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SOLUTION OF THE BOHR HAMILTONIAN FOR THE DAVIDSON POTENTIAL IN THE γ~ UNSTABLE REGION FOR ODD-EVEN NUCLEI

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HARITH AHMED SALIM AL-GBURI

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SOLUTION OF THE BOHR HAMILTONIAN FOR THE DAVIDSON POTENTIAL IN THE γ~UNSTABLE REGION FOR ODD-EVEN NUCLEI

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July 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

SOLUTION OF THE BOHR HAMILTONIAN FOR THE DAVIDSON POTENTIAL IN THE γ~UNSTABLE REGION FOR ODD-EVEN NUCLEI

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Master of Science in Physics

Advisor: Prof. Dr. İlyas İNCİ

June 2023

The energy equation giving the excitation energy spectra of odd-even nuclei exhibiting γ~unstable structures was obtained by the Nikiforov-Uvarov Method with the Davidson potential in the Collective Bohr Hamiltonian. By using the equation, the excitation energy spectra of the isotopes $^{187-195}$Ir having gamma-unstable structure were obtained and compared with the experimental data. Odd-even nuclei exhibiting structures around the critical point where the second-order phase transition occurs studied using the Collective model. For this, the Davidson potential with a free parameter in the Hamiltonian used. The Schrödinger equation was solved analytically for the first time using the Nikiforov-Uvarov Method. Potential parameters that theoretically most accurately predict the experimental spectrum of the studied nuclei have determined.

2023, 44 pages

Keywords: Nikiforov-Uvarov method, Davidson potential, γ~unstable structure, Odd-even nuclei

ÖZET

γ~ KARARSIZ BÖLGEDE DAVİDSON POTANSİYELİ İÇİN BOHR HAMİLTONİYENİNİN TEK-ÇİFT ÇEKİRDEKLER İÇİN ÇÖZÜMÜ

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Kararsız yapılar sergileyen tek-çift çekirdeklerin uyarılma enerji spektrumlarını veren enerji denklemi Collective Bohr Hamiltoniyen'deki Davidson potansiyeli ile Nikiforov-Uvarov Yöntemi ile elde edildi. Denklem kullanılarak gama kararsız yapıya sahip $^{187-195}$Ir izotoplarının uyarılma enerji spektrumları elde edilmiş ve deneysel verilerle karşılaştırılmıştır. Kolektif model kullanılarak incelenen ikinci dereceden faz geçişinin meydana geldiği kritik nokta etrafında yapılar sergileyen tek-çift çekirdekler. Bunun için Hamiltonyen'de serbest parametreli Davidson potansiyeli kullanıldı. Schrödinger denklemi ilk kez Nikiforov-Uvarov Metodu kullanılarak analitik olarak çözüldü. Çalışılan çekirdeklerin deneysel spektrumunu teorik olarak en doğru şekilde tahmin eden potansiyel parametreler belirlenmiştir.

2023, 44 sayfa

Anahtar Kelimeler: Nikiforov-Uvarov metodu, Davidson potansiyeli, γ~kararsız yapı, Tek-çift çekirdekler

PREFACE AND ACKNOWLEDGEMENTS

Praise be to God who enabled me to finish this dissertation. I dedicate this work to my dear mother, father, and brothers, and I thank Prof. İlyas İNCİ who is credited with completing this work. I do not forget my friends and loved ones who supported me in this work.

Harith Ahmed Salim AL-GBURI

Çankırı-2023

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LIST OF SYMBOLS

$\langle \beta^2 \rangle$	Average of β^2 over (β)
N	Boson number
β	Distortion factor
σ	Ductility factor
g_B	Effective boson is estimated g=Z/A
B(E2)	Electric quadrupole transition matrix elements in $e^2 \cdot b^2$ units
Rs	Energy rate value of Rs
H	Hamiltonian operator
s.d	Hypotensive effects
V	Interaction between bosons
Ir	İridium isotopes (Z=77)
M	Magnetic dipole moment
A	Mass number
δ	Mixing ratio in eb/μ_N
N_N	Normalization constant
n_d	Number of d-boson
n_s	Number of s-boson
Δ	Pairing gap energy
μ	Particle mass
h	Plank constant
χ	Quadrupole sign parameter
κ	Quadrupole-quadrupole interaction strength
ε	Reduced energy
t	Scale factor
β_0	The point where the potential is minimum
θ_i	Three Euler angles i = 1,2,3
J	Total angular momentum
W	Tungsten isotopes (Z=77)
Ψ	Wave function
$\mathcal{D}$	Wigner function of Euler angles (i = 1,2,3)
n_w	Wobbling quantum number
u(γ)	γ potential due to the variable
u(β)	βPotential due to the variable

LIST OF ABBREVIATIONS

IBM-1	The first interacting boson model
IBM	The interacting boson model
SU(3)	The rotational limit
SU(5)	The vibration limit
O(6)	γ-unstable vibrator

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1. INTRODUCTION

The equilibrium form of atomic nuclei serves as a defining characteristic. There are situations where the system is extremely unstable and goes through a phase transition between two different shapes, even though these shapes frequently correspond to stable nuclei. Here, the issue is how to characterize the nuclei's structure, particularly at the phase transition point and in the vicinity of the phase transition. The interacting boson model (IBM), an algebraic model, has been used to examine this issue (Iachello and Arima 1987, Iachello and Levine 1995). According to this method, certain forms (phases) are equivalent to dynamic symmetries of some G algebraic structures (Iachello 2001).

One of the most fascinating areas of nuclear physics is the collective motion of the atomic nucleus, which may be studied to better understand the characteristics and behavior of the atomic nucleus and thereby characterize the nuclear structure. Collective motion, a combination of rotational and vibrational motion in the atomic nucleus, is governed by the long-range quadrupole-quadropole interaction that causes deformation and the short-range coupling interaction that favors spherical forms. The atomic nucleus' form is guaranteed by the harmony of these forces. The atomic nuclei display shape-phase transition behavior as a function of the amount of protons and neutrons that make up the nucleus as a result of the degradation of the balance between these forces with an increase in the number of nucleons.

The Bohr collective model The Bohr collective model (Bohr 1952), which depicts the collective motion of the nucleus, and the Interacting boson model (IBM), which views the nucleus as a vibrating bosonic liquid droplet (Iachello and Arima 1987), both address shape phase transitions in atomic nuclei. In all methods, the spherical vibrator, axisymmetric rotor with prolest and obleyt deformed structures, and lastly the non-axially deformed rotor with triaxial (Davydov and Filippov 1958, Davydov and Rostovsky 1959) and γ-unstable (Wilets and Jean 1956) structures are all present in the atomic nucleus.

Dynamic symmetries have provided a useful tool for describing the properties of many physical systems (nuclei molecules atomic structures etc). By definition (Iachello 1979), dynamic symmetry is when the Hamilton operator H can be written in terms of the Ci Casimir operators of an algebraic chain $G \supset G' \supset G'' \supset \cdots$. The most noteworthy examples are the vibron model in molecular physics and the dynamical symmetries of the interacting boson model in nuclear physics (Iachello and Arima, 1987)(Iachello and Levine 1995). By examining the algebraic character of the issue, such dynamic symmetries can be quickly identified.

All potential dynamical symmetries of algebra G can be discovered by dissecting it into all subalgebraic chains. For instance, there are three potential dynamic symmetries in the interacting boson model labeled with the first subalgebra U(5), SU(3), and SO(6) occurring in the chain $G \equiv U(6)$. Dynamic symmetries produce all findings for observables in explicit analytical form and are specifically about solvable issues. As a result, it is very helpful in analyzing experimental data and has produced significant breakthroughs. (Cizewski *et al.* 1978, Iachello 2000).

The interacting boson model provides an alternative explanation of nuclear collective excitations of an algebraic nature as opposed to geometric models. The many electromagnetic transitions and low-energy collective states of the even-even nuclei might be accurately described by this realistic theoretical model. In the interacting boson model's original formulation (IBM1), nuclei are viewed as systems made up of bosons with angular momentum L=0 (s-bosons) or L=2 (d-bosons) (Iachello and Arima 1987).

A Hamiltonian is commonly used to describe the system of bosons up to two-body interactions, where the number of bosons is equal to half the number of valence fermions (N=n/2). The boson number is both rotationally invariant and conserved. The U(6) group structure contains the symmetries of the s and d bosons. Geometrically dynamic symmetries interacting boson model U(5) to spherical vibration, SU(3) to axisymmetric rotation and O(6) It corresponds to γ-unstable (γ-soft γ-independent) rotation (Kotb 2016).

The quantity of protons and neutrons in an atomic nucleus causes phase transitions to occur. These phase changes aren't always of the thermodynamic variety; sometimes they take the form of quantum phase changes at equilibrium between ground-level and low-energy structures. The idea of "critical point symmetries," which describes the structure of nuclei at phase transition points, became the focus of a new branch of study as a result (Casten 2006). In his study analyzing the critical point behavior of nuclei, Iachello (Iachello 2000, Iachello 2001, Iachello 2003) revealed three new dynamic symmetries termed E(5), X(5), and Y(5).

E(5) the critical point in the transition from spherical vibrator to γ-unstable U(5) $\leftrightarrow$ O(6) shapes X(5) the critical point between spherical vibrator and axially deformed U(5)$\leftrightarrow$SU(3) shapes point and Y(5) defines the critical point between axially deformed and triaxially deformed shapes. The Z(4) and Z(5) models define the critical point between the axially deformed oblate and the axially deformed prolate respectively and their solutions are obtained by taking .

2. LITERATURE REVIEW

2.1 The Interacting Boson Model

In the interacting boson model (IBM) framework (Arima 1981), It complements the Bohr Hamiltonian's algebraic description of atomic nuclei and is known that in its geometrical limit (Arima 1981) obtained by using coherent states (Arima 1981), terms of the form 22 and/or more complicated terms appear (Van Roosmalen *et al.* 1983), in addition to the typical term of the kinetic energy, 2. As a result, phrases with two or more complicated words will modify the kinetic energy term, enabling the search for a modified Bohr Hamiltonian.

This data allows for the consideration of a Bohr Hamiltonian whose mass depends on the collective variable. Recent research has focused on position-dependent effective masses (Quesne and Tkachuk 2004), and the three-dimensional harmonic oscillator is one of several Hamiltonians that have been appropriately generalized (Bagchi *et al.* 2005) to include position-dependent effective masses. These Hamiltonians are known to be soluble using methods of supersymmetric quantum mechanics (SUSYQM) (Cooper *et al.* 1995, Cooper and Sukhatme 2001).

Davidson potential in a Bohr Hamiltonian (Davidson 1932) is used to show how it can be generalized to include a mass dependent on $\beta = \beta_\circ/(1 + a^2)^2$ where B_0 and an are constants. This method will be referred to as the Davidson model of deformation-dependent mass (DDM) We'll look at three examples of potentials where exact variable separation is possible:

(i) Potentials suitable for defining vibrational and near-vibrational nuclei that are independent (Wilets and Jean 1956) of the collective variable (an angle indicating departure from axial symmetry).

(ii) Potentials of the kind (Fortunato, 2005, Wilets and Jean, 1956) $v(\beta,\gamma) = u(\beta) + w(\gamma)/\beta^2$ where $w(\gamma)$ has a deep minimum at $\gamma = 0$ and $u(\beta)$ is the Davidson potential. These potentials correspond to axially symmetric prolate deformed nuclei.

(iii) Potentials of the type $v(\beta,\gamma) = u(\beta) + w(\gamma)/\beta^2$, where $w(\gamma)$ has a deep minimum at $\gamma = \pi/6$ and $u(\beta)$ is the Davidson potential (Davidson, 1932), which corresponds to triaxial nuclei (Davydov and Filippov 1958,Davydov and Rostovsky 1959).

For all three examples, analytical findings for spectra and B(E2) transition rates will be given, using the customary approximations in each limit (Iachello 2001,Meyer-ter-Vehn 1975) whereas the first two situations, for which the vast majority of experimental data is provided, will be compared to the outcomes. In Ref. (Bonatsos *et al*. 2010) a unique solution for -unstable nuclei was previously provided.

Supersymmetric quantum mechanics methods are applied. These methods are an exact match for Infeld and Hull's factorization method (Infeld and Hull 1951) Adistorted shape invariance condition is imposed to ensure the Hamiltonian's integrability (Bagchi *et al*, 2005).

It should be noted that the idea of a nonconstant mass within the Bohr Hamiltonian framework has long been employed in relevant mean-field calculations (Libert *et al*, 1999) as well as in numerical solutions of a modified Bohr Hamiltonian (Kumar and Baranger 1967).The primary distinction between the current work and these older methods is that here, analytical solutions are obtained. Additionally in the current scenario there are only a couple or three free parameters, and SUSYQM determines how the mass functions in relation to the deformation for the chosen potential.

They introduced a brand-new class of dynamic symmetries in (Iachello 2000). It is suggested that spectra of systems undergoing a (second order) phase transition between the algebraic structures U(n-1) and SO(n), as well as, generally.

The critical point of the spherical to axially deformed shape phase transition in nuclei has an approximative solution in (Iachello 2001) The zeros of Bessel functions of irrational order are used to express the Hamiltonian's eigenvalues.

The nucleus has been compared to a drop of liquid due to the tightly packed particle configuration and relatively abrupt nuclear border (Bohr 1952) This model explains several static features of the nucleus and has been widely used in the theory of nuclear processes Therefore, if the energy of the nuclear droplet is described as a sum of surface energy volume energy and electrostatic energy the key characteristics of the empirical binding energies may be easily understood.

The spectrum of (Casten and Zamfir 2000) discusses the explored how these groups of symmetries have influenced nuclear structure evolution.

The key point of a phase transition from a vibrator to an axial rotor is proved by the first empirical manifestation of the previously anticipated analytic description of nuclei. to be ^{152}Sm and other N=90 isotones (Casten and Zamfir 2001).

The collective Bohr equation with a Coulomb- and Kratzer-like γ-unstable potential in quadrupole deformation space has an analytical solution that is shown (Fortunato and Vitturi 2003) In closed form, eigenvalues and eigenfunctions are provided, and transition rates are computed for the two scenarios It is discussed how the algebraic structure $SO(2,1) \times SO(5)$ corresponds.

In order to explain γ-soft prolate axial rotors, new analytic solutions to the quadrupole collective Bohr Hamiltonian are proposed in (Fortunato and Vitturi 2004) These solutions take advantage of a rough separation of the and variables. The model potential is made up of two terms a β-dependent term treated as a harmonic oscillator and a γ-dependent term taken as either a Coulomb-like or a Kratzer-like form. In particular, it is conceivable to provide a one-parameter paradigm for an axial rotor that is soft that can be applied to the spectrum of ^{234}U with a great deal of agreement.

It can use the well-known approximation solution, for the molecular eigenfunction $\Psi = \Phi(x, r) \cdot P(r) \cdot \Theta$ (Davidson 1932) Small terms that are easily referred to as perturbations separate the equation it satisfies from the exact molecular equation.

Fluctuation reflects vibration about a rigid non-spherical yet gamma-unstable nucleus at one extreme and a spherical equilibrium at the other (Elliott *et al*. 1986) Results from the interacting boson model are compared to spectra and E2 transition rates.

Hamiltonians, electromagnetic transition rates, and full rotation-vibrational spectra of diatomic molecules and nuclei with Davidson interactions are acquired in (Rowe and Bahri 1998) Analytical findings are produced using dynamical symmetry techniques for diatomic molecules and a liquid-drop model of the nucleus. Under the constraints of the microscopic symplectic model, numerical solutions are discovered for a many-particle nucleus with quadrupole Davidson interactions.

Differential equations' dynamic supersymmetries are defined As an illustration of a circumstance in which the dynamic supersymmetry $j = 3/2$ might take place. Here is the scenario where a liquid drop with quadrupole deformation is paired with a particle that has $OSp(5/4)$ (Iachello 2005)A unique solution, dubbed E(5/4).

Recently, it was proposed that the coupling of a j=3/2 fermion to an E(5) core, represented by the symbol E(5/4) is the first known instance of a Bose-Fermi critical point symmetry (Fetea *et al*. 2006). In order to test E(5/4) for the first time, a decay experiment was carried out to investigate levels in 135Ba where the final neutron can occupy the 2d (3/2) orbit since it has been discovered that ^{134}Ba is an empirical manifestation of E(5). Significant areas of agreement and disagreement are shown by the comparison. Also shown is a comparison of the data and calculations using the shell model and the interacting boson-fermion approximation.

It has been assumed that the system consists of a single nucleon in the $j = 3/2$ single particle orbit connected to a γ-unstable even-core in order to study the features of odd nuclei inside the collective model (Inci and Sonkaya 2018) The matching Bohr Hamiltonian for the even core makes use of the Davidson potential The electric quadrupole transition ratios and the excitation energy spectrum have been measured. The findings have been applied to forecast experimental data for a few chosen odd isotopes.

They look into boson-fermion systems' phase transitions in (Alonso *et al*. 2007a). To describe odd nuclei at the pivotal moment when spherical behavior changes to unstable behavior they offer an analytically solvable model. In the model an unpaired particle that is supposed to be travelling in the three single-particle orbitals interacts with a boson core defined within the Bohr Hamiltonian.

They look into boson-fermion systems' phase transitions in (Alonso *et al*. 2007b). To explain odd nuclei at the critical juncture where their behavior transitions from spherical to -unstable, they offer an analytically solvable model [$E(5/12)$].In the scenario, an unpaired particle that is supposed to be travelling in the three single-particle orbitals interacts with a boson core defined within the Bohr Hamiltonian $j = 1/2,3/2,5/2$.

It is looked at how the underlying even-even core nuclei transition from having spherical to distorted, unstable forms, causing a phase shift in odd nuclei (Bes 1959) The three single-particle orbitals $j = 1/2,3/2$, and 5/2 are thought to be where the odd particle is thought to be travelling. An analytical solution to the associated Bohr Hamiltonian, known as E(5/12) is developed at the phase transition's critical point. Present are energy spectrum, electromagnetic transitions, and moments. The interacting boson-fermion model is also used to approach the same issue (IBFM). There are two Hamiltonians used. The first one is improvised to reflect the circumstances in the E(5/12) model. When the boson portion approaches the O(6) requirement, the second one causes the $\mathrm{O}^{B}(6) \otimes \mathrm{U}^{F}(12)$ symmetry to occur. With this Hamiltonian, the full transition line is investigated, focusing in particular on the critical point. At the crucial point, both IBFM estimates agree with the findings of E(5/12).

The wave functions that depict -unstable vibrations have an equation for the dependent portion that can be solved (Bes 1959). A linear combination of wave functions from a "basic set" constitutes the general solution. The polynomials in cos $3y$ are used to express the coefficients in this linear combination. Some of these solutions' effects are discussed. The explicit solutions for the situations $\lambda \leqq 9$ are created, and the equations for the coefficients are given in the appendix for $I \leqq 9$.

Precise solutions of the Bohr Hamiltonian with a five-dimensional square well potential, whether they exist independently or in connection to a fermion via the five-dimensional spin-orbit relationship are thought to exhibit a novel type of dynamical symmetry known as Bose-Fermi dynamical symmetry (Caprio and Iachello 2007). The solutions act as a springboard for studies on even-even and odd-mass nuclei close to the critical point of the stable to deformed phase transition.

Valued spectroscopic data and level schemes for the isotopes ^{187}Hf, ^{187}Ta, ^{187}W, ^{187}Re, ^{187}Os, ^{187}Ir, ^{187}Pt, ^{187}Au, ^{187}Hg, ^{187}Tl, ^{187}Pb, ^{187}Bi, and ^{187}Po are presented in (Basunia 2009). This assessment for the value of 187 replaces the preceding assessments for the value of 187 made by R.B. Firestone (1991Fi02) and C.M. Baglin (1999Ba24) and made available in Nuclear Data Sheets 86 and 487, respectively (1999).

In (Johnson and Singh 2017), for 13 known mass 189 nuclides, they provide reviewed experimental data (Hf, Ta, W, Re, Os, Ir, Pt, Au, Hg, Tl, Pb, Bi, Po). The latest effort includes a new 189Hf nuclide while including structural and decay data from 25 new and primary articles since the 2003Wu02 publication.

Except for 189Au and 189Hg, all nuclides have new data. Furthermore despite the fact that there haven't been any new publications since 2003Wu02 numerous older datasets have been altered for -decay Q values and conversion coefficients.

The information supplied in earlier evaluations has been greatly improved corrected and added to in the current version of the nuclear structural characteristics of the A = 191 mass chain's nuclides (Vanin *et al*. 2007).

For the following elements: ^{193}Ta, ^{193}W, ^{193}Re, ^{193}Os, ^{193}Ir, ^{193}Pt, ^{193}Au, ^{193}Hg, ^{193}Tl, ^{193}Pb, ^{193}Bi, ^{193}Po, ^{193}At, and ^{193}Rn, Level schemes and analyzed spectroscopic data are provided in (Basunia 2017).

The ENSDF data file has been amended updated and updated data on the experimental structure and decay of every nuclide with mass A=195 (Re, Os, Ir, Pt, Au, Hg, Tl, Pb, Bi, Po, At, Rn) (Huang and Kang 2014).

We quickly go over the fundamentals of the formalism required to handle effective masses depending on the position in order to be thorough The generalization of the kinetic energy term is the major issue We provide a clear solution to this problem.

2.1.1. Boson subalgebras I

A subalgebra of $U(6)$ can be obtained by considering the 25 operators

$$\begin{aligned} G_0^{(0)}(d,d) &= [d^+ \times d]_0^{(0)} \\ G_\mu^{(1)}(d,d) &= [d^\dagger \times d]_\mu^{(1)} \\ G_\mu^{(2)}(d,d) &= [d^\dagger \times d]_\mu^{(2)} \\ G_\mu^{(3)}(d,d) &= [d^\dagger \times d]_\mu^{(3)} \\ G_\mu^{(4)}(d,d) &= [d^\dagger \times d]_\mu^{(4)} \end{aligned} \tag{2.1}$$

These operators close under the algebra U(5). The ten operators

$$\begin{aligned} G_\mu^{(1)}(d,d) &= [d^\dagger \times d]_\mu^{(1)} \\ G_\mu^{(3)}(d,d) &= [d^\dagger \times d]_\mu^{(3)} \end{aligned} \tag{2.2}$$

form a subalgebra of $U(5)$ the orthogonal algebra in five dimensions $O(5)$ The three operators

$$G_{\mu}^{(1)}(d,d) = [d^{\dagger} \times d]_{\mu}^{(1)} \tag{2.3}$$

close under the algebra of $O(3)$ the rotation algebra Finally the single component

$$G_{0}^{(1)}(d,d) = [d^{\dagger} \times d]_{0}^{(1)} \tag{2.4}$$

generates the algebra $O(2)$ of rotations around the z-axis This yields a possible chain of algebras

$$\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{O}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2), \qquad \text{(I)} \tag{2.5}$$

which we shall denote by I.

2.1.2. Boson subalgebras II

While the first chain of algebras, I, can be derived almost by inspection, the second chain is more difficult to obtain, since it involves linear combinations of the operators in Equation (2.5). Consider the operators

$$\begin{aligned}
&G_{0}^{(0)}(s,s) + \sqrt{5}G_{0}^{(0)}(d,d) = [s^{\dagger} \times \tilde{s}]_{0}^{(0)} + \sqrt{5}[d^{\dagger} \times d]_{0}^{(0)} \\
&G_{\mu}^{(1)}(d,d) = [d^{\dagger} \times d]_{\mu}^{(1)} \\
&G_{\mu}^{(2)}(d,s) + G_{\mu}^{(2)}(s,d) \pm \tfrac{1}{2}\sqrt{7}G_{\mu}^{(2)}(d,d) \\
&\ = [d^{\dagger} \times \tilde{s} + s^{\dagger} \times d]_{\mu}^{(2)} \pm \tfrac{1}{2}\sqrt{7}\,\left[d^{\dagger} \times \tilde{d}\right]_{\mu}^{(2)}
\end{aligned} \tag{2.6}$$

By using the commutation relations Equation (2.4) one can show that the nine operators Equation (2.6) close under commutation and form the algebra of $U(3)$. This occurs for both signs ($\pm$) in Equation (2.6) The operator $G_{\mu}^{(2)}(d,s)+G_{\mu}^{(2)}(s,d)\pm\frac{1}{2}\sqrt{7}G_{\mu}^{(2)}(d,d)$ is proportional to the electric quadrupole operator. In a geometric picture, this is in turn related to quadrupole deformations of the nucleus, as it will be shown later. The two signs correspond to positive $(+)$ and negative $(-)$ quadrupole moments. Since most nuclei have negative quadrupole moments, the negative sign is usually taken in Equation (2.6). Furthermore, the operator $G_{0}^{(0)}(s,s)+\sqrt{5}G_{0}^{(0)}(d,d)$ is the total number operator, $\hat{N}$. For applications to nuclei, where the total N is conserved, the eigenvalues of $\hat{N}$ are fixed. Thus, rather than the algebra $\mathrm{U}(3)$ it is more convenient to consider the algebra in which the operator $\hat{N}$ has been deleted. This is formed by the eight operators

$$\begin{aligned}
&G_{\mu}^{(1)}(d,d)=[d^{\dagger}\times d]_{\mu}^{(1)}\\
&G_{\mu}^{(2)}(d,s)+G_{\mu}^{(2)}(s,d)-\tfrac{1}{2}\sqrt{7}G_{\mu}^{(2)}(d,d)\\
&=\left[d^{\dagger}\times\tilde{s}+s^{\dagger}\times\tilde{d}\right]_{\mu}^{(2)}-\tfrac{1}{2}\sqrt{7}\left[d^{+}\times\tilde{d}\right]_{\mu}^{(2)}
\end{aligned}\tag{2.7}$$

and turns out to be the algebra $\mathrm{SU}(3)$ of special unitary transformations in three dimensions. Subalgebras of (2.16) are now that of $\mathrm{O}(3)$, formed by,

$$G_{\mu}^{(1)}(d,d)=[d^{\dagger}\times d]_{\mu}^{(1)}\tag{2.8}$$

and that of $\mathrm{O}(2)$, formed by

$$G_{0}^{(1)}(d,d)=[d^{\dagger}\times d]_{0}^{(1)}\tag{2.9}$$

Thus, a second possible chain of subalgebras is

$$\mathrm{U}(6)\supset\mathrm{SU}(3)\supset\mathrm{O}(3)\supset\mathrm{O}(2),\qquad\text{(II)}.\tag{2.10}$$

$$G_{\mu}^{(1)}(d,d)=[d^{\dagger}\times d]_{\mu}^{(1)}\tag{2.11}$$

2.1.3. Boson subalgebras III

A third possibility is to consider the 15 operators

$$\begin{aligned} G_{\mu}^{(1)}(d,d) &= [d^{\dagger} \times d]_{\mu}^{(1)} \\ G_{\mu}^{(3)}(d,d) &= [d^{\dagger} \times d]_{\mu}^{(3)} \\ G_{\mu}^{(2)}(d,s) + G_{\mu}^{(2)}(s,d) &= \Big[d^{\dagger} \times \tilde{s} + s^{\dagger} \times \chi_{\mu}^{(2)} \\ &- \end{aligned} \tag{2.12}$$

These operators close under the algebra $\mathrm{O}(6)$ of orthogonal transformations in six dimensions. Deleting the five operators $G_{\mu}^{(2)}(d,s) + G_{\mu}^{(2)}(s,d)$, one obtains the ten operators

$$\begin{aligned} G_{\mu}^{(1)}(d,d) = [d^{\dagger} \times d]_{\mu}^{(1)} \\ G_{\mu}^{(3)}(d,d) = [d^{\dagger} \times d]_{\mu}^{(3)} \end{aligned} \tag{2.13}$$

which form, as above, the algebra of $\mathrm{O}(5)$. Subalgebras of $\mathrm{O}(5)$ are that of $\mathrm{O}(3)$ and $\mathrm{O}(2)$

$$\begin{aligned} G_{\mu}^{(1)}(d,d) = [d^{\dagger} \times d]_{\mu}^{(1)} \\ G_{0}^{(1)}(d,d) = [d^{\dagger} \times d]_{0}^{(1)} \end{aligned} \tag{2.14}$$

Thus, a third chain is

$$\mathrm{U}(6) > \mathrm{O}(6) \supset \mathrm{O}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2), \qquad \text{(III).} \tag{2.15}$$

If one wants to add the rotation algebra, O(3), as a subalgebra, one may demonstrate that the three subalgebras I, II, and III are the only ones that can exist. In actuality, one has taken into account U(5), U(3), and O(6) beginning from U(6).

The algebra of $\mathrm{O}(6)$ is isomorphic to $SU(4)$ and thus all possibilities leading from $U(6)$ to the rotation algebra $\mathrm{O}(3) \approx \mathrm{SU}(2)$ have been considered. In conclusion, there are three and only three possibilities

$$\begin{array}{lll} & \nearrow \ \mathrm{U}(5) \supset \mathrm{O}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2) & \text{(I)} \\ \mathrm{U}(6) & \rightarrow \mathrm{SU}(3) \supset \mathrm{O}(3) \supset \mathrm{O}(2) & \text{(II)} \\ & \searrow \ \mathrm{O}(6) \supset \mathrm{O}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2) & \text{(III)} \end{array} \tag{2.16}$$

Before closing this section, it is worthwhile mentioning that, for operators involving linear combinations of the generators of $U(6)$, there are phase ambiguities similar to those discussed in Sect. 1.4.3. We shall call the choice made here the 'standard' choice, and keep it throughout. Other choices have also been used (Castaños *et al*. 1979).

$$H_B = H_B + H_F + V_{BF} \tag{2.17}$$

$$H_B = -\frac{h^2}{28}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\left\{\frac{1}{\sin 3x}\frac{\partial}{\partial\gamma x}\sin^{3x}\frac{\partial}{\partial s^s} - \frac{1}{4}\sum_{k=1}^{3}\frac{Q_4^2}{\sin^2\left(\gamma-\frac{4\pi}{\beta}\right)}\right\}\right] + V(\beta,\gamma) \tag{2.18}$$

For γ-unstable case $= V(\beta,\gamma) = V(\beta)$ and

$$\left[\frac{1}{\sin\ 3r}\frac{\gamma}{\partial\gamma}\sin\ 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4}\sum_{k=1}^{3}\frac{Q_i^2}{\sin^2\left(\gamma-\frac{2\pi}{\gamma}k\right)}\right]\Phi(\gamma,\theta) = -\Lambda\Phi(\gamma,\theta \tag{2.19}$$

Then

$$H_B = \frac{-\hbar^2}{20}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Delta}{\beta^2}\right] + V(\beta) \tag{2.20}$$

where $\Lambda = \tau_B(\tau_B + 3)$ and $\tau_B = 3V_\Delta + \lambda, L = \lambda, \lambda + 1, -, 2\lambda$ $(2\lambda - 1)$ maris) H_F is the hamiltonian of single particle and can taken be zero.

$$\hat{V}_{BF} = g(\beta)\hat{K}.2\langle\hat{\Lambda} \cdot \hat{\Sigma}\rangle = \hat{K}.g(\beta)[\tau_1(\tau_1 + 3) - \tau_B(\tau_B +]) - \tfrac{7}{4}] \tag{2.21}$$

Then the full Hemitisian in eq. (2.17) is.

$$H = H_B + V_{BF} = -\frac{\hbar^2}{28}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Delta}{\beta^2}\right] + \frac{R}{\beta}(\beta)[\tau_1(\varepsilon_1 + 3) - \tau_8(\tau_\theta\,\partial)] + v(\beta) \tag{2.22}$$

$$H\Psi = E4 \tag{2.23}$$

$$\psi = F(\beta) \cdot \Phi(\gamma, \theta_i; n) \tag{2.24}$$

$$H \cdot F(\beta) = EF(\beta) \tag{2.25}$$

$$\left\{\frac{-\hbar^2}{2\beta}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2}\right] + V(\beta) + Kg(\beta)\left[\tau_1(\tau_1 + 3) - \tau_8(\tau_B + 3) - \frac{z}{4}\right]\right\}_F F(\beta) = E.F(\beta) \tag{2.26}$$

$\frac{2B}{\hbar^2}E = \varepsilon, \frac{2b}{\hbar^2} \cdot V(\beta) = \nu(\beta)$ and $g(\beta) = \frac{\hbar^2}{28}$ = corrtat.

$$-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta'\frac{\partial}{\partial\beta}F(\beta) + \frac{\Delta}{\beta^2}F(\beta) + N(\beta) \cdot F(\beta) + \mathbb{E}\left[\tau_1(\tau_1 + 3) - \tau_g(\tau_\beta + 3) - \frac{7}{4}\right] f(\beta) = \varepsilon F(\beta) \tag{2.27}$$

$$-F''(\beta) - \frac{4}{\beta}F' + \left[-\varepsilon + \frac{\Lambda}{\beta^2} + U(\beta) + \underline{k}\left[\left(\tau_1(\tau_1 + 3) - \tau_B(\tau_\theta + 3) - \frac{7}{4}\right]\right] F(\beta) = 0 \tag{2.28}$$

$\Lambda = \tau_B(\tau_\beta + 3)$ and lets say. $\sigma_a = \tau_1(\tau_1 + 3) - \tau_B(\tau_B + 3) - \dfrac{7}{4}$

$$F''(\beta) + \frac{4}{\beta}F' + \left[\varepsilon - u(\beta) - \frac{\Delta}{\beta^2} - K\sigma_2\right]F(\beta) = 0 \qquad (2.29)$$

Davidsan potential is $\omega_\Delta(\beta) = \beta^2 + \frac{{\beta_0}^4}{\beta^2}$ is we use in eg.(2.29) weyet.

$$F''(\beta) + \frac{4}{\beta}F'(\beta) + \left[\varepsilon - \beta^2 - \frac{\beta_1^4}{\beta^2} - \frac{A}{\beta^2} - \bar{k}\epsilon_{]}\right]F(\beta) = 0 \qquad (2.30)$$

lets simelify wit $\beta_0^i + \Lambda = \beta_0^4 + \tau_8(\tau_0 + 3) = \alpha$

$$F''(\beta) + \frac{4}{\beta}F'(\beta) + \left[\varepsilon - \beta^2 - \frac{\alpha}{\beta^2} - \mathrm{K}\sigma F(\beta) = 0\right. \qquad (2.31)$$

$$\varepsilon' = \varepsilon - \mathrm{K}\sigma_a \Rightarrow F''(\beta) + \frac{4}{\beta}F'(\beta) + \left[\frac{\varepsilon' \cdot \beta^2 - \beta^4 - \alpha}{\beta^2}\right]F(\beta) = 0 \qquad (2.32)$$

From N.U. method:

$$\psi''(s) + \frac{\tau(\tau)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi = 0 \qquad (2.33)$$

where $\tilde{\sigma}(s)$ and $\sigma(s)$ are at mast second-degree, $\tilde{\tau}(s)$ is at mart ginst -degree edynomils.
our eguation (2.32):

$$F''(\beta) + \frac{4}{\beta}F'(\beta) + \left[\frac{\xi'\beta^2 - \beta^L - \alpha}{\beta^2}\right]F(\beta) = 0$$

If wer compare (2.32) and (2.33) whe fornd

$$\tilde{\tau}(\beta) = 4$$
$$\sigma(\beta) = \beta$$
$$\tilde{\sigma}(\beta) = -\beta^4 + \varepsilon'\beta^2 - \alpha$$

Hene $\tilde{\sigma}(\beta)$ fourth-degree polynompl, vee could aot solueit with N.U. methide We heve to depes decrease the degree of $\tilde{\delta}(\beta)$ to secend-degree. Lets chare $\beta^2 = x$ then $2\beta \cdot \alpha\beta = dx \Rightarrow \alpha\beta = \frac{1}{2\sqrt{x}} \cdot dx$

$$F''(\beta) = \frac{d^2F^2}{\partial\beta^2} \Rightarrow \frac{\partial}{\partial\beta} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial\beta} \Rightarrow \frac{\partial}{\partial\beta} = \sqrt{x} \cdot \frac{\partial}{\partial x}$$

$$\frac{d^2F}{\partial\beta^2} = \frac{d}{d\beta} \cdot \frac{d}{d\beta} \cdot F(\beta) = \left(2\sqrt{x}\frac{d}{dx}\right)\left(2\sqrt{x}\frac{d}{dx}\right)F(x)$$

$$F''(\beta) = 4xF''(x) + 2F'(x)$$

$F'(\beta) = 2\sqrt{x}F'(x)$. Wey we use $\beta^2 = x$ then eq. (2.34)

$$4xF''(x) + 2F'(x) + \frac{4}{\sqrt{x}} \cdot 2\sqrt{x}F'(x) + \left[\frac{\varepsilon' \cdot x - x^2 - \alpha}{x}\right]F(x) = 0$$

$$4F''(x) + 10F'(x) + \left[\frac{\varepsilon' \cdot x - x^2 - \alpha}{x}\right]F(x) = 0$$

$$F''(x) + \frac{10}{4x}F'(x) + \left[\frac{-x^2 + \varepsilon' \cdot x - \alpha}{4x^2}\right]F(x) = 0$$

$$F''(x) + \frac{5}{2x}F'(x) + \left[-\frac{x^2 + \varepsilon' x - \alpha}{4x^2}\right]F(x) = 0 \qquad (2.34)$$

Now, again compcotin wath eq. (2.33) we find:

$$\tilde{\tau}(x) = 5$$

$$\sigma(x) = 2x$$

$$\sigma^2(x) = 4x^2$$

$\tilde{\sigma}(x) = -x^2 + \varepsilon' x - \alpha$

$\tilde{\sigma}(x) = -x^2 + \varepsilon' x - \alpha$ (novy it is second-depree elyn.micll).

We cen now solve this equation vurth Nilliforow-Uverew methad

$$\pi(x) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}^2}{2}\right)^2 - \tilde{\sigma} + k\sigma'} \tag{2.35}$$

$$\pi(x) = \frac{2-5}{2} \pm \sqrt{\left(\frac{2-5}{2}\right)^2 - (-x^2 + \varepsilon' x - \alpha) + k.2x}$$

$$\pi(x) = \frac{-3}{2} \pm \sqrt{\frac{9}{4} + x^2 + (2k - \varepsilon')x + \alpha} \tag{2.36}$$

Discriminant of the exeression under squere-root sign shold be zero .

$$x^2 + (2k - \varepsilon')x + \left(\alpha + \frac{9}{4}\right)$$

$$\Delta = 0 = b^2 - 4ac = (2k - \varepsilon')^2 - 4 \cdot \left(\alpha + \frac{9}{4}\right) = 0$$

$$(2k - \varepsilon')^2 = 4\alpha + 9 \Rightarrow 2k - \varepsilon' = \pm\sqrt{(4\alpha + 9)}$$

$$k = \frac{\varepsilon'}{2} \pm \sqrt{\alpha + \frac{9}{4}} \tag{2.37}$$

$$\pi(x) = -\frac{3}{2} \pm \sqrt{\left(x + \frac{(2k - \varepsilon')}{2}\right)^2} = -\frac{3}{2} \pm x + \left(\frac{2kt - \varepsilon'}{2}\right) \tag{2.38}$$

Whene $k\pm=\frac{\varepsilon'}{2}\pm\sqrt{\alpha+\frac{9}{4}}$

$$\pi(x) = -\frac{3}{2}\pm x\pm\sqrt{x+\frac{4}{4}} \quad (2.39)$$

$$\tau(x) = \hat{\tau}+2\pi(x) \quad (2.40)$$

$$\tau(x) = 5+2\cdot\left(-\frac{3}{2}\pm x\pm\sqrt{\alpha+\frac{9}{4}}\right) = 2\pm 2x\pm\sqrt{4\alpha+9} \quad (2.41)$$

$\lambda = k_{-}+\pi'(s)$ and $\lambda_n = -n\cdot\tau'(s)-n(n-1)\sigma''(s)$
(2.42)

$\lambda = \frac{\varepsilon'}{2}-\sqrt{\alpha+\frac{9}{4}}-1$ and $\lambda_n = -n\cdot(-2)-n(n-1)\cdot 0$

$$\lambda_n = 2_n = \lambda \Rightarrow 2_n = \frac{s^1}{2}-\sqrt{\alpha+\frac{9}{4}}-1 \quad (2.43)$$

…(2.44) $\varepsilon' = 2\left(2n+1+\sqrt{x+\frac{9}{4}}\right)$

If we remember $\varepsilon' = \varepsilon-K\cdot\sigma_0$ then

$$\varepsilon = 4n+2\sqrt{\alpha+\frac{9}{4}}+\frac{k}{5a}+2 \quad (2.44)$$

Again remember $\alpha = \beta_0^4+\tau_B(\tau_e+3)$ ad $\epsilon_a = \tau_i(\tau_d+3)-\tau_A\left(\tau_b(\pi)-\frac{7}{4}\right.$

$$\varepsilon = 4_1+2\sqrt{p_0^4+\tau_b(\tau_a+3)+\frac{9}{4}}+K\left(\tau_1(\tau_1+3)-\tau_A(\tau_a+3)-\frac{7}{4}\right)+2 \quad (2.45)$$

2.2. Collective Model

A reliable foundation for comprehending the collective behavior of atomic nuclei has been supplied for many years by the Bohr Hamiltonian (Bohr and Lindhard 1954) and its extensions, the geometrical collective model (Herzberg 2013)(Panda *et al*. 1997). It was common practice to regard the mass in the Bohr Hamiltonian as a constant. However there was growing evidence that this approximation might not be precise enough. Namely, it is projected that the moments of inertia will rise correspondingly to 2, where is the general variable related to nuclear deformation. However, the moment of inertia that was actually determined experimentally (from the spectra) exhibits a far more mild rise in relation to the deformation that was actually determined experimentally [from the B(E2) transition rates], particularly for well-deformed nuclei (Ring and Schuck 2004). Due to this disagreement, the Bohr Hamiltonian is advocated for use in transitional and vibrating nuclei, but its applicability to deformed nuclei needs further explanation.

The mass tensor of the collective Hamiltonian should be viewed as a function of the collective coordinates, with quadrupole and hexadecapole terms present in addition to the monopole one, as recently shown (Jolos and Von Brentano 2008 , Jolos and Von Brentano 2009) by detailed comparisons to experimental data.

2.3. Bohr Hamiltonian

By placing a non-classical constraint on the orbital angular momentum of the electron, Bohr was able to determine the energy levels of a hydrogen atom for the first time. The angular momentum (rp or rmv) could only be expressed as integer multiples of the Planck constant, which is a fundamental quantity $\hbar$[rmv = n$\hbar$]. According to the hypothesis, an atom could only experience distinct orbits and energy levels.

$$\mathrm{H_B} = \mathrm{H_C} + \mathrm{H_F} + \mathrm{V_{CF}} \tag{2.46}$$

The first term $\hat{H}_B$ is given based on the global coordinates (β, γ) by the Bohr Hamiltonian (Bohr 1952) for the even-even core.

$$\hat{H}_B = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{\hat{Q}_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right] + V(\beta, \gamma) \tag{2.47}$$

When the angular momentum operator's $\hat{Q}_k$'s $(k = 1,2,3)$ components are represented in the body-fixed coordinate system. The core's potential is $V(\beta, \gamma)$ which is unaffected by γ for the

γ-instability in structures Then, assuming the eigenfunctions are also separable, the Hamiltonian $\hat{\mathrm{H}}_{\mathrm{B}}$ can be divided into the angular and radial parts. Using this premise, the angular portion is

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_{k=1,2,3}\frac{\hat{Q}_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right]\phi(\gamma, \theta) = \Lambda\phi(\gamma, \theta) \tag{2.48}$$

Because the angular component is independent of the potential Bès (Bès 1959) provides its one-of-a-kind answer. The second-order Casimir operator of $SO(5)$ constant,, has an eigenvalue of $\tau(\tau + 3)$. The values of the core angular momentum L for each of the "seniority" τ is quantum numbers $\tau = 0,1,2, ...$ are given: (Iachello and Arima 1987).

$$\tau = 3v_\Delta + \lambda\,;\; L = \lambda, \lambda + 1, \dots, 2\lambda - 2, 2\lambda \tag{2.49}$$

where v_Δ is the missing quantum number in the reduction of $SO(5) \supset \mathrm{SO}(3)$ and $2\lambda - 1$ is missing.

The second term is the Hamiltonian of a single particle, which is taken to be zero (Iachello 2005). In a spin-orbit interaction in five dimensions, $\hat{V}_{BF}$ denotes the interaction between the boson core and the single particle $\hat{\Lambda} \circ \hat{\Sigma}$. The final component is the degeneracy-breaking term proportional to the total angular momentum operator $\hat{J}$. κ and κ' the strength of the interaction is determined by $g(\beta)$ and the free parameters and'are both reversible.The linked system's whole wave function is

$$\Psi = F(\beta)\Phi(\gamma, \theta_i, \eta) \tag{2.50}$$

The quantum numbers $\tau, \tau_1, v_\Delta, J, M_J$ are used to identify the angular portion, which is produced by coupling the single-particle wave function $\chi(\eta)$ with the core wave function $\phi(\gamma, \theta_i)$.

$$\Phi_{\tau,\tau_1,v_\Delta,J,M_J} = \sum_{L,M_{L,m_j}} \left\langle \begin{matrix} [\tau,0] & \left[\frac{1}{2},\frac{1}{2}\right] & \left[\tau_1,\frac{1}{2}\right] \\ L & \frac{3}{2} & J \end{matrix} \right\rangle \left\langle \begin{matrix} L & \frac{3}{2} & J \\ M_\ell & m_j & M_J \end{matrix} \right\rangle \times \phi_{\tau,v_\Delta,L,M_L}(\gamma, \theta_i)\chi_{3/2,m_j}(\eta) \tag{2.51}$$

where the first two symbols stand for Spin(5) is reduction factors (Caprio and Iachello 2007) The result of solving the eigenvalue problem using the wave function is

$$-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2}\right]F + g(\beta)\left[\kappa 2\langle\hat{\Lambda} \circ \hat{\Sigma}\rangle + \kappa'\langle\hat{J} \cdot \hat{J}\rangle\right]F + [V(\beta) - E]F = 0 \tag{2.52}$$

with

$$\Lambda = \tau(\tau+3),\ 2\langle\hat{\Lambda} \circ \hat{\Sigma}\rangle = \tau_1(\tau_1+3) - \tau(\tau+3) - 7/4,\ \langle\hat{J} \cdot \hat{J}\rangle = J(J+1) \tag{2.53}$$

Two representations with $\tau_1 = \tau \pm 1/2$ for the entire system result from the coupling of the core representation and the fermion representation. The distance between these two spectral representations is controlled by the free parameter κ' For $(\tau_1, 1/2)$ of Spin (5) total angular momentum quantum number J is given :(Iachello and Isacker 1991) .

$$J = 2\tau_1 - 6v_\Delta + \frac{1}{2}, 2\tau_1 - 6v_\Delta - \frac{1}{2}, \dots, \tau_1 - 3v_\Delta + 1 - \frac{1}{4}[1 - (-1)^{2v_\Delta}] \quad (2.54)$$

with $v_\Delta = 0,1/2,1,3/2, \dots$. The decreased energy $\varepsilon = \frac{2B}{\hbar^2}E$ and the reduced potential $u(\beta) = \frac{2B}{\hbar^2}V(\beta)$ are introduced.

$$F''(\beta) + \frac{4}{\beta}F'(\beta) + \left[\varepsilon - \frac{\Lambda}{\beta^2} - \frac{2B}{\hbar^2}g(\beta)\left(\kappa 2\langle \hat{\Lambda} \circ \hat{\Sigma}\rangle + \kappa'\langle \hat{J} \cdot \hat{J}\rangle\right) - u(\beta)\right]F(\beta) = 0 \quad (2.55)$$

2.4. Critical Point Symmetries

For the two critical point symmetries E(5) and X(5), the analytical solutions of the Bohr Hamiltonian generated with the Sequence potential in the collective variable β are $\mathrm{E}(5) - \beta^{2n}$ for different values of n, respectively (Bonatsos and Lenis and Minkov and Raychev et al., 2004a), and $\mathrm{X}(5) - \beta^{2n}$ (Bonatsos and Lenis and Minkov and Raychev *et al*. 2004b).

Here, with increasing values of n, it is seen that the potential floor gradually becomes flatter and an infinite square well potential is obtained at the $n \to \infty$ limit. In both studies, U(5) limit structure for n=1 and E(5) critical point symmetries, and X(5) for $n \to \infty$ limit were obtained. Models $\mathrm{E}(5) - \beta^2$, $\mathrm{E}(5) - \beta^4$ and $\mathrm{E}(5) - \beta^8$, U(5) and X(5) critical point symmetry to scan the core region between U(5) and E(5) critical point symmetry For scanning the region between point symmetry $\mathrm{X}(5) - \beta^2$, $\mathrm{X}(5) - \beta^4$ and $\mathrm{X}(5) - \beta^8$ models are considered.

The ground band and other excited band energy levels obtained for each model are compared with the critical point symmetries E(5) and X(5).

On the other hand, the region between the shape phase transition point and deformed limit structures is examined with the constrained β soft (CBS) model (Bonatsos *et al*. 2006)(Pietralla and Gorbachenko, 2004). This model has been developed for X(5) and

E(5) by using the infinite square well potential bounded by two walls in the form $\beta_M > \beta_m \geq 0$ which is expressed as below, instead of the infinite square well potential.

$$u(\beta) = \begin{cases} 0, & \beta_m \leq \beta \leq \beta_M \\ \infty, & 0 \leq \beta < \beta_m, \beta > \beta_M \end{cases} \tag{2.56}$$

The evolution of the nuclear structure between X(5) and SU(3), E(5) and O(6) is discussed with the analytical solution of the Bohr Hamiltonian eigenvalue problem for this potential form.

In this context, the ratio $r_\beta = \beta_m/\beta_M$, which is a measure of the rigidity of the potential, was taken into consideration and the critical point symmetries with increasing $R_{4/2}$ energy ratios from the critical point symmetries of X(5) and E(5) for $r_\beta = 0$ to $r_\beta \to 1$ deformed limit structures. and the core region between the deformed limit structures was scanned.

From this point of view, Davidson-type potentials with a minimum at $\beta \neq 0$ and a free parameter β_o are proposed to scan the region between the O(6) and SU(3) deformed limit structures located at the other corners of the shape phase triangle from the U(5) spherical vibrator.

$$u_D^{2n}(\beta) = \beta^{2n} + \frac{\beta_0^{4n}}{\beta^{2n}} \tag{2.57}$$

Here, the above potential expression for n=1 transforms into the Davidson potential where the exact solution is obtained, and β_o corresponds to the minimum point of the potential.

$$u_D^2(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2} \tag{2.58}$$

Nuclear structure between limit structures is handled by Bohr Hamiltonian analytical solutions built with the Davidson potential in the collective variable.

Analytical solutions obtained using Davidson potential in Bohr Hamiltonian are named E(5)-D, X(5)-D (Bonatsos and Lenis and Minkov and Petrellis *et al*. 2004) for E(5) and X(5) critical point symmetries, respectively Using the Davidson potential, ES-D (Bonatsos *et al*. 2007) for the exact solution of X(5) critical point symmetry (ES-X(5)), and Z(5)-D (Yigitoglu and Bonatsos 2011) models were obtained.

Furthermore analytical solutions for the Bohr Hamiltonian depending on the deformed mass parameter are obtained with the Davidson potential in β for the γ-unstable, axisymmetric prolete and triaxially deformed structural states. Studies in which the mass parameter due to deformation is used is called DDM in the sense of mass due to deformation (Bonatsos *et al*. 2011)(Bonatsos e*t al*. 2013).

In the DDM model with the Davidson potential the increase in the moment of inertia determined by experimental observations towards the deformed nuclei in proportion was taken into account, and from here, the mass parameter in the Bohr Hamiltonian was determined as Bo(1+$\alpha\beta$2)2, taking into account the Davidson potential.

This study was carried out in order to moderate the increase of the deformation-dependent mass parameter and the moment of inertia with deformation and to eliminate the inadequacy of the model.

2.5. Davidson Potential

Electromagnetic transitions and rotational and vibrational energy spectra of diatomic molecules and nuclei are obtained using Davidson-type potentials proposed by Davidson (Davidson 1932).

The characteristic rotational-vibrational energy spectrum of the γ-soft structure (Wilets and Jean 1956) with the Davidson potential in the Bohr-Mottelson (Bohr and Mottelson 1953) collective model Hamiltonian in which the nucleus is considered as a liquid drop, using algebraic approaches is obtained (Rowe and Bahri 1998).

In the atomic phase transition Davidson-type potentials have been proposed to bridge the gap between the spherical vibrator U(5) and the deformed γ-unstable structure O(6) and the deformed axisymmetric structure SU(3) (Bonatsos and Lenis and Minkov and Raychev *et al*. 2004a) (Bonatsos and Lenis and Minkov and Raychev *et al*. 2004b).

In this context firstly Sequence potential expressed as u_2n (β)=β^2n/2, where n is an integer, is used to reach E(5) critical point symmetry and X(5) critical point symmetry from U(5) spherical vibrator structure.

2.6. Nikiforov–Uvarov Method

The Nikiforov-Uvarov (NU) approach is used to solve second-order differential equations of the hypergeometric type in terms of unique orthogonal functions. The primary equation that closely relates to the approach is presented in the following manner (Hassanabadi and Rajabi 2011, Zhangand Dong 2010).

$$\psi''(s) + \frac{\tau}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{2.59}$$

where σ(s) and $\tilde{\sigma}$ (s) are polynomials at most second-degree, τ is a first-degree polynomial, and ψ(s) is a function of the hypergeometric type. The exact solution of Eq. (2.30) can be obtained by use of following transformation

$$\psi(s) = \phi(s)y(s) \tag{2.60}$$

The transformation reduces the Equation (2.59) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{2.61}$$

The logarithm derivative can be used to define the function Φ(s).

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{2.62}$$

In this case, $\pi(s) = \frac{1}{2}[\tau(s) - \tilde{\tau}(s)]$ is only a first-degree polynomial at most. Equation (2.60)'s second ψ(s) is the hypergeometric function, and Rodrigues relation provides its polynomial solution.

$$y^{(n)}(s) = \frac{N_n d^n}{\rho(s) ds^n}[\sigma^n \rho(s)] \tag{2.63}$$

where N_n is the normalization constant and ρ(s) is the weight function which must satisfy the condition.

$$\begin{aligned} (\sigma(s)\rho(s))' &= \sigma(s)\tau(s) \\ \tau(s) &= \tilde{\tau}(s) + 2\pi(s) \end{aligned} \tag{2.64}$$

The derivative of τ(s) with regard to s should be negative, it should be emphasized. The following function π(s) and parameter λ, respectively, can be used to define the eigenfunctions and eigenvalues:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{2.65}$$

where$k \; = \; \lambda - \pi`(s)$ (2.66)

By setting the discriminant of the square root in Equation (2.65) to zero, the value of k can be determined. As a result, the following is the new eigenvalue equation:

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), n = 0,1,2 \ldots \tag{2.67}$$

3. MATERIALS AND METHODS

In this study, the excitation energy spectrum of odd-even nuclei in the U(5)-O(6) transition region will be examined with the Collective model. According to the collective model, all nucleons pair with each other to form a core (core), and the core pairs with the nucleon moving in a single-particle orbital with angular momentum. Within the scope of the study, the value of j will be 1/2, 3/2 or 5/2 depending on the nuclei to be examined. The Hamiltonian describing the system will be divided into three parts as given below. The first of these represents the paired boson core, the second represents the single remaining fermion, and the last one represents the fermion-core interaction.

$$H = H_B + H_F + V_{BF} \tag{3.1}$$

$$H_B = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma} \sin 3\gamma \frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{\kappa=1}^{3}\frac{Q_\kappa^2}{\sin^2\left(\gamma - \frac{2\pi}{3}\kappa\right)}\right] + V(\beta,\gamma) \tag{3.2}$$

$$H_F = k\frac{\hbar^2}{2B\beta^2}\left[2\hat{L}_B \circ \hat{L}_F\right] \tag{3.3}$$

$$V_{BF} = k'\frac{\hbar^2}{2B\beta^2}\hat{L}_F^2 \tag{3.4}$$

Here, B is the mass parameter, $\hat{L}_B, \hat{L}_F$ is the five-dimensional boson and fermion orbital angular momentums, respectively, and "∘" denotes the five-dimensional scalar product. The Davidson potential will then be used in this Hamiltonian. The main reason for choosing this potential is that it gives successful results in even-even nuclei exhibiting structure in the transition region. (β,γ) denotes collective variables and $V(\beta,\gamma)$ denotes potential energy. In the unstable region, the potential energy of the system depends only on the variable γ. The Davidson potential is defined as follows.

$$u_D(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2} \tag{3.5}$$

Here, $u = \frac{2B}{\hbar^2}V$ is the reduced potential represents the location of the point where β_0 is the potential is minimum.

$$H = H_B + H_F + V_{BF} \tag{3.6}$$

$$H_B = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{\kappa=1}^{3}\frac{Q_\kappa^2}{\sin^2\left(\gamma - \frac{2\pi}{3}\kappa\right)}\right] + V(\beta,\gamma) \tag{3.7}$$

$$\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 y' = \frac{1}{\beta^4}\left(\frac{\partial}{\partial\beta}(\beta^4 y')\right) = \frac{1}{\beta^4}(4\beta^3 y' + \beta^4 y'') \tag{3.8}$$

$$\frac{4\beta^3}{\beta^4}y' + \frac{\beta^4}{\beta^4}y'' = \frac{4}{\beta}y' + y'' \tag{3.9}$$

$$y''(\beta) + \frac{4}{8}y'(\beta) \tag{3.10}$$

$$-\frac{1}{\sin\ 3\gamma}\frac{\partial}{\partial\gamma}\sin\ 3\gamma\frac{\theta}{\partial\gamma} + \frac{1}{4}\sum_k \frac{Q_2^2}{\sin^2\left(\gamma - \frac{1}{3} + K\right)}\Bigg]\phi(v,\theta) = \Lambda\phi(\gamma,\theta) \tag{3.11}$$

$$H = \frac{\hbar^2}{2B}\left[y'' + \frac{4}{\beta}y' - \frac{\Lambda}{\beta^2}\right] = 0 \tag{3.12}$$

$$y''(\beta) + \frac{4}{B}y'(\beta) + \frac{\Lambda}{\beta^2}y(\beta) = Ey(\beta) \tag{3.13}$$

By comparing Equation (3.9) with N.U. method equation (3.9) we find:

$$\psi''(s) + \frac{t''(s)}{\sigma\, s}\psi'(s) + \frac{\sigma''(s)}{\sigma^2(s)}\psi(s) = 0$$

$$t'' = 4 \qquad \sigma'' = \Lambda = t(\mathrm{t}+3)$$

$H\sigma = \beta$

$\sigma^2 = \beta^2$

$$y''(\beta) + \frac{4}{\beta}\acute{y}(\beta) + \left[\left(\epsilon - \frac{\Lambda}{\beta^2} - \frac{2\beta}{\hbar^2}kg(\beta)2(\lambda' - \varepsilon') - u(\beta)\right]\right. f(\beta) = 0 \tag{3.14}$$

$\epsilon = \frac{2B}{\hbar^2}\epsilon \qquad U(\beta) = \frac{2B}{\hbar^2}V(\beta)$

By using The Devidson potetial for The reduced patcutial

$U_\beta = (\beta) = \beta^2 + \frac{\beta^2}{\beta^2}$

$$y''(\beta) + \frac{4}{\beta}\acute{y}(\beta) + \left[\epsilon - \frac{\Lambda+\beta_0^4}{\beta^2} - \frac{2\beta}{\hbar^2}kg(\beta)2(\Lambda \cdot \hat{\varepsilon}) - \beta^2\right] f(\beta) = \tag{3.15}$$

x when $g(\beta) =$ constant $g(\beta) = \frac{\hbar^2}{2\beta}$

$$y''(\beta) + \frac{4}{\beta}y'(\beta) + \epsilon - \frac{\Lambda+\beta_0^4}{\beta^2} - \frac{2\beta}{\hbar^2}k\ \frac{\hbar^2}{2\beta} = \left(\acute{\Lambda}\circ\hat{\varepsilon} - \beta^2\right)f(\beta) = 0 \tag{3.16}$$

$$y''(\beta) + \frac{4}{\beta}y'(\beta) + \left[\epsilon - \frac{\Lambda+\beta_0^4}{\beta^2} - \mathrm{k}2\left(\acute{\Lambda}\circ\hat{\varepsilon} - \beta^2\right) - \beta^2\right] f(\beta) = 0 \tag{3.17}$$

with $\epsilon - k2\left(\acute{\Lambda}\circ\hat{\varepsilon}\right) = \epsilon$ and $\Lambda + \beta_0^4 = \rho(p+3)$

$$y''(\beta) + \frac{4}{\beta}y'(\beta) + \left[\epsilon - \frac{p(\rho+3)}{\beta^2} - \beta^2\right] y(\beta) = 0 \tag{3.18}$$

$$y''(\beta) + \frac{4}{\beta} y'(\beta) + \left(\frac{\epsilon\beta^2 - p(P+3) - \beta^4}{\beta^2}\right) y(\beta) \tag{3.19}$$

When $(P(P+3) = A$

$$y''(\beta) + \frac{4}{\beta} y'(\beta) + \left(\frac{\epsilon\beta^2 - \beta^4 - A}{\beta^2}\right) y(\beta) = 0 \tag{3.20}$$

$$\psi''(s) + \frac{\tilde{t}(s)}{d(s)}\psi(s) + \frac{\sigma''(s)}{\sigma^2(s)}\psi(s) = 0 \tag{3.21}$$

$t'' = 4$

$\sigma(\beta) = \beta$

$$\sigma''(\beta) = \epsilon\beta^2 - \beta^4 - A \tag{3.22}$$

The eigen functions are given in terms of Laguerre Polyomials:

$$f_{n,r,T,J}(\beta) = \left(\frac{2n!}{r(n+\beta+5/2)}\right)^{1/2} \beta^2 \int_x^{p+3/2} (\beta^2) e^{-\beta^{2/2}} \tag{3.23}$$

x with:

$$P = -\frac{3}{2} + \sqrt{(\tau+3/2)^2 + \beta_0^4 + k[\tau_1(\tau_1+3) - \tau(\tau+3) - 7/4 + k'J(s+4)}$$

Where $r(n)$ Clearly, in this situation, denotes the r-function.

The wave functions are indepen dewt of τ_1. Then energy values ave:

$$\varepsilon^c_{n_1\tau_1,i_1} = \begin{array}{l} 2n + 1 + [\tau(\tau+3) + \beta_0^4 + 9/4]^{1/2} + \\ k[\tau_1(\tau_1+3) - \tau(\tau+3-7/4 \end{array} \tag{3.24}$$

4. RESULTS AND DISCUSSION

Table 4.1 displays the Hamiltonian parameters that were utilized to gather experimental results for a few particular isotopes.

Table 4.1 The variables in the Hamiltonian that were used to collect experimental data on a few isotopes

İzotop	β_0	κ	κ'	Δ
$^{187}_{77}$Ir	2.91	-0.25	0.32	0.00559
$^{189}_{77}$Ir	1.71	-1.6	0.17	0.46077
$^{191}_{77}$Ir	1.8	-1.73	0.31	0.01176
$^{193}_{77}$Ir	1.98	-1.39	0.4	0.03251
$^{195}_{77}$Ir	2.17	-0.51	0.59	0.02258

Comparison of experimental excitation energy data for the isotopes $^{187-195}$Ir with the results of the lowest band of the Davidson potential. The horizontal line that divides the n = 0+ (upper part) and n = 0 (lower part) band states is given in table 4.2.

Table 4.2 Comparison of the experimental (Exp) excitation energy data for the $^{187-195}$Ir isotopes with the results of the Davidson potential (Theo)

$(\tau, \tau 1)$	J^+	$^{187}_{77}Ir$		$^{189}_{77}Ir$		$^{191}_{77}Ir$		$^{193}_{77}Ir$		$^{195}_{77}Ir$	
		Theo	Exp	Theo	Exp	Theo	Exp	Theo	Exp	Theo	Exp
0,0,1⁄2	3⁄2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0,1,3⁄2	7⁄2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0,1,3⁄2	5⁄2	0.709	0.386	0.688	0.379	0.657	0.377	0.640	0.388	0.628	0.445
0,1,3⁄2	1⁄2	0.373	0.374	0.312	0.314	0.239	0.24	0.203	0.204	0.174	0.176
0,2,5⁄2	11⁄2	2.509	2.515	2.475	2.484	2.435	2.424	2.420	2.396	2.366	2.365
0,2,5⁄2	9⁄2	2.082	1.554	2.077	1.511	2.000	1.464	1.955	1.459	1.883	1.409
0,2,5⁄2	7⁄2	1.727	1.653	1.736	0.000	1.622	1.469	1.552	1.443	1.462	1.413
0,2,5⁄2	5⁄2	1.446	1.094	1.458	0.000	1.313	1.023	1.222	1.011	1.117	1.046
0,2,5⁄2	3⁄2	1.244	0.000	1.253	0.587	1.083	0.521	0.976	0.503	0.859	0.503
0,3,7⁄2	15⁄2	4.435	4.435	4.209	4.316	4.105	4.132	4.079	4.081	3.941	3.991

Comparison of experimental excitation energy data for the isotopes $^{187-195}$Ir with the results of the Davidson potential for the lowest band (n = 0). The states of n = 0+ (upper part) and n = 0− (lower part) bands are divided by the horizontal line is illustrated in table 4.3.

Table 4.3 Comparison of experimental excitation energy data for the isotopes 187–195Ir with the results of the Davidson potential for the lowest band (n = 0)

(τ , τ1)	J^-	${}^{187}_{77}Ir$		${}^{189}_{77}Ir$		${}^{191}_{77}Ir$		${}^{193}_{77}Ir$		${}^{195}_{77}Ir$	
		Theo	Exp	Theo	Exp	Theo	Exp	Theo	Exp	Theo	Exp
0,1,1⁄2	3⁄2	0.665	0.664	2.140	2.144	1.713	1.713	1.263	1.288	0.591	0.593
0,2,3⁄2	1⁄2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0,2,3⁄2	5⁄2	1.666	1.094	3.466	2.829	2.886	2.179	2.310	1.943	1.748	0.000
0,2,3⁄2	7⁄2	1.944	1.653	3.683	0.000	3.136	2.404	2.596	0.000	1.944	1.653

Oddly enough ${}^{187-195}$Ir The Morse potential will be tested using ir isotopes. The selecting process takes into account the isotopes' availability of sufficient experimental evidence. The next step is to choose the shape and intensity of the interaction between the core and a single particle. It is evident from Equation (3.24) that there is no discernible difference in energy. For the duration of the investigation, the continuous interaction term will be applied. The free parameters in the energy eigenvalue Equation (3.24) can be changed to obtain the proper spectra for each isotope. The quality factor Δ was used to determine the values of free parameters,

$$\Delta = \frac{1}{t}\sum_{i}^{t} \frac{\sqrt{(\varepsilon_i^{th} - \varepsilon_i^{\exp})}}{\varepsilon_i^{\exp}}$$

where Nd is the variable, ranging from 8 to 21, number of energy data used in the fit. The values of the free parameters for each isotope were then discovered and are shown in Table 1. Each isotope's obtained quality factor is on the order of 0.1 and is very near to zero. The anticipated spectrum and the experimental results in Table 2 have been compared using these parameters in Equation (3.24). The levels marked n_+ and n_- in this table correspond to the states $\tau_1 = \tau + 1/2$ and $\tau_1 = \tau - 1/2$, respectively. The energy of the initial $J^\pi = 7/2^+$ state has been used as the standard for all experimental and theoretical energies. For all isotopes, the concordance between theory and experiment is quite good.

The comparison between the theoretical predictions and the experimental data in Table 4.2 makes this agreement clear. The stark contrasts between theory and experiment are shown by the red arrows. The fact that the projected energy of the states with the second-largest angular momentum for each τ_1 quantum number (the second column, K=5/2-band) is higher than the experimental data for all isotopes is quite interesting. Moreover, the K=5/2-band and K=1/2-band states are degenerate. In other words, J=5/2 and J=1/2 have approximately similar energies for $\tau_1 = 3/2$. Similar results are obtained for $J = 5/2$ at J=9/2 and J=7/2 and $\tau_1 = 5/2$ at J=13/2 and J=11/2. This degeneracy proves that the structure has symmetry. This characteristic cannot potentially be created in this model because the degeneracy breaking term has been employed in the Hamiltonian.

5. CONCLUSIONS AND RECOMMENDATION

The Nikiforov-Uvarov Method will be used to obtain the γ~, which is the energy equation describing the excitation energy spectra of odd-even nuclei displaying unstable structures, utilizing the Davidson potential in the Collective Bohr Hamiltonian. The excitation energy spectra of the γ~unstable $^{187-195}Ir$ isotopes will be produced by applying the discovered equation and will then be compared with the experimental results.

The Collective model will be used to study odd-even nuclei with structures in the critical region where the second-order phase transition takes place. For this, the Hamiltonian's Davidson potential with a free parameter will be applied. The Nikiforov-Uvarov Method will be used to solve the Schrödinger equation to be produced in analytic form for the first time. We will identify potential factors that theoretically best predict the experimental spectrum of the nucleus under study. Using the established parameters, isotope excitation energy spectra will be obtained.

In this study, the Collective model will be used to explore the excitation energy spectrum of odd-even nuclei in the U(5)-O(6) transition region. The core pairs with the nucleon travelling in a single-particle orbital with angular momentum, and all nucleons pair with one another to create a core (core), according to the collective model. Depending on the nucleus being studied, the value of j within the context of the study will either be 1/2, 3/2, or 5/2. The three sections of the Hamiltonian that describes the system are listed below.

Below is a list of the key findings and recommendations reached in the current Letter:

(1) A variational method for figuring out the physical quantity values at the nuclei's shape phase transition points has been proposed. The parameter values at which the rate of change of the physical quantity becomes maximum for each value of the angular momentum separately are identified using one-parameter potentials spanning the region

between the two limiting symmetries of interest, and the corresponding values of the physical quantity at these parameter values are calculated. The physical quantity's values that are gathered in this manner describe how it behaved at the crucial point.

(2) Using one-parameter Davidson potentials and the energy ratios R L=E(L)/E(2) within the ground state band as the relevant physical quantity, the method has been applied in the shape phase transition from U(5) to O(6), resulting in a band that practically coincides with the ground state band of the E(5) model. The form phase transition from U(5) to SU(3) has also used it in the same way, producing a band that almost exactly matches the ground state band of the model.

(3) It should be noted that the Davidson potentials accurately recreate the U(5) and O(6) symmetries in the former instance (for small and high parameter values, respectively), as well as the pertinent $\mathrm{X}(5)-\beta^2$ and SU(3) symmetries in the later situation. This allowed the approach to be used (for small and large parameter values, respectively).

(4) As a by-product, Davidson potentials were used to derive the Holmberg-Lipas formula. It is obviously interesting to apply the variational method described here to physical values other than the ground state band energy ratios. The obvious candidates for energy ratios include ratios of excited band levels, ratios of intraband and interband B(E2) transition rates, and ratios of quadrupole moments. Since the Davidson potentials have the appropriate limiting behavior for both small and high parameter values, work in these directions is now being done utilizing them. However, any additional Hamiltonian or hypothetical bridge of the pertinent symmetries (U(5)-O(6) and U(5) SU(3)) should be as suitable.

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