpha t} - \alpha e^{-\beta t})^n}{(\beta - \alpha)^n} - \frac{(\beta e^{-\alpha(t+x)} - \alpha e^{-\beta(t+x)})^n}{(\beta - \alpha)^n}\right] \theta e^{-\theta t} dt \tag{2.48}$$

$$\equiv \frac{I_1 - I_2}{(\beta - \alpha)^n}, \tag{2.49}$$

for $\alpha \ne \beta$. After some computations, we obtain

$$I_1 = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{\alpha^i \beta^{n-i} \theta}{[\alpha(n - i) + \beta i + \theta]} \tag{2.50}$$

and

$$I_2 = e^{-\alpha n x} \sum_{i=0}^{n} \binom{n}{i} (-1)^i e^{-ix(\beta-\alpha)} \frac{\alpha^i \beta^{n-i} \theta}{[\alpha(n-i) + \beta i + \theta]}. \tag{2.51}$$

Hence, the cdf and pdf of ψ are given by

$$F_\psi(x) = \frac{1}{R_{SC}(\beta-\alpha)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{\alpha^i \beta^{n-i} \theta \left(1 - e^{-x(\alpha(n-i)+\beta i)}\right)}{[\alpha(n-i) + \beta i + \theta]}, \tag{2.52}$$

and

$$f_\psi(x) = \frac{1}{R_{SC}(\beta-\alpha)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{\alpha^i \beta^{n-i} \theta [\alpha(n-i) + \beta i] e^{-x(\alpha(n-i)+\beta i)}}{[\alpha(n-i) + \beta i + \theta]}, \tag{2.53}$$

for $\alpha \neq \beta$. Therefore, the MRS of our series system is obtained as

$$\Phi_{SC} = E(\psi) = \int_0^\infty x f_\psi(x) dx \tag{2.54}$$

$$= \frac{1}{R_{SC}(\beta-\alpha)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{\alpha^i \beta^{n-i} \theta}{[\alpha(n-i) + \beta i][\alpha(n-i) + \beta i + \theta]} \tag{2.55}$$

for $\alpha \neq \beta$. Moreover, Φ_{SC} can be easily derived as

$$\Phi_{SC} = \frac{1}{R_{SC}} \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \Gamma(i+1) \frac{\alpha^{j-1} \theta}{n^{i-j+1} (\alpha n + \theta)^{j+1}}, \tag{2.56}$$

for $\alpha = \beta$.

2.2.4.1 Estimation of Φ_{SC}

The ML estimate of Φ_{SC}, i.e. $\widehat{\Phi}_{SC}^{MLE}$, is computed from (2.55) by using the invariance property of MLE. It is clear that $\widehat{\Phi}_{SC}^{MLE}$ is asymptotically normal with mean Φ_{SC} and asymptotic variance is computed by using the formula below:

$$\sigma_{\Phi_{SC}}^2 = \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial \Phi_{SC}}{\partial \boldsymbol{\tau}_i}\frac{\partial \Phi_{SC}}{\partial \boldsymbol{\tau}_j} I_{ij}^{-1}, \tag{2.57}$$

where I_{ij}^{-1} is the $(i,j)^{th}$ element of the inverse of $I(\boldsymbol{\tau})$. Hence, an asymptotic $100(1-\gamma)$ percent confidence interval for Φ_{SC} is given by $(\widehat{\Phi}_{SC}^{MLE} \pm z_{\gamma/2}\widehat{\sigma}_{\Phi_{SC}})$ where $\widehat{\sigma}_{\Phi_{SC}}$ is the value of $\sigma_{\Phi_{SC}}$ at the MLE of the parameters.

Under the setup made in Subsection 2.2.3, the Bayes estimator of Φ_{SC} under the SE loss function is as follows:

$$\widehat{\Phi}_{SC}^{Bayes} = \int_0^\infty \int_0^\infty \int_0^\infty \Phi_{SC}\, \pi(\alpha,\beta,\theta \,|\, \mathbf{z},\mathbf{t})\; d\alpha d\beta d\theta. \tag{2.58}$$

Similar to the reliability case, since the above integral can not be computed analytically, the Bayes estimate of Φ_{SC} under SE loss function is derived by using Lindley's approximation. It is omitted because of a similar procedure is mentioned in the reliability case.

2.3. NUMERICAL RESULTS OF R_{SC} AND Φ_{SC}

2.3.1 Simulation Study

In this section, we carried out a Monte Carlo simulation study to examine performance of the all the proposed estimates. The point and interval estimates of R_{SC} and Φ_{SC} are computed based on the Monte Carlo simulations. The performances of the point estimates are compared via MSE and ER for ML and Bayes estimates, respectively. The ER of θ for the estimate $\widehat{\theta}$ is given by

$$ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta_i)^2, \tag{2.59}$$

under the SE loss function. Average lengths of the confidence interval and CPs provide a comparison between the interval estimates. The CP of a confidence interval is the proportion of time that the interval contains the parameter of interest. All results are obtained based on 2500 replications. The simulation studies have been carried out by using MATLAB and statistical software R [64].

In MLE case of the simulation study, we use the Broyden or Newton-Raphson methods to solve the non-linear equations using the nleqslv [65] package in software R.

In Bayesian case of the simulation study, we have selected hyperparameters as $a_1 = \alpha$, $a_2 = \beta$, $a_3 = \theta$, $b_1 = b_2 = b_3 = 1$ for the informative prior case and $a_i = b_i = 0$, $i = 1, 2, 3$ for the non-informative prior case. Bayes estimates of the stress-strength reliability and MRS of the systems are derived by using Lindley's approximation and MCMC method. In the MCMC cases, we run two MCMC chains with fairly different initial values and generate 5000 iterations for each chain. In order to reduce the effect of the starting distribution, the first 2500 results of each sequence are discarded, which is called burn-in. So as to cut off the dependence between the results in the Markov chain, only every d^{th} draw of the chain is saved, which is called thinning. In our cases, Bayesian MCMC estimates are evaluated by the means of the every 5^{th} sampled values after the discarding procedure. Moreover, the convergency of the MCMC chains has been monitored by using the scale reduction factor estimate in Gelman et al. [52]. The estimate is given by $\sqrt{Var(\psi)/W}$, where ψ is the estimand of interest, $Var(\psi) = (n-1)W/n + B/n$ with the iteration number n for each chain, the between-sequence variance B and the within-sequence variance W. In the following simulation studies, the scale factor values of all MCMC estimators are less than 1.1. It is an adequate value for their convergence.

In Tables 2.1 and 2.2, we have reported the ML and Bayesian point estimates of R_{SC} and corresponding interval estimates for different component (n) and sample (m) sizes. In this case, the ACIs are evaluated based on Fisher information matrix. Bayes estimates of R_{SC} are obtained by using both Lindley's approximation and MCMC method. The true values of the parameters are taken as $(\alpha, \beta, \theta) = (1.5, 4, 6)$ and $(5, 3, 15)$ for $n = 3, 5, 7$ and $2, 4, 6$, respectively. The results in Table 2.1 showed that Bayes estimates of R_{SC} based on informative prior have the best performance in terms of error ER. We further observe that Bayes estimate using Lindley's approximation provide relatively finer results than the MCMC method. In general, we can order the MSE and ER of estimates as $ER(\widehat{R}_{SC}^{Lindley}) < ER(\widehat{R}_{SC}^{MCMC}) < MSE(\widehat{R}_{SC})$ based on informative prior. We also note that MSE, ER, and bias of the estimates tend to decrease when the sample size goes up. Table 2.2 indicates that the AL of the asymptotic confidence interval is broader than the corresponding HPD interval. However, CPs of the HPD interval are not as close to the nominal value 0.95 as the asymptotic confidence interval.

When the cold standby is applied at component level, we have encountered convergence problems in the MCMC case for the large values of n and m. The nm is seen as a power of some terms in the marginal posterior densities of α and β given in (2.42) and (2.43). Therefore, this situation causes an indeterminate ratio in the Metropolis-Hastings algorithm. Since the performance of Bayes estimate of R_{SC} using Lindley's approximation is generally better than the MCMC method due to the results given in Table 2.1, Lindley's approximation can be considered as an adequate method. That is why Bayes estimates for large values of n and m are computed by using Lindley's approximation.

MSEs of two different ML estimates and ERs of Bayesian estimates based on the informative and non-informative priors are plotted in Figure 2.2 for several parameters and sizes. The ML and Bayes estimates of R_{SC} are evaluated by using Equations (2.10) and (2.40) at the ML estimates of (α, β, θ) which are obtained from the log-likelihood function in (2.21). Moreover, another ML estimation of R_{SC}, called MLE2, is evaluated at $\widehat{\alpha}_{MLE} = 1/\overline{X}$, $\widehat{\beta}_{MLE} = 1/\overline{Y}$ and $\widehat{\theta}_{MLE} = 1/\overline{T}$ in Equation (2.10) for comparison of other estimates. Since the ML estimates of α and β are the solution of the non-linear equation system for our model, to use the ML estimate of the unknown parameter of the random sample of the exponential distribution can be preferable in point of error performances. Bayes estimate under the informative prior has the smallest error values in most cases. The performances of two ML estimates are very similar to each other. Hence, the MLE2 estimate can be used as an alternative estimates of R_{SC}. Similar to the Table 2.1, Bayes estimate under the non-informative prior has bigger error values than other estimates. Moreover, all the error values of the estimates are large when R_{SC} is around 0.5, and it is small when R_{SC} is close to extreme values 0 and 1. This has also been obtained in similar stress-strength reliability studies in the literature, one can refer to Basirat et al. [7] and Kızılaslan [9].

In Table 2.3, we have listed the ML and Bayesian Lindley estimates of Φ_{SC} based on informative and non-informative priors. The true values of the parameters are taken as $(\alpha, \beta, \theta) = (1.5, 4, 6)$ and $(1.5, 0.5, 10)$ for $n = 8, 12, 16, 20, 24$ and $5, 10, 15, 20, 25$, respectively. From Table 2.3, we conclude that ML estimates of Φ_{SC} have relatively better results than Bayes estimates. However, ERs of Bayes estimate based on informative prior are close to MSEs of ML estimate as sample size enlarged. From Table 2.4, we observe that ALs of the ACIs for Φ_{SC} decrease as sample size grows as expected, and all the CPs are satisfactory.

Table 2.1. Estimates of R_{SC}

			MLE			$Bayes$ (Inf. prior)						$Bayes$ (Non-inf. prior)		
n	m	R_{SC}	$\widehat{R}_{SC}$	Bias	MSE	$\widehat{R}_{SC}^{Lindley}$	Bias	ER	$\widehat{R}_{SC}^{MCMC}$	Bias	ER	$\widehat{R}_{SC}^{Lindley}$	Bias	ER
$(\alpha, \beta, \theta) = (1.5, 4, 6)$														
3	10	0.81070	0.80060	-0.01011	0.00643	0.82738	0.01667	0.00577	0.77493	-0.03577	0.00416	0.78825	-0.02245	0.01133
	20		0.80149	-0.00921	0.00352	0.81933	0.00862	0.00273	0.78266	-0.02805	0.00283	0.79740	-0.01331	0.00528
	30		0.80264	-0.00806	0.00230	0.81373	0.00303	0.00239	0.78807	-0.02264	0.00207	0.80134	-0.00936	0.00389
	40		0.80636	-0.00435	0.00169	0.81332	0.00261	0.00146	0.79493	-0.01578	0.00151	0.80645	-0.00426	0.00245
5	10	0.73678	0.72864	-0.00812	0.00810	0.75074	0.01396	0.00461	0.70096	-0.03582	0.00470	0.71891	-0.01786	0.01120
	20		0.73122	-0.00556	0.00452	0.73898	0.00220	0.00292	0.71362	-0.02316	0.00333	0.72801	-0.00877	0.00578
	25		0.73088	-0.00589	0.00342	0.73575	-0.00102	0.00244	0.71706	-0.01972	0.00266	0.728037	-0.00874	0.00472
	30		0.73212	-0.00466	0.00301	0.73538	-0.00140	0.00211	0.72085	-0.01593	0.00239	0.73041	-0.00637	0.00371
7	10	0.68197	0.67817	-0.00380	0.00935	0.68675	0.00478	0.00388	0.65023	-0.03174	0.00488	0.66882	-0.01315	0.01181
	15		0.68079	-0.00118	0.00648	0.68360	0.00163	0.00334	0.65992	-0.02205	0.00393	0.67621	-0.00575	0.00808
	20		0.68079	-0.00117	0.00477	0.68130	-0.00067	0.00293	0.66647	-0.01550	0.00321	0.67757	-0.00440	0.00595
	25		0.68107	-0.00186	0.00396	0.67865	-0.00332	0.00258	0.66798	-0.01399	0.00288	0.67702	-0.00494	0.00454

(continued)

Table 2.1 Continued

			MLE			$Bayes$ (Inf. prior)						$Bayes$ (Non-inf. prior)		
n	m	R_{SC}	$\widehat{R}_{SC}$	Bias	MSE	$\widehat{R}_{SC}^{Lindley}$	Bias	ER	$\widehat{R}_{SC}^{MCMC}$	Bias	ER	$\widehat{R}_{SC}^{Lindley}$	Bias	ER
$(\alpha, \beta, \theta) = (5, 3, 15)$														
2	10	0.92298	0.92508	0.00210	0.00130	0.94869	0.02571	0.00128	0.90354	-0.01944	0.00085	0.92347	0.00049	0.00219
	20		0.91514	-0.00784	0.00103	0.94305	0.02007	0.00081	0.90287	-0.02011	0.00083	0.91793	-0.00505	0.00147
	30		0.91483	-0.00815	0.00072	0.93744	0.01446	0.00059	0.90500	-0.01798	0.00066	0.91807	-0.00491	0.00110
	40		0.91666	-0.00632	0.00050	0.93325	0.01027	0.00043	0.90820	-0.01477	0.00049	0.92064	-0.00234	0.00079
4	10	0.86536	0.85788	-0.00747	0.00374	0.88763	0.02227	0.00248	0.83667	-0.02867	0.00179	0.85605	-0.00931	0.00554
	20		0.85573	-0.00962	0.00223	0.87780	0.01244	0.00100	0.84288	-0.02247	0.00139	0.85868	-0.00667	0.00310
	30		0.85518	-0.01017	0.00158	0.87155	0.00620	0.00080	0.84690	-0.01845	0.00109	0.86057	-0.00478	0.00218
	40		0.85767	-0.00768	0.00115	0.86847	0.00312	0.00074	0.85068	-0.01468	0.00084	0.86212	-0.00323	0.00164
6	10	0.81959	0.80531	-0.01428	0.00563	0.83386	0.01426	0.00306	0.78646	-0.03313	0.00239	0.80384	-0.01575	0.00765
	20		0.81071	-0.00888	0.00293	0.82620	0.00661	0.00112	0.79933	-0.02026	0.00153	0.81554	-0.00406	0.00403
	25		0.81114	-0.00845	0.00247	0.82361	0.00401	0.00107	0.80216	-0.01743	0.00139	0.81661	-0.00299	0.00332
	30		0.81246	-0.00713	0.00198	0.82236	0.00277	0.00106	0.80315	-0.01644	0.00137	0.81782	-0.00178	0.00274

Table 2.2. Average lengths and coverage probabilities of R_{SC}

			Asymptotic		*HPD*					*Asymptotic*		*HPD*	
n	m	R_{SC}	AL	CP	AL	CP	n	m	R_{SC}	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (1.5, 4, 6)$							$(\alpha, \beta, \theta) = (5, 3, 15)$						
3	10	0.81070	0.32606	0.9220	0.17645	0.8740	2	10	0.92230	0.17323	0.9048	0.10566	0.9840
	20		0.23470	0.9372	0.14606	0.8556		20		0.13639	0.9520	0.08727	0.9372
	30		0.19238	0.9520	0.12798	0.8600		30		0.11246	0.9636	0.07693	0.9316
	40		0.16526	0.9468	0.11430	0.8668		40		0.09582	0.9672	0.06842	0.9360
5	10	0.73678	0.36759	0.9296	0.18997	0.8444	4	10	0.86536	0.25642	0.9148	0.13519	0.9380
	20		0.26308	0.9344	0.15209	0.8184		20		0.18603	0.9468	0.10784	0.8884
	25		0.23681	0.9456	0.14059	0.8268		30		0.15288	0.9472	0.09178	0.8624
	30		0.21599	0.9408	0.13353	0.8228		40		0.13120	0.9464	0.08378	0.8668
7	10	0.68197	0.38374	0.9520	0.19082	0.8380	6	10	0.81959	0.30389	0.9344	0.14710	0.9000
	15		0.31606	0.9280	0.16546	0.8032		20		0.21484	0.9420	0.11246	0.8632
	20		0.27551	0.9324	0.14867	0.8012		25		0.19232	0.9420	0.10379	0.8548
	25		0.24735	0.9328	0.15226	0.8244		30		0.17534	0.9436	0.10582	0.8680

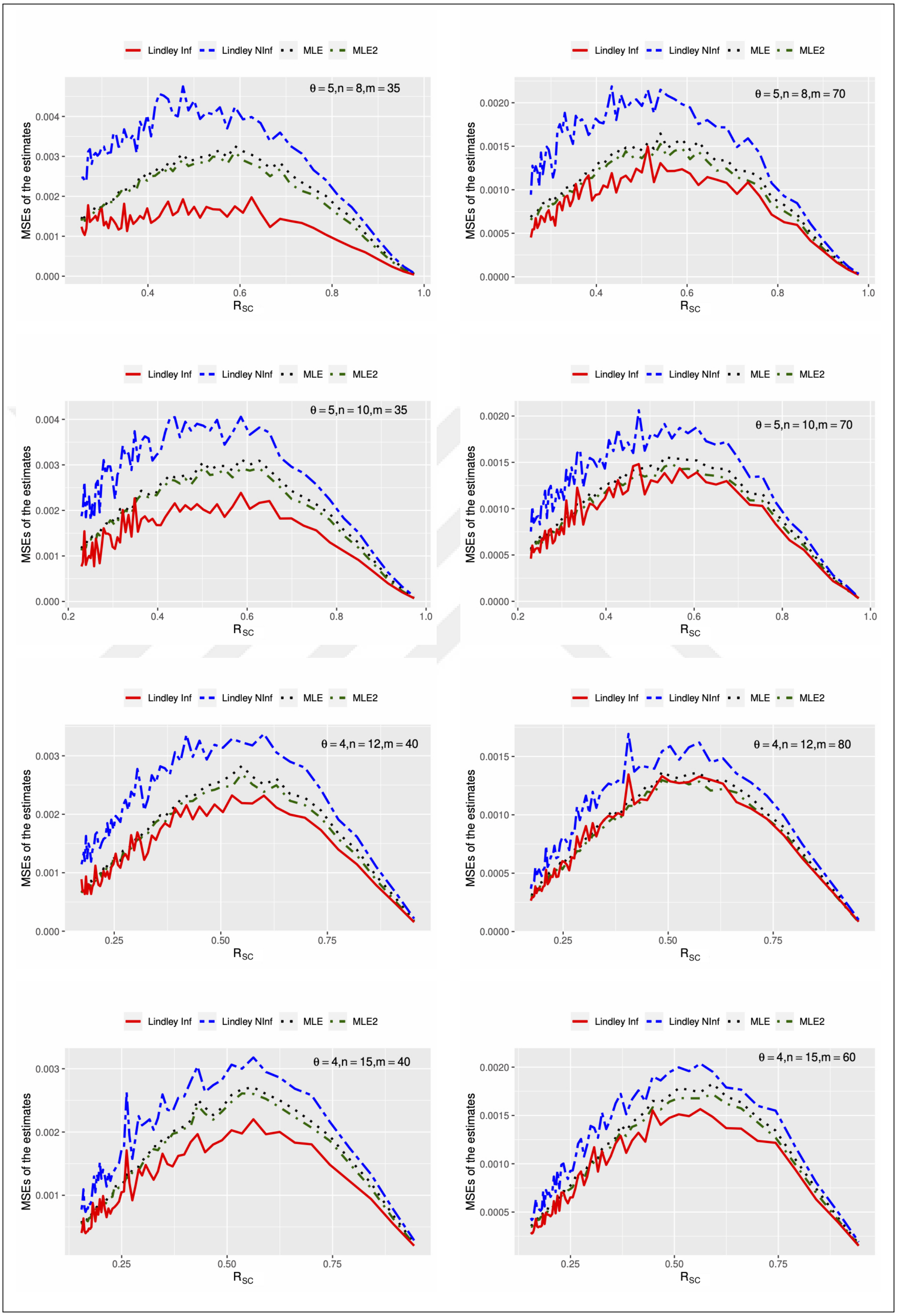

Figure 2.2. Plots for the estimates of R_{SC}

Table 2.3. Estimates of Φ_{SC}

			MLE			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	Φ_{SC}	$\hat{\Phi}_{SC}$	Bias	MSE	$\hat{\Phi}_{SC}^{Lindley}$	Bias	ER	$\hat{\Phi}_{SC}^{Lindley}$	Bias	ER
$(\alpha, \beta, \theta) = (1.5, 4, 6)$											
8	20	0.17188	0.17054	-0.00134	0.00024	0.17649	0.00461	0.00036	0.17530	0.00342	0.00062
	40		0.17156	-0.00031	0.00012	0.17338	0.00150	0.00014	0.17335	0.00147	0.00020
	60		0.17219	0.00031	0.00009	0.17311	0.00123	0.00010	0.17318	0.00130	0.00012
	80		0.17210	0.00022	0.00007	0.17283	0.00095	0.00011	0.17288	0.00100	0.00013
	100		0.17226	0.00038	0.00005	0.17264	0.00076	0.00005	0.17268	0.00080	0.00006
12	20	0.13121	0.13042	-0.00079	0.00010	0.13279	0.00158	0.00013	0.13248	0.00127	0.00021
	40		0.13151	0.00030	0.00005	0.13303	0.00182	0.00100	0.13311	0.00189	0.00105
	60		0.13153	0.00032	0.00004	0.13186	0.00065	0.00004	0.13191	0.00070	0.00005
	80		0.13140	0.00018	0.00003	0.13159	0.00037	0.00003	0.13160	0.00039	0.00003
	100		0.13131	0.00009	0.00002	0.13143	-0.00022	0.00002	0.13144	0.00023	0.00002
16	20	0.10901	0.10890	-0.00011	0.00006	0.11069	0.00167	0.00092	0.11076	0.00175	0.00097
	40		0.10922	0.00020	0.00003	0.10971	0.00070	0.00018	0.10976	0.00075	0.00020
	60		0.10908	0.00008	0.00002	0.10922	0.00021	0.00002	0.10923	0.00022	0.00002
	80		0.10910	0.00009	0.00001	0.10920	0.00018	0.00001	0.10921	0.00019	0.00002
	100		0.10918	0.00016	0.00001	0.10924	0.00022	0.00001	0.10925	0.00023	0.00001
20	20	0.09472	0.09470	-0.00002	0.00004	0.09658	0.00186	0.00304	0.09669	0.00197	0.00327
	40		0.09485	0.00013	0.00002	0.09516	0.00044	0.00006	0.09520	0.00047	0.00007
	60		0.09487	0.00015	0.00001	0.09488	0.00016	0.00002	0.09490	0.00018	0.00002
	80		0.09491	0.00019	0.00001	0.09498	0.00025	0.00001	0.09499	0.00027	0.00001
	100		0.09480	0.00007	0.00004	0.09483	0.00011	0.00000	0.09484	0.00012	0.00000
24	20	0.08461	0.08453	-0.00008	0.00002	0.08580	0.00119	0.00076	0.08590	0.00129	0.00082
	40		0.08477	0.00015	0.00001	0.08555	0.00094	0.00084	0.08559	0.00098	0.00087
	60		0.08476	0.00015	0.00001	0.08495	0.00033	0.00006	0.08497	0.00036	0.00007
	80		0.08471	0.00010	0.00001	0.08476	0.00014	0.00001	0.08476	0.00015	0.00001
	100		0.08466	0.00005	0.00000	0.08469	0.00008	0.00000	0.08470	0.00008	0.00000

(continued)

Table 2.3 Continued

			MLE			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	Φ_{SC}	$\widehat{\Phi}_{SC}$	Bias	MSE	$\widehat{\Phi}_{SC}^{Lindley}$	Bias	ER	$\widehat{\Phi}_{SC}^{Lindley}$	Bias	ER
						$(\alpha,\beta,\theta)=(1.5,0.5,10)$					
5	10	0.77878	0.76043	-0.01834	0.01219	0.81123	0.03246	0.01579	0.78802	0.00924	0.02492
	20		0.76574	-0.01304	0.00634	0.79147	0.01269	0.00837	0.78422	0.00544	0.01186
	30		0.77424	-0.00453	0.00485	0.78916	0.01039	0.00629	0.78675	0.00797	0.00811
	40		0.77578	-0.00300	0.00356	0.78563	0.00686	0.00455	0.78474	0.00597	0.00556
	50		0.77783	-0.00095	0.00305	0.78437	0.00560	0.00369	0.78398	0.00520	0.00433
10	10	0.48849	0.48010	-0.00839	0.00353	0.49410	0.00561	0.00436	0.48989	0.00140	0.00689
	20		0.48659	-0.00190	0.00201	0.49303	0.00454	0.00292	0.49276	0.00426	0.00369
	30		0.48858	0.00008	0.00133	0.49077	0.00228	0.00158	0.49079	0.00230	0.00188
	40		0.48865	0.00016	0.00104	0.48989	0.00140	0.00116	0.49001	0.00152	0.00131
	50		0.48816	-0.00003	0.00083	0.48905	0.00056	0.00089	0.48906	0.00056	0.00099
15	10	0.37502	0.37090	-0.00412	0.00172	0.37656	0.00154	0.01063	0.37658	0.00155	0.01172
	20		0.37453	-0.00049	0.00091	0.37604	0.00101	0.00194	0.37642	0.00139	0.00224
	30		0.37487	-0.00014	0.00063	0.37561	0.00059	0.00159	0.37579	0.00076	0.00167
	40		0.37578	0.00076	0.00048	0.37558	0.00056	0.00046	0.37579	0.00077	0.00052
	50		0.37540	0.00037	0.00038	0.37527	0.00025	0.00038	0.37535	0.00032	0.00041
20	10	0.31185	0.31016	-0.00169	0.00095	0.31170	-0.00016	0.00783	0.31387	0.00202	0.00834
	20		0.31184	-0.00001	0.00053	0.31263	0.00077	0.00170	0.31339	0.00153	0.00181
	30		0.31198	0.00012	0.00035	0.31214	0.00029	0.00080	0.31243	0.00057	0.00085
	40		0.31225	0.00039	0.00027	0.31192	0.00007	0.00026	0.31205	0.00020	0.00028
	50		0.31211	0.00025	0.00022	0.31185	0.00001	0.00021	0.31193	0.00007	0.00023
25	10	0.27072	0.26943	-0.00129	0.00062	0.27669	0.00596	0.08434	0.27925	0.00852	0.08576
	20		0.27080	0.00008	0.00034	0.27411	0.00339	0.00796	0.27470	0.00397	0.00791
	30		0.27174	0.00101	0.00023	0.27162	0.00090	0.00061	0.27203	0.00130	0.00062
	40		0.27144	0.00071	0.00017	0.27099	0.00027	0.00017	0.27121	0.00049	0.00019
	50		0.27129	0.00056	0.00014	0.27095	0.00023	0.00013	0.27108	0.00036	0.00015

Table 2.4. Average lengths and coverage probabilities of Φ_{SC}

n	m	Φ_{SC}	AL	CP	n	m	Φ_{SC}	AL	CP
$(\alpha, \beta, \theta) = (1.5, 4, 6)$					$(\alpha, \beta, \theta) = (1.5, 0.5, 10)$				
8	20	0.17188	0.06319	0.9504	5	10	0.77878	0.49134	0.9564
	40		0.04459	0.9544		20		0.34783	0.9600
	60		0.03649	0.9548		30		0.28497	0.9504
	80		0.03160	0.9472		40		0.24706	0.9596
	100		0.02826	0.9472		50		0.22136	0.9544
12	20	0.13121	0.04187	0.9592	10	10	0.48849	0.25506	0.9592
	40		0.02960	0.9528		20		0.18026	0.9460
	60		0.02416	0.9472		30		0.14749	0.9552
	80		0.02091	0.9436		40		0.12775	0.9496
	100		0.01871	0.9456		50		0.11424	0.9504
16	20	0.10901	0.03125	0.9648	15	10	0.37502	0.17181	0.9528
	40		0.02207	0.9540		20		0.12134	0.9572
	60		0.01797	0.9524		30		0.09924	0.9540
	80		0.01555	0.9508		40		0.08603	0.9480
	100		0.01391	0.9516		50		0.07706	0.9468
20	20	0.09472	0.02515	0.9616	20	10	0.31185	0.12999	0.9580
	40		0.01752	0.9612		20		0.09311	0.9552
	60		0.01425	0.9524		30		0.07509	0.9492
	80		0.01234	0.9440		40		0.06496	0.9468
	100		0.01103	0.9484		50		0.05808	0.9452
24	20	0.08461	0.02097	0.9644	25	10	0.27072	0.10507	0.9556
	40		0.01462	0.9520		20		0.07542	0.9524
	60		0.01190	0.9512		30		0.06064	0.9588
	80		0.01022	0.9472		40		0.05224	0.9512
	100		0.00913	0.9492		50		0.04671	0.9456

2.3.2 Real Data Analysis

In this subsection, the lifetime of steel specimens under the different stress levels has been considered for a real-life example. This data set represents the lifetime data for the steel specimens under fourteen different stress levels. All data sets are available in Lawless [66]. In the literature, these data sets have been studied many times by researchers for the stress-strength models.

We consider the series system which has n active components with corresponding n standby components. Suppose that active ($\mathbf{X}$) and standby ($\mathbf{Y}$) components are tested at 32.5 and 35 stress levels. Let us assume that an engineer wants to compare that the series system which is constructed by $\mathbf{X}$ and $\mathbf{Y}$ with another component ($\mathbf{T}$) which is tested at 33 stress level. The reliability of this series system is estimated by wielding these data sets. Then, he/she decides that which system or component is used in the production processes. Different situations can be created according to this scenario. For example, if the reliability of this series system exceeds 0.80, the series system with standby components will be preferred.

We have 20 observations for each data set. For two different (n, m) cases, the original data are divided by 100 based on the aforementioned scenario. The strength data sets $\mathbf{X}$ and $\mathbf{Y}$ are partitioned into m parts and each one has n unit. $\mathbf{T}$ is obtaining as the average value of each part. We check whether data sets $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{T}_1$ (stress data set for $n = 4, m = 5$) and $\mathbf{T}_2$ (stress data set for $n = 5, m = 4$) come from the exponential distribution or not. Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Cramer-von Mises (C-VM) tests are carried out for the goodness-of-fit test. Their test statistics values and corresponding $p-$values are listed in Table 2.5. It is seen that the exponential distribution provides a good fit to these data sets.

Table 2.5. Goodness-of-fit test for the real data set

$Data$	MLE	$K-S$	$p-value$	$A-D$	$p-value$	$C-VM$	$p-value$
$\mathbf{X}$	$\hat{\alpha}$ =0.0907	0.1629	0.6067	1.2250	0.7094	0.2313	0.6591
$\mathbf{Y}$	$\hat{\beta}$ =0.2909	0.2843	0.0635	1.2714	0.6779	0.2574	0.5719
$\mathbf{T}_1$	$\hat{\theta}$ =0.1099	0.5023	0.1090	1.4018	0.2017	0.2963	0.1372
$\mathbf{T}_2$	$\hat{\theta}$ =0.1099	0.4726	0.2396	0.8944	0.4099	0.1816	0.3141

The observed data $(\mathbf{X}, \mathbf{Y})$ and $(\mathbf{Z}, \mathbf{T}_1)$ at component level are given by

$$\mathbf{X} = \begin{bmatrix} 42.57 & 8.79 & 7.99 & 13.88 \\ 2.71 & 3.08 & 20.73 & 2.27 \\ 3.47 & 6.69 & 11.54 & 3.93 \\ 2.50 & 1.96 & 5.48 & 4.75 \\ 17.05 & 22.11 & 9.75 & 29.25 \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} 2.30 & 1.69 & 1.78 & 2.71 \\ 1.29 & 5.68 & 1.15 & 2.80 \\ 3.05 & 3.26 & 11.01 & 2.85 \\ 7.34 & 1.77 & 4.93 & 2.18 \\ 3.42 & 4.31 & 1.43 & 3.81 \end{bmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} 44.87 & 10.48 & 9.77 & 16.59 \\ 4.00 & 8.76 & 21.88 & 5.07 \\ 6.52 & 9.95 & 22.55 & 6.78 \\ 9.84 & 3.73 & 10.41 & 6.93 \\ 20.47 & 26.42 & 11.18 & 33.06 \end{bmatrix}, \ \mathbf{T}_1 = \begin{bmatrix} 6.3500 \\ 7.9750 \\ 8.9400 \\ 13.3800 \\ 8.8575 \end{bmatrix}. \tag{2.60}$$

for $n = 4, m = 5$. The observed data $(\mathbf{X}, \mathbf{Y})$ and $(\mathbf{Z}, \mathbf{T}_2)$ at component level are given by

$$\mathbf{X} = \begin{bmatrix} 42.57 & 8.79 & 7.99 & 13.88 & 2.71 \\ 3.08 & 20.73 & 2.27 & 3.47 & 6.69 \\ 11.54 & 3.93 & 2.50 & 1.96 & 5.48 \\ 4.75 & 17.05 & 22.11 & 9.75 & 29.25 \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} 2.30 & 1.69 & 1.78 & 2.71 & 1.29 \\ 5.68 & 1.15 & 2.80 & 3.05 & 3.26 \\ 11.01 & 2.85 & 7.34 & 1.77 & 4.93 \\ 2.18 & 3.42 & 4.31 & 1.43 & 3.81 \end{bmatrix}, \tag{2.61}$$

$$\mathbf{Z} = \begin{bmatrix} 44.87 & 10.48 & 9.77 & 16.59 & 4.00 \\ 8.76 & 21.88 & 5.07 & 6.52 & 9.95 \\ 22.55 & 6.78 & 9.84 & 3.73 & 10.41 \\ 6.93 & 20.47 & 26.42 & 11.18 & 33.06 \end{bmatrix}, \ \mathbf{T}_2 = \begin{bmatrix} 5.822 \\ 7.094 \\ 13.474 \\ 10.012 \end{bmatrix} \tag{2.62}$$

for $n = 5, m = 4$. Under these circumstances, the MLE of the parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}) = (0.1872, 0.1096, 0.1099)$. The ML and Bayes estimates of R_{SC} and Φ_{SC} along with 95 percent asymptotic confidence intervals and HPD credible intervals (given in bracket under the estimates) are presented in Table 2.6. We need to determine the hyperparameters for the Bayes estimate. If a practitioner has not any knowledge about the hyperparameters of the prior distributions, he/she can be use the moment estimates of the gamma distribution for each sample. Next, we present the results based on three different priors. The moment estimates of data sets $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{T}_1(\mathbf{T}_2)$ are used as Prior 1: $a_1 = 1.1229$, $b_1 = 0.1018$, $a_2 = 2.1986$, $b_2 = 0.6395$, $a_3 = 15.2068(9.5378)$, $b_3 = 1.6710(1.0480)$ for $n = 4(5)$, $m = 5(4)$. Then, Bayes estimates are calculated based on the informative priors Prior 1, Prior 2: $a_i = b_i = 1, i = 1,2,3$ and non-informative prior Prior 3: $a_i = b_i = 0$, $i = 1,2,3$.

Table 2.6. Estimates of R_{SC} and Φ_{SC} for the real data set

	(n,m)	MLE	MLE2	MCMC (Prior 1)	MCMC (Prior 2)	MCMC (Prior 3)
R_{SC}	(4,5)	0.42711	0.40395	0.83877	0.61287	0.61147
		(0.09230,0.76192)	(0.08959,0.71831)	(0.71382,0.94659)	(0.36515,0.85433)	(0.32214,0.86457)
Φ_{SC}		4.22900	4.00617	13.74654	13.56493	18.38622
		-	(3.87442,4.13791)	(5.82671,29.75366)	(5.71158,24.46485)	(6.43095,37.48032)
R_{SC}	(5,4)	0.38948	0.36612	0.80394	0.62026	0.60712
		(0.07360,0.70537)	(0.07597,0.65627)	(0.65005,0.94254)	(0.33643,0.87614)	(0.31223,0.89925)
Φ_{SC}		3.62949	3.40479	13.49258	14.40802	19.69559
		-	(3.35311,3.45648)	(5.31727,27.94549)	(5.39424,29.78039)	(6.15433,45.38128)

2.4. ESTIMATION OF R_{SS} AND Φ_{SS}

In this section, estimation of the stress-strength reliability and MRS of the series system with cold standby redundancy at system level are investigated.

2.4.1 MLE of R_{SS}

Assume that m systems are put on a test each with n original components with n cold standby components at system level in the series system. In this case, the strength data are represented as $Z_{(1),i}$, $i = 1,\dots,m$ and stress is $T_i, i = 1,\dots,m$. Here, $Z_{(1),i} = X_{(1),i} + Y_{(1),i}$ is the lifetime of i^{th} system as in the Section 2.1. Its pdf and cdf are obtained as in the Equations (2.12) and (2.14), and the pdf of $Z_{(1),i}$ is denoted by $f_{Z_{(1),i}}(z_{(1),i})$, $i = 1,\dots,m$. Then, the likelihood function of α, β and θ for the observed sample is

$$L(\alpha,\beta,\theta\,;\mathbf{z}_{(1)},\mathbf{t}) = \prod_{i=1}^{m} f_{Z_{(1),i}}(z_{(1),i}) f_T(t_i) \tag{2.63}$$

$$= \left(\frac{n\alpha\beta\theta}{\alpha-\beta}\right)^m \prod_{i=1}^{m} (e^{-\beta n z_{(1),i}} - e^{-\alpha n z_{(1),i}}) e^{-\theta t_i}. \tag{2.64}$$

The corresponding log-likelihood function is

$$\ell(\alpha,\beta,\theta\,;\mathbf{z}_{(1)},\mathbf{t}) = m[\ln(n\alpha\beta\theta) - \ln(\alpha-\beta)] + \sum_{i=1}^{m} \ln(e^{-\beta n z_{(1),i}} - e^{-\alpha n z_{(1),i}}) - \theta \sum_{i=1}^{m} t_i. \tag{2.65}$$

By partially differentiating (2.65) with respect to α and β, we attain the following non-linear equations:

$$\frac{\partial \ell}{\partial \alpha} = m\left(\frac{1}{\alpha} + \frac{1}{\beta-\alpha}\right) - \sum_{i=1}^{m} \frac{n z_{(1),i}}{1 - e^{-(\beta-\alpha) n z_{(1),i}}} = 0, \tag{2.66}$$

$$\frac{\partial \ell}{\partial \beta} = m\left(\frac{1}{\beta} - \frac{1}{\beta-\alpha}\right) + \sum_{i=1}^{m} \frac{n z_{(1),i}}{e^{-(\alpha-\beta) n z_{(1),i}} - 1} = 0. \tag{2.67}$$

The MLEs $\widehat{\alpha}$ and $\widehat{\beta}$ can be obtained by solving the non-linear equations in (2.66) and (2.67) simultaneously. This non-linear equation system can be solved by using numerical methods such as Newton–Raphson and Broyden's methods. The MLE of θ is derived as $\widehat{\theta} = m / \sum_{i=1}^{m} t_i$. After obtaining $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\theta}$, the MLE of R_{SS}, i.e. $\widehat{R}_{SS}^{MLE}$, is given by

$$\widehat{R}_{SS}^{MLE} = \frac{\widehat{\theta}[\widehat{\theta} + n(\widehat{\alpha} + \widehat{\beta})]}{(\widehat{\beta} n + \widehat{\theta})(\widehat{\alpha} n + \widehat{\theta})} \tag{2.68}$$

from Equation (2.17) using the invariance property of MLE.

2.4.2 Asymptotic Distribution and Confidence Interval for R_{SS}

The elements of the observed information matrix $J(\boldsymbol{\tau})$, in which $\boldsymbol{\tau} = (\alpha, \beta, \theta)$ and $J(\boldsymbol{\tau}) = [J_{ij}] = [-\partial^2 \ell / \partial\tau_i \partial\tau_j]$, are derived as $J_{13} = J_{31} = J_{23} = J_{32} = 0, J_{33} = m/\theta^2$,

$$J_{11} = m\left(\frac{1}{\alpha^2} - \frac{1}{(\beta-\alpha)^2}\right) + \sum_{i=1}^{m} \frac{n^2 z_{(1),i}^2\, e^{-(\beta-\alpha)nz_{(1),i}}}{(1-e^{-(\beta-\alpha)nz_{(1),i}})^2}, \tag{2.69}$$

$$J_{12} = J_{21} = \frac{m}{(\beta-\alpha)^2} - \sum_{i=1}^{m} \frac{n^2 z_{(1),i}^2\, e^{-(\beta-\alpha)nz_{(1),i}}}{(1-e^{-(\beta-\alpha)nz_{(1),i}})^2} \tag{2.70}$$

$$J_{22} = m\left(\frac{1}{\beta^2} - \frac{1}{(\beta-\alpha)^2}\right) + \sum_{i=1}^{m} \frac{n^2 z_{(1),i}^2\, e^{-(\alpha-\beta)nz_{(1),i}}}{(e^{-(\alpha-\beta)nz_{(1),i}}-1)^2}. \tag{2.71}$$

Since the Fisher information matrix $E\,(J(\boldsymbol{\tau}))$ can not be obtained analytically, it is computed by applying numerical integration methods. Then, $\hat{R}_{SS}^{MLE}$ is asymptotically normal with mean R_{SS} and asymptotic variance is computed by using the formula in Equation (2.73).

$$\sigma_{R_{SS}}^2 = \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial R_{SS}}{\partial \boldsymbol{\tau}_i} \frac{\partial R_{SS}}{\partial \boldsymbol{\tau}_j} I_{ij}^{-1}, \tag{2.72}$$

where I_{ij}^{-1} is the $(i,j)^{th}$ element of the inverse of $I(\boldsymbol{\tau})$, see Rao [61]. Afterwards,

$$\sigma_{R_{SS}}^2 = \left(\frac{\partial R_{SS}}{\partial \alpha}\right)^2 I_{11}^{-1} + 2\frac{\partial R_{SS}}{\partial \alpha}\frac{\partial R_{SS}}{\partial \beta} I_{12}^{-1} + \left(\frac{\partial R_{SS}}{\partial \beta}\right)^2 I_{22}^{-1} + \left(\frac{\partial R_{SS}}{\partial \theta}\right)^2 I_{33}^{-1}. \tag{2.73}$$

Hence, an asymptotic $100(1-\gamma)$ percent confidence interval for R_{SS} is given by $(\hat{R}_{SS}^{MLE} \pm$

$z_{\gamma/2}\hat{\sigma}_{R_{SS}}$) where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_{SS}}$ is the value of $\sigma_{R_{SS}}$ at the MLE of the parameters.

2.4.3 Bayesian Estimation of R_{SS}

In Bayesian case, we assume that α, β and θ have independent gamma priors with parameters (a_i, b_i), $i = 1, 2, 3$, respectively. Then, the joint posterior density function of α, β and θ

$$\pi(\alpha, \beta, \theta \,|\, \mathbf{z}_{(1)}, \mathbf{t}) \propto \alpha^{m+a_1-1}\beta^{m+a_2-1}(\beta-\alpha)^{-m}\theta^{m+a_3-1}e^{-\alpha b_1-\beta b_2-\theta\left(b_3+\sum_{i=1}^{m} t_i\right)} \cdot exp\left(\sum_{i=1}^{m}\ln(e^{-\alpha n z_{(1),i}}-e^{-\beta n z_{(1),i}})\right). \tag{2.74}$$

Bayes estimate of R_{SS}, i.e. $\hat{R}_{SS}^{Bayes}$, under the SE loss function is

$$\hat{R}_{SS}^{Bayes} = \int_0^\infty\int_0^\infty\int_0^\infty R_{SS}\,\pi(\alpha, \beta, \theta \,|\, \mathbf{z}_{(1)}, \mathbf{t})d\alpha d\beta d\theta. \tag{2.75}$$

Since the $\hat{R}_{SS}^{Bayes}$ can not be easily computed analytically, we use MCMC methods.

• **MCMC method**

The joint posterior density function of α, β and θ given data is stated in (2.74). Then, the marginal posterior density functions of the parameters are given respectively as

$$\theta\,|\mathbf{t} \sim Gamma\left(m+a_3, b_3+\sum_{i=1}^{m} t_i\right), \tag{2.76}$$

$$\pi(\alpha\,|\beta, \mathbf{z}_{(1)}) \propto \alpha^{m+a_1-1}(\beta-\alpha)^{-m}e^{-\alpha b_1}\exp\left(\sum_{i=1}^{m}\ln\left(e^{-\alpha n z_{(1),i}}-e^{-\beta n z_{(1),i}}\right)\right) \tag{2.77}$$

$$\pi(\beta\,|\alpha, \mathbf{z}_{(1)}) \propto \beta^{m+a_2-1}(\beta-\alpha)^{-m}e^{-\beta b_2}\exp\left(\sum_{i=1}^{m}\ln\left(e^{-\alpha n z_{(1),i}}-e^{-\beta n z_{(1),i}}\right)\right). \quad (2.78)$$

Ergo, samples of θ can be easily generated by using a gamma distribution. Since the marginal posterior distributions of α and β are not a well-known distribution, it is not possible to generate sample directly by standard methods. Hence, Bayes estimate and HPD credible interval of R_{SS} are computed by applying the hybrid Metropolis-Hastings and Gibbs sampling algorithm. Since this mechanism is similar to in Subsection 2.2.3, it is omitted.

2.4.4 Inference on Φ_{SS}

The MRS of the series system under the stress T is described by the following conditional expectation $\Phi_{SS} = E(Z_{(1)} - T\,|\,Z_{(1)} > T)$. The cdf of the conditional random variable $\psi \equiv (Z_{(1)} - T\,|\,Z_{(1)} > T)$ is

$$F_{\psi}(x) = P(Z_{(1)} - T \leq x\,|\,Z_{(1)} > T) = \frac{P(Z_{(1)} \leq T + x, Z_{(1)} > T)}{R_{SS}}. \quad (2.79)$$

Then, conditioning on $T = t$, we have

$$P(Z_{(1)} \leq T + x, Z_{(1)} > T) = \int_0^\infty P(t < Z_{(1)} \leq t + x)\, dF_T(t) \quad (2.80)$$

$$= \int_0^\infty [F_{Z_{(1)}}(t + x) - F_{Z_{(1)}}(t)]\, dF_T(t) \quad (2.81)$$

$$= \frac{\theta}{(\beta-\alpha)}\left[\frac{\alpha(e^{-\beta n x}-1)}{\beta n+\theta} - \frac{\beta(e^{-\alpha n x}-1)}{\alpha n+\theta}\right], \quad (2.82)$$

for $\alpha \neq \beta$. Hence, the cdf and pdf of ψ are given by

$$F_{\psi}(x) = \frac{\theta}{R_{SS}(\beta-\alpha)}\left[\frac{\alpha(e^{-\beta nx}-1)}{\beta n+\theta} - \frac{\beta(e^{-\alpha nx}-1)}{\alpha n+\theta}\right] \tag{2.83}$$

and

$$f_{\psi}(x) = \frac{\theta\alpha\beta n}{R_{SS}(\beta-\alpha)}\left(\frac{e^{-\alpha nx}}{\alpha n+\theta} - \frac{e^{-\beta nx}}{\beta n+\theta}\right), \tag{2.84}$$

for $\alpha \neq \beta$. Therefore, the MRS of our series system is obtained as

$$\Phi_{SS} = E(\psi) = \int_0^{\infty} x\, f_{\psi}(x)dx \tag{2.85}$$

$$= \frac{\alpha\beta\theta}{R_{SS}(\beta-\alpha)n}\left[\frac{1}{\alpha^2(\alpha n+\theta)} - \frac{1}{\beta^2(\beta n+\theta)}\right], \tag{2.86}$$

for $\alpha \neq \beta$. Moreover, Φ_{SS} is derived by using R_{SS} in (2.17) as

$$\Phi_{SS} = \frac{3\alpha n+\theta}{\alpha n(2\alpha n+\theta)} \tag{2.87}$$

for $\alpha = \beta$.

2.4.4.1 Estimation of Φ_{SS}

The MLE of Φ_{SS}, i.e. $\widehat{\Phi}_{SS}^{MLE}$, is computed from (2.86) with the help of the invariance property of MLE. It is clear that $\widehat{\Phi}_{SS}^{MLE}$ is asymptotically normal with mean Φ_{SS} and asymptotic variance is evaluated by using the formula in Equation (2.57) by replacing Φ_{SC} by Φ_{SS}. Then, an asymptotic $100(1-\gamma)$ percent confidence interval for Φ_{SS} is $(\widehat{\Phi}_{SS}^{MLE} \pm z_{\gamma/2}\widehat{\sigma}_{\Phi_{SS}})$ where $\widehat{\sigma}_{\Phi_{SS}}$ is the value of $\sigma_{\Phi_{SS}}$ at the MLE of the parameters. Under the setup made in Subsection 2.4.3, the Bayes estimate of Φ_{SS} under the SE loss function can not be computed analytically. Therefore, Bayes estimate and HPD credible interval of Φ_{SS} are computed by using the hybrid Metropolis-Hastings and Gibbs sampling algorithm as in the reliability case.

2.5. NUMERICAL RESULTS OF R_{SS} AND Φ_{SS}

2.5.1 Simulation Study

In this subsection, we carry out a Monte Carlo simulation study to compare performance of the all the proposed estimates. The point and interval estimates of R_{SS} and Φ_{SS} are computed based on the Monte Carlo simulations. The performances of the point estimates are checked by considering MSE and ER for ML and Bayes estimates, respectively. ALs of the confidence intervals and CPs provide a comparison between the interval estimates. In this case, the ACIs are evaluated based on Fisher information matrix. All results hinge on 2500 replications. The simulation studies have been carried out by using MATLAB and statistical software R [64].

In MLE case of the simulation study, we use the Broyden or Newton-Raphson methods for solving the non-linear equations using the nleqslv [65] package in software R.

In Bayesian case of the simulation study, we have selected hyperparameters as $a_1 = \alpha, a_2 = \beta, a_3 = \theta, b_1 = b_2 = b_3 = 1$ for the informative prior case and $a_i = b_i = 0, i = 1,2,3$ for the non-informative prior case. Bayes estimates of the stress-strength reliability and MRS of the systems are computed by implementing MCMC method. In the MCMC cases, we run two MCMC chains with fairly different initial values and generate 5000 iterations for each chain. In order to reduce the effect of the starting distribution, the first 2500 results of each sequence are discarded, which is called burn-in. So as to cut off the dependence between the results in the Markov chain, only every d^{th} draw of the chain is saved, which is called thinning. In our cases, Bayesian MCMC estimates are evaluated by the means of the every 5^{th} sampled values after the discarding procedure. Moreover, the convergency of the MCMC chains has been monitored by using the scale reduction factor estimate in Gelman et al. [52]. The estimate is given by $\sqrt{Var(\psi)/W}$, where ψ is the estimand of interest, $Var(\psi) = (n-1)W/n + B/n$ with the iteration number n for each chain, the between-sequence variance B and the within-sequence variance W. In the following simulation studies, the scale factor values of all MCMC estimators are less than 1.1. It is an admissible value for their convergence.

In Tables 2.7 and 2.9, we have presented the ML and Bayesian MCMC estimates of R_{SS} and Φ_{SS} based on informative and non-informative priors. In Tables 2.8 and 2.10, we have reported 95 percent ACIs and HPD credible intervals based on informative and non-informative priors. The true values of the parameters are assigned as $(\alpha, \beta, \theta) = (1.5, 2.5, 15)$, $(6, 2, 3)$ and $(3, 1, 10)$ for $n = 2, 4, 8$, and $3, 5, 7$ and $3, 8, 12, 16$, respectively. According to Table 2.7, MCMC estimate of R_{SS} with informative prior gives greater result than non-informative prior and MLE in terms of ER and MSEs. We also note that MSE, ER, and bias of the estimates tend to decrease generally as the sample size increases. Table 2.8 indicates that AL of HPD credible intervals based on informative prior are mostly narrower than the others. CPs of asymptotic confidence interval and HPD credible interval based on informative prior are quite satisfactory. However, in some cases, CPs of HPD credible interval based on non-informative prior are far away from the nominal value 0.95. All intervals tend to become narrower as the size of the sample increases.

According to Table 2.9, performances of ML and MCMC estimate of Φ_{SS} based on informative prior are similar. In some cases, MSEs of ML estimates are smaller than ERs of MCMC estimates, and vice versa. Hence, we can not give a general order for these estimates. Nevertheless, we can say that these errors are close to each other depending on the increase in the sample size. From Table 2.10, we obtain that AL of the asymptotic confidence intervals is generally smaller than that of both HPD credible intervals. Furthermore, AL of all the intervals shortens as the sample size goes up, and all CPs are fairly satisfactory.

Moreover, some plots are presented for easily comparing the performance of the estimates of R_{SS}. In Figure 2.3, MSEs of ML and ERs of MCMC Bayesian estimates (under the informative prior) of R_{SS} are plotted for different parameters and sizes. Similar to the Table 2.7, Bayes estimates under the informative prior have smaller errors than that of ML estimates.

Table 2.7. Estimates of R_{SS}

			MLE			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	R_{SS}	$\widehat{R}_{SS}$	Bias	MSE	$\widehat{R}_{SS}^{MCMC}$	Bias	ER	$\widehat{R}_{SS}^{MCMC}$	Bias	ER
						$(\alpha,\beta,\theta)=(1.5,2.5,15)$					
2	10	0.95833	0.93500	-0.02333	0.00199	0.93220	-0.02613	0.00108	0.89034	-0.06800	0.00640
	20		0.94298	-0.01536	0.00093	0.93892	-0.01941	0.00062	0.89957	-0.05876	0.00422
	30		0.94618	-0.01215	0.00061	0.94176	-0.01657	0.00046	0.90108	-0.05725	0.00382
	40		0.94889	-0.00944	0.00042	0.94445	-0.01388	0.00035	0.90361	-0.05473	0.00340
4	10	0.88571	0.85353	-0.03218	0.00551	0.84447	-0.04124	0.00317	0.79729	-0.08843	0.01214
	20		0.86370	-0.02202	0.00273	0.85547	-0.03025	0.00185	0.80856	-0.07715	0.00799
	30		0.86769	-0.01802	0.00183	0.85976	-0.02596	0.00139	0.81108	-0.07463	0.00691
	40		0.87031	-0.01540	0.00140	0.86287	-0.02285	0.00113	0.81262	-0.07309	0.00637
8	10	0.74603	0.71911	-0.02692	0.00931	0.70264	-0.04339	0.00472	0.66527	-0.08077	0.01356
	20		0.72145	-0.02459	0.00549	0.71189	-0.03415	0.00335	0.67112	-0.07492	0.00942
	30		0.72785	-0.01817	0.00374	0.71809	-0.02794	0.00257	0.67523	-0.07080	0.00764
	40		0.73065	-0.01538	0.00264	0.72167	-0.02436	0.00200	0.67667	-0.06936	0.00669
						$(\alpha,\beta,\theta)=(6,2,3)$					
3	10	0.42857	0.43662	0.00805	0.01096	0.41822	-0.01035	0.00620	0.41333	-0.01524	0.00914
	20		0.42963	0.00106	0.00564	0.42101	-0.00756	0.00404	0.41156	-0.01701	0.00491
	30		0.42767	-0.00090	0.00380	0.42076	-0.00782	0.00296	0.40928	-0.01929	0.00348
	40		0.42880	0.00023	0.00294	0.42270	-0.00587	0.00239	0.41028	-0.01829	0.00276
5	10	0.30070	0.31192	0.01122	0.00822	0.29893	-0.00177	0.00442	0.30076	0.00006	0.00685
	20		0.30523	0.00453	0.00407	0.30005	-0.00065	0.00287	0.29691	-0.00379	0.00348
	30		0.30475	0.00405	0.00276	0.30090	0.00020	0.00215	0.29600	-0.00470	0.00237
	40		0.30286	0.00216	0.00212	0.29951	-0.00119	0.00174	0.29364	-0.00706	0.00187
7	10	0.23137	0.24538	0.01401	0.00623	0.23500	0.00362	0.00333	0.23959	0.00821	0.00527
	20		0.23687	0.00550	0.00288	0.23333	0.00195	0.00204	0.23275	0.00138	0.00251
	30		0.23585	0.00448	0.00197	0.23354	0.00216	0.00155	0.23136	-0.00001	0.00174
	40		0.23389	0.00252	0.00145	0.23205	0.00068	0.00120	0.22900	-0.00237	0.00129

(continued)

Table 2.7 Continued

			MLE			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	R_{SS}	$\hat{R}_{SS}$	Bias	MSE	$\hat{R}_{SS}^{MCMC}$	Bias	ER	$\hat{R}_{SS}^{MCMC}$	Bias	ER
$(\alpha, \beta, \theta) = (3, 1, 10)$											
3	10	0.89069	0.86467	-0.02602	0.00466	0.85425	-0.03644	0.00292	0.81347	-0.07722	0.00986
	20		0.87467	-0.01601	0.00238	0.86338	-0.02730	0.00175	0.82398	-0.06671	0.00635
	30		0.87745	-0.01324	0.00161	0.86625	-0.02443	0.00134	0.82496	-0.06573	0.00553
	40		0.88015	-0.01054	0.00120	0.86943	-0.02126	0.00108	0.82728	-0.06341	0.00497
8	10	0.68627	0.66996	-0.01631	0.01055	0.65079	-0.03548	0.00556	0.62519	-0.06108	0.01225
	20		0.67488	-0.01140	0.00573	0.66104	-0.02523	0.00360	0.63176	-0.05452	0.00741
	30		0.67730	-0.00898	0.00373	0.66482	-0.02145	0.00267	0.63321	-0.05307	0.00571
	40		0.67731	-0.00897	0.00300	0.66572	-0.02056	0.00232	0.63210	-0.05417	0.00516
12	10	0.57312	0.56849	-0.00463	0.01154	0.54860	-0.02452	0.00540	0.53273	-0.04039	0.01073
	20		0.56501	-0.00811	0.00631	0.55263	-0.02049	0.00378	0.53165	-0.04147	0.00659
	30		0.56960	-0.00352	0.00414	0.55765	-0.01547	0.00278	0.53468	-0.03844	0.00469
	40		0.56853	-0.00460	0.00342	0.55865	-0.01447	0.00247	0.53365	-0.03947	0.00424
16	10	0.49071	0.49218	0.00147	0.01154	0.47313	-0.01758	0.00504	0.46464	-0.02607	0.00985
	20		0.48786	-0.00285	0.00604	0.47677	-0.01395	0.00348	0.46241	-0.02831	0.00563
	30		0.48829	-0.00243	0.00425	0.47908	-0.01164	0.00275	0.46204	-0.02867	0.00417
	40		0.48929	-0.00143	0.00326	0.48076	-0.00995	0.00230	0.46202	-0.02869	0.00340

Table 2.8. Average lengths and coverage probabilities of R_{SS}

			Asymptotic		HPD (Inf. prior)		HPD (Non-inf. prior)	
n	m	R_{SS}	AL	CP	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (1.5, 2.5, 15)$								
2	10	0.95833	0.17487	0.9640	0.11561	0.9972	0.17728	0.7904
	20		0.11381	0.9628	0.09018	0.9916	0.12449	0.5484
	30		0.08917	0.9748	0.07851	0.9884	0.10427	0.3632
	40		0.07423	0.9688	0.06998	0.9900	0.09210	0.2816
4	10	0.88571	0.29261	0.9588	0.21676	0.9932	0.28100	0.8876
	20		0.20376	0.9562	0.17112	0.9488	0.19995	0.7176
	30		0.16490	0.9616	0.14935	0.9876	0.16671	0.6116
	40		0.14170	0.9592	0.13464	0.9868	0.14730	0.4980
8	10	0.74603	0.40249	0.9484	0.31619	0.9904	0.37564	0.9308
	20		0.29071	0.9444	0.25256	0.9824	0.27272	0.8700
	30		0.23800	0.9412	0.22001	0.9784	0.22543	0.8272
	40		0.20681	0.9580	0.19859	0.9828	0.19800	0.7692
$(\alpha, \beta, \theta) = (6, 2, 3)$								
3	10	0.42857	0.42022	0.9324	0.37913	0.9712	0.41144	0.9552
	20		0.30100	0.9440	0.28668	0.9700	0.29596	0.9568
	30		0.24683	0.9448	0.24024	0.9664	0.24235	0.9548
	40		0.21460	0.9500	0.21188	0.9656	0.21070	0.9472
5	10	0.30070	0.35666	0.9340	0.32110	0.9752	0.35610	0.9572
	20		0.25214	0.9416	0.24157	0.9688	0.25390	0.9612
	30		0.20629	0.9488	0.20201	0.9700	0.20728	0.9592
	40		0.17805	0.9484	0.17691	0.9652	0.17890	0.9568
7	10	0.23137	0.30498	0.9296	0.27387	0.9720	0.30904	0.9604
	20		0.21225	0.9508	0.20387	0.9748	0.21697	0.9660
	30		0.17301	0.9428	0.17004	0.9660	0.17641	0.9592
	40		0.14889	0.9516	0.14846	0.9656	0.15193	0.9616

(continued)

Table 2.8 Continued

			Asymptotic		HPD (Inf. prior)		HPD (Non-inf. prior)	
n	m	R_{SS}	AL	CP	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (3, 1, 10)$								
3	10	0.89069	0.27973	0.9556	0.21038	0.9908	0.26180	0.9096
	20		0.19395	0.9556	0.16615	0.9852	0.18698	0.8052
	30		0.15795	0.9576	0.14597	0.9868	0.15650	0.6664
	40		0.13610	0.9576	0.13174	0.9868	0.13792	0.5804
8	10	0.68627	0.42078	0.9380	0.34216	0.9868	0.38808	0.9364
	20		0.30398	0.9456	0.27039	0.9792	0.28263	0.9228
	30		0.25102	0.9528	0.23481	0.9800	0.23486	0.9032
	40		0.21867	0.9484	0.21163	0.9760	0.20630	0.8648
12	10	0.57312	0.44064	0.9464	0.36177	0.9880	0.40890	0.9496
	20		0.31877	0.9476	0.28567	0.9788	0.29917	0.9284
	30		0.26295	0.9512	0.24700	0.9784	0.24801	0.9336
	40		0.22848	0.9400	0.22159	0.9708	0.21677	0.9048
16	10	0.49072	0.43492	0.9312	0.35790	0.9844	0.40695	0.9440
	20		0.31331	0.9428	0.28182	0.9776	0.29723	0.9468
	30		0.25773	0.9396	0.24273	0.9768	0.24588	0.9360
	40		0.22418	0.9424	0.21761	0.9736	0.21450	0.9284

Table 2.9. Estimates of Φ_{SS}

			MLE			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	Φ_{SS}	$\widehat{\Phi}_{SS}$	Bias	MSE	$\widehat{\Phi}_{SS}^{MCMC}$	Bias	ER	$\widehat{\Phi}_{SS}^{MCMC}$	Bias	ER
$(\alpha,\beta,\theta)=(1.5,2.5,15)$											
2	10	0.48986	0.50810	0.01825	0.01510	0.51325	0.02340	0.01255	0.56352	0.07366	0.02344
	20		0.50222	0.01236	0.00745	0.50699	0.01714	0.00678	0.54361	0.05376	0.01108
	30		0.49983	0.00997	0.00498	0.50423	0.01438	0.00469	0.53750	0.04765	0.00756
	40		0.49850	0.00864	0.00367	0.50211	0.01226	0.00351	0.53372	0.04386	0.00583
4	10	0.23441	0.24933	0.01493	0.00386	0.25146	0.01706	0.00327	0.28236	0.04795	0.00687
	20		0.24478	0.01037	0.00184	0.24732	0.01291	0.00170	0.27173	0.03732	0.00338
	30		0.24185	0.00744	0.00117	0.24437	0.00996	0.00114	0.26723	0.03282	0.00234
	40		0.24103	0.00662	0.00091	0.24337	0.00896	0.00089	0.26571	0.03130	0.00195
8	10	0.11206	0.12193	0.00988	0.00090	0.12288	0.01082	0.00078	0.14087	0.02881	0.00187
	20		0.11940	0.00734	0.00050	0.12048	0.00843	0.00046	0.13544	0.02338	0.00106
	30		0.11754	0.00548	0.00031	0.11890	0.00685	0.00030	0.13329	0.02124	0.00078
	40		0.11657	0.00451	0.00024	0.11795	0.00589	0.00023	0.13222	0.02016	0.00065
$(\alpha,\beta,\theta)=(6,2,3)$											
3	10	0.18518	0.20419	0.01900	0.00322	0.20540	0.02022	0.00260	0.23341	0.04823	0.00589
	20		0.19845	0.01326	0.00156	0.20227	0.01708	0.00147	0.22498	0.03980	0.00314
	30		0.19397	0.00878	0.00105	0.19935	0.01416	0.00105	0.22055	0.03536	0.00231
	40		0.19221	0.00702	0.00077	0.19780	0.01261	0.00080	0.21842	0.03323	0.00188
5	10	0.11008	0.12169	0.01161	0.00115	0.12261	0.01254	0.00095	0.13997	0.02989	0.00219
	20		0.11759	0.00751	0.00057	0.12001	0.00993	0.00053	0.13415	0.02407	0.00114
	30		0.11577	0.00569	0.00040	0.11903	0.00895	0.00041	0.13233	0.02225	0.00091
	40		0.11435	0.00427	0.00029	0.11803	0.00796	0.00031	0.13116	0.02108	0.00074
7	10	0.07829	0.08684	0.00855	0.00061	0.08742	0.00913	0.00050	0.10006	0.02178	0.00116
	20		0.08355	0.00526	0.00029	0.08545	0.00716	0.00028	0.09569	0.01740	0.00061
	30		0.08255	0.00426	0.00020	0.08483	0.00655	0.00020	0.09447	0.01618	0.00046
	40		0.08152	0.00323	0.00015	0.08417	0.00588	0.00016	0.09358	0.01529	0.00038

(continued)

Table 2.9 Continued

			MLE			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	Φ_{SS}	$\hat{\Phi}_{SS}$	Bias	MSE	$\hat{\Phi}_{SS}^{MCMC}$	Bias	ER	$\hat{\Phi}_{SS}^{MCMC}$	Bias	ER
$(\alpha, \beta, \theta) = (3, 1, 10)$											
3	10	0.39899	0.42901	0.03002	0.01292	0.44142	0.04243	0.01321	0.48432	0.08533	0.02240
	20		0.41915	0.02016	0.00650	0.42812	0.02913	0.00676	0.46161	0.06262	0.01081
	30		0.40992	0.01093	0.00402	0.41761	0.01862	0.00415	0.44913	0.05014	0.00682
	40		0.40809	0.00910	0.00311	0.41486	0.01587	0.00323	0.44566	0.04667	0.00546
8	10	0.14286	0.15399	0.01113	0.00170	0.15880	0.01595	0.00174	0.17769	0.03483	0.00320
	20		0.15168	0.00882	0.00088	0.15547	0.01262	0.00094	0.17165	0.02879	0.00177
	30		0.15009	0.00723	0.00064	0.15370	0.01084	0.00068	0.16927	0.02642	0.00134
	40		0.14696	0.00410	0.00044	0.15034	0.00748	0.00046	0.16587	0.02302	0.00100
12	10	0.09387	0.10265	0.00878	0.00079	0.10594	0.01207	0.00082	0.11950	0.02563	0.00158
	20		0.09952	0.00565	0.00039	0.10224	0.00837	0.00042	0.11366	0.01979	0.00081
	30		0.09800	0.00413	0.00026	0.10077	0.00690	0.00028	0.11194	0.01807	0.00060
	40		0.09733	0.00346	0.00020	0.09975	0.00588	0.00021	0.11096	0.01709	0.00049
16	10	0.06982	0.07641	0.00659	0.00044	0.07896	0.00914	0.00046	0.08946	0.01964	0.00091
	20		0.07417	0.00435	0.00022	0.07622	0.00640	0.00023	0.08507	0.01525	0.00046
	30		0.07345	0.00363	0.00015	0.07548	0.00566	0.00016	0.08418	0.01436	0.00036
	40		0.07261	0.00279	0.00117	0.07461	0.00479	0.00013	0.08339	0.01357	0.00303

Table 2.10. Average lengths and coverage probabilities of Φ_{SS}

			Asymptotic		HPD (Inf. prior)		HPD (Non-inf. prior)	
n	m	Φ_{SS}	AL	CP	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (1.5, 2.5, 15)$								
2	10	0.48986	0.56874	0.9620	0.58449	0.9936	0.71627	0.9964
	20		0.38614	0.9652	0.42406	0.9900	0.47991	0.9928
	30		0.30793	0.9656	0.35356	0.9916	0.38510	0.9908
	40		0.26294	0.9672	0.31009	0.9936	0.33028	0.9904
4	10	0.23441	0.28581	0.9676	0.28971	0.9916	0.36000	0.9932
	20		0.19345	0.9687	0.20992	0.9512	0.24057	0.9512
	30		0.15352	0.9772	0.17427	0.9940	0.19242	0.9872
	40		0.13109	0.9724	0.15308	0.9928	0.16569	0.9744
8	10	0.11206	0.14426	0.9796	0.14248	0.9952	0.18005	0.9924
	20		0.09849	0.9772	0.10352	0.9912	0.12056	0.9836
	30		0.07771	0.9804	0.08597	0.9924	0.09651	0.9692
	40		0.06604	0.9788	0.07536	0.9944	0.08310	0.9488
$(\alpha, \beta, \theta) = (6, 2, 3)$								
3	10	0.18518	0.25661	0.9760	0.25099	0.9928	0.31665	0.9912
	20		0.17554	0.9788	0.17901	0.9916	0.20879	0.9824
	30		0.13921	0.9728	0.14606	0.9884	0.16538	0.9592
	40		0.11918	0.9732	0.12727	0.9856	0.14127	0.9508
5	10	0.11008	0.15535	0.9772	0.15012	0.9916	0.19003	0.9908
	20		0.10559	0.9776	0.10674	0.9936	0.12488	0.9792
	30		0.08468	0.9792	0.08785	0.9860	0.09949	0.9472
	40		0.07218	0.9716	0.07652	0.9876	0.08500	0.9272
7	10	0.07829	0.11151	0.9796	0.10716	0.9952	0.13589	0.9900
	20		0.07567	0.9796	0.07616	0.9916	0.08907	0.9732
	30		0.06091	0.9764	0.06276	0.9864	0.07103	0.9512
	40		0.05205	0.9720	0.05483	0.9800	0.06084	0.9312

(continued)

Table 2.10 Continued

			Asymptotic		HPD (Inf. prior)		HPD (Non-inf. prior)	
n	m	Φ_{SS}	AL	CP	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (3, 1, 10)$								
3	10	0.39899	0.50010	0.9628	0.52019	0.9884	0.60748	0.9908
	20		0.33781	0.9640	0.36919	0.9844	0.40582	0.9820
	30		0.26611	0.9640	0.29954	0.9892	0.32152	0.9796
	40		0.22839	0.9596	0.26066	0.9836	0.27609	0.9664
8	10	0.14286	0.18717	0.9700	0.18994	0.9908	0.22384	0.9920
	20		0.12862	0.9800	0.13698	0.9880	0.15153	0.9748
	30		0.10322	0.9656	0.11319	0.9848	0.12205	0.9532
	40		0.08682	0.9632	0.09736	0.9844	0.10367	0.9464
12	10	0.09387	0.12669	0.9736	0.12742	0.9900	0.15040	0.9880
	20		0.08601	0.9744	0.09063	0.9840	0.10060	0.9748
	30		0.06858	0.9692	0.07488	0.9872	0.08088	0.9552
	40		0.05885	0.9676	0.06541	0.9876	0.06960	0.9316
16	10	0.06982	0.09541	0.9824	0.09514	0.9940	0.11296	0.9884
	20		0.06508	0.9780	0.06787	0.9872	0.07533	0.9732
	30		0.05228	0.9728	0.05638	0.9872	0.06092	0.9444
	40		0.04450	0.9696	0.04915	0.9836	0.05236	0.9172

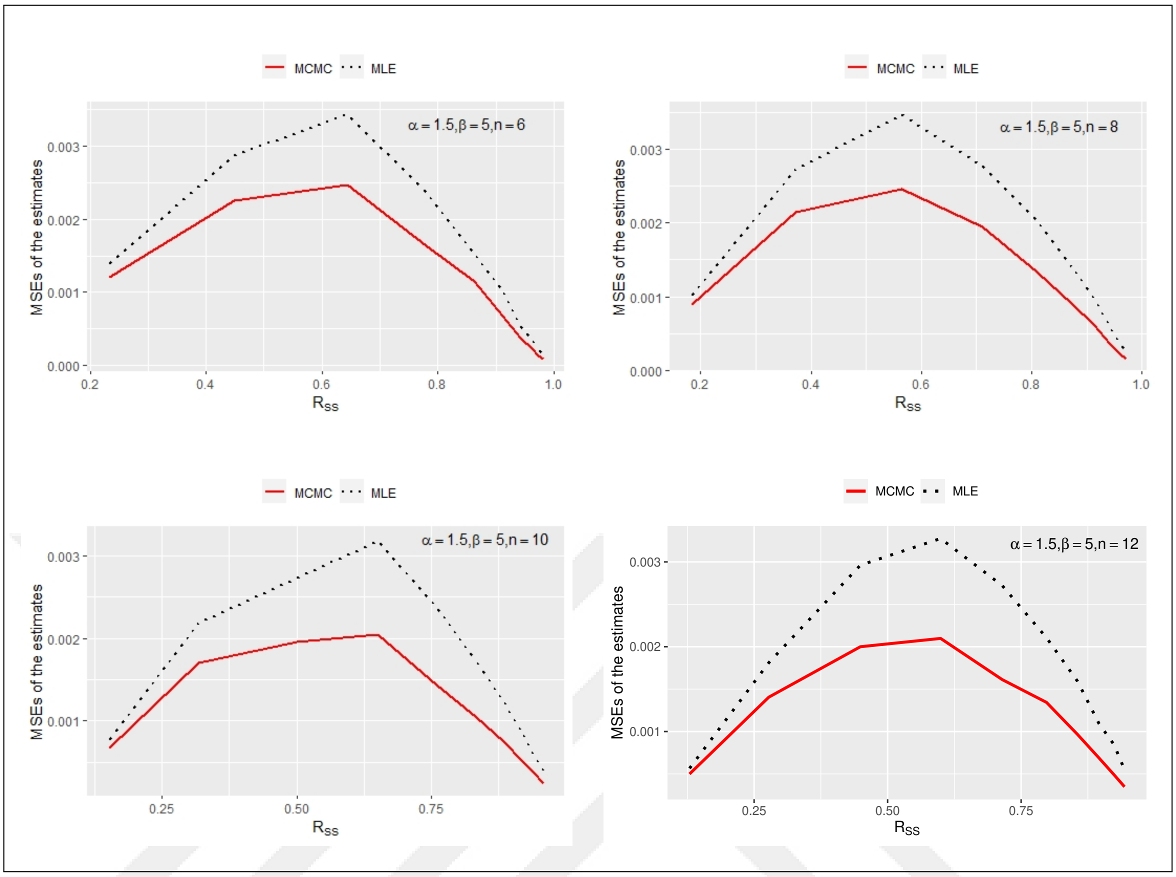

Figure 2.3. Plots for the estimates of R_{SS} when $m = 40$

2.5.2 Real Data Analysis

In this analysis, the same problem and lifetime data set given in Subsection 2.3.2 is taken into account. In that part, the goodness-of-fit tests are applied and it is seen that exponential distribution provides a good fit to these data sets. The observed data at system level $\mathbf{Z}_1$ for $n = 4, m = 5$ and $\mathbf{Z}_2$ for $n = 5, m = 4$ are given by

$$\mathbf{Z}_1 = \min(\mathbf{X}) + \min(\mathbf{Y}) = \begin{bmatrix} 9.68 \\ 3.42 \\ 6.32 \\ 3.73 \\ 11.18 \end{bmatrix}, \mathbf{Z}_2 = \begin{bmatrix} 4.00 \\ 3.42 \\ 3.73 \\ 6.18 \end{bmatrix}. \tag{2.88}$$

The stress data sets are the same as in component level case. In this case, the MLE of the parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}) = (0.07282, 0.07281, 0.10988)$ and $(0.092326, 0.092325, 0.109880)$ for $n = 4, m = 5$ and $n = 5, m = 4$, respectively, based on system level. The ML and Bayes estimates of R_{SS} and Φ_{SS} along with 95 percent asymptotic confidence and HPD credible intervals (given in bracket under the estimates) are listed in Table 2.11. Moreover, Bayes estimates are computed based on the same priors as in the component level.

Table 2.11. Estimates of R_{SS} and Φ_{SS} for the real data set

	(n, m)	MLE	MLE2	MCMC (Prior 1)	MCMC (Prior 2)	MCMC (Prior 3)
R_{SS}	(4,5)	0.47279	0.29870	0.70654	0.43288	0.42911
		(0.16274,0.78283)	(0.04816,0.54923)	(0.52938,0.87513)	(0.16807,0.72397)	(0.16960,0.70405)
Φ_{SS}		5.42190	3.00456	6.45067	6.48932	8.10311
		(1.91364,8.93015)	(0.30430,5.70483)	(2.48701,11.69452)	(0.57635,12.20563)	(2.83643,15.15230)
R_{SS}	(5,4)	0.34757	0.25158	0.55486	0.33495	0.33862
		(0.04645,0.64869)	(0.00312,0.50004)	(0.32318,0.78846)	(0.08286,0.606934)	(0.07420,0.63406)
Φ_{SS}		3.36457	2.39699	3.94688	4.12537	5.66156
		(0.95960,5.76955)	(0,4.82559)	(1.12749,7.56060)	(0.66982,8.79625)	(1.37796,11.58791)

2.6. CONCLUSIONS

In this section, statistical inference for R_{SC}, Φ_{SC}, R_{SS}, and Φ_{SS} are considered for the series system when cold standby components are handled for both component level and system level. The classical and Bayesian approaches have been applied to estimate the stress-strength reliability and MRS of the system. In Bayesian case, estimates are obtained by applying Lindley's approximation and MCMC method.

At component level, our simulation results show that in general, Lindley's method based on informative prior is adequate to estimate R_{SC} for big values of n and m. Additionally, the performances of MLE and MLE2 are similar due to the graphs. Therefore, MLE2 can be an alternative estimator of reliability. The ML estimator of Φ_{SC} generally provides relatively better results when compared with Bayes estimators. Nonetheless, the ERs of Bayes estimates based on informative prior get closer to MSEs of ML estimate as sample size enlarged.

At system level, to estimate R_{SS}, the choice of MCMC method based on informative prior is better than using a non-informative prior or MLE method. ML and Bayes estimates of Φ_{SS}, considering informative prior are found alike. Mostly, errors coming from ML estimates are smaller than ERs. When the sample size enlarges the gap between the errors of these estimates is closing and the values of ERs become smaller. It was hardly possible to make a comparison for these estimators.

From the real data analysis, we conclude that classical and Bayesian (based on non-informative prior) methods for R_{SC}, R_{SS}, Φ_{SC}, and Φ_{SS} have similar point and interval estimates. We can consider the MCMC results with Prior 1 as the closest to the stress-strength reliability value of our assumption in the scenario for both system and component levels. To estimate Φ_{SC}, MLE and MLE2 can be recommended. For Φ_{SS}, MCMC with Prior 1 and MLE can be preferable.

In our model, the total lifetime of the strength component and corresponding standby component is a convolution of the two independent and non-identical random variables. Evidently, the convolution of random variables have mixed form except for some well-known distributions under the certain conditions. For this reason, changing the underlying distributions will affect to obtain some analytical expressions.

3. STATISTICAL ANALYSIS FOR THE PARALLEL SYSTEM WITH STANDBY COMPONENTS

In this chapter, we have considered the estimation problem of stress-strength reliability and MRS of a parallel system with standby redundancy at component and system levels. The aim is to search the performance of the stress-strength reliability and MRS estimators for the parallel system under standby redundancy. The system and standby components are independent but not identically distributed random variables that follow exponential distribution.

3.1. MODEL DEFINITION

Suppose a n-component parallel system with standby components that are independent but not identically distributed with original components. We assume that $X_1, \dots, X_n$ represent the lifetimes of strength components and $Y_1, \dots, Y_n$ denote the lifetimes of independent standby strength components having exponential distributions with parameters α and β, respectively. T represents the common stress variable and follows the exponential distribution with parameter θ.

• In the case of component level, standby redundancy is applied to the components one by one. This parallel system structure is shown in Figure 1.3. The total lifetime of each strength component is $Z_i = X_i + Y_i$, $i = 1, \dots, n$. Then, the cdf and pdf of Z_i, $i = 1, \dots, n$ can be written as

$$F_{Z_i}(z) = \int_0^z F_x(z-y) f_y(y) dy \tag{3.1}$$

$$= \begin{cases} 1 + \frac{1}{\beta-\alpha}(\alpha e^{-\beta z} - \beta e^{-\alpha z}) & , \alpha \neq \beta \\ 1 - e^{-\alpha z}(1 + \alpha z) & , \alpha = \beta \end{cases} \tag{3.2}$$

$$f_{Z_i}(z) = \begin{cases} \frac{\alpha\beta}{\beta-\alpha}(e^{-\alpha z} - e^{-\beta z}) & ,\alpha \neq \beta \\ \alpha^2 z e^{-\alpha z} & ,\alpha = \beta \end{cases} \quad , \quad i = 1, \dots, n. \tag{3.3}$$

When the active strength and standby redundancy components are identical, i.e. $\alpha = \beta$, the total lifetime distributions $Z_i, i = 1, \dots, n$ follow the gamma distribution with shape parameter 2 and rate parameter α. In this circumstance, the stress-strength reliability problem is reduced to the simple stress-strength reliability of the gamma components in Nojosa and Rathie [60]. Hence, we consider the case where parameters α and β are not the same in this study. Since $Z_1, \dots, Z_n$ denote n independent total strength random variables given in (3.2), the cdf for the lifetime of the parallel system is

$$F_{Z_{(n)}}(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha e^{-\beta z} - \beta e^{-\alpha z})^k}{(\beta - \alpha)^k} \tag{3.4}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^j \frac{\alpha^{k-j}\beta^j}{(\beta - \alpha)^k} e^{-z[j(\alpha-\beta)+\beta k]} \tag{3.5}$$

where $Z_{(n)} = \max(Z_1, \dots, Z_n)$.

When the maximum strength component $Z_{(n)}$ is exposed to the common stress component T, the stress-strength reliability of the parallel system is as follows:

$$R_{PC} = \int_0^\infty P(Z_{(n)} > T \mid T = t) f_T(t) dt \tag{3.6}$$

$$= 1 - \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \frac{(-1)^j \alpha^{k-j} \beta^j \theta}{(\beta - \alpha)^k} \int_0^\infty e^{-t[\alpha j + \beta(k-j) + \theta]} dt \tag{3.7}$$

$$= 1 - \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \frac{(-1)^j}{(\beta - \alpha)^k} \frac{\alpha^{k-j}\beta^j \theta}{\alpha j + \beta(k-j) + \theta}. \tag{3.8}$$

Moreover, if we use n standby strength components as original components in the n components parallel system, we construct a new parallel system with $2n$ strength components. The stress-strength reliability of the new parallel system is derived under the common stress for comparison. Let V_i be the lifetime of i^{th} strength components $i = 1, \dots, 2n$, and follow the exponential distribution with parameter α for $1 \le i \le n$ and β for $n + 1 \le i \le 2n$. In this way, the cdf for the lifetime of the $2n$ components parallel system is

$$F_{V_{(2n)}}(t) = P(\max(V_1, \dots, V_{2n}) \le t) = (1 - e^{-\alpha t})^n (1 - e^{-\beta t})^n. \tag{3.9}$$

Then, the stress-strength reliability of this parallel system is given as

$$R_V = \int_0^\infty P(V_{(2n)} > T \mid T = t) f_T(t) dt \tag{3.10}$$

$$= 1 - \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{n}{j} (-1)^{k+j} \frac{\theta}{\alpha k + \beta j + \theta} \tag{3.11}$$

under the common stress T.

• At system level, we have two n-component parallel systems, one of them is a duplicate of the original parallel system which is constituted by standby components. A representation of standby redundancy at system level is given in Figure 1.4. Let $X_{(n)}$ and $Y_{(n)}$ denote the n^{th} order original strength and standby components, respectively. Hence, $X_{(n)} = \max(X_1, \dots, X_n)$ and $Y_{(n)} = \max(Y_1, \dots, Y_n)$ where $X_i \sim exp(\alpha)$ and $Y_i \sim exp(\beta)$, $i = 1, \dots, n$. In the case of system level, including the standby components, the total lifetime of a strength component is given by $Z_{(n)} = X_{(n)} + Y_{(n)}$. Therefore, the cdf of $Z_{(n)}$ is computed by applying the binomial formula as follows:

$$F_{Z_{(n)}}(t) = P(X_{(n)} + Y_{(n)} \le t) \tag{3.12}$$

$$= \int_0^t F_{X_{(n)}}(t-y) f_{Y_{(n)}}(y) dy \tag{3.13}$$

$$= \int_0^t (1-e^{-\alpha(t-y)})^n \beta n e^{-\beta y}(1-e^{-\beta y})^{n-1} dy \tag{3.14}$$

$$= n\beta \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} (-1)^{k+j} e^{-\alpha k t} \int_0^t e^{-y[\beta(1+j)-\alpha k]} dy \tag{3.15}$$

$$= n\beta \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} \frac{(-1)^{k+j}(e^{-\alpha k t} - e^{-\beta(1+j)t})}{\beta(1+j)-\alpha k} \tag{3.16}$$

and taking the derivative of the cdf we obtain the pdf as

$$f_{Z_{(n)}}(t) = n\beta \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} \frac{(-1)^{k+j}[\beta(1+j)e^{-\beta(1+j)t} - \alpha k e^{-\alpha k t}]}{\beta(1+j)-\alpha k}. \tag{3.17}$$

The system reliability when the strength component is under the stress is

$$R_{PS} = P(\max(X_1, \dots, X_n) + \max(Y_1, \dots, Y_n) > T) \tag{3.18}$$

$$= \int_0^\infty P(Z_{(n)} > T | T = t) f_T(t) dt \tag{3.19}$$

$$= \int_0^\infty [1 - F_{Z_{(n)}}(t)] f_T(t) dt \tag{3.20}$$

$$= 1 - \int_0^\infty F_{Z_{(n)}}(t) f_T(t) dt \tag{3.21}$$

$$= 1 - n\beta\theta \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} \frac{(-1)^{k+j}}{(\alpha k+\theta)[\beta(1+j)+\theta]}. \tag{3.22}$$

The effect of standby components for the stress-strength reliability in our case can be visualized in the following graphs. In Figure 3.1, the stress-strength reliabilities of three different series systems are plotted by using Equations (3.8), (3.11) and (3.22) based on different parameter sets. It seems that system reliability is higher when the standby components are incorporated to the system. As in the series system, here we can also add that in order to increase the reliability, standby components can be implemented under appropriate circumstances.

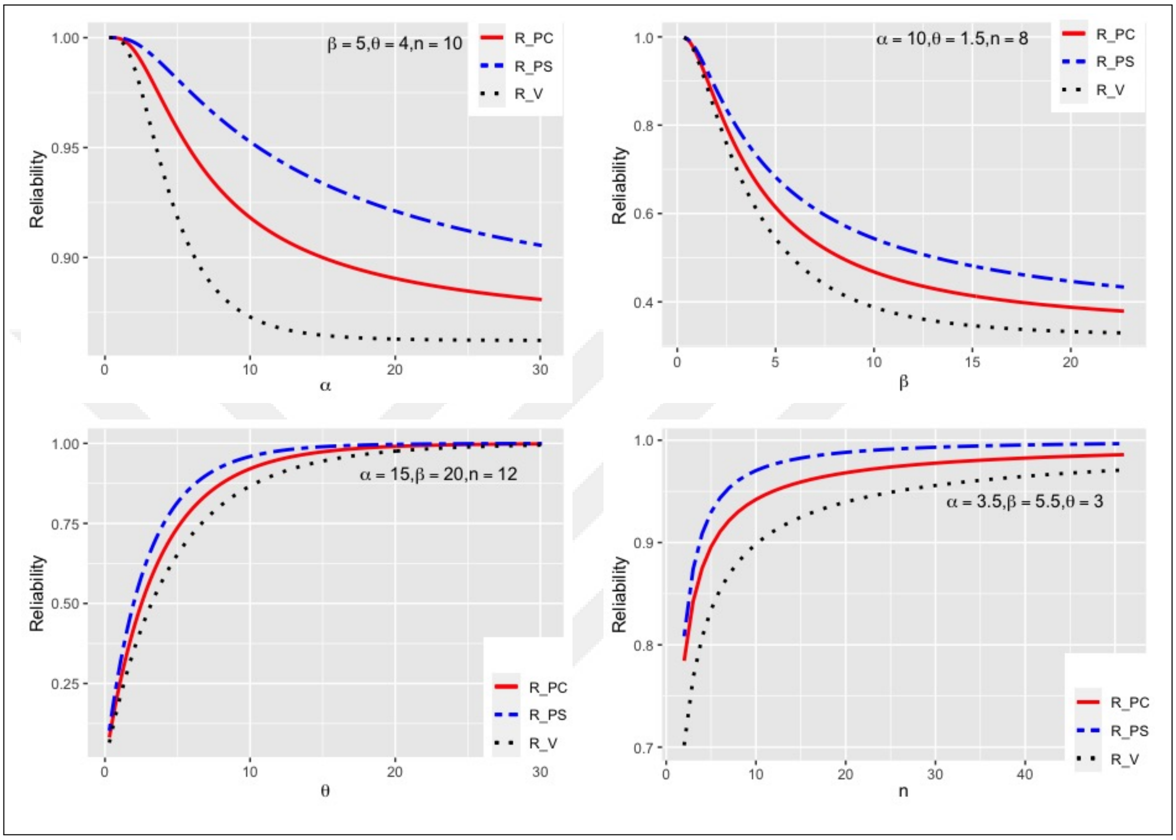

Figure 3.1. Plots for R_V, R_{PC} and R_{PS}

Now, let us give a brief framework for the rest of this chapter. First, ML estimate and Bayesian estimates of both stress-strength reliability; R_{PC} and mean remaining strength; Φ_{PC} for the parallel setting at component level are given in Section 3.2. Next, Lindley's approximation method is executed. For the purpose of comparing the estimates, a simulation study and a real data analysis are carried out in Section 3.3. Section 3.4 comprises the derivation of ML and Bayesian estimators of stress-strength reliability; R_{PS} and also Φ_{PS}, with cold standby redundancy at system level. ACI and HPD credible intervals are also obtained. Moreover, in Section 3.5 simulation results due to ML and MCMC, and a real data analysis are given for illustrative purposes. Conclusions of the parallel system are included in Section 3.6.

3.2. ESTIMATION OF R_{PC} AND Φ_{PC}

In this section, estimation of stress-strength reliability and MRS of the aforementioned parallel system at component level are investigated.

3.2.1 MLE of R_{PC}

Let m systems be put on a test each with n original components and n cold standby components in the parallel system. The strength data are represented as $Z_{i1}, \dots, Z_{in}, i = 1, \dots, m$ and stress is $T_i, i = 1, \dots, m$. Then, the likelihood function of the observed sample is

$$L(\alpha, \beta, \theta; \mathbf{z}, \mathbf{t}) = \prod_{i=1}^{m} \left(\prod_{j=1}^{n} f_Z(z_{ij}) \right) f_T(t_i) \tag{3.23}$$

$$= \left(\frac{\alpha\beta}{\beta - \alpha} \right)^{nm} \exp\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}} \right) \right) \theta^m e^{-\theta \sum_{i=1}^{m} t_i} \tag{3.24}$$

and the log-likelihood function is given by

$$\ell(\alpha, \beta, \theta; \mathbf{z}, \mathbf{t}) = nm \left(\ln(\alpha\beta) - \ln(\beta - \alpha) \right) + m \ln\theta + \sum_{i=1}^{m} \sum_{j=1}^{n} \ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}} \right) - \theta \sum_{i=1}^{m} t_i. \tag{3.25}$$

The partial derivatives with respect to three parameters are obtained in the following forms:

$$\frac{\partial \ell}{\partial \alpha} = nm \left(\frac{1}{\alpha} + \frac{1}{\beta - \alpha} \right) - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{z_{ij} e^{-\alpha z_{ij}}}{e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}}, \tag{3.26}$$

$$\frac{\partial \ell}{\partial \beta} = nm \left(\frac{1}{\beta} - \frac{1}{\beta - \alpha} \right) + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{z_{ij} e^{-\beta z_{ij}}}{e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}} \tag{3.27}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^{m} t_i. \tag{3.28}$$

Hence, the ML estimate of θ is $\hat{\theta} = 1/\bar{T}$, and the ML estimates of α and β, that is $\hat{\alpha}$ and $\hat{\beta}$, are the solutions of nonlinear equation system given in (3.26) and (3.27). $\hat{\alpha}$ and $\hat{\beta}$ can be derived with the help of numerical methods, like Newton-Raphson or Broyden's method. Therefore, $\hat{R}_{PC}$ can be obtained as

$$\hat{R}_{PC} = 1 - \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \frac{(-1)^j}{(\hat{\beta} - \hat{\alpha})^k} \frac{\hat{\alpha}^{k-j} \hat{\beta}^j \hat{\theta}}{\hat{\alpha} j + \hat{\beta}(k-j) + \hat{\theta}} \tag{3.29}$$

from (3.8) by using the invariance property of MLE.

3.2.2 Asymptotic Distribution and Confidence Interval for R_{PC}

The observed information matrix of $\boldsymbol{\tau} = (\alpha, \beta, \theta)$ is given as

$$J(\boldsymbol{\tau}) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \theta} \\ \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \beta} & \frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}. \tag{3.30}$$

Since $J_{13} = J_{31} = J_{23} = J_{32} = 0$, other entries of the matrix are

$$J_{11} = nm\left(\frac{1}{\alpha^2} - \frac{1}{(\beta - \alpha)^2}\right) + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1 - e^{-z_{ij}(\beta-\alpha)})^2}, \tag{3.31}$$

$$J_{12} = J_{21} = \frac{nm}{(\beta-\alpha)^2} - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1-e^{-z_{ij}(\beta-\alpha)})^2} \tag{3.32}$$

$$J_{22} = nm\left(\frac{1}{\beta^2} - \frac{1}{(\beta-\alpha)^2}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1-e^{-z_{ij}(\beta-\alpha)})^2}, \qquad J_{33} = \frac{m}{\theta^2}. \tag{3.33}$$

The expectations of the entries of the observed information matrix can not be obtained analytically. Therefore, the Fisher Information matrix $I(\boldsymbol{\tau}) = E(J(\boldsymbol{\tau}))$ can be obtained by using numerical methods. The MLE of R_{PC} is asymptotically normal with mean R_{PC} and asymptotic variance

$$\sigma_{R_{PC}}^2 = \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial R_{PC}}{\partial \boldsymbol{\tau}_i}\frac{\partial R_{PC}}{\partial \boldsymbol{\tau}_j} I_{ij}^{-1}, \tag{3.34}$$

where I_{ij}^{-1} is the $(i,j)^{th}$ element of the inverse of $I(\boldsymbol{\tau})$, see Rao [61]. Afterwards,

$$\sigma_{R_{PC}}^2 = \left(\frac{\partial R_{PC}}{\partial \alpha}\right)^2 I_{11}^{-1} + 2\frac{\partial R_{PC}}{\partial \alpha}\frac{\partial R_{PC}}{\partial \beta} I_{12}^{-1} + \left(\frac{\partial R_{PC}}{\partial \beta}\right)^2 I_{22}^{-1} + \left(\frac{\partial R_{PC}}{\partial \theta}\right)^2 I_{33}^{-1}. \tag{3.35}$$

Note that $I(\boldsymbol{\tau})$ can be replaced by $J(\boldsymbol{\tau})$ when $I(\boldsymbol{\tau})$ is not obtained. Therefore, an asymptotic $100(1-\gamma)$ percent confidence interval for R_{PC} is given by $\left(\widehat{R}_{PC}^{MLE} \pm z_{\gamma/2}\widehat{\sigma}_R\right)$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\widehat{\sigma}_{R_{PC}}$ is the value of $\sigma_{R_{PC}}$ at the MLE of the parameters.

3.2.3 Bayesian Estimation of R_{PC}

In this subsection, we assume that α, β and θ are random variables that follow independent gamma prior distributions with parameters (a_i, b_i), $i = 1, 2, 3$, respectively. Then, the joint posterior density function of α, β and θ is

$$\pi(\alpha,\beta,\theta|\mathbf{z},\mathbf{t}) = I(\mathbf{z},\mathbf{t})\alpha^{nm+a_1-1}\beta^{nm+a_2-1}(\beta-\alpha)^{-nm}\theta^{a_3+m-1}e^{-\alpha b_1-\beta b_2-\theta\left(b_3+\sum_{i=1}^{m} t_i\right)}z_{\alpha,\beta} \tag{3.36}$$

where $I(\mathbf{z},\mathbf{t})$ is the normalizing constant and written by

$$\frac{I(\mathbf{z},\mathbf{t})^{-1}}{\Gamma(a_3+m)}\left(b_3+\sum_{i=1}^{m} t_i\right)^{a_3+m} = \int_0^\infty\int_0^\infty\left(\frac{\alpha\beta}{\beta-\alpha}\right)^{nm}\alpha^{a_1-1}\beta^{a_2-1}e^{-\alpha b_1-\beta b_2}z_{\alpha,\beta}\,d\alpha d\beta \tag{3.37}$$

where

$$z_{\alpha,\beta} = \exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}}-e^{-\beta z_{ij}}\right)\right). \tag{3.38}$$

The Bayes estimator of R_{PC} under the SE loss function is

$$\widehat{R}_{PC}^{Bayes} = \int_0^\infty\int_0^\infty\int_0^\infty R_{PC}\,\pi(\alpha,\beta,\theta;\mathbf{z},\mathbf{t})\,d\alpha d\beta d\theta. \tag{3.39}$$

This integral can not be easily computed analytically; thus, some approximation methods are needed. In order to obtain the Bayes estimate of R_{PC}, we use Lindley's approximation and MCMC methods.

• **Lindley's approximation**

For three parameter system, let $(\theta_1,\theta_2,\theta_3)\equiv(\alpha,\beta,\theta)$ and $u\equiv u(\alpha,\beta,\theta)=R_{PC}$ given in Equation (3.8). We compute σ_{ij}, $i,j=1,2,3$ by using the following partial derivatives $L_{11}=-J_{11}, L_{12}=L_{21}=-J_{12}$ and $L_{22}=-J_{22}$. Using logarithm of the prior density, we have

$$\rho_1=\frac{(a_1-1)}{\alpha}-b_1,\quad \rho_2=\frac{(a_2-1)}{\beta}-b_2,\quad \rho_3=\frac{(a_3-1)}{\theta}-b_3. \tag{3.40}$$

Additionally, we obtain $L_{333} = \frac{2m}{\theta^3}$ and

$$L_{111} = 2nm\left(\frac{1}{\alpha^3} + \frac{1}{(\beta-\alpha)^3}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3}, \tag{3.41}$$

$$L_{121} = L_{112} = -2nm\frac{1}{(\beta-\alpha)^3} + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3}, \tag{3.42}$$

$$L_{122} = L_{221} = 2nm\frac{1}{(\beta-\alpha)^3} - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3}, \tag{3.43}$$

$$L_{222} = 2nm\left(\frac{1}{\beta^3} - \frac{1}{(\beta-\alpha)^3}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3 e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3}. \tag{3.44}$$

Let

$$S = \binom{n}{k}(-1)^k \frac{\alpha^k \beta^{n-k}\theta}{(\beta-\alpha)^n\left[\alpha(n-k)+\beta k+\theta\right]} \tag{3.45}$$

denote the common term in the first and second order partial derivatives of the parallel system reliability. We have the derivatives of reliability as given below

$$u_1 = \sum_{k=0}^{n} S\frac{\{[n\alpha + k(\beta-\alpha)][\alpha(n-k)+\beta k+\theta] - \alpha(\beta-\alpha)(n-k)\}}{\alpha(\beta-\alpha)\left[\alpha(n-k)+\beta k+\theta\right]}, \tag{3.46}$$

$$u_2 = \sum_{k=0}^{n} S\frac{(-k)[\alpha(n-k)+\beta(k+1)+\theta]}{\beta\left[\alpha(n-k)+\beta k+\theta\right]}, \tag{3.47}$$

$$u_3 = \sum_{k=0}^{n} S\frac{[\alpha(n-k)+\beta k]}{\theta\left[\alpha(n-k)+\beta k+\theta\right]}, \tag{3.48}$$

$$\begin{aligned} u_{11} = \sum_{k=0}^{n} S\Bigg[&\frac{n(n+1)}{(\beta-\alpha)^2} + \frac{2(n-k)}{\alpha(n-k)+\beta k+\theta}\left(-\frac{n}{\beta-\alpha} - \frac{k}{\alpha} + \frac{n-k}{\alpha(n-k)+\beta k+\theta}\right) \\ &+ \frac{2kn}{\alpha(\beta-\alpha)} + \frac{k(k-1)}{\alpha^2}(k-1)\Bigg], \end{aligned} \tag{3.49}$$

$$u_{12}=\sum_{k=0}^{n}S\left[\frac{n(n+1)}{(\beta-\alpha)^2}-\frac{2(n-k)}{\alpha(n-k)+\beta k+\theta}\left(\frac{n}{\beta-\alpha}+\frac{k}{\alpha}-\frac{n-k}{\alpha(n-k)+\beta k+\theta}\right)\right.$$
$$\left.+\frac{2kn}{\alpha(\beta-\alpha)}+\frac{k(k-1)}{\alpha^2}\right], \tag{3.50}$$

$$u_{22}=\sum_{k=0}^{n}S\left[\frac{n(n+1)}{(\beta-\alpha)^2}+\frac{2k}{\alpha(n-k)+\beta k+\theta}\left(\frac{n}{\beta-\alpha}-\frac{n-k}{\beta}+\frac{k}{\alpha(n-k)+\beta k+\theta}\right)\right.$$
$$\left.-\frac{2n(n-k)}{\beta(\beta-\alpha)}+\frac{(k-n)(k-n+1)}{\beta^2}\right], \tag{3.51}$$

$$u_{13}=\sum_{k=0}^{n}\frac{S}{\theta}\left[\frac{1}{\alpha(n-k)+\beta k+\theta}\left(-\theta\left(\frac{n}{\beta-\alpha}+\frac{k}{\alpha}\right)+(k-n)\left(1-\frac{2\theta}{\alpha(n-k)+\beta k+\theta}\right)\right)\right.$$
$$\left.+\frac{n}{(\beta-\alpha)}+\frac{k}{\alpha}\right], \tag{3.52}$$

$$u_{23}=\sum_{k=0}^{n}\frac{S}{\theta}\left[\frac{1}{\alpha(n-k)+\beta k+\theta}\left(\theta\left(\frac{n}{\beta-\alpha}-\frac{n-k}{\beta}\right)+k\left(\frac{2\theta}{\alpha(n-k)+\beta k+\theta}-1\right)\right)+\frac{n-k}{\beta}\right], \tag{3.53}$$

$$u_{33}=\sum_{k=0}^{n}\frac{S}{\theta}\left(\frac{\theta}{\alpha(n-k)+\beta k+\theta}-1\right). \tag{3.54}$$

Hence, we obtain $A=\sigma_{11}L_{111}+2\sigma_{12}L_{121}+\sigma_{22}L_{221}$, $B=\sigma_{11}L_{112}+2\sigma_{12}L_{122}+\sigma_{22}L_{222}$ and $C=\sigma_{33}L_{333}$. Then, Bayes estimator of R_{PC}, i.e. $\widehat{R}_{PC}^{Lindley}$, is given as

$$\widehat{R}_{PC}^{Lindley}=R+[u_1a_1+u_2a_2+u_3a_3+a_4+a_5]+\frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13})$$
$$+B(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23})+C(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33})] \tag{3.55}$$

where all the parameters are evaluated at MLEs $(\widehat{\alpha},\widehat{\beta},\widehat{\theta})$.

• **MCMC method**

The joint posterior density function of α, β and θ given data is stated in (3.36). Then, the marginal posterior density functions of the parameters are given respectively as

$$\theta \,|\mathbf{t} \sim \text{Gamma}\left(m + a_3, b_3 + \sum_{i=1}^{m} t_i\right), \tag{3.56}$$

$$\pi(\alpha \,|\beta, \mathbf{z}) \propto \alpha^{nm+a_1-1}(\beta - \alpha)^{-nm}e^{-\alpha b_1}\exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right), \tag{3.57}$$

and

$$\pi(\beta \,|\alpha, \mathbf{z}) \propto \beta^{nm+a_2-1}(\beta - \alpha)^{-nm}e^{-\beta b_2}\exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right). \tag{3.58}$$

Hence, samples of θ can be readily generated by using a gamma distribution. Since the marginal posterior distribution of α and β are not well-known distributions, it is not possible to generate sample directly as usual. Therefore, we use hybrid Metropolis-Hastings and Gibbs sampling algorithm to generate samples from $\pi(\alpha \,|\beta, \mathbf{z})$ and $\pi(\beta \,|\alpha, \mathbf{z})$, following the same algorithm as in Subsection 2.2.3. The generated posterior sample is used to compute Bayes estimate and to construct the HPD credible interval for R_{PC}. Bayes estimate of R_{PC} under a SE loss function is given by

$$\hat{R}_{PC}^{MCMC} = \frac{1}{N - M}\sum_{i=M+1}^{N} R_{PC}^{(i)}, \tag{3.59}$$

where M is the burn-in period.

3.2.4 Inference on Φ_{PC}

The MRS of a parallel system under the stress T is defined by the conditional expectation as $\Phi_{PC} = E(Z_{(n)} - T | Z_{(n)} > T)$ where $Z_{(n)} \equiv Z_{n:n}$ denotes n^{th} order statistics. Let ψ denotes the conditional random variable $\psi = Z_{n:n} - T | Z_{n:n} > T$. Then the cdf of ψ is given by

$$F_{\psi}(x) = P(Z_{n:n} - T \leq x | Z_{n:n} > T) \tag{3.60}$$

$$= P(Z_{n:n} \leq T + x | Z_{n:n} > T) \tag{3.61}$$

$$= \frac{P(Z_{n:n} \leq T + x, Z_{n:n} > T)}{P(Z_{n:n} > T)} \tag{3.62}$$

where $x > 0$ and the denominator represents R_{PC}. Thereafter, conditioning on $T = t$, gives

$$P(Z_{n:n} \leq T + x, Z_{n:n} > T) = \int_0^{\infty} P(Z_{n:n} \leq T + x, Z_{n:n} > T | T = t) f_T(t) dt \tag{3.63}$$

$$= \int_0^{\infty} P(t \leq Z_{n:n} \leq t + x) f_T(t) dt, \quad x > 0 \tag{3.64}$$

$$= \int_0^{\infty} [F_{Z_{n:n}}(t + x) - F_{Z_{n:n}}(t)] f_T(t) dt \tag{3.65}$$

$$= \int_0^{\infty} F_{Z_{n:n}}(t + x) f_T(t) dt - \int_0^{\infty} F_{Z_{n:n}}(t) f_T(t) dt \tag{3.66}$$

$$= I_1 - I_2. \tag{3.67}$$

It is obvious that $I_2 = 1 - R_{PC}$. Since this is a constant, it does not depend on x. In this manner, we calculate I_1 as given below

$$I_1 = \int_0^{\infty} F_{Z_{n:n}}(t + x) f_T(t) dt \tag{3.68}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \frac{(-1)^j \alpha^{k-j} \beta^j \theta}{(\beta - \alpha)^k} e^{-x[\alpha j + \beta(k-j)]} \int_0^{\infty} e^{-t[\alpha j + \beta(k-j) + \theta]} dt \tag{3.69}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^j\alpha^{k-j}\beta^j\theta}{(\beta-\alpha)^k}\frac{e^{-x[\alpha j+\beta(k-j)]}}{\alpha j+\beta(k-j)+\theta} \tag{3.70}$$

$$= 1+\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^j\alpha^{k-j}\beta^j\theta}{(\beta-\alpha)^k}\frac{e^{-x[\alpha j+\beta(k-j)]}}{\alpha j+\beta(k-j)+\theta}. \tag{3.71}$$

The cdf and pdf of ψ are

$$F_\psi(x) = \frac{I_1-I_2}{R_{PC}} \tag{3.72}$$

$$= \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^j\alpha^{k-j}\beta^j\theta[e^{-x[\alpha j+\beta(k-j)]}-1]}{R_{PC}(\beta-\alpha)^k[\alpha j+\beta(k-j)+\theta]} \tag{3.73}$$

and derivative of the cdf is

$$f_\psi(x) = \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^{j+1}\alpha^{k-j}\beta^j\theta[\alpha j+\beta(k-j)]e^{-x[\alpha j+\beta(k-j)]}}{R_{PC}(\beta-\alpha)^k[\alpha j+\beta(k-j)+\theta]} \tag{3.74}$$

for $\alpha \neq \beta$. By consequence, the MRS of the parallel system at component level is as follows:

$$\Phi_{PC} = E_\psi(x) = \int_0^\infty x f_\psi(x)dx \tag{3.75}$$

$$= \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^{j+1}\alpha^{k-j}\beta^j\theta[\alpha j+\beta(k-j)]}{R_{PC}(\beta-\alpha)^k[\alpha j+\beta(k-j)+\theta]}\int_0^\infty xe^{-x[\alpha j+\beta(k-j)]}dx \tag{3.76}$$

$$= \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^{j+1}\alpha^{k-j}\beta^j\theta}{R_{PC}(\beta-\alpha)^k[\alpha j+\beta(k-j)+\theta][\alpha j+\beta(k-j)]}. \tag{3.77}$$

3.2.4.1 Estimation of Φ_{PC}

The ML estimate of Φ_{PC}, i.e. $\widehat{\Phi}_{PC}^{MLE}$, which is computed from (3.77) by using the invariance property of MLE. It is clear that $\widehat{\Phi}_{PC}^{MLE}$ is asymptotically normal with mean Φ_{PC} and asymptotic variance is computed by using the formula in Equation (3.94). Hence, an asymptotic $100(1-\gamma)$ percent confidence interval for Φ_{PC} is given by $(\widehat{\Phi}_{PC}^{MLE} \pm z_{\gamma/2}\widehat{\sigma}_{\Phi_{PC}})$ where $\widehat{\sigma}_{\Phi_{PC}}$ is the value of $\sigma_{\Phi_{PC}}$ at the MLE of the parameters.

Under the setup made in Subsection 3.2.3, the Bayes estimator of Φ_{PC} under the SE loss function is written as

$$\widehat{\Phi}_{PC}^{Bayes} = \int_0^\infty \int_0^\infty \int_0^\infty \Phi_{PC}\, \pi(\alpha, \beta, \theta \,|\, \mathbf{z}, \mathbf{t})\; d\alpha d\beta d\theta. \tag{3.78}$$

Analogously to the reliability case, since the above integral can not be computed analytically, the Bayes estimate of Φ_{PC} under SE loss function is obtained by using Lindley's approximation. It is omitted because a similar procedure is mentioned in the reliability case.

3.3. NUMERICAL RESULTS OF R_{PC} AND Φ_{PC}

3.3.1 Simulation Study

In this subsection, a simulation study is carried out to examine the performance of two different ML estimates and Bayesian estimates under informative and non-informative priors with respect to MSE and estimated risk. Statistical software R [64] is executed for whole computations. The nleqslv package [65] in software R is used to solve the nonlinear equations. The interval estimates are compared with respect to ALs and CPs of 95 percent confidence intervals. The ACIs are evaluated by using the Fisher information matrix. All results are produced by conducting 2500 replications.

We have considered different sample sizes $n = 5(5)20$, $m = 25(25)100$ for $(\alpha, \beta, \theta) = (8, 2, 1.5)$ in simulation. The biases and MSEs for two different ML estimates, biases and

ERs of Bayes estimates under the informative and non-informative priors are reported in in Tables 3.1 and 3.3. The first MLE of R is computed by using (3.8) depending on the solution of the non-linear equation system given in (3.26) and (3.27). Alternatively, ML estimates of unknown parameters α, β and θ are also obtained from the random sample of exponential distributions as $\widetilde{\alpha} = 1/\overline{X}$, $\widetilde{\beta} = 1/\overline{Y}$ and $\tilde{\theta} = 1/\overline{T}$. The second MLE of R, called MLE2, is computed by using $\widetilde{\alpha}, \widetilde{\beta}$ and $\tilde{\theta}$ in (3.8). In Bayesian case, the hyperparameters are selected as $a_1 = \alpha$, $b_1 = 1$, $a_2 = \beta$, $b_2 = 1$, $a_3 = \theta$, $b_3 = 1$ for the informative prior, and $a_i = b_i = 0.0001$, $i = 1, 2, 3$ for the non-informative case. The AL for the 95 percent asymptotic confidence interval of R_{PC} and Φ_{PC}, and their corresponding CP values are given in Tables 3.2 and 3.4.

From Tables 3.1 and 3.3, MSE and ERs of all the estimates decrease as the sample size enlarges as expected. We observe that all R_{PC} estimates have very similar performance based on MSE and ER values. In our cases, since obtaining MLE2 estimate of R_{PC} is easily applicable when the underlying distributions are exponential, it can be preferable as an alternative method. MLE2 estimate shows the finest performance in the estimates of Φ_{PC}. The performance of the Bayes estimate of Φ_{PC} under the informative prior has better performance than ML and the Bayes estimate under the non-informative prior, as expected. Moreover, performances of MLE and Bayes estimate under the non-informative prior are very similar to each other. From Tables 3.2 and 3.4, we see that the AL for each interval reduces when the sample size enlarges, and all the CPs are satisfactory.

Table 3.1. Estimates of R_{PC}

			MLE			$MLE2$			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	R_{PC}	$\hat{R}_{PC}$	Bias	MSE	$\hat{R}_{PC}$	Bias	MSE	$\hat{R}_{PC}^{Lindley}$	Bias	ER	$\hat{R}_{PC}^{Lindley}$	Bias	ER
$(\alpha, \beta, \theta) = (8, 2, 1.5)$														
5	25	0.80314	0.80644	0.00330	0.00329	0.80637	0.00324	0.00328	0.81006	0.00692	0.00461	0.79152	-0.01161	0.00644
	50		0.80412	0.00098	0.00166	0.80437	0.00123	0.00164	0.80505	0.00191	0.00179	0.79790	-0.00524	0.00213
	75		0.80350	0.00037	0.00119	0.80361	0.00047	0.00118	0.80321	0.00008	0.00119	0.80006	-0.00307	0.00129
	100		0.80240	-0.00074	0.00087	0.80252	-0.00062	0.00086	0.80177	-0.00137	0.00085	0.80013	-0.00301	0.00089
10	25	0.87624	0.87549	-0.00075	0.00217	0.87583	-0.00041	0.00214	0.86996	-0.00629	0.00207	0.86478	-0.01146	0.00266
	50		0.87569	-0.00055	0.00113	0.87592	-0.00032	0.00111	0.87211	-0.00414	0.00109	0.87098	-0.00527	0.00118
	75		0.87475	-0.00149	0.00074	0.87500	-0.00125	0.00073	0.87212	-0.00412	0.00073	0.87167	-0.00457	0.00077
	100		0.87515	-0.00109	0.00057	0.87525	-0.00099	0.00056	0.87313	-0.00311	0.00056	0.87286	-0.00338	0.00058
15	25	0.90685	0.90518	-0.00167	0.00152	0.90568	-0.00117	0.00149	0.89783	-0.00901	0.00148	0.89577	-0.01108	0.00174
	50		0.90565	-0.00120	0.00080	0.90595	-0.00090	0.00078	0.90151	-0.00534	0.00080	0.90110	-0.00574	0.00085
	75		0.90605	-0.00080	0.00058	0.90622	-0.00062	0.00057	0.90321	-0.00364	0.00058	0.90302	-0.00382	0.00060
	100		0.90621	-0.00064	0.00040	0.90631	-0.00054	0.00040	0.90403	-0.00281	0.00040	0.90394	-0.00290	0.00042
20	25	0.92415	0.92178	-0.00237	0.00131	0.92213	-0.00202	0.00128	0.91399	-0.01016	0.00134	0.91305	-0.01110	0.00152
	50		0.92298	-0.00117	0.00063	0.92319	-0.00096	0.00062	0.91883	-0.00532	0.00064	0.91861	-0.00554	0.00068
	75		0.92235	-0.00179	0.00043	0.92247	-0.00168	0.00043	0.91957	-0.00461	0.00044	0.91942	-0.00473	0.00046
	100		0.92374	-0.00041	0.00034	0.92376	-0.00039	0.00033	0.92160	-0.00255	0.00034	0.92155	-0.00259	0.00035

Table 3.2. Average lengths and coverage probabilities of R_{PC}

n	m	R_{PC}	AL	CP	n	m	R_{PC}	AL	CP
$(\alpha, \beta, \theta) = (8, 2, 1.5)$									
5	25	0.80314	0.22324	0.92280	15	25	0.90685	0.15458	0.92040
	50		0.16076	0.93640		50		0.11091	0.93080
	75		0.13191	0.93637		75		0.09097	0.92800
	100		0.11492	0.94080		100		0.07900	0.93517
10	25	0.87624	0.18035	0.91520	20	25	0.92415	0.13752	0.90116
	50		0.12910	0.92397		50		0.09835	0.93515
	75		0.10656	0.94480		75		0.08145	0.92834
	100		0.09239	0.93640		100		0.06988	0.93072

Table 3.3. Estimates of Φ_{PC}

			MLE			$MLE2$			$Bayes$ (Inf. prior)			$Bayes$ (Non-inf. prior)		
n	m	Φ_{PC}	$\hat{\Phi}_{PC}$	Bias	MSE	$\hat{\Phi}_{PC}$	Bias	MSE	$\hat{\Phi}_{PC}^{Lindley}$	Bias	ER	$\hat{\Phi}_{PC}^{Lindley}$	Bias	ER
$(\alpha, \beta, \theta) = (8, 2, 1.5)$														
5	25	0.93271	0.93876	0.00605	0.00972	0.93962	0.00691	0.00794	0.90860	-0.02410	0.01927	0.94818	0.01547	0.01015
	50		0.93166	-0.00105	0.00474	0.93556	0.00285	0.00376	0.92716	-0.00555	0.00493	0.93771	0.00500	0.00472
	75		0.93186	-0.00085	0.00330	0.93389	0.00118	0.00261	0.93092	-0.00179	0.00288	0.93546	0.00275	0.00324
	100		0.93279	0.00008	0.00253	0.93481	0.00211	0.00203	0.93297	0.00026	0.00228	0.93531	0.00260	0.00253
10	25	1.16860	1.17297	0.00436	0.01011	1.17641	0.00781	0.00796	1.16671	-0.00189	0.00891	1.17776	0.00916	0.01002
	50		1.17037	0.00177	0.00493	1.17266	0.00406	0.00378	1.17027	0.00167	0.00433	1.17253	0.00393	0.00496
	75		1.16618	-0.00242	0.00314	1.16849	-0.00011	0.00244	1.16698	-0.00162	0.00289	1.16765	-0.00095	0.00315
	100		1.16864	0.00004	0.00249	1.16955	0.00094	0.00186	1.16924	0.00063	0.00232	1.16976	0.00116	0.00250
15	25	1.32142	1.32202	0.00060	0.00964	1.32719	0.00578	0.00751	1.31930	-0.00212	0.00804	1.32349	0.00207	0.00966
	50		1.32077	-0.00065	0.00513	1.32346	0.00204	0.00382	1.32091	-0.00051	0.00464	1.32164	0.00022	0.00513
	75		1.32099	-0.00042	0.00346	1.32281	0.00139	0.00272	1.32127	-0.00015	0.00325	1.32162	0.00020	0.00347
	100		1.32173	0.00031	0.00239	1.32273	0.00131	0.00187	1.32199	0.00057	0.00227	1.32222	0.00080	0.00239
20	25	1.43546	1.43949	0.00403	0.00991	1.44334	0.00788	0.00803	1.43724	0.00178	0.00848	1.43941	0.00395	0.00986
	50		1.43606	0.00060	0.00507	1.43827	0.00280	0.00392	1.43562	0.00015	0.00468	1.43613	0.00067	0.00506
	75		1.43441	-0.00106	0.00332	1.43560	0.00014	0.00248	1.43427	-0.00119	0.00314	1.43449	-0.00097	0.00332
	100		1.43682	0.00135	0.00255	1.43661	0.00115	0.00192	1.43666	0.00120	0.00244	1.43689	0.00143	0.00255

Table 3.4. Average lengths and coverage probabilities of Φ_{PC}

n	m	Φ_{PC}	AL	CP	n	m	Φ_{PC}	AL	CP
$(\alpha, \beta, \theta) = (8, 2, 1.5)$									
5	25	0.93271	0.34546	0.91539	15	25	1.32142	0.34147	0.90234
	50		0.24147	0.90394		50		0.24168	0.88826
	75		0.19798	0.90267		75		0.20010	0.89497
	100		0.17333	0.89843		100		0.17564	0.90879
10	25	1.16860	0.34496	0.89476	20	25	1.43546	0.34486	0.90813
	50		0.24123	0.90854		50		0.24580	0.90086
	75		0.19746	0.90939		75		0.20362	0.90211
	100		0.17104	0.90171		100		0.17463	0.90110

3.3.2 Real Data Analysis

In this subsection, an analysis using real-life data set is presented to illustrate the proposed methods. Nelson [67] considered the graphical methods to analyze accelerated life test data about times to breakdown of an insulating fluid exposed to several steady high test voltages. The number of times to breakdown were observed and saved for each test voltage, and all the data were given in Nelson [67] (Table 1).

An engineer has a parallel system with three components and each one is isolated by using this insulating fluid. This engineer wants to decide what voltage values should be used for these components to minimize the breakdown times. For this purpose, he/she wants to compare the low and mid voltage levels (32 Kv and 34 Kv) against the high voltage level (36 Kv). Then, three components parallel system with standby components is constructed by using 32 Kv and 34 Kv data sets. It is assumed that 32 Kv, 34 Kv and 36 Kv data sets represent the strength component (**X**), standby strength component (**Y**) and stress component (**T**) observations, respectively. For the strength data sets, **X** and **Y** are used for three components of the parallel system. Based on the previous information, if the reliability value of this system exceeds 0.90, he/she prefers the parallel system with standby components.

We use 15 observations for strength components and 5 observations for stress component. Hence, a random sample of the size 15 is chosen from 34 Kv data for **Y** and a random sample of the size 5 is chosen from 36 Kv data for $\mathbf{T}$. Every column of $\mathbf{X}$ and $\mathbf{Y}$ data sets represents the observations of each component of the parallel system. Then, the observed data $\mathbf{X}$, $\mathbf{Y}$ and $(\mathbf{Z}, \mathbf{T})$ for $n = 3$, $m = 5$ are given as

$$\mathbf{X} = \begin{bmatrix} 0.27 & 3.91 & 53.24 \\ 0.40 & 9.88 & 82.85 \\ 0.69 & 13.95 & 89.29 \\ 0.79 & 15.93 & 100.58 \\ 2.75 & 27.80 & 215.10 \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} 4.15 & 33.91 & 8.27 \\ 4.67 & 7.35 & 36.71 \\ 72.89 & 0.78 & 2.78 \\ 31.75 & 4.85 & 0.19 \\ 0.96 & 3.16 & 8.01 \end{bmatrix} \tag{3.79}$$

$$\mathbf{Z} = \begin{bmatrix} 4.42 & 37.82 & 61.51 \\ 5.07 & 17.23 & 119.56 \\ 73.58 & 14.73 & 92.07 \\ 32.54 & 20.78 & 100.77 \\ 3.71 & 30.96 & 223.11 \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} 1.97 \\ 3.99 \\ 0.99 \\ 2.58 \\ 25.50 \end{bmatrix}. \tag{3.80}$$

We check whether data sets $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{T}$ come from the exponential distribution or not. K-S, A-D and C-VM tests are carried out for the goodness-of-fit test. Test outcomes are listed in Table 3.5. According to the results, the exponential distribution provides a good fit to these data sets.

In this instance, the MLE of the parameters are obtained as $(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta})$=(0.0183, 0.8230,0.1427) for our model. The estimates of R are shared in Table 3.6 based on these ML estimates. Bayes estimates of R are obtained under three different priors like as Prior 1: $a_i = b_i = 2$, $i = 1, 2, 3$, Prior 2: $a_1 = 0.5084$, $b_1 = 0.0123$, $a_2 = 0.5539$, $b_2 = 0.0377$, $a_3 = 0.5677$, $b_3 = 0.0810$ and Prior 3: $a_i = b_i = 0.0001$, $i = 1, 2, 3$. The hyperparameters in Prior 2 are obtained by using moment estimation of Gamma distribution based on the data $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{T}$. This method can be preferable when the prior information is not available. Bayes estimates are evaluated by performing Lindley's approximation and MCMC method. In the MCMC cases, we execute two MCMC chains with fairly distinct initial values and generate 10000 iterations for every chain. The first 5000 results of each sequence are discarded for burn-in, and then every 5^{th} sampled values are saved for thinning. For our real data analysis, the convergency of the MCMC chains has been monitored by using the scale reduction factor estimate in Gelman et al. [52], and this value is observed satisfactory.

Table 3.5. Goodness-of-fit test for the real data set

$Data$	MLE	$K-S$	$p-value$	$A-D$	$p-value$	$C-VM$	$p-value$
X	$\hat{\alpha}$ =0.0243	0.3094	0.0900	3.7203	0.0123	0.4030	0.0697
Y	$\hat{\beta}$ =0.0680	0.3030	0.1020	1.5471	0.1659	0.2900	0.1439
T	$\hat{\theta}$ =0.1427	0.3658	0.4134	0.7282	0.5276	0.1336	0.4562

Table 3.6. Estimates of R_{PC} and Φ_{PC} for the real data set

Methods	R_{PC}	Intervals for R_{PC}	Φ_{PC}	Intervals for Φ_{PC}
MLE	0.99459	(0.98770,1.00)	94.95178	(43.93732,145.9662)
MLE2	0.99801	(0.98721,1.00)	85.92201	(54.06943,135.8341)
Lindley (Prior 1)	0.92702	-	97.58770	-
Lindley (Prior 2)	0.91262	-	107.47300	-
Lindley (Prior 3)	0.94527	-	106.11505	-
MCMC (Prior 1)	0.99188	(0.97169,0.99994)	91.22847	(45.9282,166.9799)
MCMC (Prior 2)	0.98898	(0.96002,0.9999)	101.6736	(47.51684,154.5611)
MCMC (Prior 3)	0.98581	(0.9479,0.99991)	101.3899	(59.82594,141.3998)

3.4. ESTIMATION OF R_{PS} AND Φ_{PS}

This section gives the estimation of the stress-strength reliability and MRS of the parallel structure at system level.

3.4.1 MLE of R_{PS}

The likelihood function of α, β and θ for the observed sample is

$$L(\alpha,\beta,\theta;\mathbf{z_{(n)}},\mathbf{t}) = \prod_{i=1}^{m} f_{Z_{(n)}}(z_{(n),i}) f_T(t_i) \tag{3.81}$$

$$=(n\beta\theta)^m \prod_{i=1}^{m}\sum_{k=0}^{n}\sum_{j=0}^{n-1}\binom{n}{k}\binom{n-1}{j}\frac{(-1)^{k+j}[\beta(1+j)e^{-\beta(1+j)z_{(n),i}}-\alpha k e^{-\alpha k z_{(n),i}}]}{\beta(1+j)-\alpha k} e^{-\theta\sum_{i=1}^{m} t_i}. \tag{3.82}$$

The log-likelihood function is

$$\ell(\alpha,\beta,\theta;\mathbf{z_{(n)}},\boldsymbol{t}) = m\ln(n\beta\theta) - \theta\sum_{i=1}^{m} t_i + \sum_{i=1}^{m}\ln\left(\sum_{k=0}^{n}\sum_{j=0}^{n-1}\binom{n}{k}\binom{n-1}{j}\frac{(-1)^{k+j}[\beta(1+j)e^{-\beta(1+j)z_{(n),i}}-\alpha k e^{-\alpha k z_{(n),i}}]}{\beta(1+j)-\alpha k}\right) \tag{3.83}$$

$$\equiv m\ln(n\beta\theta) + \sum_{i=1}^{m}\ln\left(\sum_{k=0}^{n}\sum_{j=0}^{n-1} c_{k,j}V^*[\beta(1+j)-\alpha k]^{-1}\right) - \theta\sum_{i=1}^{m} t_i \tag{3.84}$$

where $c_{k,j} = \binom{n}{k}\binom{n-1}{j}(-1)^{k+j}$ and $V^* = \beta(1+j)e^{-\beta(1+j)z_{(n),i}} - \alpha k e^{-\alpha k z_{(n),i}}$. Taking partial derivatives of (3.84) with respect to α and β, we compute following likelihood equations as

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{m}\frac{\sum_{k=0}^{n}\sum_{j=0}^{n-1}\frac{c_{k,j}k}{\beta(1+j)-\alpha k}\left[\frac{V^*}{\beta(1+j)-\alpha k} - e^{-\alpha k z_{(n),i}}(1-\alpha k z_{(n),i})\right]}{\sum_{k=0}^{n}\sum_{j=0}^{n-1}\frac{c_{k,j}V^*}{\beta(1+j)-\alpha k}}, \tag{3.85}$$

$$\frac{\partial \ell}{\partial \beta}=\frac{m}{\beta}+\sum_{i=1}^{m}\frac{\sum_{k=0}^{n}\sum_{j=0}^{n-1}\frac{c_{k,j}(1+j)}{\beta(1+j)-\alpha k}\left[-\frac{V^{*}}{\beta(1+j)-\alpha k}+e^{-\beta(1+j)z_{(n),i}}\left(1-\beta(1+j)z_{(n),i}\right)\right]}{\sum_{k=0}^{n}\sum_{j=0}^{n-1}\frac{c_{k,j}V^{*}}{\beta(1+j)-\alpha k}}. \tag{3.86}$$

The MLEs of the parameters α and β, i.e. $\widehat{\alpha}$ and $\widehat{\beta}$, can be obtained by solving the non-linear equations in (3.85) and (3.86) simultaneously. The MLE of θ is derived as $\widehat{\theta}=m/\sum_{i=1}^{m}t_i$. After obtaining $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\theta}$, $\widehat{R}_{PS}^{MLE}$, is obtained by

$$\widehat{R}_{PS}^{MLE}=1-n\widehat{\beta}\widehat{\theta}\sum_{k=0}^{n}\sum_{j=0}^{n-1}\binom{n}{k}\binom{n-1}{j}\frac{(-1)^{k+j}}{(\widehat{\alpha}k+\widehat{\theta})[\widehat{\beta}(1+j)+\widehat{\theta}]} \tag{3.87}$$

considering the invariance property of MLE.

3.4.2 Asymptotic Distribution and Confidence Interval for R_{PS}

To evaluate the asymptotic confidence interval, the second derivative of the log-likelihood function should be obtained. Since our likelihood given in (3.84) is in a complex form, to simplify the expression. Let

$$A_1(\alpha,\beta;\mathbf{z_{(n),i}})=\sum_{k=0}^{n}\sum_{j=0}^{n-1}\binom{n}{k}\binom{n-1}{j}\frac{(-1)^{k+j}[\beta(1+j)e^{-\beta(1+j)z_{(n),i}}-\alpha k e^{-\alpha k z_{(n),i}}]}{\beta(1+j)-\alpha k}. \tag{3.88}$$

The log-likelihood function in its new form is given as

$$\ell(\alpha,\beta,\theta;\mathbf{z_{(n)}},\boldsymbol{t})=m\ln(n\beta\theta)+\sum_{i=1}^{m}\ln\left(A_1(\alpha,\beta,\mathbf{z_{(n),i}})\right)-\theta\sum_{i=1}^{m}t_i \tag{3.89}$$

First and second derivatives of A_1 are denoted by $A_{1d}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}) = \frac{\partial A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})}{\partial d}$ and $A_{1dd}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}) = \frac{\partial^2 A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})}{\partial d^2}$ where $d = \alpha,\beta$.

The observed information matrix of $\boldsymbol{\tau} = (\alpha,\beta,\theta)$ is defined as

$$J(\boldsymbol{\tau}) = -\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\theta} \\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\theta} \\ \frac{\partial^2\ell}{\partial\theta\partial\alpha} & \frac{\partial^2\ell}{\partial\theta\partial\beta} & \frac{\partial^2\ell}{\partial\theta^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}. \tag{3.90}$$

The elements of our observed information matrix are derived as $J_{13} = J_{31} = J_{23} = J_{32} = 0$, $J_{33} = m/\theta^2$,

$$J_{11} = -\sum_{i=1}^{m} \frac{A_{1\alpha\alpha}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}) - (A_{1\alpha}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}))^2}{A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})^2}, \tag{3.91}$$

$$J_{12} = J_{21} = -\sum_{i=1}^{m} \frac{A_{1\alpha\beta}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}) - A_{1\alpha}(\alpha,\beta,z_n)A_{1\beta}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})}{A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})^2} \tag{3.92}$$

$$J_{22} = \frac{m}{\beta^2} - \sum_{i=1}^{m} \frac{A_{1\beta\beta}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}) - (A_{1\beta}(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}}))^2}{A_1(\alpha,\beta,\mathbf{z}_{(\mathbf{n}),\mathbf{i}})^2}. \tag{3.93}$$

The expectation of the observed information matrix $I(\boldsymbol{\tau}) = E\left(J(\boldsymbol{\tau})\right)$ can not be obtained analytically. Due to the asymptotic properties of MLE, $\widehat{R}_{PS}^{MLE}$ is asymptotically normal with mean R_{PS} and asymptotic variance

$$\sigma_{R_{PS}}^2 = \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial R_{PS}}{\partial \boldsymbol{\tau}_i}\frac{\partial R_{PS}}{\partial \boldsymbol{\tau}_j} I_{ij}^{-1}, \tag{3.94}$$

where I_{ij}^{-1} is the $(i,j)^{th}$ element of the inverse of $I(\boldsymbol{\tau})$, see Rao [61]. Afterwards,

$$\sigma_{R_{PS}}^2=\left(\frac{\partial R_{PS}}{\partial \alpha}\right)^2 I_{11}^{-1}+2\frac{\partial R_{PS}}{\partial \alpha}\frac{\partial R_{PS}}{\partial \beta}I_{12}^{-1}+\left(\frac{\partial R_{PS}}{\partial \beta}\right)^2 I_{22}^{-1}+\left(\frac{\partial R_{PS}}{\partial \theta}\right)^2 I_{33}^{-1}, \tag{3.95}$$

In this problem, the Fisher information matrix can not be obtained analytically; thus, $I(\boldsymbol{\tau})$ will be replaced by $J(\boldsymbol{\tau})$. When the solutions of the likelihood equations are unique, the observed information matrix is a consistent estimator of Fisher information matrix .Therefore, an asymptotic $100(1-\gamma)$ percent confidence interval of R_{PS} is obtained as $(\widehat{R}_{PS}^{MLE} \pm z_{\gamma/2}\widehat{\sigma}_{R_{PS}})$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\widehat{\sigma}_{R_{PS}}$ is the value of $\sigma_{R_{PS}}$ at the MLE of the unknown parameters.

3.4.3 Bayesian Estimation of R_{PS}

In this subsection, Bayesian estimation of R_{PS} will be taken into consideration. In Bayesian approach, it is assumed that we have prior information for about the unknown parameters. Suppose that α, β and θ have independent gamma priors with parameters (a_i, b_i), $i = 1, 2, 3$, respectively. The joint posterior density function of α, β and θ is given by

$$\begin{aligned}\pi(\alpha,\beta,\theta \mid \mathbf{z}_{(\mathbf{n})},\mathbf{t}) \propto\ & \alpha^{a_1-1}e^{-\alpha b_1}\beta^{m+a_2-1}e^{-\beta b_2}\theta^{m+a_3-1}e^{-\theta(b_3+\sum_{i=1}^{m}t_i)} \\ & \cdot\prod_{i=1}^{m}\sum_{k=0}^{n}\sum_{j=0}^{n-1}\binom{n}{k}\binom{n-1}{j}\frac{(-1)^{k+j}[\beta(1+j)e^{-\beta(1+j)z_{(n),i}}-\alpha k e^{-\alpha k z_{(n),i}}]}{\beta(1+j)-\alpha k}.\end{aligned} \tag{3.96}$$

Bayes estimate of R_{PS}, i.e. $\widehat{R}_{PS}^{Bayes}$, under the SE loss function is

$$\widehat{R}_{PS}^{Bayes}=\int_0^\infty\int_0^\infty\int_0^\infty R_{PS}\,\pi(\alpha,\beta,\theta \mid \mathbf{z}_{(\mathbf{n})},\mathbf{t})\,d\alpha\, d\beta\, d\theta. \tag{3.97}$$

Since this integral is not computed analytically, some approximation methods are required to obtain approximate Bayes estimate. Next, we consider the MCMC method.

• **MCMC method**

The joint posterior density function of α, β and θ given data is stated in (3.96). Then, the marginal posterior density functions of the parameters are given respectively as

$$\theta \,|\mathbf{t} \sim Gamma\left(m + a_3, b_3 + \sum_{i=1}^{m} t_i\right), \tag{3.98}$$

$$\pi(\alpha \,|\beta, \mathbf{z_{(n)}}) \propto \alpha^{a_1-1} e^{-\alpha b_1} \prod_{i=1}^{m} \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k} \binom{n-1}{j} \frac{(-1)^{k+j}}{[\beta(1+j) - \alpha k)]}$$
$$\cdot \left[\beta(1+j) e^{-\beta(1+j) z_{(n),i}} - \alpha k e^{-\alpha k z_{(n),i}}\right] \tag{3.99}$$

$$\pi(\beta \,|\alpha, \mathbf{z_{(n)}}) \propto \beta^{m+a_2-1} e^{-\beta b_2} \prod_{i=1}^{m} \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k} \binom{n-1}{j} \frac{(-1)^{k+j}}{[\beta(1+j) - \alpha k)]}$$
$$\cdot \left[\beta(1+j) e^{-\beta(1+j) z_{(n),i}} - \alpha k e^{-\alpha k z_{(n),i}}\right]. \tag{3.100}$$

The samples of θ may be easily generated by using a gamma distribution. Since the marginal posterior distributions of α and β are not well-known distributions, generating samples directly with the help of standard methods is not possible. Therefore, Bayes estimate and HPD credible interval of R_{SS} are computed by implementing the hybrid Metropolis-Hastings and Gibbs sampling algorithm. Since this process is similar to in Subsection 2.2.3, it is omitted.

3.4.4 Inference on Φ_{PS}

The MRS of a parallel system at system level under the stress T is defined by the conditional expectation as $\Phi_{PS} = E(Z_{(n)} - T|Z_{(n)} > T)$ where $Z_{(n)} = X_{(n)} + Y_{(n)}$. Let ψ denotes the conditional random variable $\psi = Z_{(n)} - T|Z_{(n)} > T$. Hence, the cdf of ψ is written as

$$F_{\psi}(x) = P(Z_{(n)} - T \leq x|Z_{(n)} > T) \tag{3.101}$$
$$= P(Z_{(n)} \leq T + x|Z_{(n)} > T) \tag{3.102}$$
$$= \frac{P(Z_{(n)} \leq T + x, Z_{(n)} > T)}{P(Z_{(n)} > T)} \tag{3.103}$$

where $x > 0$ and the denominator represents R_{PS}. Thereafter, conditioning on $T = t$, gives

$$P(Z_{(n)} \leq T + x, Z_{(n)} > T) = \int_0^{\infty} P(Z_{(n)} \leq T + x, Z_{(n)} > T|T = t) f_T(t) dt \tag{3.104}$$
$$= \int_0^{\infty} P(t \leq Z_{(n)} \leq t + x) f_T(t) dt, \quad x > 0 \tag{3.105}$$
$$= \int_0^{\infty} [F_{Z_{(n)}}(t + x) - F_{Z_{(n)}}(t)] f_T(t) dt \tag{3.106}$$
$$= \int_0^{\infty} F_{Z_{(n)}}(t + x) f_T(t) dt - \int_0^{\infty} F_{Z_{(n)}}(t) f_T(t) dt \tag{3.107}$$
$$= I_1 - I_2. \tag{3.108}$$

It is obvious that $I_2 = 1 - R_{PS}$. Since this is a constant, we obtain I_1 as below

$$I_1 = \int_0^{\infty} F_{Z_{(n)}}(t + x) f_T(t) dt \tag{3.109}$$
$$= n\beta \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k} \binom{n-1}{j} \frac{(-1)^{k+j}}{\beta(1+j) - \alpha k} \int_0^{\infty} (e^{-\alpha k(t+x)} - e^{-\beta(1+j)(t+x)}) \theta e^{-\theta t} dt \tag{3.110}$$
$$= n\beta\theta \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k} \binom{n-1}{j} \frac{(-1)^{k+j}}{\beta(1+j) - \alpha k} \left(\frac{e^{-\alpha k x}[\beta(1+j) + \theta] - e^{-\beta(1+j)x}(\alpha k + \theta)}{(\alpha k + \theta)[\beta(1+j) + \theta]} \right). \tag{3.111}$$

The cdf of ψ is

$$F_{\psi}(x) = \frac{n\beta\theta}{R_{PS}} \sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} \frac{(-1)^{k+j}}{(\alpha k+\theta)[\beta(1+j)+\theta]} \cdot \left(\frac{e^{-\alpha k x}[\beta(1+j)+\theta] - e^{-\beta(1+j)x}(\alpha k+\theta)}{\beta(1+j)+\theta} - 1 \right) \quad (3.112)$$

and derivative of the cdf gives us the pdf as below:

$$f_{\psi}(x) = \frac{1}{R_{PS}} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^{j} n\beta\theta e^{-\beta(1+j)x}}{\beta(1+j)+\theta} + \frac{n\beta\theta}{R_{PS}} \sum_{k=1}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} \frac{(-1)^{k+j}}{\beta(1+j)+\theta} \left(\frac{-\alpha k e^{-\alpha k x}}{\alpha k+\theta} + \frac{\beta(1+j) e^{-\beta(1+j)x}}{\beta(1+j)+\theta} \right). \quad (3.113)$$

Lastly, the MRS of the parallel system at system level is derived as

$$\Phi_{PS} = E_{\psi}(x) = \int_{0}^{\infty} x f_{\psi}(x)\,dx \quad (3.114)$$

$$= \frac{1}{R_{PS}} \Bigg[\sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^{j} n\beta\theta}{[\beta(1+j)]^{2}[\beta(1+j)+\theta]} + \sum_{k=1}^{n} \sum_{j=0}^{n-1} \binom{n}{k}\binom{n-1}{j} \frac{(-1)^{k+j} n\beta\theta}{\beta(1+j)-\alpha k} \left(\frac{-1}{\alpha k(\alpha k+\theta)} + \frac{1}{\beta(1+j)[\beta(1+j)+\theta]} \right) \Bigg]. \quad (3.115)$$

3.4.4.1 Estimation of Φ_{PS}

The ML estimate of Φ_{PS}, i.e. $\widehat{\Phi}_{PS}^{MLE}$, which is computed from (3.77) by considering the invariance property of MLE. It is clear that $\widehat{\Phi}_{PS}^{MLE}$ is asymptotically normal with mean Φ_{PS} and asymptotic variance is computed by using the formula in Equation (3.94).

Implementing the structure in Subsection 3.4.3, the Bayes estimator of Φ_{PS} under the SE loss function is

$$\widehat{\Phi}_{PS}^{Bayes} = \int_0^\infty \int_0^\infty \int_0^\infty \Phi_{PS}\, \pi(\alpha, \beta, \theta \mid \mathbf{z}_{(\mathbf{n})}, \mathbf{t})\; d\alpha d\beta d\theta. \tag{3.116}$$

Analogously to the reliability case, since the above integral can not be computed analytically, the Bayes estimate of Φ_{PS} is obtained by using Lindley's approximation. Since similar procedure is mentioned in the reliability case it is omitted.

3.5. NUMERICAL RESULTS OF R_{PS} AND Φ_{PS}

3.5.1 Simulation Study

This subsection contains the comparison of the presented estimates via a Monte Carlo simulation study. The point and interval estimates of R_{PS} and Φ_{PS} are computed via the Monte Carlo simulations. The comparison of the performances of the point estimates are done by considering MSE and ER for ML and Bayes estimates, respectively. ALs of the confidence intervals and CPs provide a comparison between the interval estimates. All outcomes are obtained based on 2000 replications. The simulation studies have been carried out by using statistical software R [64]. The optim function in software R is used to solve the nonlinear equations.

In Tables 3.7-3.9, we have presented the ML and Bayesian MCMC point estimates and intervals of R_{PS} and Φ_{PS} with informative prior. We have also reported an 95 percent ACI for only R_{PS}, and HPD credible interval for both R_{PS} and Φ_{PS}. Since the analytic expression of Φ_{PS} includes a ratio of the summations, obtaining the asymptotic confidence interval of Φ_{PS} is very difficult. Hence, only HPD credible interval of Φ_{PS} is reported in Table 3.9.

In Bayesian case, the the posterior distributions of α and β, i.e. $\pi(\alpha\,|\beta, \mathbf{z})$ and $\pi(\beta\,|\alpha, \mathbf{z})$ in Equations (3.99) and (3.100) have very complicated analytical forms of the data and parameters. Therefore, we can encounter some problems about the sample generations of these posterior distributions. Hence, we prefer to use MHadaptive package [68] in R for MCMC estimates of this case. The hyperparameters are selected as $a_1 = \alpha$, $a_2 = \beta$, $a_3 = \theta$, $b_1 = b_2 = b_3 = 1$. Using MHadaptive package, we generate 5000 samples Markov chain, and discard the first 2500 values as burn-in period, and take every 5^{th} variate in thinning procedure.

From Tables 3.7 and 3.9, we observe that MSE, ER, and bias of the estimates decrease generally with increasing sample sizes as expected. ML estimate of R_{PS} shows better performance than MCMC estimate except sample $m = 10$ case in terms ER ands MSEs. However, the error values of ML and Bayes estimates of R_{PS} are very close. From Table 3.8, although the average length of HPD credible interval of R_{PS} are less than that of the ACI, CP values of the asymptotic confidence interval of R_{PS} are more satisfactory than that of HPD credible interval. Moreover, the performance of ML estimate of Φ_{PS} is better than that of MCMC in all cases.

Since the sample generations in the MCMC case are very time-consuming compared to the ML case and using the different non-linear equation system solver algorithms are available in many computer programs, application of ML estimate is very easy and, hence, we can suggest using the ML method for this case.

Table 3.7. Estimates of R_{PS}

			MLE			$Bayes$		
n	m	R_{PS}	$\hat{R}_{PS}$	Bias	MSE	$\hat{R}_{PS}^{MCMC}$	Bias	ER
				$(\alpha,\beta,\theta)=(2.5,6,2)$				
5	10	0.88732	0.88379	-0.00353	0.00452	0.88414	-0.00317	0.00424
	20		0.88478	-0.00253	0.00248	0.88524	-0.00208	0.00258
	30		0.88486	-0.00245	0.00168	0.88525	-0.00206	0.00183
	40		0.88560	-0.00171	0.00127	0.88593	-0.00139	0.00142
7	10	0.91931	0.91715	-0.00216	0.00310	0.91674	-0.00257	0.00290
	20		0.91609	-0.00322	0.00182	0.91608	-0.00323	0.00186
	30		0.91737	-0.00194	0.00116	0.91745	-0.00186	0.00125
	40		0.91621	-0.00310	0.00089	0.91632	-0.00299	0.00098
9	10	0.93764	0.93323	-0.00441	0.00236	0.93331	-0.00433	0.00220
	20		0.93389	-0.00375	0.00135	0.93384	-0.00379	0.00138
	30		0.93527	-0.00236	0.00088	0.93514	-0.00250	0.00094
	40		0.93542	-0.00222	0.00064	0.93536	-0.00227	0.00070
				$(\alpha,\beta,\theta)=(8,4.5,1.5)$				
2	10	0.50505	0.51407	0.00902	0.01250	0.52010	0.01505	0.01197
	20		0.51046	0.00541	0.00619	0.51497	0.00993	0.00695
	30		0.51025	0.00520	0.00410	0.51361	0.00856	0.00485
	40		0.51001	0.00496	0.00338	0.51283	0.00778	0.00400
4	10	0.63033	0.63942	0.00909	0.01277	0.64084	0.01052	0.01199
	20		0.63558	0.00525	0.00663	0.63726	0.00694	0.00707
	30		0.63318	0.00285	0.00427	0.63454	0.00421	0.00481
	40		0.63185	0.00152	0.00321	0.63305	0.00272	0.00370
8	10	0.73240	0.73654	0.00414	0.01072	0.73580	0.00340	0.01004
	20		0.73445	0.00205	0.00559	0.73462	0.00222	0.00584
	30		0.73702	0.00462	0.00372	0.73716	0.00476	0.00405
	40		0.73653	0.00412	0.00283	0.73668	0.00428	0.00316

(continued)

Table 3.7 Continued

			MLE			$Bayes$		
n	m	R_{PS}	$\widehat{R}_{PS}$	Bias	MSE	$\widehat{R}_{PS}^{MCMC}$	Bias	ER
$(\alpha, \beta, \theta) = (10, 3, 1)$								
6	10	0.62609	0.64489	0.01880	0.01328	0.64264	0.01654	0.01316
	20		0.63525	0.00916	0.00685	0.63488	0.00878	0.00747
	30		0.62932	0.00323	0.00449	0.62933	0.00323	0.00508
	40		0.63033	0.00423	0.00340	0.63057	0.00447	0.00390
8	10	0.66636	0.67965	0.01328	0.01247	0.67710	0.01073	0.01237
	20		0.67282	0.00645	0.00629	0.67207	0.00570	0.00685
	30		0.66965	0.00328	0.00432	0.66945	0.00309	0.00483
	40		0.67189	0.00553	0.00333	0.67166	0.00529	0.00377
12	10	0.71704	0.73255	0.01550	0.01145	0.72915	0.01211	0.01127
	20		0.72718	0.01014	0.00606	0.72588	0.00884	0.00646
	30		0.72297	0.00592	0.00406	0.72225	0.00520	0.00448
	40		0.72041	0.00336	0.00304	0.71994	0.00290	0.00342

Table 3.8. Average lengths and coverage probabilities of R_{PS}

			Asymptotic		HPD (Inf. prior)	
n	m	R_{PS}	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (2.5, 6, 2)$						
5	10	0.88732	0.26649	0.8765	0.15836	0.8240
	20		0.19463	0.9045	0.11709	0.7865
	30		0.16126	0.9250	0.09684	0.7750
	40		0.14025	0.9310	0.08470	0.7650
7	10	0.91931	0.21721	0.8515	0.12701	0.7895
	20		0.16106	0.8925	0.09362	0.7500
	30		0.13267	0.9190	0.07741	0.7525
	40		0.11703	0.9360	0.06775	0.7590
9	10	0.93764	0.18965	0.8450	0.10810	0.7675
	20		0.13875	0.8820	0.07852	0.7150
	30		0.11384	0.9020	0.06449	0.7160
	40		0.09938	0.9200	0.05611	0.7320

(continued)

Table 3.8 Continued

			Asymptotic		HPD (Inf. prior)	
n	m	R_{PS}	AL	CP	AL	CP
$(\alpha, \beta, \theta) = (8, 4.5, 1.5)$						
2	10	0.50505	0.42817	0.9180	0.27659	0.8645
	20		0.30949	0.9410	0.20887	0.8415
	30		0.25426	0.9460	0.17410	0.8340
	40		0.22083	0.9395	0.15320	0.8160
4	10	0.63033	0.42977	0.9080	0.24904	0.8175
	20		0.31286	0.9275	0.18748	0.7855
	30		0.25816	0.9425	0.15687	0.7905
	40		0.22465	0.9460	0.13763	0.7810
8	10	0.73240	0.40177	0.9060	0.21423	0.7770
	20		0.29396	0.9285	0.16029	0.7455
	30		0.24192	0.9340	0.13303	0.7485
	40		0.21077	0.9315	0.11669	0.7355
$\alpha, \beta, \theta) = (10, 3, 1)$						
6	10	0.62609	0.42920	0.9000	0.24708	0.7975
	20		0.31302	0.9220	0.18348	0.7695
	30		0.25844	0.9330	0.15257	0.7635
	40		0.22477	0.9444	0.13368	0.7415
8	10	0.66636	0.42218	0.9010	0.23407	0.7735
	20		0.30922	0.9310	0.17447	0.7620
	30		0.25515	0.9415	0.14480	0.7390
	40		0.22174	0.9385	0.12617	0.7300
12	10	0.71704	0.40337	0.8870	0.20272	0.7295
	20		0.29655	0.9200	0.15069	0.6885
	30		0.24583	0.9230	0.12525	0.6990
	40		0.21459	0.9360	0.10947	0.6850

Table 3.9. Estimates, average lengths and coverage probabilities of Φ_{PS}

			MLE			$Bayes$			HPD	
n	m	Φ_{PS}	$\hat{\Phi}_{PS}$	Bias	MSE	$\hat{\Phi}_{PS}^{MCMC}$	Bias	ER	AL	CP
$(\alpha, \beta, \theta) = (2.5, 6, 2)$										
5	10	0.95821	0.97784	0.01964	0.02376	0.99433	0.03613	0.04336	0.54590	0.9380
	20		0.96885	0.01064	0.01232	0.97660	0.01839	0.02233	0.38905	0.9315
	30		0.96599	0.00778	0.00797	0.97137	0.01317	0.01461	0.31767	0.9310
	40		0.96394	0.00574	0.00582	0.96792	0.00972	0.01075	0.27531	0.9345
7	10	1.09825	1.12149	0.02324	0.02800	1.13386	0.03561	0.04791	0.56556	0.9320
	20		1.11112	0.01287	0.01365	1.11793	0.01968	0.02416	0.40197	0.9330
	30		1.10710	0.00884	0.00889	1.11081	0.01256	0.01583	0.32846	0.9285
	40		1.10203	0.00378	0.00658	1.10509	0.00684	0.01181	0.28394	0.9265
9	10	1.20970	1.24104	0.03133	0.02897	1.25050	0.04079	0.04979	0.57931	0.9360
	20		1.22367	0.01397	0.01524	1.22891	0.01921	0.02593	0.41004	0.9190
	30		1.22196	0.01226	0.00970	1.22628	0.01658	0.01710	0.33668	0.9235
	40		1.21890	0.00920	0.00706	1.22169	0.01199	0.01257	0.29213	0.9260
$(\alpha, \beta, \theta) = (8, 4.5, 1.5)$										
2	10	0.36459	0.37422	0.00963	0.00562	0.38573	0.02114	0.00888	0.25377	0.9645
	20		0.37115	0.00656	0.00267	0.37646	0.01187	0.00455	0.18503	0.9625
	30		0.37118	0.00659	0.00184	0.37438	0.00980	0.00317	0.15341	0.9570
	40		0.36959	0.00501	0.00143	0.37161	0.00702	0.00242	0.13356	0.9535
4	10	0.48096	0.49408	0.01312	0.00580	0.50214	0.02118	0.01004	0.26967	0.9580
	20		0.48957	0.00860	0.00300	0.49275	0.01179	0.00519	0.19411	0.9550
	30		0.48623	0.00526	0.00198	0.48853	0.00756	0.00349	0.15939	0.9425
	40		0.48661	0.00565	0.00145	0.48821	0.00725	0.00262	0.13845	0.9525
8	10	0.62184	0.63765	0.01582	0.00742	0.64203	0.02019	0.01194	0.28552	0.9540
	20		0.63143	0.00960	0.00344	0.63328	0.01145	0.00594	0.20442	0.9400
	30		0.63113	0.00929	0.00241	0.63191	0.01007	0.00409	0.16826	0.9340
	40		0.62909	0.00725	0.00177	0.62972	0.00789	0.00306	0.14586	0.9315

(continued)

Table 3.9 Continued

			MLE			$Bayes$			HPD	
n	m	Φ_{PS}	$\widehat{\Phi}_{PS}$	Bias	MSE	$\widehat{\Phi}_{PS}^{MCMC}$	Bias	ER	AL	CP
					$(\alpha,\beta,\theta)=(10,3,1)$					
6	10	0.69569	0.70591	0.01021	0.01128	0.71859	0.02290	0.02071	0.37595	0.9460
	20		0.69920	0.00351	0.00568	0.70442	0.00873	0.01007	0.26685	0.9355
	30		0.69823	0.00253	0.00375	0.70252	0.00682	0.00673	0.21947	0.9390
	40		0.69906	0.00337	0.00300	0.70145	0.00576	0.00513	0.19117	0.9410
8	10	0.76741	0.77876	0.01135	0.01226	0.78879	0.02138	0.02122	0.38042	0.9395
	20		0.77365	0.00624	0.00611	0.77844	0.01103	0.01067	0.27062	0.9305
	30		0.77054	0.00314	0.00407	0.77407	0.00666	0.00707	0.22176	0.9340
	40		0.77179	0.00438	0.00299	0.77405	0.00664	0.00519	0.19319	0.9400
12	10	0.87537	0.89596	0.02059	0.01433	0.90310	0.02772	0.02382	0.39145	0.9285
	20		0.89051	0.01514	0.00692	0.89409	0.01871	0.01168	0.27844	0.9300
	30		0.88441	0.00904	0.00463	0.88611	0.01073	0.00771	0.22712	0.9180
	40		0.88188	0.00650	0.00356	0.88296	0.00759	0.00589	0.19704	0.9105

3.5.2 Real Data Analysis

A real-life data set analysis is implemented to illustrate the proposed methods. Nelson [67] considered the graphical methods for analyzing accelerated life test data about the times to breakdown of an insulating fluid subjected to several steady elevated test voltages. A number of times to breakdown were observed and saved for each test voltage, and all the data were given in Nelson [67] (Table 1).

For our case, we consider only three different test voltages 32 Kv, 34 Kv and 36 Kv. It is assumed that 36 Kv, 34 Kv and 32 Kv data sets represent the strength component (X), standby strength component (Y) and stress component (T) observations, respectively. Using these data sets the stress-strength reliability of parallel system with standby components is estimated. A practitioner can decide which voltages values can be preferable for this insulating fluid based on the reliability value. For example, if the reliability of this system is not greater than 0.50, then 36 Kv voltage can be preferred instead of parallel system with standby components.

We use 15 observations for strength components and 5 observations for stress component.

Hence, 36 Kv sample is used for X, a random sample of the size 15 is selected from 34 Kv data for Y and a random sample of the size 5 is selected from 32 Kv data for T. Then, the observed data X, Y and T for $n = 3$, $m = 5$ are given as

$$\mathbf{X} = \begin{bmatrix} 0.35 & 0.59 & 0.96 \\ 0.99 & 1.69 & 1.97 \\ 2.07 & 2.58 & 2.71 \\ 2.90 & 3.67 & 3.99 \\ 5.35 & 13.77 & 25.50 \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} 1.31 & 8.01 & 6.50 \\ 12.06 & 33.91 & 4.85 \\ 7.35 & 32.52 & 0.19 \\ 4.67 & 36.71 & 8.27 \\ 2.78 & 31.75 & 3.16 \end{bmatrix}, \mathbf{T} = \begin{bmatrix} 9.88 \\ 2.75 \\ 13.95 \\ 89.29 \\ 0.40 \end{bmatrix}.$$

Since $Z = X_{(n)} + Y_{(n)}$ at system level, we have

$$\mathbf{X}_{(n)} = \begin{bmatrix} 0.96 \\ 1.97 \\ 2.71 \\ 3.99 \\ 25.50 \end{bmatrix}, \ \mathbf{Y}_{(n)} = \begin{bmatrix} 8.01 \\ 33.91 \\ 32.52 \\ 36.71 \\ 31.75 \end{bmatrix}, \ \mathbf{Z} = \begin{bmatrix} 8.97 \\ 35.88 \\ 35.23 \\ 40.70 \\ 57.25 \end{bmatrix}.$$

We check whether data sets X, Y and T come from the exponential distribution or not. K-S, A-D, and C-VM tests are carried out for the goodness-of-fit test. Test results are listed in Table 3.10. The exponential distribution fits great to these data sets.

The MLE of the parameters are obtained as $(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}) = (2.5202, 0.0515, 0.0430)$ for our model. The point and interval estimates of R_{PS} and Φ_{PS} are tabulated in Table 3.11 based on these ML and MCMC Bayesian estimates. Asymptotic confidence intervals for ML estimates and HPD credible intervals for Bayes estimates are given in the second lines of R_{PS} and Φ_{PS} results. Bayes estimates of R are computed under three different priors like as Prior 1: $a_i = b_i = 1$, $i = 1, 2, 3$, Prior 2: $a_1 = 0.5171$, $b_1 = 0.1123$, $a_2 = 1.0072$, $b_2 = 0.0779$, $a_3 = 0.4855$, $b_3 = 0.0209$ and Prior 3: $a_i = b_i = 0$, $i = 1, 2, 3$. The hyperparameters in

Prior 2 are obtained by using moment estimation of Gamma distribution based on the data X, Y and T. This method can be preferable when the prior information is not available.

Table 3.10. Goodness-of-fit test for the real data set

$Data$	MLE	$K-S$	$p-value$	$A-D$	$p-value$	$C-VM$	$p-value$
X	$\widehat{\alpha}$ =0.0243	0.2205	0.4007	0.8026	0.4769	0.1451	0.4084
Y	$\widehat{\beta}$ =0.0680	0.1943	0.5580	0.6018	0.6434	0.0919	0.6325
T	$\widehat{\theta}$ =0.1427	0.3489	0.4739	1.094	0.3080	0.1508	0.3971

Table 3.11. Estimates of R_{PS} and Φ_{PS} for the real data set

	MLE	MLE2	MCMC(Prior 1)	MCMC (Prior 2)	MCMC (Prior 3)
R_{PS}	0.70862	0.94702	0.75134	0.72449	0.73363
	(0.40594,1.00)	(0.83538,1.00)	(0.59532,0.88704)	(0.55294,0.87090)	(0.57033,0.90662)
Φ_{PS}	28.03718	78.62397	27.27251	27.05707	31.1671
			(14.75324,43.8901)	(13.03932,42.41026)	(15.53178,56.20013)

3.6. CONCLUSIONS

This section handled statistical inference for R_{PC}, Φ_{PC}, R_{PS}, and Φ_{PS} for the parallel system when cold standby components are connected at component level and system level. The classical and Bayesian approaches are implemented to estimate the stress-strength reliability and MRS of the system. In the Bayesian case, estimates are attained by applying Lindley's approximation method for component level case and MCMC method for system level under SE loss function.

According to the simulation results, at component level, all estimators of R_{PC} show similar performance when the values of MSE and ER are taken into account. Ergo, MLE2 estimate can be preferable due to the practicality of obtaining this estimate. The performance of Bayes estimator of Φ_{PC} under informative prior is found better. The ML and Bayes estimates with non-informative prior seem indistinguishable.

At system level, ML estimators of R_{PS} and Φ_{PS} perform better than the MCMC method. Additionally, when we evaluate the two estimation methods for Φ_{PS}, MLE is suggested because of the noticeable time efficiency.

In the real data analysis for component level, the results of MLE, Lindley's approximation method, and MCMC are alike, and all can be preferable. For system level, the usage of MLE and MCMC (Prior 2) can be recommended.

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