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# SOLUTION OF THE BOHR HAMILTONIAN FOR THE KRATZER POTENTIAL AROUND THE X(5) CRITICAL POINT

**IN PARTIAL FULFILLMENT OF THE REQUIREMENTS**
**FOR**
**THE DEGREE OF MASTER OF SCIENCE**
**IN**
**PHYSICS**

**BY**

**YASIR SALIH DAR DAR**

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SOLUTION OF THE BOHR HAMILTONIAN FOR THE KRATZER POTENTIAL AROUND THE X(5) CRITICAL POINT

By Yasir Salih DAR DAR

May 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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**ABSTRACT**

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Master of Science in Physics

Advisor: Assoc. Prof. Dr. İlyas İNCİ

May 2023

Obtaining the properties of nuclei exhibiting structures around the critical point theoretically is one of the subjects that nuclear structure physicists have focused on in recent years. In this thesis, the excitation energy spectra of nuclei, which are located between the critical points of U(5) and SU(3) and displaying structures around the critical point where the first-order shape phase transition takes place, indicated by X(5), were investigated. Since these nuclei deform their structures, the Collective Bohr Hamiltonian written depending on the deformation parameters was used. The Schrödinger equation, which is obtained by assuming that the potential energy of the nucleus is in the form of Kratzer potential, is solved analytically using the Nikiforov-Uvarov method. It is experimentally known that nuclei with mass numbers greater than 150 exhibit a prolate-deformed structure. Therefore, Er isotopes with mass numbers between 160 and 170 were chosen to test the suitability of the Kratzer potential. Using the Mathematica program, potential free parameters were determined for each isotope that would give the best experimental data. By using these parameters, the entire excitation energy spectrum of the isotopes was created and compared with the experimental data. However, the existence of some situations that cannot be measured experimentally has been shown theoretically and a data set has been created for future experimental studies.

**2023, 40 pages**

**Keywords:** Kratzer potential, X(5), Collective model.

# ÖZET

## X(5) KRİTİK NOKTASI CİVARINDA KRATER POTANSİYELİ İÇİN BOHR HAMİTİLTONİYENİNİN ÇÖZÜMÜ

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Mayıs 2023

Kritik nokta civarında yapı sergileyen çekirdeklerin özelliklerinin teorik olarak elde edilmesi, son yıllarda nükleer yapı fizikçilerinin yoğunlaştığı konulardan biridir. Bu tezde, U(5) ile SU(3) kritik noktaları arasında yer alan ve X(5) ile gösterilen birinci-mertebe şekil faz geçişinin gerçekleştiği kritik nokta civarında yapı sergileyen çekirdeklerin uyarılma enerji spektrumları incelenmiştir. Bu çekirdeklerin yapılarını deforme olması nedeniyle deformasyon parametrelerine bağlı olarak yazılan Kollektif Bohr Hamiltoniyeni kullanılmıştır. Çekirdeğin potansiyel enerjisi Kratzer potansiyeli formunda olduğu kabul edilerek elde edilen Schrödinger denklemi, Nikiforov-Uvarov medodu kullanılarak analitik olarak çözülmüştür. Kütle numaraları 150'den büyük olan çekirdeklerin prolate-deforme yapı sergiledikleri deneysel olarak bilinmektedir. Bu nedenle Kratzer potansiyelinin uygunluğunu test etmek için, kütle numaraları 160 ile 170 arasında değişen Er izotopları seçilmiştir. Mathematica programı kullanılarak, her bir izotop için deneysel verileri en iyi verecek potansiyel serbest parametreleri belirlenmiştir. Bu parametreler kullanılarak izotoplara ait tüm uyarılma enerji spektrumu oluşturulmuş ve deneysel verilerle karşılaştırılmıştır. Bununla birlikte deneysel olarak ölçülemeyen bazı durumların teorik olarak varlığı gösterilmiş ve ileride yapılacak deneysel çalışmalar için veri seti oluşturulmuştur.

**2023, 40 sayfa**

**Anahtar Kelimeler:** Kratzer potansiyeli, X(5), Kollektif model

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**Yasir Salih Dar DAR**
**Çankırı-2023**

**CONTENTS**

**LIST OF SYMBOLS**

| | |
|---|---|
| B | Mass parameter |
| B(E2) | Gamma transition ratio |
| C | Shielding constant |
| $c_{\xi,}$ | Normalization constant |
| h | Plank constant |
| $J_{\nu}$ | Bessel function |
| L | Angular momentum |
| $n_w$ | Wobbling quantum number |
| Rs | Energy rate value of Rs |
| t | Scale factor |
| $T^{(E2)}$ | Quadruple transition operator |
| $u(\beta)$ | $\beta$Potential due to the variable |
| $u(\gamma)$ | $\gamma$ potential due to the variable |
| $x_{sL}$ | s of the Bessel function $J\nu(z)$ |
| $\langle \beta^2 \rangle$ | Average of $\beta^2$ over ($\beta$) |
| $\varepsilon$ | Reduced energy |
| $\mu$ | Particle mass |
| $\Psi$ | Wave function |
| $\mathcal{D}$ | Wigner function of Euler angles ($i$ = 1,2,3) |
| $\beta_0$ | The point where the potential is minimum |
| $\theta_i$ | Three Euler angles $i$ = 1,2,3 |

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# 1. INTRODUCTION

In the early 2000s, Iachello proposed a solution using the Bohr Hamiltonian (Bohr 1952) to explain the structures of nuclei at critical points between two different phases with an analytical model.

This solution proposed by Iachello, in the arrangement of Bohr Hamiltonian, Morse (Boztosun *et al*. 2008), Kratzer (Fortunato *et al*. 2006; Bonatsos *et al*. 2013), Coulomb (Fortunato *et al*. 2006; Fortunato and Vitturi 2004), Davidson (Fortunato *et al*. 2006). By selecting various potential types, resolving the eigenvalue-eigenfunction puzzle, and analyzing the collective dynamics of nuclei (Yigitolu and Gökbulut 2017).

Dynamic symmetry explains the structural properties of atomic nuclei. The first of the phase transition's pivotal moments region of the shape between two different dynamic symmetries E(5) (Iachello 2000) . The crucial intersection of the SU(3) axial deformation and U(5) vibration (rotor prolate) nuclei is known as the X(5) (Iachello 2001). The Bohr Hamiltonian (Bohr 1952) provides an exact solution for the γ-soft (unstable, independent) E(5) symmetry, but the X(5) symmetry is a close approximation for 0°.

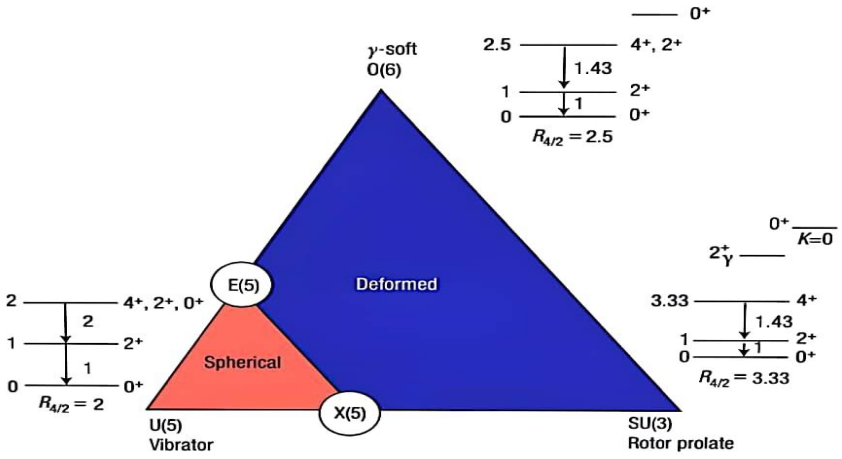


**Figure 1.1:** Casten's Triangle (Casten 2006)

The vertices of the triangle in Figure 1.1. Interacting boson model (IBM) correspond in relation to the dynamic symmetries O(6), SU(3), and U(5). These symmetries each have a distinctive quality. They represent, respectively, spherical, axially deformed, and $\gamma$-unstable core structures.The amount of deformation in the nucleus is determined based on the energy to mass ratio of the 4+ dependant on the energy of the 2+ state, or $R_{4/2}$. It does not, however, offer details regarding the extent of the deformity. For a vibrator (spherical, vibratory), the $R_{4/2}$ ratio is 2.0, for a $\gamma$ -independent (soft, unstable), 2.5, and for an axially deformed (rotor prolate), 3.33. See Figure 1.2.

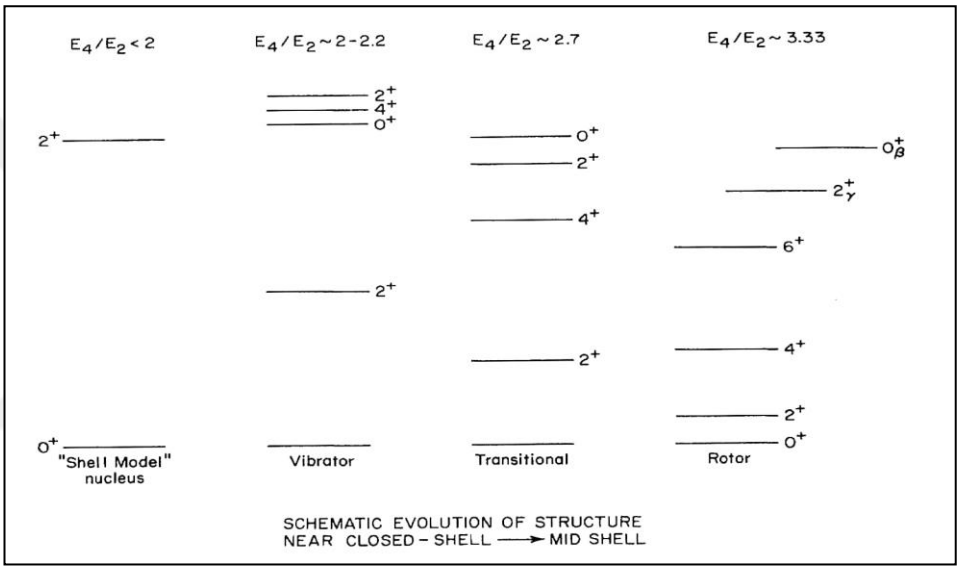


**Figure 1.2:** Level diagrams and array of nuclear structure types (Casten 1990)

The corners of the triangle in Figure 1.3, denoted by the letters U(5), SU(3), and spherical (prolate deformed), and SU(3), (oblate deformed); shows a symmetry in mathematics that is equivalent to the three-nucleus structure. First-order phase transitions are transition points with corresponding critical point symmetries. (Jolie *et al*. 2002) imply that the second-order change in nuclear shape from spherical to prolate or oblate structure is marked by a nuclear triple point (Warner, 2002).

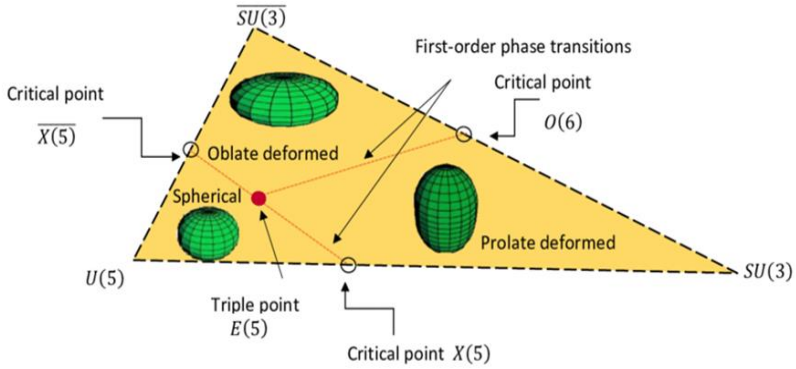


**Figure 1.3:** The Casten Triangle Extend (Iachello 1987)

It is demonstrated that There isn't a phase change between SU(3) and O(6), However, Between U(5) and SU(3), a first-order phase transition takes place, whereas between U(5) and O(6), a second-order phase transition takes place. The phase transition of the first order is distinct from the phase transition of the second order in the nuclear setting in that it changes only seldom while the Phases that are spherically distorted continue. Second-order phase transitions involve there is just one phase and the deformation grows continuously. The two internal structural variables and detach from spherical form and axial symmetry, respectively.

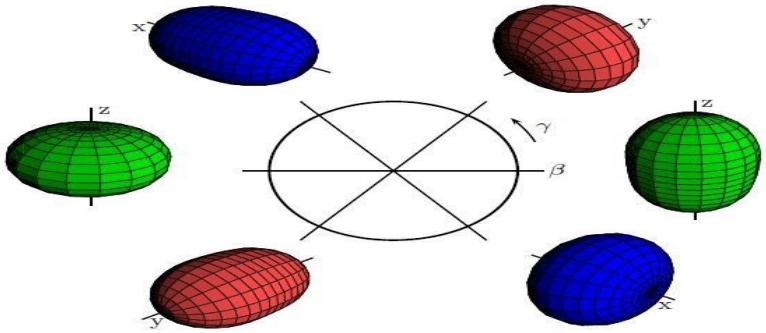


**Figure 1.4:** Hill-Wheeler coordinates $\{\beta, \gamma\}$. Quadrupole deformed shapes corresponding to $\beta = 0.4$ and $\gamma = n\pi/3$ (with $n = 0, 5$) (Fortunato 2005)

Taking into account nuclei with quadrupole or ellipsoidal forms specified by parameters, ellipsoidal deformation, and value is 0 for a circular core and approximately For standard prolate nuclei, 0.3 (Shape of an American football) nuclei. Is 0° for an axially symmetrical form, 30° find the most asymmetric form possible (like a crushed soccer ball), 60° for an oblate, and (disc-shaped) shape. It represents symmetry about the axis of symmetry of the ellipsoidal form.

## 2. INTERACTING BOSON MODEL

The physicists Francesco Iachello and Akito Arima devised a nuclear model that defines Low Lying Collective's organisational chart Nuclear Levels in 1974 (Iachello and Arima 1974). The phrase "Interacting Boson Model" describes it and it describes the structure as a collection of heavy and medium nuclei. In the IBM model of interacting bosons, nucleons that are not in shells are treated. Bosons are the name for the nucleus of even-numbered protons and neutrons, which can occupy the state s (ground state) when their angular momentum is equal to zero (L=0) and they are known as (s- bosons), excited energy when their angular momentum is equal to (L=2), and when (L=3) is an odd number, they are known as (d- bosons), and when (L=3) is a boson (f-boson), these describe the negative symmetry states. (Casten and Warner1988). The characteristics of the excitation energy spectrum have been effectively explained by Francesco Iachello and Akito Arima . When the idea was first created, there was differentiating between Atomic nuclei include neutrons and protons. The number of nucleons outside the fixed and closed shell, as shown by the formula, is equal around half of all bosons in a given nucleus (Arima and Iachello 1976).

### 2.1 U(6) group and its reductions U(5), SU(3) and O(6)

Considering the magnetic projections of the s and d bosons, one encounters six different cases, and this system is denoted in the language of group theory as the group U(6). U(6) has the subgroups Those three are U(5), SU(3), and O(6). called spherical vibrators, deformable rotors, and gamma-unstable rotor structures,respectively (Castanos *et al.*1979).The ability to build bases in which the Hamiltonian operator can be diagonalized is one of the principal uses of group (or algebra) chains. This is because the transformation qualities of the groups in which can be used to characterize the states of the basis equation (2.1). In other words, bases that represent the relevant groups can be built, and states can be assigned the correct quantum numbers.

representation [N] of U(6) being broken down into representations of subgroups poses the main challenge for the creation of a basis. This is a common group theoretic problem, and we merely offer the solution here.

If one wants to include the rotation algebra, O(3), as a subalgebra. In actuality, U(6) has been explored together with U(5), U(3), and O(6). O(6)'s algebra is isomorphic to SU(4) and thus all possibilities leading from U(6) to the rotation algebra O(3)≈SU(2) have been considered. In conclusion, there are three and only three possibilities

$$\begin{aligned} &\text{I:}\mathrm{U}(5) \supset \mathrm{O}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \\ &\text{II:}\mathrm{U}(6) \to \mathrm{SU}(3) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \\ &\text{III:}\mathrm{O}(6) \supset \mathrm{O}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \end{aligned} \tag{2.1}$$

### 2.1.1 Chain I

Representations of the groups appearing The quantum numbers characterise the links in this chain (Arima and Iachello 1976).

$$\begin{aligned} &\mathrm{U}(6) \quad [N, 0,0,0,0,0] \equiv [N] \\ &\mathrm{U}(5) \quad (n_{\mathrm{d}}, 0,0,0,0) \equiv (n_{\mathrm{d}}) \\ &\mathrm{O}(5) \quad (v, 0) \equiv v \\ &\mathrm{O}(3) \quad L \\ &\mathrm{O}(2) \quad M_L \end{aligned} \tag{2.2}$$

The values of $n_{\mathrm{d}}$ in a particular representation [N] of U(6) are as follows:

$$n_{\mathrm{d}} = N, N-1, \ldots, 1, 0 \tag{2.3}$$

The values of $v$ in a representation of U(5) called $(n_{\mathrm{d}})$ are as follows:

$$v = n_{\mathrm{d}},\ n_{\mathrm{d}} - 2, \quad \ldots, 1 \text{ or } 0 \quad (n_{\mathrm{d}} = \text{ odd or even }) \tag{2.4}$$

All three of these representations U(6), U(5), and 0(3) are completely symmetric. When going from $O(5)$ to $O(3)$ a subtle problem arises: several states with the same value L, a quantum number may appear in a given representation, $v$, of $\mathrm{O}(5)$. When this occurs in general for the group reduction $G \supset G'$, one says that $G$ is not completely reducible in

light of to $G'$. Thus, here, $O(5)$ is not completely reducible in light of to $O(3)$. Then, another quantum number is required (sometimes called the missing label) to distinguish the states individually. There are an endless amount of possible choices for this quantum number. For classification purposes, it is convenient to choose this number as the quantity of triplet bosons associated to angular momentum 0 , denoted by $n_\Delta$. The formula for calculating the values of each representation of L contains $v$ is then as follows. First, partition $n_\mathrm{d}$ as where:

$$n_\mathrm{d} = 2n_\beta + 3n_\Delta + \lambda \tag{2.5}$$

$$n_\beta = (n_\mathrm{d} - v)/2 \tag{2.6}$$

$v$ may use the values from equation 2.4, hence $n_\beta$ can take the values:

$$n_\beta = 0,1,\dots,\frac{\mathrm{nd}}{2} \text{ or } \frac{\mathrm{nd}-1}{2};\ (n_\mathrm{d} = \text{ even or odd }) \text{Then} \tag{2.7}$$

$$L = \lambda,\ \lambda+1,\ \dots, 2\lambda-2,\ 2\lambda \tag{2.8}$$

(Remember that 2-1 is absent!) Finally, $M_L$ accepts every integer between $-L$ and $+L$, as usual.

$$-L \leqslant M_L \leqslant +L \tag{2.9}$$

The basis defined in this way has the flaw that it is not orthogonal. An orthogonal basis, which we shall call the Szpikowski basis (Szpikowski and Góźdź 1980) can be obtained in the following way. For each $n_\mathrm{d}$ and $v$, let $n_{\Delta_1}, n_{\Delta_2}, \dots$ be the quantum number's values $n_\Delta$ with $n_{\Delta_1} < n_{\Delta_2} < n_{\Delta_3} < \cdots$. The quantum numbers identify the new basis $\tilde{n}_{\Delta_1}, \tilde{n}_{\Delta_2}, \dots$, with $\tilde{n}_{\Delta_1} < \tilde{n}_{\Delta_2} < \cdots$ and defined by(Iachello and Arima 1987):

$$
\begin{aligned}
&\left|n_{\mathrm{d}}, v, \tilde{n}_{\Delta_1}, L, M_L\right\rangle = \left|n_{\mathrm{d}}, v, n_{\Delta_1}, L, M_L\right\rangle, \\
&\left|n_{\mathrm{d}}, v, \tilde{n}_{\Delta_2}, L, M_L\right\rangle = x_{21}\left|n_{\mathrm{d}}, v, n_{\Delta_1}, L, M_L\right\rangle \\
&+x_{22}\left|n_{\mathrm{d}}, v, n_{\Delta_2}, L, M_L\right\rangle, \\
&\ldots, \\
&\left|n_{\mathrm{d}}, v, \tilde{n}_{\Delta_i}, L, M_L\right\rangle = \sum_{j=1}^{i} x_{ij} \left|n_{\mathrm{d}}, v, n_{\Delta_j}, L, M_L\right\rangle
\end{aligned}
\tag{2.10}
$$

The requisite: yields the coefficients$x_{ij}$ .

$$
\left\langle n_{\mathrm{d}}, v, \tilde{n}_{\Delta_i}, L, M_L \mid n_{\mathrm{d}}, v, \tilde{n}_{\Delta j}, L, M_L\right\rangle = \delta_{ij} \tag{2.11}
$$

$$
\mathrm{U}(6) \supset \mathrm{U}(5) \supset \quad \mathrm{O}(5) \supset \mathrm{O} \quad (3) \supset \quad \mathrm{O}(2\ ) \tag{2.12}
$$

### 2.1.2 Chain II

To categorise the states in this chain, terms like as (Arima and Iachello 1978).

$$
\begin{aligned}
&\mathrm{U}(6)\ \ [N, 0,0,0,0,0] \equiv [N] \\
&\mathrm{SU}(3)\ (f_1, f_2) \\
&\mathrm{O}(3)\ L \\
&\mathrm{O}(2)\ M_L
\end{aligned}
\tag{2.13}
$$

Instead of the Young tableau $(f_1, f_2)$, Introduction of the quantum numbers is typical (Elliott 1958).

$$
\begin{aligned}
&\lambda = f_1 - f_2 \\
&\mu = f_2
\end{aligned}
\tag{2.14}
$$

(Elliott numbers). The values of $(\lambda, \mu)$ symmetric depiction of it $[N]$ of $U(6)$ are given by:

$$
\begin{aligned}
(\lambda,\mu) = {} & (2N,0) \oplus (2N-4,2) \oplus (2N-8,4) \oplus \begin{Bmatrix} (0,N) \\ (2,N-1) \end{Bmatrix} \begin{Bmatrix} N = \text{ even} \\ N = \text{ odd} \end{Bmatrix} \\
& \oplus (2N-6,0) \oplus (2N-10,2) \oplus \begin{Bmatrix} (0,N-3) \\ (2,N-4) \end{Bmatrix} \begin{Bmatrix} N-3 = \text{ even} \\ N-3 = \text{ odd} \end{Bmatrix} \\
& \oplus (2N-12,0) \oplus (2N-16,2) \oplus \begin{Bmatrix} (0,N-6 \\ (2,N-7) \end{Bmatrix} \begin{Bmatrix} N-6 = \text{ even} \\ N-6 = \text{ odd} \end{Bmatrix} \\
& \oplus \cdots
\end{aligned}
\tag{2.15}
$$

Again here O(3) is not entirely converted from SU(3) transitioned to O(3) decomposable and another quantum number is required to categorise the states individually. Elliott is responsible for selecting the simplest extra quantum number in 1958. The equivalent number is known as $K$. The following procedure is then used to determine the levels of L that are present whenever it is represented$(\lambda,\mu)$:

$$
L = K,\ K+1,\ K+2, \ldots, K + \text{m } \{\lambda,\mu\} \tag{2.16}
$$

Where:

$$
\begin{aligned}
K = {} & \text{ integer } = \text{m } \{\lambda,\mu\},\ \text{m } \{\lambda,\mu\} - 2,\ \ldots,\ 1 \text{ or } 0; \\
& (\text{m } \{\lambda,\mu\} = \text{ odd or even })
\end{aligned}
\tag{2.17}
$$

unless as noted of $K = 0$ for which:

$$
\begin{aligned}
L = {} & \text{m } \{\lambda,\mu\},\ \text{m } \{\lambda,\mu\} - 2,\ \ldots, 1 \text{ or } 0; \\
& (\text{m } \{\lambda,\mu\} = \text{ odd or even })
\end{aligned}
\tag{2.18}
$$

The disadvantage of Elliott's premise is that it is not orthogonal. It is therefore practical to introduce another base, which we shall call the Vergados basis (Vergados 1968). The following may be done to create this basis using Elliott's basis. Let $K_1, K_2, \ldots, K_n$ occurrences of quantum numbers eliott in a particular representation$(\lambda,\mu)$ with $K_1 < K_2 < \cdots < K_n$. a fresh foundation is labeled by means of Quantum entanglement $\tilde{\chi}_1, \tilde{\chi}_2, \ldots, \tilde{\chi}_n$, with $\tilde{\chi}_1 < \tilde{\chi}_2 < \cdots < \tilde{\chi}_n$ and defined by:

$$
\begin{aligned}
&|(\lambda,\mu),\tilde{\chi}_1,L,M_L\rangle = |(\lambda,\mu),K_1,L,M_L\rangle_0 \\
&|(\lambda,\mu),\tilde{\chi}_2,L,M_L\rangle = x_{21}|(\lambda,\mu),K_1,L,M_L\rangle_0 + x_{22}|(\lambda,\mu),K_2,L,M_L\rangle_0 \\
&|(\lambda,\mu),\tilde{\chi}_i,L,M_L\rangle = \textstyle\sum_{j=1}^{i} x_{ij}\big|(\lambda,\mu),K_j,L,M_L\big\rangle
\end{aligned}
\tag{2.19}
$$

what states are $|(\lambda,\mu),K,L,M_L\rangle_0$ have a connection to Elliott states $|\ (\lambda,\mu),\mathrm{K},\mathrm{L},\mathrm{M_L}>$ according to phase convention.

$$
|(\lambda,\mu),K,L,M_L\rangle_0 = i^{\lambda+2\mu}|(\lambda,\mu),K,L,M_L\rangle \tag{2.20}
$$

And the coefficients $x_{ij}$ are attained by the stipulation:

$$
\big\langle(\lambda,\mu),\tilde{\chi}_i,L,M_L\ |\ (\lambda,\mu),\tilde{\chi}_j,L,M_L\big\rangle = \delta_{ij} \tag{2.21}
$$

Chain II is classified completely as follows:

$$
\mathrm{II}{:}\mathrm{U}(6) \supset \mathrm{SU}(3) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \tag{2.22}
$$

We note here in passing that, since the missing label $K$ can be chosen there are countless ways. there are several other bases, in addition to equation 2.22, that have been considered in detail. Van Isacker and Lipas in 1985 have studied recently some consequences of using a basis different from equation 2.22.

### 2.1.3 Chain III

The labels required to categorise the representations of the group in this chain are (Arima and Iachello 1979).

$$
\begin{aligned}
&\mathrm{U}(6) \quad [N,0,0,0,0,0] \equiv N \\
&\mathrm{O}(6) \quad (\sigma,0,0,0,0] \equiv \sigma \\
&\mathrm{O}(5) \quad (\tau,0) \equiv \tau \\
&\mathrm{O}(3) \quad L \\
&\mathrm{O}(2) \quad M_L
\end{aligned}
\tag{2.23}
$$

The values of the quantum number $\sigma$ contained in a given representation $N$ of $U(6)$ are:

$$\sigma = N,\ N-2, \ldots, 1 \text{ or } 0,\ (N = \text{ odd or even} \tag{2.24}$$

The values regarding the Quantum entanglement $\tau$ contained in a given representation $\sigma$ of $O(6)$ are:

$$\tau = \sigma,\ \sigma-1, \ldots, 1, \tag{2.25}$$

The representations of $U(6), O(6)$ and $O(5)$ are all totally symmetric. Once moreSince the transition It is not totally reducible from O(5) to O(3), a further quantum number is required. This quantum number will be called $v_\Delta$. The values of $L$ are then obtained by partitioning $\tau$ as:

$$\tau = 3v_\Delta + \lambda,\ v_\Delta = 0,1, \ldots, \tag{2.26}$$

and taking.

$$L = \lambda,\ \lambda+1,\ \ldots,\ 2\lambda-2,\ 2\lambda \tag{2.27}$$

(Note: 2λ-1 is absent!) It can be dealt with in a similar way by introducing a new quantum number $\tilde{v}_{\Delta_i}$, and requiring that:

$$\left\langle \sigma, \tau, \tilde{v}_{\Delta_i}, L, M_L \mid \sigma, \tau, \tilde{v}_{\Delta_j}, L, M_L \right\rangle = \delta_{ij} \tag{2.28}$$

The algorithm discussed above gives the values shown in equation (2.29). Complete categorization scheme for chain III is thus:

$$\text{III:}U(6) \supset\ O(6) \supset\ O(5) \supset\ O(3) \supset\ O(2) \tag{2.29}$$

## 2.2 Critical points E(5) and X(5)

The crucial point is represented by X (5) on the path where the change from the axial rotor to the spherical vibrator takes place. a phase transition of the first order is observed when the potential is minimum (min) ($\gamma = 0°$). Phase transition of the first order exhibits distorted and spherical phases together, and the ordering parameter is discontinuous. It has two minimum values. The phase transition's crucial moment happens when the potential passes from one minimum to another (Casten 2006).

As seen in Figure 2.1a. For a phase transition of the first order, coexistence appears as a bent minimal excited arrangement (curve2). In case there are more valence nucleons, the energy begins to decrease and a deformed equilibrium state (curve 4) occurs. It shows the critical point (curve 3) where the minimums degenerate and is likened the possibilities of the endless square well. The point of degeneration (curve3) of coexisting shapes is a critical point. The deformation changes intermittently from zero to infinity at the crucial point (curve 3). Then (curve 4), the core deforms (Casten 2006).

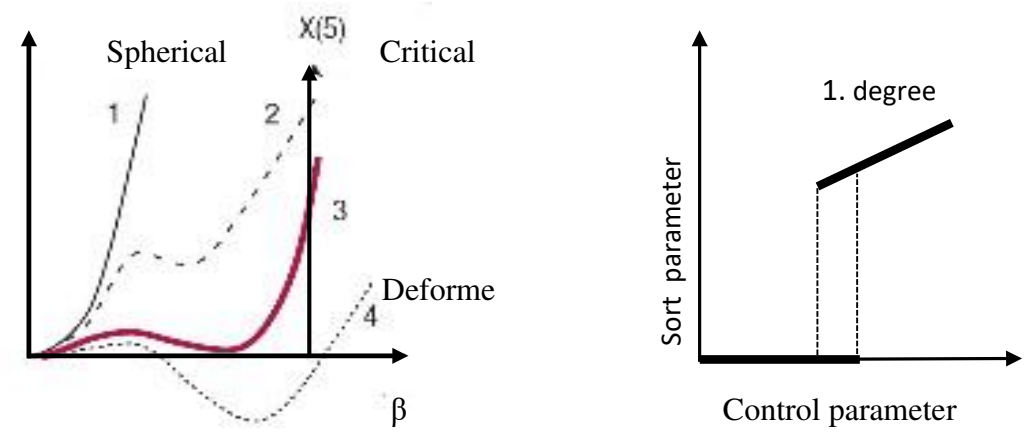


**Figure 2.1: a.** At the first-order phase transition's critical point, energy surfaces (Casten 2009). **b**. Phase transition of the first order (Casten 2007).

According to the Ehrenfest condition, a phase change of the first order occurs when it matters most because The order by control parameter's first derivative is discontinuous. There is a barrier where the phase transition takes place. As a result, the transition appears suddenly. This situation can be seen in Figure 2.1b.

In Figure 2.1a At the critical point, the curves provide typical potential energy surfaces. In general, It cannot be resolved the Analytical Schrödinger equation for these potentials. With this; Iachello (2001) offered a simplification that yields logical conclusions (Casten 2009).

The differential equation of the Bohr Hamiltonians can be analytically solved to get the structure of nuclei with X(5) dynamic symmetry. Bohr Hamiltonian equation does not support the exact answers proposed by Bohr in 1952. The answer to the dynamic symmetry problem for X(5) is as follows.

## 2.3 Collective Model

Only Immediately following the j-j-coupling shell model was first proposed in 1949 (Mayer 1949; Haxel and Suess 1949), there were collective (cooperative) effects once again incorporated into the representation of the nucleus to resolve the central problem: how the massive quadrupole moments are explained (Rainwater 1950). Since then, the development of the shell model and the scientific study of collective effects have advanced almost concurrently. One of the goals of nuclear dynamics is to combine the two facets of nuclear dynamics, single particle and collective behaviour. One of the goals of nuclear dynamics is to combine the two facets of nuclear dynamics, single particle and collective behavior field's most fascinating current topics. Collective aspects have dominated nuclear physics thought ever since Bohr and Kalckar proposed the liquid drop concept in 1937 and showed the ramifications of this and related ideas for nuclear processes in particular. In 1951, the "collective model" reappeared, although not in this form. Since then, it has become obvious that in moderate energy nuclear reactions, the redistribution of excitation energy is a considerably slower process than one had previously believed, necessitating a considerable revision of the original notion of collective motion owing to extremely strongly coupled nucleons (Feshbach et al. 1953; Friedman and Weisskopf 1955). This is unmistakable evidence up to excitation energy of ten to 15 Mev, the concept of individual nuclear "orbits" (single particle quantum states), is a valid one. This is in addition to the plethora the shell model's elegant summary of scientific evidence (See references (Flowers 1952) through (Feenberg 1955) for assessments of the shell model.) Therefore, if the "collective aspect" is restored today, it does not mean that all the evidence pointing to independent particle motion should be questioned once more. The collective phenomena that are being studied here are of a far more delicate sort, and they are most importantly closely related to the independent particle element of nuclear structure (Hill and Wheeler 1953).

## 2.4 Bohr Hamiltonian

Arima and Iachello proposed a nuclear model in 1974 (Iachello and Arima 1974) that may represent the properties of the positive parity and even-even nuclei's energy levels. As determined by the outer pairs of nucleons, of light, medium, and heavy mass numbers.

They were known as the first model of interacting bosons IBM-1 because they were the sealed shell, which was treated as a boson (Otsuka et al.1978) such that it did not take these bosons' degrees of freedom into account. Bosons were given degrees of freedom in order to create this model. The nuclei's new nuclear characteristics were disclosed as a result of this alteration, which led to the creation of the second interplay of bosons, or IBM-2 (Casten 1988).

bosons Proton $(s_\pi, d_\pi)$ likewise bosons neutron $(s_v, d_v)$ are not distinguished first interacting boson model. The total of ($N_\pi$ neutron bosons and $N_v$ proton bosons) The entire quantity of bosons (N) is produced as pairs of particles. The number of pairs of particles outside the closed shell for every nucleus (N) is equal to the number of bosons. which is a quantity that is entirely conserved and constant. Nevertheless, if it is greater compared to half the shell, then $(N_\pi)$and $(N_v)$ are considered as the number of holes Pairs (Arima and Iachello 1978).

The generic After the creation effects, formula for the Hamiltonian effect within this system is given $(s^+, d^+)$ and the repercussions of annihilation Since we learn interaction between the s and d bosons in this paradigm, (s, d) (Annihilation Operators) are defined (Iachello and Arima 1974).

We first discuss Euler angles and collective coordinates as the basis for the collective mode of the wave function and the spectrum are described by the Bohr Hamiltonian.

$$q_1 = \theta_1;\ q_2 = \theta_2; q_3 = \theta_3;\ q_4 = \beta; q_5 = \gamma \tag{2.30}$$

The Bohr Hamiltonian's kinetic energy is provided by:

$$T = \frac{B}{2}\left(\frac{ds}{dt}\right)^2 \tag{2.31}$$

is expressed as. Square of displacement in equation (2.32) $ds^2 = g_{ij}dq_i dq_j$ The Hamiltonian is obtained quantum mechanically by the Pauli-Podolsky rule (Podolsky 1928).

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g^{ij}}\,\partial_j\Phi \tag{2.32}$$

The symmetric matrix in equation (2.32). is expressed as in equation (2.33):

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix} \tag{2.33}$$

It is the determinant and inverse matrix of (Sitenko and Tartakovski 1975).

$$g = \frac{J_1 J_2 J_3}{B^3}\beta^2 \sin^2\,\theta_2 = 4\beta^8 \sin^2\,3\gamma \sin^2\,\theta_2$$

(2.34)

where the moments of inertia are shown in equation (2.35).

$$J_k = 4B\beta^2 \sin^2\left(\gamma - k\frac{2\pi}{3}\right) \tag{2.35}$$

Finally, from equation (2.32), the Bohr-Hamilton equation in equation (2.36) is obtained of any potential of the collective variables

$$H = -\frac{\hbar^2}{2B_m}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin\,3\gamma}\frac{\partial}{\partial\gamma}\sin\,3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2\pi}{3}k\right)}\right] + V(\beta,\gamma) \tag{2.36}$$

Here $B_m$ , is the mass parameter. Principal deformation coordinates β (deformation measuring coordinates the deviation from the spherical shape), γ (a metric for axial symmetry deviance) and The Bohr Hamiltonian is five-dimensional, with $\hat{Q}(k = 1,2,3)$ components in the actual structure expressed in terms of angular momentum.

$$\widehat{Q}_1 = \widehat{Q}_x = -i\left(-\frac{\cos\theta_3}{\sin\theta_2}\frac{\partial}{\partial\theta_1} + \sin\theta_3\frac{\partial}{\partial\theta_2} + \cot\theta_2\cos\theta_3\frac{\partial}{\partial\theta_3}\right) \quad (2.37)$$

$$\widehat{Q}_2 = \widehat{Q}_y = -i\left(-\frac{\sin\theta_3}{\sin\theta_2}\frac{\partial}{\partial\theta_1} + \cos\theta_3\frac{\partial}{\partial\theta_2} - \cot\theta_2\sin\theta_3\frac{\partial}{\partial\theta_3}\right)$$

(2.38)

$$\widehat{Q}_3 = \widehat{Q}_z = -i\frac{\partial}{\partial\theta_3} \quad (2.39)$$

individual a wave's behavior:

$$\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi^L_{M,K}(\gamma,\theta_i) \quad (2.40)$$

$\Phi^L_{M,K}$, to Wigner functions $D^L_{M,K}(\theta_i)$ It is defined as in .

$$\Phi^{L,\tau}_{M,K}(\gamma,\theta_i) = \sqrt{\frac{2I+1}{16\pi^2(1+\delta_{K,0})}}g^{L,\tau}_K(\gamma)\left[D^L_{M,K}(\theta_i) + (-1)^L D^L_{M,-K}(\theta_i)\right] \quad (2.41)$$

Here $\theta_i(i = 1,2,3)$ the Euler angle $D^L_{M,K}(\theta_i)$ Wigner function of Euler angle, and Langular, Angular momentum's quantum number is L, M and K are forecasts for the laboratory constant The body constant on the z axis on the z' axis, and the quantum number L, respectively (see Figure 2.2).

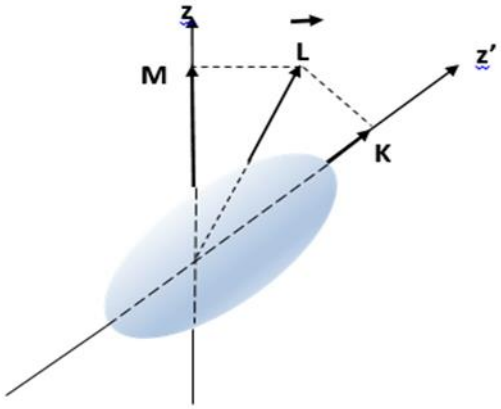


**Figure 2.2:** Total angular momentum vector projections on the laboratory fixed z-axis and the body fixed z'-axis.

condition for normalisation and volume component;

$$1 = \int_{\beta=0}^{\infty} \int_{\gamma=0}^{\pi/3} \int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{\pi} \int_{\theta_3=0}^{2\pi} \Psi^*(\beta,\gamma,\theta_1,\theta_2,\theta_3)\Psi(\beta,\gamma,\theta_1,\theta_2,\theta_3)\mathrm{dv} \qquad (2.42)$$
$$\mathrm{dv} = \beta^4 \mathrm{d}\beta|\sin\ 3\gamma|\mathrm{d}\gamma\mathrm{d}\theta_1 \sin\ \theta_2 \mathrm{d}\theta_2\mathrm{d}\theta_3$$

## 3. KRATZER POTENTIAL

There is one Kratzer potential of the most utilised possible molecular physics and chemistry models (Kratzer, 1920), which bears B. Adolf Kratzer's name. The model potential allows us to use analytical and computational techniques to characterize molecular structures and interactions. These techniques have been developed for investigating molecular systems can be anything from a single molecule (or a cluster of molecules interacting) to enormous material assemblages. They work in the computational and materials sciences domains. However, the advancement of research in modern science requires computational techniques like both using a priori and semi-empirical techniques that have complementing benefits (Herzberg, 1950).

The most basic computations can be done manually, but for any substantially complex system that requires molecular modeling, computers are inevitably needed. the level of atoms depiction the least complex level of the molecular systems is a typical aspect of molecular modeling methodologies (or an atomic cluster). In contrast, electrons are explicitly taken into account in calculations involving chemistry quantum, also referred to as computations for electrical structure. Molecular modelling has the benefit of making the system simpler, enabling simulations to take into account There are many of particles (atoms). Assume The Kratzer potential is a model potential that we have , which looks like this:

$$\mathrm{V(r)} = \mathrm{A} - \frac{\mathrm{B}}{\mathrm{r}} + \frac{\mathrm{C}}{\mathrm{r}^2} \qquad (3.1)$$

where are constants associated to the Kratzer potential by the three conditions A, B, and C. The variables A and C will be set to 0, i.e., $\mathrm{A} = \mathrm{zero}$ and $\mathrm{C} = \mathrm{zero}$, equation.(3.1) may be represented by a Coulomb possibility $\mathrm{V(r)} = -\mathrm{Ze'^2/r^2}$, where $\mathrm{B} = \mathrm{Ze'^2}$ and $\mathrm{e'} = \mathrm{e}/\sqrt{4\pi\varepsilon_0}$. The preceding part has already provided the Coulomb potential solution inside the SE framework. Coulometric potential is therefore a particular variation regarding the alleged Kratzer potential, it being possible to say. If the potential's

specifications are changed, $A = D_e, B = 2D_e r_e$ and $= D_e r_e^2$, equation.In (3.1), It is talked about the altered Kratzer potential, that is, $V(r) = D_e\left((r - r_e)/r\right)^2$ (Berkdemir *et al* .2006). The vertical distance between the dissociation limit and the dissociation energy and the interatomic separation equilibrium's lowest point on the potential curve, or $D_e$. $r = r_e$. If the interatomic distance causes possible flattening of the curve, i.e. $r \to \infty$, Its name is the maximum dissociation. The potential curve converges to zero, or this limit $V(\infty) = 0$. The energy of dissociation is so described $V(r_e) - V(\infty) = -D_e$. A relevant explanation of the term "modified" would be helpful. Include the "modified"; it is not "amazing".

Because the Kratzer-Fues are revealed by the altered Kratzer potential forming, it is into the Kratzer potential $A = 0$, i.e., $V(r) = D_e\left[\left((r - r_e)/r\right)^2 - 1\right]$, which shifts by an amount equal to $D_e$ (Fues 1926; Pliva 1999).

Let's attempt to resolve the SE using the possibility provided by equation.(3.1). replacement of the potential $(r) = A - B/r + C/r^2$ .

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \frac{2\mu}{\hbar^2}\left(E - \left[A - \frac{B}{r} + \frac{C}{r^2}\right] - \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right)R(r) = 0 \tag{3.2}$$

We may modify the equation above as follows to create further arrangements;

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \frac{1}{r^2}\left[\frac{2\mu(E-A)}{\hbar^2}r^2 + \frac{2\mu B}{\hbar^2}r - \left(\frac{2\mu C}{\hbar^2} + \ell(\ell+1)\right)\right]R(r) = 0 \tag{3.3}$$

It is useful to provide arbitrary parameters for the purpose of simplicity;

$$\begin{aligned} a &= -\frac{2\mu(E-A)}{\hbar^2} \\ \beta &= \frac{2\mu B}{\hbar^2} \\ \gamma &= \frac{2\mu C}{\hbar^2} + \ell(\ell+1) \end{aligned} \tag{3.4}$$

with $\alpha > 0$ suggests that, assuming that we are, we're taking care of solutions for bound states of energy $|E| < A, \beta > 0$ and $\gamma > 0$ more specifically, from equations. (3.3) and (3.4) the following:

$$\frac{d^2\psi(s)}{ds^2}+\frac{2}{s}\frac{d\psi(s)}{ds}+\frac{1}{s^2}(-\alpha s^2+\beta s-\gamma)\psi(s)=0 \tag{3.5}$$

where is characterised through the practical $R(r)\equiv\psi(s)$ the variable $r\to s$ . Application required the NU method, Comparing is crucial equation.(3.5) using the provided differential equation. The relevant polynomials are revealed by a straightforward comparison $\tilde{\tau}(s),\sigma(s)$ and $\tilde{\sigma}(s)$ .

$$\begin{aligned}\tilde{\tau} &= 2\\ \sigma(s) &= s\\ \tilde{\sigma} &= -\alpha s^2+\beta s-\gamma\end{aligned} \tag{3.6}$$

As a result, no more computations are required to get the solution for energy in a constrained state,

$$\alpha=\frac{\beta^2}{(1+2n+\sqrt{1+4\gamma})^2}, \tag{3.7}$$

and maintaining arbitrary parameter values $\alpha,\beta$ and $\gamma$ given by equation.(3.4) As we see it

$$-\frac{2\mu(E-A)}{\hbar^2}=\frac{\left(\frac{2\mu B}{\hbar^2}\right)^2}{\left(1+2n+\sqrt{1+4\left(\frac{2\mu C}{\hbar^2}+\ell(\ell+1)\right)}\right)^2} \tag{3.8}$$

$$E=A-\frac{\hbar^2}{2\mu}\left[\left(\frac{2\mu B}{\hbar^2}\right)^2\left(1+2n+\sqrt{1+4\left(\frac{2\mu C}{\hbar^2}+\ell(\ell+1)\right)}\right)^{-2}\right] \tag{3.9}$$

$$E=A-\frac{\frac{\mu B^2}{2\hbar^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{2\mu C}{\hbar^2}+\left(\ell+\frac{1}{2}\right)^2}\right)^2} \tag{3.10}$$

## 3.1 The Nikiforov-Uvarov method

The NU (Nikiforov-Uvarov) approach relies regarding how to resolve hypergeometric second-order differential equations utilising certain orthogonal functions (Szego 1939). Using the correct coordinate transformation, the Schrödinger equation in spherical coordinates or a Schrödinger-like equation is transformed into a generalised hypergeometric equation for a given potential energy. To find precise or particular answers, it can be systematically solved. in line with (Nikiforov and Uvarov 1988), The following form contains the key equations that closely relate to this approach.

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{3.11}$$

σs is a first-degree polynomial, σ ˜s is a function of the hypergeometric type, and τ ˜s and ψs are both at most second-degree polynomials.

Taken together, $\psi(s) = \phi(s)y(s)$ and selecting the proper function $\phi(s)$, equation.(3.11) is shaped into a form that is understandable;

$$y''(s) + \left(2\frac{\phi'(s)}{\phi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)}\right)y'(s) + \left(\frac{\phi''(s)}{\phi(s)} + \frac{\phi'(s)}{\phi(s)}\frac{\tilde{\tau}(s)}{\sigma(s)} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\right)y(s) = 0 \tag{3.12}$$

An indicator of y' (s) is calculated as $\tau(s)/\sigma(s)$, where $\tau(s)$ is a polynomial with a maximum degree of one, i.e.,

$$2\frac{\phi'(s)}{\phi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)} = \frac{\tau(s)}{\sigma(s)} \tag{3.13}$$

therefore, the form that is obtained most frequently is as follows:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{3.14}$$

where

$$\pi(s) = \frac{1}{2}[\tau(s) - \tilde{\tau}(s)] \tag{3.15}$$

The most helpful example of an equation.(3.15) is

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{3.16}$$

A polynomial makes up the new parameter s with a maximum one degree. Aside from being the word $\phi''(s)/\phi(s)$ This may be seen in equation's coefficient for y(s).It is organised as follows in (3.12)

$$\frac{\phi''(s)}{\phi(s)} = \left(\frac{\phi'(s)}{\phi(s)}\right)' + \left(\frac{\phi'(s)}{\phi(s)}\right)^2 = \left(\frac{\pi(s)}{\sigma(s)}\right)' + \left(\frac{\pi(s)}{\sigma(s)}\right)^2 \tag{3.17}$$

Equation's equality may be used to determine the percentage of y(s) in this instance is changed towards a more appropriate form.(3.14);

$$\frac{\phi''(s)}{\phi(s)} + \frac{\phi'(s)}{\phi(s)}\frac{\tilde{\tau}(s)}{\sigma(s)} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} = \frac{\bar{\sigma}(s)}{\sigma^2(s)} \tag{3.18}$$

where

$$\sigma(s) = \tilde{\sigma}(s) + \pi^2(s) + \pi(s)[\tilde{\tau}(s) - \sigma'(s)] + \pi'(s)\sigma(s) \tag{3.19}$$

replacement of the right-hand sides of incorporating equations (3.13) and (3.18) into equation (3.12),The following is how a hypergeometric equation is obtained

$$y''(s) + \frac{\tau(s)}{\sigma(s)}y'(s) + \frac{\sigma(s)}{\sigma^2(s)}y(s) = 0 \tag{3.20}$$

The functional form of the equation results from the algebraic operations indicated above.(3.11) is systematically safeguarded. the multinomial if $\bar{\sigma}(s)$ in equation.(3.20) is divided by $\sigma(s)$, i.e.,

$$\bar{\sigma}(s) = \lambda\sigma(s) \tag{3.21}$$

where a constant is $\lambda$ , equation.(3.20) is simplified to a hypergeometric equation

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{3.22}$$

Consequently, a function of hypergeometric-type is offered as its solution. Equation (3.19) and equation (3.21) are compared in order to derive the polynomial πs, and the resulting This is how the quadratic equation for s is found,

$$\pi^2(s) + \pi(s)[\tilde{\tau}(s) - \sigma'(s)] + \tilde{\sigma}(s) - k\sigma(s) = 0 \quad (3.23)$$

where

$$k = \lambda - \pi'(s) \quad (3.24)$$

The following equivalence results from solving this quadratic equation for πs

$$\pi(s) = \frac{\sigma'(s)-\tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \quad (3.25)$$

You must explicitly include the argument k following the square root symbol understood in order to determine the solutions that could be found based on plus and minus symbols in equation (3.25). A polynomial's square must fit in the equation under the square root sign to satisfy this condition. as (s) is a degree-based polynomial of no more than one. In this instance, the constant k has a quadratic equation accessible. The constant k is plainly established by setting this quadratic's discriminant to zero. π(s) is a polynomial, and derived from equation after knowing k.(3.25), then using equation, Also acquired are τ(s) and λ (3.15) and equation.(3.24) respectively.

a recurring trend that has applied to generalise equation solutions (3.22) is to demonstrate that all functions with a hypergeometric type also have a hypergeometric type in their derivatives. For this purpose, equation.(3.22) by utilising the representation to distinguish $v_1(s) = y'(s)$

$$\sigma(s)v_1''(s) + \tau_1(s)v_1'(s) + \mu_1 v_1(s) = 0 \quad (3.26)$$

Where $\tau_1(s) = \tau(s) + \sigma'(s)$ and $\mu_1 = \lambda + \tau'(s).\tau_1(s)$ is μ 1 a parameter that has no connection to the variable s, and is a polynomial with a maximum degree of one. Equation 3.26 is obviously a hypergeometric equation. by using $v_2(s) = y''(s)$ The second derivative of equation serves as a new representation.(3.22) becomes

$$\sigma(s)v_2''(s) + \tau_2(s)v_2'(s) + \mu_2 v_2(s) = 0 \quad (3.27)$$

Where

$$\tau_2(s) = \tau_1(s) + \sigma'(s) = \tau(s) + 2\sigma'(s) \quad (3.28)$$

$$\mu_2 = \mu_1 + \tau_1'(s) = \lambda + 2\tau'(s) + \sigma''(s) \quad (3.29)$$

Similar to this, a family of specific solutions may be created for a hypergeometric-type equation.(3.22) by using $v_n(s) = y^{(n)}(s)$;

$$\sigma(s)v_n''(s) + \tau_n(s)v_n'(s) + \mu_n v_n(s) = 0 \quad (3.30)$$

Furthermore shown here are the common recurrence relationships for $\tau_n(s)$ and $\mu_n$ are located, accordingly, the following,

$$\tau_n(s) = \tau(s) + n\sigma'(s) \quad (3.31)$$

$$\mu_n = \lambda + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) \quad (3.32)$$

When $\mu_n = 0$, equation.(3.32) the following occurs

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s),\ (n = 0,1,2,\dots) \quad (3.33)$$

## 4. SOLUTION

The general Collective Model Hamiltonian is given by

$$H = -\frac{h^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{\hat{Q}_k^2}{\sin\left(\gamma-\frac{2\pi k}{3}\right)^2}\right] + V(\beta,\gamma) \quad (4.1)$$

Where $\hat{Q}_k$ (k=1,1,3) , B is the mass parameter, and are the angular momentum components in the intrinsic frame where $\beta$ and $\gamma$ are the typical collective coordinates for in the appropriate Schrodinger equation solution.

$$H\psi=E\psi \quad (4.2)$$

with the for $m\psi = \psi(\beta,\gamma,\theta_i) = \Phi_k^L(\beta,\gamma).D_{M,k}^L(\theta_i)$, where $\theta_i(i = 1,2,3)$ what angles are Euler and $D(\theta_i)$ their Wigner functions in notation L are the angular momentum's eigenvalues. During M and K are, respectively, its eigenvalues, which are arguer momentum body-fixed z' axis and fixed z axis projections in the laboratory.

For the $X(5)$ critical point, The potential is at least around that $\gamma = 0$ and Equation (4.1)'s angular momentum term is expressed as

$$\sum_{k=1}^{3}\frac{\widehat{Q_k}^2}{\sin^2\left(\gamma-\frac{2\pi k}{3}\right)} \simeq \frac{4}{3}\left(\hat{Q}_1^2+\hat{Q}_2^2+\hat{Q}_3^2\right) + \hat{Q}_3^2\left(\frac{1}{\sin^2\gamma^2}-\frac{4}{3}\right) \quad (4.3)$$

If we introduce reduced energy $\varepsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $U(\beta,\gamma) = \frac{2B}{\hbar^2}V(\beta,\gamma)$, then we can separate the Schrodinger equation by assuming the reduced potentials as the form of $U(\beta,\gamma) = U(\beta) + \frac{1}{\beta^2}U(\gamma)$. Finally if we write the Wave feature

$$\psi(\beta,\gamma,\theta_i) = \xi_L(\beta)\eta_k(\gamma)D^L_{Mk}(\theta_i) \tag{4.4}$$

We obtain two schrodinger equation. The first one only depends on $\beta$, and the second depends on $\gamma$.

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{L(L+1)}{3\beta^2} + u(\beta) + \frac{\lambda}{\beta^2}\right]\xi_L(\beta) = \in \xi_L(\beta) \ldots (s) \tag{4.5}$$

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{K^2}{4}\left(\frac{1}{\sin^2\gamma} - \frac{4}{3}\right) + u(\gamma)\right]\eta_k(\gamma) = \lambda\eta_k(\gamma) \tag{4.6}$$

As seen from equation (4.5), there is a $\lambda$ parameter and this parameter is obtained from the solution of equation (4.6). So first we should solve equation (4.6). Since The potential of the form can be used if the potential in the $\gamma$-part has a minimum value of $\gamma \simeq 0$.

$$u(\gamma) = (3c)^2\gamma^2 \tag{4.7}$$

And expending equation (4.6) in powers of $\gamma$, we get

$$\left[-\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} + \frac{k^2}{4\gamma^2} - \frac{k^2}{3} + (3c)^2\gamma^2\right]\eta_k(\gamma) = \lambda\eta_k(\gamma) \tag{4.8}$$

We define new penmater $\epsilon_\gamma = \lambda + \frac{k^2}{3}$ the equation (4.8) takes the from of

$$\left[-\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} + \frac{k^2}{4\gamma^2} + (3c)^2\gamma^2\right]\eta_k(\gamma) = \epsilon_\gamma\,\eta_k(\gamma) \tag{4.9}$$

The answer is then provided using Laguerre polynomials.

$$n_{n_{\gamma,|k|}}(\gamma) = C_{n_{\gamma,|k|}} \gamma^{|k/2|} e^{-\frac{3c\gamma^2}{2}} L_{\tilde{n}}^{|k/2|}(3c\gamma^2) \quad (4.10)$$

where $\tilde{n} = \frac{n\gamma - |k/2|}{2}$ Then.

$$\in_{\gamma} = (3C)\left(n_{\gamma+1}\right) \text{ with } C = 2c, n_{\gamma} = 0,1,2, \ldots \quad (4.11)$$

The values of $n_{\gamma}$ and k are as follows

$$n_{\gamma} = 0 \ , \ k = 0 \ ; \ n_{\gamma} = 1 \ , \ k = \pm 2 \ , \ n_{\gamma} = 2 \ , \ k = 0, \pm 4; \ldots \quad (4.12)$$

$$n_{\gamma} = 3 \ , \quad k = \pm 2 \ , \pm 6 \ ; \ n_{\gamma} = 4 \ , \quad k = 0 \ , \pm 4 \ , \pm 8 \ ; \ldots \ldots$$

Now, we can start solution of equation (4.5). Defending on the selection of reduced potential $u(\beta)$ in equation (4.5) we get different solution Let us We calculate the lowered potential's Kratzer potential.

$$u_k(\beta) = -2D\left(\frac{\beta_0}{\beta} - \frac{{\beta_0}^2}{\beta^2}\right) \quad (4.13)$$

D being the depth and $\beta_0$ is the position of the prospective lowest level, as given in figure

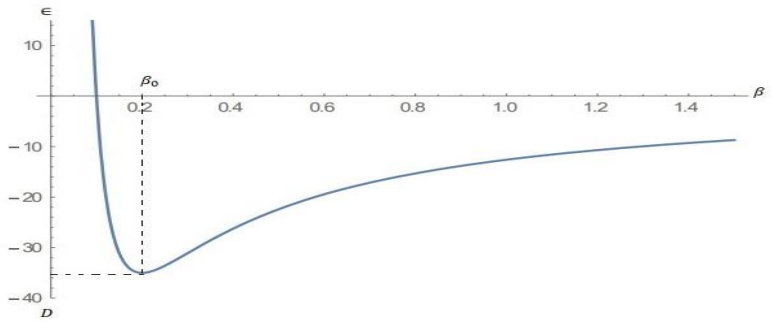


**Figure 4.1:** Figure shows Where D is the depth and β is the position of the minimum of the potential

Sometimes one can use the different form like this:

$$U_k = -2D\left(\frac{\beta_0}{\beta} - \frac{\beta_0^2}{\beta^2}\right) = -\frac{2D\beta_0}{\beta} + \frac{2D\beta_0^2}{\beta^2} = -\frac{A}{\beta} + \frac{\beta}{\beta^2} \tag{4.14}$$

Now, $A = 2D\beta_0$ and $B = 2D\beta_0^2$ and this form is more simple. If we use this form in equation (4.5), we get.

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{L(L+1)}{3\beta^2} - \frac{A}{\beta} + \frac{B}{\beta^2} + \frac{\lambda}{\beta^2}\right]\xi_L(\beta) = \epsilon\,\xi_L(\beta) \tag{4.15}$$

Then

$$-\frac{d^2\xi_L(\beta)}{d\beta^2} - \frac{4}{\beta}\frac{d\xi_L(\beta)}{d\beta} + \left(\frac{L(L+1)}{3\beta^2} - \frac{A}{\beta} + \frac{\beta}{\beta^2} + \frac{\lambda}{\beta^2}\right)\xi_L(\beta) = \epsilon\,\xi_L(\beta) \tag{4.16}$$

or

$$\xi_L''(\beta) + \frac{4}{\beta}\xi_L'(\beta) + \left(\epsilon + \frac{A}{\beta} - \frac{L(L+1)}{3\beta^2} - \frac{\beta}{\beta^2} - \frac{\lambda}{\beta^2}\right)\xi_L(\beta) = 0 \tag{4.17}$$

If we define at new parameter $\alpha$ to be $\alpha = \frac{L(L+1)}{3} + B + \lambda$ we, can rewrite equation (4.17) as

$$\xi_L''(\beta) + \frac{4}{\beta}\xi_L'(\beta) + \left(\epsilon + \frac{A}{\beta} - \frac{\alpha}{\beta^2}\right)\xi_L(\beta) = 0 \tag{4.18}$$

Finally, to get the same form of Nikiforov-Uvarov Method, we reorganize the equation (4.18) to be.

$$\xi_L''(\beta) + \frac{4}{\beta}\xi_L'(\beta) + \left(\frac{\epsilon\beta^2 + A\beta - \alpha}{\beta^2}\right)\xi_L(\beta) = 0 \tag{4.19}$$

Now, Let us remember Nikiforov-Uvarov Method:

The following from gives the method's primary equation:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{4.20}$$

where polynomials $\sigma(s)$ and $\tilde{\sigma}(s)$ are involved with a maximum degree of two, where $\tilde{\tau}(s)$ is a polynomial of the first degree

By comparing equation (4.19) and equation (4.20) , we found

$$S \to \beta : \tilde{\tau}(\beta) = 4, \sigma(\beta) = \beta, \tilde{\sigma}(\beta) = \epsilon\beta^2 + A\beta - \alpha \tag{4.21}$$

As required by the method, $\tilde{\tau}(\beta) = 4$ which is at most first-degree; $\sigma(\beta) = \beta$ and $\tilde{\sigma}(\beta) = \epsilon\beta^2 + A\beta - \alpha$ which are, at most, polynomials of second degree.

According to N.U. method, $\pi(s)$ is defined as

$$\pi(s) = \frac{\sigma^{-1}(s)-\tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma^{-}(s)-\tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{4.22}$$

then our $\pi(\beta)$ is

$$\begin{aligned}\pi(\beta) &= \frac{\sigma'(\beta)-\tilde{\tau}(\beta)}{2} \pm \sqrt{\left(\frac{\sigma'(\beta)-\tilde{\tau}(\beta)}{2}\right)^2 - \tilde{\sigma}(\beta) + k\sigma(\beta)} \\ &= \frac{1-4}{2} \pm \sqrt{\left(\frac{1-4}{2}\right)^2 - (\in \beta^2 + A\beta - \alpha) + k\beta} \\ &= -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \in \beta^2 - A\beta + \alpha + k\beta}\end{aligned} \tag{4.23}$$

since $\in < 0$ for bound state, let us change it to be $-\in = \varepsilon$ then

$$\begin{aligned}\pi(\beta) &= \frac{-3}{2} \pm \sqrt{\varepsilon\beta^2 + (k-A)\beta + \left(\frac{9}{2} + \alpha\right)} \\ \pi(\beta) &= \frac{-3}{2} \pm \sqrt{\varepsilon}\sqrt{\beta^2 + \left(\frac{k-A)}{\varepsilon}\right)\beta + \left(\frac{9+4\alpha}{4\varepsilon}\right)}\end{aligned} \tag{4.24}$$

Equation (4.24), which has a quadratic equation in solving the square root symbol by putting the discriminate the value of this is 0, i.e. $\Delta = b^2 - 4ac = 0$. The new quadratic equation that results from this approach can be resolved for using k to find the two roots

$$\begin{aligned}&\beta^2 + \frac{k-A}{\varepsilon} + \beta + \frac{9+4\alpha}{4\varepsilon} = 0 \\ &\Delta = b^2 - 4ac = \left(\frac{k-A}{\varepsilon}\right)^2 - 4 \cdot 1\left(\frac{9+4\alpha}{4\varepsilon}\right) = 0 \\ &\left(\frac{k-A}{\varepsilon}\right)^2 = \frac{9+4\alpha}{\varepsilon} \Rightarrow \\ &k = A \pm \sqrt{9+4\alpha}\sqrt{\varepsilon}\end{aligned} \tag{4.25}$$

where $k_+ = A + \sqrt{9+4\alpha}\sqrt{\varepsilon}$ and $k_- = A - \sqrt{9+4\alpha}\sqrt{\varepsilon}$

Then we can rewrite equation (4.24) as follows:

$$\pi(\beta) = \frac{-3}{2} \pm \sqrt{\varepsilon}\sqrt{\left(\beta + \left(\frac{k-A}{2\varepsilon}\right)\right)^2} = \frac{-3}{2} \pm \sqrt{\varepsilon}\left(\beta + \frac{k-A}{2\varepsilon}\right) \quad (4.26)$$

In equation (4.26), if we use $k_+$ and $k_-$ values, we obtain four passible $\pi(\beta)$ values

$$\begin{aligned}
\pi_{++}(\beta) &= \frac{-3}{2} + \sqrt{\varepsilon}\left(\beta + \frac{k_+ - A}{2\varepsilon}\right) \\
\pi_{+-}(\beta) &= \frac{-3}{2} + \sqrt{\varepsilon}\left(\beta - \frac{k_- - A}{2\varepsilon}\right) \\
\pi_{-+}(\beta) &= \frac{-3}{2} - \sqrt{\varepsilon}\left(\beta - \frac{k_+ - A}{2\varepsilon}\right) \\
\pi_{--}(\beta) &= \frac{-3}{2} - \sqrt{\varepsilon}\left(\beta - \frac{k_- - A}{2\varepsilon}\right)
\end{aligned} \quad (4.27)$$

To obtain the bound state solution we have to use $\pi_{--}(\beta)$the fourth one.

$$\begin{aligned}
\pi_{--}(\beta) = \quad & \pi(\beta) = \frac{-3}{2} - \sqrt{\varepsilon}\left(\beta - \frac{k_- - A}{2\varepsilon}\right). \\
\pi(\beta) = \quad & \frac{-3}{2} - \sqrt{\varepsilon}\left(\beta + \frac{\sqrt{9+4\alpha}}{2\sqrt{\varepsilon}}\right)
\end{aligned} \quad (4.28)$$

According to N.U. method $\tau(S)$ is defined as

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \quad (4.29)$$

Applying equation (4.29) to our solution (remember $\tilde{\tau}(\beta) = 4$),

$$\begin{aligned}
&\tau(\beta) = \tilde{\tau}(\beta) + 2\pi(\beta) \\
&\tau(\beta) = 4 + 2.\left(\frac{-3}{2} - \sqrt{\varepsilon}\left(\beta + \frac{\sqrt{9+4\alpha}}{2\sqrt{\varepsilon}}\right)\right) \\
&\tau(\beta) = 1 - 2\sqrt{\varepsilon}\beta - \sqrt{9+4\alpha}
\end{aligned} \quad (4.30)$$

Again according to N.U. method, $\lambda = k_- + \pi'(s)$ and $\lambda_n = -n\tau'(s) - n(n-1) \cdot \sigma''(s)$. Applying these equations to our Solutions =

$$\lambda = k_{-} + \pi'(\beta) \text{ and } \lambda = A - \sqrt{\varepsilon}\sqrt{9 + 4\alpha} - \sqrt{\varepsilon} \tag{4.31}$$

and

$$\lambda n = -n\tau'(\beta) - n(n-1)\sigma''(\beta) \tag{4.32}$$
$$\lambda n = -n(-2\sqrt{\epsilon})$$

Then the equality of equation (4.31) and equation (4.32), we get:

$$\begin{aligned} &\lambda = \lambda n \\ &A - \sqrt{\varepsilon}\sqrt{9 + 4\alpha} - \sqrt{\varepsilon} = 2\sqrt{\varepsilon}n \\ &A = \sqrt{\varepsilon}(2n + 1 + \sqrt{9 + 4\alpha}) \\ &\sqrt{\varepsilon} = \frac{A}{2n+1+\sqrt{9+4\alpha}} \\ &\varepsilon = \frac{A^2}{(2n+1+\sqrt{9+4\alpha})^2} \end{aligned} \tag{4.33}$$

we defined $\alpha$ to be $\alpha = \frac{L(L+1)}{3} + B + \lambda$ and using this in equation(4.33) we get:

$$\varepsilon = \frac{A^2}{\left(2n+1+\sqrt{9+\frac{4L(L+1)}{3}+4B+4\lambda}\right)^2} \tag{4.34}$$

To get the more simplified version =

$$\begin{aligned} &\mathcal{E} = \frac{A^2}{4\left(n+\frac{1}{2}+\sqrt{\frac{9}{4}+\frac{L(L+1)}{3}+B+\lambda}\right)^2} \\ &\varepsilon = \frac{A^2/4}{\left(n+\frac{1}{2}+\sqrt{\frac{L(L+1)}{3}+B+\lambda+\frac{9}{4}}\right)^2} \end{aligned} \tag{4.35}$$

## 5. RESULTS AND DISCUSSION

Equation (4.35) is the energy eigenvalue equation that gives the excitation energy spectra of nuclei having structures around X(5) critical point. To test the reliability of this result, we first selected Er isotopes. Then we entered their experimental energy data in Mathematica software. Changing B and C parameters in some range, we try to find best values of the free parameters. The quality factor of the fit is given by $\Delta$ values. From Table 5.1, one can see that the best result is obtained for $^{166}$Er isotope.

Finally, we have calculated the full excitation energy spectra of the Er isotopes and compared the experimental data. As seen from tables, some of the experimental data are missing. Our study will be a good source for experimental studies.

**Table 5.1** Hamiltonian parameters used to obtain the experimental spectrum for the selected isotopes

| Isotope | B | C | $\Delta$ |
|---|---|---|---|
| $^{160}_{68}$Er | 84 | 5.0 | 0.0620 |
| $^{162}_{68}$Er | 121 | 6.5 | 0.0844 |
| $^{164}_{68}$Er | 197 | 6.5 | 0.0709 |
| $^{166}_{68}$Er | 300 | 6.5 | 0.0418 |
| $^{168}_{68}$Er | 300 | 7.0 | 0.0797 |
| $^{170}_{68}$Er | 266 | 8.5 | 0.1025 |

**Table 5.2** Comparison of theoretical results with the experimental data (Firestone 1996) for ground state.

| Isotope | $E(4^+)$ | | $E(6^+)$ | | $E(8^+)$ | | $E(10^+)$ | |
|---|---|---|---|---|---|---|---|---|
| | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp |
| $^{160}_{68}\mathrm{E_r}$ | 3.194 | 3.099 | 6.295 | 6.086 | 9.955 | 9.772 | 13.850 | 14.000 |
| $^{162}_{68}\mathrm{E_r}$ | 3.232 | 3.230 | 6.479 | 6.533 | 10.460 | 10.750 | 14.880 | - |
| $^{164}_{68}\mathrm{E_r}$ | 3.266 | 3.276 | 6.648 | 6.722 | 10.940 | 11.210 | 15.900 | - |
| $^{166}_{68}\mathrm{E_r}$ | 3.287 | 3.289 | 6.755 | 6.776 | 11.250 | 11.310 | 16.590 | 16.750 |
| $^{168}_{68}\mathrm{E_r}$ | 3.266 | 3.309 | 6.648 | 6.876 | 10.940 | 11.630 | 15.900 | - |
| $^{170}_{68}\mathrm{E_r}$ | 3.283 | 3.307 | 6.733 | 6.873 | 11.190 | 11.600 | 16.450 | - |

**Table 5.3** Comparison of theoretical results with the experimental data (Firestone 1996) for Gamma-band (γ ).

| Isotope | $E(2^+)$ | | $E(3^+)$ | | $E(4^+)$ | | $E(5^+)$ | | $E(6^+)$ | | $E(7^+)$ | | $E(8^+)$ | | $E(9^+)$ | | $E(10^+)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp |
| $^{160}_{68}E_r$ | 6.94 | 6.79 | 7.70 | 7.84 | 8.68 | - | 9.84 | 10.47 | 11.16 | - | 12.60 | 13.86 | 14.13 | - | 15.72 | 17.84 | 17.35 | - |
| $^{162}_{68}E_r$ | 8.98 | 8.82 | 9.76 | 9.82 | 10.77 | 11.06 | 11.98 | 12.60 | 13.38 | - | 14.93 | - | 16.62 | - | 18.40 | - | 20.26 | - |
| $^{164}_{68}E_r$ | 9.33 | 9.41 | 10.18 | 10.36 | 11.28 | 11.58 | 12.62 | 13.10 | 14.19 | - | 15.96 | - | 17.91 | - | 20.00 | - | 22.23 | - |
| $^{166}_{68}E_r$ | 9.55 | 9.75 | 10.45 | 10.66 | 11.61 | 11.87 | 13.05 | 13.34 | 14.74 | 15.09 | 16.66 | - | 18.80 | - | 21.13 | - | 23.64 | 24.37 |
| $^{168}_{68}E_r$ | 9.33 | 10.29 | 10.18 | 11.22 | 11.28 | 12.46 | 12.62 | 14 | 14.19 | - | 15.96 | - | 17.91 | - | 20.00 | - | 22.23 | - |
| $^{170}_{68}E_r$ | 12.11 | 11.87 | 12.96 | 12.84 | 14.07 | - | 15.44 | - | 17.04 | - | 18.87 | - | 20.89 | - | 23.10 | - | 25.46 | - |

**Table 5.4** Comparison of theoretical results with the experimental data (Firestone 1996) for Beta-band (β).

| Isotope | $E(0^+)$ | | $E(2^+)$ | | $E(4^+)$ | | $E(6^+)$ | | $E(8^+)$ | | $E(10^+)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp | Theo | Exp |
| $^{160}_{68}\mathrm{E_r}$ | 8.96 | - | 9.73 | - | 11.41 | - | 13.80 | - | 16.65 | - | 19.71 | - |
| $^{162}_{68}\mathrm{E_r}$ | 10.79 | 10.65 | 11.59 | 11.48 | 13.36 | 13.42 | 15.96 | - | 19.15 | - | 22.73 | - |
| $^{164}_{68}\mathrm{E_r}$ | 13.58 | 13.63 | 14.40 | 14.38 | 16.28 | 16.08 | 19.09 | - | 22.66 | - | 26.81 | - |
| $^{166}_{68}\mathrm{E_r}$ | 16.68 | - | 17.53 | - | 19.49 | - | 22.45 | - | 26.31 | - | 30.90 | - |
| $^{168}_{68}\mathrm{E_r}$ | 13.58 | - | 14.40 | - | 16.28 | - | 19.09 | - | 22.66 | - | 26.81 | - |
| $^{170}_{68}\mathrm{E_r}$ | 15.89 | - | 16.74 | - | 18.67 | - | 21.61 | - | 25.40 | - | 29.89 | - |

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