SOLITON SURFACES AND SURFACES FROM A VARIATIONAL PRINCIPLE

A DISSERTATION SUBMITTED TO

THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

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July, 2007

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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In this thesis, we construct 2-surfaces in $\mathbb{R}^3$ and in three dimensional Minkowski space (M_3). First, we study the surfaces arising from modified Korteweg-de Vries (mKdV), Sine-Gordon (SG), and nonlinear Schrödinger (NLS) equations in $\mathbb{R}^3$. Second, we examine the surfaces arising from Korteweg-de Vries (KdV) and Harry Dym (HD) equations in M_3. In both cases, there are some mKdV, NLS, KdV, and HD classes contain Willmore-like and algebraic Weingarten surfaces. We further show that some mKdV, NLS, KdV, and HD surfaces can be produced from a variational principle. We propose a method for determining the parametrization (position vectors) of the mKdV, KdV, and HD surfaces.

Keywords: Soliton surfaces, Willmore surfaces, Weingarten surfaces, shape equation, integrable equations.

ÖZET

SOLİTON YÜZEYLERİ VE VARYASYONEL PRENSİBİNDEN ÇIKAN YÜZEYLER

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Temmuz, 2007

Bu tezde $\mathbb{R}^3$ de ve üç boyutlu Minkowski uzayında (M_3) 2-yüzeyler inşa ediyoruz. İlk olarak $\mathbb{R}^3$ de, modifiye edilmiş Korteweg-de Vries (mKdV), Sinüs-Gordon (SG), ve lineer olmayan Schrödinger (NLS) denklemlerinden çıkan yüzeyleri çalışıyoruz. İkinci olarak ise M_3 de, Korteweg-de Vries (KdV) ve Harry Dym (HD) denklemlerinden çıkan yüzeyleri inceliyoruz. İki durumda da; mKdV, NLS, KdV, ve HD yüzeylerinin bazıları Willmore-gibi ve cebirsel Weingarten yüzeylerini içermektedir. Bunlara ek olarak; bazı mKdV, NLS, KdV, ve HD yüzeylerinin varyasyonel prensibinden elde edilebileceğini gösteriyoruz. mKdV, KdV, ve HD yüzeylerinin parametrik temsillerini bulmak için bir metot öneriyoruz.

Anahtar sözcükler: Soliton yüzeyleri, Willmore yüzeyleri, Weingarten yüzeyleri, şekil denklemi, integrallenebilir denklemler.

Acknowledgement

It was a life-time experience and a great honor to work with Prof. Dr. Metin Gürses. I begin with expressing my sincere gratitude to him for the endless motivation and support he kept providing throughout my presence in the program.

Tremendous help came from Prof. Dr. Valery Yakhno during my junior years. He deserves many thanks and much more.

Sümeyra Tek, the candle of my life, has always been there when I need her most. I feel privileged having her.

For all the invaluable encouragement and attention they have given, my family has proven me that I cannot have any better.

Emrah and Mesut Şahin, and Murat Altunbulak considerably contributed to the emergence of this thesis. They are great friends in life, helpful colleagues at work. Thanks a bunch. Also, Tahir, Tansel, Burcu, Aslı, Uğur, Ayşegül, Mehmet, Sultan, Fatma, Ergün, Caroline and all other good friends have taken some roles in my life and work. Cheers to all.

Contents

Chapter 1

Introduction

Surface theory in three dimensional Euclidean space is widely used in different branches of science, particularly mathematics (differential geometry, topology, Partial Differential Equations (PDEs)), theoretical physics (string theory, general theory of relativity), and biology [1]-[5]. There are some special subclasses of 2-surfaces which arise in the branches of science aforementioned. For the classification of surfaces in three dimensional Euclidean space, particular conditions are imposed on the Gaussian and mean curvatures. These conditions are sometimes given as algebraic relations between curvatures and sometimes given as differential equations for these two curvatures. Here are some examples of some subclasses of 2-surfaces:

(i) Minimal surfaces: $H = 0$,

(ii) Surfaces with constant mean curvature : $H = constant$,

(iii) Surfaces with constant positive Gaussian curvature: $K = constant > 0$,

(iv) Surfaces with constant negative Gaussian curvature: $K = constant < 0$,

(v) Surfaces with harmonic inverse mean curvature: $\nabla^2(1/H) = 0$,

(vi) Bianchi surfaces: $\nabla^2(1/\sqrt{K}) = 0$ and $\nabla^2(1/\sqrt{-K}) = 0$, for positive Gaussian curvature and negative Gaussian curvature, respectively,

(vii) Weingarten surfaces: $f(H,K) = 0$. For example; linear Weingarten surfaces, $c_1\,H + c_2\,K = c_3$, and quadratic Weingarten surfaces, $c_4\,H^2 + c_5\,H\,K + c_6\,K^2 + c_7\,H + c_8\,K = c_9$, where c_i are constants, $i = 1, 2, ..., 9$,

(viii) Willmore surfaces: $\nabla^2 H + 2\,H(H^2 - K) = 0$,

(ix) Surfaces that solve the shape equation of lipid membrane:
$p - 2\,\omega H + k_c \nabla^2(2H) + k_c(2H + c_0)(2H^2 - c_0 H - 2K) = 0$,
where p, ω, k_c, and c_0 are constants.

Here, H and K are mean and Gaussian curvatures of the surface, respectively, and ω, k_c, k_0, ∇^2 will be defined later.

During the 19th and 20th centuries, many scientists studied these surfaces. Each one of these surfaces is important in different contexts. Some examples and details about these surfaces can be found in [3]-[16]. The main aim of this study is to find surfaces that solve the generalized shape equation which will be defined later. For this reason we give a brief history of the calculus of variations and minimal surfaces from [17]-[19].

History of minimal surfaces dates back to Leonhard Euler and Joseph Louis Lagrange in the 1700s. Although one dimensional variational problems were examined before Euler, he was the first to conduct a systematic study of them. In 1760, he published *'Récherchés sur la courbure des surfaces'*, which contains a new perspective on geometry by combining geometrical and differential variational methods. After L. Euler, J. L. Lagrange published his famous paper *'Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies'*. In this work, he developed his algorithm for one and higher dimensional calculus of variations, which is known today as *Euler-Lagrange differential equation.* As an example of the double integral, he developed a method to find the surface with least area for a given curve, which is the boundary of the surface in three dimensional Euclidean space. The result of his calculations for the solution surface $z = f(x,t)$ in $\mathbb{R}^3$ over a domain $\mathcal{U} \subset \mathbb{R}^2_{(x,t)}$ can be formulated as

$$(1 + f_t^2)\,f_{2x} - f_x f_t f_{xt} + (1 + f_x^2) f_{2t} = 0. \tag{1.1}$$

This equation is known as *equation of minimal surfaces.*

Heretofore, subscripts x and t denote the derivatives of the objects with respect to x and t, respectively. The subscript nx stands for n times x derivative, where n is a positive integer, e.g., u_{2x} indicates the second order derivative of u with respect to x.

In 1785, J. B. M. C. Meusnier discovered two surfaces satisfying the minimal surface equation that are the right helicoid and the catenoid. Meusnier further discovered the geometric meaning of Eq.(1.1) in the sense that it is equivalent to the vanishing of the geometric quantity H known today as mean curvature.

Although numerous scientists have studied minimal surface theory, Joseph Plateau considerably contributed to minimal surface studies with his experiments on soap films and soap bubbles. These experiments were very simple to conduct but they were completely consistent with the theory and helped to realize the minimal surfaces. Since then, the problem of finding a minimal surface bounded by a given Jordan curve has been called the *'Plateau problem'*. These experiments were also the starting point of the studies on thin structures in biology and physics. One can find the details and references of the following brief history in [13] and [20]. Plateau considered the free energy of films as

$$\mathcal{F} = \omega \oint\!\!\!\oint_S \mathrm{d}A. \tag{1.2}$$

Here S is a smooth surface, and ω, A, and H denote the surface tension, area, and mean curvature of the surface, respectively. The first variation of $\mathcal{F}$ i.e., $\delta\mathcal{F} = 0$, leads to surfaces with $H = 0$.

In 1805 and 1839, T. Young and P. S. Laplace, respectively, considered the free energy of closed soap bubbles as

$$\mathcal{F} = p \iiint \mathrm{d}V + \omega \oint\!\!\!\oint_S \mathrm{d}A, \tag{1.3}$$

where p is the pressure difference between outer and inner sides of a soap bubble and V is the volume enclosed by the bubble. The first variation of $\mathcal{F}$ leads to the surfaces with constant mean curvature $H = p/2\omega$. As A. D. Alexandrov proved that "an embedded surface with constant mean curvature in 3-dimensional Euclidian space ($\mathbb{E}^3$) must be a open subset of a sphere" in 1962, soap bubbles are spherical surfaces.

In 1833, Poisson considered the free energy of a solid shell as

$$\mathcal{F} = \oiint_S H^2 \,\mathrm{d}A. \tag{1.4}$$

In 1982, T. J. Willmore [8] found the Euler-Lagrange equation arising from Poisson's free energy $\mathcal{F}$ as

$$\nabla^2 H + 2H(H^2 - K) = 0, \tag{1.5}$$

where ∇^2 is the Laplace-Beltrami operator and K is the Gaussian curvature of the surface. Solutions of Eq. (1.5) are called *Willmore surfaces.*

In 1973, Helfrich proposed the curvature energy per unit area of the bilayer

$$\mathcal{E}_{lb} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K, \tag{1.6}$$

where k_c and $\bar{k}$ are elastic constants, and c_0 is spontaneous curvature of the lipid bilayer. Using the Helfrich curvature energy Eq. (1.6), the free energy functional of the lipid vesicle is written as

$$\mathcal{F} = \oiint_S (\mathcal{E}_{lb} + \omega)\,\mathrm{d}A + p\iiint \mathrm{d}V. \tag{1.7}$$

Taking the first variation of free energy $\mathcal{F}$, Ou-Yang and Helfrich [14] obtained the shape equation of the bilayer:

$$p - 2\omega H + k_c\nabla^2(2H) + k_c(2H + c_0)(2H^2 - c_0H - 2K) = 0. \tag{1.8}$$

Later Ou-Yang *et al.* considered the general energy functional

$$\mathcal{F} = \oiint_S \mathcal{E}(H, K)\,\mathrm{d}A + p\iiint_V \mathrm{d}V \tag{1.9}$$

which arises both in red blood cells and liquid crystals [2], [11]-[16]. Here $\mathcal{E}$ is function of H and K, p is a constant, and V is the volume enclosed within the surface S. For open surfaces, we let $p = 0$. The first variation (Euler-Lagrange) of $\mathcal{F}$ gives a highly nonlinear PDE of K and H on surface S. It is given by [2], [11]-[13]

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \tag{1.10}$$

where ∇^2 and $\nabla \cdot \bar{\nabla}$ will be defined in Chapter 2.

As we see in **(i)**-**(ix)** on pages 1 and 2, certain subclasses of surfaces arise as solutions of some differential equations. That is there are some relations between surfaces and PDEs. Since these equations are high order nonlinear PDEs, these equations are not so easy to solve. For this reason some indirect methods [21]-[34] have been developed for the construction of 2-surfaces in $\mathbb{R}^3$ and three dimensional Minkowskian geometry (M_3). Let us first examine some relations between surfaces and PDEs, then we will come back to the construction of surfaces.

Let $F = (F^1, F^2, F^3) : \mathcal{O} \to \mathbb{R}^3$ be an immersion of a domain $\mathcal{O} \subset \mathbb{R}^2_{(x,t)}$ into $\mathbb{R}^3$. Let the first and second fundamental forms of the surface $F(x,t)$ be in forms $(ds_I)^2 \equiv g_{ij}dx^i\,dx^j$, and $(ds_{II})^2 \equiv h_{ij}dx^i dx^j$, respectively, $i, j = 1, 2$ where $x^1 = x$, $x^2 = t$.

In this thesis, we use Einstein's summation convention on repeated indices over their range. Let $N(x,t)$ be the normal vector field defined at each point of the surface $F(x,t)$. Then let $\{F_x, F_t, N\}$ defines a basis for T_pS at a point $p \in S$, where S is the graph of $F(x,t)$. For sufficiently smooth surfaces, coefficients of the first and second fundamental forms satisfy a system of nonlinear PDEs which are known as Gauss-Minardi-Codazzi (GMC) equations. GMC equations constitute the compatibility conditions for the pair of linear equations known as Gauss-Weingarten (GW) equations for moving frames. So from the differential geometry we get a pair of linear equations and its compatibility condition i.e. a system of nonlinear PDEs. This idea was known to Gauss. In some cases these equations are reduce to a single equation. Some well known examples are the sine-Gordon equations for pseudo-spherical surfaces and sinh-Gordon equation for surfaces with constant mean curvature. In these examples, the equations are integrable. In the literature, surface theory, in the aforementioned sense, has been used to find new integrable systems. These kinds of surfaces (i.e. GMC is an integrable equation) are called integrable (soliton) surfaces [7].

Soliton equations play a crucial role for the construction of surfaces. The theory of nonlinear soliton equations was developed in 1960s. Lax representation of integrable equations should exist in order to apply inverse scattering method for finding solutions of these integrable equations. For details of integrable equations one may look [35], [36], and the references therein. Lax representation of nonlinear PDEs consists of two linear equations which are called Lax equations

$$\Phi_x = U\,\Phi, \quad \Phi_t = V\,\Phi, \tag{1.11}$$

and their compatibility condition

$$U_t - V_x + [U, V] = 0, \tag{1.12}$$

where x and t are independent variables. Here U and V are called Lax pairs. They depend on independent variables x and t, and a spectral parameter λ. For our cases, U and V are 2×2 matrices and they are in a given Lie algebra $\mathfrak{g}$. Eq. (1.12) is also called the zero curvature condition. Integrable equations arise as the compatibility conditions, Eq. (1.12), of the Lax equations [Eq. (1.11)]. Since GMC equations are compatibility conditions of GW equations, there is a close relationship between surfaces and Lax equations. GW equations and Lax equations play similar roles but they are not exactly the same. While Lax equations depend on spectral parameters, GW equations do not. Moreover GW equations are written in terms of 3×3 matrices whereas Lax pairs are 2×2 matrices. The former problem can be solved easily by inserting spectral parameters in GW equations using the one dimensional symmetry group of GW equations. The latter problem was solved by Sym [23]. By making use of the isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, he rewrote the GW equations in terms of 2×2 matrices. So for integrable surfaces, GW equations can be written in terms of 2×2 matrices using the conformal parametrization.

2-surfaces and integrable equations can be related by the analogy between GW equations and Lax equations. Such a relation is established by the use of Lie groups and Lie algebras. Using this relation, soliton surface theory was first developed by Sym [21]-[23]. He studied the surface theory in both directions: from geometry to solitons and from solitons to geometry. In the first direction, he obtained some well known soliton equations as a consequence of GMC equations.

In the second direction, he obtained the following formula using the deformation of Lax equations for integrable equations

$$F = \Phi^{-1}\frac{\partial \Phi}{\partial \lambda}, \tag{1.13}$$

which gives a relation between a family of immersions (F) into the Lie algebra and the Lax equations for given Lax pairs. Fokas and Gel'fand [24] generalized Sym's formula as

$$\begin{aligned} F = {} & \alpha_1 \Phi^{-1} U\, \Phi + \alpha_2 \Phi^{-1} V\, \Phi + \alpha_3 \Phi^{-1}\frac{\partial \Phi}{\partial \lambda} + \alpha_4\, x\, \Phi^{-1} U\, \Phi \\ & + \alpha_5\, t\, \Phi^{-1} V\, \Phi + \Phi^{-1} M\, \Phi, \end{aligned} \tag{1.14}$$

where α_i, $i = 1,2,3,4,5$ and $M \in \mathfrak{g}$ are constants. So by this technique, which is called *the soliton surface technique*, using the symmetries of the integrable equations and their Lax equations we can find a large class of soliton surfaces for given Lax pairs. One may find 2-surfaces developed by soliton surface technique, which belong to subclasses of the surfaces, mentioned in **(i)**-**(ix)** on pages 1 and 2, in the references [6], [7], [21]-[34].

On the other hand, there are some surfaces that arise from a variational principle for a given Lagrange function free energy, which is a polynomial of degree less than or equal to two in the mean curvature of the surfaces. Examples of this type are minimal surfaces, constant mean curvature surfaces, linear Weingarten surfaces, Willmore surfaces, and surfaces solving the shape equation for the Lagrange functions. Taking more general Lagrange function of the mean and Gaussian curvatures of the surface, we may find more general surfaces that solve the generalized shape equation [Eq. (2.29)]. Examples for this type of surfaces can be found in [28] and [29].

The principal purpose of this thesis is to find new classes of 2-surfaces using the deformations of Lax equations for mKdV, KdV, NLS, SG and HD equations and to obtain solutions of the generalized shape equation [Eq. (2.29)] for polynomial Lagrange functions of the curvatures H and K

$$\mathcal{E} = a_{N0}\, H^N + \ldots + a_{11}\, H\, K + a_{21}\, H^2\, K + \ldots + a_{01}\, K + \ldots \tag{1.15}$$

For each N, we find the constants a_{nl} in terms of the parameters of the surface, where $n, l = 0, 1, 2, \ldots$ and $N = 3, 4, 5, \ldots$

This thesis is organized as follows: Chapter 2 gives the general theory for construction of 2-surfaces by using the integrable equations. Here, Lie groups SU(2), SL(2,$\mathbb{R}$) and their Lie algebras $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, respectively, are used. General formulation of generalized shape equation for closed and for open surfaces are given.

In Chapter 3, we construct, as an application of Section 2.1, 2-surfaces in $\mathbb{R}^3$ which correspond to mKdV, NLS, and SG equations. Here, Lie group SU(2) and the corresponding Lie algebra $\mathfrak{su}(2)$ are used. Spectral deformation, spectral-Gauge deformation, and deformations of parameters are employed for the construction of mKdV, NLS and SG surfaces. This chapter shows that there are some algebraic Weingarten and Willmore-like mKdV and NLS surfaces. We find some surfaces that solve Eq. (2.29). Free energies (Lagrangians) are polynomial functions of the Gaussian and mean curvatures of mKdV and NLS surfaces. The general form of the functional is also studied. For the mKdV surfaces associated with spectral deformation and deformation of parameters, starting with one soliton solution and solving the Lax equations the chapter ends with finding the position vectors of mKdV surfaces. We plot some of these surfaces for some special values of constants.

In Chapter 4, we construct 2-surfaces in M_3 corresponding to KdV and HD equations. In this case Lie group is $SL(2,\mathbb{R})$ and the corresponding Lie algebra is $\mathfrak{sl}(2,\mathbb{R})$. We use spectral deformation and spectral-Gauge deformation for the construction of classes of KdV and HD surfaces. We show that there are some algebraic Weingarten and Willmore-like KdV and HD surfaces. We find some KdV and HD surfaces that solve Eq. (2.29), where general form of the functional is similar to Chapter 3. Using the one soliton solution of the KdV equation and a solution of the HD equation, we find the position vectors of the KdV and HD surfaces for spectral deformation case.

In Conclusion chapter, we give a summary of the study to construct 2-surfaces in $\mathbb{R}^3$ and in M_3. We point out some new algebraic Weingarten surfaces, Willmore-like surfaces, and surfaces solving the generalized shape equation.

Appendix includes some maple codes for fundamental forms and curvatures of the surfaces arising from spectral deformation. We consider Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$. We further give the codes for Willmore-like surfaces, surfaces that solve the generalized shape equation, and a method to find the position vector of the surfaces.

Chapter 2

General Theory

Ou-Yang *et al.* considering the general energy functional Eq. (1.9), obtained the generalized shape equation Eq. (1.10), [2], [11]-[13]. It is the highly nonlinear partial differential equations (PDEs) of the curvatures K and H of the surfaces S. Solving this equation is very difficult. For this reason we do not try to solve it directly. We instead construct some surfaces by using the soliton technique, then we search for the surfaces, which solve the generalized shape equation for suitable functionals. Thus this Chapter first introduces soliton surface technique, then analyzes the general formulation of surfaces arising from variational principle.

2.1 Theory of Integrable Surfaces

As mentioned in Introduction, we use the soliton surface technique in order to construct 2-surfaces. In this technique, we use the deformations of Lax equations of integrable equations. In the literature, there are certain surfaces corresponding to certain integrable equations like SG, sinh-Gordon, KdV, mKdV, and NLS equations [6], [7], [10], [21]-[29]. Symmetries of the integrable equations for given Lax pairs play the crucial role in this method. This method was first developed by Sym [21]-[23]. Then it was generalized by Fokas and Gel'fand [24], Fokas *et al.* [25] and Cieśliński [34]. Now by considering surfaces in a Lie group and in the corresponding Lie algebra, we give the general theory.

Let G be a Lie group and $\mathfrak{g}$ be the corresponding Lie algebra. We give the theory for dim $\mathfrak{g} = 3$, it is possible to generalize it for finite dimension n. Assume that there exists an inner product $\langle , \rangle$ on $\mathfrak{g}$ such that for g_1, $g_2 \in \mathfrak{g}$ as $\langle g_1, g_2 \rangle$. Let $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ be the orthonormal basis in $\mathfrak{g}$ such that $\langle \mathrm{e}_i, \mathrm{e}_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Let Φ be a G valued differentiable function of x, t, and λ for every $(x, t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. So a map can be defined from tangent space of G to the Lie algebra $\mathfrak{g}$ as

$$\Phi_x \, \Phi^{-1} = U, \quad \Phi_t \, \Phi^{-1} = V, \tag{2.1}$$

where Φ_x and Φ_t are the tangent vectors of Φ; U and V are functions of x, t and λ, and take values in $\mathfrak{g}$.

The function Φ, which is defined by Eq. (2.1) exists if and only if U and V satisfy the following equation

$$U_t - V_x + [U, V] = 0, \tag{2.2}$$

where $[,]$ is the Lie algebra commutator such that $[\mathrm{e}_i, \mathrm{e}_j] = c_{ij}^k \, \mathrm{e}_k$, $k = 1, 2, 3$, and c_{ij}^k are structural constants of $\mathfrak{g}$.

Indeed, Φ exists if and only if the equations Eq. (2.1) are compatible. To prove that, we take t and x derivatives of the first and second equations in Eq. (2.1), respectively and that makes

$$\Phi_{xt} \, \Phi^{-1} = U_t + \Phi_x \, \Phi^{-1} \, \Phi_t \, \Phi^{-1}, \tag{2.3}$$

$$\Phi_{tx} \, \Phi^{-1} = V_x + \Phi_t \, \Phi^{-1} \, \Phi_x \, \Phi^{-1}. \tag{2.4}$$

Since left hand sides of Eqs. (2.3) and (2.4) are equal, equating right hand sides of these equations and using Eq. (2.1) we obtain Eq. (2.2).

Φ is a surface in G defined by Eq. (2.1) with the compatibility condition Eq. (2.2). Now let us introduce a surface in Lie algebra $\mathfrak{g}$. Let F be a $\mathfrak{g}$ valued differentiable function of x, t, and λ for every $(x, t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. The first and second fundamental forms of F are defined as

$$(ds_I)^2 \equiv g_{ij} dx^i\, dx^j = \langle F_x, F_x \rangle dx^2 + 2\langle F_x, F_t \rangle dx\, dt + \langle F_t, F_t \rangle dt^2, \quad (2.5)$$

$$(ds_{II})^2 \equiv h_{ij} dx^i\, dx^j = \langle F_{xx}, N \rangle dx^2 + 2\langle F_{xt}, N \rangle dx\, dt + \langle F_{tt}, N \rangle dt^2, \quad (2.6)$$

where g_{ij} and h_{ij} are the components of the first and second fundamental forms in a respective way. Here $i, j = 1, 2$, $x^1 = x$ and $x^2 = t$, and $N \in \mathcal{G}$ is defined as

$$\langle N, N \rangle = 1, \quad \langle F_x, N \rangle = \langle F_t, N \rangle = 0. \quad (2.7)$$

Here $\{F_x, F_t, N\}$ forms a frame at each point of the surface.

We are studying on a finite dimensional Lie algebra $\mathfrak{g}$. Therefore it has a matrix representation by Ado's theorem. We use matrices, so the adjoint map is of the form $\Phi^{-1}\, A\, \Phi$, for $\Phi \in \mathrm{G}$ and $A \in \mathfrak{g}$.

By using the adjoint representation, we can relate the surfaces in G to the surfaces in $\mathfrak{g}$ as

$$F_x = \Phi^{-1}\, A\, \Phi, \quad F_t = \Phi^{-1}\, B\, \Phi, \quad (2.8)$$

where A and B are $\mathfrak{g}$ valued differentiable functions of x, t, and λ for every $(x, t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

The equations in Eq. (2.8) define a surface F if and only if A and B satisfy the following equation

$$A_t - B_x + [A, V] + [U, B] = 0. \quad (2.9)$$

Indeed, the equations in Eq. (2.8) have no meaning unless they are compatible.

N can also appear as the following form by using the adjoint representation

$$N = \Phi^{-1}\, C\, \Phi, \quad (2.10)$$

where $C \in \mathcal{G}$.

Inner product is invariant under adjoint representation. If Eqs. (2.8) and (2.10) are used, we can write the components of the first and second fundamental forms of the surface F as

$$g_{11}=\langle A,A\rangle,\ g_{12}=g_{21}=\langle A,B\rangle,\ g_{22}=\langle B,B\rangle, \tag{2.11}$$
$$h_{11}=\langle A_x+[A,U],C\rangle,\ h_{12}=h_{21}=\langle A_t+[A,V],C\rangle,\ h_{22}=\langle B_t+[B,V],C\rangle,$$

where

$$C=\frac{[A,B]}{\|[A,B]\|},\ \ \|A\|=\sqrt{|\langle A,A\rangle|}. \tag{2.12}$$

The following theorem summarizes the results above.

Theorem 2.1 *Let U, V, A, and B be $\mathfrak{g}$ valued differentiable functions of x, t, and λ for every $(x,t)\in\mathcal{O}\subset\mathbb{R}^2$ and $\lambda\in\mathbb{R}$. Assume that U, V, A, and B satisfy the following equations*

$$U_t-V_x+[U,V]=0, \tag{2.13}$$

and

$$A_t-B_x+[A,V]+[U,B]=0. \tag{2.14}$$

Then the following equations

$$\Phi_x=U\,\Phi,\quad \Phi_t=V\,\Phi, \tag{2.15}$$

and

$$F_x=\Phi^{-1}\,A\,\Phi,\quad F_t=\Phi^{-1}\,B\,\Phi, \tag{2.16}$$

define surfaces $\Phi\in G$ and $F\in\mathfrak{g}$, respectively. The first and second fundamental forms of the surface F are of the following form, respectively

$$(ds_I)^2\equiv g_{ij}dx^i\,dx^j,\quad (ds_{II})^2\equiv h_{ij}dx^i\,dx^j \tag{2.17}$$

where $i,j=1,2$, $x^1=x$, $x^2=t$, g_{ij} and h_{ij} are of the form that appear in Eqs. (2.11) and (2.12). The Gaussian and mean curvatures of the surface are, respectively, shown by

$$K=det(g^{-1})h,\quad H=\frac{1}{2}trace(g^{-1}h), \tag{2.18}$$

where g and h denote the matrices (g_{ij}) and (h_{ij}), respectively, and g^{-1} stands for the inverse of the matrix g.

As it is seen in Theorem 2.1, we need to know the fundamental forms and curvatures to characterize a surface . In order to calculate them, it is sufficient to know U, V, A, and B. Since the main aim is to find a class of surfaces, which corresponds to integrable equations, we need here to find A and B from Eq. (2.14). But in general, solving this equation is difficult. However, there are some deformations that provide A and B directly. We use '*deformation*' in the following sense. By replacing U and V by $U + \varepsilon\, A$ and $V + \varepsilon B$ in Eq. (2.13), respectively, we get

$$U_t - V_x + [U,V] + \varepsilon\,(A_t - B_x + [A,V] + [U,B]) + O(\varepsilon^2) = 0, \tag{2.19}$$

which produces Eq. (2.14). This means that for every symmetry of Eq. (2.13) we have a solution for Eq. (2.14). We give four types of deformations below. The first three were given by Sym [21]-[23], Fokas and Gel'fand [24], Fokas *et al.* [25] and Cieśliński [34]. The last one is introduced in [30].

- Spectral parameter λ invariance of the equation:

$$A = \mu_1\,\frac{\partial U}{\partial \lambda},\ \ B = \mu_1\,\frac{\partial V}{\partial \lambda},\ \ F = \mu_1\,\Phi^{-1}\frac{\partial \Phi}{\partial \lambda}, \tag{2.20}$$

  where μ_1 is an arbitrary function of λ. Since the integrable equation, obtained form Eq. (2.13), is independent of the spectral parameter λ, Eq. (2.13) is invariant under λ translation. That gives the Eq. (2.20). That kind of deformation was first used by Sym [21]-[23].

- Symmetries of the (integrable) differential equations:

$$A = \delta U,\ \ B = \delta V,\ \ F = \Phi^{-1}\delta\Phi, \tag{2.21}$$

  where δ represents the classical Lie symmetries and (if integrable) the generalized symmetries of the nonlinear PDE's [24], [25], [34].

- The Gauge symmetries of the Lax equation:

$$A = M_x + [M,U],\ \ B = M_t + [M,V],\ \ F = \Phi^{-1}M\Phi, \tag{2.22}$$

  where M is any traceless 2×2 matrix. Since Eq. (2.13) is invariant under the gauge transformation

$$U \mapsto R\,U\,R^{-1} + R_x\,R^{-1},\quad V \mapsto R\,V\,R^{-1} + R_t\,R^{-1}. \tag{2.23}$$

and letting $R = I + \varepsilon M$ in Eq. (2.23), $O(\varepsilon)$ terms give Eq. (2.22). [24], [25], [34].

- The deformation of parameters for solution of integrable equation:

$$A = \mu_2 \left(\partial U / \partial \xi_i\right), \; B = \mu_2 \left(\partial V / \partial \xi_i\right), \; F = \mu_2 \Phi^{-1} \left(\partial \Phi / \partial \xi_i\right), \tag{2.24}$$

where $i = 0, 1$ and ξ_i are parameters of the solution $u(x, t, \xi_0, \xi_1)$ of the PDEs, μ_2 is constant. Here x and t are independent variables. [30]

By using the λ deformation, Sym constructs a family of surfaces F, which is given by Eq. (1.13). By using the linear combination of the aforementioned first three deformations Fokas *et al.* [25] state the following theorem, which is the generalization of Sym's result.

Theorem 2.2 *Let U, V, and Φ be differentiable functions of x, t, and λ, where λ is the spectral parameter. Assume that the Lax pairs U and V satisfy Eq. (2.13), which gives an integrable equation, and Φ satisfy the Lax equations [Eq. (2.15)]. Let A and B be defined by* ($i = 0, 1$)

$$A = \mu_1 \frac{\partial U}{\partial \lambda} + \delta U + M_x + [M, U] + \mu_2 \frac{\partial U}{\partial \xi_i}, \tag{2.25}$$

$$B = \mu_1 \frac{\partial V}{\partial \lambda} + \delta V + M_t + [M, V] + \mu_2 \frac{\partial V}{\partial \xi_i}, \tag{2.26}$$

where μ_1 is an arbitrary function of λ and μ_2 is an arbitrary function of ξ_i. Here ξ_i are the parameters of the solution, u, of the PDEs. δ is generalized symmetry of the integrable equation which is obtained form Eq. (2.13) and M is a $\mathfrak{g}$ valued function of x and t. Then the equations in Eq. (2.16) define a family of surfaces with the immersion function F is given by

$$F = \Phi^{-1} \left(\mu_1 \frac{\partial \Phi}{\partial \lambda} + \mu_2 \frac{\partial \Phi}{\partial \xi_i} + \delta \Phi + M\, \Phi \right). \tag{2.27}$$

2.2 Surfaces from a Variational Principle

Let S be a 2-surface (either in M_3 or in $\mathbb{R}^3$) with the mean and Gaussian curvatures H and K, respectively.

Definition 2.3 *A free energy $\mathcal{F}$ of S is defined by*

$$\mathcal{F} = \oiint_S \mathcal{E}(H,K)\,\mathrm{d}A + p\iiint_V \mathrm{d}V \tag{2.28}$$

where $\mathcal{E}$ is some function of H and K, p is a constant and V is the volume enclosed within the surface S. For open surfaces, we let $p = 0$.

The following proposition gives the first variation of the functional $\mathcal{F}$.

Proposition 2.4 *Let $\mathcal{E}$ be a twice differentiable function of H and K. Then the Euler-Lagrange equation for $\mathcal{F}$ reduces to [2], [11]-[13]*

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial\mathcal{E}}{\partial H} + 2(\nabla\cdot\bar{\nabla} + 2KH)\frac{\partial\mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0. \tag{2.29}$$

where ∇^2 and $\nabla\cdot\bar{\nabla}$ are defined as

$$\nabla^2 = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\sqrt{\tilde{g}}g^{ij}\frac{\partial}{\partial x^j}\right),\ \nabla\cdot\bar{\nabla} = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\sqrt{\tilde{g}}Kh^{ij}\frac{\partial}{\partial x^j}\right), \tag{2.30}$$

and $\tilde{g} = \det(g_{ij})$, g^{ij} and h^{ij} are inverse components of the first and second fundamental forms, respectively, and $i,j = 1,2$, where $x^1 = x$, $x^2 = t$. Eq. (2.29) is called generalized shape equation.

Some of the subclasses of the surfaces given in **(i)**-**(ix)** on pages 1 and 2, can be derived from a variational principle for a suitable $\mathcal{E}$. These are given as:

(a) Minimal surfaces: $\mathcal{E} = 1$, $p = 0$;

(b) Surfaces with constant mean curvature: $\mathcal{E} = 1$;

(c) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants;

(d) Willmore surfaces: $\mathcal{E} = H^2$ [8], [9];

(e) Surfaces that solve the shape equation of lipid membrane: $\mathcal{E} = (H-c)^2$, where c is a constant [2], [11]-[16];

(f) Shape equation of closed lipid bilayer: $\mathcal{E} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K$, where k_c and $\bar{k}$ are elastic constants, and c_0 is the spontaneous curvature of the lipid bilayer [14];

Definition 2.5 *Surfaces that solve the equation*

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{2.31}$$

are called Willmore-like surfaces, where a and b are arbitrary constants.

Remark 2.6 *$a = 2$, $b = -2$ case corresponds to the Willmore surfaces which arise from a variational problem. For other values of a and b Willmore-like surfaces cannot be derived from a variational problem.*

For compact 2-surfaces, the constant p in Eq. (2.28) may be different than zero, but for noncompact surfaces we assume it to be zero. For the latter, asymptotic conditions are required, so K goes to a constant and H goes to zero asymptotically. These conditions are obtained if the integrable equations like mKdV, NLS, and KdV equations have solutions decaying rapidly to zero as $|x| \to \infty$. Soliton solutions of these integrable equations satisfy this requirement. For this purpose, we shall use the Euler-Lagrange equation [Eq. (2.29)] for surfaces obtained by mKdV, NLS, KdV, HD equations and search for solutions (surfaces) of the Euler-Lagrange equation [Eq. (2.29)].

For open or noncompact 2-surfaces, the following definition and proposition can be given.

Definition 2.7 *Let S be an open 2-surface with its curvatures H and K. A free energy $\mathcal{F}$ of S is defined by*

$$\mathcal{F} = \oiint_S \mathcal{E}(H,K)\,\mathrm{d}A + \oint_C \Gamma(k_n, k_g)\,\mathrm{d}s, \tag{2.32}$$

where $\mathcal{E}$ is some function of H and K, C is an edge of S as shown in Figure 2.1. k_n and k_g are normal and geodesic curvatures of the surface S.

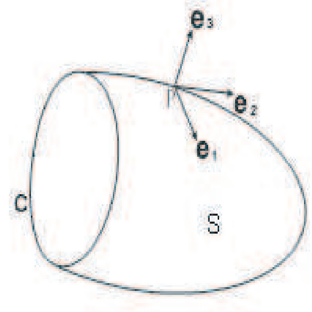

Figure 2.1: A smooth and orientable surface S with an edge C

The first variation of the functional $\mathcal{F}$ given in Eq. (2.32) reads the following proposition.

Proposition 2.8 *Let $\mathcal{E}$ be a twice differentiable function of H and K. Then the Euler-Lagrange equation for $\mathcal{F}$ given in Eq. (2.32) reduces to [2], [11]-[13]*

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} = 0, \tag{2.33}$$

with the following equations

$$\begin{aligned}
&\mathbf{e}_2 \cdot \nabla\left(\frac{\partial \mathcal{E}}{\partial H}\right) + 2\,\mathbf{e}_2 \cdot \bar{\nabla}\left(\frac{\partial \mathcal{E}}{\partial K}\right) - 2\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\tau_g\frac{\partial \mathcal{E}}{\partial K}\right) + 2\frac{\mathrm{d}^2}{\mathrm{d}\,s^2}\left(\frac{\partial \Gamma}{\partial k_n}\right) + 2\frac{\partial \Gamma}{\partial k_n}(k_n^2 - \tau_g^2)\\
&\quad + 2\tau_g\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{\partial \Gamma}{\partial k_g}\right) + 2\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\tau_g\frac{\partial \Gamma}{\partial k_g}\right) - 2\left(\Gamma - \frac{\partial \Gamma}{\partial k_g}k_g\right)k_n\Bigg|_C = 0, \qquad (2.34)\\
&-\frac{\partial \mathcal{E}}{\partial H} - 2k_n\frac{\partial \mathcal{E}}{\partial K} - 2\frac{\partial \Gamma}{\partial k_n}k_g\Bigg|_C = 0, \qquad (2.35)\\
&\frac{\mathrm{d}^2}{\mathrm{d}\,s^2}\left(\frac{\partial \Gamma}{\partial k_g}\right) + K\frac{\partial \Gamma}{\partial k_g} - k_g\left(\Gamma - \frac{\partial \Gamma}{\partial k_g}k_g\right) + 2(k_n - H)k_g\frac{\partial \Gamma}{\partial k_n}\\
&\quad - \tau_g\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{\partial \Gamma}{\partial k_n}\right) - \frac{\mathrm{d}}{\mathrm{d}\,s}\left(\tau_g\frac{\partial \Gamma}{\partial k_n}\right) - \mathcal{E}\Bigg|_C = 0, \qquad (2.36)
\end{aligned}$$

where τ_g is the geodesic torsion of the boundary curve, $\mathbf{e}_2$ is a unit vector which is perpendicular to the tangent vector of edge C and normal vector of surface S.

In this proposition, Eq. (2.33) gives the shape of the surface S as before, and boundary conditions, Eqs. (2.34)-(2.36), give the position of the curve C in S.

Taking $\mathcal{E} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K + \omega$, Eq. (1.8) gives the equation of closed bilayer. For the shape equation of open lipid bilayer we have additional equations. Consider $\mathcal{E}$ as in closed lipid bilayer with $\Gamma = k_b\left(k_n^2 + k_g^2\right)/2 + \gamma$, where k_b and γ are constants. In additional to Eq. (1.8) we have the following equations which are the lipid bilayer versions of the Eqs. (2.34)-(2.36)

$$k_b\left[\mathrm{d}^2 k_n/\,\mathrm{d}\,s^2 + k_n\,\left(\kappa^2/2 - \tau_g^2\right) + \tau_g\,\mathrm{d}\,k_g/\,\mathrm{d}\,s + \mathrm{d}(\tau_g k_g)/\,\mathrm{d}\,s\right] \tag{2.37}$$

$$+\, k_c\mathbf{e}_2 \cdot \nabla(2H) - \bar{k}\,\mathrm{d}\,\tau_g/\,\mathrm{d}\,s - \gamma k_n\Big|_C = 0, \tag{2.38}$$

$$k_c\,(2\,H + c_0) + \bar{k}\,k_n\Big|_C = 0, \tag{2.39}$$

$$k_b\left[\mathrm{d}\,k_g/\,\mathrm{d}\,s^2 + k_g(\kappa^2/2 - \tau_g^2) - \tau_g\,\mathrm{d}\,k_n/\,\mathrm{d}\,s - \mathrm{d}(\tau_g k_n)/\,\mathrm{d}\,s\right] \tag{2.40}$$

$$-\left[(k_c/2)(2\,H + c_0)^2 + \bar{k}K + \mu + \gamma k_g\right]\Big|_C = 0, \tag{2.41}$$

where k_g, τ_g and k_n are respectively the geodesic curvature, the geodesic torsion, and the normal curvature of the curve; k_c and $\bar{k}$ are elastic constants; and c_0 is the spontaneous curvature of the surface, and $\kappa^2 = k_n + k_g$.

Chapter 3

Immersions in $\mathbb{R}^3$

In Section 2.1, we introduce the theory that gives a connection between integrable equations and surfaces. For that connection, a Lie group G and the corresponding Lie algebra $\mathfrak{g}$ are employed. In this chapter, we investigate the immersions of some 2-surfaces in $\mathbb{R}^3$. For this purpose, we use Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$ with basis $\mathrm{e}_j = -i\,\sigma_j,\ j = 1, 2, 3$, where σ_j denote the usual Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

Define an inner product on $\mathfrak{su}(2)$ as

$$\langle X, Y \rangle = -\frac{1}{2}\,\mathrm{trace}(XY), \tag{3.2}$$

where X, $Y \in \mathfrak{su}(2)$. $[.,.]$ denotes the usual commutator.

By using the method given in Section 2.1, we construct the families of 2-surfaces which correspond to mKdV, SG, and NLS equations. The surfaces corresponding to these equations are called mKdV, SG, and NLS surfaces, respectively. Spectral deformations, Gauge deformations, and combinations of these deformations are used for the construction of the surfaces. These deformations have been used so far in previous studies. In addition to these studies, we use the deformation of parameters [30]. By the 'parameters', we mean the parameters of the solution of integrable equations. For the construction of these surfaces, we begin with the $\mathfrak{su}(2)$ valued Lax pairs U and V. We find different A's and B's, that satisfy Eq. (2.14), and first fundamental forms, second fundamental forms, Gaussian curvatures, and mean curvatures of the surfaces by using different deformations. $\{F_x, F_t, N\}$ defines a frame on this surface, where N, F_x and F_t are given by Eqs. (2.10) and (2.16), respectively. For a given solution of the integrable nonlinear differential equation, we find the solution of the Lax equations. Inserting these solutions, A and B in Eq. (2.16), we construct $\mathfrak{su}(2)$ valued position vector F of the surface as

$$y_j = F^j,\ j = 1, 2, 3, \quad F = \sum_{k=1}^{3} F^k\ \mathrm{e}_k\,. \tag{3.3}$$

In the next section, we find the position vector of the mKdV surfaces by using one soliton solution of mKdV equation. We plot some of these surfaces for some special values of parameters. We generate new algebraic Weingarten surfaces, Willmore-like surfaces, and the surfaces which solve the generalized shape equation [Eq. (2.29)].

3.1 mKdV Surfaces from Spectral Deformations

In this section, we develop surfaces which arise from the spectral deformation of Lax pair for the mKdV equation [29].

Let $u(x, t)$ satisfy the mKdV equation

$$u_t = u_{3x} + \frac{3}{2}u^2 u_x. \tag{3.4}$$

Substituting the travelling wave ansatz $u_t - \alpha\, u_x = 0$ in Eq. (3.4), we get

$$u_{2x} = \alpha u - \frac{u^3}{2}, \tag{3.5}$$

where α is an arbitrary real constant and integration constant is taken to be zero.

Eq. (3.5) can be obtained from Lax pairs U and V, where

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix}, \tag{3.6}$$

$$V = -\frac{i}{2}\begin{pmatrix} \frac{1}{2}u^2 - (\alpha + \alpha\lambda + \lambda^2) & (\alpha+\lambda)u - iu_x \\ (\alpha+\lambda)u + iu_x & -\frac{1}{2}u^2 + (\alpha + \alpha\lambda + \lambda^2) \end{pmatrix}, \tag{3.7}$$

and λ is a spectral parameter.

The following proposition gives a family of 2-surfaces corresponding to mKdV equation by using spectral parameter deformations.

Proposition 3.1 *Let u (which describes a travelling mKdV wave) satisfy Eq. (3.5). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by Eqs. (3.6) and (3.7), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = \frac{i}{2}\begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \tag{3.8}$$

$$B = -\frac{i}{2}\begin{pmatrix} -(\alpha\,\mu + 2\,\mu\,\lambda) & \mu\,u \\ \mu\,u & \alpha\,\mu + 2\,\mu\,\lambda \end{pmatrix}, \tag{3.9}$$

where $A = \mu\,\partial U/\partial\lambda$, $B = \mu\,\partial V/\partial\lambda$, μ is a constant and λ is the spectral parameter. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms ($j, k = 1, 2$)

$$(ds_I)^2 \equiv g_{jk}\,dx^j\,dx^k = \frac{\mu^2}{4}\big([dx + (\alpha + 2\,\lambda)dt]^2 + u^2\,dt^2\big), \tag{3.10}$$

$$(ds_{II})^2 \equiv h_{jk}\,dx^j\,dx^k = \frac{\mu\,u}{2}\big(dx + (\alpha+\lambda)dt\big)^2 + \frac{\mu\,u}{4}(u^2 - 2\,\alpha)dt^2, \tag{3.11}$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{2}{\mu^2}\big(u^2 - 2\,\alpha\big), \quad H = \frac{1}{2\mu\,u}\big(3\,u^2 + 2\,(\lambda^2 - \alpha)\big), \tag{3.12}$$

where $x^1 = x$, $x^2 = t$.

The following proposition presents a class of mKdV surfaces, which are called Willmore-like.

Proposition 3.2 *Let u satisfy $u_x^2 = \alpha\, u^2 - u^4/4$. Then surface S, defined in Proposition 3.1, is a Willmore-like surface, i.e., the Gaussian and mean curvatures satisfy Eq. (2.31), where*

$$a = \frac{4}{9}, \quad b = 1, \quad \alpha = \lambda^2, \tag{3.13}$$

and λ is an arbitrary constant.

There are also mKdV surfaces arising from variational principles. The following proposition proposes a class of mKdV surfaces that solve the Euler-Lagrange equation [Eq. (2.29)].

Proposition 3.3 *Let u satisfy $u_x^2 = \alpha\, u^2 - u^4/4$. Then there are mKdV surfaces, defined in Proposition 3.1, that satisfy the generalized shape equation [Eq. (2.29)] when $\mathcal{E}$ is a polynomial function of curvatures H and K.*

Here are several examples:

Example 3.4

Let $\deg(\mathcal{E}) = N$, then

i) for $N = 3$:
$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H$,
$$\alpha = \lambda^2, \;\; a_1 = -\frac{p\, \mu^4}{72\, \lambda^4}, \;\; a_2 = a_3 = a_4 = 0, \;\; a_6 = \frac{p\, \mu^4}{32\, \lambda^4},$$
where $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants;

ii) for $N = 4$:

$$\mathcal{E} = a_1\,H^4 + a_2\,H^3 + a_3\,H^2 + a_4\,H + a_5 + a_6\,K + a_7\,K\,H + a_8\,K^2 + a_9\,K\,H^2,$$

$$\alpha = \lambda^2,\;\; a_2 = -\frac{p\,\mu^4}{72\,\lambda^4},\;\; a_3 = -\frac{8\,\lambda^2}{15\,\mu^2}\,(27\,a_1 - 8\,a_8),\;\; a_4 = 0,$$

$$a_5 = \frac{\lambda^4}{5\,\mu^4}\,(81\,a_1 + 16\,a_8),\;\; a_7 = \frac{p\,\mu^4}{32\,\lambda^4},\;\; a_9 = -\frac{1}{120}\,(189\,a_1 + 64\,a_8),$$

where $\lambda \neq 0$, $\mu \neq 0$, and p, a_1, a_6, and a_8 are arbitrary constants;

iii) for $N = 5$:

$$\mathcal{E} = a_1\,H^5 + a_2\,H^4 + a_3\,H^3 + a_4\,H^2 + a_5\,H + a_6 + a_7\,K + a_8\,K\,H + a_9\,K^2 + a_{10}\,K\,H^2 + a_{11}\,K^2\,H + a_{12}\,K\,H^3,$$

$$\alpha = \lambda^2,\;\; a_3 = -\frac{1}{504\,\mu^2\,\lambda^4}\,\big(\lambda^6[4212\,a_1 + 256\,a_{11}] + 7\,p\,\mu^6\big),$$

$$a_4 = -\frac{8\,\lambda^2}{15\,\mu^2}\,(27\,a_2 - 8\,a_9)\,,\;\; a_5 = \frac{6\,\lambda^4}{7\,\mu^4}\,(135\,a_1 - 88\,a_{11}),$$

$$a_6 = \frac{\lambda^4}{5\,\mu^4}\,(81\,a_2 + 16\,a_9)\,,\;\; a_8 = \frac{1}{32\,\mu^2\,\lambda^4}\,\big(\lambda^6[-324\,a_1 + 512\,a_{11}] + p\,\mu^6\big),$$

$$a_{10} = -\frac{1}{120}\,(189\,a_2 + 64\,a_9)\,,\;\; a_{12} = -\frac{1}{756}\,(1053\,a_1 + 512\,a_{11}),$$

where $\lambda \neq 0$, $\mu \neq 0$, and p, a_1, a_2, a_7, a_9, and a_{11} are arbitrary constants;

iv) for $N = 6$:

$$\mathcal{E} = a_1\,H^6 + a_2\,H^5 + a_3\,H^4 + a_4\,H^3 + a_5\,H^2 + a_6\,H + a_7 + a_8\,K + a_9\,K\,H + a_{10}\,K^2 + a_{11}\,K\,H^2 + a_{12}\,K^2\,H + a_{13}\,K\,H^3 + a_{14}\,K^3 + a_{15}\,K^2\,H^2 + a_{16}\,K\,H^4,$$

$$\alpha = \lambda^2,$$

$$a_4 = -\frac{1}{504\,\mu^2\,\lambda^4}\,\big(\lambda^6[4212\,a_2 + 256\,a_{12}] + 7\,p\,\mu^6\big),$$

$$a_5 = -\frac{\lambda^4}{900\,\mu^4}\,(-359397\,a_1 + 191488\,a_{14} - 203472\,a_{16}) - \frac{8\lambda^2}{15\mu^2}\,(27a_3 - 8a_{10}),$$

$$a_6 = \frac{6\,\lambda^4}{7\,\mu^4}\,(135\,a_2 - 88\,a_{12})\,,$$

$$
\begin{aligned}
a_7 &= \frac{\lambda^6}{25\,\mu^6}\,(29889\,a_1 - 9856\,a_{14} + 11664\,a_{16}) \\
&\quad + \frac{\lambda^4}{5\mu^4}\,(81a_3 + 16a_{10}), \\
a_9 &= \frac{1}{32\,\mu^2\,\lambda^4}\,\big(\lambda^6[-324\,a_2 + 512\,a_{12}] + p\,\mu^6\big), \\
a_{11} &= -\frac{\lambda^2}{1800\,\mu^2}\,(59778\,a_1 - 13312\,a_{14} + 23328\,a_{16}) \\
&\quad - \frac{1}{120}\,(189a_3 + 64a_{10}), \\
a_{13} &= -\frac{1}{756}\,(1053\,a_2 + 512\,a_{12})\,, \\
a_{15} &= -\frac{1}{2880}\,(5103\,a_1 + 2048\,a_{14} + 3888\,a_{16})\,,
\end{aligned}
$$

where $\lambda \neq 0$, $\mu \neq 0$, and p, a_1, a_2, $a_3, a_8, a_{10}, a_{12}, a_{14}$, and a_{16} are arbitrary constants;

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor\frac{(N-n)}{2}\rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and a_{nl} are constants.

3.1.1 The Parameterized Form of the Three Parameter Family of mKdV Surfaces

In the previous section, we constructed mKdV surfaces satisfying certain equations. We found the first and second fundamental forms, Gaussian and mean curvatures of the surfaces. But we did not find the position vectors of these surfaces. In this section, we explore the position vector

$$\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t)) \tag{3.14}$$

of the mKdV surfaces for a given solution of the mKdV equation and the corresponding Lax pairs. Actually, immersion function F is given implicitly by Eq. (2.16). In order to find F explicitly, we need to find the solution (Φ) of the Lax equations [Eq. (2.15)]. If Lax equations are solved, then we can find F by solving the Eq. (2.16). The method for determining the position vector of the mKdV surfaces from spectral deformation comes with the following steps:

i) Find a solution u of the mKdV equation with a given symmetry:

Here we consider Eq. (3.5) which is produced from the mKdV equation by using the travelling wave solutions $u_t = \alpha u_x$, where $\alpha = -1/c$ and $c \neq 0$ are arbitrary constants.

ii) Find the matrix Φ, which is a solution of the Lax equations [Eq. (2.15)] for given U and V:

In our case, the corresponding $\mathfrak{su}(2)$ valued U and V of the mKdV equation are given by Eqs. (3.6) and (3.7). Consider the 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{3.15}$$

By using Eqs. (3.15) and (3.6), we write $\Phi_x = U\Phi$ in matrix form as

$$\begin{pmatrix} (\Phi_{11})_x & (\Phi_{12})_x \\ (\Phi_{21})_x & (\Phi_{22})_x \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\, i\,\lambda\,\Phi_{11} - \frac{1}{2}\, i\, u\,\Phi_{21} & \frac{1}{2}\, i\,\lambda\,\Phi_{12} - \frac{1}{2}\, i\, u\,\Phi_{22} \\ -\frac{1}{2}\, i\,\lambda\,\Phi_{21} - \frac{1}{2} iu\,\Phi_{11} & -\frac{1}{2}\, i\,\lambda\,\Phi_{22} - \frac{1}{2}\, i\, u\,\Phi_{12} \end{pmatrix}. \tag{3.16}$$

Combining $(\Phi_{11})_x = \frac{1}{2}\, i\lambda\,\Phi_{11} - \frac{1}{2}\, i\, u\,\Phi_{21}$ and $(\Phi_{21})_x = -\frac{1}{2}\, i\lambda\,\Phi_{21} - \frac{1}{2}\, i\, u\,\Phi_{11}$, we obtain a second order equation for Φ_{21}

$$(\Phi_{21})_{xx} - \frac{u_x}{u}\,(\Phi_{21})_x + \left[\frac{1}{4\,u}\Big(u\,(\lambda^2 + u^2) - 2\,i\,\lambda\,u_x\Big)\right]\Phi_{21} = 0. \tag{3.17}$$

It is enough to find Φ_{11} and Φ_{21} because of the symmetry of Eq. (3.16).

By solving the second order equation [Eq. (3.17)] of Φ_{21}, we determine the explicit x dependence of Φ_{21}. And by using $(\Phi_{21})_x = -\frac{1}{2}\, i\lambda\, \Phi_{21} - \frac{1}{2}\, i\, u\, \Phi_{11}$ we also determine the x dependence of Φ_{11}. By substituting these solutions into the equations obtained from $\Phi_t = V\,\Phi$, the following equations are obtained

$$(\Phi_{11})_t = -\frac{i}{2}\left[\frac{u^2}{2} - \alpha - \alpha\,\lambda - \lambda^2\right]\Phi_{11} - \frac{i}{2}\left[(\alpha+\lambda)\,u - i\,u_x\right]\Phi_{21}, \tag{3.18}$$

$$(\Phi_{21})_t = \frac{i}{2}\left[\frac{u^2}{2} - \alpha - \alpha\lambda - \lambda^2\right]\Phi_{21} - \frac{i}{2}\left[(\alpha+\lambda)u + iu_x\right]\Phi_{11}. \tag{3.19}$$

We find Φ_{11} and Φ_{21} explicitly by solving these equations. Because of the symmetry, Φ_{12} and Φ_{22} are found easily. Thus we find the solution of the Lax equations [Eq. (2.15)].

iii) We use Eq. (2.16) to find F. By inserting the solution of Lax equations, A and B into Eq. (2.16) and solving the resultant equation, we find the immersion function F explicitly. Here A and B are given by Eqs. (3.8) and (3.9), respectively.

Now by using a given solution of the mKdV equation, we find the position vector of the mKdV surface. Let $u = k_1\,\mathrm{sech}\,\xi$, $\xi = k_1\left(k_1^2\,t + 4\,x\right)/8$, be one soliton solution of the mKdV equation, where $\alpha = k_1^2/4$. By substituting u into the second order equation [Eq. (3.17)] and using the notation $u_x = k_1\,u_\xi/2, (\Phi_{21})_x = k_1\,(\Phi_{21})_\xi/2$, we find the solution of $\Phi_x = U\,\Phi$ as follows:

$$\begin{aligned}\Phi_{21} \;=\;& iA_1(t)\,(\tanh\xi+1)^{i\lambda/2k_1}\,(\tanh\xi-1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi \\ &+B_1(t)\,(k_1\tanh\xi+2\,i\,\lambda)\,(\tanh\xi-1)^{i\lambda/2k_1}\,(\tanh\xi+1)^{-i\lambda/2k_1}\,,\end{aligned} \tag{3.20}$$

$$\begin{aligned}\Phi_{22} \;=\;& iA_2(t)\,(\tanh\xi+1)^{i\lambda/2k_1}\,(\tanh\xi-1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi \\ &+B_2(t)\,(k_1\tanh\xi+2\,i\,\lambda)\,(\tanh\xi-1)^{i\lambda/2k_1}\,(\tanh\xi+1)^{-i\lambda/2k_1}\,,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\Phi_{11} \;=\;& -\frac{i}{k_1}A_1(t)\,(2\lambda+i\,k_1\tanh\,\xi)\,(\tanh\,\xi+1)^{i\lambda/2k_1}\,(\tanh\xi-1)^{-i\lambda/2k_1} \\ &+i\,k_1\,B_1(t)\,(\tanh\xi-1)^{i\lambda/2k_1}\,(\tanh\xi+1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi,\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Phi_{12} &= -\frac{i}{k_1} A_2(t)\,(2\lambda + ik_1 \tanh\xi)\,(\tanh\xi + 1)^{i\lambda/2k_1}\,(\tanh\xi - 1)^{-i\lambda/2k_1} \\ &\quad + i\,k_1\,B_2(t)\,(\tanh\xi - 1)^{i\lambda/2k_1}\,(\tanh\xi + 1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi. \end{aligned} \tag{3.23}$$

Hence one part ($\Phi_x = U\Phi$) of the Lax equations has been solved. Using these solutions in Eqs. (3.18) and (3.19), which are obtained from $\Phi_t = V\Phi$, we find

$$A_1(t) = A_1\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8} \quad \text{and} \quad B_1(t) = B_1\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}, \tag{3.24}$$

$$A_2(t) = A_2\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8} \quad \text{and} \quad B_2(t) = B_2\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}, \tag{3.25}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants. We solved the Lax equations for given U, V and a solution u of the mKdV equation [Eq. (3.5)]. The components of Φ are

$$\begin{aligned}\Phi_{11} &= -\frac{i}{k_1} A_1\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8}\,(2\,\lambda + i\,k_1\,\tanh\xi)\,(\tanh\xi + 1)^{i\lambda/2k_1}\,(\tanh\xi - 1)^{-i\lambda/2k_1} \\ &\quad + i\,k_1\,B_1\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\,(\tanh\xi - 1)^{i\lambda/2k_1}\,(\tanh\xi + 1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi, \end{aligned} \tag{3.26}$$

$$\begin{aligned}\Phi_{12} &= -\frac{i}{k_1} A_2\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8}\,(2\,\lambda + i\,k_1\,\tanh\xi)\,(\tanh\xi + 1)^{i\lambda/2k_1}\,(\tanh\xi - 1)^{-i\lambda/2k_1} \\ &\quad + i\,k_1\,B_2\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\,(\tanh\xi - 1)^{i\lambda/2k_1}\,(\tanh\xi + 1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi. \end{aligned} \tag{3.27}$$

$$\begin{aligned}\Phi_{21} &= i\,A_1\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8}\,(\tanh\xi + 1)^{i\lambda/2k_1}\,(\tanh\xi - 1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi \\ &\quad + B_1\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\,(k_1 \tanh\xi + 2i\lambda)\,(\tanh\xi - 1)^{i\lambda/2k_1}\,(\tanh\xi + 1)^{-i\lambda/2k_1}, \end{aligned} \tag{3.28}$$

$$\begin{aligned}\Phi_{22} &= i\,A_2\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8}\,(\tanh\xi + 1)^{i\lambda/2k_1}\,(\tanh\xi - 1)^{-i\lambda/2k_1}\,\mathrm{sech}\,\xi \\ &\quad + B_2\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\,(k_1 \tanh\xi + 2i\lambda)\,(\tanh\xi - 1)^{i\lambda/2k_1}\,(\tanh\xi + 1)^{-i\lambda/2k_1}. \end{aligned} \tag{3.29}$$

Here we find that $\det(\Phi) = [(k_1^2 + 4\lambda^2)/k_1]\,(A_1 B_2 - A_2 B_1) \neq 0$.

First we insert A, B, and Φ in Eq. (2.16), then we solve the resultant equation. Let $A_1 = A_2$, $B_1 = \left(A_1\,\mathrm{e}^{\pi\lambda/k_1}\right)/k_1$, $B_2 = -B_1$, then we obtain a three parameter (λ, k_1, μ) family of surfaces, which are parameterized by

$$y_1 = \frac{1}{4\,k_1\,(\mathrm{e}^{2\xi} + 1)} R_1\left(E_1\,(\mathrm{e}^{2\xi} + 1) + 32k_1\right), \tag{3.30}$$

$$y_2 = -4\,R_1\,\cos G_1\,\mathrm{sech}\,\xi, \tag{3.31}$$

$$y_3 = -4\,R_1\,\sin G_1\,\mathrm{sech}\,\xi, \tag{3.32}$$

where

$$R_1 = -\frac{\mu\, k_1}{2\,(k_1^2 + 4\,\lambda^2)}, \tag{3.33}$$

$$G_1 = t\left(\lambda^2 + \frac{1}{4}\,k_1^2[1+\lambda]\right) + x\,\lambda, \tag{3.34}$$

$$E_1 = \left(t\,[8\,\lambda + k_1^2] + 4\,x\right)\left(k_1^2 + 4\,\lambda^2\right), \tag{3.35}$$

$$\xi = \frac{k_1^3}{8}(t + \frac{4x}{k_1^2}). \tag{3.36}$$

This surface has the following first and second fundamental forms

$$(ds_I)^2 = \frac{1}{4}\,\mu^2\left[\left(dx + [\frac{1}{4}\,k_1^2 + 2\lambda]\,dt\right)^2 + k_1^2\,\mathrm{sech}^2\,\xi\,dt^2\right], \tag{3.37}$$

$$(ds_{II})^2 = \frac{1}{2}\mu\,k_1\,\mathrm{sech}\,\xi\left[dx + \left(\frac{1}{4}\,k_1^2 + \lambda\right)dt\right]^2 + \frac{1}{8}\,\mu\,k_1^3\,\mathrm{sech}\,\xi\,\left[2\,\mathrm{sech}^2\,\xi - 1\right]dt^2.$$

and the Gaussian and mean curvatures, respectively, are

$$K = \frac{k_1^2}{\mu^2}\left(2\,\mathrm{sech}^2\,\xi - 1\right), \tag{3.38}$$

$$H = \frac{1}{4\,\mu\,k_1\,\mathrm{sech}\,\xi}\left(6\,k_1^2\,\mathrm{sech}^2\,\xi + \,(4\,\lambda^2 - k_1^2)\right). \tag{3.39}$$

Proposition 3.5 *The surface which is parameterized by Eqs. (3.30)-(3.32) is a cubic Weingarten surface, i.e.,*

$$4\,\mu^2\,H^2\left(2[\mu^2\,K + k_1^2]\right) - 9\,\mu^4\,K^2 - 12\,\mu^2\,\left(k_1^2 + 2\,\lambda^2\right)\,K - \left(k_1^2 + 2\,\lambda^2\right)^2 = 0. \tag{3.40}$$

When $k_1 = 2\,\lambda$ in Eqs. (3.38) and (3.39), it reduces to a quadratic Weingarten surface, i.e.,

$$8\,\mu^2\,H^2 - 9\,\mu^2\,K - 36\,\lambda^2 = 0. \tag{3.41}$$

3.1.2 The Analysis of the Three Parameter Family of mKdV Surfaces

Generally speaking, y_2 and y_3 are asymptotically decaying functions, and y_1 approaches $\pm\infty$ as ξ tends to $\pm\infty$. For some small intervals of x and t, we plot some of the three parameter family of surfaces for some special values of the parameters k_1, λ, and μ in Figs. 3.1-3.4.

Example 3.6 *Taking* $k_1 = 2$, $\lambda = 1$, *and* $\mu = -8$ *in Eqs. (3.30)-(3.32), we get the surface (Fig. 3.1).*

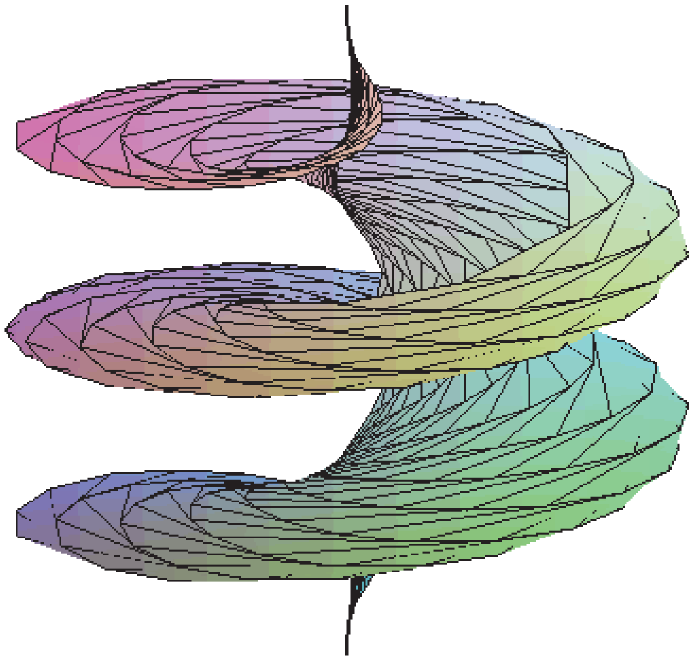

Figure 3.1: $(x, t) \in [-3, 3] \times [-3, 3]$

The components of the position vector of the surface are

$$y_1 = E_2 + 8/(\mathrm{e}^{2\xi} + 1), y_2 = -4 \cos G_1 \operatorname{sech} \xi, y_3 = -4 \sin G_1 \operatorname{sech} \xi, \tag{3.42}$$

where $E_2 = 4(x+3t)$, $G_1 = x+3t$, *and* $\xi = x+t$. *As* ξ *tends to* $\pm\infty$, y_1 *approaches* $\pm\infty$, *and* y_2 *and* y_3 *approach zero. This can also be seen in Fig. 3.1. For small values of* x *and* t, *the surface has a twisted shape.*

Example 3.7 *Taking* $k_1 = 2$, $\lambda = 0$, *and* $\mu = -4$ *in Eqs. (3.30)-(3.32), we get the surface (Fig. 3.2).*

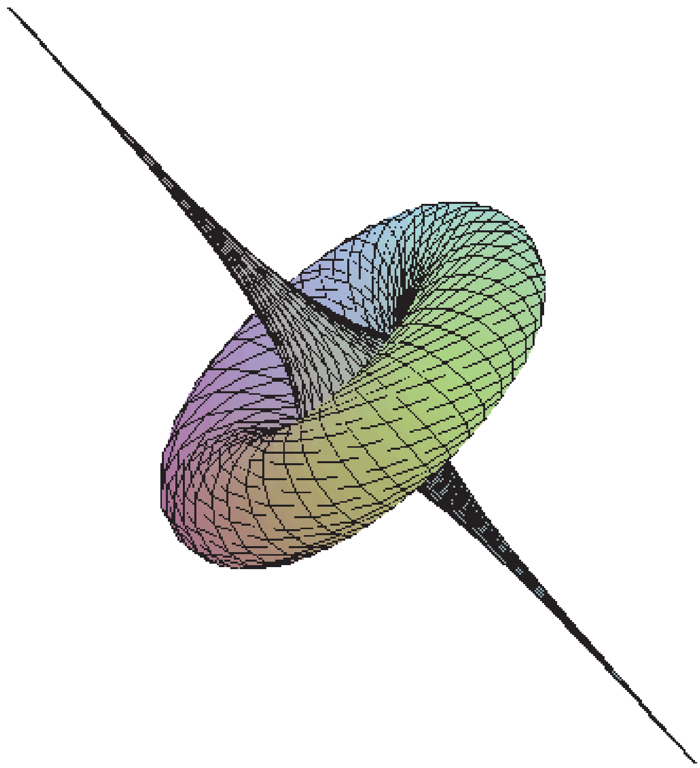

Figure 3.2: $(x,t) \in [-6,6] \times [-6,6]$

The components of the position vector of the surface are

$$y_1 = E_2 + 8/(\mathrm{e}^{2\xi}+1), y_2 = -4\cos G_1 \operatorname{sech}\xi, y_3 = -4\sin G_1 \operatorname{sech}\xi, \tag{3.43}$$

where $E_2 = 2(x+t)$, $G_1 = t$, *and* $\xi = x+t$. *As* ξ *tends to* $\pm\infty$, y_1 *approaches* $\pm\infty$, *and* y_2 *and* y_3 *tend to zero. This can also be seen in Fig. 3.2. Asymptotically, this surface and the surface given in Example 2 are the same. However, for small values of* x *and* t, *they are different.*

Example 3.8 *Taking* $k_1 = 3$, $\lambda = 1/10$, *and* $\mu = -452/75$ *in Eqs. (3.30)-(3.32), we get the surface (Fig. 3.3).*

The components of the position vector of the surface are

$$y_1 = E_2 + 8/(\mathrm{e}^{2\xi}+1), y_2 = -4\cos G_1 \operatorname{sech}\xi, y_3 = -4\sin G_1 \operatorname{sech}\xi, \tag{3.44}$$

where $E_2 = (5537\,t+2260\,x)/750$, $G_1 = (497\,t+20\,x)/200$, *and* $\xi = (12\,x+27\,t)/8$. *Asymptotically, this surface is similar to the previous two surfaces.*

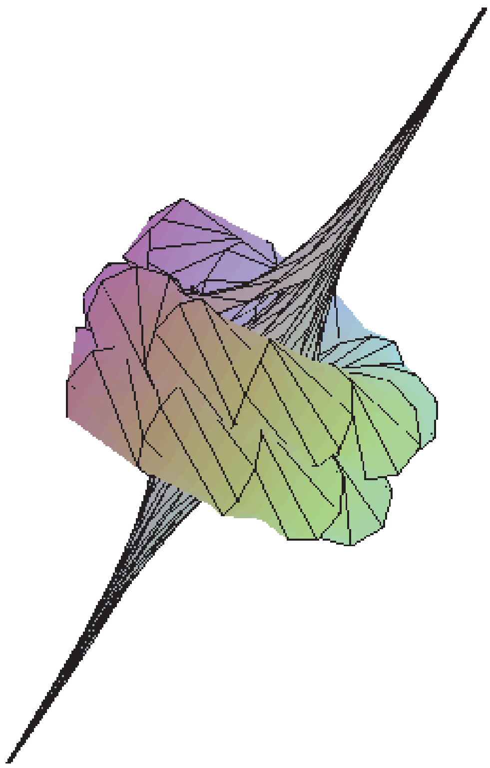

Figure 3.3: $(x, t) \in [-6, 6] \times [-6, 6]$

Example 3.9 *Taking* $k_1 = 1$, $\lambda = -1/10$, *and* $\mu = -52/25$ *in Eqs. (3.30)-(3.32), we get the surface (Fig. 3.4).*

The components of the position vector of the surface are

$$y_1 = E_2 + 8/(e^{2\xi} + 1), y_2 = -4 \cos G_1 \operatorname{sech} \xi, y_3 = -4 \sin G_1 \operatorname{sech} \xi, \tag{3.45}$$

where $E_2 = 13(20\,x + t)/250$, $G_1 = (47\,t - 20\,x)/200$, *and* $\xi = (4\,x + t)/8$. *Asymptotically, this surface is similar to the previous three surfaces.*

3.2 mKdV Surfaces from the Spectral-Gauge Deformations

This section attempts to find surfaces arising from a combination of the spectral and Gauge deformations of the Lax pairs for the mKdV equation.

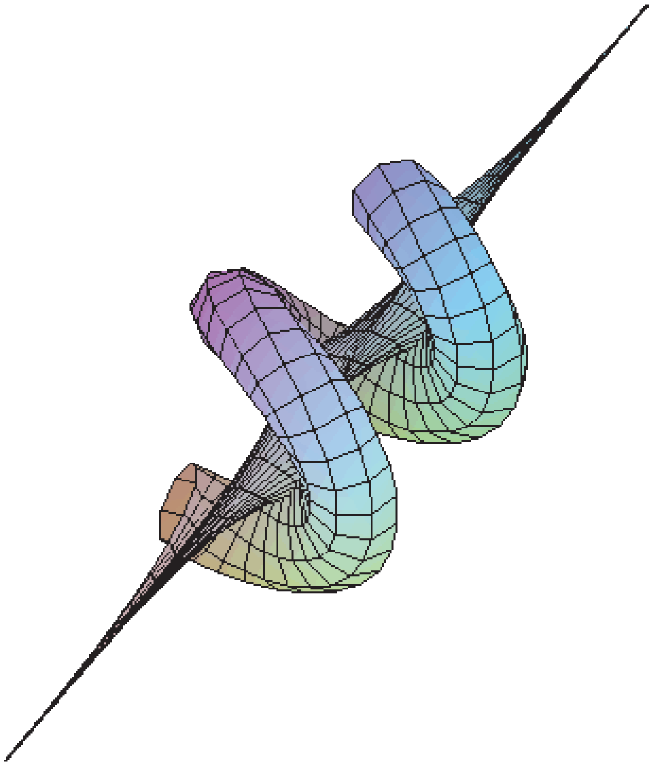

Figure 3.4: $(x,t) \in [-20,20] \times [-20,20]$

Proposition 3.10 *Let u (which describes a traveling mKdV wave) satisfy Eq. (3.5). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by Eqs. (3.6) and (3.7), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = i \begin{pmatrix} (\frac{1}{2}\,\mu - \nu\,u) & -\nu\lambda \\ -\nu\lambda & -(\frac{1}{2}\,\mu - \nu\,u) \end{pmatrix}, \tag{3.46}$$

$$B = i \begin{pmatrix} \frac{1}{2}\,\mu\,(\alpha + 2\,\lambda) - \nu\,(\alpha+\lambda)\,u & -\frac{1}{2}\,\mu\,u + \nu\,(\frac{1}{2}\,u^2 - \alpha - \alpha\,\lambda - \lambda^2) \\ -\frac{1}{2}\,\mu\,u + \nu\,(\frac{1}{2}\,u^2 - \alpha - \alpha\,\lambda - \lambda^2) & -\frac{1}{2}\,\mu\,(\alpha+2\lambda) + \nu\,(\alpha+\lambda)\,u \end{pmatrix}, \tag{3.47}$$

where $A = \mu\,\partial U/\partial\lambda + \nu\,[\sigma_2, U]$, $B = \mu\,\partial V/\partial\lambda + \nu\,[\sigma_2, V]$, λ is a spectral parameter, μ and ν are constants, and σ_2 is the Pauli sigma matrix. Then the surface S, generated by U, V, A, and B, has the following first and second fundamental forms ($j,k = 1,2$)

$$(ds_I)^2 \equiv g_{jk}\,dx^j\,dx^k, \tag{3.48}$$

$$(ds_{II})^2 \equiv h_{jk}\,dx^j\,dx^k, \tag{3.49}$$

where

$$g_{11} = \frac{1}{4}\,\mu^2 + \nu\,(\nu\,[u^2+\lambda^2] - \mu\,u), \tag{3.50}$$

$$g_{12} = \frac{1}{4}(\alpha+2\,\lambda)\mu^2 + \frac{1}{4}\nu\Big(\nu\big[2(\lambda+2\,\alpha)u^2 + 4\,(\lambda^3+\alpha\,\lambda+\lambda^2\,\alpha)\big] - 4\,\mu\,(\alpha+\lambda)\,u\Big), \tag{3.51}$$

$$g_{22} = \frac{1}{4}(u^2+(2\,\lambda+\alpha)^2)\mu^2 + \nu\,\Bigg(\nu\,\big[\frac{1}{4}\,u^4 + \alpha\,(\alpha-1+\lambda)u^2 + ((1+\lambda)\alpha+\lambda^2)^2\big] - \frac{1}{2}\,\mu\,u^3 - \mu\,(\alpha^2+(2\,\lambda-1)\,\alpha+\lambda^2)\,u\Bigg), \tag{3.52}$$

$$h_{11} = \frac{1}{2}\,\mu\,u - \nu(u^2+\lambda^2), \tag{3.53}$$

$$h_{12} = \frac{1}{2}\,\mu\,(\alpha+\lambda)\,u - \nu\,\Big(\lambda(\lambda^2+\alpha\,\lambda+\alpha) + \frac{1}{2}\,(\lambda+2\,\alpha)u^2\Big), \tag{3.54}$$

$$h_{22} = \frac{1}{4}\,\mu\,\Big(u^3 + 2\big[\alpha^2+(2\,\lambda-1)\alpha+\lambda^2\big]\,u\Big) - \nu\,\Big(\frac{1}{4}\,u^4 + \alpha(\alpha-1+\lambda)u^2 + ((1+\lambda)\,\alpha+\lambda^2)^2\Big), \tag{3.55}$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{2\,u\,(u^2-2\,\alpha)}{\nu\Big(2\,\nu u[u^2-2\,\alpha] - 3\,\mu u^2 - 2\mu(\lambda^2-\alpha)\Big) + \mu^2 u}, \tag{3.56}$$

$$H = \frac{\mu(3\,u^2+2(\lambda^2-\alpha)) - 4\,u\,\nu(u^2-2\,\alpha)}{2\,\nu\Big(2\,\nu\,u[u^2-2\,\alpha] - 3\,\mu\,u^2 - 2\,\mu(\lambda^2-\alpha)\Big) + 2\,\mu^2\,u}, \tag{3.57}$$

where $x^1 = x$ and $x^2 = t$.

mKdV surfaces defined in Proposition 3.10 solve neither Willmore-like equation Eq. [(2.31)] nor generalized shape equation [Eq. (2.29)].

3.2.1 The Parameterized Form of the Four Parameter Family of mKdV Surfaces

We apply the same technique that is used in previous section in order to find the position vector of the mKdV surfaces given in Proposition 3.10.

Let

$$u = k_1 \operatorname{sech}\xi, \quad \xi = \frac{k_1}{8}\left(k_1^2\, t + 4x\right), \tag{3.58}$$

be the one soliton solution of the mKdV equation, where $\alpha = k_1^2/4$. The Lax pair U and V are given by Eqs. (3.6) and (3.7), respectively, which are same as in the spectral deformation case. So we can use the solution of the Lax equation [Eq. (2.15)] that we have found in the spectral deformation case. We obtain the position vector by solving Eq. (2.16), where the components of Φ are given by Eqs. (3.26)-(3.29) and A, B are given by Eqs. (3.46) and (3.47), respectively. Here we choose $A_1 = A_2$, $B_1 = \left(A_1\, e^{\pi\lambda/k_1}\right)/k_1$, $B_2 = -B_1$ in order to write F in such a form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$. Hence we obtain a four parameter (λ, k_1, μ, ν) family of surfaces parameterized by

$$\begin{aligned}
y_1 &= -R_2\, \frac{e^{2\xi} - 1}{(e^{2\xi} + 1)} \operatorname{sech}\xi - R_3\, E_3 - R_4 \frac{1}{e^{2\xi} + 1}, \\
y_2 &= \left[\frac{1}{2}\, R_4 \operatorname{sech}\xi + R_5\, \frac{\left(e^{4\xi} + 1\right)}{(e^{2\xi} + 1)^2} - R_6 \operatorname{sech}^2\xi\right] \cos G_1 + R_7\, \frac{\left(e^{2\xi} - 1\right)}{\left(e^{2\xi} + 1\right)} \sin G_1, \\
y_3 &= \left[\frac{1}{2}\, R_4 \operatorname{sech}\xi + R_5\, \frac{\left(e^{4\xi} + 1\right)}{(e^{2\xi} + 1)^2} - R_6 \operatorname{sech}^2\xi\right] \sin G_1 - R_7\, \frac{\left(e^{2\xi} - 1\right)}{\left(e^{2\xi} + 1\right)} \cos G_1,
\end{aligned} \tag{3.59}$$

where

$$R_2 = \frac{2\, k_1^2\, \nu}{k_1^2 + 4\lambda^2}, \quad R_3 = \frac{\mu}{8}, \tag{3.60}$$

$$R_4 = \frac{4\, \mu\, k_1^2}{k_1^2 + 4\lambda^2}, \quad R_5 = \frac{\nu\, (k_1^2 - 4\lambda^2)}{k_1^2 + 4\lambda^2}, \tag{3.61}$$

$$R_6 = \frac{\nu\, (4\, \lambda^2 + 3\, k_1^2)}{2(k_1^2 + 4\lambda^2)}, \quad R_7 = \frac{4\, \lambda\, k_1^2\, \nu}{k_1^2 + 4\lambda^2}, \tag{3.62}$$

$$G_1 = t\left(\lambda^2 + k_1^2\, [1 + \lambda]/4\right) + x\lambda, \tag{3.63}$$

$$E_3 = \left(t\, [8\, \lambda + k_1^2] + 4\, x\right), \quad \xi = \frac{k_1^3}{8}(t + \frac{4\, x}{k_1^2}). \tag{3.64}$$

Thus the position vector $\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of the surface is given by Eq. (3.59). This surface has the following first and second fundamental forms $(j, k = 1, 2)$

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k, \tag{3.65}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k, \tag{3.66}$$

where

$$g_{11} = \frac{1}{4}\,\mu^2 + \nu\,(\nu\,[k_1^2\,\mathrm{sech}^2\,\xi + \lambda^2] - \mu\,k_1\,\mathrm{sech}\,\xi),$$
$$g_{12} = \frac{1}{4}(\alpha + 2\,\lambda)\mu^2 + \frac{1}{4}\,\nu\,\Big(\nu\,\big[2\,k_1^2(\lambda + 2\,\alpha)\,\mathrm{sech}^2\,\xi + (4\,\lambda^3 + 4\,\alpha\,\lambda + 4\,\lambda^2\,\alpha)\big] - 4\,\mu\,(\alpha + \lambda)\,k_1\,\mathrm{sech}\,\xi\Big),$$

$$g_{22} = \frac{1}{4}\,(k_1^2\,\mathrm{sech}^2\,\xi + (2\,\lambda + \alpha)^2)\mu^2 + \nu\,\Bigg(\nu\,\Big[\frac{1}{4}\,k_1^4\,\mathrm{sech}^4\,\xi + \alpha\,k_1^2(\alpha - 1 + \lambda)\,\mathrm{sech}^2\,\xi + ((1 + \lambda)\alpha + \lambda^2)^2\Big] - \frac{1}{2}\,\mu\,k_1^3\,\mathrm{sech}^3\,\xi - \mu\,k_1\,(\alpha^2 + (2\,\lambda - 1)\alpha + \lambda^2)\,\mathrm{sech}\,\xi\Bigg),$$
$$h_{11} = \frac{1}{2}\,\mu\,k_1\,\mathrm{sech}\,\xi - \nu\,(k_1^2\,\mathrm{sech}^2\,\xi + \lambda^2),$$
$$h_{12} = \frac{1}{2}\,\mu(\alpha + \lambda)\,k_1\,\mathrm{sech}\,\xi - \nu\,\Big(\lambda(\lambda^2 + \alpha\,\lambda + \alpha) + \frac{1}{2}\,k_1^2\,(\lambda + 2\,\alpha)\,\mathrm{sech}^2\,\xi\Big),$$
$$h_{22} = \frac{1}{4}\,\mu(k_1^3\,\mathrm{sech}^3\,\xi + 2\,k_1^2\,[\alpha^2 + (2\,\lambda - 1)\alpha + \lambda^2]\,\mathrm{sech}^2\,\xi) - \nu\,\Big(\frac{1}{4}\,k_1^4\,\mathrm{sech}^4\,\xi + \alpha\,k_1^2\,(\alpha - 1 + \lambda)\,\mathrm{sech}^2\,\xi + ((1 + \lambda)\alpha + \lambda^2)^2\Big),$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{2\,k_1\,\mathrm{sech}\,\xi(k_1^2\,\mathrm{sech}^2\,\xi - 2\,\alpha)}{\nu\,\Big(2\,\nu\,k_1\,\mathrm{sech}\,\xi[k_1^2\,\mathrm{sech}^2\,\xi - 2\,\alpha] - 3\,\mu\,k_1^2\,\mathrm{sech}^2\,\xi - 2\,\mu\,(\lambda^2 - \alpha)\Big) + \mu^2\,k_1\,\mathrm{sech}\,\xi},$$

$$H = \frac{\mu\,(3\,k_1^2\,\mathrm{sech}^2\,\xi + 2\,(\lambda^2 - \alpha)) - 4\,\nu\,k_1\,\mathrm{sech}\,\xi(k_1^2\,\mathrm{sech}^2\,\xi - 2\,\alpha)}{2\,\nu\,\Big(2\,\nu\,k_1\,\mathrm{sech}\,\xi[k_1^2\,\mathrm{sech}^2\,\xi - 2\,\alpha] - 3\,\mu\,k_1^2\,\mathrm{sech}^2\,\xi - 2\,\mu(\lambda^2 - \alpha)\Big) + 2\,\mu^2\,k_1\,\mathrm{sech}\,\xi},$$

where $x^1 = x$, $x^2 = t$, and $\alpha = \dfrac{1}{4}k_1^2$.

3.2.2 The Analysis of the Four Parameter Family of Surfaces

Asymptotically, y_1 approaches $\pm\infty$, y_2 approaches $R_5\,\cos G_1 \pm R_7\,\sin G_1$, and y_3 approaches $-R_5\,\sin G_1 \pm R_7\,\cos G_1$ as ξ tends to $\pm\infty$. For some small intervals of x and t, we plot some of the four parameter family of surfaces for some special values of the parameters k_1, λ, μ, and ν in Figs. 3.5-3.7.

Example 3.11 *Taking $k_1 = 2, \lambda = 0, \mu = -4,$ and $\nu = 1$ in Eq. (3.59), we get the surface (Fig. 3.5).*

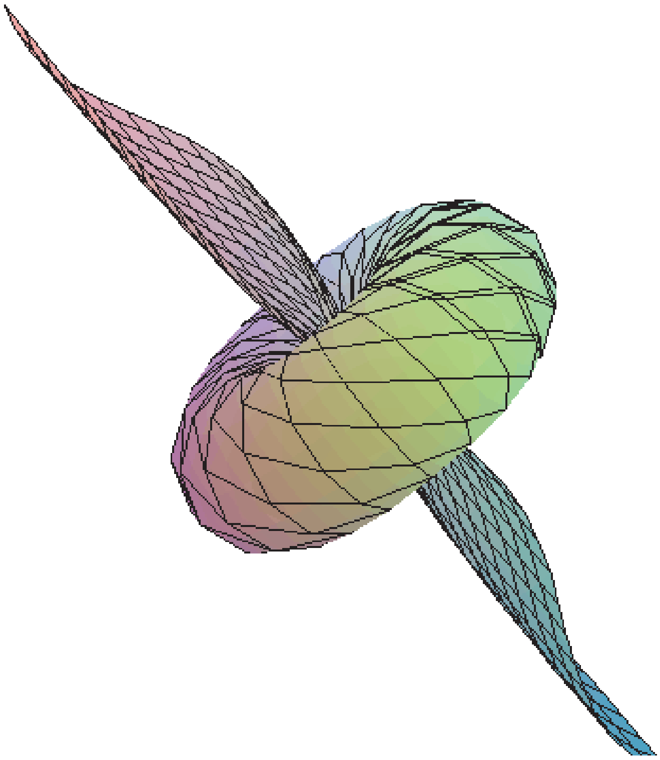

Figure 3.5: $(x,t) \in [-4,4] \times [-4,4]$

The components of the position vector are

$$y_1 = -2\,\mathrm{sech}\,\xi\,(\mathrm{e}^{2\xi} - 1)/(\mathrm{e}^{2\xi} + 1) + E_4 + 8/(\mathrm{e}^{2\xi} + 1), \tag{3.67}$$

$$y_2 = \big[-4\,\mathrm{sech}\,\xi + \big(\mathrm{e}^{4\xi} + 1\big)/(\mathrm{e}^{2\xi} + 1)^2 - (3/2)\,\mathrm{sech}^2\,\xi\big]\cos G_1, \tag{3.68}$$

$$y_3 = \big[-4\,\mathrm{sech}\,\xi + \big(\mathrm{e}^{4\xi} + 1\big)/(\mathrm{e}^{2\xi} + 1)^2 - (3/2)\,\mathrm{sech}^2\,\xi\big]\sin G_1, \tag{3.69}$$

where $E_4 = 2(x+t)$, $G_1 = t$, and $\xi = x + t$. As ξ tends to $\pm\infty$, y_1 approaches $\pm\infty$, y_2 approaches $\cos G_1$, and y_3 approaches $-\sin G_1$.

Example 3.12 *Taking $k_1 = 2, \lambda = 1, \mu = 1/10,$ and $\nu = 1$ in Eq. (3.59), we get the surface (Fig. 3.6).*

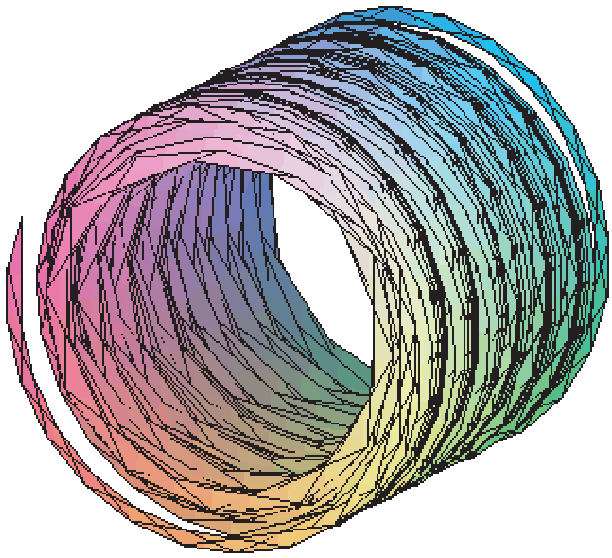

Figure 3.6: $(x,t) \in [-6,6] \times [-6,6]$

The components of the position vector are

$$y_1 = -\operatorname{sech}\xi\,(\mathrm{e}^{2\xi}-1)/(\mathrm{e}^{2\xi}+1) - E_4 - 1/(10(\mathrm{e}^{2\xi}+1)), \tag{3.70}$$

$$y_2 = \left[(1/20)\operatorname{sech}\xi - \operatorname{sech}^2\xi\right]\cos G_1 + \sin G_1\left(\mathrm{e}^{2\xi}-1\right)/\left(\mathrm{e}^{2\xi}+1\right), \tag{3.71}$$

$$y_3 = \left[(1/20)\operatorname{sech}\xi - \operatorname{sech}^2\xi\right]\sin G_1 - \cos G_1\left(\mathrm{e}^{2\xi}-1\right)/\left(\mathrm{e}^{2\xi}+1\right), \tag{3.72}$$

where $E_4 = (x+3t)/20$, $G_1 = (x+3t)$, *and* $\xi = x+t$. *As* ξ *tends to* $\pm\infty$, y_1 *tends to* $\pm\infty$, y_2 *approaches* $\sin G_1$, *and* y_3 *approaches* $\cos G_1$.

Example 3.13 *Taking* $k_1 = 1, \lambda = -1/10, \mu = -52/25$, *and* $\nu = -1$ *in Eq. (3.59), we get the surface (Fig. 3.7).*

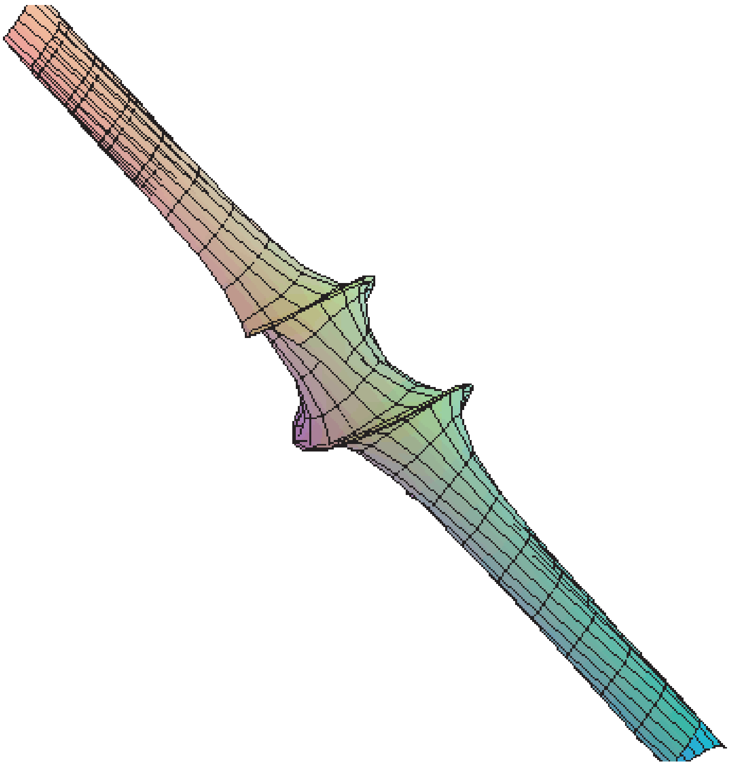

Figure 3.7: $(x,t) \in [-20,20] \times [-20,20]$

The components of the position vector are

$$y_1 = (25/13)\,\mathrm{sech}\,\xi\,(\mathrm{e}^{2\xi}-1)/(\mathrm{e}^{2\xi}+1) + E_4 + 8/(\mathrm{e}^{2\xi}+1), \tag{3.73}$$

$$\begin{aligned} y_2 = {} & \big[-4\,\mathrm{sech}\,\xi - (12/13)\,\big(\mathrm{e}^{4\xi}+1\big)/(\mathrm{e}^{2\xi}+1)^2 + (19/13)\,\mathrm{sech}^2\,\xi\big]\cos G_1 \\ & + (5/13)\,\sin G_1\,\big(\mathrm{e}^{2\xi}-1\big)/\big(\mathrm{e}^{2\xi}+1\big), \end{aligned} \tag{3.74}$$

$$\begin{aligned} y_3 = {} & \big[-4\,\mathrm{sech}\,\xi - (12/13)\,\big(\mathrm{e}^{4\xi}+1\big)/(\mathrm{e}^{2\xi}+1)^2 + (19/13)\,\mathrm{sech}^2\,\xi\big]\sin G_1 \\ & - (5/13)\,\cos G_1\,\big(\mathrm{e}^{2\xi}-1\big)/\big(\mathrm{e}^{2\xi}+1\big), \end{aligned} \tag{3.75}$$

where $E_4 = 13(20\,x+t)/250$, $G_1 = (47\,t - 20\,x)/200$ *and* $\xi = (4x+t)/8$. *As* ξ *tends to* $\pm\infty$, y_1 *tends to* $\pm\infty$, y_2 *approaches* $[5\,\sin G_1 - 12\,\cos G_1]\,/13$ *and* y_3 *to* $-[5\,\cos G_1 + 12\,\sin G_1]\,/13$.

3.3 mKdV Surfaces from Deformation of Parameters

In sections 4.5 and 3.2, and in [29], we considered spectral parameter deformation and combination of spectral and Gauge deformations. In this section, we consider the mKdV surfaces arising from deformations of parameters of the integrable equations' solution [30].

Consider the one soliton solution of mKdV equation [Eq. (3.5)] as

$$u = k_1 \operatorname{sech} \xi, \tag{3.76}$$

where $\alpha = k_1^2/4$, $\xi = k_1(k_1^2 t + 4x)/8 + \xi_0$, and ξ_0 and k_1 are arbitrary constants. The following Proposition gives the mKdV surfaces arising from deformation of parameter ξ_0.

Proposition 3.14 *Let u (which describes a travelling mKdV wave), given by Eq. (3.76), satisfy Eq. (3.5). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by Eqs. (3.6) and (3.7), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = -\frac{i\mu}{2}\begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}, \tag{3.77}$$

$$B = -\frac{i\mu}{2}\begin{pmatrix} u\,\phi & (k_1^2/4+\lambda)\phi - i\,\phi_x \\ (k_1^2/4+\lambda)\phi + i\,\phi_x & -u\,\phi \end{pmatrix}, \tag{3.78}$$

where $A = \mu\,(\partial U/\partial \xi_0)$, $B = \mu\,(\partial V/\partial \xi_0)$, $\phi = \partial\, u/\partial \xi_0$; ξ_0 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms ($j, k = 1, 2$)

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k, \tag{3.79}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k, \tag{3.80}$$

where

$$g_{11} = \frac{1}{4}\mu^2\phi^2, \;\; g_{12} = g_{21} = \frac{1}{16}\mu^2(k_1^2 + 4\,\lambda)\phi^2, \tag{3.81}$$

$$g_{22} = \frac{1}{64}\mu^2\left(16\,\phi_x^2 + \phi^2\left[16\,u^2 + (k_1^2 + \lambda)^2\right]\right), \tag{3.82}$$

$$h_{11} = -16\,\Delta\,\lambda\,u\,\phi^2, \tag{3.83}$$

$$h_{12} = 4\,\Delta\phi\left(4\,\phi_x\,u_x + u\,\phi\left[2\,u^2 - k_1^2(\lambda+1) - 4\,\lambda^2\right]\right), \tag{3.84}$$

$$h_{22} = \Delta\Big(4\,\phi\,\phi_x\left[(k_1^2 + 4\,\lambda)u_x + 4\,u_t\right] + 64\,u\Big[\phi\phi_{xt} - \phi_x\phi_t\Big] \tag{3.85}$$

$$+\,u\,(k_1^2 + 4\,\lambda)\Big[\phi^2\left(2\,u^2 + 4\,\lambda^2 + k_1^2(\lambda+1) + 4\,\phi_x^2\right)\Big]$$

$$\Delta = \frac{\mu}{32\left(\phi_x^2 + u^2\phi^2\right)^{1/2}} \tag{3.86}$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{16\lambda^2}{k_1^2\mu^2}, \quad H = -\frac{4\lambda}{k_1\mu}, \tag{3.87}$$

where $x^1 = x$, $x^2 = t$.

mKdV surfaces constructed by ξ_0 deformation are sphere.

Another parameter of the solution is k_1. It is also possible to use k_1 parameter to construct new mKdV surfaces. These classes of surfaces are given by the following proposition.

Proposition 3.15 *Let u, given by Eq. (3.76), satisfy Eq. (3.5). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by Eqs. (3.6) and (3.7), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = -\frac{i\mu}{2}\begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}, \tag{3.88}$$

$$B = -\frac{i\mu}{2}\begin{pmatrix} u\,\phi & (k_1^2/4 + \lambda)\phi - i\,\phi_x \\ (k_1^2/4 + \lambda)\phi + i\,\phi_x & -u\,\phi \end{pmatrix}, \tag{3.89}$$

where $A = \mu\,(\partial U/\partial k_1)$, $B = \mu\,(\partial V/\partial k_1)$, and $\phi = \partial\,u/\partial k_1$; k_1 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by

U, V, A and B, has the following first and second fundamental forms ($j, k = 1, 2$)

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k, \tag{3.90}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k, \tag{3.91}$$

where

$$g_{11} = \frac{1}{4}\mu^2\phi^2,\ \ g_{12} = g_{21} = \frac{1}{16}\,\phi\left(2\,k_1\,u + \phi[k_1^2 + 4\,\lambda]\right), \tag{3.92}$$

$$g_{22} = \frac{1}{64}\mu^2\Big(4\left[k_1^2 + 4\,\phi^2\right]u^2 + 4\,k_1\,(k_1^2 - 4)u\,\phi \tag{3.93}$$

$$+\,\phi_x^2 + (k_1^2 + 4\,\lambda)^2\,\phi^2 + 4\,k_1^2(\lambda + 1)^2\Big), \tag{3.94}$$

$$h_{11} = \frac{1}{16}\Delta\,\mu^3\,\lambda\,\phi^2\big(k_1[\lambda + 1] - 2\,u\,\phi\big), \tag{3.95}$$

$$h_{12} = \frac{1}{4}\Delta\,\mu^3\phi^2\Big(8\,\phi_x u_x + \Big[k_1(\lambda + 1) - 2u\phi\Big]\Big[2(2\lambda^2 - u^2) + k_1^2(\lambda + 1)\Big]\Big),$$

$$\begin{aligned} h_{22} = \frac{1}{256}\Delta\,\mu^3\,\phi\Big(&8\,\phi_x\Big[2\,k_1\,u\,u_x + (k_1 + 4\,\lambda)(\phi\,u_x - u) + 4(u\phi)_t\Big] \\ &+ \Big[k_1(\lambda + 1) - 2\,u\,\phi\Big]\Big[16\,\phi_{xt} \\ &+ \phi(k_1^2 + 4\,\lambda)(2[u^2 + 2\,\lambda] + k_1^2[\lambda + 1]) - 4\,k_1\,u(u^2 - 2\,\lambda)\Big]\Big), \end{aligned} \tag{3.96}$$

and the corresponding Gaussian and mean curvatures are

$$\begin{aligned} K = \frac{1}{262144}\Delta^6\,\mu^{10}\,\phi^5&\left(4\,\phi\,u[\phi\,u - k_1(\lambda + 1)] + 4\,\phi_x^2 + k_1^2(\lambda + 1)^2\right) \\ &\cdot\Bigg(\big(k_1(\lambda + 1) - 2\,u\,\phi\big)\Big\{\big(k_1(\lambda + 1) - 2\,u\,\phi\big)\Big[16\,\lambda\,\phi_{xt} \\ &+ 2\,\phi\,u^2(k_1^2[3\,\lambda + 2] + 12\,\lambda^2 - 2\,u^2) - 4\,k_1\,\lambda\,u(u^2 + 2\,\lambda) \\ &- k_1^2\,\phi(k_1^2[\lambda + 1] + 4\,\lambda^2)\Big] - 8\,\lambda\,u\,\phi_x^2(k_1^2 + 4\,\lambda) + 8\,u_x\,\phi\Big[2\,k_1\,\lambda\,u \\ &+ 4\,\phi(u^2 + \lambda^2) + \phi\,k_1^2(\lambda + 2)\Big] - \phi_x\,\lambda(u\phi)_t\Big\} - 48\,\phi\,u_x^2\,\phi_x^2\Bigg), \end{aligned} \tag{3.97}$$

$$\begin{aligned} H = \frac{1}{2048}\Delta^3\mu^5\,\phi^2\Bigg(&-8\,\phi\,\phi_x\Big[2\,k_1\,u\,u_x + (k_1^1 + 4\,\lambda)(\phi\,u_x + u) - (u\phi)_t\Big] \\ &+ \big(k_1(\lambda + 1) - 2\,u\,\phi\big)\Big[16\,\phi\,\phi_{xt} + 2\,k_1\,u\,\phi\big(2\,u^2 - 4\,\lambda^2 - 6\,\lambda k_1^2\big) + 4\,\lambda\,\phi_x^2 \\ &+ 2\,u^2\big(2\,k_1^2\,\lambda + \phi^2[3\,k_1^2 + 20\,\lambda]\big) + 4\,\lambda\,k_1^2(\lambda + 1)^2 - k_1^2\,\phi^2(k_1^2 + 4\,\lambda)\Big]\Bigg) \end{aligned} \tag{3.98}$$

where $x^1 = x$, $x^2 = t$. *Here* Δ *is*

$$\Delta = \frac{1}{\sqrt{det\, g}} = \frac{8}{\mu^2 \phi \sqrt{4\,\phi_x^2 + \left(2\,u\,\phi - k_1[\lambda+1]^2\right)}} \tag{3.99}$$

When u*, given by Eq. (3.76), is substituted into* K *and* H*, they take the following forms*

$$K = \frac{1}{\mu^2\,\eta_0\,(4\,\eta_4^2 + \eta_3^2)^2} Q_l\,(\mathrm{sech}\ \xi)^l,\ \ l = 1, 2, ..., 7 \tag{3.100}$$

$$H = \frac{1}{4\,\mu^2\,\eta_0^{1/2}\,(4\,\eta_4^2 + \eta_3^2)^{3/2}} Z_m\,(\mathrm{sech}\ \xi)^m,\ \ m = 0, 1, ..., 6 \tag{3.101}$$

where

$$\eta_0 = \mathrm{sech}\ \xi\,(2 - k_1\,\eta_1\ \mathrm{sech}\ \xi)\,/2, \tag{3.102}$$

$$\eta_1 = (3\,k_1^2\,t + 4\,x)/4, \quad \eta_2 = 3\,k_1^3\,\pi\,t(k_1^2\,t + 8\,x)/16, \tag{3.103}$$

$$\eta_3 = k_1(\pi - k_1)/\pi - 2\,k_1 \mathrm{sech}^2\,\xi\,(1 - k_1\,\eta_1 \mathrm{sech}\ \xi\ \sinh \xi)\,, \tag{3.104}$$

$$\eta_4 = k_1^2\,\eta_1 \mathrm{sech}\ \xi\,\left(1 - 2\ \mathrm{sech}^2\,\xi\right)/2 - k_1\ \mathrm{sech}^2\,\xi\ \sinh \xi, \tag{3.105}$$

$$Q_1 = -k_1^6\,\left(\pi - k_1\right)^2 \left(\pi[\pi + 4\,k_1] + 12\right)/\pi^4, \tag{3.106}$$

$$Q_2 = k_1^7\,\left(4 + \pi^2\right)\left(\pi - k_1\right)^2 \eta_1\ \sinh \xi/(32\,\pi^4), \tag{3.107}$$

$$Q_3 = 4\,k_6\,(k_1\,\pi\,[\pi - 2\,k_1] + 6\,[\pi - k_1])\,/\pi^4, \tag{3.108}$$

$$Q_4 = -\left(k_1\,\pi - 6\,k_1 + 8\,\pi\right) Q_2/\left(4[\pi - k_1]\right), \tag{3.109}$$

$$Q_5 = 2\,k_1^7\,\left([\pi\,k_1\,x^2 + \eta_2][7\,k_1 + \pi] + 6\,\pi[\pi\,k_1 + 4]\right)/\pi^4, \tag{3.110}$$

$$Q_6 = -2\,k_1^8\,\left(\pi\,k_1[3 + x^2] + 24 + \eta_2\right)\,\eta_1\ \sinh \xi, \pi^3, \tag{3.111}$$

$$Q_7 = -12\,k_1^8\,\left(k_1\,\pi\,x^2 + \eta_2\right)/\pi^4, \quad Z_0 = 4\,\mu\,k_1^4(k_1 - \pi)^3/\pi^4, \tag{3.112}$$

$$Z_1 = 0, \quad Z_2 = 4\,\mu\,k_1^4(k_1 - \pi)(6\,k_1 + [k_1^2 - 5]\pi)/\pi^3, \tag{3.113}$$

$$Z_3 = -12\,\mu\,k_1^5(k_1 - \pi)^2\eta_1\ \sinh \xi/\pi^3, \tag{3.114}$$

$$Z_4 = \mu\,k_1^5\,\left(3\,\eta_1^2\,k_1[6\,k_1 + \pi] + 8[k_1\,\pi + 7]\right)/\pi^2, \tag{3.115}$$

$$Z_5 = -2\,\mu\,k_1^6\,\left(28 + k_1\,\pi[2 + \eta_1^2]\right)/\pi^2, \quad Z_6 = -14\mu\,k_1^7\,\eta_1^2/\pi^2 \tag{3.116}$$

3.3.1 The Parameterized Form of the Four Parameter Family of mKdV Surfaces

We apply the same technique that we have used in previous section in order to find the position vector of the surface. Let

$$u = k_1 \operatorname{sech} \xi, \quad \xi = \frac{k_1}{8}\left(k_1^2\, t + 4x\right), \tag{3.117}$$

be the one soliton solution of the mKdV equation, where $\alpha = k_1^2/4$. The Lax pairs U and V are given by Eqs. (3.6) and (3.7), respectively, which are same as in the spectral deformation case. So we can use the solution of the Lax equations [Eq. (2.15)] that we found in the spectral deformation case. By solving the following equation

$$F = \mu\, \Phi^{-1} \frac{\partial \Phi}{\partial k_1} \tag{3.118}$$

we obtain the position vector, where the components of Φ are given by Eqs. (3.26)-(3.29), respectively. Here we choose $A_2 = \mathrm{e}\, k_1\, B_1$, $B_2 = -A_1/\left(\mathrm{e}\; k_1\right)$, $\lambda = -k_1/\pi$ to write F in the form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$. Hence we obtain a four parameter (k_1, μ, A_1, B_1) family of surfaces parameterized by

$$\begin{aligned} y_1 = {} & -\frac{1}{\left(\mathrm{e}^{2\xi}+1\right)^2}\Big[R_{10}\, \mathrm{e}^{\xi}\Big(\eta_{10} \sin G_2 + \eta_{11}\, \cos G_2\Big) \\ & + R_{12} + R_{13}\Big(\eta_6 \left[\mathrm{e}^{4\xi}+1\right] + \eta_7\, \mathrm{e}^{2\xi}\Big)\Big], \end{aligned} \tag{3.119}$$

$$y_2 = \frac{1}{\left(\mathrm{e}^{2\xi}+1\right)^2} R_9\, \mathrm{e}^{\xi}\Big(\eta_{11} \sin G_2 - \eta_{10}\, \cos G_2\Big), \tag{3.120}$$

$$\begin{aligned} y_3 = {} & -\frac{1}{\left(\mathrm{e}^{2\xi}+1\right)^2}\Big[R_{11}\, \mathrm{e}^{\xi}\Big(\eta_{10} \sin G_2 + \eta_{11}\, \cos G_2\Big) \\ & - 2\, R_{10} + \frac{R_8}{\pi}\Big(\eta_{12} \left[\mathrm{e}^{4\xi}+1\right] + \eta_{13}\, \mathrm{e}^{2\xi}\Big)\Big], \end{aligned} \tag{3.121}$$

where

$$G_2 = \frac{1}{4\,\pi^2} k_1 \left(t\left[{k_1}^2 \pi - k_1 \pi^2 - 4\, k_1\right] + 4\, x\pi\right), \tag{3.122}$$

$$\xi = \frac{k_1}{8}\left(k_1^2\, t + 4\, x\right), \quad \eta_5 = k_1 x - 3\, \xi, \tag{3.123}$$

$$\eta_6 = \left(\pi^2+4\right)\left(\pi - k_1\right)\left(\eta_1 + \xi\right) + 2\, k_1\, \pi^2, \tag{3.124}$$

$$\eta_7 = 2\, \eta_6 + \left(-4\, \pi^2 - 8\, \pi^2 \eta_5\right) k_1, \tag{3.125}$$

$$\eta_8 = \left(\pi^2+4\right)\left[{k_1}^2\, \pi\, t + 4\, \left(\xi + \eta_5\right)\right] - 8\, \pi^2, \tag{3.126}$$

$$\eta_9 = 16\,\pi^2 + 2\,\eta_8 + 32\,\pi^2\eta_5, \tag{3.127}$$

$$\eta_{10} = 2\,(\eta_5 + 1)\,\mathrm{e}^{2\xi} - 2\,\eta_5 + 2, \quad \eta_{11} = \pi\,\left(e^{2\xi} + 1\right)\eta_5, \tag{3.128}$$

$$\eta_{12} = {A_1}^2\eta_8 + 4\,k_1\,{B_1}^2\,\mathrm{e}^2\,\eta_6, \quad \eta_{13} = {A_1}^2\,\eta_9 + 4\,\mathrm{e}^2\,k_1\,{B_1}^2\,\eta_7, \tag{3.129}$$

$$R_8 = \mu/\big(4\,k_1\left[\pi^2 + 4\right]\left[{k_1}^2{B_1}^2\mathrm{e}^2 + {A_1}^2\right]\big), \tag{3.130}$$

$$R_9 = -2\,\pi/\big(k_1\left[\pi^2 + 4\right]\big), \quad R_{10} = 8\,\pi\,\left({k_1}^2{B_1}^2\mathrm{e}^2 - {A_1}^2\right)\,R_8, \tag{3.131}$$

$$R_{11} = 16\,R_8\,\pi\,\mathrm{e}\,k_1, \quad R_{12} = 32\,\mathrm{e}\,k_1\,\pi\,A_1\,B_1\,R_8, \tag{3.132}$$

$$R_{13} = -8\,\mathrm{e}\,A_1\,B_1\,R_8/\pi. \tag{3.133}$$

Thus the position vector $\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of the surface is given by Eqs. (3.146)-(3.146).

3.3.2 The Analysis of the Four Parameter Family of Surfaces

Example 3.16 *Taking $k_1 = -1/2, B_1 = 0,$ and $\mu = 1$ in Eqs. (3.119)-(3.121), we get the surface (Fig. 3.8).*

The components of the position vector are

$$y_1 = -\frac{1}{\left(\mathrm{e}^{2\xi} + 1\right)^2}\Big[R_{10}\,\mathrm{e}^{\xi}\Big(\eta_{10}\sin G_2 + \eta_{11}\,\cos G_2\Big)\Big], \tag{3.134}$$

$$y_2 = \frac{1}{\left(\mathrm{e}^{2\xi} + 1\right)^2}R_9\,\mathrm{e}^{\xi}\Big(\eta_{11}\sin G_2 - \eta_{10}\,\cos G_2\Big), \tag{3.135}$$

$$\begin{aligned} y_3 = -\frac{1}{\left(\mathrm{e}^{2\xi} + 1\right)^2}\Big[&R_{11}\,\mathrm{e}^{\xi}\Big(\eta_{10}\sin G_2 + \eta_{11}\,\cos G_2\Big) \\ &- 2\,R_{10} + \frac{R_8}{\pi}\Big(\eta_{12}\left[\mathrm{e}^{4\xi} + 1\right] + \eta_{13}\,\mathrm{e}^{2\xi}\Big)\Big],, \end{aligned} \tag{3.136}$$

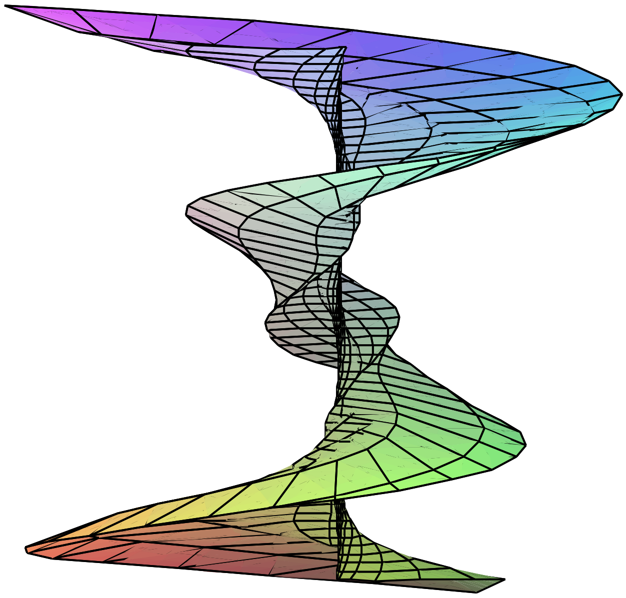

Figure 3.8: $(x,t) \in [-95,95] \times [-95,95]$

where

$$G_2 = -\frac{1}{32\,\pi^2}\left(16\,\pi\,x + t\left[2\,\pi^2 + \pi + 8\right]\right), \tag{3.137}$$

$$\xi = -\frac{1}{64}\,(t + 16\,x)\,, \quad \eta_5 = -\frac{1}{2}\,(x + 6\,\xi)\,, \tag{3.138}$$

$$\eta_6 = \frac{1}{2}\left(\pi^2 + 4\right)(2\,\pi + 1)\,(\eta_5 + \xi) - \pi^2, \tag{3.139}$$

$$\eta_7 = 2\,\eta_6 + 2\,\pi^2\;(1 + 2\,\eta_5)\,, \tag{3.140}$$

$$\eta_8 = \left(\pi^2 + 4\right)\left[\pi\,t + 16\;(\xi + \eta_5)\right]/4 - 8\,\pi^2, \tag{3.141}$$

$$\eta_9 = 16\,\pi^2\,(1 + 2\,\eta_5) + 2\,\eta_8, \quad \eta_{10} = 2\;(\eta_5 + 1)\,\mathrm{e}^{2\xi} - 2\,\eta_5 + 2, \tag{3.142}$$

$$\eta_{11} = \pi\;\left(\mathrm{e}^{2\xi} + 1\right)\eta_5, \quad \eta_{12} = {A_1}^2\eta_8, \quad \eta_{13} = {A_1}^2\,\eta_9, \tag{3.143}$$

$$R_8 = 1/\left(2\left[\pi^2 + 4\right]{A_1}^2\right), \quad R_9 = 4\,\pi/\left(\left[\pi^2 + 4\right]\right), \tag{3.144}$$

$$R_{10} = -8\,\pi\,{A_1}^2\,R_8, \quad R_{11} = -8\,R_8\,\pi\,\mathrm{e}. \tag{3.145}$$

As ξ tends to $\pm\infty$, y_1 and y_2 approach zero, y_3 approaches $\mp\infty$. This can also be seen in Fig. 3.8. For small values of x and t, the surface has a twisted shape around a line.

Example 3.17 *Taking $k_1 = 1/2, B_1 = 0$, and $\mu = 1$ in Eqs. (3.119)-(3.121), we get the surface (Fig. 3.9).*

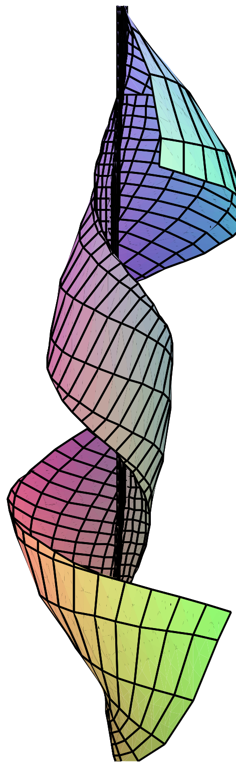

Figure 3.9: $(x,t) \in [-60,60] \times [-60,60]$

The components of the position vector are

$$y_1 = -\frac{1}{(\mathrm{e}^{2\xi}+1)^2}\Big[R_{10}\,\mathrm{e}^{\xi}\Big(\eta_{10}\sin G_3 + \eta_{11}\,\cos G_3\Big)\Big], \tag{3.146}$$

$$y_2 = \frac{1}{(\mathrm{e}^{2\xi}+1)^2} R_9\,\mathrm{e}^{\xi}\Big(\eta_{11}\sin G_3 - \eta_{10}\,\cos G_3\Big), \tag{3.147}$$

$$\begin{aligned} y_3 = &-\frac{1}{(\mathrm{e}^{2\xi}+1)^2}\Big[R_{11}\,\mathrm{e}^{\xi}\Big(\eta_{10}\sin G_3 + \eta_{11}\,\cos G_3\Big) \\ &- 2\,R_{10} + \frac{R_8}{\pi}\Big(\eta_{12}\left[\mathrm{e}^{4\xi}+1\right] + \eta_{13}\,\mathrm{e}^{2\xi}\Big)\Big],, \end{aligned} \tag{3.148}$$

where

$$G_3 = \frac{1}{32\,\pi^2}\left(16\,\pi\,x - t\left[2\,\pi^2+\pi+8\right]\right), \tag{3.149}$$

and η_i, R_{10}, $i = 7, 8, ..., 13$ are same those that are given in Example 3.16, and η_5, η_6, R_8, R_9, R_{11}, ξ are same those that are given in Example 3.16 with negative sign. As ξ tends to $\pm\infty$, y_1 and y_2 approach zero, y_3 approaches $\mp\infty$.

3.4 Sine-Gordon (SG) Surfaces

Let $u(x,t)$ satisfy the sine-Gordon (SG) equation

$$u_{xt} = \sin u. \tag{3.150}$$

Eq. (3.150) can be obtained from a Lax pairs U and V where

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u_x \\ -u_x & -\lambda \end{pmatrix}, \tag{3.151}$$

$$V = \frac{1}{2\,\lambda}\begin{pmatrix} -i\,\cos u & \sin u \\ -\sin u & i\,\cos u \end{pmatrix}, \tag{3.152}$$

were λ is a spectral constant.

3.4.1 SG Surfaces from Spectral Deformation and Symmetries

Proposition 3.18 *Let u satisfy Eq. (3.150). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the SG equation are given by Eqs. (3.151) and (3.152), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = \frac{i\,\mu}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.153}$$

$$B = \frac{\mu}{2\,\lambda}\begin{pmatrix} i\,\cos u & -\sin u \\ \sin u & -i\,\cos u \end{pmatrix}, \tag{3.154}$$

where $A = \mu\,\partial U/\partial\lambda$, $B = \mu\,\partial V/\partial\lambda$, μ is a constant and λ is a spectral parameter. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j,k = 1,2)$

$$\begin{aligned}(ds_I)^2 &\equiv g_{jk}\,dx^j\,dx^k = \frac{\mu^2}{4}\Big(dx^2 + \frac{2}{\lambda^2}\,\cos u\,dx\,dt + \frac{1}{\lambda^4}\,dt^2\Big),\\ (ds_{II})^2 &\equiv h_{jk}\,dx^j\,dx^k = -\frac{\mu}{\lambda}\sin u\,dx\,dt,\end{aligned} \tag{3.155}$$

and the corresponding Gaussian and mean curvatures are

$$K = -\frac{4\,\lambda^2}{\mu^2}, \quad H = \frac{2\,\lambda}{\mu}\cot u, \tag{3.156}$$

where $x^1 = x$, $x^2 = t$.

Proposition 3.19 *Let u satisfy Eq. (3.150). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the SG equation are given by Eqs. (3.151) and (3.152), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = -\frac{i}{2}\varphi_x\sigma_1, \quad B = \frac{i}{2\lambda}\,\varphi(\cos u\sigma_2 + \sin u\sigma_3) \tag{3.157}$$

where $A = U'\varphi$ and $B = V'\varphi$, where λ is constant and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli sigma matrices. Here primes denote Frèchet differentiation and φ is a symmetry of (3.150), i.e. φ is a solution of

$$\varphi_{xt} = \varphi\,\cos u \tag{3.158}$$

Then the surface S, generated by U, V, A, B, has the following first and second fundamental forms $(j,k = 1,2)$

$$(ds_I)^2 \equiv g_{jk}dx^j\,dx^k = \frac{1}{4}\Big(\varphi_x^2\,dx^2 + \frac{1}{\lambda^2}\,\varphi^2 dt^2\Big), \tag{3.159}$$

$$(ds_{II})^2 \equiv g_{jk}dx^j\,dx^k = \frac{1}{2}\Big(\lambda\,\varphi_x\,\sin u\,dx^2 + \frac{1}{\lambda}\,\varphi\,u_t\,dt^2\Big), \tag{3.160}$$

with the corresponding Gaussian and mean curvatures

$$K = \frac{4\lambda^2 u_t \sin u}{\varphi\varphi_x}, \quad H = \frac{\lambda(\varphi_x u_t + \varphi\sin u)}{\varphi\varphi_x} \tag{3.161}$$

Indeed, Eq. (3.158) has infinitely many explicit solutions in terms of u and its derivatives. The following corollary gives the surfaces corresponding to $\varphi = u_x$ which is special case of Proposition 3.19.

Corollary 3.20 *Let $\varphi = u_x$ in Proposition 3.19, then the surface turns out to be a sphere with first and second fundamental forms*

$$(ds_I)^2 = \frac{1}{4}\Big(\sin^2 u\, dx^2 + \frac{1}{\lambda^2}\, u_t^2 dt^2 \Big), \tag{3.162}$$

$$(ds_{II})^2 = \frac{1}{2}\Big(\lambda\, \sin^2 u\, dx^2 + \frac{1}{\lambda}\, u_t^2\, dt^2 \Big), \tag{3.163}$$

and the corresponding Gaussian and mean curvatures are

$$K = 4\,\lambda^2, \;\; H = 2\,\lambda. \tag{3.164}$$

For the solutions

$$\varphi = u_x, \varphi = u_{3x} + \frac{u_x^3}{2}, \varphi = u_{3t} + \frac{u_t^3}{2}, \varphi = u_{5x} + \frac{5}{2}\, u_x^2\, u_{3x} + \frac{5}{2}\, u_x\, u_{2x}^2 + \frac{3}{8}\, u_x^5 \tag{3.165}$$

of the Eq. (3.158), the Gaussian and mean curvatures of the surfaces which are constructed in Proposition 3.19 are not constant directly like in Corollary 3.20. But they are constant for one soliton solution of the SG equation, where one soliton solution is given by

$$u = 4\, \arctan\big(e^{\xi}\big), \tag{3.166}$$

where $\xi = \big(k_1\,[t+x] + [k_1^2 - 1]^{1/2}\,[t-x] + \xi_0\big)$.

3.4.2 SG Surfaces from Deformation of Parameters

In the previous section we used the deformation of parameters to construct mKdV surfaces. It is possible to construct SG surfaces by using the same technique. In this section, we use the parameters of one soliton solution of SG equation which is given by Eq. (3.166). By using one of these parameter, namely ξ_0 deformation, the following proposition gives the SG surfaces.

Proposition 3.21 *Let u, given by Eq. (3.166), satisfy Eq. (3.150). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the SG equation are given by Eqs. (3.151) and (3.152), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = \frac{i\,\mu}{2} \begin{pmatrix} \lambda & -\phi_x \\ -\phi_x & -\lambda \end{pmatrix}, \tag{3.167}$$

$$B = \frac{\mu}{2\,\lambda} \begin{pmatrix} i\,\sin(u)\,\phi & \cos(u)\,\phi \\ -\cos(u)\,\phi & -i\,\sin(u)\,\phi \end{pmatrix}, \tag{3.168}$$

where $A = \mu\,(\partial U/\partial\xi_0)$, $B = \mu\,(\partial V/\partial\xi_0)$, $\phi = \partial\,u/\partial\xi_0$, ξ_0 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j,k=1,2)$

$$(ds_I)^2 \equiv g_{jk}dx^j\,dx^k = \mu^2\,\mathrm{sech}^2\xi\Big(\Big[(k_1^2-1)^{1/2}-k_1\Big]^2 \sinh\xi\,dx^2 \tag{3.169}$$

$$+\frac{1}{\lambda^2}\,dt^2\Big), \tag{3.170}$$

$$(ds_{II})^2 \equiv g_{jk}dx^j\,dx^k = 2\,\mu\,\mathrm{sech}^2\xi\Big(\lambda\,\Big[k_1-(k_1^2-1)^{1/2}\Big]\,\tanh\xi\,dx^2 \tag{3.171}$$

$$+\frac{1}{\lambda}\,\Big[k_1+(k_1^2-1)^{1/2}\Big]\,dt^2\Big), \tag{3.172}$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{4\,\lambda^2\,\big(1+[k_1^2-1]^{1/2}\big)^2}{\mu},\quad H = \frac{2\,\lambda\,\big(1+[k_1^2-1]^{1/2}\big)}{\mu}, \tag{3.173}$$

where $x^1 = x$, $x^2 = t$, and u is the one soliton solution of SG given by Eq. (3.166).

These surfaces are also sphere in $\mathbb{R}^3$.

As in the ξ_0 deformation case, the following proposition gives new SG surfaces by using deformation of the other parameter k_1.

Proposition 3.22 *Let u, given by Eq. (3.166), satisfy Eq. (3.150). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the SG equation are given by Eq. (3.151) and (3.152), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = \frac{i\,\mu}{2} \begin{pmatrix} \lambda & -\phi_x \\ -\phi_x & -\lambda \end{pmatrix}, \tag{3.174}$$

$$B = \frac{\mu}{2\,\lambda} \begin{pmatrix} i\,\sin(u)\,\phi & \cos(u)\,\phi \\ -\cos(u)\,\phi & -i\,\sin(u)\,\phi \end{pmatrix}, \tag{3.175}$$

where $A = \mu\,(\partial U/\partial k_1)$, $B = \mu\,(\partial V/\partial k_1)$, $\phi = \partial\, u/\partial k_1$, k_1 *is a parameter of the one soliton solution* u*,and* μ *is a constant. Then the surface* S*, generated by* U, V, A *and* B*, has the following first and second fundamental forms* $(j, k = 1, 2)$

$$(ds_I)^2 \equiv g_{jk} dx^j\, dx^k = W_1\, \mathrm{sech}^4\xi\, \left(\xi_2\, \sinh\xi + \cosh\xi\right)^2\, dx^2 + W_2\, \xi_2\, \mathrm{sech}^2\xi\, dt^2, \tag{3.176}$$

$$(ds_{II})^2 \equiv g_{jk} dx^j\, dx^k = W_3 \tanh\xi\, \mathrm{sech}^3\xi\, \left(\xi\, \sinh\xi + \cosh\xi\right)\, dx^2 + W_4\, \xi_2\, \mathrm{sech}^2\xi\, dt^2, \tag{3.177}$$

and the corresponding Gaussian and mean curvatures are

$$K = W_5 \frac{\sinh\xi}{\xi_2\,(\xi_2\, \sinh\xi + \cosh\xi)}, \tag{3.178}$$

$$H = W_6 \frac{(2\,\xi_2\, \sinh\xi + \cosh\xi)}{\xi_2\; (\xi_2\, \sinh\xi + \cosh\xi)} \tag{3.179}$$

where

$$\xi = \left(k_1\,[t + x] + [k_1^2 - 1]^{1/2}\,[t - x] + \xi_0\right), \tag{3.180}$$

$$\xi_2 = (t + x)[k_1^2 - 1]^{1/2} + k_1(t - x), \tag{3.181}$$

$$W_1 = \frac{\mu^2\left([k_1^2 - 1]^{1/2} - k_1\right)^2}{k_1^2 - 1}, \quad W_2 = \frac{\mu^2}{\lambda^2\,(k_1^2 - 1)}, \tag{3.182}$$

$$W_3 = -\frac{2\,\mu\,\lambda\left([k_1^2 - 1]^{1/2} - k_1\right)}{[k_1^2 - 1]^{1/2}}, \tag{3.183}$$

$$W_4 = \frac{2\,\mu\,\lambda\left([k_1^2 - 1]^{1/2} + k_1\right)}{\lambda\,[k_1^2 - 1]^{1/2}}, \tag{3.184}$$

$$W_5 = \frac{4\,\lambda^2\left([k_1^2 - 1]^{1/2} + k_1\right)^2 (1 - k_1^2)}{\mu^2}, \tag{3.185}$$

$$\tag{3.186}$$

3.5 Nonlinear Schrödinger (NLS) Surfaces from Spectral Deformation

Let complex function $u(x,t) = r(x,t) + is(x,t)$ satisfy the nonlinear Schrödinger (NLS) equation

$$r_t = s_{2x} + 2s(r^2 + s^2), \tag{3.187}$$

$$s_t = -r_{2x} - 2r(r^2 + s^2), \tag{3.188}$$

where r, s are real functions. The corresponding $\mathfrak{su}(2)$ valued Lax pairs are

$$U = \frac{i}{2}\begin{pmatrix} -2\lambda & 2(s - ir) \\ 2(s + ir) & 2\lambda \end{pmatrix}, \tag{3.189}$$

$$V = -\frac{i}{2}\begin{pmatrix} -4\lambda^2 + 2(r^2 + s^2) & v_1 - iv_2 \\ v_1 + iv_2 & 4\lambda^2 - 2(r^2 + s^2) \end{pmatrix}, \tag{3.190}$$

where $v_1 = 2r_x + 4\lambda s, \quad v_2 = -2s_x + 4\lambda r$ and λ is a constant. By changing the variables r and s, Eqs. (3.187) and (3.187) take the following form

$$r = q\cos\phi, \quad s = q\sin\phi, \tag{3.191}$$

and NLS becomes

$$q\phi_t = -q_{2x} - 2q^3 + q\phi_x^2, \tag{3.192}$$

$$q_t = q\phi_{2x} + 2q_x\phi_x. \tag{3.193}$$

The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of these equations are

$$U = \frac{i}{2}\begin{pmatrix} -2\lambda & 2\,q\,(\sin\phi - i\,\cos\phi) \\ 2\,q\,(\sin\phi + i\,\cos\phi) & 2\lambda \end{pmatrix}, \tag{3.194}$$

$$V = -\frac{i}{2}\begin{pmatrix} -2\,(2\,\lambda^2 - q^2) & w_1 + iw_2 \\ w_1 - iw_2 & 2\,(2\,\lambda^2 - q^2) \end{pmatrix}, \tag{3.195}$$

where

$$w_1 = 2\,(q_x + 2\,\lambda\,q)\cos\phi - 2\,q\,\phi_x\sin\phi, \tag{3.196}$$

$$w_2 = 2\,(q_x + 2\,\lambda\,q)\sin\phi - 2\,q\,\phi_x\cos\phi, \tag{3.197}$$

and λ is a constant.

The following proposition gives the NLS surfaces arising from spectral deformation.

Proposition 3.23 *Let q and ϕ satisfy Eqs. (3.192) and (3.193). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the NLS equation are given by Eqs. (3.194) and (3.195), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = \frac{i}{2} \begin{pmatrix} -2\mu & 0 \\ 0 & 2\mu \end{pmatrix}, \tag{3.198}$$

$$B = -\frac{i}{2} \begin{pmatrix} -8\lambda\mu & 4\mu\, q(\cos\phi - i\,\sin\phi) \\ 4\mu\, q(\cos\phi + i\,\sin\phi) & 8\lambda\mu \end{pmatrix}, \tag{3.199}$$

where $A = \mu\,\partial U/\partial\lambda$, $B = \mu\,\partial V/\partial\lambda$, λ is spectral parameter and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms ($j, k = 1, 2$)

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k = \mu^2 \Big([dx - 4\lambda\, dt]^2 + 4\, q^2\, dt^2 \Big), \tag{3.200}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k = -2\,\mu\, q\, \Big(dx - [2\,\lambda - \phi_x]dt \Big)^2 + 2\,\mu\, q_{2x}\, dt^2 \tag{3.201}$$

and the corresponding Gaussian and mean curvatures are

$$K = -\frac{q_{2x}}{\mu^2\, q}, \quad H = \frac{q_{2x} - q\,(\phi_x + 2\,\lambda^2) - 4\, q^3}{4\,\mu\, q^2}, \tag{3.202}$$

where $x^1 = x$, $x^2 = t$.

The following proposition gives a class of NLS surfaces which are Willmore-like.

Proposition 3.24 *Let $\phi = \alpha\, t$ and $q = q(x)$ satisfies $q_x^2 = -q^4 - \alpha\, q^2$. Then the surface S, defined in Proposition 3.23, is a Willmore-like surface, i.e., the Gaussian and mean curvatures satisfy Eq. (2.31), where*

$$a = \frac{4}{3}, \quad b = 0, \quad \alpha = -2\,\lambda^2, \tag{3.203}$$

and λ is an arbitrary constant.

There are also NLS surfaces arising from a variational principle. In following proposition, we present a class of NLS surfaces that solve the Euler-Lagrange equation [Eq. (2.29)].

Proposition 3.25 *Let u satisfy $u_x^2 = \alpha\, u^2 - u^4/4$. Then there are NLS surfaces, defined in Proposition 3.23, that satisfy the generalized shape equation [Eq. (2.29)] when $\mathcal{E}$ is a polynomial function of H and K.*

Here are several examples:

Example 3.26

Let $\deg(\mathcal{E}) = N$, then

i) for $N = 3$:
$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H,$
$$\alpha = -2\,\lambda^2,\ \ a_1 = -\frac{p\,\mu^4}{18\,\lambda^4},\ \ a_2 = a_4 = 0,\ a_3 = \frac{p\,\mu^2}{16\,\lambda^2},\ \ a_6 = \frac{p\,\mu^4}{8\,\lambda^4},$$
where $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants;

ii) for $N = 4$:
$\mathcal{E} = a_1\, H^4 + a_2\, H^3 + a_3\, H^2 + a_4\, H + a_5 + a_6\, K + a_7\, K\, H + a_8\, K^2 + a_9\, K\, H^2,$
$$\alpha = -2\,\lambda^2,\ \ a_1 = -\frac{8}{189}\,(8\,a_8 + 15\,a_9),\ \ a_2 = -\frac{p\,\mu^4}{18\,\lambda^4},$$
$$a_3 = \frac{2\,\lambda^2}{7\,\mu^2}\,(32\,a_8 + 25\,a_9),\ a_4 = \frac{p\,\mu^2}{16\,\lambda^2},$$
$$a_5 = -\frac{2\,\lambda^4}{21\,\mu^4}\,(38\,a_8 + 45\,a_9)\ a_7 = \frac{p\,\mu^4}{8\,\lambda^4},$$
where $\lambda \neq 0$, $\mu \neq 0$, and $p, a_6, a_8,$ and a_9 are arbitrary constants;

iii) for $N = 5$:
$$\mathcal{E} = a_1\, H^5 + a_2\, H^4 + a_3\, H^3 + a_4\, H^2 + a_5\, H + a_6 + a_7\, K + a_8\, K\, H + a_9\, K^2 + a_{10}\, K\, H^2 + a_{11}\, K^2\, H + a_{12}\, K\, H^3,$$

$$\alpha = -2\,\lambda^2, \quad a_1 = -\frac{4}{1053}\,(128\,a_{11} + 189\,a_{12}),$$
$$a_2 = -\frac{8}{169}\,(8\,a_9 + 15\,a_{10})\,,$$
$$a_3 = \frac{1}{936\,\mu^2\,\lambda^4}\,\big(\lambda^6\,[784\,a_{11} + 2313\,a_{12}] - 52\,p\,\mu^6\big),$$
$$a_4 = \frac{2\,\lambda^2}{7\,\mu^2}\,(32\,a_9 + 25\,a_{10}),$$
$$a_5 = \frac{1}{832\,\mu^4\,\lambda^2}\,\big(52\,p\,\mu^6 - \lambda^6\,[111248\,a_{11} + 6449\,a_{12}]\big),$$
$$a_6 = -\frac{2\,\lambda^4}{21\,\mu^4}\,(38\,a_9 + 45\,a_{10}),$$
$$a_8 = \frac{1}{416\,\mu^2\,\lambda^4}\,\big(\lambda^6\,[8048\,a_{11} + 3591\,a_{12}] + 52\,p\,\mu^6\big)$$

where $\lambda \neq 0$, $\mu \neq 0$, and p, a_7, a_9, a_{10}, a_{11}, and a_{12} are arbitrary constants;

iv) for $N = 6$:

$$\mathcal{E} = a_1\,H^6 + a_2\,H^5 + a_3\,H^4 + a_4\,H^3 + a_5\,H^2 + a_6\,H + a_7 + a_8\,K + a_9\,K\,H + a_{10}\,K^2 + a_{11}\,K\,H^2 + a_{12}\,K^2\,H + a_{13}\,K\,H^3 + a_{14}\,K^3 + a_{15}\,K^2\,H^2 + a_{16}\,K\,H^4,$$

$\alpha = -2\,\lambda^2$, Some of the constants can be written in terms of some others.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor\frac{(N-n)}{2}\rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

Chapter 4

Immersions in M_3

In this chapter, similar to Chapter 3, we give a connection between integrable equations like KdV and HD equations and surfaces in M_3. For the KdV case, this connection has been studied in [28]. In this chapter Lie group G is $\mathrm{SL}(2,\mathbb{R})$ and the corresponding Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}(2,\mathbb{R})$ with basis e_j, $j = 1, 2, 3$, given by

$$\mathrm{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{e}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.1}$$

Define an inner product on $\mathfrak{sl}(2,\mathbb{R})$ as

$$\langle X, Y \rangle = \frac{1}{2}\,\mathrm{trace}(XY), \tag{4.2}$$

for X,$Y \in \mathfrak{sl}(2,\mathbb{R})$.

We construct families of surfaces corresponding to KdV and HD equations by using the method given in Section 2.1. The surfaces corresponding to KdV and HD equations are called KdV and HD surfaces, respectively. Using the deformations of Lax equations [Eq. (2.15)] for given $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V, we find fundamental forms and curvatures of the surfaces. Let $\{F_x, F_t, N\}$ defines a frame on this surface, where N, F_x and F_t are given by Eqs. (2.10) and (2.16), respectively. Substituting a solution of the Lax equation, A, and B in Eq.

(2.16), we find the $\mathfrak{sl}(2,\mathbb{R})$ valued position vector F of the surface as

$$y_j = F^j,\ j = 1,2,3, \quad F = \sum_{k=1}^{3} F^k\ \mathrm{e}_k\,. \tag{4.3}$$

We generate some new algebraic Weingarten surfaces, Willmore-like surfaces, and surfaces solving the generalized shape equation.

4.1 Surfaces from the KdV Hierarchy

In this section, we investigate some surfaces arising from the KdV equation. KdV surfaces are embedded in a three dimensional Minkowski space with signature +1. The following theorem gives the KdV hierarchy [37].

Theorem 4.1 *[37] Let $u(x,t_m) = u$ satisfy the evolution equations*

$$u_{t_m} = Y_m(u) = \varphi^m Y_0, m = 0,1,2,... \tag{4.4}$$

and set

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, V_m = \begin{pmatrix} \tau_m & \kappa_m \\ \rho_m & -\tau_m \end{pmatrix}, \tag{4.5}$$

then we have

$$U_{t_m} + (V_m)_x + [U, V_m] = 0, \quad m \geq 0 \tag{4.6}$$

where $U, V_m \in \mathfrak{sl}(2,\mathbb{R})$, $Y_0 = u_x$, $\varphi(u) = \frac{1}{4}D^2 + u + \frac{1}{2}u_x D^{-1}$

$$\tau_m = -\frac{1}{4}\sum_{i=1}^{m} \lambda^{m-i} Y_{i-1}, \tag{4.7}$$

$$\kappa_m = \lambda^m + \frac{1}{2}\sum_{i=1}^{m} \lambda^{m-i}\gamma_{i-1}, \tag{4.8}$$

$$\rho_m = \lambda^m(\lambda - u) - \frac{1}{4}\sum_{i=1}^{m} \lambda^{m-i}\big((Y_{i-1})_x - 2(\lambda - u)\gamma_{i-1}\big), \tag{4.9}$$

$$D\gamma_j = Y_j, \tag{4.10}$$

Remark 4.2 *U and V_m in Eq. (4.5) define Lax pairs of the KdV hierarchy Eq. (4.4).*

In the following proposition we present classes of surfaces corresponding to the KdV hierarchy .

Proposition 4.3 *Let u satisfy Eq. (4.4). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued hierarch of Lax pairs U and V_m are given by (4.5). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B_m are*

$$A = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \quad B_m = \begin{pmatrix} \mu\tau'_m & \mu\kappa'_m \\ \mu\rho'_m & -\mu\tau'_m \end{pmatrix}, \tag{4.11}$$

where $A = \mu\,\partial U/\partial\lambda$, $B = \mu\,\partial V_m/\partial\lambda$, μ and λ are arbitrary constants. Then the surfaces S_m, generated by U, V_m, A and B_m, have the following first and second fundamental forms (m=0,1,2,...)

$$(ds_I)^2 = \mu^2\kappa'_m dx\,dt + \mu^2((\tau'_m)^2 + \kappa'_m\rho'_m)dt^2, \tag{4.12}$$

$$\begin{aligned}(ds_{II})^2 = -\mu dx^2 - \mu\kappa_m dxdt - \frac{\mu}{\kappa'_m}\Big(\kappa'_m\big[(\tau'_m)_t + \kappa'_m\rho_m \\ - \kappa_m\rho'_m + 2\tau_m\tau'_m\big] - \tau'_m(\kappa'_m)_t - 2(\tau_m)^2\kappa_m\Big)dt^2,\end{aligned} \tag{4.13}$$

and the corresponding Gaussian and mean curvatures are

$$\begin{aligned}K_m = \frac{4}{\mu^2(\kappa'_m)^3}\Big(\kappa'_m\Big[(\kappa_m)^2 + (\tau'_m)_t - \kappa_m\rho'_m + \kappa'_m\rho_m \\ + 2\tau_m\tau'_m\Big] - 2\kappa_m(\tau'_m)^2 - \tau'_m(\kappa'_m)_t\Big),\end{aligned} \tag{4.14}$$

$$H_m = -\frac{2}{\mu(\kappa'_m)^2}\Big(\kappa'_m(\kappa_m - \rho'_m) - (\tau'_m)^2\Big). \tag{4.15}$$

where τ_m, κ_m, ρ_m are respectively (4.7), (4.8), (4.9), primes denote λ partial derivatives and $(\tau'_m)_t = \partial\tau'_m/\partial t$, $(\kappa'_m)_t = \partial\kappa'_m/\partial t$.

Example 4.4 *For $m = 1$ in Theorem 4.1, we have the Korteweg-de Vries (KdV) equation*

$$u_t = \frac{1}{4}u_{3x} + \frac{3}{2}uu_x = Y_1(u). \tag{4.16}$$

$sl(2,R)$ valued Lax pair U and V (we use V notation instead of V_1) are

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \tag{4.17}$$

$$V = \begin{pmatrix} -\frac{1}{4}u_x & \frac{1}{2}u + \lambda \\ -\frac{1}{4}u_{2x} + \frac{1}{2}\,(2\,\lambda + u)\,(\lambda - u) & \frac{1}{4}u_x \end{pmatrix}. \tag{4.18}$$

4.2 KdV Surfaces from Spectral Deformations

The following proposition gives a class of surfaces that correspond to KdV equation [Eq. (4.16)] arising from spectral deformations of Lax pairs.

Proposition 4.5 *Let u satisfy Eq. (4.16). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V of KdV equation are given by Eqs. (4.17) and (4.18). $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B are*

$$A = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \tag{4.19}$$

$$B = \begin{pmatrix} 0 & \mu \\ \frac{\mu}{2}(4\lambda - u) & 0 \end{pmatrix}, \tag{4.20}$$

where $A = \mu(\partial U/\partial \lambda)$, $B = \mu(\partial V/\partial \lambda)$, λ is spectral parameter, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(i, j = 1, 2)$

$$(ds_I)^2 \equiv g_{ij} dx^i\, dx^j = \mu^2 dx\, dt + \frac{\mu^2}{2}(4\lambda - u) dt^2, \tag{4.21}$$

$$\begin{aligned}(ds_{II})^2 \equiv h_{ij} dx^i dx^j &= -\mu\, dx^2 - \mu(2\,\lambda + u) dx\, dt \\ &\quad - \frac{\mu}{4}\Big(u_{2x} + (u + 2\,\lambda)^2\Big) dt^2,\end{aligned} \tag{4.22}$$

and the corresponding Gaussian and mean curvatures are

$$K = -\frac{u_{2x}}{\mu^2}, \quad H = \frac{2(\lambda - u)}{\mu},$$

where $x^1 = x$, $x^2 = t$.

The following proposition contains the quadratic Weingarten surfaces which are obtained by considering the travelling wave solutions of the KdV equation i.e. $u_t + u_x/c = 0$, where c is a constant.

Proposition 4.6 *Let S be the surface obtained in Proposition 4.5 and u satisfy*

$$u_{2x} = -3u^2 - \frac{4}{c} u + 4\beta. \tag{4.23}$$

Then S is a quadratic Weingarten surface satisfying the relation

$$4\,c\,\mu^2\,K + 4\,\mu\,(2+3\,c\,\lambda)\,H - 3\,c\,\mu^2\,H^2 - 4\,(3\,c\,\lambda^2 + 4\,\lambda - 4\,\beta\,c) = 0, \tag{4.24}$$

where c and β are constants.

The following proposition gives a class of surfaces which are called Willmore-like surfaces.

Proposition 4.7 *Let u be the travelling wave solution ($u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma$) of the KdV equation, then surface S defined in Proposition 4.5 is a Willmore-like surface, i.e. Gaussian and mean curvatures satisfy [Eq. (2.31)], where*

$$a = \frac{7}{4}, b = 1, \tag{4.25}$$

$$\beta = \frac{1}{20}\big(28\lambda\alpha - 16\alpha^2 - 21\lambda^2\big), \tag{4.26}$$

$$\gamma = \frac{1}{5}\big(16\alpha^3 - 56\lambda\alpha^2 + 70\alpha\lambda^2 - 28\lambda^3\big). \tag{4.27}$$

$\alpha = -1/c$ ($c \neq 0$), λ and c are arbitrary constants.

Proposition 4.8 *By using the travelling wave solution of the KdV equation and Proposition 4.5, mean curvature of the KdV surface S satisfies a more general differential equation*

$$\begin{aligned} \nabla^2 H \;=\; & -\frac{1}{2\mu^3}\Big[5\mu^3 H^3 + 2\mu^2(2\alpha - 3\lambda)H^2 \\ & +4\mu(12\alpha\lambda - 9\lambda^2 - 8\alpha^2 + 12\beta)H \\ & +56\lambda^3 - 112\lambda^2\alpha + 64\alpha^2\lambda - 32\lambda\beta + 64\alpha\beta + 16\gamma\Big]. \end{aligned} \tag{4.28}$$

There are some KdV surfaces arising from variational principle. The following proposition gives a class of the KdV surfaces that solve the Euler-Lagrange equation [Eq. (2.29)].

Proposition 4.9 *Let $u_x^2 = -2u^3+4\alpha u^2+8\beta u+2\gamma$. Then there are KdV surfaces, defined in Proposition 4.5, that satisfy the generalized shape equation [Eq. (2.29)] when $\mathcal{E}$ is a polynomial function of H and K.*

We have several examples:

Example 4.10

Let $\deg(\mathcal{E}) = N$, then

i) for $N = 3$:

$$\begin{aligned}
\mathcal{E} &= a_1H^3 + a_2H^2 + a_3H + a_4 + a_5K + a_6KH,\\
a_1 &= -\frac{11\,p\,\mu^4}{64\,\Omega_1},\\
a_2 &= -\frac{15}{32\,\Omega_1}\,p\,\mu^3\big(2\,\alpha - 3\,\lambda\big),\\
a_3 &= -\frac{p\,\mu^2}{16\,\Omega_1}\big(33\,\lambda^2 - 44\,\alpha\,\lambda + 8\,\alpha^2 - 20\,\beta\big),\\
a_4 &= \frac{p\,\mu}{8\,\Omega_1}\,\big(47\,\lambda^3 - 94\,\alpha\,\lambda^2 + 4\,(10\,\alpha^2 - 17\,\beta)\,\lambda + 40\,\alpha\,\beta - 2\,\gamma\big)\\
a_6 &= \frac{7\,p\,\mu^4}{16\,\Omega_1}
\end{aligned}$$

where $\Omega_1 = 12\,\lambda^4 - 32\,\alpha\,\lambda^3 + (20\,\alpha^2 - 36\,\beta)\lambda^2 + (40\,\alpha\,\beta - 3\,\gamma)\lambda + 2\,\alpha\,\gamma + 16\,\beta^2$, $\mu \neq 0$, $p \neq 0$, λ, α, β, γ and a_5 are arbitrary constants, but λ, α, β and γ can not be zero at the same time.

ii) for $N = 4$:

$$\mathcal{E} = a_1H^4 + a_2H^3 + a_3H^2 + a_4H + a_5 + a_6K + a_7KH + a_8K^2 + a_9KH^2,$$

$$\begin{aligned}
a_1 &= -\frac{1}{64}\,(34\,a_9 + 15\,a_8)\\
a_2 &= \frac{1}{56\,\mu}\,\Big[(210\,\lambda - 140\,\alpha)\,a_9 + (195\,\lambda - 130\,\alpha)\,a_8 - 22\,\mu\,a_7\Big]
\end{aligned}$$

$$
\begin{aligned}
a_3 =& \frac{1}{56\,\mu^2}\Big[\Big(1512\,\alpha\,\lambda - 308\,\alpha^2 - 1134\,\lambda^2 + 588\,\beta\Big)a_9 \\
&+ \Big(546\,\beta - 718\,\alpha^2 - 2025\,\lambda^2 + 2700\,\alpha\,\lambda\Big)a_8 \\
&+ 60\,\mu\,\Big(3\,\lambda - 2\,\alpha\Big)a_7\Big] \\
a_4 =& \frac{1}{14\,\mu^3}\Big[\Big(1414\,\lambda^3 - 2828\,\lambda^2\alpha + (1652\,\alpha^2 - 700\,\beta)\,\lambda + 392\,\beta\,\alpha \\
&- 28\,\gamma - 280\,\alpha^3\Big)a_9 + \Big(2265\,\lambda^3 - 4530\,\lambda^2\alpha + (2702\,\alpha^2 - 954\,\beta)\,\lambda \\
&- 42\,\gamma + 524\,\beta\,\alpha - 484\,\alpha^3\Big)a_8 \\
&- 2\,\Big(33\lambda^2 - 20\,\beta + 8\,\alpha^2 - 44\,\alpha\,\lambda\Big)\mu\,a_7\Big], \\
a_5 =& \frac{1}{28\,\mu^3}\Big[\Big(19960\,\lambda^3\alpha - 7485\,\lambda^4 + (2844\,\beta - 19012\,\alpha^2)\,\lambda^2 \\
&+ (96\,\gamma + 7664\,\alpha^3 - 3536\,\beta\,\alpha)\,\lambda + 784\,\beta^2 - 1008\,\alpha^4 + 1616\,\alpha^2\beta \\
&- 64\,\alpha\,\gamma\Big)a_8 + \Big(9744\,\lambda^3\alpha - 3654\,\lambda^4 + (168\,\beta - 9688\,\alpha^2)\,\lambda^2 \\
&+ (4256\,\alpha^3 - 224\,\beta\,\alpha)\,\lambda + 224\,\beta^2 + 224\,\alpha^2\beta - 672\,\alpha^4\Big)a_9 \\
&+ 8\,\mu\,a_7\Big(47\,\lambda^3 - 94\,\lambda^2\alpha + (-68\,\beta + 40\,\alpha^2)\,\lambda - 2\,\gamma + 40\,\beta\,\alpha\Big)\Big], \\
a_7 =& \frac{1}{16\,\mu\,\Omega_2}\Big[-672\,\Big(4\,\alpha\,\lambda - \alpha^2 + \beta - 3\,\lambda^2\Big)\Big(7\,\lambda^3/6 - 7\,\lambda^2\alpha/3 \\
&+ (\alpha^2 - 5\,\beta/3)\,\lambda + \beta\,\alpha - \gamma/24\Big)a_9 + \Big(4680\,\lambda^5 - 15600\,\lambda^4\alpha \\
&+ \big(17576\,\alpha^2 - 9672\,\beta\big)\lambda^3 - \big(7664\,\alpha^3 + 414\,\gamma - 18240\,\beta\,\alpha\big)\lambda^2 \\
&+ (552\,\alpha\,\gamma + 1008\,\alpha^4 + 3216\,\beta^2 - 9280\,\alpha^2\beta)\,\lambda - 170\,\alpha^2\gamma \\
&+ 42\,\gamma\,\beta - 2032\,\alpha\,\beta^2 + 1008\,\beta\,\alpha^3\Big)a_8 + 7\,p\,\mu^5\Big]
\end{aligned}
$$

where $\Omega_2 = 4\,\lambda^3\,(3\,\lambda - 8\,\alpha) + 4\,(5\,\alpha^2 - 9\,\beta)\,\lambda^2 + (-3\,\gamma + 40\,\beta\,\alpha)\,\lambda + 2\,\alpha\,\gamma + 16\,\beta^2$, and $\mu \neq 0$, $p \neq 0$, λ, α, β, γ, a_6, a_8 and a_9 are arbitrary constants, but λ, α, β and γ can not be zero at the same time.

iii) for $N = 5$:
$\mathcal{E} = a_1\,H^5 + a_2\,H^4 + a_3\,H^3 + a_4\,H^2 + a_5\,H + a_6 + a_7\,K + a_8\,K\,H + a_9\,K^2 + a_{10}\,K\,H^2 + a_{11}\,K^2\,H + a_{12}\,K\,H^3$,
$a_1, a_2, a_3, , a_4, a_5, a_6, a_8$ can be written in terms of $a_9, a_{10}, a_{11}, a_{12}$, α, β, γ, μ, p and λ.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

4.2.1 The Parameterized Form of the KdV Surfaces

In the previous section, possible surfaces satisfying certain equations are found without giving the F functions explicitly. In this section, we find the position vector

$$\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t)) \tag{4.29}$$

of the KdV surfaces for a given solution of KdV equation and the corresponding Lax pairs. Our method of constructing the position vector $\overrightarrow{y}$ of integrable surfaces consists of the following steps:

i) Find a solution $u = u(x,t)$ of the KdV equation with a given symmetry:

Here, we use travelling wave solutions $u_t = -u_x/c$. By using this assumption we get

$$u_x^2 = -2u^3 - \frac{4}{c}u^2 + 8\beta u + 2\gamma, \tag{4.30}$$

where $c \neq 0$, β and γ are arbitrary constants.

ii) Find a solution of the Lax equations [Eq. (2.15)] for given U and V:

In our case, corresponding $\mathfrak{sl}(2, R)$ valued Lax pairs of the KdV equation U and V are given by Eqs. (4.17) and (4.18). Consider 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{4.31}$$

By using Φ and U we can write $\Phi_x = U\Phi$ in matrix form as

$$\begin{pmatrix} (\Phi_{11})_x & (\Phi_{12})_x \\ (\Phi_{21})_x & (\Phi_{22})_x \end{pmatrix} = \begin{pmatrix} \Phi_{21} & \Phi_{22} \\ (\lambda - u)\Phi_{11} & (\lambda - u)\Phi_{12} \end{pmatrix}. \tag{4.32}$$

Using $(\Phi_{11})_x = \Phi_{21}$ and $(\Phi_{21})_x = (\lambda - u)\Phi_{11}$, we have

$$(\Phi_{11})_{xx} - (\lambda - u)\Phi_{11} = 0. \tag{4.33}$$

Similarly we have an equation for Φ_{12} as

$$(\Phi_{12})_{xx} - (\lambda - u)\Phi_{12} = 0. \tag{4.34}$$

By solving Eqs. (4.33) and (4.34), we determine the explicit x dependence of Φ_{11}, Φ_{12} and also Φ_{21}, Φ_{22}. By using $\Phi_t = V\Phi$, we get

$$(\Phi_{11})_t = -\frac{1}{4}u_x\Phi_{11} + (\frac{1}{2}u + \lambda)\Phi_{21}, \tag{4.35}$$

$$(\Phi_{21})_t = \left[-\frac{1}{4}u_{2x} + \frac{1}{2}\,(2\,\lambda + u)\,(\lambda - u)\right]\Phi_{11} + \frac{1}{4}u_x\Phi_{21}, \tag{4.36}$$

and

$$(\Phi_{12})_t = -\frac{1}{4}u_x\Phi_{12} + (\frac{1}{2}u + \lambda)\Phi_{22}, \tag{4.37}$$

$$(\Phi_{22})_t = \left[-\frac{1}{4}u_{2x} + \frac{1}{2}\,(2\,\lambda + u)\,(\lambda - u)\right]\Phi_{12} + \frac{1}{4}u_x\Phi_{22}. \tag{4.38}$$

Hence solving these equations, we determine the explicit t dependence of Φ_{11}, Φ_{21}, Φ_{12} and $\Phi_{22.}$ Thus we find a solution Φ of the Lax equations.

iii) Find F:

If Φ depends on λ explicitly, F can be found directly from

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} = y_1 \, \mathrm{e}_1 + y_2 \, \mathrm{e}_2 + y_3 \, \mathrm{e}_3 \,. \tag{4.39}$$

If Φ has been found for a fixed value of λ, we can use Eq. (2.16) to find F. For our case, A and B are given by Eqs. (4.82) and (4.20), respectively. Integrating the equations [Eqs. (2.16)] we obtain F. We get the components of the vector $\overrightarrow{y}$ by writing F as a linear combination of e_1, e_2 and e_3, and collecting the coefficients of e_i.

Example 4.11

Let $u = u_0 = \frac{2}{3}(\alpha \pm \sqrt{\alpha^2 + 3\beta})$ be a constant solution of the integrated form $u_x^2 + 2u^3 - 4\alpha u^2 - 8\beta u - 2\gamma = 0$ of the KdV equation [Eq. (4.16)], where $\alpha = -1/c$, $c \neq 0$. Denoting $\lambda - u_0 = m^2$, we find the solutions of Eqs. (4.33) and (4.34) as

$$\Phi_{11} = A_1(t) \, \mathrm{e}^{mx} + B_1(t) \, \mathrm{e}^{-mx}, \tag{4.40}$$

$$\Phi_{12} = A_2(t) \, \mathrm{e}^{mx} + B_2(t) \, \mathrm{e}^{-mx}, \tag{4.41}$$

and

$$\Phi_{21} = (\Phi_{11})_x = m(A_1(t) \, \mathrm{e}^{mx} - B_1(t) \, \mathrm{e}^{-mx}), \tag{4.42}$$

$$\Phi_{22} = (\Phi_{12})_x = m(A_2(t) \, \mathrm{e}^{mx} - B_2(t) \, \mathrm{e}^{-mx}). \tag{4.43}$$

Since u is a constant solution, the equations obtained from $\Phi_t = V \, \Phi$ are of the following forms

$$(\Phi_{11})_t = (\frac{1}{2}u_0 + \lambda)\Phi_{21}, \quad (\Phi_{21})_t = \left[\frac{1}{2} \, (2\,\lambda + u_0) \, m\right] \Phi_{11}, \tag{4.44}$$

and

$$(\Phi_{12})_t = (\frac{1}{2}u_0 + \lambda)\Phi_{22}, \quad (\Phi_{22})_t = \left[\frac{1}{2} \, (2\,\lambda + u_0) \, m\right] \Phi_{12}. \tag{4.45}$$

Denoting $(2\lambda + u_0)/2 = n$ and using Eqs. (4.40)-(4.43) in Eqs. (4.44) and (4.45), we find

$$A_1(t) = C_1 \, \mathrm{e}^{nmt}, \quad B_1(t) = D_1 \, \mathrm{e}^{-nmt}, \tag{4.46}$$

$$A_2(t) = C_2 \, \mathrm{e}^{nmt}, \quad B_2(t) = D_2 \, \mathrm{e}^{-nmt}, \tag{4.47}$$

where C_1, C_2, D_1 and D_2 are arbitrary constants. Thus the components of Φ are

$$\Phi_{11} = C_1 \, \mathrm{e}^{m(nt+x)} + D_1 \, \mathrm{e}^{-m(nt+x)}, \tag{4.48}$$

$$\Phi_{12} = C_2 \, \mathrm{e}^{m(nt+x)} + D_2 \, \mathrm{e}^{-m(nt+x)}, \tag{4.49}$$

$$\Phi_{21} = m(C_1 \, \mathrm{e}^{m(nt+x)} - D_1 \, \mathrm{e}^{-m(nt+x)}), \tag{4.50}$$

$$\Phi_{22} = m(C_2 \, \mathrm{e}^{m(nt+x)} - D_2 \, \mathrm{e}^{-m(nt+x)}) \tag{4.51}$$

Here $\det(\Phi) = 2m(C_2 D_1 - C_1 D_2) \neq 0$.

By using Eqs. (4.48)-(4.51), we can write F directly as

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} = y_1 \, \mathrm{e}_1 + y_2 \, \mathrm{e}_2 + y_3 \, \mathrm{e}_3, \tag{4.52}$$

where e_1, e_2, e_3 are basis elements of $sl(2, R)$ and

$$y_1 = -\left(\frac{D_1 C_2 + C_1 D_2}{D_1 C_2 - C_1 D_2} \right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}, \tag{4.53}$$

$$y_2 = \left(\frac{D_1 C_1 - D_2 C_2}{D_1 C_2 - D_2 C_1} \right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}, \tag{4.54}$$

$$y_3 = -\left(\frac{D_1 C_1 + D_2 C_2}{D_1 C_2 - D_2 C_1} \right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}. \tag{4.55}$$

Thus we find the position vector $\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$, where y_1, y_2 and y_3 are given by Eqs. (4.53)-(4.55), respectively. This solution corresponds to a plane in $\mathbb{R}^3$.

Example 4.12

Let $u = 2\,k^2\,c^2 \mathrm{sech}^2 k(t - cx)$ be one soliton solution of the KdV equation, where $k^2 = -1/c^3$. By denoting $k(t - cx) = \xi$, we find the solutions of Eqs. (4.33) and (4.34) as

$$\Phi_{11} = A_1(t) \mathrm{sech}\, \xi + B_1(t)[\sinh \xi + \xi \mathrm{sech}\, \xi], \tag{4.56}$$

$$\Phi_{12} = A_2(t) \mathrm{sech}\, \xi + B_2(t)[\sinh \xi + \xi \mathrm{sech}\, \xi], \tag{4.57}$$

and

$$\begin{aligned}\Phi_{21} = (\Phi_{11})_x = & \, kc\, A_1(t)\, \operatorname{sech}\xi \tanh\xi \\ & + kc\, B_1(t)\, [\xi \operatorname{sech}\xi \tanh\xi - \cosh\xi - \operatorname{sech}\xi], \end{aligned} \quad (4.58)$$

$$\begin{aligned}\Phi_{22} = (\Phi_{12})_x = & \, kc\, A_2(t)\, \operatorname{sech}\xi \tanh\xi \\ & + kc\, B_2(t)\, [\xi \operatorname{sech}\xi \tanh\xi - \cosh\xi - \operatorname{sech}\xi], \end{aligned} \quad (4.59)$$

for $\lambda = k^2c^2$. By using these functions and considering Eqs. (4.35)-(4.38) with $u_x = 4k^3c^3 \operatorname{sech}^2\xi \tanh\xi$, $u_{2x} = 4k^3c^3 \left(2\operatorname{sech}^2\xi \tanh^2\xi - \operatorname{sech}^4\xi\right)$, we get

$$B_1(t) = B_1 \quad \text{and} \quad A_1(t) = 2B_1kt + C_1, \quad (4.60)$$

$$B_2(t) = B_2 \quad \text{and} \quad A_2(t) = 2B_2kt + C_2, \quad (4.61)$$

where B_1, B_2, C_1 and C_2 are arbitrary constants. Thus components of Φ are

$$\Phi_{11} = B_1 \left(2kt \operatorname{sech}\xi + \sinh\xi + \xi \operatorname{sech}\xi\right) + C_1 \operatorname{sech}\xi, \quad (4.62)$$

$$\Phi_{12} = B_2 \left(2kt \operatorname{sech}\xi + \sinh\xi + \xi \operatorname{sech}\xi\right) + C_2 \operatorname{sech}\xi, \quad (4.63)$$

$$\begin{aligned}\Phi_{21} = kc\Bigg[B_1\bigg(& 2kt \operatorname{sech}\xi \tanh\xi - \cosh\xi - \operatorname{sech}\xi \\ & + \xi \operatorname{sech}\xi \tanh\xi\bigg) + C_1 \operatorname{sech}\xi \tanh\xi\Bigg], \end{aligned} \quad (4.64)$$

$$\begin{aligned}\Phi_{22} = kc\Bigg[B_2\bigg(& 2kt \operatorname{sech}\xi \tanh\xi - \cosh\xi - \operatorname{sech}\xi \\ & + \xi \operatorname{sech}\xi \tanh\xi\bigg) + C_2 \operatorname{sech}\xi \tanh\xi\Bigg]. \end{aligned} \quad (4.65)$$

Here $\det(\Phi) = 2\,k\,c(C_2B_1 - C_1B_2) \neq 0$.

By substituting solution (Φ) of the Lax equations, A, and B into the Eq. (2.16), we can write the components of F_x and F_t in such form

$$F_x = \Phi^{-1}A\Phi = \begin{pmatrix} F_x^{11} & F_x^{12} \\ F_x^{21} & F_x^{22} \end{pmatrix}, \quad F_t = \Phi^{-1}B\Phi = \begin{pmatrix} F_t^{11} & F_t^{12} \\ F_t^{21} & F_t^{22} \end{pmatrix}, \quad (4.66)$$

where

$$F_x^{11} = -\frac{\mu}{2kc\cosh^2\xi\ (C_2B_1 - C_1B_2)}\Big[B_1B_2\Big(2\xi\sinh\xi\cosh\xi + 4kt\sinh\xi\cosh\xi + \cosh^4\xi + 4kt\xi + \xi^2 + 4k^2t^2 - \cosh^2\xi\Big) + \Big(B_1C_2 + B_2C_1\Big)\Big(\sinh\xi\cosh\xi + 2kt + \xi\Big) + C_1C_2\Big], \tag{4.67}$$

$$F_x^{12} = -\frac{\mu}{2kc\cosh^2\xi\ (C_2B_1 - C_1B_2)}\Big[B_2^2\Big(2\xi\sinh\xi\cosh\xi + 4kt\sinh\xi\cosh\xi + \cosh^4\xi + 4kt\xi + \xi^2 + 4k^2t^2 - \cosh^2\xi\Big) + 2B_2C_2\Big(\sinh\xi\cosh\xi + 2kt + \xi\Big) + C_2^2\Big], \tag{4.68}$$

$$F_x^{21} = -\frac{\mu}{2kc\cosh^2\xi\ (C_2B_1 - C_1B_2)}\Big[B_1^2\Big(2\xi\sinh\xi\cosh\xi + 4kt\sinh\xi\cosh\xi + \cosh^4\xi + 4kt\xi + \xi^2 + 4k^2t^2 - \cosh^2\xi\Big) + 2B_1C_1\Big(\sinh\xi\cosh\xi + 2kt + \xi\Big) + C_1^2\Big], \tag{4.69}$$

$$F_x^{22} = -F_x^{11}, \tag{4.70}$$

$$F_t^{11} = -\frac{\mu}{2kc\cosh^2\xi\ (C_2B_1 - C_1B_2)}\Big[B_1B_2\Big(6\xi\sinh\xi\cosh\xi + 12kt\sinh\xi\cosh\xi + \cosh^4\xi + 4kt\xi + 4k^2t^2 + \xi^2 - 5\cosh^2\xi\Big) + \Big(B_1C_2 + B_2C_1\Big)\Big(3\sinh\xi\cosh\xi + 2kt + \xi\Big) + C_1C_2\Big], \tag{4.71}$$

$$F_t^{12} = -\frac{\mu}{2kc\cosh^2\xi\ (C_2B_1 - C_1B_2)}\Big[B_2^2\Big(6\xi\sinh\xi\cosh\xi + 12kt\sinh\xi\cosh\xi + \cosh^4\xi + 4kt\xi + 4k^2t^2 + \xi^2 - 5\cosh^2\xi\Big) + 2B_2C_2\Big(3\sinh\xi\cosh\xi + 2kt + \xi\Big) + C_2^2\Big], \tag{4.72}$$

$$F_t^{21} = -\frac{\mu}{2kc\cosh^2\xi\ (C_2B_1 - C_1B_2)}\Big[B_1^2\Big(6\xi\sinh\xi\cosh\xi + 12kt\sinh\xi\cosh\xi + \cosh^4\xi + 4kt\xi + 4k^2t^2 + \xi^2 - 5\cosh^2\xi\Big) + 2B_1C_1\Big(3\sinh\xi\cosh\xi + 2kt + \xi\Big) + C_1^2\Big], \tag{4.73}$$

$$F_t^{22} = -F_t^{11}. \tag{4.74}$$

Solving these equations we find the immersion function F explicitly. Using

$$F = y_1\,\mathrm{e}_1 + y_2\,\mathrm{e}_2 + y_3\,\mathrm{e}_3, \tag{4.75}$$

we get the position vector of the surface in M_3 corresponding to the KdV equation with non-constant solution as

$$y_1 = 2\zeta_1\left(B_1B_2\zeta_2 + \zeta_3(C_1B_2 + C_2B_1) + 16c^3C_1C_2\right), \tag{4.76}$$

$$y_2 = \zeta_1\left(\zeta_2(B_2^2 - B_1^2) + 2\zeta_3(B_1C_1 - B_2C_2) - 16c^3(C_1^2 - C_2^2)\right), \tag{4.77}$$

$$y_3 = \zeta_1\left(\zeta_2(B_1^2 + B_2^2) + 2\zeta_3(B_1C_1 + B_2C_2) + 16c^3(C_1^2 + C_2^2)\right), \tag{4.78}$$

where e_1, e_2, e_3 are basis elements of $\mathfrak{sl}(2,\mathbb{R})$, B_1, B_2, C_1, and C_2 are arbitrary constants, and

$$\zeta_1 = \frac{\mu}{32c^2(B_1C_2 - B_2C_1)(1+\mathrm{e}^{2\xi})}, \tag{4.79}$$

$$\begin{aligned}\zeta_2 = {}& -8\left(1-\mathrm{e}^{2\xi}\right)\left(3t - cx\right)^2 + 4k\,c^3(9t - c\,x)(1+\mathrm{e}^{2\xi}) \\ & + c^3(1-\mathrm{e}^{4\xi}) - 2c^3\sinh 2\xi,\end{aligned} \tag{4.80}$$

$$\zeta_3 = 8\,k\,c^3\,(3\,t - c\,x)\left(1-\mathrm{e}^{2\xi}\right). \tag{4.81}$$

4.3 KdV Surfaces from Spectral-Gauge Deformations

The following proposition gives a class of surfaces that correspond to KdV equation [Eq. (4.16)] arising from spectral-Gauge deformations.

Proposition 4.13 *Let u satisfy Eq. (4.16). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V of KdV equation are given by Eqs. (4.17) and (4.18). $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B are*

$$A = \begin{pmatrix} 0 & 2\nu \\ 2\nu(u-\lambda)+\mu & 0 \end{pmatrix}, \tag{4.82}$$

$$B = \begin{pmatrix} 0 & \nu(2\lambda+u)+\mu \\ \frac{\nu}{2}\left(u_{xx} - 2(2\lambda-u)(u+\lambda)\right) + \frac{\mu}{2}(4\lambda-4) & 0 \end{pmatrix},$$

where $A = \mu(\partial U/\partial \lambda) + \nu[e_1, U]$, $B = \mu(\partial V/\partial \lambda) + \nu[e_1, V]$ *and* μ *and* ν *are arbitrary constants. Then the surface* S*, generated by* U, V, A *and* B*, has the following first and second fundamental forms* $(i, j = 1, 2)$

$$
\begin{aligned}
(ds_I)^2 \equiv g_{ij} dx^i\, dx^j &= 2\,\nu\Big(2\nu(u-\lambda)+\mu\Big)dx^2 \qquad (4.83)\\
&+ \Big(\nu\Big[\nu\, u_{2x} - 2(u+2\,\lambda)(\mu - 2\,\nu\,[\lambda - u])\Big] + \mu^2\Big)dxdt\\
&- \frac{1}{2}\Big(2[u+2\lambda]+\mu\Big)\Big(2\nu[u+2\lambda][\lambda-u] - \mu[4\lambda - u] - \nu u_{xx}\Big)dt^2,\\
(ds_{II})^2 \equiv h_{ij} dx^i\, dx^j &= \Big(4\nu(\lambda-u)-\mu\Big)dx^2 \qquad (4.84)\\
&- \Big(\nu\, u_{2x} + \big[\mu - 4\,\nu\,(\lambda-u)\big]\big[2\,\lambda + u\big]\Big)dx\,dt\\
&- \frac{1}{4}\Big(\big[\mu + 2\,\nu(2\,\lambda+u)\big]u_{2x} + \big[\mu - 4\,\nu(\lambda-u)\big]\big[u+2\,\lambda\big]^2\Big)dt^2,
\end{aligned}
$$

and the corresponding Gaussian and mean curvatures are

$$
K_1 = \frac{u_{xx}}{\nu^2 u_{xx} + \mu\,(4\,\nu[\lambda-u]-\mu)}, \quad H_1 = \frac{2\mu(\lambda-u)+\nu u_{xx}}{\nu^2 u_{xx} + \mu\,(4\,\nu[\lambda-u]-\mu)},
$$

where $x^1 = x$, $x^2 = t$.

Example 4.14 *For* $m = 2$ *in Theorem 4.1 we have the fifth order Korteweg-de Vries (fKdV) equation*

$$
u_t = \frac{1}{16}u_{5x} + \frac{5}{6}uu_{3x} + \frac{7}{24}u_x u_{2x} + \frac{5}{6}u^2 u_x = Y_2(u). \qquad (4.85)
$$

$\mathfrak{sl}(2, R)$ *valued Lax pair* U *and* V *(instead of* V_2 *we write* V*) are*

$$
U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, V = \begin{pmatrix} \tau_2 & \kappa_2 \\ \rho_2 & -\tau_2 \end{pmatrix}, \qquad (4.86)
$$

where

$$
\begin{aligned}
\tau_2 &= -\frac{1}{16}\lambda u_{3x} - \frac{1}{6}\lambda u u_x, \qquad (4.87)\\
\kappa_2 &= \frac{1}{8}\lambda u_{2x} + \frac{1}{6}\lambda u^2 + \frac{1}{2}u + \lambda^2, \qquad (4.88)\\
\rho_2 &= -\frac{1}{16}\lambda u_{4x} + \frac{1}{24}\Big(3(\lambda^2-2) - 7u\Big)u_{2x} \qquad (4.89)\\
&\quad - \frac{1}{6}\lambda u_x^2 + \frac{1}{6}(\lambda^2 - 3 - \lambda u)u^2 + \frac{1}{2}\lambda(2\lambda^2 - u).
\end{aligned}
$$

4.4 Surfaces from the Higher KdV Equations

The following proposition gives a class of surfaces that correspond to fifth order KdV equation [Eq. (4.85)] arising from spectral deformations.

Proposition 4.15 *Let u satisfy Eq. (4.85). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V of fKdV equation are given by Eq. (4.86). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued matrices of A and B are*

$$A = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \mu\,\tau'_2 & \mu\,\kappa'_2 \\ \mu\,\rho'_2 & -\mu\,\tau'_2 \end{pmatrix}, \tag{4.90}$$

where $A = \mu\,\partial U/\partial\lambda$, $B = \mu\,\partial V/\partial\lambda$, μ and λ are arbitrary constants. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms

$$(ds_I)^2 = \mu^2\,\kappa'_2 dx\,dt + \mu^2((\tau'_2)^2 + \kappa'_2\,\rho'_2)dt^2, \tag{4.91}$$

$$\begin{aligned}(ds_{II})^2 = -\mu\,dx^2 - \mu\,\kappa_2\,dx\,dt - \frac{\mu}{\kappa'_2}\Big(\kappa'_2\big[(\tau'_2)_t + \kappa'_2\,\rho_2 \\ - \kappa_2\,\rho'_2 + 2\,\tau_2\,\tau'_2\big] - \tau'_2(\kappa'_2)_t - 2(\tau_2)^2\kappa_2\Big)dt^2,\end{aligned} \tag{4.92}$$

and the corresponding Gaussian and mean curvatures are

$$\begin{aligned}K_2 = \frac{4}{\mu^2(\kappa'_2)^3}\Big(\kappa'_2\Big[(\kappa_2)^2 + (\tau'_2)_t - \kappa_2\,\rho'_2 + \kappa'_2\,\rho_2 \\ + 2\tau_2\,\tau'_2\Big] - 2\kappa_2(\tau'_2)^2 - \tau'_2(\kappa'_2)_t\Big),\end{aligned} \tag{4.93}$$

$$H_2 = -\frac{2}{\mu\,(\kappa'_2)^2}\Big(\kappa'_2(\kappa_2 - \rho'_2) - (\tau'_2)^2\Big). \tag{4.94}$$

where τ_2, κ_2, ρ_2 are (4.87), (4.88), (4.89), respectively, primes denote λ partial derivatives and $(\tau'_2)_t = \partial\tau'_2/\partial t$, $(\kappa'_2)_t = \partial\kappa'_2/\partial t$.

4.5 Harry Dym Surfaces from Spectral Deformations

Let $u(x,t)$ satisfy the Harry Dym (HD) equation

$$u_t = -u^3\, u_{3x}. \tag{4.95}$$

Assuming the travelling wave ansatz $u_t - \alpha\, u_x = 0$ in Eq. (4.95), we get

$$u_{2x} = \frac{\alpha}{2}\,\frac{1}{u} - C_1. \tag{4.96}$$

where α and C_1 are arbitrary constants. Eq. (4.95) can be obtained from $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V where

$$U = \begin{pmatrix} 0 & 1 \\ \dfrac{\lambda^2}{u^2} & 0 \end{pmatrix}, \tag{4.97}$$

$$V = 2\,\lambda^2 \begin{pmatrix} u_x & -2\,u \\ u_{2x} - \dfrac{2\,\lambda^2}{u} & -u_x \end{pmatrix}, \tag{4.98}$$

were λ is a spectral constant. The Lax equations are given by Eq. (2.15), where the integrability of these equations are guaranteed by the HD equation or the zero curvature condition Eq. (2.13) for given U and V as Eqs. (4.97) and (4.98), respectively. The following proposition gives HD surfaces from spectral deformations.

Proposition 4.16 *Let u satisfy Eq. (4.95). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V of the HD equation are given by Eq. (4.97) and (4.98), respectively. $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B are*

$$A = 2\,\mu\,\lambda \begin{pmatrix} 0 & 0 \\ \dfrac{1}{u^2} & 0 \end{pmatrix}, \tag{4.99}$$

$$B = 4\,\mu\,\lambda \begin{pmatrix} u_x & -2\,u \\ u_{2x} - \dfrac{4\,\lambda}{u} & -u_x \end{pmatrix} \tag{4.100}$$

where $A = \mu\, \partial U/\partial\lambda$, $B = \mu\, \partial V/\partial\lambda$, μ is a constant and λ is a spectral parameter. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms ($j, k = 1, 2$)

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k = -16\,\mu^2\,\lambda^2\Big(\frac{1}{u} dx\, dt + [u_x^2 - 2\,u\,u_{2x} + 8\,\lambda^2] dt^2\Big), \tag{4.101}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k = -\frac{2\,\mu\,\lambda}{u^2}\Big(dx^2 - 8\,\lambda^2\,u\,dx\,dt + 2\,u^2\left[2\,u^2\,u_x\,u_{3x} + u^3\,u_{4x} + 8\,\lambda^4\right] dt^2\Big) \tag{4.102}$$

and the corresponding Gaussian and mean curvatures are

$$K = -\frac{u^2}{8\,\mu^2\,\lambda^2}\left(2\,u_x\,u_{3x} + u\,u_{4x}\right), \quad H = \frac{1}{4\,\mu\,\lambda}\left(u_x^2 - 2\,u\,u_{2x} + 4\,\lambda^2\right), \tag{4.103}$$

where $x^1 = x$, $x^2 = t$.

The following proposition gives Willmore-like HD surfaces.

Proposition 4.17 *Let u satisfy $u_x^2 = -\alpha\frac{1}{u} - 2\,C_1\,u + 2\,C_2$. Then the surface S, defined in Proposition 4.16, is a Willmore-like surface, i.e. the Gaussian and mean curvatures satisfy the equation [Eq. (2.31)], where*

$$a = -2, \quad b = 6, \quad C_1 = \frac{16\,\lambda^4}{\alpha}, \quad C_2 = -6\,\lambda^2 \tag{4.104}$$

and λ is an arbitrary constant.

In order to study the HD surfaces arising from variational principle, it is enough to know fundamental forms and curvatures for such surfaces. The following proposition gives a class of the HD surfaces that solve the Euler-Lagrange equation [Eq. (2.29)].

Proposition 4.18 *Let u satisfy $u_x^2 = -\alpha\frac{1}{u} - 2\,C_1\,u + 2\,C_2$. Then there are HD surfaces, defined in Proposition 4.16, that satisfy the generalized shape equation [Eq. (2.29)] when $\mathcal{E}$ is a polynomial function of H and K.*

We have several examples:

Example 4.19

Let $\deg(\mathcal{E}) = N$, then

i) for $N = 3$:
$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H,$

$$a_1 = -\frac{11\,\mu\, a_2}{30\,\lambda},\; a_3 = -\frac{4\,\lambda\, a_2}{15\,\mu},\; a_6 = \frac{14\,\mu\, a_2}{15\,\lambda},$$
$$a_4 = 0, C_1 = p = 0, C_2 = 2\,\lambda,$$

where $\lambda \neq 0$, μ, and a_5 are arbitrary constants.

ii) for $N = 4$:
$\mathcal{E} = a_1\, H^4 + a_2\, H^3 + a_3\, H^2 + a_4\, H + a_5 + a_6\, K + a_7\, K\, H + a_8\, K^2 + a_9\, K\, H^2,$

$$a_1 = -\frac{1}{64}\,(15\, a_8 + 34\, a_9),$$
$$a_2 = \frac{1}{480\,\mu\,\lambda}\,\big(\lambda^2\,[358\, a_9 - 7\, a_8] - 176\,\mu^2\, a_3\big),$$
$$a_4 = \frac{4\,\lambda}{15\,\mu^3}\,\big(\lambda^2\,[13\, a_8 + 8\, a_9] - \mu^2\, a_3\big),$$
$$a_5 = -\frac{3\,\lambda^4}{4\,\mu^4}\,(3\, a_8 + 2\, a_9),$$
$$a_7 = \frac{1}{120\,\mu\,\lambda}\,\big(\lambda^2\,[359\, a_8 + 154\, a_9] + 112\,\mu^2\, a_3\big),$$
$$C_1 = p = 0, C_2 = 2\,\lambda,$$

where $\lambda \neq 0$, $\mu \neq 0$, and a_6 are arbitrary constants.

iii) for $N = 5$:
$\mathcal{E} = a_1\, H^5 + a_2\, H^4 + a_3\, H^3 + a_4\, H^2 + a_5\, H + a_6 + a_7\, K + a_8\, K\, H + a_9\, K^2 + a_{10}\, K\, H^2 + a_{11}\, K^2\, H + a_{12}\, K\, H^3,$

$$a_1 = -\frac{3}{464}\,(51\, a_{11} + 92\, a_{12}),$$

$$
\begin{aligned}
a_2 &= -\frac{1}{1856\,\mu}\,(\mu\,[435\,a_9 986 +\ a_{10}] - \lambda\,[2590\,a_{11} + 2268\,a_{12}]),\\
a_3 &= -\frac{1}{13920\,\mu^2\,\lambda}\Big(\mu\,\lambda^2\,[203\,a_9 - 10382\,a_{10}]\\
&\quad - \lambda^3\,[14486\,a_{11} + 14220\,a_{12}] + 5104\,\mu^3\,a_4\Big),\\
a_5 &= -\frac{4\,\lambda}{435\,\mu^4}\Big(- \mu\,\lambda^2\,[377\,a_9 + 232\,a_{10}]\\
&\quad + \lambda^3\,[1544\,a_{11} + 720\,a_{12}] + 29\,\mu^3\,a_4\Big),\\
a_6 &= -\frac{3\,\lambda}{116\,\mu^5}\Big(\mu\,[87\,a_9 + 58\,a_{10}] - \lambda\,[494\,a_{11} + 252\,a_{12}]\Big),\\
a_8 &= \frac{1}{3480\,\mu^2\,\lambda}\Big(\mu\,\lambda^2\,[10411\,a_9 + 4466\,a_{10}]\\
&\quad - \lambda^3\,[34582\,a_{11} + 17100\,a_{12}] + 3248\,\mu^3\,a_4\Big),\\
C_1 &= p = 0, C_2 = 2\,\lambda,
\end{aligned}
$$

where $\lambda \neq 0$, $\mu \neq 0$, a_7 are arbitrary constants.

iv) for $N = 6$:
$\mathcal{E} = a_1\,H^6 + a_2\,H^5 + a_3\,H^4 + a_4\,H^3 + a_5\,H^2 + a_6\,H + a_7 + a_8\,K + a_9\,K\,H + a_{10}\,K^2 + a_{11}\,K\,H^2 + a_{12}\,K^2\,H + a_{13}\,K\,H^3 + a_{14}\,K^3 + a_{15}\,K^2\,H^2 + a_{16}\,K\,H^4$,
where $a_1, a_2, a_3, a_4, a_6, a_7, a_9$ can be written in terms of a_5, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16} and $C_1 = p = 0, C_2 = 2\,\lambda$, where $\lambda \neq 0$, $\mu \neq 0$, are arbitrary constants.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

4.5.1 The Parameterized Form of the HD Surfaces

In order to find the position vector

$$\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t)), \tag{4.105}$$

of the HD surfaces for a given solution of the HD equation and the corresponding Lax pairs, we use the following steps:

i) Find a solution u of the HD equation with a given symmetry:

Here we consider Eq. (4.95) the travelling wave solutions $u_t = \alpha u_x$, where α is arbitrary constant.

ii) Find the matrix Φ of the Lax equations [Eq. (2.15)] for given U and V:

The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued U and V of the HD equation are given by Eqs. (4.97) and (4.98). Consider 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{4.106}$$

By using this form and Eq. (4.97) for U, we can write $\Phi_x = U\Phi$ in matrix form as

$$\begin{pmatrix} (\Phi_{11})_x & (\Phi_{12})_x \\ (\Phi_{21})_x & (\Phi_{22})_x \end{pmatrix} = \begin{pmatrix} \Phi_{21} & \Phi_{22} \\ \dfrac{\lambda^2}{u}\Phi_{11} & \dfrac{\lambda^2}{u}\Phi_{11} \end{pmatrix}. \tag{4.107}$$

Combining equations $(\Phi_{11})_x = \Phi_{21}$ and $(\Phi_{21})_x = \dfrac{\lambda^2}{u}\Phi_{11}$, we get

$$(\Phi_{11})_{xx} - \frac{\lambda^2}{u}\Phi_{11} = 0. \tag{4.108}$$

Similarly, by using the coupled first order equations Φ_{12} and Φ_{12}, a second order equation can be obtained for Φ_{22}. By solving the second order equation (4.108) of Φ_{11} and the equation for Φ_{12}, we determine the explicit x dependence of Φ_{11}, Φ_{12}

and also Φ_{21}, Φ_{22}. The components of $\Phi_t = V\Phi$ read

$$(\Phi_{11})_t = 2\,\lambda^2\,u_x\,\Phi_{11} - 4\,\lambda^2\,u\,\Phi_{21}, \tag{4.109}$$

$$(\Phi_{21})_t = 2\,\lambda^2\left(u_{2x} - \frac{2\,\lambda^2}{u}\right)\Phi_{11} - 2\,\lambda^2\,u_x\,\Phi_{21} \tag{4.110}$$

and

$$(\Phi_{12})_t = 2\,\lambda^2\,u_x\,\Phi_{12} - 4\,\lambda^2\,u\,\Phi_{22}, \tag{4.111}$$

$$(\Phi_{22})_t = 2\,\lambda^2\left(u_{2x} - \frac{2\,\lambda^2}{u}\right)\Phi_{12} - 2\,\lambda^2\,u_x\,\Phi_{22}. \tag{4.112}$$

Solving these equations, we determine the explicit t dependence of Φ_{11}, Φ_{21}, Φ_{12} and $\Phi_{22.}$ This way we completely determine the solution Φ of the Lax equations.

iii) We use

$$F = \mu\,\Phi^{-1}\,\frac{\partial\Phi}{\partial\lambda}, \tag{4.113}$$

to find F, where λ is a spectral parameter.

Now by using a given solution of the HD equation, we find the position vector of the HD surface. Let $u = -(\alpha/2)\,18^{1/3}\,\xi^{2/3}$, $\xi = t + x/\alpha$, be a solution of the HD equation, where $\alpha \neq 0$ is a constant. By inserting u into the second order equation [Eq. (4.108)] and using $(\Phi_{11})_x = (\Phi_{11})_\xi/\alpha$, we find the solution of $\Phi_x = U\,\Phi$ as

$$\begin{aligned}\Phi_{11} = \frac{1}{\lambda^{3/2}}\Big(&A_1(t)\,\big[18^{1/3} - 6\,\lambda\,\xi^{1/3}\big]\exp\big\{\lambda\,18^{2/3}\,\xi^{1/3}/3\big\}\\ &+ B_1(t)\,\big[18^{1/3} + 6\,\lambda\,\xi^{1/3}\big]\exp\big\{-\lambda\,18^{2/3}\,\xi^{1/3}/3\big\}\Big)\end{aligned} \tag{4.114}$$

$$\begin{aligned}\Phi_{21} = -\frac{2\,\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(&A_1(t)\,\exp\big\{\lambda\,18^{2/3}\,\xi^{1/3}/3\big\}\\ &+ B_1(t)\,\exp\big\{-\lambda\,18^{2/3}\,\xi^{1/3}/3\big\}\Big)\end{aligned} \tag{4.115}$$

$$\Phi_{12} = \frac{1}{\lambda^{3/2}}\Big(A_2(t)\left[18^{1/3} - 6\,\lambda\,\xi^{1/3}\right]\exp\left\{\lambda\,18^{2/3}\,\xi^{1/3}/3\right\}$$
$$+ B_2(t)\left[18^{1/3} + 6\,\lambda\,\xi^{1/3}\right]\exp\left\{-\lambda\,18^{2/3}\,\xi^{1/3}/3\right\}\Big) \qquad (4.116)$$
$$\Phi_{22} = -\frac{2\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_2(t)\,\exp\left\{\lambda\,18^{2/3}\,\xi^{1/3}/3\right\}$$
$$+ B_2(t)\,\exp\left\{-\lambda\,18^{2/3}\,\xi^{1/3}/3\right\}\Big) \qquad (4.117)$$

Hence one part ($\Phi_x = U\Phi$) of the Lax equations has been solved. By using these solutions in the equations obtained from $\Phi_t = V\Phi$, we find

$$A_1(t) = A_1\exp\{4\,\lambda^3\,t\} \quad \text{and} \quad B_1(t) = B_1\exp\{-4\,\lambda^3\,t\}, \qquad (4.118)$$

$$A_2(t) = A_2\exp\{4\,\lambda^3\,t\} \quad \text{and} \quad B_2(t) = B_2\exp\{-4\,\lambda^3\,t\}, \qquad (4.119)$$

where A_1, A_2, B_1 and B_2 are arbitrary constants. We solved the Lax equations for a given U, V and a solution u of the HD equation [Eq. (4.95)]. The components of Φ are

$$\Phi_{11} = \frac{1}{\lambda^{3/2}}\Big(A_1\left[18^{1/3} - 6\,\lambda\,\xi^{1/3}\right]\exp\left\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}$$
$$+ B_1\left[18^{1/3} + 6\,\lambda\,\xi^{1/3}\right]\exp\left\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}\Big) \qquad (4.120)$$
$$\Phi_{21} = -\frac{2\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_1\,\exp\left\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}$$
$$+ B_1\,\exp\left\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}\Big) \qquad (4.121)$$

$$\Phi_{12} = \frac{1}{\lambda^{3/2}}\Big(A_2\left[18^{1/3} - 6\,\lambda\,\xi^{1/3}\right]\exp\left\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}$$
$$+ B_2\left[18^{1/3} + 6\,\lambda\,\xi^{1/3}\right]\exp\left\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}\Big) \qquad (4.122)$$
$$\Phi_{22} = -\frac{2\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_2\,\exp\left\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}$$
$$+ B_2\,\exp\left\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\right\}\Big) \qquad (4.123)$$

Here $\det(\Phi) = \dfrac{(k_1^2 + 4\lambda^2)}{k_1}\,(A_1B_2 - A_2B_1) \neq 0.$

By using Eq. (4.113), we get the family of the surfaces which are parameterized by

$$y_1 = \zeta_4\Big(\zeta_5\, B_1\, B_2 + \zeta_6\, A_1\, A_2 + \frac{1}{2}\,\zeta_7\,(A_1\, B_2 + A_2\, B_1)\Big), \tag{4.124}$$

$$y_2 = \frac{\zeta_4}{2}\Big(\zeta_5(B_2^2 - B_1^2) + \zeta_6\,(A_2^2 - A_1^2) + \zeta_7\,(A_2\, B_2 - A_1\, B_1)\Big), \tag{4.125}$$

$$y_3 = \frac{\zeta_4}{2}\Big(\zeta_5(B_2^2 + B_1^2) + \zeta_6\,(A_2^2 + A_1^2) + \zeta_7\,(A_2\, B_2 + A_1\, B_1)\Big), \tag{4.126}$$

where

$$\zeta_4 = \frac{\mu}{3\,\alpha^{2/3}\,\lambda^2\,(A_1\, B_2 - A_2\, B_1)\,(\alpha\, t + x)^{1/3}}, \tag{4.127}$$

$$\zeta_5 = \frac{1}{2}\Big(3\,\lambda\,\alpha^{2/3}(\alpha\, t + x)^{1/3} + \alpha\, 18^{1/3}\Big)\, \exp\{-2\,\lambda\Big(12\,\lambda^2\,\alpha^{1/3}t + 18^{2/3}\,(\alpha\, t + x)^{1/3}\Big)/(3\,\alpha^{1/3})\} \tag{4.128}$$

$$\zeta_6 = \frac{1}{2}\Big(-3\,\lambda\,\alpha^2/3(\alpha\, t + x)^{1/3} + \alpha\, 18^{1/3}\Big)\, \exp\{2\,\lambda\Big(12\,\lambda^2\,\alpha^{1/3}t + 18^{2/3}\,(\alpha\, t + x)^{1/3}\Big)/(3\,\alpha^{1/3})\} \tag{4.129}$$

$$\zeta_7 = 2\,\lambda^2\, 18^{2/3}\,\alpha^{1/3}\,(\alpha\, t + x)^{2/3} + 72\,\lambda^4\,\alpha^{2/3}\, t\,(\alpha\, t + x)^{1/3} + 18^{1/3}\,\alpha.$$

This surface has the following first and second fundamental forms $(j, k = 1, 2)$

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k, \quad (ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k, \tag{4.130}$$

where

$$\begin{aligned}
g_{11} &= 0, \quad g_{12} = \frac{8\,\mu^2\,\lambda^2 18^{2/3}}{9\,\alpha^{1/3}(\alpha\, t + x)^{2/3}},\\
g_{22} &= \frac{32\,\mu^2\,\lambda^2\,\big[36\,\lambda^2\,\alpha^{1/3}\,(\alpha\, t + x)^{2/3} + \alpha\, 18^{2/3}\big]}{9\,\alpha^{1/3}(\alpha\, t + x)^{2/3}},\\
h_{11} &= -\frac{4\,\mu\,\lambda\, 18^{1/3}}{9\,\alpha^{2/3}(\alpha\, t + x)^{4/3}}, \quad h_{12} = -\frac{8\,\mu\,\lambda^3\, 18^{2/3}}{9\,\alpha^{1/3}(\alpha\, t + x)^{2/3}},\\
h_{22} &= -\frac{16\,\mu\,\lambda\, 18^{1/3}\,\big[18^{2/3}\,\lambda^4\,(\alpha^{2/3}\, x + \alpha^{5/3}\, x)\,(\alpha\, t + x)^{1/3} - (3/4)\,\alpha^2\big]}{9\,\alpha^{2/3}(\alpha\, t + x)^{4/3}},
\end{aligned} \tag{4.131}$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{18^{1/3}\,\alpha^{4/3}}{24\,\mu^2\,\lambda^2(\alpha\, t + x)^{4/3}}, \;\; H = \frac{(18\,\alpha)^{2/3}\,(\alpha\, t + x)^{1/3} + 18\,\lambda^2\,(\alpha\, t + x)}{18\,\mu\,\lambda\,(\alpha\, t + x)}, \tag{4.132}$$

where $x^1 = x$, $x^2 = t$.

Proposition 4.20 *The HD surface, given by Eqs. (4.124)-(4.126), is a quadratic Weingarten surface, i.e.*

$$3\,\mu^2\,H^2 - 6\,\mu\,\lambda\,H - 4\,\mu^2\,K + 3\,\lambda^2 = 0. \tag{4.133}$$

Chapter 5

Conclusion

This thesis intends to construct 2-surfaces in $\mathbb{R}^3$ and in M_3 by using the soliton surface technique. For its purpose, the relation between these surfaces and surfaces in Lie groups and Lie algebras are used.

Using the Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$, we find the mKdV, SG, and NLS surfaces using the spectral deformation, deformation of parameters, and spectral-Gauge deformation.

We introduce a new deformation which is deformation of parameters of integrable nonlinear PDEs' solution. We find new mKdV and NLS surfaces that solve the generalized shape equation. Here Lagrange function is polynomial of the Gaussian and mean curvature of the corresponding surfaces. Some new algebraic Weingarten and Willmore-like surfaces are introduced for the mKdV case. Using the solution of the Lax equations we determine the position vectors of mKdV surfaces. Some of them are plotted for some special values of constants.

Using the Lie group $SL(2,\mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, KdV and HD surfaces are constructed through using spectral and spectral-Gauge deformations. We find the parameterized form of the KdV and HD surfaces that arise from spectral deformation by using one soliton solution of KdV equation and a solution of HD equation. In addition, we find new algebraic Weingarten and Willmore-like surfaces and obtain some KdV and HD surfaces from variational principle.

Chapter 6

Appendix A: Maple Codes

Owing to the fact that the codes are similar for the surfaces in the thesis, we do not give all maple codes. We give only codes for surfaces from spectral deformation in the case of Lie group SU(2) and the corresponding Lie algebra $\mathfrak{su}(2)$.

A.1 First and Second Fundamental Forms, Gaussian and Mean Curvatures

The following codes give the first and second fundamental forms, Gaussian and mean curvatures of the surface in $\mathbb{R}^3$.

```
>  with(LinearAlgebra):
>  l:=lambda
>  U:=;
>  UL11:=diff(U[1,1],l);
>  UL12:=diff(U[1,2],l);
>  UL21:=diff(U[2,1],l);
>  UL22:=diff(U[2,2],l);
>  UL:=<<UL11|UL12>,<UL21|UL22>>;
```

```
> A:=mu*UL;
> A11x:=diff(A[1,1],x);
> A12x:=diff(A[1,2],x);
> A21x:=diff(A[2,1],x);
> A22x:=diff(A[2,2],x);
> Ax:=<<A11x|A12x>,<A21x|A22x>>:
> A11t:=diff(A[1,1],t);
> A12t:=diff(A[1,2],t);
> A21t:=diff(A[2,1],t);
> A22t:=diff(A[2,2],t);
> At:=<<A11t|A12t>,<A21t|A22t>>:
> V:=;
> VL11:=diff(V[1,1],l);
> VL12:=diff(V[1,2],l);
> VL21:=diff(V[2,1],l);
> VL22:=diff(V[2,2],l);
> VL:=<<VL11|VL12>,<VL21|VL22>>;
> B:=mu*VL;
> B11x:=diff(B[1,1],x);
> B12x:=diff(B[1,2],x);
> B21x:=diff(B[2,1],x);
> B22x:=diff(B[2,2],x);
> Bx:=<<B11x|B12x>,<B21x|B22x>>:
> B11t:=diff(B[1,1],t);
> B12t:=diff(B[1,2],t);
> B21t:=diff(B[2,1],t);
> B22t:=diff(B[2,2],t);
> Bt:=<<B11t|B12t>,<B21t|B22t>>:
> AB:=simplify(MatrixMatrixMultiply(A,B)):
```

```
> BA:=simplify(ScalarMultiply(MatrixMatrixMultiply(B,A),-1)):
> ABBA:=simplify(MatrixAdd(AB,BA));
> NABBA:=-(1/2)*Trace(MatrixMatrixMultiply(ABBA,ABBA)):
> C:=simplify(ABBA/((NABBA)^(1/2)),symbolic);
> AU:=MatrixMatrixMultiply(A,U):
> UA:=ScalarMultiply(MatrixMatrixMultiply(U,A),-1):
> AUUA:=MatrixAdd(AU,UA):
> AV:=MatrixMatrixMultiply(A,V):
> VA:=ScalarMultiply(MatrixMatrixMultiply(V,A),-1):
> AVVA:=MatrixAdd(AV,VA):
> BV:=simplify(MatrixMatrixMultiply(B,V)):
> VB:=ScalarMultiply(MatrixMatrixMultiply(V,B),-1):
> BVVB:=MatrixAdd(BV,VB):
> AxAUUA:=MatrixAdd(Ax,AUUA):
> AtAVVA:=MatrixAdd(At,AVVA):
> BtBVVB:=MatrixAdd(Bt,BVVB):
> h[1,1]:=-simplify((1/2)*Trace(MatrixMatrixMultiply(AxAUUA,C)));
> h[1,2]:=-simplify((1/2)*Trace(MatrixMatrixMultiply(AtAVVA,C)));
> h[2,2]:=-simplify((1/2)*Trace(MatrixMatrixMultiply(BtBVVB,C)));
> h:=<<h[1,1]|h[1,2]>,<h[1,2]|h[2,2]>>;
> g[1,1]:=-simplify(1/2)*Trace(MatrixMatrixMultiply(A,A));
> g[1,2]:=-simplify((1/2)*Trace(MatrixMatrixMultiply(A,B)));
> g[2,2]:=-simplify((1/2)*Trace(MatrixMatrixMultiply(B,B)),size);
> g:=<<g[1,1]|g[1,2]>,<g[1,2]|g[2,2]>>;
> ginv := LinearAlgebra:-MatrixInverse( g ):
> ginvh:=MatrixMatrixMultiply(ginv,h):
> H:=(1/2)*(LinearAlgebra:-Trace( ginvh)):
> K:= simplify(LinearAlgebra:-Determinant( ginvh ));
```

A.2 Willmore-like Surfaces

The following codes is used to find the Willmore-like classes of the surfaces in $\mathbb{R}^3$.

```
> H:=;
> K:=;
> g:=;
> ginv:= LinearAlgebra:-MatrixInverse(g);
> detg:= LinearAlgebra:-Determinant( g);
> sqrtdetg:=sqrt(abs( detg));

> Hx:=diff(H,x);
> Ht:=diff(H,t);
> Ginv11 := ginv[1,1];
> Ginv12 := ginv[1,2];
> Ginv22 := ginv[2,2];
> Ginv11Hx:=simplify(sqrtdetg*Ginv11*Hx,size):
> Ginv12Ht:=simplify(sqrtdetg*Ginv12*Ht,size):
> Ginv12Hx:=simplify(sqrtdetg*Ginv12*Hx,size):
> Ginv22Ht:=simplify(sqrtdetg*Ginv22*Ht,size):
> DGinv11Hx:=simplify((1/(sqrtdetg ))*diff(Ginv11Hx,x),size):
> DGinv12Ht:=simplify((1/(sqrtdetg ))*diff(Ginv12Ht,x),size):
> DGinv12Hx:=simplify((1/(sqrtdetg ))*diff(Ginv12Hx,t),size):
> DGinv22Ht:=simplify((1/(sqrtdetg ))*diff(Ginv22Ht,t),size):
> eq1:=simplify(DGinv11Hx+DGinv12Ht,size):
> eq2:=simplify(DGinv12Hx+DGinv22Ht,size):
> eq3:=a*H^3+b*H*K:
> Willmorelike:=simplify(eq1+eq2+eq3,size):
> solve(Willmorelike):
```

A.3 Generalized Shape Equation

By using the following codes one can find the surfaces solving the shape equation for the Lagrange function which is a third order polynomial of Gaussian and mean curvature.

```
> H:=;
> K:=;
> N2H2:=;
> N2K:=;
> NdotNH:=;
> N2H:=;
> N2H3:=;
> N2KH:=;
> NdotNH2:=;
> NdotNK:=;
> eq1:=3*a[1]*N2H2+12*a[1]*H^4-6*a[1]*K*H^2+2*a[2]*N2H+8*a[2]*H^3
> -4*a[2]*K*H+4*a[3]*H^2-2*a[3]*K+a[6]*N2K+4*a[6]*K*H^2-2*a[6]*K^2;
> eq2:=4*a[5]*K*H+2*a[6]*NdotNH+4*a[6]*K*H^2;
> eq3:=-4*a[1]*H^4-4*a[2]*H^3-4*a[3]*H^2-4*a[4]*H
> -4*a[5]*K*H-4*a[6]*K*H^2+2*p;
> generalizedshape:=simplify(eq1+eq2+eq3);
> solve(generalizedshape);
```

Here `N2` and `NdotN` stand for ∇^2 and $\nabla\cdot\bar{\nabla}$, respectively. Calculation of them for H is given by the following codes.

```
> N2H:
> H:=
```

```
> K :=
> g:=
> ginv:= LinearAlgebra:-MatrixInverse(g);
> detg:= LinearAlgebra:-Determinant( g);
> sqrtdetg1:=sqrt(abs( detg));

> Hx:=diff(H,x);
> Ht:=diff(H,t);
> Ginv11 := ginv[1,1]:
> Ginv12 := ginv[1,2]:
> Ginv22 := ginv[2,2]:
> Ginv11Hx:=simplify(sqrtdetg*Ginv11*Hx,size):
> Ginv12Ht:=simplify(sqrtdetg*Ginv12*Ht,size):
> Ginv12Hx:=simplify(sqrtdetg*Ginv12*Hx,size):
> Ginv22Ht:=simplify(sqrtdetg*Ginv22*Ht,size):
> DGinv11Hx:=simplify((1/(sqrtdetg ))*diff(Ginv11Hx,x),size):
> DGinv12Ht:=simplify((1/(sqrtdetg ))*diff(Ginv12Ht,x),size):
> DGinv12Hx:=simplify((1/(sqrtdetg ))*diff(Ginv12Hx,t),size):
> DGinv22Ht:=simplify((1/(sqrtdetg ))*diff(Ginv22Ht,t),size):
> eq1:=simplify(DGinv11Hx+DGinv12Ht,size):
> eq2:=simplify(DGinv12Hx+DGinv22Ht,size):
> N2H:=simplify(eq1+eq2,size):


> NdotNH:
> h:=
> hinv:=LinearAlgebra:-MatrixInverse(h);
> H:=
> K:=
```

```
> g:=
> ginv:= LinearAlgebra:-MatrixInverse(g);
> detg:= LinearAlgebra:-Determinant( g);
> sqrtdetg:=sqrt(abs( detg));

> Hx:=diff(H,x);
> Ht:=diff(H,t);
> Ginv11 := hinv[1,1]:
> Ginv12 := hinv[1,2]:
> Ginv22 := hinv[2,2]:
> Ginv11Hx:=simplify(K*sqrtdetg*Ginv11*Hx,size):
> Ginv12Ht:=simplify(K*sqrtdetg*Ginv12*Ht,size):
> Ginv12Hx:=simplify(K*sqrtdetg*Ginv12*Hx,size):
> Ginv22Ht:=simplify(K*sqrtdetg*Ginv22*Ht,size):
> DGinv11Hx:=simplify((1/(sqrtdetg ))*diff(Ginv11Hx,x),size):
> DGinv12Ht:=simplify((1/(sqrtdetg ))*diff(Ginv12Ht,x),size):
> DGinv12Hx:=simplify((1/(sqrtdetg ))*diff(Ginv12Hx,t),size):
> DGinv22Ht:=simplify((1/(sqrtdetg ))*diff(Ginv22Ht,t),size):
> eq1:=simplify(DGinv11Hx+DGinv12Ht,size):
> eq2:=simplify(DGinv12Hx+DGinv22Ht,size):
> NdotNH:=simplify(eq1+eq2,size):
```

A.4 Position Vectors of the Surface

Here we give the codes that give the position vectors of the surface in $\mathbb{R}^3$.

```
> with(LinearAlgebra):
> Phi[1,1]:=;
> Phi[1,2]:=;
```

```
> Phi[2,1]:=;
> Phi[2,2]:=;
> Phi:= <<Phi[1,1]|Phi[1,2]>,<Phi[2,1]|Phi[2,2]>>:
> detPhi0:= LinearAlgebra:-Determinant( Matrix(Phi) ):
> detPhi:=simplify(%,symbolic);
> PhiInv:=Adjoint(Phi)/detPhi;
> A:=;
> B:=;
> PhiInvA:=MatrixMatrixMultiply(PhiInv,A):
> PhiInvAPhi:=MatrixMatrixMultiply(PhiInvA,Phi):
> Fx:=simplify(%,'size'):
> Fx11:=Fx[1,1];
> Fx12:=Fx[1,2];
> Fx21:=Fx[2,1];
> Fx22:=Fx[2,2];
> FiInvBB:=MatrixMatrixMultiply(FiInv,BB):
> FiInvBBFi:=MatrixMatrixMultiply(FiInvBB,Fi):
> Ft:=simplify(%,'size'):
> Ft11:= Ft[1,1];
> Ft12:=Ft[1,2];
> Ft21:=Ft[2,1];
> Ft22:=Ft[2,2];

> F11W:=int(Fx11, x)+W(t);
> F11Wt:=diff(%,t);
> eqW11:=F11Wt-Ft11:
> W11:==dsolve(eqW11,{W(t)});
> F11:=subs(W(t)=W11,F11W):
> F12W:=int(Fx12, x)+W(t);
> F12Wt:=diff(%,t);
```

```
> eqW12:=F12Wt-Ft12:
> W12:==dsolve(eqW12,{W(t)});
> F12:=subs(W(t)=W12,F12W):
> F21W:=int(Fx21, x)+W(t);
> F21Wt:=diff(%,t);
> eqW21:=F21Wt-Ft21:
> W21:==dsolve(eqW21,{W(t)});
> F21:=subs(W(t)=W21,F21W):
> F22W:=int(Fx22, x)+W(t);
> F22Wt:=diff(%,t);
> eqW22:=F22Wt-Ft22:
> W22:==dsolve(eqW22,{W(t)});
> F22:=subs(W(t)=W22,F22W):

> eqF11F22:=solve(F11+F22);
> eqRF11:=solve(Re(F11));
> eqCF12F21:=solve(Im(F12)-Im(F21));
> eqRF12F21:=solve(Re(F12)+Re(F21));

> y1:=-Im(F12);
> y2:=-Re(F12);
> y3:=-Im(F11);

> plot3d([y1,y2,y3],x=-a..a,t=-b..b);
```

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